GALOIS AND SIMPLE CURRENT SYMMETRIES IN CONFORMAL FIELD THEORY

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Chapter 1

Introduction

1.1 Conformal Field Theory in Physics and Mathematics

Conformal quantum field theory in two dimensions has stimulated many new developments in both mathematics and physics and has catalyzed much fruitful interaction between these two fields. There are various good reasons to study these theories, and we will list a few of them.

One motivation for physicists to investigate these theories comes from statistical mechanics: in the description of critical phenomena the renormalization 'group' has turned out to be a very powerful tool. Applying it in the infra-red direction, the key idea is to look at the system at a larger and larger scale, taking in a sense a poorer and poorer magnifying glass. Technically this is achieved by integrating out degrees of freedom belonging to a larger and larger length scale. This way one obtains a flow on the space of all (effective) theories. At the critical point fluctuations occur at all length scales, hence the trajectory of the critical point under the renormalization group contains a point which the renormalization transformations leave fixed. Evidently, at the fixed point the effective theory is covariant under rescaling; in fact, one can argue [17] that this is not only true for rescalings, but also for general conformal mappings: at this point the theory can be described by a conformal field theory. We will see that in the case of two dimensions, this leads to particularly powerful tools; in this thesis we will restrict ourselves to this case.

A second important motivation from physics is string theory: any vacuum configuration of the string corresponds to a conformal field theory [77]. In string theory, point particles are replaced by one-dimensional objects, *strings*. Evolving in time, the string sweeps out a two-dimensional surface, the *world sheet*, which is the analogue of the *world line* of a point particle. The coordinates of space-time (or some internal space) can then be interpreted as fields defined on this world sheet. Any action now inherits – at least at the classical level – not only symmetries from those of the target space, but symmetries also arise from the fact that the action should be invariant under reparametrizations of the world sheet. The latter include in particular conformal transformations, so that one ends up with a conformal field theory, defined on the world sheet.

Quantum field theories in two dimensions show various particularly interesting features. In four dimensions a quantum field theory can only describe particles with either bosonic or fermionic statistics: exchanging two particles introduces a phase of ± 1 to the wavefunction. In two dimensions the situation is more involved: the phase depends in general on the path along which the particles have been exchanged. Therefore, it is not the permutation group which governs the statistics but rather the braid group, leading to anyons which generalize the notion of bosons and fermions in higher dimensions. Braid group statistics [38,40] which is in particular realized in two-dimensional conformal field theories has been proposed e.g. as one clue to the understanding of phenomena like high T_C superconductivity or the fractional quantum Hall effect.

A slightly more abstract motivation for the study of conformal field theories is that they provide specific examples of quantum field theories which are by far more tractable than conventional quantum field theories are. Quantum field theory has turned out to be a key concept in many branches of physics, ranging from the standard model of elementary particle physics to applications in condensed matter physics, e.g. in the description of superconductivity.

Also for pure mathematics, quantum field theory in general, and conformal field theory in particular, has turned out to provide a most inspiring source for intuition. It has not only provided new insight in existing fields (a good example for this is mirror symmetry [144], which, for the quantum field theory, simply corresponds to two different conventions for the assignment of u(1) charges, but gives surprisingly deep insight into the problem of 'counting' curves on certain varieties), but has also provided new links between hitherto (nearly) unrelated areas of mathematics, involving e.g. C^*-algebras, their representation theory, infinite-dimensional Lie algebras, commutative algebra, the theory of modular forms, number theory, differential and algebraic geometry, singularities and catastrophe theory, link polynomials, just to name a few fields.

One other spin-off of conformal field theories are topological field theories which can be obtained e.g. by twisting certain $N = 2$ superconformal theories; topological quantum field theories have turned out to be a powerful tool to address various issues in algebraic topology and geometry. We will construct explicit examples for $N = 2$ superconformal theories in this thesis. In a different way, conformal field theories are also closely related to topological field theories in three dimensions, e.g. Chern–Simons theories which allow for the construction of link invariants and invariants of three-manifolds. Also, there are close connections to other types of quantum field theories which are of mathematical interest, e.g. Toda field theories, and to integrable systems.

We have been careful to call quantum field theory only a source for intuition; unfortunately the mathematical status of quantum field theory at present is far from being satisfactory. In fact, many fundamental aspects remain to be clarified: e.g. a rigorous definition of a path integral is still missing for many classes of quantum field theories, e.g. for gauge theories. In general, all non-perturbative features deserve a better understanding. We therefore feel that it is very important to have examples in which on can *calculate* many quantities at a non-perturbative level. One part of this thesis will therefore be devoted to a special construction of conformal field theories, the so-called coset construction, which allows for an exact description of many interesting conformal field theories.

We want to emphasize that looking at examples is not a luxury, or simply a remedy for physicists' lacking knowledge of higher mathematics. Experience has shown that this way sometimes also surprising new insight can be obtained; a good example for this phenomenon is again the discovery of mirror symmetry [144].

The rest of this introduction is organized as follows: we will explain some of the aspects of conformal field theory which are relevant for this thesis. Then we will present several concrete examples of conformal field theories which will be basic ingredients and fundamental examples: free bosons compactified on a circle, WZW theories, and coset conformal field theories. We conclude the introduction by giving an outline of the rest of this thesis.

1.2 Conformal field theory

1.2.1 The chiral symmetry algebra

A conformal field theory is by definition a (two-dimensional) field theory that is conformally invariant in the sense that the space of all fields carries a representation of the conformal group respectively of the conformal algebra. This should be compared e.g. to the definition of a Lorentz covariant theory in which the fields carry representations of the Lorentz group and in which, as a consequence, the excitations can be classified by representations of the relevant group respectively algebra.

At this point, one ought to define what 'fields' are. For a rigorous definition – which, however, is beyond the scope of this introduction – one should therefore first choose an axiomatic framework like e.g. the Wightman axioms [136], or the algebraic theory of superselection sectors [79]. However, each of these systems has to be slightly modified to be applied to two-dimensional conformal field theories, in the sense of the bootstrap approach of Belavin, Polyakov and Zamolodchikov [12], so we refrain from presenting a more careful analysis at this place and encourage the reader to think of 'fields' as operator valued distributions over (Euclidean) space-time. Assuming that space-time is compact, we are led to consider theories which are defined on two-dimensional compact Riemann surfaces. The operators themselves are linear mappings of some Hilbert space; the construction of these spaces in concrete examples will be addressed in this thesis.

Although we deal with quantum field theories we will never use path integrals. In fact, even if we disregard the problem of defining path integrals rigorously and decide to work at a purely formal level, it is not clear at all whether for all of the models we are going to discuss a path integral formulation does exist. The purely algebraic formulation of quantum field theories we are going to use may not directly appeal to geometric intuition; however, it has the crucial advantage that it allows for many exact calculations and that it therefore enables us to explore quantum field theories with non-perturbative methods.

In particular, we do not describe these theories by a Lagrangian density. In fact, there is no general argument known why any quantum field theory should be associated to an action or even be a 'quantization' of a classical field theory. In the case of coset conformal field theories there are several different proposals for Lagrangians and we are, at present, far from having reached a complete understanding. In practice, in the Lagrangian approach mostly results have been reproduced which had already been derived in the algebraic approach (see e.g. [143] where some of the results of [103] are rederived): this nicely demonstrates that the algebraic approach is superior in its computational power to the geometric approaches using Lagrangian densities. Many 'global' questions are especially difficult to address in a Lagrangian framework. One example for such global issues is modular invariance, which e.g. leads, as we will see below, to the phenomenon of field identification in coset conformal field theories; it is therefore not surprising that this effect has not been noticed in the description of these models as gauged sigma-models [64, 132].

Let us now have a closer look at conformal field theories in two dimensions; since the description of local quantum field theories which are also conformally invariant involves several subtleties [58, 107] we will use here for simplicity a setting that is tailored to the application of conformal field theory to string theory and statistical mechanics. Then we can assume that space-time is Euclidean and compact; we restrict ourselves for the time being to the case of a Riemann surface of genus 0. From complex analysis it is well known that the mappings of the (compactified) complex plane that preserve angles are precisely the holomorphic mappings. Given the real coordinates x^1, x^2 it is therefore natural to introduce complex coordinates $z, \bar{z}$ by $z, \bar{z} = x^1 \pm \mathrm{i} x^2$. All infinitesimal conformal transformations are then generated by mappings

$$z \to z + \epsilon_n \quad \text{where} \quad \epsilon_n = -z^{n+1}. \tag{1.2.1}$$

The corresponding generators on functions

$$l_n = -z^{n+1}\partial_z \quad \text{resp.} \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{1.2.2}$$

then span a Lie algebra with commutation relations

$$[l_n, l_m] = (n-m)\, l_{m+n}\,, \tag{1.2.3}$$

and analogously for $\bar{l}$. In the sequel, our strategy will be to consider z and $\bar{z}$ as two independent complex variables, much in the spirit of 'Wirtinger calculus', and to set $\bar{z}$ equal to the complex conjugate of z only at the end of our calculations.

It has been shown that the only anomaly the algebra (1.2.3) can develop in a quantum field theory can be described by a central element C, which by definition commutes with any other element of the algebra. This assertion, known as the Lüscher–Mack theorem [105], only assumes that the Wightman axioms hold, that the system is invariant under dilatations and that there exists a conserved symmetric energy-momentum tensor. The algebra which reflects the conformal symmetry in a two-dimensional quantum field theory is therefore the Virasoro algebra $\mathcal{V}ir$:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{C}{12}(n^3 - n)\delta_{n+m,0}\,, \qquad [C, L_n] = 0\,. \tag{1.2.4}$$

The same facts can also be described using the energy-momentum tensor: conformal invariance of the theory implies that it is traceless. It has a purely holomorphic zz-component $T(z)$ and a purely anti-holomorphic $\bar{z}\bar{z}$-component $\overline{T}(\bar{z})$. These components can be thought of as generating functionals for the generators L_n:

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}. \tag{1.2.5}$$

The eigenvalue of L_0 on a representation plays an important role: it is called the conformal dimension. Its fractional part has the same value for all eigenvectors of L_0 in one irreducible representation i; it will be denoted by $\Delta(i)$.

It follows that the fields should carry a representation of $\mathcal{V}ir \oplus \overline{\mathcal{V}ir}$ for two complex variables z and $\bar{z}$. Due to this direct sum structure, it is sufficient for many aspects to restrict oneself to z and objects depending holomorphically on z. Again, there are more rigorous arguments in one of the axiomatic frameworks [107]. Each of these halves is called a *chiral conformal field theory*, which is effectively a one-dimensional theory, more precisely a one-dimensional theory on the circle, $z \in S^1$: the fact that the fields depend holomorphically on z implies that, to describe the theory on the complex plane, it is sufficient to describe it on the unit circle.

However, we will see that even from a purely mathematical point of view conformal field theory is not just representation theory of the Virasoro algebra. For instance, the value c of the central element C appearing in (1.2.4), the conformal or Virasoro anomaly, is the same for any representation occurring in a given conformal field theory. Nonetheless, the representation theory of the Virasoro algebra plays an important role and, fortunately, this theory is well developed. As it turns out, certain values of the central charge c are especially interesting: here only finitely many inequivalent unitary irreducible representations exist: in this case a theory is called *rational.*

From a physical point of view rationality is not a fundamental property of a conformal field theory. However, it is of utmost practical importance, since it allows to perform many explicit calculations. This way one can explore many structures which are highly interesting from a mathematical point of view, e.g. rational fusion rings or rational Hopf algebras [137, 46], a structure closely related to quantum groups.

For the Virasoro algebra rationality requires [71] that the value of the central charge c is $c = 1 - 6/m(m+1)$, where m is $m = 3, 4, \ldots$. The series of rational conformal field theories with these values of the central charge is called the Virasoro-minimal series. However, for many interesting applications, e.g. in string theory, higher values of c are required. In order to have still rational theories, one extends the Virasoro algebra to some larger algebra, a so-called $\mathcal{W}$-algebra, and looks for the irreducible representations of this algebra. The latter is also called the *chiral symmetry algebra* or symmetry algebra $\mathcal{W}$ of the conformal field theory. However, not any extension of the Virasoro algebra defines a conformal field theory; rather, locality of the theory requires that also on the modules of $\mathcal{W}$ the conformal weight is uniquely defined $\bmod \mathbb{Z}$. One important example in this thesis are WZW theories, in which the symmetry algebra is the semi-direct sum of the Virasoro algebra $\mathcal{V}ir$ and an affine Lie algebra; they will be discussed in some detail below. Let us remark that minimal series also occur for chiral algebras larger than the Virasoro algebra.

Analogously to the conformal algebra, which is covered by the Lüscher–Mack theorem, in a quantum conformal field theory with symmetry algebra $\mathcal{W}$ there can arise central elements in the chiral algebra. In any representation that occurs in a quantum field theory these central elements have to be represented by numbers; the fundamental reason for this is that central charges are never local. There is also a practical argument: we will see below that frequently one needs expressions in terms of these charges which only make sense for numbers.

Since in physics one imposes the condition that the energy is bounded from below, the class of irreducible highest weight representations is singled out in our considerations. The corresponding fields which transform like the highest weight under the symmetry algebra are usually called primary fields. Primary fields are thus in one-to-one correspondence with the irreducible representations present in a conformal field theory.

1.2.2 Characters and modular invariance

An important tool for the description an irreducible representation of a chiral algebra is its character. Characters are functions of one or several complex variables, which are defined as traces of operators over the vector space which carries the irreducible representation. The most important character, which can be defined for any chiral algebra, is the Virasoro specialized character. It is a function of one complex variable τ, which is convergent in the upper complex half-plane. It is defined as

$$\chi_R(\tau) := \mathrm{tr}_R \, \mathrm{e}^{2\pi\mathrm{i}\tau(L_0 - \frac{c}{24})} \,. \qquad (1.2.6)$$

A crucial observation is that the space of characters of the relevant representations of the chiral algebra carries a representation of $SL_2(\mathbb{Z})$, the double covering of the modular group.

On the modular parameter τ the modular group acts as

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}, \quad a,b,c,d \in \mathbb{Z} \quad , \quad ad-bc=1\,. \tag{1.2.7}$$

The covering $SL_2(\mathbb{Z})$ of the modular group is freely generated by two elements S, T modulo the relations $S^2 = (ST)^3$, $S^4 = 1$, for the modular group $PSL_2(\mathbb{Z})$ this is supplemented by the relation $S^2 = \mathbf{1}$. S and T are represented on the modular parameter τ as

$$T : \tau \mapsto \tau + 1 \quad \text{and} \quad S : \tau \mapsto -\frac{1}{\tau}\,; \tag{1.2.8}$$

the corresponding unitary matrices on the space of characters are correspondingly referred to as S-matrix and T-matrix:

$$\chi_R(\tau+1) = \sum_{R'} T_{RR'}\,\chi_{R'}(\tau) \quad \text{and} \quad \chi_R(-\frac{1}{\tau}) = \sum_{R'} S_{RR'}\,\chi_{R'}(\tau)\,. \tag{1.2.9}$$

From the definition (1.2.6) we see that T is a diagonal unitary matrix while S turns out to be a symmetric unitary matrix. We will see below that knowing the characters as functions of τ typically is not sufficient to determine S and T; one has rather to use the full characters.

Recall that as a consequence of the 'Wirtinger' split into z and $\bar{z}$ we decomposed a conformal field theory into two chiral halves and consequently we obtain characters as functions of τ and $\bar{\tau}$ for the two chiral halves. To recover the full conformal field theory, one has to match both halves: for the coordinates this is done by the prescription that $\bar{z}$ should be the complex conjugate of z. Fields of the conformal field theory now carry a representation of the direct sum of both chiral algebras $\mathcal{W} \oplus \overline{\mathcal{W}}$. For the conformal field theory we have to specify how often the irreducible representation labelled by i of $\mathcal{W}$ is combined with the irreducible representation j of $\overline{\mathcal{W}}$: these non-negative integer numbers Z_{ij} can be combined into a matrix Z. The partition function of the conformal field theory is then

$$Z(\tau) = Z(\tau,\bar{\tau})\big|_{\bar{\tau}^*=\tau} := \sum_{ij} \chi_i(\tau) Z_{ij} \chi_j(\bar{\tau})\big|_{\bar{\tau}^*=\tau}, \tag{1.2.10}$$

where ultimately we have to set $\bar{\tau}$ to the complex conjugate of τ.

To qualify as a partition function of a physical theory, the matrix Z has to fulfill a number of consistency requirements. It has to be positive and, since in a physical theory the representation with lowest eigenvalue of L_0, the vacuum, has to be unique, the corresponding matrix element Z_{00} has to be $Z_{00} = 1$. We will always use the index '0' to refer to the vacuum. So far we have considered the conformal field theory only on the complex plane, i.e., after compactification, on Riemannian surfaces of genus 0. Consistency of the theory at higher genus can be shown to imply that the partition (1.2.10) is invariant under the modular transformations (1.2.7). This is equivalent to the requirement that both the S-matrix and the T-matrix commute with Z:

$$[Z,S] = [Z,T] = 0. \tag{1.2.11}$$

There is, of course, always a trivial solution to these constraints: simply set Z to the identity matrix. We will refer to this modular invariant as the trivial, diagonal or A-type invariant. The fact that the space of characters should carry a unitary representation of $SL_2(\mathbb{Z})$ turns out to be quite a powerful restriction. The example of coset conformal field theories shows that its implementation is a highly non-trivial task. Also the problem of constructing a modular invariant partition function for a given set of characters turns out to be very difficult: all modular invariant partition functions have been classified only in very few cases, e.g. for the WZW theories based on $A_1^{(1)}$, the Virasoro minimal models [65,19,20], and, recently, also for $A_2^{(1)}$ [60]. Part of this thesis is therefore devoted to the development of new tools for their construction.

1.2.3 Fusion rings

We will now introduce the last piece of structural information about conformal field theory needed in this thesis: the fusion rules. Field theory provides us with an associative product of the fields, the operator product: upon forming radially ordered products, the fields realize a closed associative operator product algebra. A large amount of information about the operator product algebra is already contained in the fusion rules of primary fields $\Phi_i \equiv i$, which can be written as formal products, $i \star j = \sum_k \mathcal{N}_{ij}{}^k\, k$. $\mathcal{N}_{ij}{}^k$ counts the number of times that k appears in the operator product of i and j. It is important to realize that this product of two representations of the chiral algebra is *not* the usual tensor product. This is in fact quite easy to see: otherwise e.g. the central charges would add up, $c_{\rm tot} = c_1 + c_2$, whereas the fusion product yields fields in the same conformal field theory, having the same central charge.

Depending on whether one considers this product over the ring of integer numbers $\mathbb{Z}$ or over the field of rational numbers $\mathbb{Q}$, one obtains the structure of a fusion ring respectively of a fusion algebra. These are associative and commutative algebras (respectively rings) with a conjugation and unit, for which a distinguished basis exists (containing the vacuum '0') in which the structure constants are non-negative integers: $\mathcal{N}_{ij}^k \in \mathbb{Z}_{\geq 0}$.

We will deal with fusion rings in more detail in Section 5.1 of this thesis. One can show that the so-called fusion matrices $\mathcal{N}_i$ with entries $(\mathcal{N}_i)_j^k = \mathcal{N}_{ij}^k$ can be simultaneously diagonalized by a unitary matrix S. For a conformal field theory it can be argued [111,138] that S is just the symmetric matrix that implements the modular transformation $\tau \mapsto -\frac{1}{\tau}$ on the characters that was introduced in the previous section, leading to the Verlinde formula [138]

$$\mathcal{N}_{ij}^k = \sum_l \frac{S_{il} S_{jl} S_{kl}^*}{S_{0l}} \,. \qquad (1.2.12)$$

One finds that charge conjugation is an involutive automorphism $i \mapsto i^+$ of the fusion ring. It is non-trivial precisely in the case when it is not the modular group $PSL_2(\mathbb{Z})$, but rather its twofold cover $SL_2(\mathbb{Z})$ that acts on the space of primary fields. The S-matrix elements involving conjugate fields are complex conjugates: $S_{i^+j} = S_{ij}^*$.

Automorphisms of fusion rings play an important role, since a deeper analysis of the consistency requirements on higher genus [110, 28] shows that any modular invariant partition function that belongs to a fully consistent conformal field theory can be described by an automorphism of the fusion rules on top of an extension of the chiral algebra. It should be noted that more requirements than just modular invariance are necessary for the existence and consistency of a conformal field theory; it is thus not surprising that we will encounter modular invariant partition functions which cannot correspond to any conformal field theory.

1.3 Simple currents

As it turns out, the units of a fusion ring, i.e. those elements which possess an inverse, are of considerable importance. They are called *simple currents.* We will sketch in this section their most important properties; for reviews see [45, 130].

It is easy to see that a simple current J can be equivalently characterized as a primary fields for which the fusion product with the conjugate field just yields the vacuum $\Phi_0 \equiv \mathbf{1}$, $J \star J^+ = \mathbf{1}$, or for which the fusion rules are simple in the sense that on the right hand side of $J \star i$ for any primary field i there occurs just one primary field with multiplicity one.

Due to the associativity of the fusion product, the product of two simple currents is again a simple current. Simple currents thus form an abelian group under multiplication, which is called the *center* of a conformal field theory. Since there are only finitely many simple currents present in a rational theory, we can define the order of a simple current to be the smallest positive integer N such that $J^N = \mathbf{1}$. Any simple current organizes the primary fields into orbits; the length of any orbit divides the order N of the simple current.

For any simple current J we can associate to any primary field i a rational number $\mathrm{mod}\,\mathbb{Z}$, the monodromy charge $Q_J(i)$:

$$Q_J(i) := \Delta(J) + \Delta(i) - \Delta(J \star i) \bmod \mathbb{Z}. \tag{1.3.1}$$

Here $\Delta(i)$ is defined as in Subsection (1.2.1). It can be shown that the monodromy charge is additive under the operator product; note that the monodromy charge describes relations between T-matrix elements of fields on the same simple current orbit.

Simple currents can also be shown to provide relations between S-matrix elements of fields on the same simple current orbit. In a unitary theory the following relation holds true [130, 86]:

$$S_{J^p i J^q k} = \mathrm{e}^{2\pi \mathrm{i} p Q_J(k)} \mathrm{e}^{2\pi \mathrm{i} q Q_J(i)} \mathrm{e}^{2\pi \mathrm{i} pq Q_J(J)} S_{ik} \,. \tag{1.3.2}$$

The relations between S-matrix as well as T-matrix elements can be combined and used to construct modular invariants.

In fact, one can construct modular invariants for any simple current in a subgroup of the center, the *effective center.* The effective center is the group of all simple currents whose conformal dimension multiplied by the order of the current is an integer; this condition has to be imposed to guarantee T-invariance.

Assume that the simple current J of order N is an element of the effective center. Then the following matrix Z describes a modular invariant [126, 130]: the only vanishing matrix elements are between fields on the same simple current orbit; the non-vanishing elements are given by

$$Z_{i,J^n i} = \mathrm{Mult}(i)\, \delta^{(1)}(Q_J(i) + \frac{n}{2} Q_J(J)). \tag{1.3.3}$$

Here $\delta^{(1)}(x)$ is equal to one if x is an integer and zero otherwise; $\mathrm{Mult}(i)$ is the multiplicity of the orbit, i.e. the order N of the simple current divided by the length N_i of the orbit of i.

It can be shown that if the center is the cyclic group generated by the simple current J, then this is the only simple current invariant, i.e modular invariant which has non-zero matrix elements only for fields which are on the same orbit of some simple current. If the center is not cyclic, the situation is more involved and there is an additional freedom in choosing Z, parametrized by the so-called discrete torsion; for more details we refer to [100].

The form of the modular invariant (1.3.3) becomes particularly simple if J has integer conformal dimension; it reduces to

$$Z = \sum_{n,Q(i)=0} \frac{N}{N_i} \left| \sum_{n=0}^{N_i-1} \chi_{J^n i} \right|^2 , \qquad (1.3.4)$$

where N denotes again the order of J and N_i the length of the orbit of i. Note that, since N_i divides N, there is always a positive integer in front of the complete square; this observation will be crucial in many applications. Also note that only fields with vanishing monodromy charge occur. We will refer to invariants of this type as integer spin simple current invariants.

The modular invariant given in (1.3.4) has the following interpretation: the chiral algebra $\mathcal{W}$ is enlarged by adding the simple currents to it. Any irreducible representation of the larger algebra $\mathcal{W}'$ decomposes into irreducible representations of $\mathcal{W}$, what explains the complete squares. Not any irreducible representation of $\mathcal{W}$ will be contained in an irreducible representation of $\mathcal{W}'$: this is encoded in the requirement that only irreducible representations of $\mathcal{W}$ occur for which the monodromy charge vanishes.

The interpretation of the multiplicities is slightly more involved: on general grounds [110, 28] any inequivalent irreducible representation of $\mathcal{W}'$ has to appear precisely once; a multiplicity in front of the complete square indicates that there are several inequivalent representations of $\mathcal{W}'$ which reduce to the same representation of $\mathcal{W}$. Therefore, the corresponding expression should be interpreted as several distinct primary fields in the conformal field theory with the enlarged chiral algebra. Fields with higher multiplicities are termed 'fixed points' of the simple current.

Whether such an invariant describes a fully consistent conformal field theory has not been proven rigorously up to now; however, there are arguments from the comparison to orbifolds [100] that the theory should be consistent. In that case, one would like to compute the S-matrix (and as a consequence also the fusion ring) of the new theory. The S-matrix of the original theory provides some constraints (in particular it already determines the S-matrix elements which involve at least one field of multiplicity 1); however determining those elements involving two fixed points, what is usually called 'resolving the fixed points', is a problem that is not fully solved up to now.

1.4 Examples

In this section we present a few basic examples of conformal field theories: the free boson compactified on a circle and WZW theories. Both are not only interesting in themselves, but they also serve as building blocks for the coset conformal field theories, which will be introduced in the next section.

1.4.1 WZW Theories

A Wess–Zumino–Witten (WZW) theory is a conformal field theory whose chiral symmetry algebra is the semidirect sum of the Virasoro algebra with an untwisted affine Lie algebra; its energy-momentum tensor is quadratic in the currents, i.e., in the generators of the affine algebra.

Affine Lie algebras can be constructed as follows: for any reductive complex Lie algebra $\bar{\mathbf{g}}$ with generators J^a and commutation relations

$$[J^a, J^b] = f_c^{ab} J^c \tag{1.4.1}$$

the corresponding untwisted affine Lie algebra $\mathbf{g}$ can be constructed by extending the loop algebra with generators J_n^a, $n \in \mathbb{Z}$ and commutation relations

$$[J_m^a, J_n^b] = f_c^{ab} J_{m+n}^c \tag{1.4.2}$$

by one central element K and a derivation $D = -L_0$. This gives the commutation relations

$$[J_m^a, J_n^b] = f_c^{ab} J_{m+n}^c + Km\kappa^{ab}\delta_{m+n,0}\,, \qquad [J_m^a, K] = 0\,, \qquad [-D, J_m^a] = mJ_m^a \tag{1.4.3}$$

where κ denotes the Killing form of $\bar{\mathbf{g}}$.

Any untwisted affine Lie algebra contains the corresponding reductive Lie algebra as a subalgebra: the generators of the form J_0^a, the zero-modes, form the horizontal subalgebra. Many quantities of interest of a WZW theory can be described entirely in terms of this subalgebra and of the eigenvalue k of the central element K. k is related to the level $k^\vee$ by $k^\vee = \frac{2}{(\theta_{\bar{\mathbf{g}}},\theta_{\bar{\mathbf{g}}})}k$ where $\theta_{\bar{\mathbf{g}}}$ is the highest root of $\bar{\mathbf{g}}$. The level $k^\vee$ does not depend on the normalization of the Killing form of $\bar{\mathbf{g}}$; for unitary theories $k^\vee$ is a non-negative integer.

One possibility to realize the Virasoro algebra explicitly in terms of the affine Lie algebra is the Sugawara construction which uses the quadratic Casimir operator:

$$L_n := \frac{1}{2(k^\vee + g^\vee)} \sum_m \kappa_{ab} : J_{m+n}^a J_{-m}^b :\,, \tag{1.4.4}$$

where :: denotes a normal ordering prescription and $g^\vee$ the dual Coxeter number of $\bar{\mathbf{g}}$ which is essentially the eigenvalue of the quadratic Casimir operator in the adjoint representation. The factor $(k^\vee + g^\vee)^{-1}$ in the definition of L_n makes sense in a quantum field theory only if $k^\vee$ is a number rather than an operator, as was mentioned in Section 1.2. The Virasoro central charge c can now be expressed in terms of $k^\vee$; it is

$$c(\mathbf{g}, k^\vee) = \frac{k^\vee \dim \bar{\mathbf{g}}}{k^\vee + g^\vee}\,. \tag{1.4.5}$$

The primary fields of a unitary WZW theory with diagonal modular invariant are in one-to-one correspondence with the integrable highest weights, i.e., with the dominant integral weights Λ of $\bar{\mathbf{g}}$ that satisfy

$$(\Lambda, \theta_{\bar{\mathbf{g}}}) \leq k\,. \tag{1.4.6}$$

Only finitely many weights of $\bar{\mathbf{g}}$ fulfill these conditions: WZW theories are therefore rational conformal field theories; this feature will also carry over to the coset conformal field theories to be discussed in the next section. To illustrate rationality we have depicted in Figure 1.1 the dominant affine Weyl chamber of the affine Lie algebra $A_2^{(1)}$ at various levels $k^\vee$. The figure shows the weight space of the simple Lie algebra A_2 which describes the horizontal projection of the weights of the affine Lie algebra $A_2^{(1)}$. The six arrows represent the six roots of A_2. The dominant affine Weyl chamber at level $k^\vee = 5$ is shaded in light grey; integral weights are marked by dots. Due to condition (1.4.6) the Weyl chamber contains only finitely many integrable highest weights which are in one-to-one correspondence to the primary fields of the WZW theory. In darker grey we have shaded the 'interior' of the dominant Weyl chamber; this is nothing but the translate of the Weyl chamber at level $k^\vee = 2$ by the Weyl vector $\rho = \sum_{i=1}^{n} \Lambda_{(i)}$, with $\Lambda_{(i)}$ the fundamental weights of $\bar{\mathbf{g}}$ (note that in the case of A_2 the Weyl vector is equal to the highest root $\theta_{\bar{\mathbf{g}}}$). For many purposes, e.g. the considerations in Part II of this thesis, it will be convenient to use the weights shifted by the Weyl vector.

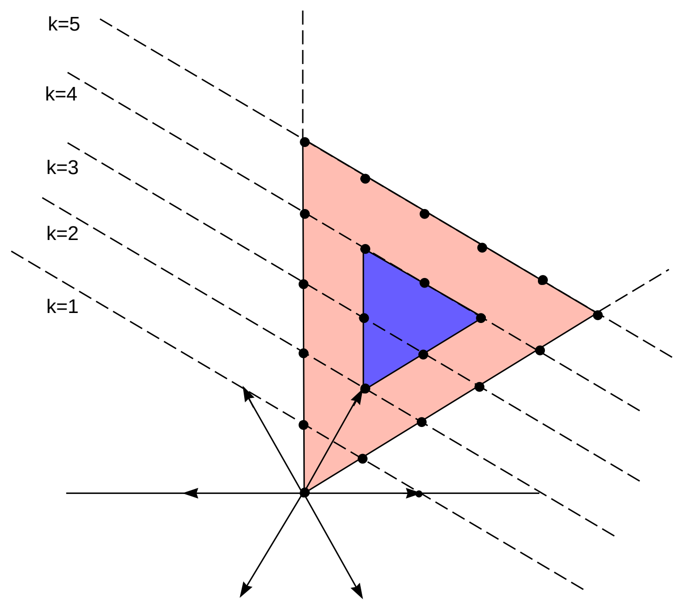


Figure 1.1: Dominant affine Weyl chamber of $A_1^{(1)}$ at various levels $k^\vee$.

The conformal dimension of a primary field with highest weight Λ is

$$h_\Lambda \equiv h_{(\mathbf{g})}(\Lambda) = \frac{(\Lambda, \Lambda + 2\rho)}{2(k^\vee + g^\vee)} . \tag{1.4.7}$$

This immediately gives the T-matrix; the S-matrix is given by the Kac–Peterson formula [92]

$$S_{\Lambda\Lambda'} = \mathcal{N} \sum_{w \in W} \text{sign}(w) \exp[-\frac{2\pi \mathrm{i}}{k^\vee + g^\vee} (w(\Lambda + \rho), \Lambda' + \rho)]. \tag{1.4.8}$$

Here the summation is over the Weyl group $\overline{W}$ of the horizontal subalgebra $\bar{\mathbf{g}}$; the normalization $\mathcal{N}$ follows from the requirement that S should be unitary.

Let us give one example to which we will refer later on frequently: the situation is particularly simple for $\bar{\mathbf{g}} = D_d$ at level one. Then there are four primary fields corresponding to the singlet (0), vector (v), spinor (s), and conjugate spinor (c) representation of D_d, or, in other words, to the conjugacy classes of the D_d weight lattice; their conformal dimension is

$$h = \begin{cases} 0 & \text{for} \quad 0\,, \\ 1/2 & \text{for} \quad \mathrm{v}\,, \\ d/8 & \text{for} \quad \mathrm{s,\,c}\,. \end{cases} \tag{1.4.9}$$

The modular matrix S of D_d at level $k^\vee = 1$ reads

$$S((D_d)_1) = \tfrac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \mathrm{i}^{-d} & -\mathrm{i}^{-d} \\ 1 & -1 & -\mathrm{i}^{-d} & \mathrm{i}^{-d} \end{pmatrix} . \tag{1.4.10}$$

The simple currents of all WZW theories have been classified in [48,43]. Except for the case of E_8 at level $k^\vee = 2$, they are of the form $k^\vee \Lambda_{(i)}$, where $k^\vee$ is the level and $\Lambda_{(i)}$ a cominimal weight, i.e. the Coxeter label a_i is equal to one. (Coxeter labels are defined as the coefficients of the highest root $\theta_{\bar{\mathfrak{g}}}$ in a basis of simple roots:

$$\theta_{\bar{\mathfrak{g}}} = \sum_{i=1}^{\operatorname{rank}\bar{\mathfrak{g}}} a_i \alpha^{(i)} \,.) \tag{1.4.11}$$

The action of simple currents on the set of integrable highest weight representations corresponds in these cases to the automorphisms of the affine Dynkin diagrams; the monodromy charge is proportional to the conjugacy class of the representation.

To fix the notation, let us describe the simple currents of WZW theories (for the numbering of the simple roots we use the convention of [90]): For A_r the center is isomorphic to the cyclic group $\mathbb{Z}_{r+1}$; it is generated e.g. by $J = k\Lambda_{(1)}$. For B and C type theories, there is a single simple current besides the identity primary field; this current will be denoted by J (the corresponding highest weight is $k\Lambda_{(1)}$ for B_r, and $k\Lambda_{(r)}$ for C_r theories). For D_r type theories, there are three non-trivial simple currents, corresponding to the highest weights $k\Lambda_{(1)}$, $k\Lambda_{(r)}$, and $k\Lambda_{(r-1)}$; they are denoted by J_v, J_s, and J_c, as their fusion rules are isomorphic to the multiplication of the vector (v), spinor (s), and conjugate spinor (c) conjugacy classes. For E_6 there is a simple current of order three, $J = k\Lambda_{(1)}$ and $J^2 = k\Lambda_{(5)}$, and for E_7 a simple current of order 2, $J = k\Lambda_{(6)}$. E_8 has only a simple current at level $k^\vee = 2$, while G_2 and F_4 do not have simple currents at all.

1.4.2 The free boson

Our second example is a single free boson. If it is compactified on a circle of rational radius squared, the corresponding conformal field theory turns out to be rational. For simplicity we will refer to these theories as WZW theories with horizontal subalgebra u(1).

The primary fields ϕ_Q of these theories are labelled by u(1)-charges $Q \in \{0, 1, , \ldots , \mathcal{N} - 1\}$, where the number $\mathcal{N}$ of primaries is related to the radius of the circle. The conformal dimension of a u(1)-primary of charge Q is $Q^2/2\mathcal{N}$. The S-matrix elements of a u(1) WZW theory are

$$S_{PQ} = \frac{1}{\sqrt{\mathcal{N}}} \exp(-2\pi\mathrm{i}\, PQ/\mathcal{N}). \tag{1.4.12}$$

For u(1) WZW theories, the fusion rules read $\phi_P \star \phi_Q = \phi_{P+Q \bmod \mathcal{N}}$, and hence any primary field is a simple current. The conformal central charge of a single free boson is $c = 1$.

1.5 Coset conformal field theories

1.5.1 The Coset Construction

The subclass of WZW models has the big advantage that it presents examples of conformal field theories which are particularly manageable, since the representation theory of the chiral algebra is known. However, it is not too hard to see that for many purposes this subclass is not comprehensive enough.

The expression (1.4.5) for the conformal anomaly c shows that c is greater or equal to the rank of $\bar{\mathsf{g}}$, in particular $c \geq 1$. However, there are interesting conformal field theories for which c is smaller than one: the Virasoro minimal models. These models do not only play a special role in the representation theory of the Virasoro algebra, but they also have interesting applications in physics, especially in statistical mechanics. In fact, one finds the following central charges for the models listed below:

critical Ising model	$c = \frac{1}{2}$
tricritical Ising model	$c = \frac{7}{10}$
critical three states Potts model	$c = \frac{4}{5}$
tricritical three states Potts model	$c = \frac{6}{7}$

A more abstract reason which makes it desirable to enlarge the class of models under consideration is that WZW models have a rather special chiral algebra. For some applications, e.g. the construction of superstring vacua (cf. Chapter 4 of this thesis), one would also like to realize superconformal algebras and their extensions. More complicated $\mathcal{W}$-algebras also play an important role in the programme of classifying all rational conformal field theories: the programme is first to classify all $\mathcal{W}$-algebras and then to work out their representation theory.

From the practical side, one frequently studies concrete examples in order to get hints on more general structures; the discovery of Galois symmetry of conformal field theories described in Part II of this thesis is a good example. Now, it is difficult to get control on whether observations made within such a limited framework like WZW theories can be generalized to other conformal field theories. Fortunately, as we will see, many observations directly apply also to the larger class of coset conformal field theories.

The coset construction [72, 73] allows within the framework of affine Lie algebras to obtain an explicit description of a large class of conformal field theories. The idea is to associate to any pair $\bar{\mathsf{g}}$, $\bar{\mathsf{h}}$ of reductive Lie algebras for which $\bar{\mathsf{h}}$ is a subalgebra of $\bar{\mathsf{g}}$, a conformal field theory called the coset theory and denoted by

$$\mathcal{C}[\bar{\mathsf{g}}/\bar{\mathsf{h}}]_k . \qquad (1.5.1)$$

The embedding $\bar{\mathsf{h}} \hookrightarrow \bar{\mathsf{g}}$ induces an embedding of the corresponding untwisted affine Lie algebras $\mathsf{h} \hookrightarrow \mathsf{g}$. In practical calculations, one has at this point to determine the precise form of the affinization: e.g. if $\bar{\mathsf{g}}$ is simple, then the level k_i of any simple summand $\bar{\mathsf{h}}_i$ of $\bar{\mathsf{h}}$ is related to the level $k^\vee$ of $\bar{\mathsf{g}}$ by $k_i = I_i k^\vee$, where I_i is the Dynkin index of the embedding $\bar{\mathsf{h}}_i \hookrightarrow \bar{\mathsf{g}}$.

By definition [73], the Virasoro generators of the coset theory are obtained by subtracting the Virasoro generators of the WZW theory based on h from the ones of the WZW theory based on g: $L^{\mathsf{g/h}} := L^{\mathsf{g}} - L^{\mathsf{h}}$. Since the Virasoro generators of the coset theory commute with any generator J^a_n of h,

$$[L^{\mathsf{g/h}}_m, J^a_n] = [L^{\mathsf{g}}_m - L^{\mathsf{h}}_m, J^a_n] = 0\,, \tag{1.5.2}$$

we see that

$$[L^{\mathsf{g/h}}_m, L^{\mathsf{g/h}}_n] = [L^{\mathsf{g}}_m, L^{\mathsf{g}}_n] - [L^{\mathsf{h}}_m, L^{\mathsf{h}}_n], \tag{1.5.3}$$

and that, as a consequence, the generators $L^{\mathsf{g/h}}_n$ span a Virasoro algebra with central charge $c_{\mathsf{g/h}} = c_{\mathsf{g}} - c_{\mathsf{h}}$.

1.5.2 Branching rules

So far we have only dealt with algebras; in order to check whether the definition of the coset Virasoro algebra leads to a well-defined conformal field theory, one also has to specify the spectrum of primary fields of the theory. As it turns out, to obtain the primary fields of the coset theory is a somewhat delicate issue. However, equation (1.5.2) shows that the coset Virasoro algebra acts in the same way on all vectors of the same module of h. Therefore the branching spaces $\mathcal{H}^\Lambda_\lambda$ which arise in the decomposition of the g-module $\mathcal{H}^{\mathsf{g}}_\Lambda$ into h-modules $\mathcal{H}^{\mathsf{h}}_\lambda$,

$$\mathcal{H}^{\mathsf{g}}_\Lambda = \bigoplus_\lambda \left(\mathcal{H}^\Lambda_\lambda \otimes \mathcal{H}^{\mathsf{h}}_\lambda\right) \tag{1.5.4}$$

are natural candidates for the modules of the coset Virasoro algebra. (Here Λ and λ stand for integrable highest weights of g and h, respectively, if g and h are simple, and similarly in the general case.)

The candidates for the characters are therefore the so-called branching functions b^Λ_λ, which are the coefficient functions in the decomposition

$$\mathcal{X}_\Lambda(\tau) = \sum_\lambda b^\Lambda_\lambda(\tau)\,\chi_\lambda(\tau) \tag{1.5.5}$$

of the characters $\mathcal{X}_\Lambda$ of g with respect to the characters χ_λ of h.

Branching functions have a definite behaviour under modular transformations which suggests that the coset theory associated to the embedding $\mathsf{h} \hookrightarrow \mathsf{g}$ might be essentially something like $\mathsf{g} \oplus \mathsf{h}^*$, where the notation '*' indicates that the complex conjugates of the modular transformation matrices of the WZW theory based on h should be used. Note that if S and T generate a representation of the modular group, the same is true for S^* and T^*. If there exists a conformal field theory whose characters transform according to this complex conjugate representation, it is called the *complement* of the h theory [130].

To check whether this guess can be correct, it is instructive to look at a simple example: the critical Ising model with $c = \frac{1}{2}$ which can be realized with $\mathsf{h} = A^{(1)}_1$ at level 2 diagonally embedded into $\mathsf{g} = A^{(1)}_1 \oplus A^{(1)}_1$, both algebras at level 1. Any candidate for a primary field can now be described by three labels, Φ^{lm}_n. l, m and n are the Dynkin labels of the highest weights; the fact that all representations are unitary restricts l and m to the values $l, m = 0, 1$ and n to $n = 0, 1, 2$. Since c is less than one, the field contents of this theory follows already from the representation theory of the Virasoro algebra: we expect the following primary fields and find them realized as

Field	Δ	realized as	
vacuum	0	Φ_0^{00}	Φ_2^{11}
twist field	1/16	Φ_1^{01}	Φ_1^{10}
energy operator	1/2	Φ_0^{11}	Φ_2^{00}

Any field that does not appear in this table turns out to have vanishing branching function. In our example, this can be explained by the group theoretical selection rules for the couplings of two spins which can be expressed by the condition $l + m - n = 0 \bmod 2$. We do not only find that some of the 'fields' we would naively expect to be present vanish, but we also realize that all fields we expect from the representation theory of the Virasoro algebra seem to appear twice. This is clearly in conflict with the requirement of a unique vacuum. However, this requirement is absolutely crucial for the consistency for a conformal field theory: on the level of the fusion ring, the vacuum gives the unital element which must be unique; on the level of representation spaces, the vacuum corresponds to the identity operator which also is unique. Closer inspection shows that also modular invariance is spoiled and the restriction of the S-matrix to non-vanishing fields is not unitary any more, since certain rows respectively columns coincide.

As it turns out, this situation generalizes for arbitrary coset conformal field theories: several branching functions vanish, and several of the non-vanishing branching functions coincide. One can imagine three different reasons why branching functions vanish:

- Group theoretical selection rules, as we have seen in the case of the Ising model.
- The occurrence of 'unexpected' null states in the Verma module: they certainly occur in conformal embeddings. (Conformal embeddings [9, 125] are by definition those embeddings for which the coset central charge c vanishes; hence they describe a trivial coset conformal field theory.)

  There are, however, a few exceptional cosets known where null vectors occur 'unexpectedly': the so-called mavericks cosets [30, 31].
- There is also a more technical combination argument [129]; however, since there is no example known where this applies, we refrain from explaining it at this place.

One might hope to cure the situation by simply forgetting about the 'fields' with vanishing branching function. But this inevitably leads to inconsistencies since, in general, S-matrix elements between vanishing and non-vanishing branching functions do not vanish, so that one would spoil the unitarity of the S-matrix this way.

1.5.3 Field identification

For the generic case, when only group theoretical selection rules have to be implemented, there is, fortunately, a conceptual framework to address the situation at least on the level of characters and representations of the modular group: simple currents. Recall that the monodromy of a simple current of a WZW theory is proportional to the conjugacy class of the corresponding representation. Therefore, non-vanishing 'fields' can be characterized by the fact that their monodromy charge vanishes for a subgroup of the center of $\mathsf{g} \oplus \mathsf{h}^*$. We will call this subgroup the identification group; its elements are called *identification* *currents.* They are specific tensor products, to be denoted as $(J_{(\mathsf{g})}\,/\,J_{(\mathsf{h})})$, of the simple currents of the WZW theories that underly the coset theory. To determine the identification group explicitly is the first step in the process of setting up a coset conformal field theory in practice.

An important property of identification currents is that branching functions related by them are identical. This can be easily seen:

$$\frac{b_i(-1/\tau)}{b_{Ji}(-1/\tau)} = \frac{\sum_j S_{ij} b_j(\tau)}{\sum_j S_{Jij} b_j(\tau)} = \frac{\sum_j S_{ij} b_j(\tau)}{\sum_j S_{ij} \mathrm{e}^{2\pi \mathrm{i} Q(j)} b_j(\tau)}\,, \tag{1.5.6}$$

which is equal to 1 since only allowed fields, i.e. precisely the fields for which the monodromy charge $Q(j)$ vanishes, contribute to the sum.

This calculation shows that the conformal dimension is constant on each orbit and that in particular all identification currents have integer conformal weight since they are on the orbit of the identity. It is now natural to use the modular invariant expression (1.3.4) as a candidate for a partition function; notice that due to the selection rule $Q(i)=0$ only non-zero fields occur in (1.3.4). Since now within one complete square all branching functions are identical, we are led to the following consequence:

> The true primary fields of a coset conformal field theory are defined as *equivalence classes*, they are the orbits of the identification group.

This is commonly referred to as *field identification* [68]. In other words, the coset theory is in fact rather different from $\mathsf{g} \oplus \mathsf{h}^*$: we have to associate physical fields not with individual branching functions, but rather with certain equivalence classes of them.

To conclude let us remark that for the maverick coset conformal field theories [30,31] the situation is far from being understood; interestingly enough, there are in all cases modular invariants which can be used to implement a consistent field identification. However, these are exceptional invariants rather than simple current invariants as in the case of ordinary coset conformal field theories.

1.5.4 Field Identification Fixed Points

As long as all orbits of the identification currents have equal size, the orbits are precisely the physical primary fields we are after. The situation is more involved if the orbits have different lengths [103]; the number of representatives on a orbit is in any case a divisor of the length N of the orbit of the identity field. Orbits with less than N representatives are referred to as 'fixed points' of the identification currents.

These fixed points cause a serious problem: recall that all branching functions in a complete square in (1.3.4) are identical. So, in order to have a unique vacuum, we want to keep just one representative of every orbit, i.e. divide Z by N^2. Now doing this naively would entail non-integer coefficients in the putative character of the shorter orbits, the fixed points; clearly this is inadmissible for a character, which counts states and therefore must have integer coefficients.

Let us pause at this point to make some general remarks: the phenomenon of 'field identification' can be placed in a broader context: the coset construction can be seen as a special example of a reduction procedure. As a common feature of reduction processes, we observe that first class constraints (i.e. constraints that generate gauge transformations) can be required by consistency conditions. E.g. in gauge theories, in a Hamiltonian formulation, one such consistency requirement is the uniqueness of the time evolution. In the example at hand, coset conformal field theories, modular invariance is required for consistency. The analogy to gauge theories can be pushed even further: in both cases the constraints 'generate' the identifications. There is even an analogue to the fixed points: reducible connections (for some of the properties of reducible connections see e.g. [98,81]), i.e. connections which have non-trivial stabilizer under the action of the gauge group. Unfortunately only little is known about the effects of these connections in physical theories (for a closer study of these connections in the case of an SU(2) gauge theory on the four-dimensional sphere S^4 we refer the reader to [53]).

In the case of coset conformal field theories, the prefactors in (1.3.4) suggest – as in the case of D-invariants – that every fixed point f of length $N_f < N$ should correspond to N/N_f distinct physical primary fields. Again there is the problem of fixed point resolution: in this case not only the full S-matrix, but also the characters for the individual physical fields are unknown. Some information is already contained in S, the S-matrix obtained by the action of the identification currents from the original S-matrix. This leads to the following ansatz [129,123] for the S-matrix elements in a coset theory with fixed points:

$$\tilde{S}_{e_i f_j} = \frac{N_e N_f}{N} S_{ef} + \Gamma^{ef}_{ij} \,, \tag{1.5.7}$$

where $i = 1, 2, \ldots, N/N_e$ and $j = 1, 2, \ldots, N/N_f$ label the fields into which the naive fields e and f are to be resolved if they are fixed points.

After having resolved the fixed points, $\tilde{S}$ and the corresponding extension of T must form an unitary representation of $SL_2(\mathbb{Z})$; this implies sum rules

$$\sum_{i=1}^{N/N_e} \Gamma^{ef}_{ij} = 0 = \sum_{j=1}^{N/N_f} \Gamma^{ef}_{ij} \tag{1.5.8}$$

for the S-matrix elements of the fields f_i. Note that if either e or f is not a fixed point, the sum rule tells us that Γ vanishes. So Γ is non-zero only for pairs of resolved fixed points, in which case it has also to be symmetric under simultaneous exchange of (e,i) and (f,j), since the total S-matrix $\tilde{S}$ must have this property.

It is a surprising empirical observation [130] that in most cases consistent Γ matrices can be described in terms of a different WZW theory, the so-called *fixed point theory*. We will see in Chapter (3.7) that even in the case when the fixed point theory is not a WZW theory the structure of Γ is surprisingly close to that of a WZW theory.

Resolving a fixed point also amounts to considering fields having different characters χ_{f_i}, i.e., the naive branching function χ_f of the 'unresolved fixed point' must be modified to have integer coefficients after dividing by N^2. It turns out that this can be done by adding an appropriate multiple of a character $\check{\chi}_f$ of the fixed point theory. Again, modular invariance implies a sum rule, namely

$$\sum_{i=1}^{N/N_f} \chi_{f_i} = \chi_f \,. \tag{1.5.9}$$

Only after having found a consistent solution for Γ and the character modifications $\check{\chi}_f$, one can speak of a conformal field theory given by a coset. Unfortunately, no general results concerning existence or uniqueness of a resolution procedure are known. Also, fixed point resolution has been implemented in practice only in very few cases: for the minimal series of the $N = 1$ superconformal algebra and for $N = 2$ superconformal coset theories, the Kazama–Suzuki models [96,97]. For the latter models, the fixed point resolution procedure has been worked out in [123] for the special case of hermitian symmetric cosets; the general case [54] will be presented in Chapter 2 of this thesis.

We conclude this introduction to coset conformal field theories with two remarks: from what we have explained it is clear that a 'Lie algebraic coset' (1.5.1) as it stands is far from defining a conformal field theory. We point out that the correspondence between Lie algebraic cosets and coset conformal field theories is also not one-to-one: first, also for cosets a modular invariant has to be chosen; each choice will describe a different conformal field theory. The problem of classifying all these modular invariants is a particularly hard one: in principle any modular invariant of the tensor product $\mathsf{g} \oplus \mathsf{h}^*$ that is compatible with the field identification is admissible. Unfortunately, the modular invariants for g and h separately have not been classified, and, even worse, the problem of classifying all invariants of $\mathsf{g} \oplus \mathsf{h}^*$ does not even factorize to that of classifying the invariants of g and h separately.

Conversely, it also turns out that different combinations of algebras g and subalgebras h can describe one and the same conformal field theory. One example of this phenomenon which will be explained in this thesis in some detail are level-rank dualities: cosets for which level and rank (or some simple functions thereof) are interchanged turn out to describe the same conformal field theory. The problem of counting all conformal field theories that can be described by a Lie algebraic coset is therefore rather different from the problem of counting all cosets.

1.6 Outline of the thesis

We now give an outline of the rest of this thesis. A particularly interesting subclass of conformal field theories are those which are not only invariant under the conformal symmetry described by the Virasoro algebra, but even under a larger algebra, the $N = 2$ superconformal algebra.

The study of $N = 2$ superconformal theories was initially motivated by string theory: a tensor product of these theories with central charge $c = 9$ can be used as the inner sector in a heterotic string compactification. In this application, $N = 2$ superconformal symmetry on the world sheet – together with charge quantization – implies that the spectrum of the string is space-time supersymmetric [10].

An independent motivation to study these models comes from their beautiful intrinsic structure and their deep connection to other objects in mathematical physics. For example, these theories can be 'twisted' to obtain two-dimensional topological quantum field theories.

In Part I of this thesis we will use the coset construction to construct concrete examples for $N = 2$ superconformal theories. In Chapter 2 we classify all coset models of the Kazama–Suzuki form that have $N = 2$ supersymmetry and can be used as the inner sector in a string compactification. The field identification and fixed point resolution structure of these models is worked out, and several general properties of $N = 2$ coset theories are proven (e.g. the formula (2.3.14) for the number of elements of the identification group).

An important structure present in $N = 2$ superconformal theories is the chiral ring [103]: for appropriately chosen $N = 2$ coset conformal field theories the ring structure of the chiral ring determines the number of massless generations in the corresponding compactification of the heterotic string; also the relation to topological field theories is mainly through this ring. The structure of the chiral ring of $N = 2$ coset theories is explored in Section 2.4; Hasse diagrams which are described in Appendix 2.A are a useful tool for these calculations.

As it turns out, several cosets lead to the same conformal field theory; a systematic reason for this phenomenon are level-rank dualities which are proven in Chapter 3. They make heavy use of level-rank dualities for WZW theories which are also described in Chapter 3. The results of these two chapters are applied in Chapter 4 to the construction of string vacua. To this end several projections have to be implemented; we describe how this can be done using simple currents and how this prescription leads to the definition of the extended Poincaré polynomial. The extended Poincaré polynomial is then used to compute the massless spectra of all string vacua that can be constructed by the use of $N = 2$ coset models.

To show that the cosets described in Part I give rise to consistent conformal field theories, field identification fixed points have to be resolved using fixed point theories. In some cases, the fixed point theory is not a WZW theory; however, we will see in Chapter 3 that level-rank dualities allow to determine an S-matrix which implements the resolution at the level of representations of the modular group. To identify a conformal field theory that might correspond to this matrix (and hence the character modifications) it is natural to try to determine which of the fields should be the identity, and then use the Verlinde formula (1.2.12) to compute the corresponding fusion ring. A priori one would expect that using the wrong primary field as the identity '0' in (1.2.12) would lead to non-integer fusion rule coefficients. Surprisingly, many choices seem to be equally good. In many cases the fusion rule coefficients turned out to be integer, though in no case were they positive. However, it was always possible to find a set of signs $\epsilon(i)$ and a new matrix $S'_{ij} = \epsilon(i)\epsilon(j)S_{ij}$ that made all the coefficients positive. There is a second surprise: the fusion rules obtained from (1.2.12) for different choices of the vacuum '0' turned out to be identical up to some permutation of the fields.

Inspired by this observation one can investigate WZW models in a similar way, and finds that some of the primary fields other than the identity could play the role of the identity in the above sense. It is natural to look now for some underlying symmetry of the modular matrix S, and it turns out that indeed such a symmetry exists. It can be described using the Galois group of the algebraic number fields which contain the generalized quantum dimensions. This new symmetry of rational quantum field theories which is the subject of Part II of this thesis turned out to be extremely powerful. We will see in Chapter 5 that it can be used to construct automorphisms of fusion rings as well as modular invariant partition functions.

In Chapter 6 these new tools will be applied systematically to the fusion ring of WZW theories. We will see that both exceptional and simple current modular invariants can be explained by Galois theory. Using Galois symmetry several infinite series of previously unknown exceptional automorphism invariants for WZW theories based on algebras of type B and D are found.

In these investigations, it turned out that – at least in the case of WZW theories – Galois symmetry can be further generalized. These symmetries which we call quasi-Galois symmetries will be the subject of Chapter 7. These quasi-Galois symmetries have various applications: they lead to sum rules for the elements of the modular matrix S which can be used for the construction of modular invariants. Moreover, they relate WZW theories at different levels and provide a powerful algorithm for the computation of the branching rules of conformal embeddings.

The chapters of this thesis are based on the following publications:

- Chapter 2:
  J. Fuchs and C. Schweigert, *Non-hermitian symmetric $N = 2$ coset models, Poincaré polynomials, and string compactification*, Nucl. Phys. B 411 (1994) 181

- Chapter 3:
  J. Fuchs and C. Schweigert, *Level-rank duality of WZW theories and isomorphisms of $N = 2$ coset models*, Ann. Phys. 234 (1994) 102

- Chapter 4:
  J. Fuchs and C. Schweigert, *Non-hermitian symmetric $N = 2$ coset models, Poincaré polynomials, and string compactification*, Nucl. Phys. B 411 (1994) 181
  C. Schweigert, *Poincaré polynomials and level rank dualities in the $N = 2$ coset construction*, Theor. Math. Phys. 98 (1994) 326

- Chapter 5:
  J. Fuchs, B. Gato-Rivera, A.N. Schellekens, and C. Schweigert, *Modular invariants and fusion rule automorphisms from Galois theory*, Phys. Lett. B 334 (1994) 113

- Chapter 6:
  J. Fuchs, A.N. Schellekens, and C. Schweigert, *Galois modular invariants of WZW models*, Nucl. Phys. B 437 (1995) 667

- Chapter 7:
  J. Fuchs, A.N. Schellekens, and C. Schweigert, *Quasi-Galois symmetries of the modular S-matrix*, to appear in Commun. Math. Phys.

Part I

$N = 2$ Superconformal Coset Models

In Part I of this thesis we investigate a particularly interesting subclass of conformal field theories: $N = 2$ superconformal coset theories. In Chapter 2 the field identification problem, including fixed point resolution, is solved for these models and some general results for generic $N = 2$ coset theories are proven. In Chapter 3 level-rank dualities between several infinite series of these models are shown; they make use of level-rank dualities for B, C, and D type WZW theories, which are also described in some detail. Finally, in Chapter 4 we use the coset theories introduced in Chapter 2 as subtheories in $N = 2$ tensor products with $c = 9$, which are taken as the inner sector of heterotic superstring compactifications.

Chapter 2

Non-Hermitian Symmetric $N = 2$ Coset Models

In this chapter, the field identification problem, including fixed point resolution, is solved for the non-hermitian symmetric $N = 2$ superconformal coset theories. Thereby these models are finally identified as well-defined modular invariant conformal field theories. Further, some general results for generic $N = 2$ coset theories are proven: a simple formula for the number of identification currents is found, and it is shown that the set of Ramond ground states of any $N = 2$ coset model is invariant under charge conjugation.

2.1 $N = 2$ superconformal theories

While the conditions necessary for the consistency of a superstring theory seem to be too weak to pinpoint a 'theory of everything', string theory remains an interesting approach to unify the fundamental interactions including gravity. Furthermore, the study of strings has given new and deep insight in various topics in mathematics and physics so that there are good reasons, beyond possible direct application to phenomenology, to have a closer look at the structures arising in string theory.

A class of two-dimensional field theories for which this point of view is particularly justified are the $N = 2$ superconformal theories which are needed for the inner sector of (heterotic) string theories. The enlarged ($N = 2$) world sheet supersymmetry for the right-moving part of the theory is in this case dictated [135,10] by the requirement of space-time supersymmetry, a property imposed for phenomenological reasons, such as to 'solve' the gauge hierarchy problem. In the present chapter, we consider theories for which $N = 2$ supersymmetry is present in the left-moving part as well; just like in the generic case, these $N = 2$ theories are interesting in their own right, as they are singled out by the presence of new structures such as the ring of chiral primary fields and the connection with Calabi–Yau manifolds [103]. Furthermore, there exist deep relations between $N = 2$ superconformal field theories and conformal field theories in general, including the interpretation of the fusion ring of any rational conformal field theory as a deformation of the chiral ring of some $N = 2$ theory [69,87,104,24].

There exist several approaches to construct the inner sector of a heterotic string theory: non-linear sigma models with a Calabi–Yau manifold as their target space [18], the description in terms of Landau–Ginzburg potentials [78,108], and exactly solvable models (these approaches are closely interrelated, but the question to which extent they are equivalent has not yet been resolved completely). By exactly solvable we mean that all correlation functions can (at least in principle) be calculated exactly. Among the solvable superconformal field theories there are free field constructions employing the Coulomb gas approach [29], and theories constructed by algebraic methods. In the algebraic approach the *coset construction* [73] plays a prominent rôle, for it allows to obtain many superconformal theories within the framework of affine Kac–Moody algebras.

In [97] Kazama and Suzuki considered coset models $\mathcal{C}$ of the form

$$\mathcal{C}[\bar{\mathsf{g}} \oplus \mathrm{so}(2d)/\bar{\mathsf{h}}]_k \,. \tag{2.1.1}$$

Here $\bar{\mathsf{g}}$ stands for a semi-simple Lie algebra, and $\bar{\mathsf{h}}$ is a reductive subalgebra of $\bar{\mathsf{g}}$; the integer d is defined as $2d = \dim \bar{\mathsf{g}} - \dim \bar{\mathsf{h}}$, while the integer k denotes the level of the affinization $\mathsf{g} = \bar{\mathsf{g}}^{(1)}$ of $\bar{\mathsf{g}}$. As shown in [97], the symmetry algebra of such coset models always contains the $N=1$ superconformal algebra.

It was also investigated [96] for which models the symmetry algebra is indeed enlarged to an $N=2$ superconformal algebra. Although the proof of the classification turned out to be not quite complete [133], a complete list of all $N=2$ coset theories of the form (2.1.1) was obtained. Indeed the following conditions are necessary and sufficient for a coset theory of the form (2.1.1) to have $N=2$ superconformal symmetry:

1. The embedding $\bar{\mathsf{h}} \hookrightarrow \bar{\mathsf{g}}$ has to be regular.

2. The number
$$n := \tfrac{1}{2}\,(\mathrm{rank}\,\bar{\mathsf{g}} - \mathrm{rank}\,\bar{\mathsf{h}}) \tag{2.1.2}$$
must be an integer.

3. Denoting the simply connected compact Lie groups having $\bar{\mathsf{g}}$ and $\bar{\mathsf{h}}$ as their Lie algebras by G and H, respectively, the coset *manifold*
$$\frac{G}{H \times \mathrm{U}(1)^{2n}} \tag{2.1.3}$$
has to be Kählerian.

Up to now the following theories solving these constraints have been considered in the literature:

- Tensor products of $N=2$ minimal models [66], including models which employ non-diagonal and non-product modular invariants [128,49,131,106,50].

- Tensor products of the so-called projective cosets [37], corresponding to coset theories of the form
$$\mathcal{C}[\mathrm{su}(n+1) \oplus \mathrm{so}(2n)_1/\mathrm{su}(n) \oplus \mathrm{u}(1)]_k \,. \tag{2.1.4}$$
For these models non-diagonal modular invariants have been investigated, too [3].

- Tensor products of arbitrary hermitian symmetric coset theories ('HSS-cosets') with the diagonal modular invariant [123].

Note that $N=2$ minimal models can be considered as projective cosets with $n=1$, and projective cosets are a subclass of the hermitian symmetric cosets.

From the classification [96,133] of $N=2$ superconformal coset models in the Kazama–Suzuki framework it is well known that there exist even more models that possess $N=2$ superconformal symmetry; the hermitian symmetric coset theories constitute only a subclass. In this chapter we shall consider the general case. The chapter is organized as follows. First, we recall in Section 2.2 the classification of $N=2$ superconformal coset models obtained in the Kazama–Suzuki framework. As a by-product we prove a simple characterization of hermitian symmetric spaces which differs from the one given in the standard literature. Based on the general classification, we then provide a complete list of all non-hermitian symmetric cosets that can be used in tensor products with conformal central charge $c = 9$.

We proceed by specifying the conformal field theories defining the cosets of our interest. This is necessary because a 'Lie-algebraic coset' $\mathcal{C}$ as it stands in (2.1.1) is in itself far from defining a consistent modular invariant conformal field theory. We emphasize that although the theories described in this chapter have been introduced as formal cosets (2.1.1) already in 1989, they have previously *not* been shown to describe consistent conformal field theories. (By consistency of a conformal field theory we understand among other requirements that the characters of the theory carry a (projective) unitary representation of the modular group. Note that up to now it is even unknown whether a conformal field theory can be associated with every coset, and if so, whether this theory is unique.) To define the theories, we first determine the precise form of the affinization of the subalgebras involved. In particular we identify, in Subsection 2.3.1 the level of the u(1)-subalgebra that is present in each of the models.

Moreover, as is well known [68, 129], in order to obtain a modular invariant partition function, 'fields' in the coset theory have to be 'identified'. Problems arise when the length of the identification orbits is not constant; orbits of non-maximal length have to be 'resolved' [129, 130], which in general is a rather delicate issue. In Subsection 2.3.2 of this chapter we determine these identification rules. Furthermore we derive a formula, valid for any $N = 2$ coset theory of the form (2.1.1), for the order of the abelian group that is generated by the identification currents; this provides a convenient check for the completeness of the identification rules. The resolution of fixed points is dealt with in Subsection 2.3.3.

Finally, we derive in Section 2.4 a formula giving the full superconformal u(1)-charge of any Ramond ground state in terms of the length of an associated element of the Weyl group. (This length is conveniently calculated by means of Hasse diagrams; the diagrams corresponding to our models are described in Appendix 2.A.) This result is used to show that the set of Ramond ground states of any $N = 2$ coset model is invariant under charge conjugation.

2.2 Classification

In [97] a supersymmetric extension of the coset construction [73] was used to obtain a large class of superconformal coset models. By bosonizing the fermions of the super WZW theories involved in the construction of these models, one arrives at a level one $\mathrm{so}(2d)_1$ WZW theory. As a consequence, the models can be written as

$$\mathcal{C}[\bar{\mathsf{g}} \oplus \mathrm{so}(2d)_1/\bar{\mathsf{h}}]_k \,. \tag{2.2.1}$$

In the sequel we will adopt the notation of [96] and denote indices referring to generators of the algebra $\bar{\mathsf{g}}$ by capital letters $A, B, \ldots$, indices referring to the subalgebra $\bar{\mathsf{h}}$ by $a, b, \ldots$, and indices referring to the set $\bar{\mathsf{g}}\backslash\bar{\mathsf{h}}$, and hence also to $\mathrm{so}(2d)$, by $\bar{a}, \bar{b}, \ldots$. Thus in particular the currents generating g are denoted by $\hat{J}^A$, and the $\mathrm{so}(2d)$ algebra is generated by $\dim \bar{\mathsf{g}}/\bar{\mathsf{h}}$ fermions $j^{\bar{a}}$. Denoting the structure constants of $\bar{\mathsf{g}}$ by $f^{AB}_{\ \ C}$, the currents

$$\tilde{J}^a = \hat{J}^a - \frac{\mathrm{i}}{k} f^a_{\ \bar{b}\bar{c}} j^{\bar{b}} j^{\bar{c}} \tag{2.2.2}$$

then specify the embedding of h in $\bar{\mathsf{g}} \oplus \mathrm{so}(2d)$. [1] From the embedding (2.2.2) we can read off the levels of the simple subalgebras $\bar{\mathsf{h}}_i$ of

$$\bar{\mathsf{h}} = \hat{\bar{\mathsf{h}}} \oplus \mathrm{u}(1)^m = \bigoplus_i \bar{\mathsf{h}}_i \oplus \mathrm{u}(1)^m. \tag{2.2.3}$$

Namely,

$$k(\bar{\mathsf{h}}_i) = I_i\,(k + g^\vee) - h_i^\vee, \tag{2.2.4}$$

where $g^\vee$ and $h_i^\vee$ denote the dual Coxeter numbers of $\bar{\mathsf{g}}$ and $\bar{\mathsf{h}}_i$, respectively, and where I_i is the Dynkin index of the embedding $\bar{\mathsf{h}}_i \hookrightarrow \bar{\mathsf{g}}$, i.e. the relative length squared of the highest roots $\theta_{\bar{\mathsf{g}}}$ of $\bar{\mathsf{g}}$ and θ_i of $\bar{\mathsf{h}}_i$,

$$I_i := \frac{(\theta_{\bar{\mathsf{g}}}, \theta_{\bar{\mathsf{g}}})}{(\theta_i, \theta_i)}. \tag{2.2.5}$$

(Here and below we refer to an untwisted affine Kac–Moody algebra with horizontal algebra $\bar{\mathsf{f}}$ as $\mathsf{f} = \bar{\mathsf{f}}^{(1)}$ and to the Heisenberg algebra $\hat{\mathrm{u}}(1)$ by its horizontal subalgebra $\mathrm{u}(1)$. Also, we use the short hand notation f_k if $\mathsf{f} = \bar{\mathsf{f}}^{(1)}$ is at level k.)

With (2.2.4), the conformal central charge of the coset theory becomes

$$c = \frac{3}{2}\,(\dim \bar{\mathsf{g}} - \dim \bar{\mathsf{h}}) - \frac{(\theta_{\bar{\mathsf{g}}}, \theta_{\bar{\mathsf{g}}})\, g^\vee \dim \bar{\mathsf{g}} - \sum_i (\theta_i, \theta_i)\, h_i^\vee \dim \bar{\mathsf{h}}_i}{(\theta_{\bar{\mathsf{g}}}, \theta_{\bar{\mathsf{g}}})(k + g^\vee)}. \tag{2.2.6}$$

The symmetry algebra of the models (2.2.1) always contains the $N = 1$ supersymmetry algebra. To find a second supercurrent G^2, one starts with the most general ansatz expressing a spin 3/2 current of the coset theory (2.1.1) in terms of the currents $\hat{J}^A$ and the fermions $j^{\bar{a}}$ [96],

$$G^2(z) = \frac{2}{k}\,(h_{\bar{a}\bar{b}} :j^{\bar{a}} \hat{J}^{\bar{b}}: - \frac{\mathrm{i}}{3k} S_{\bar{a}\bar{b}\bar{c}} :j^{\bar{a}} j^{\bar{b}} j^{\bar{c}}:)\,. \tag{2.2.7}$$

Here the colons denote normal ordering, and S is a totally antisymmetrical tensor. This ansatz mimics the structure of the first supercurrent G^1 for which $h_{\bar{a}\bar{b}}$ and $S_{\bar{a}\bar{b}\bar{c}}$ are given by the Killing form $\kappa_{\bar{a}\bar{b}}$ and by the structure constants $f_{\bar{a}\bar{b}\bar{c}}$, respectively.

The calculation of the relevant operator products that involve $G^2(z)$ shows that the following set of equations for h and S is necessary and sufficient for enlarged supersymmetry:

$$\begin{aligned}
h_{\bar{a}\bar{b}} &= -h_{\bar{b}\bar{a}}, \qquad h_{\bar{a}\bar{b}} h_{\bar{b}\bar{c}} = -\delta_{\bar{a}\bar{c}}, && (2.2.8)\\
h^{\bar{a}\bar{b}} f_{\bar{b}\bar{c}e} &= f^{\bar{a}\bar{b}}{}_e\, h_{\bar{b}\bar{c}}, && (2.2.9)\\
f_{\bar{a}\bar{b}\bar{c}} &= h_{\bar{a}\bar{p}} h_{\bar{b}\bar{q}} f^{\bar{p}\bar{q}}{}_{\bar{c}} + \text{cyclic permutations in } \bar{a}, \bar{b} \text{ and } \bar{c}, && (2.2.10)\\
S_{\bar{a}\bar{b}\bar{c}} &= h_{\bar{a}\bar{p}} h_{\bar{b}\bar{q}} h_{\bar{c}\bar{r}} f^{\bar{p}\bar{q}\bar{r}}. && (2.2.11)
\end{aligned}$$

The condition (2.2.8) means that h is a complex structure on G/H, which is $\bar{\mathsf{h}}$-invariant by (2.2.9). (2.2.10) is a consistency condition, while (2.2.11) can be used to eliminate S from the problem.

This set of equations can also be understood in more geometrical terms. Namely, let $\bar{\mathsf{t}}$ denote the orthogonal complement of $\bar{\mathsf{h}}$ with respect to the Killing form κ of $\bar{\mathsf{g}}$ (this is well defined since, $\bar{\mathsf{g}}$ being semi-simple, κ is non-degenerate). Then the model $\mathcal{C}[\bar{\mathsf{g}} \oplus \mathrm{so}(2d)_1/\bar{\mathsf{h}}]_k$

[1] Unless stated otherwise, we use the summation convention, i.e. equal upper and lower indices should be contracted.

is $N = 2$ supersymmetric if and only if there exists a direct sum decomposition of vector spaces,

$$\bar{\mathsf{t}} = \bar{\mathsf{t}}_+ \oplus \bar{\mathsf{t}}_-, \tag{2.2.12}$$

which obeys the conditions that $\dim \bar{\mathsf{t}}_+ = \dim \bar{\mathsf{t}}_-$, that $\bar{\mathsf{t}}_+$ and $\bar{\mathsf{t}}_-$ separately form closed Lie algebras, and that the restriction of the Killing form to $\bar{\mathsf{t}}_+$ and to $\bar{\mathsf{t}}_-$ vanishes,

$$\kappa_{|\mathsf{t}_\pm} \equiv 0. \tag{2.2.13}$$

This geometric characterization is in fact rather easy to prove [96]. Suppose first that the theory $\mathcal{C}[\bar{\mathsf{g}} \oplus \mathrm{so}(2d)_1/\bar{\mathsf{h}}]_k$ is $N = 2$ supersymmetric. Define $\bar{\mathsf{t}}_\pm$ to be the eigenspaces corresponding to the eigenvalues $\pm\mathrm{i}$ of the complex structure h. Then the relations $\bar{\mathsf{t}} = \bar{\mathsf{t}}_+ \oplus \bar{\mathsf{t}}_-$ and $\dim \bar{\mathsf{t}}_+ = \dim \bar{\mathsf{t}}_-$ are immediate. Using (2.2.8) to (2.2.11), it is also easy to show that

$$[t_\pm^{\bar a}, t_\pm^{\bar b}] = \frac{1}{2}\,(\mathrm{i} f^{\bar a \bar b}{}_{\bar c} \pm S^{\bar a \bar b}{}_{\bar c})\; t_\pm^{\bar c}, \tag{2.2.14}$$

where $t_\pm^{\bar a}$ denotes the component of $t^{\bar a}$ in $\bar{\mathsf{t}}_\pm$. Thus the elements of $\bar{\mathsf{t}}_\pm$ close under the Lie bracket. Finally, for arbitrary $r_\pm,\, s_\pm \in \bar{\mathsf{t}}_\pm$ the antisymmetry (2.2.8) of h implies $\kappa(r_\pm, s_\pm) = \mp\mathrm{i}\,\kappa(hr_\pm, s_\pm) = \pm\mathrm{i}\,\kappa(r_\pm, hs_\pm) = -\kappa(r_\pm, s_\pm) = 0$, so that (2.2.13) holds. Conversely, given a decomposition like (2.2.12), define h by requiring $\bar{\mathsf{t}}_\pm$ to be the eigenspaces of h corresponding to the eigenvalues $\pm\mathrm{i}$, assuring that the second equation of (2.2.8) is fulfilled. Then (2.2.9), (2.2.10) can be shown to follow from the fact that $\bar{\mathsf{t}}_\pm$ are subalgebras, while (2.2.13) implies the first part of (2.2.8). Namely, for arbitrary $r, s \in \bar{\mathsf{t}}$ one has $r = r_+ + r_-$ and $s = s_+ + s_-$ with $r_\pm, s_\pm \in \bar{\mathsf{t}}_\pm$, and therefore $\kappa(hr, s) = \kappa(\mathrm{i}r_+ - \mathrm{i}r_-, s_+ + s_-) = \mathrm{i}\kappa(r_+, s_-) - \mathrm{i}\kappa(r_-, s_+) = -\kappa(r_+ + r_-, \mathrm{i}s_+ - \mathrm{i}s_-) = -\kappa(r, hs)$.

Our task is now to classify embeddings satisfying (2.2.8) to (2.2.11), or, equivalently, (2.2.12) and (2.2.13). As the following remarks show, we can assume that $\bar{\mathsf{g}}$ and $\bar{\mathsf{h}}$ are of equal rank. In [96] a sequential method has been introduced which allows us to reduce $N = 2$ coset theories with $\mathrm{rank}\,\bar{\mathsf{h}} < \mathrm{rank}\,\bar{\mathsf{g}}$ to the equal rank case. (It is worthwhile mentioning that the validity of this sequential algorithm has been proven in [96] only as far as the $N = 2$ superconformal algebras of the models are concerned. As for the field contents, the general belief is that for a chain of embeddings $\bar{\mathsf{f}} \hookrightarrow \bar{\mathsf{h}} \hookrightarrow \bar{\mathsf{g}}$ the coset theory $\mathcal{C}[\bar{\mathsf{g}}/\bar{\mathsf{f}}]$ carries the structure of the tensor product of the theories $\mathcal{C}[\bar{\mathsf{g}}/\bar{\mathsf{h}}]$ and $\mathcal{C}[\bar{\mathsf{h}}/\bar{\mathsf{f}}]$, albeit a non-product modular invariant must be used. This is easy to see if no field identification is necessary, and should also hold in the case when the identification currents do not have fixed points.) To apply the sequential method, one needs an intermediate subalgebra satisfying

$$\bar{\mathsf{h}} \subseteq \bar{\mathsf{h}} \oplus \mathrm{u}(1)^{\mathrm{rank}\,\bar{\mathsf{g}} - \mathrm{rank}\,\bar{\mathsf{h}}} \subseteq \bar{\mathsf{g}} \tag{2.2.15}$$

(direct sum of Lie algebras). Such an intermediate algebra exists only [133] for the so-called regular subalgebras. A regular subalgebra $\bar{\mathsf{h}} \hookrightarrow \bar{\mathsf{g}}$ is by definition (see e.g. [85]) a subalgebra for which every generator associated to a root of the subalgebra $\bar{\mathsf{h}}$ is also associated to a root of the overlying algebra $\bar{\mathsf{g}}$; all other subalgebras are called special. In [133] it was shown that the cosets derived from special subalgebras never have enlarged supersymmetry; correspondingly we can restrict ourselves in the sequel to regular subalgebras, and hence the sequential algorithm is applicable. Regular subalgebras have been classified by Dynkin [32]; their Dynkin diagram must be a subdiagram of the extended Dynkin diagram of the overlying algebra (the extended Dynkin diagram of a simple Lie algebra $\bar{\mathsf{g}}$ coincides with the Dynkin diagram of its affinization $\mathsf{g} = \bar{\mathsf{g}}^{(1)}$).

In short, we can restrict our attention to regular embeddings satisfying $\mathrm{rank}\,\bar{\mathsf{g}} = \mathrm{rank}\,\mathsf{h}$. We now turn to the classification of such embeddings generating $N=2$ superconformal coset theories. From the $N=2$ conditions (2.2.8) to (2.2.11), one easily deduces that

$$f^{cde}\, h_{\bar{a}\bar{b}}\, f^{\bar{a}\bar{b}}_{\ \ e} = 0 \tag{2.2.16}$$

for all c, d. We will denote by Δ_+, Δ_-, and $\Delta_{\bar{\mathsf{h}}}$ the sets of roots of $\bar{\mathsf{t}}_+$, $\bar{\mathsf{t}}_-$, and $\bar{\mathsf{h}}$, respectively, and define

$$\tilde{v}_\circ := \sum_{\bar{\alpha}\in\Delta_+} \bar{\alpha}. \tag{2.2.17}$$

Writing (2.2.16) in a Cartan–Weyl basis and comparing prefactors, we find

$$(\tilde{v}_\circ, \gamma) = 0 \quad \text{iff} \;\; \gamma \in \Delta_{\bar{\mathsf{h}}}. \tag{2.2.18}$$

This relation implies that

$$[\sum_{\bar{\alpha}\in\Delta_+} \bar{\alpha}_i H^i, T^a] = 0 \quad \text{for all} \quad T^a \in \bar{\mathsf{h}}\,, \tag{2.2.19}$$

where by H^i we denote the generators of the Cartan subalgebra, i.e. that $\bar{\mathsf{h}}$ contains a u(1) ideal with generator $\sum_{\bar{\alpha}\in\Delta_+} \bar{\alpha}_i H^i$. Thus the embedding $\bar{\mathsf{h}} \hookrightarrow \bar{\mathsf{g}}$ is such that the Dynkin diagram of $\bar{\mathsf{h}}$ is obtained from the extended Dynkin diagram of $\bar{\mathsf{g}}$ by removing at least two nodes. One can also show [96] that

$$(\tilde{v}_\circ, \bar{\beta}) \geq (\bar{\beta}, \bar{\beta}) > 0 \tag{2.2.20}$$

for all $\bar{\beta} \in \Delta_+$.

We claim that the subalgebras yielding $N=2$ superconformal cosets are precisely *diagram subalgebras*, i.e. subalgebras whose Dynkin diagram is contained in the *non-extended* Dynkin diagram of $\bar{\mathsf{g}}$. Moreover, if the Dynkin diagram of $\bar{\mathsf{h}}$ is obtained from that of $\bar{\mathsf{g}}$ by removing more than one node, then the sequential method alluded to above can be applied to reduce the theory to a tensor product; hence we can assume that only a single node is deleted. We will denote by $i_\circ$ the label of this distinguished node of the Dynkin diagram of $\bar{\mathsf{g}}$; thus, for example, $\alpha^{(i_\circ)}$ is the corresponding simple $\bar{\mathsf{g}}$-root that is not a root of $\bar{\mathsf{h}}$. Note that the notation $\tilde{v}_\circ$ introduced in (2.2.17) was chosen with foresight; for instance, denoting the fundamental $\bar{\mathsf{g}}$-weights by $\Lambda_{(i)}$, the relation (2.2.18) can be rephrased as

$$\tilde{v}_\circ \propto \Lambda_{(i_\circ)} \tag{2.2.21}$$

(the constant of proportionality, obtainable with the help of the strange formula, reads

$$\frac{(\theta_{\bar{\mathsf{g}}}, \theta_{\bar{\mathsf{g}}})\, g^\vee \dim \bar{\mathsf{g}} - \sum_i (\theta_i, \theta_i)\, h_i^\vee \dim \bar{\mathsf{h}}_i}{12 \sum_j G_{i_\circ j}}\,, \tag{2.2.22}$$

where $G_{ij} = (\Lambda_{(i)}, \Lambda_{(j)})$ denotes the metric on the weight space of $\bar{\mathsf{g}}$, i.e. the inverse of the symmetrized Cartan matrix).

To prove the above claim, we have to show that the highest root $\theta_{\bar{\mathsf{g}}}$ of $\bar{\mathsf{g}}$ is not a root of $\bar{\mathsf{h}}$. If $\theta_{\bar{\mathsf{g}}}$ were a root of $\bar{\mathsf{h}}$, then according to (2.2.18) it would satisfy $(\tilde{v}_\circ, \theta_{\bar{\mathsf{g}}}) = 0$. But this is not allowed, as can be seen with the help of the decomposition of $\theta_{\bar{\mathsf{g}}}$ in terms of the simple $\bar{\mathsf{g}}$-roots $\alpha^{(i)}$,

$$\theta_{\bar{\mathsf{g}}} = \sum_{i=1}^{\mathrm{rank}\,\bar{\mathsf{g}}} a_i\, \alpha^{(i)}. \tag{2.2.23}$$

Namely, the coefficients a_i on the right hand side of (2.2.23), known as the Coxeter labels of $\bar{\mathfrak{g}}$, are positive integers, and hence the inequality (2.2.20) implies $(\tilde{v}_\circ, \theta_{\bar{\mathfrak{g}}}) = \sum_i a_i\,(\tilde{v}_\circ, \alpha^{(i)}) \geq \sum_{\bar{\alpha}\in\Delta_+} a_i(\bar{\alpha}^{(i)}, \bar{\alpha}^{(i)}) > 0$. Thus $\theta_{\bar{\mathfrak{g}}}$ is not a root of $\bar{\mathfrak{h}}$, so that $\bar{\mathfrak{h}}$ is a diagram subalgebra of $\bar{\mathfrak{g}}$.

The converse is seen as follows. Given a diagram subalgebra $\bar{\mathfrak{h}}$ of $\bar{\mathfrak{g}}$, assign the root $\bar{\alpha}$ of $\bar{\mathfrak{t}}$ to belong to Δ_+ and Δ_-, respectively, iff it is a positive respectively a negative root of $\bar{\mathfrak{g}}$. Since we assumed that $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{h}}$ have equal rank, this prescription yields a decomposition of $\bar{\mathfrak{t}}$ of the form (2.2.12). It is now straightforward to check that the vector spaces generated by the elements corresponding to $\Delta_\pm$ satisfy the geometrical formulation of the $N=2$ conditions. Namely, nilpotency (2.2.13) is immediate from the well-known properties of the Killing form in a Cartan–Weyl basis; the dimensions of $\bar{\mathfrak{t}}_+$ and $\bar{\mathfrak{t}}_-$ coincide because positive and negative roots of $\bar{\mathfrak{g}} \setminus \bar{\mathfrak{h}}$ come in pairs; and the assertion that $\bar{\mathfrak{t}}_\pm$ close under the Lie bracket can be verified by using the fact that $(\tilde{v}_\circ, \bar{\alpha}) > 0$ iff $\bar{\alpha} \in \Delta_+$.

Clearly, the $N=2$ conditions (2.2.8) to (2.2.11) are particularly simple if the structure constants $f_{\bar{a}\bar{b}\bar{c}}$ vanish. As we will see shortly, the corresponding coset manifold is then a hermitian symmetric space. In this case we automatically have $\operatorname{rank}\bar{\mathfrak{h}} = \operatorname{rank}\bar{\mathfrak{g}}$. Moreover using the Jacobi identity together with the relation

$$2f^{\bar{a}\bar{c}d} f_{\bar{b}\bar{c}d} = f^{\bar{a}CD} f_{\bar{b}CD} = g^\vee \delta^{\bar{a}}_{\bar{b}}, \tag{2.2.24}$$

it is easy to show that

$$f_{\bar{c}\bar{d}e}\, h_{\bar{a}\bar{b}}\, f^{\bar{a}\bar{b}e} = g^\vee\, h_{\bar{c}\bar{d}}. \tag{2.2.25}$$

Similarly as with (2.2.18), another useful relation is obtained by writing (2.2.25) in a Cartan–Weyl basis; comparing prefactors one finds

$$(\tilde{v}_\circ, \bar{\gamma}) \equiv \sum_{\bar{\alpha}\in\Delta_+} (\bar{\alpha}, \bar{\gamma}) = g^\vee \quad \text{iff } \bar{\gamma} \in \Delta_+. \tag{2.2.26}$$

With these results, we are in a position to classify all subalgebras yielding hermitian symmetric spaces. Let us first sketch the way these spaces are usually described in the mathematical literature (see e.g. [82]). Given $f_{\bar{a}\bar{b}\bar{c}} = 0$, it is possible to define an involutive automorphism σ of the Lie algebra $\bar{\mathfrak{g}}$ such that the subalgebra left invariant by σ is equal to $\bar{\mathfrak{h}}$, namely $\sigma(T^a) := T^a$, $\sigma(T^{\bar{a}}) := -T^{\bar{a}}$. Lie algebras admitting such an automorphism are called orthogonal involutive Lie algebras and have been classified by Cartan; a complete list can be found e.g. in [82, p. 354]. Because of (2.2.19), among the orthogonal involutive Lie algebras one only has to consider those whose fixed algebra contains a u(1) ideal. Finally, one verifies by inspection that for all such Lie algebras the $N=2$ conditions are fulfilled.

(The nomenclature used above arises from the following geometrical setting. The fact that $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{h}}$ form an orthogonal involutive Lie algebra can be shown to be equivalent to the property that the homogeneous space G/H, with G and H the compact simply connected Lie groups corresponding to $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{h}}$, is a *Riemannian globally symmetric space*. These spaces are defined as follows. For a Riemannian manifold, a neighbourhood of any point p of the manifold can be described by mapping a sphere in the tangent space at p on the neighbourhood; via this map the reflection about the origin of the tangent space (the pre-image of p) induces a mapping τ of this neighbourhood. If τ is an isometry, the manifold is called a locally symmetric space; if in addition τ can be extended to a global isometry, the manifold is called a globally symmetric space. It can be shown that all globally symmetric spaces are homogeneous spaces, i.e. isomorphic to the quotient of a simply connected Lie group by a closed subgroup. In this geometrical context the condition (2.2.19) means that G/H carries in addition an almost complex structure J which is *hermitian*, i.e. the metric g

Table 2.1: Hermitian symmetric coset theories (HSS) and their Virasoro charges

$\mathcal{C}[\bar{\mathsf{g}}/\bar{\mathsf{h}}]_k$	c	name
$\mathcal{C}[A_{m+n-1}/A_{m-1}\oplus A_{n-1}\oplus \mathrm{u}(1)]_k$	$3kmn/(k+m+n)$	$(A,\, m,\, n,\, k)$
$\mathcal{C}[B_{n+1}/B_n\oplus \mathrm{u}(1)]_k$	$3k(2n+1)/(k+2n+1)$	$(B,\, 2n+1,\, k)$
$\mathcal{C}[D_{n+1}/D_n\oplus \mathrm{u}(1)]_k$	$6kn/(k+2n)$	$(B,\, 2n,\, k)$
$\mathcal{C}[C_n/A_{n-1}\oplus \mathrm{u}(1)]_k$	$3kn(n+1)/2(k+n+1)$	$(C,\, n,\, k)$
$\mathcal{C}[D_n/A_{n-1}\oplus \mathrm{u}(1)]_k$	$3kn(n-1)/2(k+n-2)$	$(D,\, n,\, k)$
$\mathcal{C}[E_6/D_5\oplus \mathrm{u}(1)]_k$	$48k/(k+12)$	$(E6,\, k)$
$\mathcal{C}[E_7/E_6\oplus \mathrm{u}(1)]_k$	$81k/(k+18)$	$(E7,\, k)$

satisfies $\mathrm{g}(JX,JY)=\mathrm{g}(X,Y)$ for all elements X, Y of the tangent space. It can be shown that for homogeneous spaces this automatically implies that J is Kählerian, i.e. covariantly constant. In the general case where $f_{\bar{a}\bar{b}\bar{c}}$ is non-vanishing (which is the situation in which we are interested in the present chapter), the homogeneous space G/H is no longer a Riemannian globally symmetric space, but as was shown in [96], it is nonetheless still a Kählerian space iff the $N=2$ conditions are fulfilled. We remark that for our purposes these geometric characterizations are of little use. In fact, one of the main achievements of the theory of homogeneous spaces was precisely to recast the problems in purely Lie algebraic terms, which finally provided a powerful handle on the geometric objects.)

Alternatively, the classification of hermitian symmetric spaces can be found by the following simple prescription [33]: the hermitian symmetric spaces are obtained by deleting a node of the Dynkin diagram of $\bar{\mathsf{g}}$ that corresponds to a so-called [48] *cominimal* fundamental weight, i.e. a fundamental $\bar{\mathsf{g}}$-weight $\Lambda_{(i)}$ such that $a_i=1$ in the decomposition (2.2.23) of the highest $\bar{\mathsf{g}}$-root $\theta_{\bar{\mathsf{g}}}$. To prove this characterization, we proceed as follows. Multiplying both sides of (2.2.23) with $\tilde{v}_\circ$ as defined in (2.2.17), one obtains

$$(\tilde{v}_\circ,\theta_{\bar{\mathsf{g}}}) = \sum_{i=1}^{\mathrm{rank}\,\bar{\mathsf{g}}} \sum_{\bar{\alpha}\in\Delta_+} a_i\,(\bar{\alpha},\alpha^{(i)}). \tag{2.2.27}$$

Now suppose that $\theta_{\bar{\mathsf{g}}}$ is a root of $\bar{\mathsf{h}}$. Then according to (2.2.18) one has $\sum_{\bar{\alpha}\in\Delta_+}(\bar{\alpha},\theta_{\bar{\mathsf{g}}})=0$. Given the fact that the Coxeter labels a_i are positive, we thus learn from (2.2.27) that $(\tilde{v}_\circ,\alpha^{(i)})=0$ for all simple roots. But then (2.2.18) and (2.2.26) imply that all simple roots of $\bar{\mathsf{g}}$ are contained in $\bar{\mathsf{h}}$, and hence $\bar{\mathsf{g}}=\bar{\mathsf{h}}$, showing that the coset would be trivial in this case. Thus again we conclude that $\theta_{\bar{\mathsf{g}}}$ cannot be a root of $\bar{\mathsf{h}}$. From (2.2.26) we then learn that the left hand side of (2.2.27) equals $g^\vee$. The right hand side can take this value only in the case when exactly one simple root of $\bar{\mathsf{g}}$ with Coxeter label equal to 1 is not contained in $\bar{\mathsf{h}}$. Now using the classification of regular subalgebras [32], it is straightforward to check that one obtains in this way exactly the same list as before.

In Table 2.1 we recall the list of all HSS models and their Virasoro charges (the shorthand notation displayed in column 3 is taken from [37]). We now return to the general case. Let us stress that we are in a position to give a complete list of *all* $N=2$ coset models. However, even when grouping these theories (of which there are infinitely many)

Table 2.2: Non-hermitian symmetric coset theories relevant for $c = 9$ tensor products

$\mathcal{C}[\bar{\mathfrak{g}}/\bar{\mathfrak{h}}]_k$	c	name
$\mathcal{C}[B_n/A_{n-1} \oplus \mathrm{u}(1)]_k$	$\frac{3}{2}n(n+1) - \frac{3n^3}{k+2n-1}$	(BA, n, k)
$\mathcal{C}[B_n/B_{n-2} \oplus A_1 \oplus \mathrm{u}(1)]_k$	$12n - 15 - \frac{24(n-1)^2}{k+2n-1}$	(BB, n, k)
$\mathcal{C}[C_n/C_{n-1} \oplus \mathrm{u}(1)]_k$	$6n - 3 - \frac{6n^2}{k+n+1}$	(CC, n, k)
$\mathcal{C}[C_3/A_1 \oplus A_1 \oplus \mathrm{u}(1)]_k$	$21 - \frac{75}{k+4}$	$(C3, k)$
$\mathcal{C}[C_4/A_2 \oplus A_1 \oplus \mathrm{u}(1)]_k$	$36 - \frac{162}{k+5}$	$(C4, k)$
$\mathcal{C}[D_4/A_1 \oplus A_1 \oplus A_1 \oplus \mathrm{u}(1)]_k$	$27 - \frac{150}{k+6}$	$(D4, k)$
$\mathcal{C}[D_5/A_2 \oplus A_1 \oplus A_1 \oplus \mathrm{u}(1)]_k$	$45 - \frac{324}{k+8}$	$(D5_1, k)$
$\mathcal{C}[D_5/A_3 \oplus A_1 \oplus \mathrm{u}(1)]_k$	$39 - \frac{294}{k+8}$	$(D5_2, k)$
$\mathcal{C}[F_4/C_3 \oplus \mathrm{u}(1)]_k$	$45 - \frac{384}{k+9}$	$(F4, k)$
$\mathcal{C}[G_2/A_1^{>} \oplus \mathrm{u}(1)]_k$	$15 - \frac{50}{k+4}$	$(G2_1, k)$
$\mathcal{C}[G_2/A_1^{<} \oplus \mathrm{u}(1)]_k$	$15 - \frac{54}{k+4}$	$(G2_2, k)$

into a finite number of series, this list still remains rather long, and we will not present it here in full detail. Rather, we list only those models that can be used as factor theories in tensor products with conformal central charge $c = 9$ (as well as some other models which fall into infinite series that contain models relevant for $c = 9$). The interest in these models comes from superstring theory where they can be used for the inner sector of heterotic string vacua [67], and from the possible relation with Calabi–Yau manifolds and with Landau–Ginzburg theories.

The result of our classification is presented in Table 2.2, where we supply the coset theories together with their conformal central charge (as calculated according to (2.2.6)) and with a short-hand name that derives from the Lie algebras involved. From the classification of regular subalgebras described above, the relevant embedding $\bar{\mathfrak{h}} \hookrightarrow \bar{\mathfrak{g}}$ is determined uniquely by the pair $\bar{\mathfrak{g}}$, $\bar{\mathfrak{h}}$ of Lie algebras for all entries in Table 2.2 except for the two models with $\bar{\mathfrak{g}} = G_2$. In the latter cases we use the superscripts '$<$' and '$>$' to indicate that the A_1-subalgebra corresponds to the short and long simple root of G_2, respectively.

For convenience we have grouped some models in the table in three series. From the above remarks it should be clear that there is no physical distinction between the models within these series and the other models. The different appearance is a mere artifact of our string theory-oriented condition on the central charges. We also emphasize that the list in Table 2.2 does *not* contain all $N=2$ coset theories with central charge $c \leq 9$. Their number is much larger, but most of them cannot be combined with other known $N=2$ theories to obtain $c=9$ tensor product theories. For instance, we have not included the model $\mathcal{C}[D_6/D_4 \oplus A_1 \oplus \mathrm{u}(1)]_k$, which has $c = 51 - \frac{486}{k+10}$. For level $k=1$ the conformal central charge is $c = \frac{75}{11} < 9$, but there does not exist any $N=2$ model with $c = \frac{24}{11}$ which could be tensored with this theory to arrive at a $c=9$ conformal field theory.

Note that the number of the models so obtained is relatively small. This can be traced back to two simple facts. First, if $\bar{\mathfrak{g}}$ is a Lie algebra of A type, all subalgebras lead to coset theories of the HSS type. Second, for any fixed Lie algebra $\bar{\mathfrak{g}}$, the central charge of the coset theory grows rather fast when one moves the node with label $i_\circ$ away from the 'margin' of the Dynkin diagram of $\bar{\mathfrak{g}}$ towards the inner part (note that except for A_r all cominimal fundamental weights, i.e. those leading to hermitian symmetric cosets, correspond to marginal nodes).

2.3 Specification of the coset theories

As already emphasized, the 'Lie-algebraic coset' as it stands in (2.1.1) is in itself far from defining a consistent modular invariant conformal field theory. In this section we will provide a detailed specification of the conformal field theory.

In fact, the first step to do so was already taken in the previous section when we computed the levels (2.2.4) of the semi-simple part of the subalgebra $\bar{\mathfrak{h}}$, i.e. of the simple ideals in the decomposition (2.2.3), which in the case of our interest reads

$$\bar{\mathfrak{h}} = \hat{\bar{\mathfrak{h}}} \oplus \mathrm{u}(1) = \bigoplus_i \bar{\mathfrak{h}}_i \oplus \mathrm{u}(1). \tag{2.3.1}$$

But the abelian ideal of $\bar{\mathfrak{h}}$ must be specified as well.

2.3.1 The u(1) subalgebra

The conformal field theory corresponding to a u(1) algebra has Virasoro charge $c=1$. As all $c=1$ conformal field theories have been classified [27, 15] and their field contents is known, it is sufficient to have a look at the conformal dimensions occurring in the conformal field theory we are after, which, as we shall show now, in turn are fixed by the embedding.

The direction of the u(1) in root space is given by $\tilde{v}_\circ$. From the embedding (2.2.2) we read off the precise form of the u(1)-generator $\mathcal{Q}$; it is proportional to

$$\tilde{\mathcal{Q}}(z) := (\tilde{v}_\circ, H(z)) + \sum_{\bar{\alpha}\in\Delta_+} (\tilde{v}_\circ, \bar{\alpha})\, {:}\Psi^{\bar{\alpha}}\Psi^{-\bar{\alpha}}{:}(z)\,. \tag{2.3.2}$$

Here ${:}\Psi^{\bar{\alpha}}\Psi^{-\bar{\alpha}}{:}$ denotes the fermion number operator for the complex fermion that is associated to the root $\bar{\alpha}$; it takes integer values in the Neveu–Schwarz sector and half-integer values in the Ramond sector; H stands for the Cartan subalgebra currents of $\mathfrak{g}$.

By replacing $\tilde{v}_\circ$ in (2.3.2) by an appropriate multiple $v_\circ$ of $\tilde{v}_\circ$, all eigenvalues of $\mathcal{Q}$ can be taken to be integers. We will assume that we have chosen the smallest multiple fulfilling this requirement (otherwise we would be forced later on to introduce additional identification currents that have a non-trivial component only in the u(1) part), and write

$$\tilde{v}_\circ \equiv \sum_{\bar{\alpha}\in\Delta_+} \bar{\alpha} = \xi_\circ\, v_\circ; \tag{2.3.3}$$

the number $\xi_\circ$ turns out to be an integer or half integer in all cases except for the model of type $G2_1$ for which $\xi_\circ = 5/3$. The operator product of $\mathcal{Q}$ with itself then reads

$$\mathcal{Q}(z)\,\mathcal{Q}(w) \sim \frac{\mathcal{N}}{(z-w)^2}, \tag{2.3.4}$$

with

$$\mathcal{N} = (v_\circ, v_\circ)\,k + \sum_{\bar\alpha\in\Delta_+} (v_\circ, \bar\alpha)^2 = (v_\circ, v_\circ)(k + g^\vee). \tag{2.3.5}$$

Denote by φ a canonically normalized free boson, satisfying $\mathrm{i}\partial\varphi(z)\,\mathrm{i}\partial\varphi(w) \sim (z-w)^{-2}$. Expressing $\mathcal{Q}$ in terms of φ, i.e. $\mathcal{Q} = \sqrt{\mathcal{N}}\mathrm{i}\partial\varphi$, we obtain the energy-momentum tensor

$$T = \frac{1}{2} : \mathrm{i}\partial\varphi\,\mathrm{i}\partial\varphi := \frac{1}{2\mathcal{N}} : \mathcal{Q}\mathcal{Q} : \ . \tag{2.3.6}$$

Thus the conformal dimension Δ of a primary field is

$$\Delta = \frac{Q^2}{2\mathcal{N}}, \tag{2.3.7}$$

with Q the u(1)-charge of the field, i.e. the eigenvalue of $\mathcal{Q}$.

Thus the u(1) theory in question is the conformal field theory of a free boson compactified on a circle whose radius is adjusted (or, in other words, the chiral algebra is enlarged) precisely in such a manner that the charges are identified modulo $\mathcal{N}$. [2] In the sequel we will denote this theory by $\mathrm{u}(1)_{\mathcal{N}}$. The relevant values of the integer $\mathcal{N}$ (as well as the explicit values of the levels of the simple ideals $\bar{\mathsf{h}}_i$ computed according to (2.2.4)) for the cases of our interest are provided in Table 2.3.

For hermitian symmetric cosets it was noticed [123] that $\mathcal{N}$ is always a divisor of $\mathcal{N}_0(\mathsf{g},\mathsf{h})$, where

$$\mathcal{N}_0(\mathsf{g},\mathsf{h}) = I_c(\bar{\mathsf{g}}) \cdot I_c(\hat{\bar{\mathsf{h}}}) \cdot (k + g^\vee). \tag{2.3.8}$$

Here I_c stands for the index of connection (i.e. the number of conjugacy classes, which is equal to the order of the center Z of the corresponding universal covering Lie group) of a Lie algebra, and $I_c(\hat{\bar{\mathsf{h}}}) \equiv \prod_i I_c(\bar{\mathsf{h}}_i)$, where $\bar{\mathsf{h}}_i$ are the simple algebras which appear in the decomposition $\hat{\bar{\mathsf{h}}} = \bigoplus_i \bar{\mathsf{h}}_i$ of $\hat{\bar{\mathsf{h}}}$ into simple ideals. In fact, in most cases one even has $\mathcal{N} = \mathcal{N}_0(\mathsf{g},\mathsf{h})$; also, by introducing additional identification currents with a non-trivial component only in the u(1) part one could use (as has been done in [123]) $\mathcal{N}_0(\mathsf{g},\mathsf{h})$ in place of $\mathcal{N}$. For non-hermitian symmetric cosets, however, we encounter two cases, namely $(G2_1, k)$ and the models (BA, n, k) with n odd, where the value of $\mathcal{N}$ is larger than $\mathcal{N}_0(\mathsf{g},\mathsf{h})$.

2.3.2 Selection rules and field identification

Our next task is to identify the physical fields of the theories of our interest, as described in Section 1.5. In the case $N = 2$ cosets, the branching functions $b^{\Lambda,\mathrm{x}}_{\lambda,Q}$ are the coefficient functions in the decomposition

$$\mathcal{X}_{\Lambda,\mathrm{x}}(\tau) = \sum_\lambda b^{\Lambda,\mathrm{x}}_{\lambda,Q}(\tau)\,\chi_{\lambda,Q}(\tau) \tag{2.3.9}$$

---

[2] Thus e.g. $\mathrm{u}(1)_2$ is the theory for which the extended algebra is the level one $A_1^{(1)}$ Kac–Moody algebra, and $\mathrm{u}(1)_4 \cong \mathrm{so}(2)_1$.

Table 2.3: The values of the levels k_i and of $\mathcal{N}$ for non-hermitian symmetric coset theories

name	$\mathcal{C}[\bar{\mathsf{g}}_k \oplus \mathrm{so}(2d)_1 / \bigoplus_i (\bar{\mathsf{h}}_i)_{k_i} \oplus \mathrm{u}(1)_{\mathcal{N}}]$
(BA,n,k) , n even	$\mathcal{C}[(B_n)_k \oplus \mathrm{so}(n^2+n)_1 \,/\, (A_{n-1})_{k+n-1} \oplus \mathrm{u}(1)_{n(k+2n-1)}]$
(BA,n,k) , n odd	$\mathcal{C}[(B_n)_k \oplus \mathrm{so}(n^2+n)_1 \,/\, (A_{n-1})_{k+n-1} \oplus \mathrm{u}(1)_{4n(k+2n-1)}]$
$(BB,3,k)$	$\mathcal{C}[(B_3)_k \oplus \mathrm{so}(14)_1 \,/\, (A_1)_{2k+8} \oplus (A_1)_{k+3} \oplus \mathrm{u}(1)_{2(k+5)}]$
(BB,n,k) , $n>3$	$\mathcal{C}[(B_n)_k \oplus \mathrm{so}(8n-10)_1 \,/\, (B_{n-2})_{k+4} \oplus (A_1)_{k+2n-3} \oplus \mathrm{u}(1)_{2(k+2n-1)}]$
(CC,n,k)	$\mathcal{C}[(C_n)_k \oplus \mathrm{so}(4n-2)_1 \,/\, (C_{n-1})_{k+1} \oplus \mathrm{u}(1)_{2(k+n+1)}]$
$(C3,k)$	$\mathcal{C}[(C_3)_k \oplus \mathrm{so}(14)_1 \,/\, (A_1)_{k+2} \oplus (A_1)_{2k+6} \oplus \mathrm{u}(1)_{4(k+4)}]$
$(C4,k)$	$\mathcal{C}[(C_4)_k \oplus \mathrm{so}(24)_1 \,/\, (A_2)_{2k+7} \oplus (A_1)_{k+3} \oplus \mathrm{u}(1)_{6(k+5)}]$
$(D4,k)$	$\mathcal{C}[(D_4)_k \oplus \mathrm{so}(18)_1 \,/\, (A_1)_{k+4} \oplus (A_1)_{k+4} \oplus (A_1)_{k+4} \oplus \mathrm{u}(1)_{2(k+6)}]$
$(D5_1,k)$	$\mathcal{C}[(D_5)_k \oplus \mathrm{so}(30)_1 \,/\, (A_2)_{k+5} \oplus (A_1)_{k+6} \oplus (A_1)_{k+6} \oplus \mathrm{u}(1)_{12(k+8)}]$
$(D5_2,k)$	$\mathcal{C}[(D_5)_k \oplus \mathrm{so}(26)_1 \,/\, (A_3)_{k+4} \oplus (A_1)_{k+6} \oplus \mathrm{u}(1)_{2(k+8)}]$
$(F4,k)$	$\mathcal{C}[(F_4)_k \oplus \mathrm{so}(30)_1 \,/\, (C_3)_{k+5} \oplus \mathrm{u}(1)_{2(k+9)}]$
$(G2_1,k)$	$\mathcal{C}[(G_2)_k \oplus \mathrm{so}(10)_1 \,/\, (A_1)_{k+2} \oplus \mathrm{u}(1)_{6(k+4)}]$
$(G2_2,k)$	$\mathcal{C}[(G_2)_k \oplus \mathrm{so}(10)_1 \,/\, (A_1)_{3k+10} \oplus \mathrm{u}(1)_{2(k+4)}]$

of the product of the characters of g and $\mathrm{so}(2d)$ with respect to the characters of h. Here Λ and λ stand for integrable highest weights of g and $\hat{\mathsf{h}}$, respectively, and Q for an allowed u(1)-charge, while x denotes an integrable highest weight of $\mathrm{so}(2d)$ at level one, i.e. the singlet (0), vector (v), spinor (s), or conjugate spinor (c) highest weight.

We have seen in Section 1.5 that the correct way to arrive at a modular invariant theory is to interpret the primary fields of the coset theory in terms of equivalence classes of branching functions [68, 129, 83]; this prescription is usually referred to as *field identification.* Under the assumptions mentioned in Section 1.5.2, the equivalence relation is uniquely determined by the conjugacy class selection rules. If all equivalence classes have the same number of elements, one can simply define a primary field as an equivalence class of branching functions. Its character is then just any of the (identical) branching functions of its representatives, and accordingly the primary field can be denoted as $\Phi^{\Lambda,\mathrm{x}}_{\lambda,Q}$, where $(\Lambda,\mathrm{x},\lambda,Q)$ is a representative combination of the relevant highest weights Λ of g, x of $\mathrm{so}(2d)$, λ of $\hat{\bar{\mathsf{h}}}$, and Q of u(1). If, on the other hand, several distinct sizes of equivalence classes are present, fixed points have to resolved; this will be addressed in the next subsection.

Our task is thus to find the relevant selection rules and deduce the identifications implied by them. This is a straightforward exercise in group theory, but is still somewhat involved owing to the non-trivial embedding of $\bar{\mathsf{h}}$ in $\mathrm{so}(2d)$. A convenient way to state these selection rules is to characterize the non-vanishing branching functions by the fact that their monodromy charge with respect to the *identification group* $\mathcal{G}_{\mathrm{id}}$ of the coset theory vanishes. The identification group contains all identification currents; we will denote its order by $|\mathcal{G}_{\mathrm{id}}|$. Its orbits on the branching functions are just the equivalence classes we are looking for. To qualify as an identification current, a simple current must have integer conformal weight [130] (this allows for a simple check of our results for the identification currents); this condition must be met because any identification current is a representative of the equivalence class describing the identity primary field, and conformal weights are constant modulo integers on each identification orbit.

To begin the description of identification currents for the theories of our interest, we derive a formula for the order $|\mathcal{G}_{\rm id}|$ of the identification group of any $N=2$ coset theory of the form (2.1.1). This provides an important check for the completeness of the selection rules that will be listed below. Our starting point is the formula [103]

$$|\mathcal{G}_{\rm id}| = \left| \frac{L^*_{\bar{\mathfrak{g}}}}{L^\vee_{\bar{\mathfrak{h}}}} \right| . \tag{2.3.10}$$

Here L denotes the root lattice of a reductive algebra, and $L^\vee$ the corresponding coroot lattice. The symbol '$*$' is used to indicate the dual lattice; in particular $(L^\vee)^* = L^W$, where L^W is the weight lattice. Writing the relation (2.3.10) in terms of the dual lattices and denoting the volume of the unit cell by 'vol', we see that

$$|\mathcal{G}_{\rm id}| = \left| \frac{(L^\vee_{\bar{\mathfrak{h}}})^*}{(L^*_{\bar{\mathfrak{g}}})^*} \right| = \left| \frac{L^W_{\bar{\mathfrak{h}}}}{L_{\bar{\mathfrak{g}}}} \right| = \frac{{\rm vol}(L_{\bar{\mathfrak{g}}})}{{\rm vol}(L^W_{\bar{\mathfrak{h}}})} . \tag{2.3.11}$$

Since the direction of the u(1) is orthogonal to $\hat{\bar{\mathfrak{h}}}$ in weight space, it follows that

$$ {\rm vol}(L^W_{\bar{\mathfrak{h}}}) = {\rm vol}(L^W_{\hat{\bar{\mathfrak{h}}}}) \cdot 1 = {\rm vol}(L^W_{\hat{\bar{\mathfrak{h}}}}) \tag{2.3.12}$$

and

$$ {\rm vol}(L_{\bar{\mathfrak{g}}}) = {\rm vol}(L_{\hat{\bar{\mathfrak{h}}}}) \cdot Q_{i_\circ} , \tag{2.3.13}$$

where $Q_{i_\circ}$ is the u(1)-charge of the simple root $\alpha^{(i_\circ)}$. Thus

$$|\mathcal{G}_{\rm id}| = Q_{i_\circ} \frac{{\rm vol}(L_{\hat{\bar{\mathfrak{h}}}})}{{\rm vol}(L^W_{\hat{\bar{\mathfrak{h}}}})} = Q_{i_\circ} \left| \frac{L^W_{\hat{\bar{\mathfrak{h}}}}}{L_{\hat{\bar{\mathfrak{h}}}}} \right| \equiv Q_{i_\circ}\, I_{\rm c}(\hat{\bar{\mathfrak{h}}}). \tag{2.3.14}$$

Here $I_{\rm c}(\hat{\bar{\mathfrak{h}}}) = \prod_i I_{\rm c}(\bar{\mathfrak{h}}_i)$ as in (2.3.8), and we made use of the fact that $I_{\rm c}(\bar{\mathfrak{h}}) = |L^W_{\bar{\mathfrak{h}}}/L_{\bar{\mathfrak{h}}}|$ for any simple Lie algebra $\bar{\mathfrak{h}}$.

While the result (2.3.14) is completely general, the precise form of the group theoretical selection rules must be determined in a case by case study. To do so, a rather tedious investigation of the way $\bar{\mathfrak{h}}$ is embedded in $\bar{\mathfrak{g}} \oplus {\rm so}(2d)$ is necessary. In particular a careful handling of the embedding of $\bar{\mathfrak{h}}$ in so$(2d)$ (best to be described in an orthogonal basis which corresponds to the free fermion realization of so$(2d)_1$), which is a special [3] embedding, is required. We list in Table 2.4 our results for the identification currents ϕ_J of all non-hermitian symmetric $N=2$ coset theories that can be used in $c=9$ tensor products. We use the notation $J \equiv (J^{(\mathfrak{g})}, J^{({\rm so}(2d))} \,/\, J^{(\mathfrak{h}_1)}, J^{(\mathfrak{h}_2)}, \dots, J^{({\rm u}(1))})$. In the individual entries, we write $J_{\rm v}$ for the vector simple current, and $J_{\rm s}$ and $J_{\rm c}$ for the spinor and conjugate spinor simple currents, respectively, of B and D type algebras, while for A type algebras, J stands for the simple current that acts as $\mu^i \mapsto \mu^{i+1 \bmod (r+1)}$ on the Dynkin labels of a A_r-weight (this current is associated with a marginal node of the Dynkin diagram; it has maximal order, and hence generates all simple currents of the theory); finally, for the u(1) part a

Table 2.4: The identification groups for non-hermitian symmetric coset theories

name	$\lvert\mathcal{G}_{\rm id}\rvert$	generators of $\mathcal{G}_{\rm id}$	fixed p.
(BA,n,k) , n even	n	$(J,1\,/\,J,k+2n-1)$	–
(BA,n,k) , n odd	$2n$	$(J,J_{\rm v}\,/\,J,2(k+2n-1))$	–
(BB,n,k)	4	$\left\{\begin{array}{c}(J,1\,/\,J,1,0)\\(J,1\,/\,1,J,\pm(k+2n-1))\end{array}\right.$	$\left\{\begin{array}{c}++\\-\end{array}\right.$
(CC,n,k)	2	$(J,(J_{\rm v})^n\,/\,J,\pm(k+n+1))$	–
$(C3,k)$	4	$\left\{\begin{array}{c}(J,1\,/\,J,J,0)\\(J,J_{\rm v}\,/\,J,1,\pm 2(k+4))\end{array}\right.$	$\left\{\begin{array}{c}+\\-\end{array}\right.$
$(C4,k)$	6	$(J,J_{\rm v}\,/\,J,J,-(k+5))$	–
$(D4,k)$	8	$\left\{\begin{array}{c}(J_{\rm c},1\,/\,J,J,1,0)\\(J_{\rm s},1\,/\,J,1,J,0)\\(1,J_{\rm v}\,/\,J,J,J,\pm(k+6))\end{array}\right.$	$\left\{\begin{array}{c}+\\+\\-\end{array}\right.$
$(D5_1,k)$	24	$\left\{\begin{array}{c}(J_{\rm s},J_{\rm v}\,/\,J,J,1,-(k+8))\\(J_{\rm v},1\,/\,1,J,J,0))\end{array}\right.$	$\left\{\begin{array}{c}-\\+\end{array}\right.$
$(D5_2,k)$	8	$\left\{\begin{array}{c}(J_{\rm v},J_{\rm v}\,/\,1,J,\pm(k+8))\\(J_{\rm s},1\,/\,J,J,0))\end{array}\right.$	$\left\{\begin{array}{c}-\\+\end{array}\right.$
$(F4,k)$	2	$(1,1\,/\,J,\pm(k+9))$	–
$(G2_1,k)$	2	$(1,J_{\rm v}\,/\,J,\pm 3(k+4))$	–
$(G2_2,k)$	2	$(1,J_{\rm v}\,/\,J,\pm(k+4))$	–

field is simply denoted by its u(1)-charge Q. Notice that in Table 2.4 we only give a set of generators of the group $\mathcal{G}_{\rm id}$ rather than all of its elements.

The way in which we arrived at these results is best described by giving an example. Thus let us have a look at the coset theory denoted by $(C4,k)$. We denote the Dynkin labels of weights of $\bar{\mathfrak{g}} = C_4$ by Λ^i, $i=1,2,3,4$, of weights of $\bar{\mathfrak{h}}_1 = A_2$ by λ^1 and λ^2, of weights of $\bar{\mathfrak{h}}_2 = A_1$ by λ^4, and the u(1)-charge by Q. By analysing the embedding, we find that these numbers must be related by

$$3\Lambda^1 + 3\Lambda^3 - 6N + 2\lambda^1 + 4\lambda^2 + 3\lambda^4 + Q \equiv 0 \bmod 6, \tag{2.3.15}$$

where $6N$ stands for the sum of six different eigenvalues of the Cartan generators of so(24), which have integer values in the Neveu–Schwarz sector and half integer values in the Ramond sector. We want to interpret this result as a relation for monodromy charges, namely

$$Q_{\rm m}[C_4] + Q_{\rm m}[{\rm so}(24)] + Q_{\rm m}[A_2] + Q_{\rm m}[A_1] + Q_{\rm m}[{\rm u}(1)] \equiv 0 \bmod 1. \tag{2.3.16}$$

It is easily checked that $N \bmod \mathbb{Z}$ is the monodromy charge with respect to the vector current $J_{\rm v}$ of so$(2d)$, and that Q/p is the monodromy charge with respect to the current

[3] This is not in conflict with the previously mentioned result [133] that $N=2$ symmetry requires regular embeddings. The part of the embedding that must be regular is $\bar{\mathfrak{h}} \hookrightarrow \bar{\mathfrak{g}}$ rather than $\bar{\mathfrak{h}} \hookrightarrow$ so$(2d)$.

with u(1)-charge $-\mathcal{N}/p$ of the u(1) theory. For the g and h_i parts, the identification currents can also be fixed uniquely, simply because all simple currents, as well as the associated monodromy charges, of the corresponding WZW theories are known. We then arrive at the combination

$$\phi_J = (J, J_{\rm v}\,/\,J, J, -(k+5)) \tag{2.3.17}$$

of simple currents that has (2.3.16) as its monodromy charge. This current has order 6. This coincides with the result of formula (2.3.14) for the order of the identification group, and hence we have already found all identification currents.

2.3.3 Fixed points

If the equivalence classes described in the previous subsection have different sizes N_i, the identification procedure becomes more complicated. Note that the maximal size of a class is equal to the size $N = |\mathcal{G}_{\rm id}|$ of the equivalence class of the identity primary field, and that any other allowed size is a divisor of N. The equivalence classes of size $N_f < N$ should correspond to N/N_f distinct physical fields [129, 130]. The required resolution of classes of non-maximal size into primary fields is problematic because not all necessary pieces of information are directly supplied by the embedding; in other words, the resolution potentially introduces some arbitrariness in the description of primary fields. In particular we do not know the characters of the individual primary fields into which such a class f is resolved. We do know, however, their sum, since modular invariance imposes the constraint

$$\sum_i \mathcal{X}_{f_i} = \mathcal{X}_f, \tag{2.3.18}$$

where $\mathcal{X}_f$ denotes the original branching function of the class f.

Now given the naive S-matrix element S_{fg} between two fixed points f and g, one can make the ansatz (1.5.7)

$$\tilde{S}_{f_i\,g_j} = \frac{N_f N_g}{N}\, S_{fg} + \Gamma^{fg}_{ij} \tag{2.3.19}$$

for the full S-matrix between different fields f_i, g_i into which the fixed points are to be resolved. The matrix Γ introduced here must be symmetric (with respect to the double index (f,i)), but a priori is otherwise arbitrary. Modular invariance can be shown to imply the sum rules (1.5.8)

$$\sum_i \Gamma^{fg}_{ij} = 0 = \sum_j \Gamma^{fg}_{ij}. \tag{2.3.20}$$

To find a solution for Γ we assume that with respect to the individual entries of the multi-index $(f,i) \equiv (\Lambda, {\rm x}, \lambda, Q, i)$ it factorizes as

$$\Gamma_{ij}^{\Lambda,{\rm x},\lambda,Q;\Lambda',{\rm x}',\lambda',Q'} = \Gamma_{(\mathsf{g})}^{\Lambda\,\Lambda'}\,\Gamma_{({\rm so}(2d))}^{{\rm x}\,{\rm x}'}\,\Gamma_{(\hat{\mathsf{h}})}^{\lambda\,\lambda'}\,\Gamma_{({\rm u}(1))}^{Q\,Q'}\,P_{ij}, \tag{2.3.21}$$

where

$$P_{ij} = \delta_{ij} - \frac{N_f}{N}\,. \tag{2.3.22}$$

Since in all cases of our interest the fixed points f have order $N/N_f = 2$ and must therefore be resolved into two fields, the fact that (2.3.22) can be factored out is an immediate consequence of the sum rules (2.3.20). Following [130], with the factorization assumption (2.3.21) we can identify in all cases a so-called fixed point conformal field theory, whose characters can be added to the branching functions to get the full collection of primary fields; these characters are nothing but the summands $\mathcal{X}_{f_i}(\tau)$ in the decomposition (2.3.18) above.

This procedure of fixed point resolution is certainly quite important, because it is only after having accomplished this task that we really deal with a well-defined conformal field theory (it is even unknown whether the prescription works for an arbitrary coset theory, and whether the conformal field theory it provides is unique). However, it is not difficult to see that some important quantities we will be interested in, namely the number of generations and anti-generations in a four-dimensional string compactification, can be obtained in our case without a detailed knowledge of the resolution procedure (see also the comments in Chapter 4).

In the third column of Table 2.4 we marked whether identification fixed points occur in the theories in question. The following notation is used: '–' indicates that fixed points never occur in the corresponding theory; '+' means that fixed points can occur, but not at any of the levels that are relevant for $c = 9$ tensor products (this typically happens when we are only interested in low levels where the associated outer automorphisms of g act freely on the integrable representations of g); finally '++' is used to indicate that fixed points occur and have to be resolved. Note that an identification current can possess a fixed point only if it has vanishing u(1)-charge.

2.3.4 Modular invariants

It should be noted that the discussion of field identification in the previous subsections refers only to one chiral half of the conformal field theory. For the full theory, one has to use all fields as identification currents, i.e. as representatives of the identity primary field, that have non-vanishing branching functions and are identification currents with respect to both the holomorphic and the anti-holomorphic part. For example, for the $N = 2$ minimal models this prescription implies the presence of left-right asymmetric identification currents if the $D_{\rm even}$, E_6, or E_8 type invariants of the associated A_1 WZW theory are chosen.

For the $N = 2$ theories of our present interest, we will confine ourselves to analyse only the situation where the diagonal modular invariants of g, h and $\mathrm{so}(2d)_1$ are used. As a consequence, the identification currents are just the left-right symmetric version of the chiral currents listed in Table 2.4. The extension to any known non-diagonal modular invariant is immediate; recall however that the classification of modular invariants of simple Lie algebras (other than A_1 and A_2 and their tensor products) is far from being complete.

2.4 Chiral ring and Poincaré polynomials

In this section we present some results concerning the chiral ring of our theories. The chiral ring is spanned by the collection of *chiral primary fields* of the theory; these fields are by definition those primary fields which satisfy $q_{\rm suco} = h/2$. They generate the *chiral ring* [103] of the theory; this is a finite-dimensional nilpotent ring $\mathcal{R}$ whose product is the naive operator product $\lim_{z\to w} \phi(z)\phi'(w)$. The reader should note that this product is different from the one defined by the fusion ring: the conjugation is not the evaluation with respect to the vacuum any more; rather the conjugate of the vacuum is the chiral primary field with highest u(1) charge.

The information contained in the chiral ring is crucial for many applications; e.g. the relation to topological field theories is mainly through this ring. It can also be used e.g. to determine quantities relevant to string compactification: in Chapter 4 we will compute the quantities which are the most relevant ones for the phenomenological aspects, namely the number of (anti-)generations for a compactification of the heterotic string to four space-time dimensions.

For the models under consideration, it is in fact easier to work with the ground states of the Ramond sector, which owing to spectral flow [103] provide equivalent information on the theory. Namely, the chiral primary fields (with superconformal charge [4] $q_{\rm suco}$) are via spectral flow in one to one correspondence with Ramond ground states (with superconformal charge $q_{\rm suco} - c/6$). In all $N = 2$ coset models of the form (2.1.1) we can identify the simple current in the Ramond sector which generates the flow; it is the unique Ramond ground state with highest superconformal charge, which has been termed *spinor current* in [123]. It is easily seen that one representative of the spinor current is the field

$$S = \Phi^{0,{\rm s}}_{0,Q_s}, \tag{2.4.1}$$

with

$$Q_s = (v_\circ, \rho_{\bar{\mathsf g}} - \rho_{\bar{\mathsf h}}). \tag{2.4.2}$$

Here $\rho_{\bar{\mathsf g}} = \sum_i \Lambda_{(i)}$ and $\rho_{\bar{\mathsf h}}$ are the Weyl vectors, i.e. half the sum of positive roots, of $\bar{\mathsf g}$ and of $\hat{\bar{\mathsf h}}$, respectively.

The information on the multiplicities of chiral states with a given superconformal charge is encoded in the *Poincaré polynomial* [103], which can be defined as a trace over the chiral ring $\mathcal{R}$,

$$P(t,\bar t\,) := {\rm Tr}_{\mathcal R}\, t^{J_0} \bar t^{\bar J_0}. \tag{2.4.3}$$

Here J_0 denotes the generator of the superconformal u(1), and the barred quantities refer to the second chiral half of the theory. In the sequel we will only consider the left-right symmetric diagonal modular invariant; correspondingly we can restrict ourselves to one chiral half and replace $t\bar t$ for the sake of simplicity by t.

2.4.1 Ramond ground states

To determine the ground states of the Ramond sector one can use a simple formula for the $\mathsf g$- and $\mathsf h$-weights of these states which can be derived [103] by means of an index argument. The advantage of this formula is twofold. First, in coset models it is usually difficult to calculate the integer part of the conformal weight h of a primary field; for Ramond ground states (which all have $h = c/24$), however, the index argument makes it possible to identify the state without having to evaluate a formula for h. Second, the formula automatically takes care of possibly arising null states; again, this is a rather delicate issue in the general case. [5]

Denote by $\overline{W}_{\bar{\mathsf g}}$ the Weyl group of $\bar{\mathsf g}$, by $|\overline{W}_{\bar{\mathsf g}}|$ its order, and by $\overline{W}_{\bar{\mathsf h}}$ and $|\overline{W}_{\bar{\mathsf h}}|$ the analogous quantities for $\hat{\bar{\mathsf h}}$. For any integrable $\mathsf g$-weight Λ, the recipe of [103] provides $|\bar W_{\bar{\mathsf g}}|/|\bar W_{\bar{\mathsf h}}|$ Ramond ground states. The $\mathsf h$-weight $\tilde\lambda$ of each of these Ramond ground states is related to its $\mathsf g$-weight by

$$\tilde\lambda = w(\Lambda + \rho_{\bar{\mathsf g}}) - \rho_{\bar{\mathsf h}}\,. \tag{2.4.4}$$

Here the weight $\tilde\lambda \equiv (\lambda, Q)$ incorporates both the weight λ of the semi-simple part $\hat{\mathsf h}$ of $\mathsf h$ and the u(1)-charge Q. Also, the map w in (2.4.4) is the representative of any class of the

[4] In the literature sometimes a normalization is chosen where $\frac12 q_{\rm suco}$ is the superconformal charge.

[5] Compare the remark about E_6 singlets in Section 4.4 below.

coset $\overline{W}_{\bar{g}}/\overline{W}_{\bar{h}}$ possessing the property that λ is a dominant integral highest weight of h; each class of $\overline{W}_{\bar{g}}/\overline{W}_{\bar{h}}$ contains a unique representative w satisfying this requirement [99]. If $\mathrm{sign}(w) = 1$, then the highest weight of so$(2d)$ that is associated to $\tilde{\lambda}$ is the spinor (s), while for $\mathrm{sign}(w) = -1$, it is the conjugate spinor (c). Note that $\rho_{\bar{g}} - \rho_{\bar{h}} \propto \Lambda_{(i_\circ)}$, the constant of proportionality being $\sum_j(\Lambda_{(j)}, \Lambda_{(i_\circ)})/(\Lambda_{(i_\circ)}, \Lambda_{(i_\circ)})$ as can be deduced from $(\rho_{\bar{h}}, \rho_{\bar{g}} - \rho_{\bar{h}}) = 0$.

For an $N = 2$ coset theory $\mathcal{C}[\mathsf{g} \oplus D_d/\bar{\mathsf{h}} \oplus \mathrm{u}(1)]_K$ without fixed points, the number of chiral primary fields is correspondingly [103]

$$\mu = \frac{N(\mathsf{g})}{|Z(\bar{\mathsf{g}})|} \frac{|\overline{W}_{\bar{g}}|}{|\overline{W}_{\bar{h}}|}, \tag{2.4.5}$$

where N is the number of primary fields of the WZW theory based on g at level K, and $Z(\bar{\mathsf{g}})$ is the center of the universal covering group whose Lie algebra is $\bar{\mathsf{g}}$ (which is isomorphic to the group of simple currents of the WZW theory). The factor $1/|Z(\bar{\mathsf{g}})|$ takes care of the necessary field identifications among representatives. In contrast, if an $N = 2$ coset theory has fixed points, the number of Ramond ground states is larger than (2.4.5). Namely, each primary field of g still gives rise to $|\overline{W}_{\bar{g}}|\overline{W}_{\bar{h}}|$ representatives of chiral primaries, but in addition for fixed points it is still true that (after resolution of fixed points) every Ramond ground state has a representative whose $\bar{\mathsf{g}}$- and $\bar{\mathsf{h}}$-weights fulfill (2.4.4) and that every equivalence class containing one representative with $\tilde{\lambda} = w(\Lambda + \rho_{\bar{g}}) - \rho_{\bar{h}}$ yields precisely one Ramond ground state.

To implement the formula (2.4.4) on a computer, it is convenient not to start with the weights of g, but rather to scan all dominant weights of h that are allowed by the selection rules. For each such weight $\tilde{\lambda}$ one determines the unique dominant integral g-weight which lies on the same $\overline{W}_{\bar{g}}$-orbit as $\tilde{\lambda} + \rho_{\bar{h}}$ (if this g-weight is not integrable at the relevant level of the affine algebra $\bar{\mathsf{g}}^{(1)}$, then the corresponding state has to be rejected). To do so, one only has to know the action of the fundamental reflections $w_i \in \overline{W}_{\bar{g}}$ (see e.g. [57]). This method has the advantage that one needs not know the whole $\overline{W}_{\bar{g}}$-orbit of a highest $\bar{\mathsf{g}}$-weight which, especially for large rank algebras, would require a lot of memory.

2.4.2 Poincaré polynomials

Having found the Ramond ground states, we can proceed to compute the Poincaré polynomial of an $N = 2$ coset theory. To do so, we also need the superconformal charge of the Ramond ground states. This charge is given by [97]

$$q_{\mathrm{suco}} = \sum_{\bar{\alpha}\in\Delta_+} \tilde{\Lambda}^{\bar{\alpha}} - \frac{\xi_\circ Q}{k + g^\vee}. \tag{2.4.6}$$

Here $\xi_\circ = \sqrt{(\tilde{v}_\circ, \tilde{v}_\circ)/(v_\circ, v_\circ)} = \sqrt{(k + g^\vee)(\tilde{v}_\circ, \tilde{v}_\circ)/\mathcal{N}}$ is the number defined by (2.3.3), Q is the u(1)-charge of the Ramond ground state, and $\tilde{\Lambda}^{\bar{\alpha}} \in \{\frac{1}{2}, -\frac{1}{2}\}$ are the components of its so$(2d)$-weight in the orthogonal basis. Unfortunately the index argument [103] leading to (2.4.4) does not provide the full weight $\tilde{\Lambda}$, but only yields the information whether it is a weight of the spinor or of the conjugate spinor module of so$(2d)$, or in other words, only the value of $\sum_{\bar{\alpha}\in\Delta_+} \tilde{\Lambda}^{\bar{\alpha}}$ modulo 2. To translate (2.4.6) into a more convenient formula, we proceed as follows. [6] Denote by $\Delta_+^{\bar{g}}$, $\Delta_-^{\bar{g}}$, and $\Delta^{\bar{g}}$ the sets of positive roots, of negative

[6] An analogous result has been obtained in [104] for simply laced hermitian symmetric cosets at level one, and in [33, 70] for all hermitian symmetric cosets in their free field realization.

roots, and of all roots, respectively, of the Lie algebra $\bar{\mathfrak{g}}$, and by $\Delta_\pm^{\mathfrak{h}}$, $\Delta^{\mathfrak{h}}$ the corresponding quantities for $\hat{\bar{\mathfrak{h}}}$. For an arbitrary element w of the Weyl group $\overline{W}_{\bar{\mathfrak{g}}}$ define

$$\Delta_\pm^{[w]} := \{\alpha \in \Delta^{\bar{\mathfrak{g}}} \mid w^{-1}(\alpha) \in \Delta_\pm^{\bar{\mathfrak{g}}}\}. \tag{2.4.7}$$

For any $w \in \overline{W}_{\bar{\mathfrak{g}}}$, $\Delta^{\bar{\mathfrak{g}}}$ is the disjoint union of $\Delta_+^{[w]}$ and $\Delta_-^{[w]}$. We can express the image of the Weyl vector $\rho_{\bar{\mathfrak{g}}}$ under w as

$$w(\rho_{\bar{\mathfrak{g}}}) = \tfrac{1}{2}\Big[\sum_{\alpha\in\Delta_+^{[w]}} \alpha - \sum_{\alpha\in\Delta_-^{[w]}} \alpha\Big]\,, \tag{2.4.8}$$

as is easily verified by applying w^{-1} to both sides of the equation.

Given a subalgebra $\bar{\mathfrak{h}}$ of $\bar{\mathfrak{g}}$, we call $w \in \overline{W}_{\bar{\mathfrak{g}}}$ $\bar{\mathfrak{h}}$-*positive* [104], iff

$$\Delta_+^{\bar{\mathfrak{h}}} \subset \Delta_+^{[w]}. \tag{2.4.9}$$

We claim that in order to compute $\sum_{\bar\alpha\in\Delta_+}\tilde\Lambda^{\bar\alpha}$, we only need to identify the $\bar{\mathfrak{h}}$-positive representative w of the coset $\overline{W}_{\bar{\mathfrak{g}}}/\overline{W}_{\bar{\mathfrak{h}}}$ that appears in (2.4.4), and that the components $\tilde\Lambda^{\bar\alpha}$ of the so$(2d)$-weight $\tilde\Lambda$ are given by

$$\tilde\Lambda^{\bar\alpha} = \tilde\Lambda^{\bar\alpha}_{[w]} := \begin{cases} \tfrac{1}{2} & \text{if} \quad \bar\alpha \in \Delta_+^{[w]}, \\ -\tfrac{1}{2} & \text{if} \quad \bar\alpha \in \Delta_-^{[w]}. \end{cases} \tag{2.4.10}$$

This can be seen as follows. Let α be an arbitrary element of $\Delta_+^{\bar{\mathfrak{h}}}$. For any highest $\bar{\mathfrak{h}}$-weight $\tilde\lambda$ we have $(\tilde\lambda + \rho_{\bar{\mathfrak{h}}}, \alpha) > 0$; as a consequence,

$$0 < (\tilde\lambda + \rho_{\bar{\mathfrak{h}}}, \alpha) = (w(\Lambda + \rho_{\bar{\mathfrak{g}}}), \alpha) = (\Lambda + \rho_{\bar{\mathfrak{g}}}, w^{-1}(\alpha)). \tag{2.4.11}$$

This shows that $w^{-1}(\alpha) \in \Delta_+^{\bar{\mathfrak{g}}}$, or in other words, that $\Delta_+^{\bar{\mathfrak{h}}} \subset \Delta_+^{[w]}$. Now the general form of the Cartan currents of $\hat{\mathfrak{h}}$ reads

$$H^i_{\mathfrak{h}} = H^i_{\mathfrak{g}} + \sum_{\bar\alpha\in\Delta_+} \bar\alpha^i \,{:}\Psi^{\bar\alpha}\Psi^{-\bar\alpha}{:}\,. \tag{2.4.12}$$

As a consequence, under the embedding $\mathfrak{h} \hookrightarrow \mathfrak{g} \oplus \mathrm{so}(2d)_1$ the state with weight $(w(\Lambda), \tilde\Lambda^{\bar\alpha}_{[w]})$ branches to

$$\begin{aligned} \tilde\lambda &= w(\Lambda) + \tfrac{1}{2}\sum_{\bar\alpha\in\Delta_+^{[w]}} \bar\alpha - \tfrac{1}{2}\sum_{\bar\alpha\in\Delta_-^{[w]}} \bar\alpha \\ &= w(\Lambda) + w(\rho_{\bar{\mathfrak{g}}}) - \tfrac{1}{2}\sum_{\alpha\in\Delta_+^{[w]}\cap\Delta_+^{\bar{\mathfrak{h}}}} \alpha + \tfrac{1}{2}\sum_{\alpha\in\Delta_-^{[w]}\cap\Delta_+^{\bar{\mathfrak{h}}}} \alpha\,. \end{aligned} \tag{2.4.13}$$

This reduces to $w(\Lambda + \rho_{\bar{\mathfrak{g}}}) - \rho_{\bar{\mathfrak{h}}}$, i.e. yields the correct result (2.4.4), iff w is $\bar{\mathfrak{h}}$-positive. Note that the weight $(w(\Lambda), \tilde\Lambda^{\bar\alpha}_{[w]})$ is always present in the weight system of the $\mathfrak{g} \oplus \mathrm{so}(2d)$-module with highest weight (Λ, s) or (Λ, c), because the Weyl group orbit of any weight of a highest weight module with dominant integral highest weight is contained in the weight system of the module.

Inserting our result (2.4.10) into the formula (2.4.6) for the superconformal charge $q_{\rm suco}$, we obtain

$$q_{\rm suco} = \tfrac{1}{2}\left(|\Delta_+^{[w]} \cap \Delta_+| - |\Delta_-^{[w]} \cap \Delta_+|\right) - \frac{\xi_\circ Q}{k+g^\vee}\,. \qquad (2.4.14)$$

To simplify this formula further, we recall that the length $l(w)$ of a Weyl group element w, which is defined as the minimal number of fundamental reflections needed to obtain w, obeys [84, sect. 1.7]

$$l(w) = |\Delta_-^{[w]} \cap \Delta_+|. \qquad (2.4.15)$$

Using the identity

$$|\Delta_+^{[w]} \cap \Delta_+| + |\Delta_-^{[w]} \cap \Delta_+| = d, \qquad (2.4.16)$$

we finally obtain

$$q_{\rm suco} = \tfrac{1}{2}d - l(w) - \frac{\xi_\circ Q}{k+g^\vee}\,. \qquad (2.4.17)$$

The length of the relevant elements of $\overline{W}_{\bar{\mathfrak g}}/\overline{W}_{\bar{\mathfrak h}}$ can be obtained conveniently via the so-called Hasse diagram of the embedding $\bar{\mathfrak h} \hookrightarrow \bar{\mathfrak g}$ (for some details, see the Appendix), and hence the formula (2.4.17) is easily implemented in a computer program. For the spinor current (2.4.1), one has $w = id$ so that (2.4.17) reduces to

$$q_{\rm suco}(S) = \tfrac{1}{2}d - \frac{(\tilde v_\circ, \rho_{\bar{\mathfrak g}} - \rho_{\bar{\mathfrak h}})}{k+g^\vee} = \frac{c}{6}\,, \qquad (2.4.18)$$

where the last equality follows with (2.2.6) and the strange formula.

We are now in a position to compute the Poincaré polynomials of the theories listed in Section 2.2. For notational simplicity, we will present the Poincaré polynomials in the form $P(t^\ell)$, with ℓ the smallest positive integer for which all values of $\ell q_{\rm suco}$ of chiral primary fields are integers. We find that for the three series $(BB, m+2, 1)$, $(CC, 2, 2m+1)$, and $(CC, 2m+2, 1)$ with $m \in \mathbb{Z}_{\geq 0}$, the Poincaré polynomials are given by a common formula, namely $\ell = m+2$ and

$$P(t^{m+2}) = \sum_{j=0}^{m} (j+1)\,(t^j + t^{3m+2-j}) + (3m+4) \sum_{j=m+1}^{2m+1} t^j. \qquad (2.4.19)$$

The Poincaré polynomials of the remaining models are listed in Table 2.5.

To conclude this subsection, we remark that the resolution of fixed points does not alter the number of Ramond ground states. In other words, independently of its length each identification orbit that contains a representative satisfying (2.4.4) provides exactly one Ramond ground state [123].

Table 2.5: Poincaré polynomials for non-hermitian symmetric coset theories

name	ℓ	$P(t^\ell)$
$(BA,3,1)$	4	$1+3t^2+4t^3+3t^4+t^6$
$(BA,3,2)$	14	$1+t^6+t^8+t^9+2t^{10}+t^{11}+3t^{12}+t^{13}+2(t^{14}+t^{15}+t^{16})$ $+t^{17}+3t^{18}+t^{19}+2t^{20}+t^{21}+t^{22}+t^{24}+t^{30}$
$(BA,3,4)$	6	$1+(t^2+t^3)+3t^4+2t^5+9t^6+7t^7+14t^8+12t^9$ $+14t^{10}+7t^{11}+9t^{12}+2t^{13}+3t^{14}+t^{15}+t^{16}+t^{18}$
$(BA,4,1)$, $(C3,1)$, $(G2_2,2)$	2	$1+4t+14t^2+4t^3+t^4$
$(BA,5,1)$	4	$1+5t^2+10t^4+16t^5+10t^6+5t^8+t^{10}$
$(BA,6,1)$	2	$1+6t+15t^2+52t^3+15t^4+6t^5+t^6$
$(BB,3,3)$	2	$1+4t+17t^2+40t^3+17t^4+4t^5+t^6$
$(BB,4,2)$	3	$1+2t+8t^2+14t^3+35t^4+35t^5+14t^6+8t^7+2t^8+t^9$
$(CC,2,2)$, $(CC,3,1)$, $(G2_2,1)$	5	$1+2t^2+3(t^3+t^4)+2t^5+t^7$
$(CC,2,4)$, $(CC,5,1)$	7	$1+2t^2+3t^4+5t^5+4(t^6+t^7)+5t^8+3t^9+2t^{11}+t^{13}$
$(CC,2,6)$, $(CC,7,1)$	9	$1+2t^2+3t^4+4t^6+7t^7+5t^8+6(t^9+t^{10})$ $+5t^{11}+7t^{12}+4t^{13}+3t^{15}+2t^{17}+t^{19}$
$(CC,3,2)$,	2	$1+6t+16t^2+6t^3+t^4$
$(CC,3,5)$, $(CC,6,2)$	3	$1+3t+12t^2+20t^3+48(t^4+t^5)+20t^6+12t^7+3t^8+t^9$
$(CC,4,3)$	2	$1+8t+29t^2+64t^3+29t^4+8t^5+t^6$
$(C4,1)$	2	$1+4t+15t^2+40t^3+15t^4+4t^5+t^6$
$(D4,1)$	7	$1+t^2+3t^4+4t^5+3(t^6+t^7)+4t^8+3t^9+t^{11}+t^{13}$
$(D5_1,1)$	3	$1+t+4t^2+12t^3+22(t^4+t^5)+12t^6+4t^7+t^8+t^9$
$(D5_2,1)$	9	$1+t^2+2t^4+3t^6+5t^7+4(t^8+t^9+t^{10}+t^{11})$ $+5t^{12}+3t^{13}+2t^{15}+t^{17}+t^{19}$
$(F4,1)$	5	$1+t+2(t^2+t^3)+9(t^4+t^5+t^6+t^7)$ $+2(t^8+t^9)+t^{10}+t^{11}$
$(G2_1,1)$	3	$1+5(t^2+t^3)+t^5$
$(G2_1,2)$	18	$1+t^{10}+t^{12}+t^{13}+t^{14}+t^{15}+2t^{16}+t^{17}+t^{18}+t^{19}+2t^{20}$ $+t^{21}+t^{22}+t^{23}+2t^{24}+t^{25}+t^{26}+t^{27}+t^{28}+t^{30}+t^{40}$
$(G2_2,5)$	3	$1+t+4t^2+8t^3+22(t^4+t^5)+8t^6+4t^7+t^8+t^9$

2.4.3 Charge conjugation

From (2.4.19) and the results in Table 2.5 one can read off that the superconformal charges of chiral primary fields lie between zero and $c/3$, as it must be. One also notes that according to the results the Poincaré polynomials obey

$$P(t) = t^{c/3}\, P(t^{-1}). \tag{2.4.20}$$

In terms of the Ramond sector this means that the collection of Ramond ground states is symmetric with respect to the charge conjugation $q_{\rm suco} \mapsto -q_{\rm suco}$.

In fact, using the formulæ (2.4.4) and (2.4.17) it is possible to show that this is a generic feature of all $N=2$ coset theories of the form (2.1.1). To show this, consider along with an arbitrary Ramond ground state $\Phi^{\Lambda,{\rm x}}_{\tilde\lambda} \equiv \Phi^{\Lambda,{\rm x}}_{\lambda,Q}$ also the field represented by $\Phi^{\Lambda^+,{\rm x}'}_{\tilde\lambda^+}$, with Λ^+, ${\rm x}'$ and $\tilde\lambda^+$ defined as follows. As before, x stands for either the spinor or conjugate spinor, and we define ${\rm x}'$ to be equal to x if d is even, and to belong to the opposite conjugacy class if d is odd. Moreover,

$$\Lambda^+ \;:=\; -w^{\bar{\mathfrak g}}_{\rm max}(\Lambda), \tag{2.4.21}$$

$$\tilde\lambda^+ \;:=\; -w^{\bar{\mathfrak h}}_{\rm max}(\tilde\lambda) \tag{2.4.22}$$

(recall that $w_{\rm max}$, denoting the longest element of a Weyl group $\overline{W}$, acts as the negative of the conjugation in the representation ring of a Lie algebra). In the definition (2.4.22), $w^{\bar{\mathfrak h}}_{\rm max}$ is to be considered as an element of the Weyl group $\overline{W}_{\bar{\mathfrak g}}$. As a consequence, $w^{\bar{\mathfrak h}}_{\rm max}$ acts on the $\hat{\bar{\mathfrak h}}$-weights like the usual conjugation of weights and maps Q to $-Q$. Namely, by virtue of (2.2.18) each fundamental Weyl reflection of $\overline{W}_{\bar{\mathfrak h}}$, and thus any element of $\overline{W}_{\bar{\mathfrak g}}$, acts on $v_\circ$ as the identity.

Using the identities $\rho_{\bar{\mathfrak g}} = -w^{\bar{\mathfrak g}}_{\rm max}(\rho_{\bar{\mathfrak g}})$ and $\rho_{\bar{\mathfrak h}} = -w^{\bar{\mathfrak h}}_{\rm max}(\rho_{\bar{\mathfrak h}})$, we see that the highest weight $\tilde\lambda^+ + \rho_{\bar{\mathfrak h}}$ of $\bar{\mathfrak h}$ can be written as

$$\begin{aligned}\tilde\lambda^+ + \rho_{\bar{\mathfrak h}} &= -w^{\bar{\mathfrak h}}_{\rm max}(\tilde\lambda + \rho_{\bar{\mathfrak h}}) \;=\; -w^{\bar{\mathfrak h}}_{\rm max} w(\Lambda + \rho_{\bar{\mathfrak g}}) \\ &= w^+(\Lambda^+ + \rho_{\bar{\mathfrak g}})\,,\end{aligned} \tag{2.4.23}$$

where

$$w^+ := w^{\bar{\mathfrak h}}_{\rm max} w\, w^{\bar{\mathfrak g}}_{\rm max}\,, \tag{2.4.24}$$

and where w is the Weyl group element introduced in (2.4.4). To calculate the sign of w^+, which determines the so$(2d)$ conjugacy class, we observe (by inspection) that for all simple Lie algebras $\bar{\mathfrak g}$ the relation

$$\text{sign}\,(w^{\bar{\mathfrak g}}_{\rm max}) = (-1)^{n_+} \tag{2.4.25}$$

is satisfied, where $n_+ = |\Delta^{\bar{\mathfrak g}}_+|$ is the number of positive $\bar{\mathfrak g}$-roots. Therefore

$$\text{sign}\,(w^+) = \text{sign}\,(w^{\bar{\mathfrak h}}_{\rm max})\, \text{sign}\,(w^{\bar{\mathfrak g}}_{\rm max})\, \text{sign}\,(w) = (-1)^{(\dim\bar{\mathfrak g}-\dim\bar{\mathfrak h})/2}\, \text{sign}\,(w), \tag{2.4.26}$$

and (2.4.4) now clearly implies that the state $\Phi^{\Lambda^+,{\rm x}'}_{\tilde\lambda^+}$ is again a Ramond ground state. Also note that as a by-product we proved that along with w also w^+ is $\bar{\mathfrak h}$-positive.

So far we have seen that the set of Ramond ground states is symmetric in the u(1)-charge. The symmetry in the superconformal charge then follows from (2.4.17) together with the identity

$$l(w) + l(w^+) = d. \tag{2.4.27}$$

This relation arises as follows. Let $\bar\alpha$ be an arbitrary root in Δ_+. Then either $w^{-1}(\bar\alpha) \in \Delta_-^{\bar{\mathfrak g}}$ or $(w^+)^{-1}(\bar\alpha) \in \Delta_-^{\bar{\mathfrak g}}$, because the map $w \mapsto w^+$ swaps exactly from negative to positive roots of $\bar{\mathfrak g} \setminus \bar{\mathfrak h}$. Thus Δ_+ is the disjoint union of $\Delta_+ \cap \Delta_-^{[w]}$ and $\Delta_+ \cap \Delta_-^{[w^+]}$, which by (2.4.15) and (2.4.16) proves the assertion.

Let us also note that the unique Ramond ground state with minimal superconformal charge $q_{\rm suco} = -c/6$ (which via spectral flow corresponds to the identity primary field) is obtained by applying the above prescription to the spinor current (2.4.1), and hence is given by $\Phi^{0,x'}_{0,-Q_s}$. For this field the relevant $\bar{\mathfrak h}$-positive Weyl group element is $w = w^{\bar{\mathfrak h}}_{\max} w^{\bar{\mathfrak g}}_{\max}$ so that (2.4.27) implies

$$l(w^{\bar{\mathfrak h}}_{\max} w^{\bar{\mathfrak g}}_{\max}) = d\,, \tag{2.4.28}$$

while by setting $\Lambda = \lambda = 0$ in (2.4.4), one obtains $\rho_{\bar{\mathfrak h}} = \frac12(\rho_{\bar{\mathfrak g}} + w^{\bar{\mathfrak h}}_{\max} w^{\bar{\mathfrak g}}_{\max}(\rho_{\bar{\mathfrak g}})) = \frac12(\rho_{\bar{\mathfrak g}} - w^{\bar{\mathfrak h}}_{\max}(\rho_{\bar{\mathfrak g}}))$. Since the $\bar{\mathfrak h}$-positive representative $w^{\bar{\mathfrak g}/\bar{\mathfrak h}}_{\max}$ of $\overline W_{\bar{\mathfrak g}}/\overline W_{\bar{\mathfrak h}}$ with largest length d is unique [84], (2.4.28) shows that this representative is given by

$$w^{\bar{\mathfrak g}/\bar{\mathfrak h}}_{\max} = w^{\bar{\mathfrak h}}_{\max} w^{\bar{\mathfrak g}}_{\max}\,. \tag{2.4.29}$$

2.5 Conclusions

In this chapter we have presented a detailed analysis of non-hermitian symmetric $N = 2$ superconformal coset theories; in addition, we have proven some general statements on the structure of any $N = 2$ coset theory. Concerning the non-hermitian symmetric coset theories themselves, we have shown that they indeed allow for an interpretation as a consistent conformal field theory; this lends further support to the expectation that *any* coset theory, naively 'defined' as $\mathcal C[\bar{\mathfrak g}/\bar{\mathfrak h}]_k$, possesses such an interpretation. In particular, it was shown that the fixed points that arise in the process of field identification can be resolved by the methods of [129].

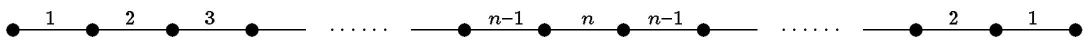
To conclude, let us come back to the hypothesis that, given a chain of subalgebras $\bar{\mathfrak h}_1 \hookrightarrow \bar{\mathfrak h}_2 \hookrightarrow \bar{\mathfrak g}$, the coset theory $\mathcal C[\bar{\mathfrak g}/\bar{\mathfrak h}_1]_k$ should correspond to the tensor product of the two cosets $\mathcal C[\bar{\mathfrak g}/\bar{\mathfrak h}_2]_k$ and $\mathcal C[\bar{\mathfrak h}_2/\bar{\mathfrak h}_1]_{k'}$, with a suitably chosen non-product modular invariant. We emphasize that in the presence of fixed points this hypothesis is far from being proven. With the methods employed in the present chapter it should be straightforward to examine the structure of both $\mathcal C[\bar{\mathfrak g}/\bar{\mathfrak h}_1]_k$ and the tensor product of $\mathcal C[\bar{\mathfrak g}/\bar{\mathfrak h}_2]_k$ and $\mathcal C[\bar{\mathfrak h}_2/\bar{\mathfrak h}_1]_{k'}$ in detail, and thereby test the hypothesis for any given chain of embeddings. To prove the equivalence in full generality, however, still a deeper understanding of the structure of coset conformal field theories seems to be necessary.

2.A Appendix: Hasse diagrams

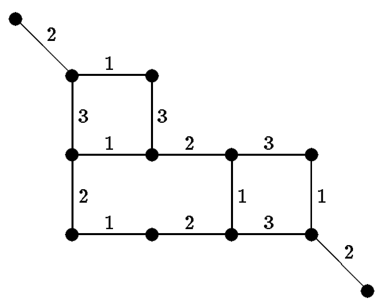
The Hasse diagram [11] for an embedding $\bar{\mathfrak h} \hookrightarrow \bar{\mathfrak g}$ of a reductive Lie algebra in a simple Lie algebra is the graph of the coset $\overline W_{\bar{\mathfrak g}}/\overline W_{\bar{\mathfrak h}}$, interpreted as a subgraph of the graph of $\overline W_{\bar{\mathfrak g}}$, with the edges as prescribed by the Bruhat ordering of $\overline W_{\bar{\mathfrak g}}$. (Hasse diagrams also arise in the description of the topological structure of generalized flag manifolds and of the structure of the Bernstein–Gelfand–Gelfand-resolution of Verma modules.) The nodes of the Hasse diagram correspond to those representatives of elements of $\overline W_{\bar{\mathfrak g}}/\overline W_{\bar{\mathfrak h}}$ that send a dominant $\bar{\mathfrak g}$-weight Λ to a dominant $\hat{\bar{\mathfrak h}}$-weight λ, i.e. to $\bar{\mathfrak h}$-positive elements of $\overline W_{\bar{\mathfrak g}}$, and the integer i attached to an edge indicates that the two nodes connected by the edge correspond to Weyl group elements w and w' related by $w' = w_{(i)} w$, with $w_{(i)}$ the ith fundamental reflection. For an embedding $\bar{\mathfrak{h}} \hookrightarrow \bar{\mathfrak{g}}$ for which the Dynkin diagram of $\bar{\mathfrak{h}}$ is obtained by deleting the node with label $i_\circ$ from the Dynkin diagram of $\bar{\mathfrak{g}}$, the Hasse diagram is isomorphic to the $\overline{W}_{\bar{\mathfrak{g}}}$-orbit of $\Lambda_{(i_\circ)}$, i.e. to the 'restricted weight diagram' that one obtains when acting successively on the weight $\Lambda_{(i_\circ)}$ with the fundamental reflections.

The Hasse diagrams for the embeddings relevant to hermitian symmetric cosets have been described in [33]. Below we present the Hasse diagrams for some of the non-hermitian symmetric cases which appear in Table 2.2 (the diagrams for the remaining cases, i.e. the BA and BB series and the two $D5$ theories look more complicated, and we refrain from drawing them here). [7]

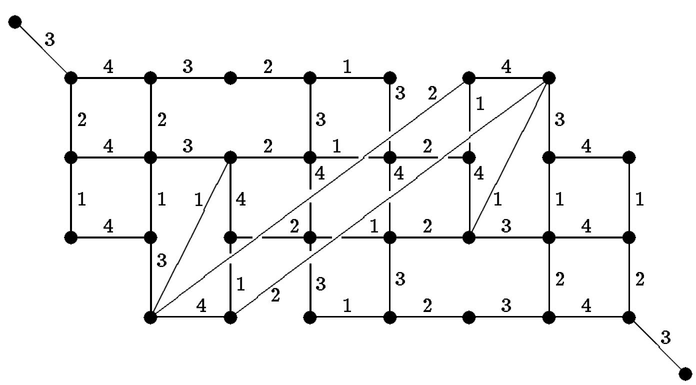
Hasse diagram of $\overline{W}(C_n)/\overline{W}(C_{n-1})$:



Hasse diagram of $\overline{W}(C_3)/\overline{W}(A_1 \oplus A_1)$:

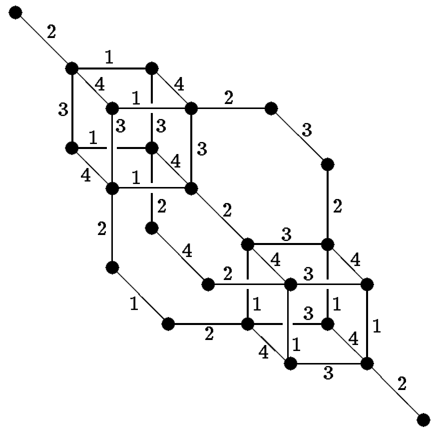


Hasse diagram of $\overline{W}(C_4)/\overline{W}(A_2 \oplus A_1)$:

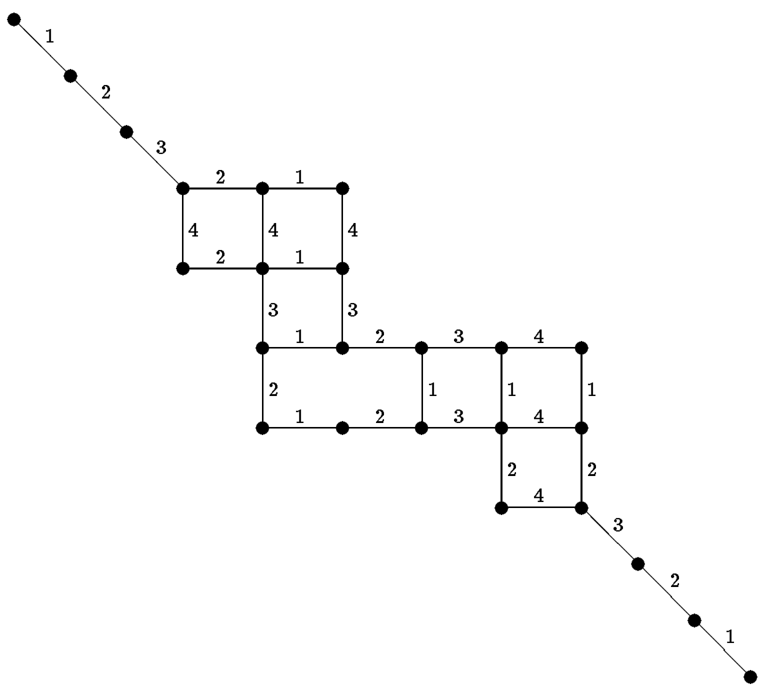


[7] The Hasse diagram of $\overline{W}(F_4)/\overline{W}(C_3)$ can also be found in [22, p. 86].

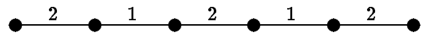
Hasse diagram of $W(D_4)/W(A_1 \oplus A_1 \oplus A_1)$:



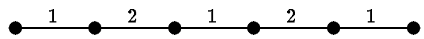
Hasse diagram of $\overline{W}(F_4)/\overline{W}(C_3)$:



Hasse diagram of $W(G_2)/W(A_1^>)$:



Hasse diagram of $\overline{W}(G_2)/\overline{W}(A_1^<)$:



Chapter 3

Level-Rank Duality of WZW Theories and of $N = 2$ Coset Models

As it turns out, the same conformal field theory can be described by different Lie algebraic cosets. In this chapter we show that certain infinite series of $N = 2$ superconformal coset models introduced in Chapter 2 coincide; to this end we construct mappings between these series. They make use of level-rank dualities for B, C, and D type WZW theories, which are described in some detail. The WZW level-rank dualities do not constitute isomorphisms of the theories; for example, for B and D type WZW theories, only simple current orbits rather than individual primary fields are mapped onto each other. Nevertheless they lead to level-rank dualities of $N = 2$ coset models that preserve the conformal field theory properties in such a manner that the coset models related by duality are expected to be, in fact, isomorphic as conformal field theories; in particular, the representation of the modular group on the characters and the ground states of the Ramond sector are shown to coincide. The construction also gives some further insight into the nature of the resolution of field identification fixed points of coset theories.

3.1 Level-rank dualities

Level-rank dualities relate objects that are present in two different structures that are connected to each other by exchanging the level (or possibly some simple function thereof) and the rank of an affine Lie algebra (or some closely related algebraic structure). They emerge in various areas of physics and mathematics: in WZW conformal field theories [114, 115, 140, 109, 116] and the theories obtained from them via the coset construction [6]; in three-dimensional Chern–Simons theories [109, 113]; in the representation theory of quantum groups with deformation parameter a root of unity [56,121] and of Hecke algebras whose parameter is a root of unity [76]; and in the description of edge variables in fusion-RSOS models [102].

Usually, level-rank duality merely implies certain non-trivial relations among quantities of different theories, such as correlation functions or fusion rules of WZW models. In this chapter, we describe several level-rank dualities which go much beyond such relations in that they provide an isomorphism between the respective theories. We show that there exist several such equivalences among infinite series of $N = 2$ superconformal coset theories. More specifically, we describe the identifications

$$\begin{aligned} &(B, 2n+1, 2k+1) \cong (B, 2k+1, 2n+1)\,, \\ &(B, 2n, 2k+1) \cong (B, 2k+1, 2n)_{|D}\,, \\ &(BB, n+2, 1) \cong (CC, 2, 2n+1)\,, \\ &(CC, n, k) \cong (CC, k+1, n-1)\,. \end{aligned} \tag{3.1.1}$$

Here the notations are taken from [37] and Tables 2.1 and 2.2 , compare also Tables 3.1 and 3.2 below. Let us note that isomorphisms between infinite series of coset conformal field theories have been observed previously. For instance, the $c < 1$ minimal conformal models can be described [73] as $\mathcal{C}[(A_1)_{m-2} \oplus (A_1)_1 \,/\, (A_1)_{m-1}]$, but also [4] as $\mathcal{C}[(C_{m-1})_1 \,/\, (C_{m-2})_1 \oplus (C_1)_1]$; in this case the field contents is tightly constrained by the representation theory of the chiral algebra, so that it is relatively easy to construct an isomorphism as a mapping between primary fields. Our result (3.1.1) demonstrates for the first time the presence of such isomorphisms for $N = 2$ superconformal theories of arbitrarily high central charge.

The identifications (3.1.1) are constructed as one-to-one maps between the primary fields of the respective theories. Both at the level of the representation of the modular group and, hence, for the fractional part of the conformal dimensions and for the fusion rules, and (by identifying Ramond ground states) at the level of the ring of chiral primary fields we verify that these maps possess the properties needed for an isomorphism of conformal field theories. Clearly one would like to extend the proof from the fusion rules to the full operator product algebra. Because of the technical difficulties arising in the conformal bootstrap (compare, e.g., [44]), this would be a quite formidable task. However, it is reasonable to expect that any two $N = 2$ superconformal field theories that possess the same value of the conformal central charge, the same fusion rules, and the same conformal dimensions modulo integers are in fact isomorphic. [1] We are therefore convinced that the two coset theories in question furnish merely two different descriptions of one and the same conformal field theory. In this context note that in general the conformal dimensions of primary fields change with the 'moduli' of some class of conformal field theories. For compatibility with the fusion rules, the number of primary fields must then depend on the moduli as well (in fact, when deforming a rational conformal field theory by a massless modulus one generically obtains an irrational theory, compare the situation at $c = 1$). The arguments in favor of the interpretation of the relations (3.1.1) as isomorphisms seem to us already conclusive for any fixed choice of a pair of theories from the list (3.1.1); they become even more convincing when one realizes that our identifications always come in infinite series.

Similar remarks apply to the structure of the chiral ring. We can substantiate our expectation that there is not only a one-to-one map between the chiral primary fields of the theories, but that the sets of chiral primaries also possess isomorphic ring structures, by various arguments. First note that the identification of the sets of Ramond ground states of two $N = 2$ theories implies that they possess the same Poincaré polynomial. From the experience with coset constructions, the observation that there exist coset theories with coinciding Poincaré polynomials is not very spectacular. However, we will see in Chapter 4 that not only the ordinary Poincaré polynomials, but also the *extended* Poincaré polynomials (introduced in [123]) of the relevant theories appearing in (3.1.1) coincide [123, 54]; [2] note that the extended Poincaré polynomial describes explicitly part of the structure of the chiral ring, whereas the ordinary Poincaré polynomial essentially counts multiplicities. Second, the mapping between Ramond ground states, and thus also between chiral primary fields, leaves the superconformal charge q invariant. When proving this, it is important that (in contrast to the case of generic primary fields of a coset theory) for Ramond ground

[1] In the non-supersymmetric case, examples are known [124] where conformal field theories for which these data coincide are nevertheless distinct theories. These theories have conformal central charge a multiple of 8 and contain only a single primary field.

[2] Surprisingly, it seems that in fact for *all* $N = 2$ coset theories for which the ordinary Poincaré polynomials are identical, the same holds for the extended Poincaré polynomials as well.

states not only can we easily compute the conformal weight exactly (and not just modulo integers), but also the superconformal charge q (cf. formula (2.4.17)). In addition, the ring product of the chiral ring is highly constrained by the fusion rules. Namely, since the ring product is defined as the operator product at coinciding points, the fusion rules (together with naturality [110]) determine which of the structure constants of the chiral ring are non-zero. Finally, the charge conjugation on the fusion ring is implemented by the fusion coefficients $\mathcal{N}_{ij}{}^{0}$, and thus our map respects charge conjugation, too. In particular, the charge conjugation behavior of the Ramond ground states is respected. Since conjugation on the chiral ring is induced by the ordinary charge conjugation on the Ramond ground states via spectral flow (which means that there is a highly non-trivial interplay between the chiral ring and the representation of the modular group on the characters), it follows that the map is compatible with the conjugation of the chiral ring.

As it turns out, the identifications (3.1.1) are also interesting in the context of the field identification problem that arises in coset conformal field theories. Namely, field identification fixed points are mapped on non-fixed points, so that the duality provides additional insight into the procedure of fixed point resolution. (The resolution procedure for field identification fixed points shows up in two different ways: for models of BB type, or of B type with rank and level odd, fixed points are mapped on longer orbits, while for B type theories at odd level and even rank the resolution is accomplished by mapping on pairs of so-called spinor-conjugate orbits.)

The plan of this chapter is as follows. The various level-rank dualities (3.1.1) of coset theories are consecutively dealt with in Sections 3.6 to 3.9 (the isomorphism statements are made in the equations (3.6.1), (3.7.1), (3.8.1), and (3.9.1), respectively). These sections make heavy use of underlying level-rank dualities for the WZW theories [112,109] the coset models are composed of. For the benefit of the reader we describe the relevant aspects of these dualities in some detail in Sections 3.2 to 3.4, in a formulation that is adapted to the needs in $N = 2$ theories, making in particular frequent use of simple current terminology.

To conclude this introduction to the subject, let us mention that level-rank dualities for $N = 2$ coset theories have first been conjectured, for hermitian symmetric cosets, in [97]; this conjecture just relied on the symmetry of the conformal central charges of the relevant coset theories. Calculations of the spectra of $N = 2$ coset theories were first performed in [37,123] for hermitian symmetric cosets, and in [54] for non-hermitian symmetric cosets. The results of [123] provided some evidence that the dualities indeed exist; in particular, it was realized that for B type theories at odd level and even rank the D type modular invariant must be used rather than the diagonal one. In the present chapter, we combine the level-rank dualities of WZW theories with the properties of simple current symmetries to construct a map between the primary fields of the $N = 2$ coset theories in question that makes the level-rank duality explicit and is expected to be an isomorphism of the two conformal field theories. It is worthwhile to stress that the underlying level-rank dualities of WZW theories are definitely not isomorphisms of conformal field theories. In particular, these WZW dualities are typically *not* mappings between primary fields, but rather between simple current orbits of (part of) the primary fields. As we will see, this fits perfectly to the application to coset theories, because owing to the necessary field identifications the physical fields of a coset theory can be characterized in terms of combinations of simple current orbits only. In some cases this technical complication makes the formulation of the mapping somewhat awkward (and adds to the length of this chapter), but, nonetheless, the mappings are based on simple current symmetries, and hence on natural objects of the underlying WZW theories. We shall show in the sequel that these mappings have the properties required for isomorphisms of conformal field theories.

In [97] it was conjectured that a relation between B type theories at even rank and even level should exist, too. In this case non-diagonal modular invariants must be chosen, but up to now it is not yet clear which of them could do the job. [3] Finally, based on a free field realization of the symmetry algebra, a level-rank duality for the A type hermitian symmetric cosets has been shown to be present at the level of symmetry algebras [97]. It would be interesting to explore these dualities by the techniques developed in the present chapter.

3.2 B type WZW theories at odd level

In this section we will describe a map τ between the WZW theories $(B_n)_{2k+1}$ and $(B_k)_{2n+1}$ that has simple behavior with respect to the modular matrices T (i.e., with respect to conformal dimensions modulo integers) and S. Thus the two theories that are connected by τ are related by exchanging twice the rank plus one (recall that $B_n \cong \mathrm{so}(2n+1)$) with the level of a B type affine Lie algebra; a relation of this type is called *level-rank duality.* As mentioned in Section 3.1, such dualities emerge in various different contexts; here we will concentrate on those aspects that are needed for the identifications of $N=2$ coset theories in Sections 3.6 to 3.9 below. The level-rank duality in question was first realized in [109]; in the notation of [109], our map τ corresponds to the map 'tilde' for B weights that are tensors, and to the map 'hat' for spinor weights, respectively. To be more precise, τ will be a one-to-one map between orbits with respect to the relevant simple currents J of the two theories. Thus, to start, we note that the number of primaries of the $(B_n)_{2k+1}$ WZW theory, i.e., the number of integrable representations of the affinization of B_n at level $2k+1$, is

$$N^B_{n,2k+1} = \sum_{l=0}^{k} \binom{2k-2l+3}{2} \binom{l+n-3}{l} = (4k+3n)(n+k-1)!/n!k!\,; \qquad (3.2.1)$$

of these,

$$F^B_{n,2k+1} = \binom{n+k-1}{k} \qquad (3.2.2)$$

are fixed points, so that the number of orbits is $2\binom{n+k}{k}$. This is invariant under $n \leftrightarrow k$, so that indeed a one-to-one map between the respective sets of orbits is conceivable.

For any integrable highest weight $\Lambda = \sum_{i=1}^{n} \Lambda^i \Lambda_{(i)}$ of $(B_n)_{2k+1}$, denote by

$$c_\Lambda = \Lambda^n \bmod 2 \qquad (3.2.3)$$

the conjugacy class of Λ. For brevity, we will often refer to Λ as a 'tensor' and as 'spinor' weight if $c_\Lambda = 0$ and $c_\Lambda = 1$, respectively. Consider now the components of Λ in the orthonormal basis of the weight space; they read

$$\ell_i(\Lambda) = \sum_{j=i}^{n-1} \Lambda^j + \tfrac{1}{2}\,\Lambda^n. \qquad (3.2.4)$$

[3] Also, none of these $N=2$ models is relevant to string compactification. For us this is another reason to refrain from investigating these dualities here.

Adding to these numbers the components of the Weyl vector as well as a term $\frac{1}{2}(1-c_\Lambda)$ such as to make the result integer-valued, one defines

$$\tilde\ell_i(\Lambda) := \ell_i(\Lambda+\rho) + \tfrac{1}{2}(1-c_\Lambda) = \sum_{j=i}^{n-1} \Lambda^j + n + 1 - i + \tfrac{1}{2}(\Lambda^n - c_\Lambda). \qquad (3.2.5)$$

Under the action of the simple current J that carries the highest weight $(2k+1)\Lambda_{(1)}$, the numbers ℓ_i, $i=2,3,\dots,n$, are invariant, while ℓ_1 becomes replaced by $2k+1-\ell_1$. As a consequence, we may characterize any orbit of J by a set of n positive integers $\tilde\ell_i$, $i=1,2,\dots,n$ subject to $\tilde\ell_i > \tilde\ell_j$ for $i<j$ as well as $\tilde\ell_1 \le k+n$, or in other words, by a subset M_Λ of cardinality $|M_\Lambda| = n$ of the set

$$M := \{1,2,\dots,k+n\}\,. \qquad (3.2.6)$$

Each such subset describes precisely one tensor and one spinor orbit (in particular, there are as many spinor orbits as tensor orbits if the level of B_n is odd), and conversely, any integrable highest weight of $(B_n)_{2k+1}$ corresponds to precisely one of these subsets.

We are now in a position to present the map τ. First consider spinor weights Λ of $(B_n)_{2k+1}$. Given the associated subset $M_\Lambda \subset M$, define the complementary set

$$\{\tilde\ell^{(\tau)}_i\} \equiv M_{\tau(\Lambda)} := M \setminus M_\Lambda, \qquad (3.2.7)$$

where the numbers $\tilde\ell^{(\tau)}_i$ are to be ordered according to $\tilde\ell^{(\tau)}_i > \tilde\ell^{(\tau)}_j$ for $i<j$. Since this subset of M again satisfies $\tilde\ell^{(\tau)}_1 \le k+n$, and is of cardinality k, it describes precisely one orbit $\{\tau(\Lambda), J\star\tau(\Lambda)\}$ of integrable highest spinor weights of $(B_k)_{2n+1}$. Also note that M_Λ describes a spinor fixed point iff $k+n \in M_\Lambda$ (in contrast, there do not exist tensor fixed points at odd level); thus spinor fixed points are mapped to spinor orbits of size two, and vice versa.

Let us now check how the modular matrix T transforms under the map τ. By combining the formulæ (1.4.7) and (3.2.7), and inserting the strange formula for the length of the Weyl vectors, one finds

$$\begin{aligned}\Delta_\Lambda + \Delta_{\tau(\Lambda)} &= [\textstyle\sum_{j\in M_\Lambda} j^2 - (\rho,\rho) + \sum_{j\in M_{\tau(\Lambda)}} j^2 - (\rho^{(\tau)},\rho^{(\tau)})]/[4(k+n)] \\ &= [\textstyle\sum_{j=1}^{k+n} j^2 - \frac{1}{12}(4n^3-n+4k^3-k)]/[4(k+n)] \\ &= \tfrac{1}{8}(k+n+2kn+\tfrac{1}{2}),\end{aligned} \qquad (3.2.8)$$

where ρ and $\rho^{(\tau)}$ denote the Weyl vectors of B_n and B_k respectively. (Recall that we choose the representatives Λ and $\Lambda^{(\tau)}$ such that $\tilde\ell_1 \le k+n$ and $\tilde\ell_1^{(\tau)} \le k+n$; as the conformal dimensions of the elements of a spinor orbit differ by an integer, this means that for the other member of a length-two orbit, the formula holds true modulo $\mathbb{Z}$).

For tensors we will have to consider a definition of τ that is different from that for spinors [109]. Namely, while again the complement of M_Λ in M plays a role, we now define $M_{\tau(\Lambda)}$ by

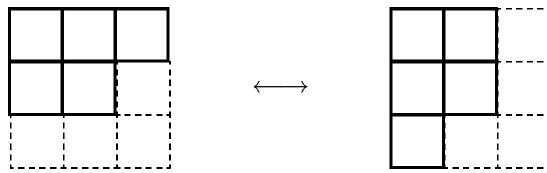
$$\{\tilde\ell^{(\tau)}_i\} \equiv M_{\tau(\Lambda)} := \{k+n+1-l \mid l \in M\setminus M_\Lambda\}. \qquad (3.2.9)$$

By definition, this maps tensor orbits to tensor orbits, and again the image covers all such orbits of $(B_k)_{2n+1}$ precisely once. For the sum of conformal dimensions we now obtain

$$\begin{aligned}\Delta_\Lambda + \Delta_{\tau(\Lambda)} &= [\textstyle\sum_{j\in M_\Lambda}(j-\frac{1}{2})^2 - (\rho,\rho) + \sum_{j\in M\setminus M_\Lambda}(k+n+\frac{1}{2}-j)^2 \\ &\qquad - (\rho^{(\tau)},\rho^{(\tau)})]\,/[4(k+n)] \\ &= \tfrac{1}{4}k(k+2n+1) - \tfrac{1}{2}\textstyle\sum_{j\in M\setminus M_\Lambda} j\,,\end{aligned} \qquad (3.2.10)$$

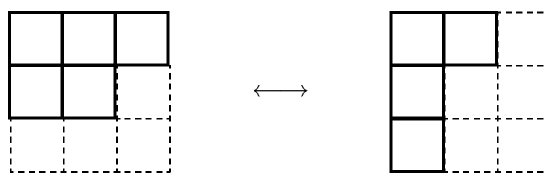
which is a half integer. (Again this result is true for Λ such that $\ell_1 \le k+n$, and analogously for $\tau(\Lambda)$; the conformal dimensions of the elements of a tensor orbit differ by $\frac{1}{2}$ plus an integer, so that for the other members of the orbits, the formula still holds modulo $\mathbb{Z}/2$).

One can visualize the map τ in terms of Young tableaux $Y(\Lambda)$, defined as having $\ell_i(\Lambda) - \frac{1}{2}c_\Lambda$ boxes in the ith row. The prescription (3.2.7) corresponds to forming the complement with respect to the rectangular Young tableau $Y(k\Lambda_{(n)})$, followed by reflection at an axis perpendicular to the main diagonal. Similarly, the map (3.2.9) corresponds just to reflection at the main diagonal. For example, consider the following mapping between tensor orbits of the (self-dual) $(B_3)_7$ WZW theory (for better readability, we display, with dotted lines, also the missing boxes that are needed to extend a tableau $Y(\Lambda)$ to $Y(k\Lambda_{(n)})$):



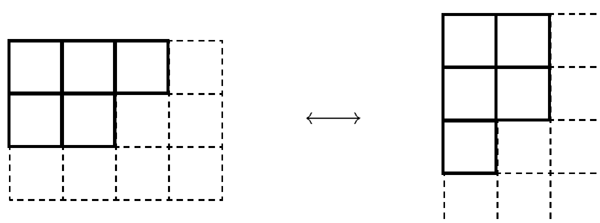
(3.2.11)

According to the previous prescriptions, the corresponding orbits are $\{(1,2,0),(2,2,0)\}$ for the left hand side, and $\{(0,1,2),(3,1,2)\}$ for the right hand side (here we write the weights in the basis of fundamental highest weights), and indeed these orbits are mapped onto each other by (3.2.9). Considering, instead, the left hand side as a Young tableau for a spinor orbit, namely for the fixed point $(1,2,1)$, it is mapped via (3.2.7) to the spinor orbit $\{(1,0,3),(3,0,3)\}$, i.e.



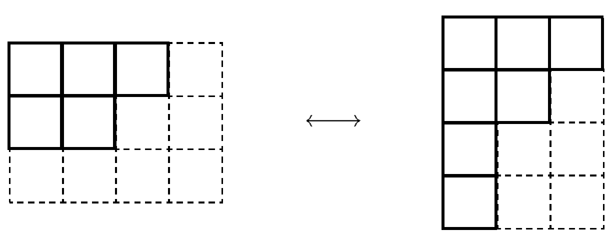
(3.2.12)

As further examples, consider the mappings



(3.2.13)

and



(3.2.14)

between orbits of $(B_3)_9$ (left) and $(B_4)_7$ (right). The first of these corresponds to the tensor orbits $\{(1,2,0),(4,2,0)\} \leftrightarrow \{(0,1,1,0),(3,1,1,0)\}$, and the second to the spinor orbits $\{(1,2,1),(3,2,1)\} \leftrightarrow \{(1,1,0,3)\}$.

Above, we have already obtained all information that we need about the modular matrix T. Next we want to determine the behavior of the S-matrix under the map τ.

We first recall that the Weyl group W of B_n acts in the orthonormal basis by all possible permutations and sign changes of the components. This implies that

$$\sum_{w\in W} \mathrm{sign}\,(w)\, \exp\,[\frac{\pi\mathrm{i}}{k+n}\,(w(\Lambda+\rho),\Lambda'+\rho)] = (2\mathrm{i})^n \,\mathrm{det}_{i,j}\mathcal{M}_{ij}(\Lambda,\Lambda'), \tag{3.2.15}$$

where

$$\mathcal{M}_{ij}(\Lambda,\Lambda') := \sin\,[\frac{\pi\,\ell_i(\Lambda+\rho)\,\ell_j(\Lambda'+\rho)}{k+n}]\,. \tag{3.2.16}$$

Inserting this identity into the Kac–Peterson formula (1.4.8) for the S-matrix, one arrives at

$$S_{\Lambda,\Lambda'} = (-1)^{n(n-1)/2} 2^{n/2-1} (k+n)^{-n/2}\, \mathrm{det}_{i,j}\mathcal{M}_{ij}(\Lambda,\Lambda'). \tag{3.2.17}$$

Now of course this result for the S-matrix refers to particular highest weights Λ and Λ'. However, what we really would like to compare are not the S-matrix elements for individual weights, but S-matrix elements for orbits with respect to simple currents. Now within an orbit, the sign of S depends on the choice of the representative (except if only tensor weights are involved). Thus if we want to interpret (3.2.17) as an equation for orbits, we have to keep in mind that when evaluating the equation we have to employ specific representatives (namely, those with the smaller value of ℓ_1). For the application to coset theories it will be crucial that the sign in (3.2.17) is correlated with the alternative whether the relation (3.2.10) between conformal weights holds exactly or only modulo $\frac12\mathbb{Z}$.

An analogous computation as for (3.2.17) yields

$$S_{\tau(\Lambda),\tau(\Lambda')} = (-1)^{k(k-1)/2} 2^{k/2-1} (k+n)^{-k/2}\, \mathrm{det}_{i,j}\tilde{\mathcal{M}}_{ij}(\Lambda,\Lambda') \tag{3.2.18}$$

with

$$\tilde{\mathcal{M}}_{ij}(\Lambda,\Lambda') := \sin\,[\frac{\pi\,\ell_i^{(\tau)}(\Lambda^{(\tau)}+\rho^{(\tau)})\,\ell_j^{(\tau)}(\Lambda'^{(\tau)}+\rho^{(\tau)})}{k+n}]\,. \tag{3.2.19}$$

To relate the numbers (3.2.17) and (3.2.18), we first note that $\mathcal{M}_{ij}(\Lambda,\Lambda')$ can be viewed as a $n\times n$ sub-matrix of the $(k+n)\times(k+n)$ matrix

$$A_{ij} := \begin{cases} A^{(\mathrm{tt})}_{ij} := \sin[(\pi\,(i-\frac12)(j-\frac12)/(k+n)] & \text{for}\quad c_\Lambda = c_{\Lambda'} = 0\,, \\ A^{(\mathrm{ts})}_{ij} := \sin[(\pi\,(i-\frac12)j)/(k+n)] & \text{for}\quad c_\Lambda = 0,\ c_{\Lambda'} = 1\,, \\ A^{(\mathrm{st})}_{ij} := \sin[(\pi\,i(j-\frac12)/(k+n)] & \text{for}\quad c_\Lambda = 1,\ c_{\Lambda'} = 0\,, \\ A^{(\mathrm{ss})}_{ij} := \sin[(\pi\,ij)/(k+n)] & \text{for}\quad c_\Lambda = c_{\Lambda'} = 1\,, \end{cases} \tag{3.2.20}$$

$i,j=1,2,\ldots,k+n$. Similarly, $\tilde{\mathcal{M}}_{ij}(\Lambda,\Lambda')$ is a $k\times k$ sub-matrix of

$$\tilde A_{ij} := \begin{cases} \sin[(\pi\,(k+n+\frac12-i)(k+n+\frac12-j))/(k+n)] & \\ \qquad\qquad = (-1)^{i+j+k+n+1}\,A^{(\mathrm{tt})}_{ij} & \text{for}\quad c_\Lambda = c_{\Lambda'} = 0\,, \\ \sin[(\pi\,(k+n+\frac12-i)j)/(k+n)] = (-1)^{j+1}\,A^{(\mathrm{ts})}_{ij} & \text{for}\quad c_\Lambda = 0,\ c_{\Lambda'} = 1\,, \\ \sin[(\pi\,i(k+n+\frac12-j))/(k+n)] = (-1)^{i+1}\,A^{(\mathrm{st})}_{ij} & \text{for}\quad c_\Lambda = 1,\ c_{\Lambda'} = 0\,, \\ A^{(\mathrm{ss})}_{ij} & \text{for}\quad c_\Lambda = c_{\Lambda'} = 1\,. \end{cases} \tag{3.2.21}$$

More precisely, the two submatrices are such that together they cover each value of i and j precisely once. As a consequence, one can use (a simple case of) the so-called

Jacobi-theorem [89] to relate $S_{\Lambda,\Lambda'}$ to $S_{\tau(\Lambda),\tau(\Lambda')}$. The theorem states that for any invertible matrix A whose rows and columns are labelled by a set H, one has for $I,\,J\subset H$ with $I\cup J=H$, $I\cap J=\emptyset$, that

$$\det\,[(A^{-1})^t]_{IJ} = (-1)^{\Sigma_I+\Sigma_J}(\det A)^{-1}\,(\det A)_{\overline{IJ}} \tag{3.2.22}$$

with $\overline{I}=H\setminus I$, $\overline{J}=H\setminus J$, and

$$\Sigma_I=\sum_{j\in I} j,\quad \Sigma_J=\sum_{j\in J} j. \tag{3.2.23}$$

Writing $S_{\Lambda,\Lambda'}=\alpha\det A_{IJ}$, $S_{\tau(\Lambda),\tau(\Lambda')}=\beta\det A_{\overline{IJ}}$ and $\det\,[(A^{-1})^t]_{IJ}=\delta\det A_{IJ}$, application of this theorem yields

$$S_{\Lambda,\Lambda'}=(-1)^{\Sigma_I+\Sigma_J}\alpha\,(\beta\gamma\delta)^{-1}S_{\tau(\Lambda),\tau(\Lambda')} \tag{3.2.24}$$

with $I=M_\Lambda$, $J=M_{\Lambda'}$, and A as defined in (3.2.20). (Actually, the definition of δ implies the assumption that $\det A_{IJ}\neq 0$ for all choices of I and J. This turns out to be true for all cases we are interested in. Moreover, in some cases in fact δ does not depend on the choice of I and J at all.)

An explicit expression for the number α can be read off (3.2.17), while when determining the parameters β, γ, δ, one has to distinguish between tensors and spinors. If both Λ and Λ' are tensors, then by straightforward calculation one finds

$$\begin{aligned}&\beta=(-1)^{k(k-1)/2}(-1)^{k(k+n+1)+\Sigma_{\overline{I}}+\Sigma_{\overline{J}}}\,2^{k/2-1}(k+n)^{-k/2},\\ &\gamma=(-1)^{(k+n)(k+n-1)/2}((k+n)/2)^{(k+n)/2},\qquad \delta=(2/(k+n))^n\,.\end{aligned} \tag{3.2.25}$$

When inserted into (3.2.24), this yields, upon use of the identity $\Sigma_{\overline{I}}+\Sigma_I=\sum_{j=1}^{k+n}j=(k+n)(k+n+1)/2$ [109],

$$S_{\Lambda,\Lambda'}=S_{\tau(\Lambda),\tau(\Lambda')}. \tag{3.2.26}$$

Note that this implies that τ connects tensor orbits with identical quantum dimension. (Since simple currents have quantum dimension 1 and quantum dimensions behave multiplicatively under the fusion product, the quantum dimension is constant on simple current orbits.)

If Λ is a tensor and Λ' is a spinor, one obtains [4]

$$\begin{aligned}&\beta=(-1)^{k+\Sigma_{\overline{I}}}2^{k/2-1}(k+n)^{-k/2},\\ &\gamma=(-1)^{(k+n)(k+n-1)/2}\,2^{(1-k-n)/2}(k+n)^{(k+n)/2},\qquad \delta=2^{n-f(\Lambda')}\,(k+n)^{-n}\,,\end{aligned} \tag{3.2.27}$$

where

$$f(\Lambda'):=\begin{cases}1 & \text{for }\ \Lambda'\ \text{ a fixed point,}\\ 0 & \text{for }\ \Lambda'\ \text{ an orbit of length two.}\end{cases} \tag{3.2.28}$$

Thus in this case [109]

$$S_{\Lambda,\Lambda'}=(-1)^{\Sigma_I+n(n+1)/2}\,2^{f(\Lambda')-1/2}\,S_{\tau(\Lambda),\tau(\Lambda')}. \tag{3.2.29}$$

---

[4] Notice that if Λ is a tensor, then the order of the rows of $\tilde{A}_{ij}$ is actually to be read backwards such as to satisfy the requirement that the numbers obey $\tilde{\ell}_i^{(\tau)}>\tilde{\ell}_j^{(\tau)}$ for $i<j$; this contributes a factor $(-1)^{k(k-1)/2}$ to β. If Λ' is a tensor, the same factor arises from an analogous re-ordering of columns. In particular, for both Λ and Λ' tensors, these factors cancel out.

(Again, the sign depends on the choice of representative of the tensor orbit. It is as given in (3.2.29) if the representative with smaller value of ℓ_1 and $\tilde{\ell}_1$ is taken.) In particular, for spinors the quantum dimensions of the orbits of Λ and $\tau(\Lambda)$ differ by a factor $\sqrt{2}$ for orbits of length 2, and by a factor $1/\sqrt{2}$ for fixed points.

Analogously, for Λ a spinor and Λ' a tensor, one obtains (3.2.29) with Σ_J replaced by Σ_I and $f(\Lambda')$ replaced by $f(\Lambda)$. Finally, if both Λ and Λ' are spinors, we again have to distinguish between several cases. Observing that Λ is a fixed point iff $k+n \in M_\Lambda$, and that $A^{(\mathrm{ss})}_{j,k+n} = A^{(\mathrm{ss})}_{k+n,j} = \sin(\pi j) = 0$, we conclude that

$$S_{\Lambda,\Lambda'} = S_{\tau(\Lambda),\tau(\Lambda')} = 0 \tag{3.2.30}$$

if Λ is a fixed point and Λ' belongs to a length-two spinor orbit or vice versa. In contrast, if both Λ and Λ' are fixed points, $S_{\Lambda,\Lambda'}$ vanishes but $S_{\tau(\Lambda),\tau(\Lambda')}$ does not, and the other way round for both Λ and Λ' belonging to length-two spinor orbits.

3.3 B type theories at even level versus D type at odd level

In this section we present a map τ relating $(B_k)_{2n}$ and $(D_n)_{2k+1}$ that behaves similarly as the one described in the previous section. However, for $(B_k)_{2n}$ we now have to restrict ourselves to tensor weights; for these, we define ℓ_i and $\tilde{\ell}_i$ as in (3.2.4) and (3.2.5). In contrast to odd level, now the map is no longer one-to-one on the simple current orbits. Rather, some of the orbits of $(B_k)_{2n}$ (namely, those tensors which are fixed points; in contrast to odd level, fixed points now must be tensors) correspond to two distinct orbits of $(D_n)_{2k+1}$.

For $(D_n)_{2k+1}$, the components of a weight Λ in the orthonormal basis are

$$\begin{aligned} &\ell_i(\Lambda) = \textstyle\sum_{j=i}^{n-2} \Lambda^j + \frac{1}{2}\,(\Lambda^{n-1}+\Lambda^n) \quad \text{for } i=1,2,\ldots,n-2\,, \\ &\ell_{n-1} = \tfrac{1}{2}\,(\Lambda^{n-1}+\Lambda^n)\,, \qquad \ell_n = \tfrac{1}{2}\,(-\Lambda^{n-1}+\Lambda^n)\,. \end{aligned} \tag{3.3.1}$$

At odd level, all orbits (with respect to the full set of simple currents, which is generated by J_s for odd n, and by J_s and J_v for even n) consist of four fields. Each such orbit of integrable highest weights contains precisely one representative that satisfies $\Lambda^0 \geq \Lambda^1$ and $\Lambda^{n-1} - \Lambda^n \in 2\mathbb{Z}$, implying that $\ell_i(\Lambda) \in \mathbb{Z}$ and $k \geq \ell_1 \geq \ell_2 \geq \ldots \geq \ell_n$. From now on we restrict our attention to this particular representative. Thus the numbers

$$\tilde{\ell}_i(\Lambda) := \ell_i(\Lambda+\rho) = \ell_i(\Lambda) + n - i \tag{3.3.2}$$

satisfy $0 < \tilde{\ell}_i(\Lambda) \leq k+n-1$ for $i=1,2,\ldots,n-1$, and $|\tilde{\ell}_n| < k$. As it turns out, a special role is played by those orbits for which $\ell_n = 0$; we will refer to such orbits as *spinor-symmetric*. Analogously, orbits that are transformed into each other upon changing the sign of ℓ_n ($\neq 0$) are called 'spinor-conjugate' to each other.

We define now a map τ between the orbits of $(D_n)_{2k+1}$ and the tensor orbits of $(B_k)_{2n}$ as follows. To an orbit of $(D_n)_{2k+1}$ with representative Λ we associate the subset M_Λ of $M = \{1,2,\ldots,k+n\}$ by

$$M_\Lambda := \{\, |\tilde{\ell}_i(\Lambda)| + 1 \mid i = 1,2,\ldots,n\}\,. \tag{3.3.3}$$

Then the (tensor) weight $\tau(\Lambda)$ of $(B_k)_{2n}$ is defined by the requirement that the set $M_{\tau(\Lambda)}$ (the connection between Λ and M_Λ for $(B_k)_{2n}$ is defined in the same way as for $(B_k)_{2n+1}$ in Section 3.2) is given by

$$\{\tilde{\ell}_i^{(\tau)}\} \equiv M_{\tau(\Lambda)} := \{k+n+1-l \mid l \in M \setminus M_\Lambda\}\,. \tag{3.3.4}$$

Note that we have chosen our conventions for $(D_n)_{2k+1}$ (in particular the constant term '+1' in (3.3.3)) in such a manner that the prescription (3.3.4) is formally the same as (3.2.9) in Section 3.2. Furthermore, $\tau(\Lambda)$ is a fixed point iff $k+n \in M_{\tau(\Lambda)}$, i.e., iff $1 \notin M_\Lambda$, i.e., iff Λ is not spinor-symmetric. Note also that this map is *not* one-to-one on the orbits. Rather, non-spinor-symmetric $(D_n)_{2k+1}$weights which transform into each other upon interchanging ℓ^{n-1} and ℓ^n are mapped on the same weight of $(B_k)_{2n}$. (As we will see later on, this is precisely the behavior we need in coset theories in order to implement the fixed point resolution.)

We now consider the behavior of the modular matrices T and S under the map τ. For the sum of conformal dimensions one finds

$$\begin{aligned} \Delta_\Lambda + \Delta_{\tau(\Lambda)} &= [\textstyle\sum_{j\in M_\Lambda} j^2 - (\rho,\rho) + \sum_{j\in M_{\tau(\Lambda)}} j^2 - (\rho^{(\tau)},\rho^{(\tau)})]\,/[4(k+n-\frac12)] \\ &= \tfrac14\, k(k+2n+1) - \tfrac12 \textstyle\sum_{j\in M\setminus M_\Lambda} j\,, \end{aligned} \tag{3.3.5}$$

which is always a half integer. The Weyl group of $(D_n)_{2k+1}$ corresponds in the orthonormal basis to permutations and to even numbers of sign changes of the components, so that the Kac–Peterson formula for the S-matrix leads to

$$S_{\Lambda,\Lambda'} = (-1)^{n(n-1)/2} 2^{n/2-2} (k+n-\tfrac12)^{-n/2}\, [\det_{i,j} \mathcal{M}^+_{ij}(\Lambda,\Lambda') + \mathrm{i}^n \det_{i,j} \mathcal{M}^-_{ij}(\Lambda,\Lambda')]\,, \tag{3.3.6}$$

where

$$\begin{aligned} \mathcal{M}^+_{ij}(\Lambda,\Lambda') &:= \cos\,[\frac{2\pi\, \ell_i(\Lambda+\rho)\, \ell_j(\Lambda'+\rho)}{2k+2n-1}], \\ \mathcal{M}^-_{ij}(\Lambda,\Lambda') &:= \sin\,[\frac{2\pi\, \ell_i(\Lambda+\rho)\, \ell_j(\Lambda'+\rho)}{2k+2n-1}]\,. \end{aligned} \tag{3.3.7}$$

Note that $\det(\mathcal{M}^-_{ij}(\Lambda,\Lambda')) = 0$ whenever Λ or Λ' are spinor-symmetric. For later convenience we denote by $S^{(+)}_{\Lambda,\Lambda'}$ the numbers obtained from (3.3.6) when replacing $\mathcal{M}^-_{ij}(\Lambda,\Lambda')$ by zero, i.e.,

$$S^{(+)}_{\Lambda,\Lambda'} = (-1)^{n(n-1)/2} 2^{n/2-2} (k+n-\tfrac12)^{-n/2} \det_{i\in M_\Lambda, j\in M_{\Lambda'}} \cos\,[\frac{\pi\,(i-1)(j-1)}{k+n-\frac12}]\,. \tag{3.3.8}$$

The S-matrix of $(B_k)_{2n}$ can be calculated analogously as described in the previous section for $(B_k)_{2n+1}$. The result is

$$\begin{aligned} S_{\tau(\Lambda),\tau(\Lambda')} &= (-1)^{k(k-1)/2} 2^{k/2-1} (k+n-\tfrac12)^{-k/2} \\ &\qquad \cdot \det_{i\in M\setminus M_\Lambda, j\in M\setminus M_{\Lambda'}} \sin\,[\frac{\pi\,(k+n+\frac12-i)(k+n+\frac12-j)}{k+n-\frac12}]\,. \end{aligned} \tag{3.3.9}$$

Combining (3.3.8) with (3.3.9), we can use the Jacobi-theorem together with the identity $\sin[\pi\,(k+n+\frac12-i)(k+n+\frac12-j)/(k+n-\frac12)] = (-1)^{i+j+k+n+1} \cos[\pi\,(i-1)(j-1)/(k+n-\frac12)]$ to obtain again a relation like (3.2.24), namely

$$S^{(+)}_{\Lambda,\Lambda'} = (-1)^{\Sigma_I+\Sigma_J}\, \alpha\, (\beta\gamma\delta)^{-1} S_{\tau(\Lambda),\tau(\Lambda')}. \tag{3.3.10}$$

The parameters are this time calculated as [109]

$$\begin{aligned} \alpha &= (-1)^{k(k-1)/2} (-1)^{k(n+k+1)+\Sigma_I+\Sigma_J} 2^{k/2-1} (k+n-\tfrac12)^{-k/2}, \\ \beta &= (-1)^{n(n-1)/2} 2^{n/2-2} (k+n-\tfrac12)^{-n/2}, \\ \gamma &= 2(-1)^{(k+n)(k+n-1)/2} ((k+n-\tfrac12)/2)^{(k+n)/2}, \\ \delta &= 2^{-\mathrm{s}(\Lambda)-\mathrm{s}(\Lambda')}\, (2/(k+n-\tfrac12))^k\,, \end{aligned} \tag{3.3.11}$$

where

$$s(\Lambda) := \begin{cases} 0 & \text{if } \Lambda \text{ is spinor-symmetric,} \\ 1 & \text{else.} \end{cases} \tag{3.3.12}$$

This leads to

$$S^{(+)}_{\Lambda,\Lambda'} = 2^{s(\Lambda)+s(\Lambda')} S_{\tau(\Lambda),\tau(\Lambda')} . \tag{3.3.13}$$

(When interpreting this equation as a relation between simple current orbits, one must take the specific representative of the orbit of the D type WZW theory described above. Otherwise (3.3.13) becomes modified by a phase. However, as only tensors of the B type WZW theory are involved, the phase does not depend on the representative of the orbits of the B theory.) Recalling that $\det(\mathcal{M}^{-}_{ij}(\Lambda,\Lambda')) = 0$, i.e., $S_{\Lambda,\Lambda'} = S^{(+)}_{\Lambda,\Lambda'}$, if Λ or Λ' are spinor-symmetric, this means in more detail that

$$S_{\tau(\Lambda),\tau(\Lambda')} = \begin{cases} S_{\Lambda,\Lambda'} & \text{for } \Lambda \text{ and } \Lambda' \text{ spinor-symmetric,} \\ S_{\Lambda,\Lambda'}/2 & \text{for } \Lambda \text{ spinor-symmetric, } \Lambda' \text{ non-spinor-symmetric,} \\ & \qquad\qquad\qquad \text{or vice versa,} \\ S^{(+)}_{\Lambda,\Lambda'}/4 & \text{for } \Lambda \text{ and } \Lambda' \text{ non-spinor-symmetric.} \end{cases} \tag{3.3.14}$$

3.4 *C* type WZW theories

When considering C type WZW theories, we are in a more convenient position than previously. Namely, one can construct a map τ between individual fields and not just between simple current orbits. [5]

We consider again the components of Λ in an orthogonal basis of the weight space. However, for convenience we multiply the components of the ortho*normal* basis by a factor $\sqrt{2}$, because we then have to deal with integral coefficients only. The components of a weight Λ in this non-normalized basis read $\ell_i(\Lambda) = \sum_{j=i}^{n} \Lambda^j$. Again we add to these numbers the components of the Weyl vector, i.e., we define

$$\tilde{\ell}_i(\Lambda) := \ell_i(\Lambda+\rho) = \sum_{j=i}^{n} \Lambda^j + n + 1 - i . \tag{3.4.1}$$

This time the integrability condition (1.4.6) implies, for $(C_n)_k$, that

$$k+n \geq \tilde{\ell}_1 > \ldots > \tilde{\ell}_i > \tilde{\ell}_{i+1} > \ldots > \tilde{\ell}_n \geq 1 . \tag{3.4.2}$$

Thus we can describe every weight Λ uniquely by a set of n positive integers $\tilde{\ell}_i$, $i = 1,2,\ldots,n$, subject to $\tilde{\ell}_i > \tilde{\ell}_j$ for $i<j$ as well as $\tilde{\ell}_1 \leq k+n$, that is, by a subset M_Λ of cardinality n of the set $M = \{1,2,\ldots,k+n\}$. Given such a subset M_Λ, we define $\tau(\Lambda)$ through the complementary set $\{\tilde{\ell}^{(\tau)}_i\} \equiv M_{\tau(\Lambda)} := M \setminus M_\Lambda$, where again the numbers $\tilde{\ell}^{(\tau)}_i$ are to be ordered according to $\tilde{\ell}^{(\tau)}_i > \tilde{\ell}^{(\tau)}_j$ for $i<j$. Since this subset of M again satisfies $\tilde{\ell}^{(\tau)}_1 \leq k+n$, and is of cardinality k, it describes precisely one integrable highest weight $\tau(\Lambda)$ of $(C_k)_n$. (In terms of Young tableaux, the map corresponds to forming the complement with respect to the rectangular Young tableau $Y(k\Lambda_{(n)})$, followed by reflection at an axis perpendicular to the main diagonal.)

[5] In the notation of [109], our map τ is the composition of the maps 'ρ' of Section 2 and 'tilde' of Section 1 of [109].

As in the previous sections, it is straightforward to calculate the quantity $\Delta_\Lambda + \Delta_{\tau(\Lambda)}$. Taking care of the extra factor $\frac{1}{2}$ in the scalar product that is caused by our normalization of the ℓ_i, one obtains

$$\begin{aligned}
\Delta_\Lambda + \Delta_{\tau(\Lambda)} &= [\tfrac{1}{2}\textstyle\sum_{j\in M_\Lambda} j^2 - (\rho,\rho) + \tfrac{1}{2}\sum_{j\in M_{\tau(\Lambda)}} j^2 - (\rho^{(\tau)},\rho^{(\tau)})]\,/[2(k+n+1)] \\
&= [\tfrac{1}{2}\textstyle\sum_{j=1}^{k+n} j^2 - \tfrac{1}{12}(2n^3+3n^2+n+2k^3+3k^2+k)]\,/[2(k+n+1)] \\
&= \tfrac{1}{4}\,kn\,,
\end{aligned} \tag{3.4.3}$$

where ρ and $\rho^{(\tau)}$ denote the Weyl vectors of C_n and C_k, respectively.

Proceeding to the modular matrix S, we note that the Weyl group of C_n acts in the orthogonal basis by permutations and arbitrary sign changes, implying that

$$\sum_{w\in W} \text{sign}\,(w)\,\exp\big[\frac{\pi\text{i}}{k+n}\,(w(\Lambda+\rho),\Lambda'+\rho)\big] = (2\text{i})^n\,\det_{i,j}\mathcal{M}_{ij}(\Lambda,\Lambda') \tag{3.4.4}$$

with

$$\mathcal{M}_{ij}(\Lambda,\Lambda') := \sin\big[\frac{\pi\,\ell_i(\Lambda+\rho)\,\ell_j(\Lambda'+\rho)}{k+n+1}\big]\,. \tag{3.4.5}$$

Thus the Kac–Peterson formula for the S-matrix yields

$$S_{\Lambda,\Lambda'} = (-1)^{n(n-1)/2}2^{n/2}(k+n+1)^{-n/2}\,\det_{i,j}\mathcal{M}_{ij}(\Lambda,\Lambda') \tag{3.4.6}$$

and similarly,

$$S_{\tau(\Lambda),\tau(\Lambda')} = (-1)^{k(k-1)/2}2^{k/2}(k+n+1)^{-k/2}\,\det_{i,j}\tilde{\mathcal{M}}_{ij}(\Lambda,\Lambda'). \tag{3.4.7}$$

Now $\mathcal{M}_{ij}(\Lambda,\Lambda')$ can be viewed as a $n\times n$ sub-matrix, and $\tilde{\mathcal{M}}_{ij}(\Lambda,\Lambda')$ as a $k\times k$ submatrix, of the $(k+n)\times(k+n)$ matrix $A_{ij} := \sin[(\pi\,ij/(k+n+1)]$, $i,j\in\{1,2...,k+n\}$, such that the two submatrices together cover each value of i and j precisely once. As a consequence, the Jacobi-theorem is again applicable, leading to the relation (3.2.24) between $S_{\Lambda,\Lambda'}$ and $S_{\tau(\Lambda),\tau(\Lambda')}$. The numbers α, β, γ, δ in that relation are this time found to be

$$\begin{aligned}
&\alpha = (-1)^{n(n-1)/2}\,2^{n/2}(k+n+1)^{-n/2}, \qquad \beta = (-1)^{k(k-1)/2}2^{k/2}(k+n+1)^{-k/2}, \\
&\gamma = (-1)^{(k+n)(k+n-1)/2}((k+n+1)/2)^{(k+n)/2}, \qquad \delta = (2/(k+n+1))^n\,.
\end{aligned} \tag{3.4.8}$$

When inserting this into (3.2.24), we make use of the identities $\Sigma_I = n(n+1)/2 + r(\Lambda)$ and $\Sigma_{\overline{I}} = k(k+1)/2 + r(\Lambda')$, where

$$r(\Lambda) := \sum_{i=1}^{n} \ell_i(\Lambda)\,, \tag{3.4.9}$$

which is modulo 2 the conjugacy class of the C_n-weight Λ (also, r equals the number of boxes in the Young tableau $Y(\Lambda)$ that is associated to Λ). One then obtains

$$S_{\Lambda,\Lambda'} = (-1)^{r(\Lambda)+r(\Lambda')+kn}\,S_{\tau(\Lambda),\tau(\Lambda')}. \tag{3.4.10}$$

Table 3.1: Some $N=2$ superconformal coset theories and their Virasoro charges

name	$\mathcal{C}[\bar{\mathsf{g}}_K \oplus (D_d)_1 / \bigoplus_i (\bar{\mathsf{h}}_i)_{K_i} \oplus (\mathrm{u}(1))_{\mathcal{N}}]$	c
$(B, 2n+1, K)$	$\mathcal{C}[(B_{n+1})_K \oplus (D_{2n+1})_1 \,/\, (B_n)_{K+2} \oplus (u_1)_{4(K+2n+1)}]$	$\dfrac{3K(2n+1)}{K+2n+1}$
$(B, 2n, K)$	$\mathcal{C}[(D_{n+1})_K \oplus (D_{2n})_1 \,/\, (D_n)_{K+2} \oplus (u_1)_{4(K+2n)}]$	$\dfrac{6Kn}{K+2n}$
$(BB, 3, K)$	$\mathcal{C}[(B_3)_K \oplus (D_7)_1 \,/\, (A_1)_{2K+8} \oplus (A_1)_{K+3} \oplus (u_1)_{2(K+5)}]$	$21 - \dfrac{96}{K+5}$
(BB, n, K), $n>3$	$\mathcal{C}[(B_n)_K \oplus (D_{4n-5})_1 \,/\, (B_{n-2})_{K+4} \oplus (A_1)_{K+2n-3} \oplus (\mathrm{u}_1)_{2(K+2n-1)}]$	$12n - 15 - \dfrac{24(n-1)^2}{K+2n-1}$
(CC, n, K)	$\mathcal{C}[(C_n)_K \oplus (D_{2n-1})_1 \,/\, (C_{n-1})_{K+1} \oplus (u_1)_{2(K+n+1)}]$	$6n - 3 - \dfrac{6n^2}{K+n+1}$

3.5 Coset theories

In this section we collect some information on $N=2$ coset theories; for those theories which correspond to non-hermitian symmetric coset, part of the information was already displayed in the previous chapter. We will repeat it here to fix the notation and to give some information on hermitian symmetric coset cosets we will use below as well.

The theories of interest in this chapter are listed in Table (3.1), together with their central charges. The identification currents of these theories (including the simple current that implements the D type modular invariant in the case of $(B, 2n+1, 2k)$) are displayed in Table (3.2) together with their order N.

Among the theories of our interest, only the cosets of CC type and $(B, 2n, 2k+1)$ do not possess any fixed points. If fixed points are present, one has to give a prescription for finding the physical fields, i.e. 'fixed point resolution.' Every fixed point of length N_f has to be resolved in N/N_f distinct physical fields.

We will label the primary fields Φ of a $N=2$ coset theory $\mathcal{C}[\bar{\mathsf{g}} \oplus D_d / \bar{\mathsf{h}} \oplus \mathrm{u}(1)]_K$ by the weights carried by the primaries of the WZW theories it is composed of, i.e.,

$$\Phi \mathrel{\hat{=}} (\Lambda, \mathrm{x} \,/\, \lambda, Q) \tag{3.5.1}$$

with Λ and λ integrable highest weights of the g and h algebras, x a conjugacy class of D_d, and $Q \in \{0, 1, , \dots, \mathcal{N}-1\}$ a u(1)-charge. However, as a consequence of the necessary field identification, this labelling is not one-to-one. Rather, all combinations of labels that are connected via the action of the identification currents describe one and the same primary field; moreover, fixed point resolution introduces an additional label i according to

$$\Phi_{\text{fix}} \mathrel{\hat{=}} (\Lambda, \mathrm{x} \,/\, \lambda, Q)_i \tag{3.5.2}$$

The conformal dimension of the field Φ is modulo integers

$$\Delta(\Phi) = \Delta_{(\mathsf{g})}(\Lambda) + \Delta_{(D_d)}(\mathrm{x}) - \Delta_{(\mathsf{h})}(\lambda) - \Delta_{(\mathrm{u}_1)}(Q), \tag{3.5.3}$$

Table 3.2: Identification currents for $N=2$ coset theories

name	N	Independent identification currents
$(B,2n+1,2k+1)$	4	$\begin{cases} J_{(1)} := (J,0\,/\,J,0) \\ J_{(2)} := (J,J_{\rm v}\,/\,0,\pm 4(k+n+1)) \end{cases}$
$(B,2n+1,2k)_{\vert D}$	8	$\begin{cases} J_{(1)} := (J,0\,/\,J,0) \\ J_{(2)} := (J,J_{\rm v}\,/\,0,\pm 2(2k+2n+1)) \\ J_{(3)} := (J,0\,/\,0,0)) \end{cases}$
$(B,2n,K)$	8	$\begin{cases} J_{(1)} := (J_{\rm v},0\,/\,J_{\rm v},0) \\ J_{(2)} := (J_{\rm s},(J_{\rm v})^n\,/\,J_{\rm s},(K+2n)) \end{cases}$
(BB,n,K)	4	$\begin{cases} J_{(1)} := (J,0\,/\,J,0,0) \\ J_{(2)} := (J,0\,/\,0,J,\pm(K+2n-1)) \end{cases}$
(CC,n,K)	2	$J_{(1)} := (J,(J_{\rm v})^n\,/\,J,\pm(K+n+1))$

where $\Delta_{(\mathsf{g})}(\Lambda)$ and $\Delta_{(\mathsf{h})}(\lambda)$ are defined as in (1.4.7), $\Delta_{(D_d)}(\mathrm{x})$ is given in (1.4.9), and $\Delta_{(\mathrm{u}_1)}(Q)=Q^2/2\mathcal{N}$. The superconformal charge q is modulo 2 given by

$$q(\Phi)=\sum_\alpha \mathrm{x}^\alpha-\frac{\xi_\circ Q}{K+g^\vee}\,. \tag{3.5.4}$$

Here x^α are the components of x in the orthonormal basis of the D_d weight space, For the theories of our interest, one has $\xi_\circ=n$ for $(B,2n,K)$ and (CC,n,K), $\xi_\circ=n+\frac{1}{2}$ for $(B,2n+1,K)$, and $\xi_\circ=2(n-1)$ for (BB,n,K).

In order to identify the chiral rings of $N=2$ coset theories, we will again look at the Ramond ground states. Any Ramond ground state has at least one representative

$$\Phi_R=(\Lambda,\mathrm{x},\tilde\lambda) \tag{3.5.5}$$

for which Λ and $\tilde\lambda$ are related through a Weyl group element $w\in\overline{W}_{\bar{\mathsf{g}}}$ according to [103]

$$\tilde\lambda+\rho_{\bar{\mathsf{h}}}=w(\Lambda+\rho_{\bar{\mathsf{g}}})\,. \tag{3.5.6}$$

Here $\tilde\lambda$ incorporates both the weight λ of the semi-simple part h of $\tilde{\mathsf{h}}$ and the u(1)-charge Q, and

$$\mathrm{x}=\begin{cases}\mathrm{s} & \text{for sign}\,(w)=1\,,\\ \mathrm{c} & \text{for sign}\,(w)=-1\,.\end{cases} \tag{3.5.7}$$

Furthermore, the Weyl group element w has to be chosen in such a manner that λ is a highest weight of $\bar{\mathsf{h}}$ (this fixes uniquely one representative of each element of the coset $\overline{W}_{\bar{\mathsf{g}}}/\overline{W}_{\bar{\mathsf{h}}}$). The superconformal charge q (including the integer part) of a Ramond ground state is given by the formula (2.4.17)

$$q(\Phi_R)=\frac{d}{2}-l(w)-\frac{\xi_\circ Q}{K+g^\vee} \tag{3.5.8}$$

that relates q to the u(1)-charge Q and to the length $l(w)$ of the Weyl group element w that appears in (3.5.6) ($\xi_\circ$ is the number introduced in (3.5.4)).

3.6 $N=2$ coset models of type B; odd values of rank and level

3.6.1 The map $\mathcal{T}$

We are now going to describe a one-to-one map $\mathcal{T}$ between the primary fields of the $N=2$ superconformal coset theories $(B,2n+1,2k+1)$ and $(B,2k+1,2n+1)$. We will show that this map leaves the modular matrices S and T invariant and, moreover, provides a one-to-one map between chiral primary fields. Correspondingly we consider the two coset theories as isomorphic conformal field theories and write

$$(B,2n+1,2k+1) \overset{\mathcal{T}}{\cong} (B,2k+1,2n+1)\,. \tag{3.6.1}$$

This is in contrast to the level-rank duality of the underlying WZW theories which is far from providing an isomorphism of conformal field theories.

To start, let us mention two simple necessary requirements for such an identification to exist. First, from Table 3.1 we read off that the Virasoro central charge of $(B,2n+1,2k+1)$ is $c_{2n+1,2k+1}=\frac{3(2k+1)(2n+1)}{2(k+n+1)}$, which is invariant under exchanging n and k. It was precisely this observation [97] that led to the idea of level-rank duality of these theories. Second, we see that the two theories possess the same number of (Virasoro and u(1)) primary fields. Namely, for the coset theory $(B,2n+1,2k+1)$ the number of primaries can be expressed as

$$\begin{aligned}\nu^{BB}_{2n+1,2k+1} = N^{D}_{2n+1,1}\, N^{1}_{8(k+n+1)}\, \big(\tfrac{1}{16}\,[N^{B}_{n+1,2k+1}\, N^{B}_{n,2k+3} - F^{B}_{n+1,2k+1}\, F^{B}_{n,2k+3}] \\ + 2\cdot\tfrac{1}{4}\, F^{B}_{n+1,2k+1}\, F^{B}_{n,2k+3}\big)\end{aligned} \tag{3.6.2}$$

in terms of the numbers $N^{B}_{m,K}$ of primary fields and $F^{B}_{m,K}$ of fixed points of the B type WZW theories. Here the first two factors come from D_{2n+1} at level one and from the u(1) theory, respectively. The numbers in the bracket refer to the theories B_{n+1} at level $2k+1$ and B_n at level $2k+3$; the term in square brackets corresponds to the orbits of length four, with the factor $\frac{1}{16}$ taking care of the selection rule and the identification of order four (one quarter of the possible combinations of quantum numbers of the individual theories gets projected out, and each identification orbit has four members), and the second term corresponds to the fixed points, the factor of 2 being due to the resolution procedure (for the fixed points, the factor of $\frac{1}{16}$ is replaced by $\frac{1}{4}$ because none of the fixed points is projected out by the selection rule encoded in $J_{(1)}$). Inserting $N^{D}_{d,1}=4$ and $N^{1}_{\mathcal{N}}=\mathcal{N}$ as well as the formulæ (3.2.1) and (3.2.2) for $N^{B}_{m,K}$ and $F^{B}_{m,K}$, (3.6.2) becomes

$$\nu^{BB}_{2n+1,2k+1} = 2\left(4n+4k+3-\frac{2kn}{k+n+1}\right)\binom{k+n+1}{k}\binom{k+n+1}{n}. \tag{3.6.3}$$

Obviously, for $(B,2k+1,2n+1)$ one obtains the same number of primaries.

After these preliminaries, we now present the map $\mathcal{T}$ alluded to above. Suppose we are given a specific representative $(\Lambda,\mathrm{x}\,/\,\lambda,Q)$ of a field Φ as described in (3.5.1); then we map the simple current orbits of Λ and λ on their images under the map τ that was introduced in Section 3.2. Thus

$$\mathcal{T}(\Phi) \,\hat{=}\, (\tau(\lambda), \mathrm{x}_{\mathcal{T}}\,/\,\tau(\Lambda), Q_{\mathcal{T}})\,, \tag{3.6.4}$$

with τ as defined in (3.2.7) and (3.2.9), and with $\mathrm{x}_{\mathcal{T}}$ and $Q_{\mathcal{T}}$ to be specified below. Now the objects on the right hand side of (3.6.4) are just representatives of primary fields, and not yet the primary fields themselves. In particular, the quantities $\mathrm{x}_{\mathcal{T}}$ and $Q_{\mathcal{T}}$ are to be considered as orbits, and only after fixing representatives of the orbits of $\tau(\lambda)$ and $\tau(\Lambda)$ are they fixed as well, so that $\mathrm{x}_{\mathcal{T}}$ becomes an element of $\{0,\mathrm{v},\mathrm{s},\mathrm{c}\}$ and $Q_{\mathcal{T}}$ an integer between 0 and $\mathcal{N}$. To describe the physical fields, we have to implement the identification currents. According to Table 3.2, in the present case there are two independent identification currents $J_{(1)}$ and $J_{(2)}$. As $J_{(1)}=(J,1\,/\,J,0)$ acts trivially on the D_d and u(1) parts, it is convenient to first restrict the attention to $J_{(1)}$-orbits and to implement $J_{(2)}$ later on. Provided that no fixed points are present, [6] for fixed choice of $\mathrm{x}_{\mathcal{T}}$ and $Q_{\mathcal{T}}$ we have to deal with a total of four representatives of two $J_{(1)}$-orbits.

Now observe that, owing to the selection rule implemented by $J_{(1)}$, the conjugacy classes of Λ and λ coincide, so that we only need to consider combinations of tensors with tensors, or of spinors with spinors. In the case of tensors of both $(B_{n+1})_{2k+1}$ and $(B_n)_{2k+3}$, fixed points do not occur. Further, modulo $\mathbb{Z}$, the conformal dimensions of the two $J_{(1)}$-orbits differ by $\frac{1}{2}$, precisely as the conformal dimensions of the corresponding fields of $(B,2n+1,2k+1)$. To start with the definition of $\mathcal{T}$, we now simply choose the $J_{(1)}$-orbit that has conformal weight equal to $\Delta_\Lambda-\Delta_\lambda$ modulo $\mathbb{Z}$. Due to the identification current $J_{(2)}$, this choice actually does not constitute any loss of generality (but it simplifies some formulæ further on). Namely, each of the $J_{(1)}$-orbits $\mathcal{O}$ lies on the same orbit with respect to $J_{(2)}$ as another $J_{(1)}$-orbit whose values of $\mathrm{x}_{\mathcal{T}}$ and $Q_{\mathcal{T}}$ differ from those of $\mathcal{O}$ in such a manner that the values of $\Delta_\Lambda-\Delta_\lambda$ differ by $\frac{1}{2}$ mod $\mathbb{Z}$.

For spinors, both $J_{(1)}$-orbits in question have identical conformal weight. The freedom to choose one of the orbits turns out to be closely connected with the issue of fixed point resolution. Namely, the property of τ to map WZW fixed points on WZW-orbits of length two and vice versa, translates into the following property of $\mathcal{T}$: any 'unresolved fixed point' is mapped on two distinct fields, and vice versa, such that the non-fixed points of one theory precisely describe the resolved fixed points of the other theory. In case that just one of the orbits in either the 'numerator' or the 'denominator' of the coset theory is a fixed point, we have exactly the reversed situation in the dual theory.

Having fixed the B parts of the theory, we extend the definition of $\mathcal{T}$ to the u(1) and D_d parts by the following definitions: the D_d part remains unchanged, i.e. $\mathrm{x}_{\mathcal{T}}=\mathrm{x}$, while the u(1)-charge is transformed according to

$$Q_{\mathcal{T}}=-Q+\begin{cases} QL & \text{for}\quad c_\Lambda=0 \quad\text{and}\quad \mathrm{x}\in\{0,\mathrm{v}\}\,,\\ Q(2n+1)L & \text{for}\quad c_\Lambda=0 \quad\text{and}\quad \mathrm{x}\in\{\mathrm{s},\mathrm{c}\}\,,\\ Q(2k+1)L & \text{for}\quad c_\Lambda=1 \quad\text{and}\quad \mathrm{x}\in\{0,\mathrm{v}\}\,,\\ (2n-2k-Q)L & \text{for}\quad c_\Lambda=1 \quad\text{and}\quad \mathrm{x}\in\{\mathrm{s},\mathrm{c}\}\,. \end{cases} \tag{3.6.5}$$

Here, for convenience, we use the abbreviation

$$L=2(k+n+1)\,, \tag{3.6.6}$$

and all u(1)-charges are understood modulo $\mathcal{N}=4L$. (Thus L is one quarter of the u(1)-charge of the primary field that extends the chiral algebra of the u(1) theory, and hence the appearance of this number in (3.6.5) is quite natural.)

The definition of $\mathcal{T}$ is not yet complete, of course, as we still have to make precise its meaning when acting on, or mapping to, resolved fixed points. Nevertheless already at this

[6] Note that in order to have a fixed point of the coset theory, we must have a fixed point in all WZW theories that make up the coset.

stage we can verify that $\mathcal{T}$ as defined above satisfies the following properties:

1. The result is independent of the particular choice of the representative of the original field Φ. [7]

2. The conformal weights Δ of fields related by $\mathcal{T}$ are equal modulo $\mathbb{Z}$, which implies that the modular T-matrices of the two theories coincide. This is in fact already the maximal information about conformal dimensions that we could hope to prove in the general case, because for primary fields of a coset theory (other than Ramond ground states of an $N=2$ theory), it is very hard to compute the integer part of the conformal weight.

3. The superconformal u(1)-charges coincide modulo 2 (again, except for Ramond ground states it is hard to show that the charges coincide exactly).

Actually, the two last-mentioned properties (together with a prescribed choice of the orbits of $\tau(\Lambda)$ and $\tau(\lambda)$, such as the one discussed above) already specify uniquely $\mathrm{x}_{\mathcal{T}}$ and $Q_{\mathcal{T}}$ for fields that are not fixed points. Thus our choice $\mathrm{x}_{\mathcal{T}} = \mathrm{x}$ and $Q_{\mathcal{T}}$ as in (3.6.5) is the only possibility that allows for $\mathcal{T}$ to possess the required properties.

4. The elements of the modular S-matrices corresponding to non-fixed points coincide after properly taking into account the field identification. As we will show in the next subsection, the same is true for fixed points; it follows that both theories possess the same fusion rules and, together with the first observation, that their characters realize isomorphic representations of $\mathrm{SL}(2,\mathbb{Z})$. If the B weights of one field are tensors and those of the other field are spinors, (1.5.7) implies that the corresponding S-matrix element of the full theory is simply the product of the respective WZW S-matrix elements if the spinors are fixed points, and twice this product if the spinors are not fixed points. For the dual theory, the corresponding factor of two is provided by our map $\mathcal{T}$ through the factor $\sqrt{2}$ that appears (both for the 'numerator' and the 'denominator' of the coset theory) in the transformation (3.2.29) of S-matrix elements of the B type WZW theories under τ.

5. $\mathcal{T}$ maps the unique Ramond ground state $\Phi_R^{\max}$ with highest superconformal charge $q = \frac{c}{6}$ of one theory to the corresponding Ramond ground state of the dual theory. (This check is particularly important, as this field is the simple current that generates spectral flow.) Namely, for this field there is a standard representative (compare Section 2.4.1) $\Phi_R^{\max} \hat{=} (0, \mathrm{s} \,/\, \rho_{\bar{\mathfrak{g}}} - \rho_{\bar{\mathfrak{h}}})$, and $\mathcal{T}$ maps this particular representative to the analogous representative of the highest Ramond ground state of the dual theory.

3.6.2 Fixed points

In order to prove that these statements pertain to the full coset theories, [8] we now come to the more detailed description of the action of $\mathcal{T}$ on fixed points, as promised. (The fixed point resolution will be interesting also from a different point of view, see the remarks after (3.6.13) below.) As it turns out, this is a somewhat subtle issue. We will first deal with the case when an unresolved fixed point is mapped on a pair of non-fixed points. In fact, we have so far only specified on what pair of fields a fixed point gets mapped, and noticed that the number of the fields is the right one. But each unresolved fixed point gives rise to

[7] Also, applying the analogous prescription $\mathcal{T}_{\mathcal{T}}$ to the transformed field $\mathcal{T}(\Phi)$ brings us back to the field Φ of the original theory, thus justifying the name *du*ality.

[8] Recall that only after fixed point resolution, we are allowed to interpret the object $\mathcal{C}[\tilde{\mathfrak{g}}/\tilde{\mathfrak{h}}]_K$ as a genuine conformal field theory.

two distinct physical fields, and so we have to describe which of the resolved fixed points is mapped to which field. To settle this question, it is not sufficient to look at the fractional part of the conformal dimensions Δ and superconformal charges q, because for the two resolved fixed points the conformal dimensions and superconformal charges must coincide modulo $\mathbb{Z}$ and $2\mathbb{Z}$, respectively. Thus we have to resort to the modular matrix S.

In order to simplify notation, we first look at those parts of the theory which behave non-trivially under the identification current that has fixed points, which is $J_{(1)} = (J, 1 \,/\, J, 0)$. In other words, we restrict our attention to the theory $(B_{n+1})_{2k+1}[(B_n)_{2k+3}]^*$, where we use the symbol '*' to indicate that the complex conjugates of the modular S- and T-matrices are to be considered (compare the remarks after (1.5.5)). As has been shown in [130], the matrices $\Gamma_{(\cdot)}$ appearing in (1.5.7) and in the factorization (2.3.21) are given, up to certain phases, by the S-matrices of the WZW theories $(C_n)_k$ and $(C_{n-1})_{k+1}$. We denote these phases, to be determined below, by ω_n and ω_{n-1}, respectively.

In terms of the components $\tilde{\ell}_i$, the relation between fixed points and the corresponding fields of the fixed point theory is given by

$$\tilde{\ell}_i^{(C)} = \tilde{\ell}_{i+1}^{(B)} \tag{3.6.7}$$

for $i = 1, 2, \ldots, n$. In other words, for the S-matrices the resolution of fixed points amounts to simply deleting the row and the column with $i = k + n + 1$ of the matrix A as defined in (3.2.20). But it was precisely this row that made the S-matrix elements vanish if fixed points were involved. Now once more we can use the Jacobi-theorem for the $(k+n) \times (k+n)$ matrix $M_{ij} = \sin[(\pi ij)/(k + n + 1)]$ to relate the S-matrix of the fixed point resolution to the S-matrix of the images of the fixed points. We find that

$$\tilde{S}_{\Lambda\Lambda'} \tilde{S}_{\lambda\lambda'} = \varepsilon \omega_{n-1} \omega_n \, (-1)^{\Sigma_{\Lambda\Lambda'} + \Sigma_{\lambda\lambda'} + 1} S_{\tau(\Lambda)\tau(\Lambda')} S_{\tau(\lambda)\tau(\lambda')} \,. \tag{3.6.8}$$

Here $\tilde{S}$ denotes the S-matrix of the fixed point resolution, while

$$\Sigma_{\Lambda\Lambda'} = \sum_{i \in M_\Lambda} i + \sum_{i \in M_{\Lambda'}} i \,, \tag{3.6.9}$$

and $\Sigma_{\lambda\lambda'}$ is the sum of the analogous numbers for the theory in the 'denominator.' Further, $\varepsilon \equiv \varepsilon_{\Lambda\lambda\Lambda'\lambda'} \in \{1, -1\}$ depends on the particular action of $\mathcal{T}$ on resolved fixed points. Namely, the left hand side of (3.6.8) is to be multiplied with the matrix $P = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. On the right hand side of (3.6.8) this is reflected by the fact that the subscripts actually do not refer to an orbit, but to a specific representative; the sign of the right hand side changes when one changes from one representative to the other representative of the orbit. The two representatives, which will be denoted by $\tau(\Lambda, \lambda)_>$ and $\tau(\Lambda, \lambda)_<$, can be described as follows. For any orbit $\{\phi_\Lambda, J\phi_\Lambda\}$ of a B type WZW theory denote by $\Lambda_<$ the representative with smaller values of $\tilde{\ell}_1$, and by $\Lambda_>$ the other one; then $\tau(\Lambda, \lambda)_> := (\tau(\Lambda)_<, \tau(\lambda)_<) \cong (\tau(\Lambda)_>, \tau(\lambda)_>)$, while $\tau(\Lambda, \lambda)_< := (\tau(\Lambda)_<, \tau(\lambda)_>) \cong (\tau(\Lambda)_>, \tau(\lambda)_<)$, with the two equivalent states mapped onto one another by the action of the identification current $J_{(1)}$. Now the value of ε depends on whether the first of the resolved fixed points is mapped to $\tau(\Lambda, \lambda)_>$ and the second to $\tau(\Lambda, \lambda)_<$, or the other way round. As we will see, a consistent prescription for this choice can be given for which ε precisely cancels the further possible signs in (3.6.8).

To compute the phases ω_n and ω_{n-1}, we first note that, given a representation $(ST)^3 = S^2$, $S^4 = \mathbb{1}$ of $\mathrm{SL}(2, \mathbb{Z})$, the only rescalings of S and T which again lead to a representation of $\mathrm{SL}(2, \mathbb{Z})$ are

$$T \mapsto \mathrm{e}^{\pi \mathrm{i} m/6}\, T \,, \qquad S \mapsto \mathrm{e}^{-\pi \mathrm{i} m/2}\, S. \tag{3.6.10}$$

We can determine the integer m in the first of these rescalings from the global shift in the conformal dimensions that is present in the fixed point theories as compared to the C type WZW theories. In the case of our interest we have for $(B_{n+1})_{2k+1}$ the shift $\Delta_{(B)} - \Delta_{(C)} = (6k+2n+3)/24$, and analogously for $(B_n)_{2k+3}$. Subtracting the two shifts, one finds $m = -2$. With (3.6.10), this implies that for the resolution one should take minus the product of the S-matrices of the C type theories rather than simply their product. In other words, $\omega_{n-1}\omega_n = -1$, and hence (3.6.8) reduces to

$$\tilde{S}_{\Lambda\Lambda'}\tilde{S}_{\lambda\lambda'} = \varepsilon(-1)^{\Sigma_{\Lambda\Lambda'}+\Sigma_{\lambda\lambda'}} S_{\tau(\Lambda)\tau(\Lambda')} S_{\tau(\lambda)\tau(\lambda')} \,. \tag{3.6.11}$$

To complete the construction of $\mathcal{T}$, we first investigate the restrictions that are obtained from requiring that the S-matrix is left invariant. Let us choose an arbitrary fixed point $\Phi_f \hat{=} (\Lambda,\lambda)$ to start with, and denote the resolved fixed points by $\Phi_{f\pm}$, as in (2.4.10). We can now map Φ_{f_+} either to $\tau(\Lambda,\lambda)_>$ or to $\tau(\Lambda,\lambda)_<$ (and, correspondingly, Φ_{f_-} to $\tau(\Lambda,\lambda)_<$ and to $\tau(\Lambda,\lambda)_>$, respectively). After fixing this choice, the requirement that the S-matrix should be invariant already fixes $\mathcal{T}(\Phi_{f'})$ for any fixed point $\Phi_{f'}$ uniquely. Namely, assume that the first possibility, $\Phi_{f_+} \mapsto \tau(\Lambda,\lambda)_>$, is chosen; then we have to map $\Phi_{f'_+} \mapsto \tau(\Lambda',\lambda')_>$, $\Phi_{f'_-} \mapsto \tau(\Lambda',\lambda')_<$ if the number $\Sigma_{ff'} \equiv \Sigma_{\Lambda\Lambda'} + \Sigma_{\lambda\lambda'}$ computed according to (3.6.9) is even, while if $\Sigma_{ff'}$ is odd, the map must be $\Phi_{f'_+} \mapsto \tau(\Lambda',\lambda')_<$, $\Phi_{f'_-} \mapsto \tau(\Lambda',\lambda')_>$. With this prescription, one obtains $\varepsilon_{ff'} = (-1)^{\Sigma_{ff'}}$, and hence (3.6.11) reduces to the desired equality

$$\tilde{S}_{\Lambda\Lambda'}\tilde{S}_{\lambda\lambda'}\, P_{ij} = \left(S_{\tau(\Lambda)\tau(\Lambda')} S_{\tau(\lambda)\tau(\lambda')}\right)_{\mathcal{T}(i)\mathcal{T}(j)} , \tag{3.6.12}$$

where on the left hand side $i,j \in \{+,-\}$, while on the right hand side $\mathcal{T}(i),\mathcal{T}(j) \in \{<,>\}$. This not only works for any fixed choice of f', but also for all S-matrix elements $S_{f'f''}$, because $\Sigma_{f'f''} = \Sigma_{ff'} + \Sigma_{ff''}$. The latter identity also implies that the choice of reference fixed point Φ_f is immaterial.

As long as we only take care of the S-matrix, the alternative to choose $\Phi_{f_+} \mapsto \tau(\Lambda,\lambda)_>$ or $\Phi_{f_+} \mapsto \tau(\Lambda,\lambda)_<$ means that there are two different allowed mappings on the fixed points. But according to (2.4.10) the characters of Φ_{f_+} and Φ_{f_-} are different; Φ_{f_+} has more states with minimal conformal weight. Therefore by looking at the characters one can remove the ambiguity in the definition of $\mathcal{T}$. However, since this reasoning can be applied to any fixed point, it has also to be checked whether the constraints obtained from different fixed points are compatible. In practice, this is quite difficult to check, as it requires a detailed analysis of the characters. But there is a rather general argument that the consistency conditions coming from the characters are compatible with those originating from the S-matrix. Namely, defining for any fixed point f the function

$$\mathcal{X}_{\tau(f)} := \chi_{\tau(f)_>} - \chi_{\tau(f)_<} , \tag{3.6.13}$$

it is easy to verify that the functions $\mathcal{X}$ transform under the modular group exactly like the character modifications $\breve{\chi}_f \equiv (\chi_{f_+} - \chi_{f_-})/2v$. In itself, this does not yet imply that $\mathcal{X}_{\tau(f)}$ and $\breve{\chi}_f$ are necessarily equal, but the fact that the result holds for an infinite series is a rather strong hint that they indeed coincide. (Note that it directly follows from (2.4.10) that only $\mathcal{X}$ as defined in (3.6.13), and not $-\mathcal{X}$ can be a sensible character; thus there is in particular no sign ambiguity in defining $\mathcal{X}$.)

In principle, we should perform the same kind of reasoning as above also for resolved fixed points that occur as the images of non-fixed points. However, due to the duality property of the map $\mathcal{T}$ the arguments needed for this analysis closely parallel the arguments given above, so that we refrain from repeating them here.

At this point it is worth recalling that there does not exist a general proof that the fixed points of a coset theory can be resolved in a unique way [130]. In the present case, the manner in which the resolution procedure described in [123] fits the duality map $\mathcal{T}$ is, however, so non-trivial that it is hard to imagine that there could exist another prescription for the resolution that would be compatible with duality as well. Note that the extended Poincaré polynomials of the theories considered here should obey level-rank duality for *any* possible resolution, because according to quite general arguments [123] (compare also Chapter 2) the extended Poincaré polynomial of an $N=2$ coset theory does not depend on the details of the resolution procedure.

3.6.3 Ramond ground states

Finally we turn our attention to the chiral ring of the theories. According to the formula (2.4.5), the number of representatives of Ramond ground states with a fixed $(B_{n+1})_{2k+1}$ weight is given by the relative size

$$\frac{|\overline{W}_{\bar{\mathsf{g}}}|}{|\overline{W}_{\bar{\mathsf{h}}}|}=\frac{2^{n+1}\,(n+1)!}{2^n\,n!}=2\,(n+1) \tag{3.6.14}$$

of the Weyl groups. After implementing the resolution of fixed points, one finds that the dimension of the ring is indeed invariant under the exchange of n and k; this is a direct consequence of the much stronger result [123] that the (ordinary, and also even the extended) Poincaré polynomials of the theories coincide.

Our goal is now to show that the map $\mathcal{T}$ defined above maps every Ramond ground state to a Ramond ground state of the dual theory with identical superconformal charge. To do so, we first note that the relation (3.5.6) between Λ and λ can be reformulated in terms of the sets M_Λ and M_λ, and of the charge Q, as follows. Take a highest g-weight Λ described by the set M_Λ, and consider it as ordered with respect to the magnitude of the elements. The action of any Weyl group element w is then to permute the elements of M_Λ and to multiply them with a sign: the $2(n+1)$ special elements of the classes of $\overline{W}_{\bar{\mathsf{g}}}/\overline{W}_{\bar{\mathsf{h}}}$ that appear in (3.5.6) are characterized by the property that they choose among the $n+1$ elements of M_Λ a particular element $\tilde{\ell}_i$ which gets placed before all the other elements and change its sign or not, leaving all other signs unchanged. We will denote such a Weyl group element that maps the ith basis vector e_i of the orthonormal basis on $\pm e_1$ and respects the ordering of all other basis vectors by $w_i^{(\pm)}$. By inserting the explicit form of the roots α in the orthonormal basis into (2.4.15), it is straightforward to calculate the length of the elements $w_i^{(\pm)}$. We find

$$l(w_i^{(+)})=i-1 \qquad \text{and} \qquad l(w_i^{(-)})=2(n+1)-i\,, \tag{3.6.15}$$

where $n+1$ is the rank of the algebra. This result reflects the linear structure of the associated Hasse diagram of the embedding $B_n\hookrightarrow B_{n+1}$ [33].

For the Ramond ground state defined by acting with $w_i^{(\pm)}$ on Λ, the u(1)-charge Q is given by $\pm 2\tilde{\ell}_i$ for spinors and $\pm(2\tilde{\ell}_i-1)$ for tensors. Opposite sign choices correspond to choosing charge-conjugate Ramond ground states. As a consequence, the map $\mathcal{T}$ automatically respects the charge conjugation properties of the Ramond ground states and, hence, is compatible with the conjugation isomorphisms of the chiral rings of the theories. As mentioned in Section 3.1, this compatibility must in fact hold on rather general grounds.

Next we remark that not all representatives of a Ramond ground state are of the form (3.5.6) (recall that (3.5.6) is a formula for representatives, and not for physical fields). To be able to employ the relation (3.5.6), we therefore pick a specific representative of any combination of simple current orbits of weights Λ and λ that describes a Ramond ground state. After applying the map $\mathcal{T}$ in the form (3.6.4), (3.6.5) to this specific representative of a Ramond ground state Φ_R, we obtain a specific representative of the primary field $\mathcal{T}(\Phi_R)$ of the dual theory. What we have to show is that $\mathcal{T}(\Phi_R)$ is again a Ramond ground state, and we will do this by employing the formula (3.5.6). Of course, generically the particular representative of $\mathcal{T}(\Phi_R)$ with which we are dealing in the first place cannot be expected to be of the form (3.5.6). As we will see, it is indeed sometimes not of this form, but as was shown in [103] there is always at least one representative of the Ramond ground state fulfilling (3.5.6).

Suppose, to start with, that Λ and λ are both spinor weights and that the Ramond ground state is given by the Weyl group element $w_i^{(+)}$ acting on Λ. Recalling that the index i of $w_i^{(+)}$ refers to the fact that $M_\Lambda \setminus M_\lambda = \{\tilde{\ell}_i\}$, and observing that via the map τ on the WZW theories, i.e., upon forming the complement relative to $\{1,2,\ldots,k+n+1\}$, this is transformed to the relation $M_{\tau(\lambda)} \setminus M_{\tau(\Lambda)} = \{\tilde{\ell}_i\}$, we learn that there exists a Weyl group element $w_{\mathcal{T}}$ of the dual theory that relates $\tau(\lambda)$ and $\tau(\Lambda)$ in the correct manner and is given by one of the two elements $w_{i_{\mathcal{T}}}^{(\pm)}$, with $i_{\mathcal{T}}$ determined by the requirement $\tilde{\ell}_{i_{\mathcal{T}}}^{(\tau)} = \tilde{\ell}_i$. To decide which of these two elements is the correct one, we observe that owing to the latter relation $Q_{\mathcal{T}}$ must be equal either to Q or to $-Q$; from (3.6.5) (together with the explicit form of the identification currents) it follows that in fact $Q_{\mathcal{T}} = -Q$. In summary, using the sets $M_{\tau(\lambda)}$ and $M_{\tau(\Lambda)}$, and the sign of $Q_{\mathcal{T}}$ relative to the sign of Q, we fix a unique Weyl group element $w_{\mathcal{T}}$ of $\overline{W}(B_{k+1})$; in fact, a more detailed analysis shows that $i_\tau = k+n-Q/2-i+3$, i.e. $w_{\mathcal{T}} = w_{k+n-Q/2-i+3}^{(-)}$. To verify that this Weyl group element indeed provides us with a Ramond ground state, the only thing that we still have to do is to check that it yields the proper D_d part. [9] While in the foregoing discussion we fixed the representative with respect to $J_{(2)}$ by $\mathrm{x}_{\mathcal{T}} = \mathrm{x}$, the present choice of representative for the charge $Q_{\mathcal{T}}$ implies that $\mathrm{x}_{\mathcal{T}}$ must be given by

$$\mathrm{x}_{\mathcal{T}} = (J_{\mathrm{v}})^{n-k-Q/2}\mathrm{x}\,. \tag{3.6.16}$$

Now the formulæ (3.6.15) for the length of Weyl group elements tell us that

$$l(w) - l(w_{\mathcal{T}}) = n-k-Q/2 \tag{3.6.17}$$

and, hence, recalling that the sign of w is equal to $(-1)^{l(w)}$,

$$\mathrm{sign}\,(w)\,\mathrm{sign}\,(w_{\mathcal{T}}) = (-1)^{l(w)+l(w_{\mathcal{T}})} = (-1)^{k+n+Q/2}. \tag{3.6.18}$$

In view of (2.4.4), this shows that (3.6.16) is indeed fulfilled. Furthermore, plugging (3.6.17) into the formula (2.4.17) for the superconformal charge of Ramond ground states, it follows that Φ_R and $\mathcal{T}(\Phi_R)$ have the same superconformal charge (exactly, and not just modulo 2).

The reasoning above applies also to the case $w = w_i^{(-)}$, as the two cases are clearly dual to each other. If both Λ and λ are tensor weights, the situation is slightly more

[9] In some cases we also must show that the correct $J_{(1)}$-orbit out of two possibilities is chosen. This happens when an 'unresolved fixed point' is resolved into two fields whose conformal weights differ by an integer. The discussion of fixed points in the previous subsection shows that indeed the right orbit is chosen.

complicated. This is because $\tilde{\ell}_i$ gets mapped under τ to $\tilde{\ell}^{(\tau)}_{i_{\mathcal{T}}} = \frac{L}{2} + 1 - \tilde{\ell}_i$. If the Ramond ground state is defined by $w = w_i^{(+)}$, this shows that $Q = 2\tilde{\ell}_i - 1$ should be mapped on $Q_{\mathcal{T}} = L - Q$, implying that $w_{\mathcal{T}}$ involves no minus sign. While in the foregoing discussion we always chose the representative of the field by requiring that $\Delta_\Lambda - \Delta_\lambda$ should be an integer, we now have to fix the representative by requiring that $Q_{\mathcal{T}} = -Q + L$, which, owing to the second identification current $J_{(2)}$, is always possible. This choice of representative leads to

$$\mathrm{x}_{\mathcal{T}} = (J_{\mathrm{v}})^{n-(Q-1)/2}\mathrm{x}\,. \tag{3.6.19}$$

Again, a Weyl group element $w_{\mathcal{T}}$ for the dual theory is completely fixed and can be shown to be given by $w_{\mathcal{T}} = w^{(+)}_{(Q+1)/2-n+i-1}$. It follows that $l(w) - l(w_{\mathcal{T}}) = n - \frac{Q+1}{2} + 1$, so that

$$\mathrm{sign}\,(w)\,\mathrm{sign}\,(w_{\mathcal{T}}) = (-1)^{n+(1-Q)/2}\,, \tag{3.6.20}$$

implying that the correct mapping (3.6.19) of the D_d-weights is reproduced and, also, that the superconformal charge is left invariant. It is also clear that we have chosen the right $J_{(1)}$-orbit, because Δ is conserved modulo $\mathbb{Z}$ under $\mathcal{T}$ and because the relevant different $J_{(1)}$-orbits differ in their conformal weight by $\frac{1}{2}$ modulo $\mathbb{Z}$.

For $w = w_i^{(-)}$, the discussion must be slightly changed. This time $Q = -(2\tilde{\ell}_i - 1)$ is mapped on $Q_{\mathcal{T}} = -L - Q$, i.e., we have to choose a different representative, leading to $\mathrm{x}_{\mathcal{T}} = (J_{\mathrm{v}})^{n+2k+(Q+1)/2}\,\mathrm{x}$. Explicit calculation shows that $w_{\mathcal{T}} = w^{(-)}_{i-n-(Q+1)/2}$, leading to $l(w) - l(w_{\mathcal{T}}) = n - 2k - \frac{Q+1}{2}$, which gives the right transformation of the D_d part and implies identity of superconformal charges.

Thus we have proven that $\mathcal{T}$ always maps Ramond ground states to Ramond ground states with identical superconformal charge.

3.7 Type B coset models with level and rank not congruent modulo 2

In the same spirit as before, we can deal with the other level-rank dualities mentioned in the introduction. As the discussion often closely parallels the one of the previous section, we will usually be rather brief and shall only mention some new features. In the present section we use the map τ for B type algebras at even level to relate the coset theory $(B, 2k+1, 2n)$ with the D type modular invariant to $(B, 2n, 2k+1)$ with the diagonal modular invariant, i.e.,

$$(B, 2k+1, 2n)_{|D} \overset{\mathcal{T}}{\cong} (B, 2n, 2k+1)\,. \tag{3.7.1}$$

According to Subsection 3.5, taking the D-invariant amounts to incorporating the integer spin simple current $J_{(3)} := (J, 1\,/\,1, 0)$ into the chiral algebra. This introduces further fixed points which can have order 2 or 4 and which have to be resolved, but it also has the crucial advantage that it leaves us with tensors of the B algebras only, so that the map τ constructed in Section 3.3 is applicable.

The choice of the $J_{(1)}$-orbits is now immaterial. This is because the presence of $J_{(3)}$ implies that $\tau(\Lambda, \lambda)_< \cong \tau(\Lambda, \lambda)_>$, so that any pair of tensor orbits of the B type WZW theories, combined with a D_d-weight and a u(1)-charge, corresponds to a single physical field. However, we still have to take into account the additional identification current $J_{(2)} \star J_{(3)} = (1, J_{\mathrm{v}}\,/\,1, \pm 2L)$, where $L := 2k + 2n + 1$.

Again the general form of the map $\mathcal{T}$ is given by (3.6.4) (recall that on the right hand side of (3.6.4) only a representative of $\mathcal{T}(\Phi)$ is given). Starting from a fixed representative of a field or a fixed point Φ of the B type coset theory at even level and odd rank, we obtain all representatives of $\mathcal{T}(\Phi)$ by using the map τ and the identification currents of the coset theory at even rank and odd level. Moreover, with the help of the identification currents we can also fix uniquely a representative of $\mathcal{T}(\Phi)$ for which $\tau(\Lambda)$ and $\tau(\lambda)$ are tensors and which has the same conformal weight as the chosen representative of Φ. Note that fixed points are mapped on a spinor-conjugate pair of orbits, which reflects the resolution of fixed points. In particular fixed points of order two and of order four are mapped on two and four fields, respectively.

One can now show again that there is a unique mapping $\mathcal{T}$ that preserves both the superconformal charge q modulo 2 and the conformal dimension Δ modulo integers; it is given by

$$Q_{\mathcal{T}} = \begin{cases} -Q + QL & \text{for} \quad \mathrm{x} \in \{0, \mathrm{v}\}, \\ -Q + (2k+1)QL & \text{for} \quad \mathrm{x} \in \{\mathrm{s}, \mathrm{c}\}, \end{cases} \tag{3.7.2}$$

and

$$\mathrm{x}_{\mathcal{T}} = \begin{cases} (J_{\mathrm{v}})^{Q/2}\mathrm{x} & \text{for} \quad \mathrm{x} \in \{0, \mathrm{v}\}, \\ (J_{\mathrm{v}})^{-k+(Q-1)/2}\mathrm{x} & \text{for} \quad \mathrm{x} \in \{\mathrm{s}, \mathrm{c}\}. \end{cases} \tag{3.7.3}$$

To check this, one has to make use of the fact that the representatives of the orbits of the D type WZW theories that were chosen above always have vanishing monodromy charge relative to $(J_{\mathrm{s}}, 1 \,/\, J_{\mathrm{s}}, 0)$.

Of course, again $\mathcal{T}$ must be complemented by a prescription on the fixed points. This time the fixed point theory is not a WZW theory; rather, it is closely related to certain conformal field theories, denoted by the symbol $\mathcal{B}$, that were described in [130]. In fact, the existence of the map $\mathcal{T}$ suggests that the S-matrix and characters of the $\mathcal{B}$ theories are related to a D type WZW theory, and it should be interesting to explore the level-rank duality further to gain deeper insight in the structure of these peculiar conformal field theories. Finally, it is again possible to prove that the modular S-matrices are identical and that Ramond ground states are mapped on Ramond ground states with equal superconformal charge.

3.8 BB versus CC theories

In this section we present the isomorphism

$$(BB, n+2, 1) \stackrel{\mathcal{T}}{\cong} (CC, 2, 2n+1). \tag{3.8.1}$$

To relate the non-hermitian symmetric cosets $(BB, n+2, 1)$ and $(CC, 2, 2n+1)$ we first notice the isomorphism $C_2 \cong B_2$ of simple Lie algebras. This allows us to make use once again of the map τ of Section 3.2 to relate the $(B_n)_5$ theory appearing in $(BB, n+2, 1)$ with the $(B_2)_{2n+1} \cong (C_2)_{2n+1}$ part of $(CC, 2, 2n+1)$. The $(B_{n+2})_1$ part, on the other hand, is comparatively easy to deal with, because it has only three integrable highest weights, and because the identification current $J_{(1)}$ strongly restricts their combination with weights of the other parts. Namely, $(B_n)_5$-weights that are tensors must be combined with either the tensor weight $\Lambda = 0$ or the tensor weight $\Lambda = \Lambda_{(1)}$ of $(B_{n+2})_1$, while spinors are to be combined with the spinor weight $\Lambda_{(n+2)}$ of $(B_{n+2})_1$; furthermore, $J_{(1)}$ introduces an additional identification, implying that in the case of tensors we can characterize the B part completely by a $(B_n)_5$-weight and by the difference $\Delta_\Lambda - \Delta_\lambda$ of the conformal dimensions.

Also, by using the identification current $J_{(1)}$ of the CC models, we can choose without loss of generality for a fixed representative of Φ the representative of the C_2-orbit in such a way that it has conformal dimension $\Delta_\Lambda - \Delta_\lambda$ modulo integers. For spinor fixed points we have again an ambiguity which is connected to the issue of fixed point resolution.

This time, the mapping τ has to be complemented not only by a mapping on the D_d and u(1) parts, but also on the $(A_1)_{2n+2}$ part of the theory. Thus

$$\begin{aligned} \Phi &\hat{=} (\Lambda, \mathrm{x}\,/\,\lambda, \mu, Q)\,, \\ \mathcal{T}(\Phi) &\hat{=} (\tau(\lambda), \mathrm{x}_{\mathcal{T}}\,/\,\mu_{\mathcal{T}}, Q_{\mathcal{T}})\,, \end{aligned} \tag{3.8.2}$$

where μ and $\mu_{\mathcal{T}}$ are A_1-weights (recall that $C_1 \cong A_1$). It is easy to see that equality of the superconformal charges modulo 2 is equivalent to the relation $\mathrm{x}_{\mathcal{T}} = (J_\mathrm{v})^Q \mathrm{x}$. In fact one can show again that there is a unique mapping that preserves the fractional part of Δ, as well as q modulo 2. Namely, choosing the weights of the B parts in the manner described above, for tensors in the B parts one needs

$$Q_{\mathcal{T}} = \begin{cases} -Q + QL & \text{for} \quad \mathrm{x} \in \{0, \mathrm{v}\}\,, \\ -Q + (Q+1)L & \text{for} \quad \mathrm{x} \in \{\mathrm{s}, \mathrm{c}\} \end{cases} \tag{3.8.3}$$

with $L = 2n + 4$, while for spinor weights in the B parts we must set

$$Q_{\mathcal{T}} = \begin{cases} -Q + L & \text{for} \quad \mathrm{x} \in \{0, \mathrm{v}\}\,, \\ -Q & \text{for} \quad \mathrm{x} \in \{\mathrm{s}, \mathrm{c}\}\,. \end{cases} \tag{3.8.4}$$

The corresponding prescription for the weight μ of $(A_1)_{2m+2}$ is, independent of the value of x,

$$\mu_{\mathcal{T}} = \begin{cases} J^\mu \mu & \text{for } c_\Lambda = c_\lambda = 0\,, \\ \mu & \text{for } c_\Lambda = c_\lambda = 1\,. \end{cases} \tag{3.8.5}$$

Fixed points have to be dealt with more carefully again. Using general simple current arguments, it is easy to see that the S-matrix element between a fixed point and *any* other spinor has to vanish. At first sight, this might seem inconsistent, because the S-matrix element between two non-fixed point spinors of $(B_n)_5$ does not vanish in general, whereas both are mapped on fixed points with respect to $J_{(1)}$ of the C_2-theory, and the S-matrix of the image vanishes. However, spinors of $(B_n)_5$ are always combined with the spinor weight $\Lambda_{(n+2)}$ of $(B_{m+2})_1$; now $S_{\Lambda_{(n+2)}\Lambda_{(n+2)}}$ vanishes and, hence, the same is true for the corresponding S-matrix element of the coset theory.

We can use the Jacobi-theorem to relate the S-matrix arising in the resolution of the fixed points to the S-matrix of the CC theory. The resolution is this time accomplished by mapping the fixed point on an orbit of length two. Calculation shows that the product of the S-matrix elements of A_1, D_d, and u(1) differs from the corresponding S-matrix-element of the CC coset theory by a factor of $\varepsilon(-1)^{P+Q}$, where P and Q are the u(1)-charges of the BB theory and where the sign ε depends on the specific action of $\mathcal{T}$ on fixed points analogously as discussed after (3.6.9). In a similar manner as we dealt with the factor $(-1)^\Sigma$ in Section 3.6, it can be shown that the action of $\mathcal{T}$ can be chosen in such a way that $\varepsilon(-1)^{P+Q}$ is the correct sign for obtaining equality of the full S-matrices. A parallel argument also shows that this definition of $\mathcal{T}$ reproduces the correct identification between the characters of the resolved fixed points and those of the corresponding fields of the CC theory. Let us also mention that the factors stemming from the S-matrix of $(B_{m+2})_1$ precisely compensate the different size of the identification group in the case of non-fixed points; for fixed points they assure, together with the factors of $\sqrt{2}$ appearing in (3.2.29), the equality of the S-matrices.

It is by now not too difficult to verify that the mapping $\mathcal{T}$ fulfills the same properties as in the cases treated in the previous sections. Besides preserving q and Δ as well as the modular S-matrix, we see that $\mathcal{T}$ maps again the Ramond ground states with highest superconformal charge onto each other, proving again the isomorphisms of the spectral flows. It is also possible to check that Ramond ground states are mapped on Ramond ground states with the same superconformal charge quite in the same way we did before. Owing to the presence of the additional A_1-subalgebra, the arguments are, however, slightly more complicated, and we refrain from presenting the technical details here.

3.9 Duality in the CC series

Here we construct a map $\mathcal{T}$ between the $N=2$ superconformal coset models (CC, n, k) and $(CC, k+1, n-1)$, which as in the previously discussed cases leaves S and T invariant and identifies the rings of chiral primary fields,

$$(CC, n, k) \overset{\mathcal{T}}{\cong} (CC, k+1, n-1)\,. \tag{3.9.1}$$

The definition of the map $\mathcal{T}$ will be such that

$$\mathcal{T}((\Lambda, \mathrm{x} \,/\, \lambda, Q)) = (\tau(\lambda), \mathrm{x}_{\mathcal{T}} \,/\, \tau(\Lambda), Q_{\mathcal{T}})\,. \tag{3.9.2}$$

This is formally very similar to the analogous definition (3.6.4) in Section 3.6, but its contents is quite different. Namely, this time the underlying map τ of the C type WZW theories was defined on representatives of simple current orbits rather than on the orbits themselves (see Section 3.4). Correspondingly, (3.9.2) is a map between representatives as well, and hence we will have to check that the relevant quantities of the coset theories do not depend on the choices of representatives. Therefore we will be a bit more explicit than in the two previous sections.

We begin again by checking the conformal central charge and the number of primaries. According to Table 3.1, the Virasoro charge of (CC, n, k) is equal to $-3 + 6n(k+1)/(k+n+1)$, and hence is invariant under exchanging $n \leftrightarrow k+1$. The number of primaries of the $(C_n)_k$ WZW theory is $N^C_{n,k} = \begin{pmatrix} n+k \\ k \end{pmatrix}$. Furthermore, the coset theory does not have any fixed points, and hence the number of primary fields of (CC, n, k) is

$$\nu^{CC}_{n,k} = \frac{1}{4}\, N^C_{n,k}\, N^D_{2n-1,1}\, N^C_{n-1,k+1}\, N^1_{2(k+n+1)} = 2(k+n+1) \begin{pmatrix} k+n \\ n \end{pmatrix} \begin{pmatrix} k+n \\ n-1 \end{pmatrix}, \tag{3.9.3}$$

where the first factor $\frac{1}{4}$ takes care of the selection rule and the identification of order two. Obviously, the number of primaries of the $(CC, k+1, n-1)$ theory is given by (3.9.3), too.

Next we present the map $\mathcal{T}$. In (3.9.2), τ is to be taken as the map defined after (3.4.2), and $\mathrm{x}_{\mathcal{T}}$ and $Q_{\mathcal{T}}$ are defined by

$$\mathrm{x}_{\mathcal{T}} = \begin{cases} (J_{\mathrm{v}})^{k+n+Q+1}\mathrm{x} & \text{for} \quad \mathrm{x} \in \{\mathrm{s}, \mathrm{c}\}\,, \\ (J_{\mathrm{v}})^{k+1-Q}\mathrm{x} & \text{for} \quad \mathrm{x} \in \{\mathrm{v}, 0\}\,, \end{cases} \tag{3.9.4}$$

and

$$Q_{\mathcal{T}} = \begin{cases} -Q & \text{for} \quad \mathrm{x} \in \{\mathrm{s},\mathrm{c}\}\,, \\ -Q+k+n+1 & \text{for} \quad \mathrm{x} \in \{\mathrm{v},0\}\,. \end{cases} \tag{3.9.5}$$

Note that already in terms of representatives, the map $\mathcal{T}$ squares to the identity, $\mathcal{T}_{\mathcal{T}} \circ \mathcal{T} = id$. Also, combining the expression (3.5.3) for the conformal dimension of Φ with the result (3.4.3) for the conformal dimensions of the C type WZW theories, one can again show that $\mathcal{T}$ is the only map that preserves q modulo 2 and the fractional part of the conformal weight Δ, as well as the S-matrix. To check the last-mentioned property, it is important to make use of the selection rules encoded in the identification current $J_{(1)}$.

As already emphasized, the map $\mathcal{T}$ must provide a mapping between fields rather than only a mapping between formal combinations of weights of the underlying Lie algebras. The following remarks show that the mapping is indeed well defined on physical fields.

1. The map (3.9.2) is consistent with the selection rules, i.e., it maps allowed fields to allowed fields. Note that the dependence of $Q_{\mathcal{T}}$ on x is necessary to fulfill the selection rule encoded in $J_{(1)}$, (explicitly, the selection rule reads $r(\Lambda)+r(\lambda)+n\sigma+Q \equiv 0 \bmod 2$, where $r(\Lambda)$ is the number defined in (3.4.9), which modulo 2 is equal to the conjugacy class of Λ, and where σ is 0 in the Neveu–Schwarz sector and 1 in the Ramond sector).

2. Identification currents are mapped onto identification currents: [10]

$$\begin{aligned} &(0,0\,/\,0,0) \stackrel{\mathcal{T}}{\longleftrightarrow} ((n-1)\Lambda_{(k+1)}, (J_{\mathrm{v}})^{k+1}\,/\,n\Lambda_{(k)}, \pm(k+n+1))\,, \\ &(k\Lambda_{(n)}, (J_{\mathrm{v}})^{n}\,/\,(k+1)\Lambda_{(n-1)}, \pm(k+n+1)) \stackrel{\mathcal{T}}{\longleftrightarrow} (0,0\,/\,0,0)\,. \end{aligned} \tag{3.9.6}$$

Computation shows that the products of S-matrices of the respective WZW theories coincide (one has to make use once again of the selection rules, which imply cancellation of the factors $(-1)^{r(\Lambda)}$ that are present in equation (3.4.10)). This implies that in fact the two representatives of one physical field Φ are mapped on the representatives of the corresponding physical field $\mathcal{T}(\Phi)$ of the dual theory, or, in other words, that we can interpret $\mathcal{T}$ also as a mapping of physical fields.

3. The two representatives of the Ramond ground state with highest u(1)-charge are exchanged:

$$\begin{aligned} &(0,\mathrm{s}\,/\,0,n) \stackrel{\mathcal{T}}{\longleftrightarrow} ((n-1)\Lambda_{(k+1)}, (J_{\mathrm{v}})^{k+1}\mathrm{s}\,/\,n\Lambda_{(k)}, -n)\,, \\ &(k\Lambda_{(n)}, (J_{\mathrm{v}})^{n}\mathrm{s}\,/\,(k+1)\Lambda_{(n-1)}, -k-1) \stackrel{\mathcal{T}}{\longleftrightarrow} (0,\mathrm{s}\,/\,0,k+1)\,. \end{aligned} \tag{3.9.7}$$

In other words, in terms of fields we have proven compatibility of the map $\mathcal{T}$ with spectral flow.

To show that $\mathcal{T}$ maps Ramond ground states on Ramond ground states, again we first check the dimension of the chiral ring. We have to use the formula (2.4.5) with $N = N^{C}_{n,k}$, $|Z| = |Z(C_n)| = 2$, and

$$\frac{|\overline{W}_{\bar{\mathfrak{g}}}|}{|\overline{W}_{\bar{\mathfrak{h}}}|} = \frac{2^n\, n!}{2^{n-1}(n-1)!} = 2n\,. \tag{3.9.8}$$

Thus $\mu^{CC}_{n,k} = n\,N^{C}_{n,k} = (n+k)!/((n-1)!\,k!)$, which is invariant under $n \leftrightarrow k+1$. Of course, this also follows from the observation that the (ordinary and extended) Poincaré

[10] This does not furnish a group isomorphism between the groups that describe the fusion rules of the identification currents. Since these groups are isomorphic to $\mathbb{Z}_2$, such an isomorphism would necessarily be trivial.

polynomials of the theories (CC, n, k) and $(CC, k+1, n-1)$ are identical, as we will see in Chapter 4.

To analyse the Ramond ground states in more detail, first recall that in the orthogonal basis the action of the Weyl group is given by permuting the components and multiplying them with a sign, and has thus the same structure as in the case of B type Lie algebras. This allows us to use the same notation for Weyl group elements as in Section 3.6. Furthermore, the roots of B type and C type algebras differ only by normalization factors, and these are irrelevant for the determination of the length of Weyl group elements. As a consequence, the formulæ (3.6.15) are valid for C type Lie algebras, too (and the Hasse diagram of the embedding $C_{n-1} \hookrightarrow C_n$ is again linear, compare Appendix 2.A). Correspondingly, the reasoning below will be very similar to that of Section 3.6. The relation (3.5.6) between the weights Λ and $\tilde{\lambda}$ implies that in terms of the numbers $\tilde{\ell}_i$ introduced in (3.4.1), the C_{n-1}-weight λ of a Ramond ground state Φ_R is related to the C_n-weight Λ by

$$\tilde{\ell}_i(\lambda) = \tilde{\ell}_{i+1}(w(\Lambda)), \tag{3.9.9}$$

and also

$$|Q| = \tilde{\ell}_1(w(\Lambda)) \tag{3.9.10}$$

for some Weyl group element w. When we characterize Λ and λ by the sets M_Λ and M_λ, this translates into

$$M_\lambda = M_\Lambda \setminus \{\tilde{\ell}_\circ\}, \tag{3.9.11}$$

where $\tilde{\ell}_\circ = \pm Q$ is an arbitrary element of M_Λ (recall that $\tilde{\ell}_\circ > 0$). Again the freedom in the choice of the sign of Q reflects the invariance of the set of Ramond ground states under charge conjugation. An analogous description applies to the image $\mathcal{T}(\Phi_R)$ of the Ramond ground state. Now $\mathcal{T}$ fixes uniquely the transformation of all weights, and

$$M_{\tau(\lambda)} = M \setminus M_\lambda = (M \setminus M_\Lambda) \cup \{\tilde{\ell}_\circ\} = M_{\tau(\Lambda)} \cup \{\tilde{\ell}_\circ\}, \tag{3.9.12}$$

so that $\tau(\Lambda)$ and $Q_{\mathcal{T}}$ are related to $\tau(\lambda)$ by the formula (3.5.6) with a suitably chosen Weyl group element $w_{\mathcal{T}}$.

To verify that $\mathcal{T}(\Phi_R)$ is again a Ramond ground state, it is now sufficient to check that $\mathcal{T}$ gives the correct weight in the D_d part of the theory. The Weyl group elements w and $w_{\mathcal{T}}$ are uniquely fixed by the weights Λ and $\tilde{\lambda}$, respectively by their images under τ; for $w = w_i^{(+)}$ the Weyl group element $w_{\mathcal{T}}$ is given by $w^{(-)}_{k+n-Q-i+2}$, which implies that $l(w) - l(w_{\mathcal{T}}) = n - k - 1 - Q$. From this equation we can derive not only the equality of superconformal charges, but also the behavior on the D_d part; we have

$$\text{sign}\,(w)\;\text{sign}\,(w_{\mathcal{T}}) = (-1)^{l(w)+l(w_{\mathcal{T}})} = (-1)^{k+n+1+Q}, \tag{3.9.13}$$

which reproduces the prescription given in (3.9.4). This shows that $\mathcal{T}$ maps Ramond ground states on Ramond ground states, as claimed, and thus completes our arguments that the map $\mathcal{T}$ fulfills the requirements for the isomorphism (3.9.1) of conformal field theories, analogously as for the other isomorphisms of (3.1.1).

Chapter 4

Applications to String Theory

4.1 Introduction

In this chapter we apply the results of Chapter 2 to string compactification: we use the non-hermitian symmetric coset $N = 2$ coset theories introduced in Chapter 2 as subtheories in $N = 2$ tensor products with $c = 9$, which are taken as the inner sector of heterotic superstring compactifications.

To this end several projections have to be implemented on the tensor product; in Section 4.2 we describe these projections in the language of simple currents. The information on the chiral ring which is necessary to perform these projections can be conveniently encoded in the so-called extended Poincaré polynomial [123] which is described in Section 4.3.

The extended Poincaré polynomial can be deduced from the ordinary Poincaré polynomial and the action of the so-called spinor current; it can be used to compute the massless spectra of these compactifications, i.e. the number of massless generations and anti-generations. In Section 4.4 we present the complete list of all tensor products of coset theories that involve at least one non-hermitian symmetric coset theory and have central charge $c = 9$, providing thus consistent vacua for heterotic string compactification to four space-time dimensions [67]. When combined with the list of tensor products involving only minimal models [49] and with the corresponding list for hermitian symmetric spaces [37], this completes the list of all tensor products of $N = 2$ coset theories that can be obtained from cosets of the type (2.1.1). Note that the set of all string vacua is much bigger than the set of all tensor products of coset theories, as in general by choosing different modular invariants of the g- and h- WZW theories one gets different string vacua. However, to obtain this set is, at present, beyond reach, as a complete classification of modular invariants is still lacking for WZW theories based on simple Lie algebras other than A_1 and A_2.

Finally, in Section 4.5 we conclude with a brief summary and an outlook on possible further work.

4.2 Heterotic string compactification and simple currents

To build out of a tensor product $\mathcal{C}_9$ of $N = 2$ superconformal field theories with $c = 9$ a heterotic string theory, one has to perform several projections. We will sketch in this section how this can be described in terms of simple currents and explain the resulting prescription encoded in the 'extended Poincaré polynomial'. In a second step we shall comment on the case $c = 3 + 6n$. This case is of much practical interest as we have to resort to it in some cases to remove ambiguities in the resolution of fixed points. Note that in this section we do not make any assumption on how the $N = 2$ theories have been constructed. The results we will derive in this section are therefore valid for any $N = 2$ theory.

After splitting off the contribution of the bosonic space-time coordinates and applying the bosonic string map [35], we can describe the heterotic string in a conformal field theory language as the tensor product

$$(D_5)_1 \oplus (E_8)_1 \oplus \mathcal{C}_9 \ . \tag{4.2.1}$$

The first two factors will provide for the right movers the gauge multiplet, for the left movers they describe the contribution of the fermions. As the only purpose of the E_8 factor is to provide a phase in the S-matrix such that the fermions are correctly reproduced, we will drop it in our discussion from now on.

Any superconformal field theory has a simple current T_F – the generator of world sheet supersymmetry – which of order two and conformal dimension $h = 3/2$. We say that primary fields which have monodromy charge 0 with respect to T_F are in the Neveu–Schwarz sector, fields with monodromy charge 1/2 in the Ramond sector. By the 'superpartner' of a primary field i we will denote the primary field $T_F \star i$. Note that i and $T_F \star i$ are distinct primary fields and that in particular T_F itself is a primary field.

To obtain supersymmetry on the world sheet also for the tensor product of supersymmetric theories, we have to align the boundary conditions in the various theories such that the fields are either all in the Ramond or all in the Neveu–Schwarz sector. This alignment is precisely achieved by enlarging the chiral algebra by all bilinears ${T_F}^{(i)}{T_F}^{(j)}$ which have conformal dimension 3. In the D_5 part we set ${T_F}^{(0)} := J_\mathrm{v}$, which is, just like any other primary field of $(D_5)_1$, a simple current as well.

Space-time supersymmetry requires the projection on even [1] values of the u(1) charges [66]. To implement this projection we use the fact that any $N = 2$ superconformal theory has a second simple current: the Ramond ground state R_0 with highest u(1) charge. It has conformal dimension $h = \frac{c}{24}$, its monodromy charge is half of the superconformal charge. The desired projection it is thus equivalent to including the integer spin simple current $S_\mathrm{tot} := (J_\mathrm{s}, R_0)$ in the chiral algebra. Here J_s is the spinor simple current of $(D_5)_1$. S_tot has been termed spinor current in [123]. We will see below that its presence in the chiral algebra assures the existence of a space-time gravitino in the corresponding heterotic string spectrum.

In conformal field theory language a heterotic string theory thus amounts to a conformal field theory (4.2.1) with the modular invariant generated by the integer spin simple current S_tot and all bilinear combinations ${T_F}^{(i)}{T_F}^{(j)}$.

We are now in a position to recover the massless spectrum of the heterotic string. To obtain the proper interpretation we recall that in one chiral sector of the theory, e.g. for left movers, we have to apply the bosonic string map: the $D_5 \oplus E_8$ part is mapped on a $\mathrm{so}(2)_1$ theory by interchanging vector and scalar and changing the sign of the spinor and conjugate spinor representation in the partition function. This map preserves the modular transformation properties and allows for a description of the fermionic coordinates of the string.

In a purely bosonic description, massless fields are characterized by the property $h = \bar{h} = 1$. Let us first explain how in this formulation the generic part of the string spectrum arises which provides the supergauge- and supergravity-multiplets. Two fields that occur in any $N = 2$ theory in the inner sector are the vacuum and the two Ramond ground states with highest and lowest u(1)-charge. The massless right moving fields that are tensored with the vacuum of the inner sector have conformal weight $h = 1$ and, due to the charge

[1] Here we formulate the condition *after* applying the bosonic string map, what explains the difference to what the reader might expect, namely projection on odd values [66].

selection rule, $q = 0, \pm 2$. These conditions are fulfilled for the currents of $E_8 \oplus D_5$ and the transverse bosons. In the modular invariant described above these fields are paired with the following left movers: $(J_v, 0)$ what yields for the transverse bosons the graviton (as well as an antisymmetric tensor and the dilaton as the trace) and for the currents the gauge multiplets. Applying S_{tot} in the left moving sector yields the superpartners of the gauge bosons and the graviton.

In the right moving sector, we also find in the complete square of the identity the fields S_{tot} and ${S_{\text{tot}}}^\dagger$, as well as the 0 of D_5 tensored with the u(1)-current of the $N = 2$ algebra. According to the branching of the adjoint representation of E_6 to the adjoint representation, the spinor, conjugate spinor and scalar of D_5, these fields extend the gauge symmetry from $E_8 \oplus D_5$ to $E_8 \oplus E_6$. In particular cases, if more fields are present, one can even further extend both the gauge symmetry for right movers and the supersymmetry for the left movers.

To explain how massless (anti-)generations transforming in the 27 respectively $\overline{27}$ representations of E_6 arise, we remark that massless states that are vectors of D_5 have $h = \frac{1}{2}$ and $q = \pm 1$ in $\mathcal{C}_9$, i.e. they are (anti-)chiral fields. Acting twice with ${S_{\text{tot}}}^\dagger$ on the vector tensored with a chiral primary field with $q = 1$ yields a spinor tensored with a Ramond ground state and in a second step 0 tensored with an anti-chiral state with $q = -2$; these states combine in a 27 of E_6. Starting with an anti-chiral field and applying S_{tot} instead we obtain states transforming in a $\overline{27}$ of E_6. These states can be paired with spinors or conjugate spinors in the left moving sector; together they give rise to the generations and anti-generations and their CPT conjugates.

To extract information on the spectra we introduce the following notation: denote by $h^{p,q}$ the number of fields which are in both the left and the right moving part of $\mathcal{C}_9$ chiral primaries and have superconformal charge p respectively q; p, q are integers smaller than $d := c/3$. These numbers can be seen as analogues to the Hodge numbers of a Calabi-Yau threefold. In fact, we find the usual symmetries: $h^{p,q} = h^{q,p}$, as we started from a left right symmetric invariant, and $h^{p,q} = h^{d-q,d-p}$, due to the conjugation symmetry on the chiral ring. Note that if the vacuum is not paired with any chiral primary field other than the unique chiral primary field with $q = \frac{c}{3}$, we have $h^{0,1} = h^{0,2} = 0$; in the corresponding heterotic string compactification neither gauge symmetry nor space-time supersymmetry is extended. As this is the most interesting case we will restrict ourselves to it from now on. The 'Euler number' is given by $\chi := \sum (-1)^{p+q} h^{p,q}$.

The discussion above shows that the number N_{27} of massless generations transforming in the 27 representation of E_6 is equal [2] to $h^{1,1}$, or equivalently to the number of fields in the theory, which are in both sectors spinors of D_5 tensored with a Ramond ground state with superconformal charge $-\frac{1}{2}$. The massless anti-generations $N_{\overline{27}}$ transforming in the $\overline{27}$ of E_6 can be correspondingly characterized by the fields which are spinors and Ramond ground states with charge $-\frac{1}{2}$ in one sector and conjugate spinors and Ramond ground states with charge $+\frac{1}{2}$ in the other sector.

4.3 The extended Poincaré polynomial

To compute the massless spectrum of a heterotic string compactification we have to keep track of the relevant simple current orbits. Let us first look at the orbits of S_{tot}: as we are

[2] Our notation is different from the one used for Calabi-Yau manifolds: there the superconformal charge in both sectors is defined with a relative minus sign, so the number of generations corresponds to the Hodge number $h^{1,-1} = h^{1,2}$ of the manifold.

only interested in the massless spectrum we start with an arbitrary Ramond ground state $(J_s, R^{(1)}, \ldots)$. Suppose now that, on the orbit, we encounter $((J_v)^{\epsilon_0} v, (T_F{}^{(i)})^{\epsilon_i} R'_i)$, where ϵ_i is 0 or 1. This state – which is massive unless all ϵ_i vanish – is paired in the simple current invariant with the original state in the other sector of the theory. But the chiral algebra contains also all bilinears of the form $(J_v, T_F{}^{(i)})$: we thus find within the same complete square of the partition function the corresponding massless state, for which all ϵ_i vanish, too. If the D_5 part is a spinor this yields a generation; conjugate spinors correspond to anti-generations.

The information on the orbit of $S_{\rm tot}$ is very conveniently encoded in the extended Poincaré polynomial [123]. To start with, we define it on each factor of the tensor product separately. As any simple current has finite order, the orbit has some periodicity which we first factor out for convenience: for any Ramond ground state R we define N_R to be the smallest power of the spinor current such that $(S^{N_R})R$ is equal to R or $T_F R$. We define $\epsilon(R)$ to be $+1$ in the first and -1 in the second case. The extended Poincaré polynomial is now defined as :

$$\mathcal{P}(t,x) = \sum_R \frac{t^q}{1-\epsilon(R)x^{N_R}}[\sum_{m\in\mathcal{F}_+} x^m - \sum_{n\in\mathcal{F}_-} x^n]. \tag{4.3.1}$$

The sum is over all Ramond ground states R, q is the superconformal charge of the chiral primary field connected via spectral flow. The sets $\mathcal{F}_\pm$ are defined by the prescription: $m \in \mathcal{F}_+$ iff $(S_{\rm tot})^m R$ is a Ramond ground state and $n \in \mathcal{F}_-$ iff $(S_{\rm tot})^n R$ is T_F applied to a Ramond ground state; in particular all m, n are even. The extended Poincaré polynomial is *not* a polynomial in the new variable x, but rather a series with periodic coefficients. We remark that we recover the ordinary Poincaré polynomial as $\mathcal{P}(t, 0)$.

In fact, typically one deals with a tensor product of $N = 2$ coset theories rather than with a single theory. The ordinary Poincaré polynomial $P_{\rm tot}(t)$ of a tensor product is just the product of the ordinary Poincaré polynomials $P_i(t)$ of the factor theories. The extended Poincaré polynomial for a tensor product can be obtained by the following multiplication: given the extended Poincaré polynomials $\mathcal{P}_i(t, x_i)$ of the factors, first perform the ordinary product of polynomials and then delete all terms in which the powers of the x_i do not coincide. This procedure implements the simple observation that, in order to have a Ramond ground state of the tensor product, we need Ramond ground states in each factor of the theory.

To compute the spectrum of the corresponding string compactification we have first to check whether the gauge symmetry is extended or not. If it is not extended, then the polynomial in x multiplying t^0 is equal to $1 + x^2$. The numbers N_{27} and $N_{\overline{27}}$ can then be read off [123] from the extended Poincaré polynomial of a $c = 9$ theory. Namely, if $\mathcal{P}$ is written as

$$\mathcal{P}(t,x) = \sum_i \sum_{m=0}^{\infty} a_m^{(q_i)}\, t^{q_i} x^{2m}, \tag{4.3.2}$$

then we find

$$N_{27} + N_{\overline{27}} = \sum_{m=0}^{M_s/2-1} |a_m^{(1)}| \tag{4.3.3}$$

and

$$N_{27} - N_{\overline{27}} = \sum_{m=0}^{M_s/2-1} (-1)^m a_m^{(1)}\,. \tag{4.3.4}$$

Here M_s denotes the smallest positive integer such that the $(2M_s+1)$st power of the spinor current is either equal to the spinor current itself or to its superpartner. The formulae (4.3.3) and (4.3.4) implement the observation in the previous section that, since the action of any of the bilinears $(J_v, {T_F}^{(i)})$ and of $(S_{tot})^2$ changes the conjugacy class in the D_5 theory we find generations if $a_m > 0$ and $m = 0 \bmod 4$ or $a_m < 0$ and $m = 2 \bmod 4$; the other cases correspond to anti-generations. Note that in formula (4.3.3) it is assumed that a fixed twisted sector m contributes either only generations or anti-generations so that the contributions to $a_m^{(1)}$ do not cancel. A proof of this assumption can be found in [101].

For later application, it is necessary to slightly generalize this formalism such that it can be applied to tensor products of cosets with conformal charge $c = 3 + 6n$. Here in general, we have no string interpretation at hand and we have to replace the D_5 factor by some other D_r factor. However, we have to require that the current (J_s, R_0), with conformal weight $r/8 + c/24$ has integer spin. This fixes r to $r = -2n - 1 \bmod 8$. (We recover the previous situation for $r = 5, n = 1$.) It is important to note that, as the S matrices of D_r and D_{r+4} coincide, the choice of r does not affect the fusion rules. (Note however, that the T matrices coincide only for D_r and D_{r+24}.)

We now implement analogous projections, i.e. take the simple current invariant induced by all bilinears in the ${T_F}^{(i)}$ and (J_s, R_0), and obtain the extended Poincaré polynomial by exactly the same prescription as in the $c = 9$ case. Again massless states that are spinors or conjugate spinors in D_d are Ramond ground states of $\mathcal{C}_{3+6n}$. The charge selection rule implies that states paired with spinors have superconformal charge $q \equiv -\frac{1}{2} \bmod 2$ and for conjugate spinors $+\frac{1}{2} \bmod 2$. The chiral primary fields connected via spectral flow have thus charge $q \equiv n \bmod 2$ for spinors respectively $n + 1 \bmod 2$ for conjugate spinors. This shows that we can recover the Euler number from the polynomials multiplying all odd powers of t in the extended Poincaré polynomial and summing up all contributions. We remark that in general we can only read off $\sum_q h^{p,q}(\pm 1)^{p+q}$ from the extended Poincaré polynomial; this is sufficient to determine all Hodge numbers separately only for $n \leq 1$, if the symmetry is not extended.

4.4 String spectra of $N = 2$ coset models

Knowing the exact form of the Ramond ground states in $N = 2$ coset theories (cf. Section 2.4), we can calculate the massless spectrum of the string theory that employs a tensor product of $N = 2$ coset model as its inner part, or more precisely, the numbers N_{27} of 'generations' and $N_{\overline{27}}$ of 'anti-generations' which carry the two inequivalent 27-dimensional representations of the E_6 part of the space-time gauge group of the string theory. One possibility to find these numbers is, of course, the extended Poincaré polynomial introduced in the previous section.

Another method is the 'method of beta vectors' that was introduced [66] in the context of $N = 2$ minimal models; in principle this method has the additional benefit to provide in addition the number N_1 of E_6 singlets. In practise, however, this is not the most convenient approach, as the dimensionality and structure of the lattice spanned by the beta vectors depends strongly on the algebras involved, so that one would be forced into a lengthy case by case analysis. (However, for the calculation of the number of massless states carrying the singlet representation of the space-time gauge group E_6, the method of beta vectors is still the only known algorithm. Unfortunately the knowledge of the Ramond ground states is *not* sufficient to get the singlets. While for Ramond ground states the correct treatment of null states is already implemented through (2.4.4), for general $N = 2$ coset theories the presence of null states makes the determination of the singlets a hard problem. In fact, the singlet numbers have so far not been determined for (tensor products of) $N=2$ coset models other than the minimal ones. For the latter theories, the representation theory of the $N=2$ algebra gives a good handle on null states.)

To determine the exact form of the extended Poincaré polynomial in the case of $N=2$ coset models is a somewhat tricky issue, as in fact we only know some specific representatives of the fields which are Ramond ground states (the formula (2.4.4) does not provide all members of an equivalence class), while for the calculation of $\mathcal{P}(t,x)$ in principle all representatives are required. Fortunately, one can show that the following procedure yields the full result. Take a single representative for each Ramond ground state, and act on it with all even powers of all representatives of S that have $\bar{\mathfrak{g}}$-weight $\Lambda=0$. This is sufficient because of the fact, proven in the appendix of [103], that for any representative R of a Ramond ground state there exists at least one representative R' that belongs to the set obtained via (2.4.4) and that has the same $\bar{\mathfrak{g}}$-weight as R.

It follows from the considerations in the previous section that, if in $\mathcal{P}(t,x)$ the highest (and, due to charge conjugation invariance proven in Section 2.4.3, also the lowest) power in t gets multiplied with more than two distinct powers of x, then additional gravitinos that lead to extended space-time supersymmetry (respectively, additional gauge bosons, yielding an extension of the space-time gauge group E_6 to E_7 or E_8) are present. In the tables below we have marked all models where this happens by an asterisk on the net generation number. Note that in the tables we display the number of E_6 multiplets even if the gauge group gets extended. (All models of this type that appear in our list describe in fact string propagation on the manifold $K3\times T^2$, and hence have $N_{27}=N_{\overline{27}}=21$. The number N_{56} of the associated E_7 multiplets is in these cases $N_{56}=N_{27}-1=20$, as one generation-antigeneration pair becomes part of the gauge boson multiplet.)

As an illustration, we present one example of an extended Poincaré polynomial, namely for the theory $(G2_1,2)$. This has the (somewhat atypical) property that to some powers of t other than the highest and the lowest ones there are associated more than two different powers of x. The 'polynomial' reads

$$\begin{aligned}\mathcal{P}(t^{18},x) = \{ & (1+x^2)+t^{10}(1+x^{10})+t^{12}(1+x^8+x^{18}+x^{26})+t^{13}(1-x^{16}) \\ & +t^{14}(1+x^6-x^{12}-x^{30})+t^{15}(1-x^{32})+t^{16}(2+x^4+x^{18}+2\,x^{22}) \\ & +t^{17}(1-x^{12})+t^{18}(1+x^2+x^{18}+x^{20})+t^{19}(1-x^{28}) \\ & +t^{20}(2-x^6-x^{12}+2\,x^{18}-x^{24}-x^{30})+t^{21}(1-x^8) \\ & +t^{22}(1+x^{16}+x^{18}+x^{34})+t^{23}(1-x^{24}) \\ & +t^{24}(2+2\,x^{14}+x^{18}+x^{32})+t^{25}(1-x^4)+t^{26}(1-x^6-x^{24}+x^{30}) \\ & +t^{27}(1-x^{20})+t^{28}(1+x^{10}+x^{18}+x^{28}) \\ & +t^{30}(1+x^{26})+t^{40}(1+x^{34})\}\,(1-x^{36})^{-1}\,. \end{aligned} \tag{4.4.1}$$

In the presence of fixed points the above prescription for obtaining the extended Poincaré polynomial is not yet quite complete, since from the quantum numbers of a fixed point alone it cannot be decided whether a field into which the fixed point is resolved and which appears in the orbit of another Ramond ground state is a Ramond ground state (or the superpartner of a Ramond ground state) or not. In principle one could resolve this ambiguity by using the full S-matrix of the theory to calculate the fusion rules which, in turn, determine the orbits of the spinor current. But again, there is a way to avoid this involved calculation, which has the additional benefit of showing that the results for the extended Poincaré polynomial do not depend as strongly on the details of the resolution as one might imagine. To this end we note that an important check of the spectra obtained via the extended Poincaré polynomial is provided by the results of [16], where an independent way to calculate the net generation number δN by means of the ordinary Poincaré polynomial $P(t)$ was found. Namely,

$$\delta N \equiv N_{27} - N_{\overline{27}} = \frac{1}{M_s} \sum_{r,s=0}^{M_s/2-1} P(\mathrm{e}^{2\pi \mathrm{i}\, \mathrm{d}(r,s)/M_s}) , \tag{4.4.2}$$

where $\mathrm{d}(r,s)$ stands for the largest common divisor of the integers r and s.

Now since (4.4.2) determines the net generation number δN from the ordinary Poincaré polynomial alone, δN cannot depend on the resolution procedure [123]. To determine the correct extended Poincaré polynomial, we thus simply have to start with the most general ansatz compatible with the prescriptions given above and calculate, for all possible values of the unknown parameters that arise from the orbits containing resolved fixed points, the net generation number for string vacua that involve the model under investigation as one factor theory. If the net generation number generated this way does not fit the value prescribed by (4.4.2), we can exclude the corresponding set of values for the unknown parameters. To resolve all ambiguities uniquely, it is sometimes necessary to take into account the result of the previous section that we can apply all formulas not only to tensor products with $c = 9$, but to tensor products with $c = 3 + 6n$ for any positive integer n as well.

As an example, let us have a look at the theory $(BB, 4, 2)$ which has $c = 9$. The analysis of the Ramond ground states shows that the coefficient of t in the extended Poincaré polynomial is the polynomial

$$14 + a_1 x^2 - a_2 x^4 \tag{4.4.3}$$

(multiplied with the irrelevant factor $(1 + x^6)^{-1}$). Here a_1 and a_2 are parameters arising from the fixed point ambiguities just described; they must be integers between 4 and 7. Now (4.4.2) shows that $\delta N = 0$, so that (4.3.4) yields $a_1 + a_2 = 14$, which in the given range has the unique solution $a_1 = a_2 = 7$. Once the exact form of the extended Poincaré polynomial is known, we can read off the number of generations and antigenerations separately, namely $N_{27} = N_{\overline{27}} = 14$.

We present the results of our calculations in Tables 4.1 to 4.3. In Table 4.1 we list all tensor products that can be written as the tensor product of a $c = 6$ and of a $c = 3$ theory and in which at least one factor is neither a hermitian symmetric coset nor the model $(CC, 2, 1)$ that will be dealt with separately. The un-numbered lines contain the relevant non-hermitian symmetric theories, while the numbered lines provide the spectra for those $c = 9$ theories that are obtained by tensoring the $c = 6$ part with the following $c = 3$ models, respectively:

$$\begin{array}{lll} 1-1-1\,, & & \\ 1-4 & \text{or} & (A,1,2,3)\,, \\ 2-2\ , & & \\ (CC,2,1) & & \end{array} \tag{4.4.4}$$

(the theories $1 - 4$ and $(A, 1, 2, 3)$ possess the same extended Poincaré polynomial and therefore yield the same spectrum). Here and below, the symbol '–' is used to indicate the tensor product, and a single integer k stands for the $N = 2$ minimal model at level k.

Next we display, in Table 4.2, all tensor products that contain the model $(CC, 2, 1)$ which has $c = 3$, but do not contain any other non-hermitian symmetric coset theory. We can tensor this model twice and use the five $c = 3$ models listed in (4.4.4); as the model $(CC, 2, 1)$ itself occurs in that list, this includes tensoring three copies of the model. We can also tensor it with 17 different combinations of minimal models and 27 other combinations of hermitian symmetric cosets with $c = 6$. [3] Altogether, this yields $15\times 5+4+17+27 = 123$ models with $c = 9$ that involve non-hermitian symmetric cosets and contain a $c = 3$ part.

Finally, in Table 4.3 we list all tensor products having $c = 9$ in which at least one factor is not a hermitian symmetric coset and which do not contain a tensor product with $c = 3$. We find 75 models of this type. The number of theories that we count as different gets reduced by various identifications among the total of 198 theories. We have taken care of these identifications, thereby reducing the number of entries in the Tables 4.1 to 4.3 to 112.

As it turns out, the extended Poincaré polynomials for several theories that are defined as distinct naive coset theories coincide. From the experience with coset constructions, the observation that there exist a priori distinct coset theories with coinciding extended Poincaré polynomials is not very spectacular. What is surprising, however, is that in fact for *all* non-hermitian $N = 2$ coset theories for which the ordinary Poincaré polynomials are identical (compare table 2.5 above), the same is true for the extended Poincaré polynomials.

The cases where this happens can be easily read off the tables as follows. If the extended Poincaré polynomials of some theories are identical, these theories are listed together in an un-numbered line; the numbered line(s) following this line then contain the theories with which each of them can be tensored to obtain a $c = 9$ theory. For instance, the line preceding the lines numbered from 25 to 29 in Table 4.3 shows that the theories $(CC, 2, 2)$, $(CC, 3, 1)$ and $(G2_2, 1)$ have identical extended Poincaré polynomials.

One systematic reason why the extended Poincaré polynomials of distinct Lie algebraic cosets coincide are the level-rank dualities proven in Chapter 3. We have also taken into account the known [123] fact that the extended Poincaré polynomials of the hermitian symmetric cosets $(A, 1, 2, 3)$, $(A, 2, 2, 2)$, [4] $(C, 3, 1)$, and $(D, 5, 1)$ coincide with those of the tensor products $1-4$, $1-1-1-1$, $3-3$, and $1-7$ of minimal models, respectively.

4.5 Conclusions

In this chapter we have presented a detailed analysis of compactifications of the heterotic string that contain non-hermitian symmetric $N = 2$ superconformal coset theories in their inner sector. The spectra of string compactifications that we obtained are certainly not spectacular, but rather similar to those obtained for previously analyzed classes of compactifications. This confirms the by now common lore that extending the set of string compactifications does not have a very large impact on the set of known spectra. The results also confirm the experience that when employing more complicated conformal field

[3] The list in [37], containing 28 hermitian symmetric cosets with $c = 6$, is incomplete in several respects. First, rather than $(D, 5, 2) - 16$, one must use the combinations $(D, 5, 2)$ and $(D, 5, 1) - 16$. Further, it was not realized that the coset theory $(A, 1, 2, 2)$ (appearing in three of the 28 theories) coincides with the minimal model at level 8. Finally, the theories $(B, 3, 6)$ and $(B, 6, 3)$ which in [37] were supposed to be identical, are in fact [123] distinct conformal field theories. Implementing these corrections, the number of the models gets reduced by one, leading to the correct number of 27 models.

[4] However, in Table 4.3 we have nevertheless kept the entries # 17 and # 42 containing $(A, 2, 2, 2)$, because after identification with $1-1-1-1$, they would correspond to entries in a different table, namely Table 4.1.

theories, the numbers of generations and anti-generations tend to be smaller than in the case of simpler (say, $N=2$ minimal) theories.

There still remain several directions for further work on the subject. First, one may consider modular invariant combinations of characters of the g- and h- WZW theories other than the diagonal one, in particular non-diagonal invariants of tensor product theories that are not obtained from products of the invariants of the affine Lie algebras associated to the individual factor theories. One may also investigate whether the coset theories, or at least their tensor products with $c=3n$, might have a description in terms of Landau–Ginzburg potentials or Calabi–Yau manifolds, or of orbifolds thereof (while it is generally assumed that such a connection should exist, the arguments supporting this expectation are far from being rigorous). To identify these different descriptions it would be very useful to have a more detailed knowledge of the discrete symmetries of the models. One of these discrete symmetries is obvious, namely the symmetry of the operator products induced by conservation of the superconformal u(1)-charge; but generically there may be further symmetries, and it is not clear how one could find all of them. Of course, once discrete symmetries are known, one can divide out some of them so as to obtain orbifolds of our models.

We also mention that a complete computation of massless string spectra, i.e. including the fields that are singlets under E_6, would clearly be welcome. To this end one would have to compute the character decompositions by means of the Kac–Weyl character formula (in order to identify null states and to obtain the integer part of the conformal weight of a field), and implement the beta vector method known from tensor products of minimal models. It is evident that this is a laborious procedure, and any alternative method would be of great interest.

Another interesting aspect of the string spectra obtained in the chapter is that the extended Poincaré polynomials $\mathcal{P}(t,x)$, and hence the generation numbers N_{27} and $N_{\overline{27}}$ of the associated string compactifications, of two theories are identical whenever the ordinary Poincaré polynomials $P(t)=\mathcal{P}(t,0)$ are. This indicates that the structure of the extended Poincaré polynomial is to a large extent already dictated by the information contained in the *ordinary* Poincaré polynomial; in particular (compare [123]), in the presence of fixed points the numbers of massless generations and anti-generations do not depend at all on the details of the resolution procedure. A general proof of this observation is however still lacking.

Table 4.1: $c = 9$ tensor product theories that contain a $c = 6$ part combined with a non-hermitian symmetric factor (different from $(CC, 2, 1)$), and the associated generation and anti-generation numbers

#	Model	N_{27}	$N_{\overline{27}}$	δN
	$(BA, 3, 1)$			
1	$-2-1-1-1$	21	21	* 0
2	$-2-1-4$	31	7	24
3	$-2-2-2$	39	3	36
4	$-2-(CC, 2, 1)$	31	7	24
	$(BA, 4, 1)$ or $(C3, 1)$ or $(G2_2, 2)$			
5	$-1-1-1$	21	21	* 0
6	$-1-4$	31	7	24
7	$-2-2$	31	7	24
8	$-(CC, 2, 1)$	31	7	24
	$(BB, 3, 1)$ or $(CC, 2, 3)$ or $(CC, 4, 1)$			
9	$-1-1-1-1$	51	3	48
10	$-1-1-4$	51	3	48
11	$-1-2-2$	21	21	* 0
12	$-1-(CC, 2, 1)$	21	21	* 0
	$(BB, 4, 1)$ or $(CC, 2, 5)$ or $(CC, 6, 1)$			
13	$-1-1-1$	21	21	* 0
14	$-1-4$	23	23	0
15	$-2-2$	44	8	36
16	$-(CC, 2, 1)$	23	23	0
	$(CC, 2, 2)$ or $(CC, 3, 1)$ or $(G2_2, 1)$			
17	$-3-1-1-1$	21	21	* 0
18	$-3-1-4$	21	21	* 0
19	$-3-2-2$	21	21	* 0
20	$-3-(CC, 2, 1)$	21	21	* 0
	$(CC, 3, 2)$			
21	$-1-1-1$	21	21	* 0
22	$-1-4$	41	5	36
23	$-2-2$	41	5	36
24	$-(CC, 2, 1)$	41	5	36
	$(G2_1, 1)$			
25	$-1-1-1-1$	29	5	24
26	$-1-1-4$	29	5	24
27	$-1-2-2$	21	21	* 0
28	$-1-(CC, 2, 1)$	21	21	* 0

Table 4.2: $c = 6$ tensor products, and the net generation number δN for the $c = 9$ models obtained by tensoring in addition with $(CC, 2, 1)$

#	Model ($c = 6$ part)	N_{27}	$N_{\overline{27}}$	δN
1	$1 - 1 - 1 - 1 - 1 - 1$	21	21	* 0
2	$1 - 1 - 1 - 1 - 4$	35	11	24
3	$1 - 1 - 1 - 2 - 2$	21	21	* 0
4	$1 - 1 - 2 - 10$	35	11	24
5	$1 - 1 - 4 - 4$	51	3	48
6	$1 - 2 - 2 - 4$	51	3	48
7	$2 - 2 - 2 - 2$	61	1	60
8	$1 - 5 - 40$	35	35	0
9	$1 - 6 - 22$	43	19	24
10	$1 - 7 - 16$	43	19	24
11	$1 - 8 - 13$	27	27	0
12	$1 - 10 - 10$	59	11	48
13	$2 - 3 - 18$	39	15	24
14	$2 - 4 - 10$	45	9	36
15	$2 - 6 - 6$	55	7	48
16	$3 - 3 - 8$	39	15	24
17	$4 - 4 - 4$	60	6	54
18	$(A, 1, 2, 4) - 12$	38	20	18
19	$(A, 1, 2, 5) - 6$	55	7	48
20	$(A, 1, 2, 6) - 1 - 1$	21	21	* 0
21	$(A, 1, 2, 6) - 4$	23	23	0
22	$(A, 1, 2, 7) - 3$	39	15	24
23	$(A, 1, 2, 9) - 2$	45	9	36
24	$(A, 1, 2, 15) - 1$	43	19	24
25	$(A, 1, 3, 3) - 5$	21	21	* 0
26	$(A, 1, 3, 4) - 2$	51	3	48
27	$(A, 1, 3, 5) - 1$	21	21	* 0
28	$(A, 1, 3, 8)$	45	9	36
29	$(A, 1, 4, 5)$	41	5	36
30	$(A, 2, 2, 4)$	51	3	48
31	$(B, 6, 3)$	21	21	* 0
32	$(C, 2, 3) - 2$	51	3	48
33	$(C, 2, 6)$	23	23	0
34	$(C, 3, 2)$	21	21	* 0
35	$(C, 4, 1) - 1$	35	11	24
36	$(D, 5, 2)$	21	21	* 0
37	$(CC, 2, 1) - 1 - 1 - 1$	21	21	* 0
38	$(CC, 2, 1) - 1 - 4$	51	3	48
39	$(CC, 2, 1) - 2 - 2$	51	3	48
40	$(CC, 2, 1) - (CC, 2, 1)$	51	3	48

Table 4.3: $c = 9$ tensor products that contain a non-hermitian symmetric coset and cannot be decomposed in the tensor product of a $c = 3$ and a $c = 6$ theory

#	Model	N_{27}	$N_{\overline{27}}$	δN
	$(BA,3,1)$			
1	$-1-1-10$	19	19	0
2	$-3-18$	23	23	0
3	$-4-10$	27	15	12
4	$-6-6$	35	11	24
5	$-(A,1,2,9)$	27	15	12
6	$-(A,1,3,4)$	31	7	24
7	$-(B,6,2)$	35	11	24
8	$-(C,2,3)$	31	7	24
9	$-(BA,3,1)$	15	15	0
10	$(BA,3,2)-12$	12	8	4
11	$(BA,3,4)$	14	2	12
12	$(BA,5,1)-2$	15	15	0
13	$(BA,6,1)$	15	15	0
	$(BB,3,1)$ or $(CC,2,3)$ or $(CC,4,1)$			
14	$-2-10$	35	11	24
15	$-4-4$	43	7	36
16	$-(A,1,2,6)$	35	11	24
17	$-(A,2,2,2)$	51	3	48
18	$(BB,3,3)$	17	5	12
19	$(BB,4,2)$	14	14	0
	$(BB,5,1)$ or $(CC,2,7)$ or $(CC,8,1)$			
20	-8	39	15	24
	$(BB,6,1)$ or $(CC,2,9)$ or $(CC,10,1)$			
21	$-1-1$	29	29	0
22	-4	44	14	30
	$(BB,8,1)$ or $(CC,2,13)$ or $(CC,14,1)$			
23	-2	34	34	0
	$(BB,12,1)$ or $(CC,2,21)$ or $(CC,22,1)$			
24	-1	43	43	0
	$(CC,2,2)$ or $(CC,3,1)$ or $(G2_2,1)$			
25	$-1-1-28$	23	23	0
26	$-4-28$	29	29	0
27	$-8-8$	47	11	36
28	$-(A,1,2,12)$	35	17	18
29	$-(B,8,2)$	41	5	36
	$(CC,2,4)$ or $(CC,5,1)$			
30	$-(A,1,2,4)$	29	14	15

Table 4.3: *continued.*

#	Model	N_{27}	$N_{\overline{27}}$	δN
	$(CC,2,6)$ or $(CC,7,1)$			
31	-16	27	27	0
32	$(CC,3,5)$ or $(CC,6,2)$	20	20	0
33	$(CC,4,3)$	29	9	20
34	$(C4,1)$	15	7	8
35	$(D4,1)-(A,1,2,4)$	23	11	12
36	$(D5_1,1)$	12	0	12
37	$(D5_2,1)-16$	19	19	0
38	$(F4,1)-8$	25	13	12
	$(G2_1,1)$			
39	$-2-10$	17	17	0
40	$-4-4$	23	11	12
41	$-(A,1,2,6)$	17	17	0
42	$-(A,2,2,2)$	29	5	24
43	$(G2_1,2)-7$	9	9	0
44	$(G2_2,5)$	8	8	0

Part II

(Quasi-) Galois Symmetries in Conformal Field Theory

In Part II of this thesis we will develop new algebraic tools for the study of fusion rings. After shortly reviewing the structure of fusion rings and their applications in physics and mathematics, we will show that Galois theory of cyclotomic number fields provides a powerful tool to construct automorphisms of a fusion ring and modular invariant partition functions. In Chapter 6 these tools are applied to WZW theories; several new series of exceptional modular invariants are found. In the case of WZW theories Galois symmetries admit for a generalization which we call quasi-Galois symmetries. They are the subject of Chapter 7.

Chapter 5

Galois Symmetry in Rational Conformal Field Theory

5.1 Fusion Rings

Among the wealth of structures quantum field theory in two dimensions has revealed up to now, *fusion rings* can be considered as objects of central importance for the description of low-dimensional physics. We have already seen that they describe the coupling of primary fields of $\mathcal{W}$-algebras in conformal field theory; closely connected to that, they also describe the composition of superselection sectors in the C^*-algebraic approach to quantum field theory. Moreover, they describe how tensor products of finite-dimensional representations of reductive Lie algebras, of finite groups, or of associative (bi-)algebras decompose into irreducible representations. The set of unitary representations of quantum groups with deformation parameter a root of unity is turned by the truncated tensor product into a fusion ring as well. Finally, after slightly relaxing the properties of the conjugation involution, one can also describe the multiplication of (classes of) polynomials in any quotient of a polynomial ring, e.g. the ring of chiral primary fields in $N = 2$ superconformal field theories, and, closely connected to that, operator products in topological field theory.

The axioms describing a fusion ring can be abstracted from the structure present in the family of all finite-dimensional representations of a compact Lie group. Let us therefore have a look at this family. There are two operations: an addition, since the direct sum of finite-dimensional representations is again a finite-dimensional representation, and a multiplication, the tensor product. Any tensor product can be fully reduced into a direct sum over irreducible representations:

$$L_\Lambda \otimes L_{\Lambda'} = \bigoplus_{\Lambda''} \mathcal{N}_{\Lambda\Lambda'}^{\Lambda''} L_{\Lambda''} \,. \tag{5.1.1}$$

These operations are associative, commutative and distributive. The one-dimensional module which carries the trivial representation acts as the identity under multiplication. There is a distinguished basis, the irreducible representations, which contains the identity and in which the structure constants $\mathcal{N}_{\Lambda\Lambda'}^{\Lambda''}$ are non-negative integers.

In the formal treatment one also allows for negative multiplicities of irreducible representations and thus obtains a unital ring over the integer numbers $\mathbb{Z}$. One can also extend the structure a little bit by admitting rational coefficients and obtain the closely related structure of a fusion algebra over the field of rational numbers $\mathbb{Q}$. Introducing negative multiplicities is necessary to make contact to structures like rings and algebras for which results from classical algebra are available. However, one has to pay a price for that: we will see that it is quite difficult to take the positivity into account, which nonetheless is essential for most physical issues.

Finally, taking the conjugate representation yields an involutive automorphism of the fusion ring. The trivial module is self-conjugate; it plays a special role in the sense that the tensor product of two irreducible representations L_Λ and $L_{\Lambda'}$ contains the identity if and only if $L_{\Lambda'}$ is conjugate to L_Λ, in which case the identity appears just once. The conjugation can therefore be described by the evaluation of the tensor product with respect to the unit element.

In this special example the distinguished basis contains infinitely many elements; however, any product of elements of the distinguished basis can be decomposed into only finitely many irreducible representations: such fusion rings are called quasi-rational.

Axiomatically, a fusion ring can be described as an associative and commutative ring over the integers $\mathbb{Z}$ with unit element for which a distinguished basis exists which contains the unit element and in which all structure constants are non-negative integers. The evaluation of the product with respect to to the identity is required to describe an involutive automorphism, the conjugation. We write the element conjugate to i as i^+. Let us remark that this system of axioms is not minimal; for a detailed discussion we refer the reader to the review [45]. We call a fusion ring *rational* if it is finite-dimensional.

For any rational fusion ring with generators ϕ_i, $i \in I$ (I some finite index set), and fusion product $\phi_i \star \phi_j = \sum_{k\in I} \mathcal{N}_{ij}{}^k \phi_k$ with $\mathcal{N}_{ij}{}^k \in \mathbb{Z}_{\geq 0}$, the structure constants $\mathcal{N}_{ij}^k$ can be grouped into matrices

$$(\mathcal{N}_i)_j^k := \mathcal{N}_{ij}^k \,, \tag{5.1.2}$$

the *fusion matrices.* They form a representation of the fusion ring, the regular representation. The axioms of the fusion ring imply that the fusion matrices $\mathcal{N}_i$ commute among each other and that the fusion matrix of i^+ is the transpose of the fusion matrix of i, $\mathcal{N}_{i^+} = (\mathcal{N}_i)^{\rm t}$. Hence fusion matrices are in particular normal and can be simultaneously diagonalized by a unitary matrix S.

Motivated by conformal field theory we single out a particularly interesting subclass of fusion rings: *modular fusion rings.* A fusion ring is called modular if and only if the matrix S which diagonalizes the fusion rules can be chosen to be symmetric and there is a diagonal unitary matrix T with entries $T_{ij} = T_i\,\delta_{ij} := {\rm e}^{2\pi{\rm i}(\Delta_i - c/24)}\delta_{ij}$, such that T and S generate a finite-dimensional representation of $SL_2(\mathbb{Z})$, the twofold covering of the modular group. In particular, $S^2 = C\,,\ (ST)^3 = C\,,\ C^2 = \mathbb{1}$. Here C denotes the charge conjugation matrix, which is a permutation of order 2 and which can be written as $C_{ij} = \delta_{i,j^+}$.

The diagonalization of the fusion rules then takes the form

$$\mathcal{N}_{ij}{}^k = \sum_{\ell\in I} \frac{S_{i\ell}S_{j\ell}S^*_{k\ell}}{S_{0\ell}} \,. \tag{5.1.3}$$

This is the Verlinde formula; in the case of rational conformal field theory it was argued in [111,138] that S is precisely the matrix which describes the transformation of the characters under the modular transformation $\tau \mapsto -\frac{1}{\tau}$. This is a far reaching insight which transcends the mere framework of fusion rings.

The condition of modularity relates the eigenvectors of the fusion matrices to the primary fields and from (5.1.3) we can read off the eigenvalues,

$$\frac{S_{ij}}{S_{0j}}, \tag{5.1.4}$$

which are labelled by primary fields j. These eigenvalues are called the (generalized) *quantum dimensions.* Since all entries of the fusion matrices are non-negative integers, there exists a unique eigenvector, the Perron-Frobenius eigenvector, for which all eigenvalues are *positive* real numbers. These eigenvalues, the so-called *main quantum dimensions*, are labelled by the vacuum $j = 0$; they will play an important role.

Since the quantum dimensions are eigenvalues of the fusion matrices (which form the regular representation of the fusion ring), it is clear that they form themselves one-dimensional irreducible representations of the fusion ring:

$$\frac{S_{i\ell}}{S_{0\ell}}\frac{S_{j\ell}}{S_{0\ell}} = \sum_{k\in I} \mathcal{N}_{ij}{}^{k} \frac{S_{k\ell}}{S_{0\ell}} . \tag{5.1.5}$$

Fusion rings have the important property that the irreducible representations given by the generalized quantum dimensions already exhaust all inequivalent irreducible representations.

From their definition as roots of the characteristic polynomial

$$\det(\lambda \mathbb{1} - \mathcal{N}_i)\,,$$

it follows that quantum dimensions are algebraic numbers. (The transcendental number π, e.g., could not be a quantum dimension.) In addition, since characteristic polynomials are normalized, in the sense that they have integral coefficients and that their leading coefficient is 1, quantum dimensions are even algebraically integer numbers in some algebraic number field L over the rational numbers $\mathbb{Q}$. We will use these facts to employ number theoretical tools in the study of fusion rings.

5.2 Modular invariance

One central problem in the application of fusion rings to rational conformal field theories is that of finding all modular invariant partition functions. Unfortunately, despite a lot of efforts in the last decade, this problem remains to a large extent unsolved. It is part of the programme of classifying all rational conformal field theories, which in turn is part of the even more ambitious programme of classifying all string theories.

The aim is to find a matrix Z that commutes with the generators S and T of the modular group, and that furthermore is integer-valued, non-negative and has $Z_{00} = 1$, where 0 represents the identity primary field. The partition function of the theory has then the form $\sum_{ij} \mathcal{X}_i Z_{ij} \overline{\mathcal{X}}_j^*$, where $\mathcal{X}_i$ are the characters of the left chiral algebra and $\overline{\mathcal{X}}_j$ those of the right one (the left and right algebras need not necessarily coincide).

At present the classification is complete only for very few rational conformal field theories. All modular invariant partition functions are known e.g. for the simplest RCFT's, whose left and right chiral algebra consist only of the Virasoro algebra [39, 19, 20]. The next simplest case is that of WZW models, whose chiral algebra has in addition to the Virasoro algebra further currents of spin 1. In general such a theory can be 'heterotic' (i.e. it may have different left and right Kac-Moody algebras) and both the left and right chiral algebra may have more than one affine factor, but even in the simplest case – equal left and right simple affine algebras – the classification is complete at arbitrary level only for the cases A_1 [19, 20] and A_2 [60]. Several other partial classification results have been presented, see for example [88, 59, 61].

Although there is no complete classification, many methods are known for finding at least a substantial number of solutions, for example simple currents [126] (see also [13, 7, 2, 86, 36]), conformal embeddings [14], level-rank duality [94, 141, 5, 114, 115, 139, 55], supersymmetric index arguments [142], selfdual lattice methods [119], orbifold constructions using discrete subgroups of Lie groups [1], and the elliptic genus [124].

In this chapter we show that Galois theory of cyclotomic number fields provides a new powerful tool to construct systematically integer-valued matrices commuting with the modular matrix S, as well as automorphisms of the fusion rules. Both prescriptions allow the construction of modular invariants and offer new insight in the structure of known exceptional invariants.

5.3 The Galois group and the modular matrix S

In this section we will show that Galois theory of cyclotomic number fields can be used to construct fusion rule automorphisms and modular invariants of rational fusion rings. Galois theory was proposed as a tool to study fusion rings in [26]; in [23] it was observed that it can be applied to the elements of the modular matrix S to provide selection rules for positive modular invariants.

The starting point is the last observation in Section 5.1, the fact that the generalized quantum dimensions S_{il}/S_{0l} are algebraically integer numbers in some algebraic number field L over the rational numbers $\mathbb{Q}$. The extension $L/\mathbb{Q}$ is normal [26], and hence (using also the fact that the field $\mathbb{Q}$ has characteristic zero) a *Galois extension*; its *Galois group* $\mathcal{G}al(L/\mathbb{Q})$ is abelian. Invoking the theorem of Kronecker and Weber, it follows [26] that L is contained in some cyclotomic field $\mathbb{Q}(\zeta_n)$, where ζ_n is a primitive nth root of unity.

Let us describe a few elementary facts about Galois theory of cyclotomic fields. Denote by $\mathbb{Z}_n^*$ the multiplicative group of all elements of $\mathbb{Z}_n \equiv \mathbb{Z}/n\mathbb{Z}$ that are coprime with n. Note that precisely these elements have an inverse with respect to multiplication. (For example, the group $(\mathbb{Z}_{10}^*, \cdot) \cong (\{\pm 1, \pm 3\}, \cdot \mod 10)$ is isomorphic to the additive group $(\mathbb{Z}_4, +)$.) The number $\varphi(n)$ of elements of $\mathbb{Z}_n^*$ is given by Euler's φ function, which can be computed as follows. If $n = \prod_i p_i^{n_i}$ is a decomposition of n into distinct primes p_i, then one has

$$\varphi(n) = \varphi(\prod_i p_i^{n_i}) = \prod_i \varphi(p_i^{n_i}) = \prod_i p_i^{n_i - 1}(p_i - 1)\,. \tag{5.3.1}$$

The Galois automorphisms (relative to $\mathbb{Q}$) of the cyclotomic field $\mathbb{Q}(\zeta_n)$ in which $\mathcal{G}al(L/\mathbb{Q})$ is contained are in one-to-one correspondence with the elements $\ell \in \mathbb{Z}_n^*$. The automorphism associated to each such ℓ simply acts as

$$\sigma_{(\ell)}: \quad \zeta_n \mapsto (\zeta_n)^\ell\,. \tag{5.3.2}$$

This implies in particular that $\ell = -1$ corresponds to complex conjugation. Thus if the fusion ring is self-conjugate in the sense that $i^+ = i$ for all $i \in I$, so that the S-matrix is real, then the automorphism $\sigma_{(-1)}$ acts trivially. In this case the relevant field L is already contained in the maximal real subfield $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$ of the cyclotomic field $\mathbb{Q}(\zeta_n)$, which is the field that is fixed under complex conjugation.

Applying an element $\sigma_L \in \mathcal{G}al(L/\mathbb{Q})$ on equation (5.1.5) and using the fact that the fusion coefficients ${\mathcal{N}_{ij}}^k$ are integers and hence invariant under σ_L, we learn that the numbers $\sigma_L(S_{ij}/S_{0j})\,, i \in I$, again realize a one-dimensional representation of the fusion ring. As the generalized quantum dimensions exhaust all inequivalent one-dimensional representations of the fusion ring [95, 25], there must exist some permutation of the labels j which we denote by $\dot\sigma$, such that

$$\sigma_L(\frac{S_{ij}}{S_{0j}}) = \frac{S_{i\,\dot\sigma(j)}}{S_{0\,\dot\sigma(j)}}\,. \tag{5.3.3}$$

The field M defined as the extension of $\mathbb{Q}$ that is generated by all S-matrix elements extends L. The extension $M/\mathbb{Q}$ is again normal and has abelian Galois group [23], so that $\mathcal{G}al(M/L)$ is a normal subgroup of $\mathcal{G}al(M/\mathbb{Q})$. Elementary Galois theory then shows that

$$0 \to \mathcal{G}al(M/L) \xrightarrow{\imath} \mathcal{G}al(M/\mathbb{Q}) \xrightarrow{r} \mathcal{G}al(L/\mathbb{Q}) \to 0\,, \tag{5.3.4}$$

with $\imath$ the canonical inclusion and r the restriction map, is an exact sequence, and hence

$$\mathcal{G}al(L/\mathbb{Q}) \cong \mathcal{G}al(M/\mathbb{Q}) \,/\, \mathcal{G}al(M/L)\,. \tag{5.3.5}$$

In particular any $\sigma_M \in \mathcal{G}al(M/\mathbb{Q})$, when restricted to L, maps L onto itself and equals some element $\sigma_L \in \mathcal{G}al(L/\mathbb{Q})$. Conversely, any $\sigma_L \in \mathcal{G}al(L/\mathbb{Q})$ can be obtained this way. Therefore by a slight abuse of notation we will frequently use the abbreviation σ for both σ_M and its restriction σ_L.

Working in the field M, it follows from (5.3.3) that for any $\sigma_L \in \mathcal{G}al(L/\mathbb{Q})$ there exist signs $\epsilon_\sigma(i) \in \{\pm 1\}$ such that the relation

$$\sigma_M(S_{ij}) = \epsilon_\sigma(i) \cdot S_{\dot\sigma(i)\,j} \tag{5.3.6}$$

is fulfilled for all $i,j \in I$ [23]. We note that the Galois group element σ and the permutation $\dot\sigma$ of the labels that is induced by σ need not necessarily have the same order. However, it is easily seen (see the remarks around (5.5.11) below) that an extra factor of 2 is the only difference that can appear.

These observations can be extended in two directions: first, we show that Galois theory can be used to construct automorphisms of the fusion rules. Second, we derive from Galois theory a prescription for the systematic construction of integral-valued matrices in the commutant of the modular matrix S, and hence of candidate modular invariants. We describe how this method is implemented for WZW theories. As it turns out, our general prescription is able to explain many of the modular invariants that are usually referred to as 'exceptional'.

5.4 Fusion rule automorphisms

We first show that, if the permutation $\dot\sigma$ induced by the Galois group element σ leaves the identity fixed,

$$\dot\sigma(0) = 0\,, \tag{5.4.1}$$

then $\dot\sigma$ is an automorphism of the fusion rules. To prove this, we first calculate

$$\frac{S_{0i}}{S_{00}} = \sigma_L(\frac{S_{0i}}{S_{00}}) = \frac{\sigma_M(S_{0i})}{\sigma_M(S_{00})} = \frac{\epsilon_\sigma(i)\, S_{0\,\dot\sigma(i)}}{\epsilon_\sigma(0)\, S_{00}}\,. \tag{5.4.2}$$

Since S_{0j}/S_{00}, the main (i.e., zeroth) quantum dimensions, are positive, we learn that the sign $\epsilon_\sigma(i)$ is the same for all $i \in I$,

$$\epsilon_\sigma(i) = \epsilon_\sigma(0) =: \epsilon_\sigma = \text{const}\,. \tag{5.4.3}$$

Applying σ on the Verlinde formula (5.1.3), we then find

$$\mathcal{N}_{ij}^{\ \ k} = \sigma(\mathcal{N}_{ij}^{\ \ k}) = \sum_{l\in I} \frac{\epsilon_\sigma^3\, S_{\dot\sigma(i)\,l} S_{\dot\sigma(j)\,l} S^*_{\dot\sigma(k)\,l}}{\epsilon_\sigma\, S_{0l}} = \mathcal{N}_{\dot\sigma(i)\,\dot\sigma(j)}^{\qquad \dot\sigma(k)}\,. \tag{5.4.4}$$

Next we note that in terms of the cyclotomic field $\mathbb{Q}(\zeta_n) \supseteq M \supseteq L$, the elements $\sigma_{(\ell)} \in \mathcal{G}al(L/\mathbb{Q})$ are simply the restrictions of elements $\tilde\sigma_{(\ell)} \in \mathcal{G}al(\mathbb{Q}(\zeta_n)/\mathbb{Q})$; the latter act as $\zeta_n \mapsto (\zeta_n)^\ell$, and $\mathcal{G}al(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong \mathbb{Z}_n^*$ is the set of all such maps with ℓ coprime to n. In particular, $\ell = -1$ corresponds to complex conjugation; the associated permutation of the generators of the fusion ring is the charge conjugation C. As the Galois group is abelian, it follows that $\dot\sigma$ is compatible with charge conjugation,

$$\dot\sigma(i^+) = (\dot\sigma(i))^+ . \tag{5.4.5}$$

Together with (5.4.1), the results (5.4.4) and (5.4.5) show that, as claimed, $\dot\sigma$ is an automorphism of the fusion rules.

The presence of such automorphisms of the fusion rules can be understood as follows. The 'main' quantum dimensions

$$\frac{S_{i0}}{S_{00}} \tag{5.4.6}$$

all lie in a real field $L_{(0)}$ that is contained in the field L generated by all (generalized) quantum dimensions S_{ij}/S_{0j}. The elements of the group $\mathcal{G}al(L/L_{(0)})$ leave the main quantum dimensions invariant, and hence the associated permutations $\dot\sigma$ are fusion rule automorphisms. These automorphisms are thus a manifestation of the fact that the main quantum dimensions do *not* exhaust the field spanned by all generalized quantum dimensions.

The general result is nicely illustrated by the example of complex conjugation. Suppose that the fusion ring is non-selfconjugate, i.e. there is at least one $i \in I$ such that $i^+ \neq i$. Then the modular matrix S is complex, and as already mentioned the charge conjugation C which acts as $i \mapsto i^+$ is induced by $\sigma_C = \sigma_{(-1)} \in \mathcal{G}al(L/\mathbb{Q})$, i.e. $i^+ = \dot\sigma_C(i)$. As the main quantum dimensions are real (which is equivalent to $(0)^+ = 0$), G contains at least σ_C as a nontrivial element, and charge conjugation is the corresponding non-trivial automorphism.

As a second illustration, consider the extremal case $G = \mathcal{G}al(L/Q)$. This means that all main quantum dimensions are rationals (and, since they are algebraic integers, in fact even ordinary integers). This situation is realized e.g. for $c = 1$ conformal field theories, both for compactification of the free boson on a circle and for compactification on those $\mathbb{Z}_2$ orbifolds for which the number of fields is $m^2 + 7$ for some $m \in \mathbb{Z}$, as well as for the $(\mathrm{so}(N^2))_2$ and $(\mathrm{su}(3))_3$ WZW theories. Consider e.g. the theory of a free boson on the circle, with $N \in 2\mathbb{Z}$ primary fields. The fusion rules read $p \star q = p + q \bmod N$, and the modular matrix S has entries $S_{pq} = \mathrm{e}^{-2\pi \mathrm{i}\, pq/N}$. The permutations induced by the Galois group are parametrized by l, with l and N coprime, and act like $p \mapsto lp \bmod N$. This is invertible just because l and N are coprime, and clearly an automorphism. Thus G is the full Galois group, $G \cong \mathbb{Z}_N^*$. Analogous considerations hold for the orbifolds and for the WZW theories just mentioned.

Note that a permutation automorphism of generic order N does not directly lead to a modular invariant since the corresponding permutation matrix Π_σ generically does not commute with S, but rather obeys $S^{-1}\Pi_\sigma S = \Pi_\sigma^{-1}$. For $N = 2$ (such as e.g. charge conjugation), Π_σ does commute with S, and hence provides a candidate modular invariant. For being indeed a modular invariant, Π_σ also has to commute with the modular matrix T; it is not difficult to establish (see the remarks around (5.5.20) below) that any automorphism of the fusion rules that fulfills (5.3.6) and commutes with the T-matrix has order two.

Sometimes there also exist automorphisms of the fusion rules that *cannot* be obtained from elements of the Galois group. This happens for instance if the S-matrix elements of all fields that are permuted are rational numbers; in this situation, any element of the Galois group necessarily leaves these fields fixed, and hence cannot induce the fusion rule automorphism.

5.5 The construction of S-matrix invariants

As an easy consequence of the relation (5.3.6) between $S_{\dot\sigma(i)\,j}$ and S_{ij}, it follows that for any matrix Z which satisfies

$$[Z,S]=0\,,\qquad Z_{ij}\in\mathbb{Z}\;\;\forall\,i,j\in i\,, \tag{5.5.1}$$

the relation $Z_{\dot\sigma(i)\,\dot\sigma(j)}=\epsilon_\sigma(i)\epsilon_\sigma(j)\,Z_{ij}$ holds [23]. This leads to a selection rule for those matrices Z which obey $Z_{ij}\geq 0$ in addition to (5.5.1), and which hence provide a candidate modular invariant $\mathcal{Z}(\tau,\bar\tau)=\sum_{i,j}\chi_i^*(\bar\tau)Z_{ij}\chi_j(\tau)$ for the associated conformal field theory (this restriction is a generalization of the 'parity rule' of [60] and the 'arithmetical symmetry' of [120]).

Here we will go beyond the level of mere selection rules and show that Galois theory can be used to *construct* modular invariants. Let us apply σ^{-1} to the relation (5.3.6) and permute the second label of S on the right hand side; then we have

$$S_{ij}=\sigma^{-1}\sigma(S_{ij})=\sigma^{-1}(\epsilon_\sigma(i)S_{\dot\sigma(i)\,j})=\epsilon_\sigma(i)\epsilon_{\sigma^{-1}}(j)\,S_{\dot\sigma(i)\,\dot\sigma^{-1}(j)}\,, \tag{5.5.2}$$

where in the last equality one uses the fact that $\epsilon_\sigma(i)=\pm1$ is rational and hence fixed under σ. Using (5.5.2) l times, we obtain

$$S_{ij}=\epsilon_l(i)\epsilon_{-l}(j)\,S_{\dot\sigma^l(i)\,\dot\sigma^{-l}(j)}\,, \tag{5.5.3}$$

where the signs $\epsilon_l(i)\equiv\epsilon_{\sigma^l}(i)\in\{\pm1\}$ are determined by $\epsilon_1\equiv\epsilon_\sigma$ through

$$\epsilon_l(i)=\prod_{m=0}^{l-1}\epsilon_1(\dot\sigma^m(i))\,. \tag{5.5.4}$$

We will employ the simple result (5.5.2), respectively (5.5.3), to show that to any element of the Galois group one can associate a matrix Z which obeys (5.5.1).

Before proceeding, we should point out that a relation of the form (5.5.3) need not necessarily stem from Galois theory. In the proof we actually use only this relation, but not the information whether it is derived from Galois theory or not. [1] In particular, we need not assume that the signs ϵ_l are prescribed by some Galois group element σ, but only use that they are determined by the permutation $\dot\sigma$. However, Galois theory constitutes the only systematic tool that is known so far to derive such relations, even though it does not provide an exhaustive list. (A situation where the symmetry property (5.5.3) of the modular matrix S is satisfied in the absence of Galois symmetries is provided by mutually local simple currents [130] of order two.) [2]

Thus assume that $\dot\sigma$ is a permutation, of order $\dot N$, of the index set I of a fusion ring and satisfies a relation of the type (5.5.3), and define the integer N to be the order of the associated map $S_{ij}\mapsto\epsilon_\sigma(i)\,S_{\dot\sigma(i)\,j}$. We can then show that for any set $\{f_l\mid l=1,2,\dots,N\}\subset\mathbb{Z}$ of integers that satisfy

$$f_l=f_{-l}\equiv f_{N-l}\,, \tag{5.5.5}$$

the matrix Z with integral entries

$$Z_{jk}:=\sum_{l=0}^{N-1}f_l\,\epsilon_l(k)\,\delta_{j,\dot\sigma^l(k)} \tag{5.5.6}$$

[1] This remark applies in fact equally to the considerations about fusion rule automorphisms above.

[2] Considering simple currents of general order would amount to allow the ϵ's in (5.5.3) to be arbitrary phases instead of signs. Unfortunately there are no nontrivial cases with $N>2$ and (5.5.6) being real-valued.

commutes with the modular matrix S. Namely, by direct calculation we have

$$(SZ)_{ik} = \sum_{j\in I}\sum_{l=0}^{N-1} S_{ij}\cdot f_l\,\epsilon_l(k)\,\delta_{j,\dot\sigma^l(k)} = \sum_{l=0}^{N-1} f_l\epsilon_l(k)\,S_{i\,\dot\sigma^l(k)} \tag{5.5.7}$$

as well as

$$\begin{aligned}(ZS)_{ik} &= \sum_{j\in I}\sum_{l=0}^{N-1} f_l\epsilon_l(j)\,\delta_{i,\dot\sigma^l(j)}\cdot S_{jk} = \sum_{l=0}^{N-1} f_l\epsilon_l(\dot\sigma^{-l}(i))\,S_{\dot\sigma^{-l}(i)\,k}\\ &= \sum_{l=0}^{N-1} f_l\epsilon_l(\dot\sigma^{-l}(i))\cdot\epsilon_l(\dot\sigma^{-l}(i))\epsilon_{-l}(k)\,S_{i\,\dot\sigma^{-l}(k)} = \sum_{l=0}^{N-1} f_l\epsilon_{-l}(k)\,S_{i\,\dot\sigma^{-l}(k)}\,,\end{aligned} \tag{5.5.8}$$

where in the transition to the second line we employed (5.5.3). Now one merely has to replace the sum on l in (5.5.8) by one on $-l$ and use (5.5.5) to conclude that indeed S and Z commute. The terms in the sum of (5.5.6) correspond to the elements of the cyclic group that is generated by the element σ appearing in (5.5.2); considering more generally an arbitrary abelian group G whose generators satisfy (5.5.2), one proves that the prescription (5.5.6) generalizes to

$$Z_{jk} = \sum_{\sigma\in G} f_\sigma\,\epsilon_\sigma(k)\,\delta_{j,\dot\sigma(k)}\,, \tag{5.5.9}$$

with f_σ restricted by

$$f_\sigma = f_{\sigma^{-1}} \tag{5.5.10}$$

for all $\sigma\in G$.

Returning to the interpretation in terms of the Galois group, we note that according to (5.3.6) the upper limit N of the summation in equation (5.5.6) is precisely the order of the Galois group element σ (in particular, Galois theory provides a relation of the type (5.5.3) with $-l\equiv N-l$), and recall that this order need not necessarily coincide with the order of the permutation $\dot\sigma$ of the labels that σ induces. However, the following consideration shows that the distinction between N and $\dot N$ is actually not very relevant to applications. First, at most a relative factor of 2 can be present; namely, since $\dot\sigma$ is of order $\dot N$, one has in particular $\dot\sigma^{\dot N}(0)=0$, which by (5.4.3) implies that the sign $\epsilon_{\sigma^{\dot N}}$ is universal, and hence

$$\sigma^{2\dot N}(S_{ij}) = \sigma^{\dot N}(\epsilon_{\sigma^{\dot N}}\,S_{ij}) = (\epsilon_{\sigma^{\dot N}})^2\,S_{ij} = S_{ij}\,, \tag{5.5.11}$$

so that $\sigma^{2\dot N}=id$ on M; thus either $N=\dot N$ or else $N=2\dot N$. Furthermore, for $N=2\dot N$ the terms in the formula for Z are easily seen to cancel out pairwise, so that the proposed invariant is identically zero, and hence the case $N\neq\dot N$ is rather uninteresting.

This result can also be obtained in a slightly different formulation: for any Galois transformation σ we define the orthogonal matrix

$$(\Pi_\sigma)_{ij} := \epsilon_\sigma(i)\,\delta_{j,\dot\sigma i} = \epsilon_{\sigma^{-1}}(j)\,\delta_{i,\dot\sigma^{-1}j}\,, \tag{5.5.12}$$

where in the second equality we used the relation

$$\epsilon_\sigma(\dot\sigma^{-1}(i)) = \epsilon_{\sigma^{-1}}(i) \tag{5.5.13}$$

which is obtained from the identity $\sigma\sigma^{-1}S_{ij}=S_{ij}$ when acting twice on the first label of S.

These orthogonal matrices can easily be shown to satisfy the identities

$$(\Pi_\sigma)^{-1} = \Pi_{\sigma^{-1}} = (\Pi_\sigma)^T \ , \tag{5.5.14}$$

and they implement the Galois transformations (5.3.6) in the following way:

$$\sigma S = \Pi_\sigma \cdot S = S \cdot \Pi_\sigma^{-1} \ . \tag{5.5.15}$$

Now we can write (5.5.2) in matrix notation as (omitting the subscript σ of Π_σ)

$$S = \Pi S \Pi \ , \tag{5.5.16}$$

or $\Pi^{-1} S = S\Pi$. Obviously the same identity holds with Π replaced by its inverse, and by adding these two relations we see that the matrix $\Pi + \Pi^{-1} = \Pi + \Pi^T$ commutes with S. If Π is equal to its own inverse one can take half this matrix, i.e. Π itself.

The full Galois commutant is obtained by considering all sums and products of these matrices. Because the matrices Π form a representation of the Galois group $\mathcal{G}al(L/\mathbb{Q})$, it is easy to see that the product of any two matrices of the form $\Pi + \Pi^{-1}$ is a linear combination of such matrices with integral coefficients. Hence the most general integer-valued S-invariant that can be obtained in this way is

$$Z = \sum_{(\sigma,\sigma^{-1})\in G} f_\sigma (\Pi_\sigma + \Pi_\sigma^{-1}) \ , \tag{5.5.17}$$

where the sum is over all elements of the Galois group G modulo inversion, and $f_\sigma \in \mathbb{Z}$.

Note that this derivation of S-invariants goes through for any matrix Π that satisfies (5.5.16), even if it did not originate from Galois symmetry. If such a new matrix Π commutes with all matrices Π_G that represent Galois symmetries, one may extend the Galois group G to a larger group $\tilde{G} \supset G$ by including all matrices $\Pi \cdot \Pi_G$. The most general S-invariant related to $\tilde{G}$ is then obtained by extending the sum in (5.5.6) to $\tilde{G}$.

As was observed in [23], Galois symmetry implies a relation that any modular invariant Z, irrespective of whether it is itself a Galois invariant, should satisfy. Indeed, using $\sigma Z = Z$ and $\sigma S^{-1} = (\sigma S)^{-1}$, one derives $Z = \sigma Z = \sigma(SZS^{-1}) = \Pi_\sigma Z \Pi_\sigma^{-1}$, i.e. Z commutes with Π. If Z is an automorphism of order 2, then we have in addition the relation $S = ZSZ$, and hence Z is a 'Galois-like' automorphism that can be used to extend the Galois group as described above. If Z is an automorphism of higher order or corresponds to an extension of the chiral algebra, then it has different commutation properties with S, and it cannot be used to extend the Galois group, but one can still enlarge the commutant by multiplying all matrices (5.5.6) with the new invariant Z and its higher powers. In this case the full commutant is considerably harder to describe, however.

We can make another statement about $\dot\sigma$ by assuming that it commutes with the T-matrix, $T_{\dot\sigma(i)} = T_i$. Applying this property together with the relation (5.5.2) to the identity

$$T_i^{-1} S_{ik} T_k^{-1} = \sum_{j\in I} S_{ij} T_j S_{jk} \tag{5.5.18}$$

which follows from $(ST)^3 = S^2 = C$, we obtain

$$\begin{aligned} \epsilon_\sigma(i)\epsilon_{\sigma^{-1}}(k)\, T_i^{-1} S_{\dot\sigma(i)\,\dot\sigma^{-1}(k)} T_k^{-1} &= \epsilon_{\sigma^{-1}}(i)\epsilon_{\sigma^{-1}}(k) \sum_{j\in I} S_{\dot\sigma^{-1}(i)\,\dot\sigma(j)} T_{\dot\sigma(j)} S_{\dot\sigma(j)\,\dot\sigma^{-1}(k)} \\ &= \epsilon_{\sigma^{-1}}(i)\epsilon_{\sigma^{-1}}(k)\, T^{-1}_{\dot\sigma^{-1}(i)} S_{\dot\sigma^{-1}(i)\,\dot\sigma^{-1}(k)} T^{-1}_{\dot\sigma^{-1}(k)} \ . \end{aligned} \tag{5.5.19}$$

Thus

$$S_{\dot\sigma^{-1}(i)\,j} = \epsilon_\sigma(i)\epsilon_{\sigma^{-1}}(i)\,S_{\dot\sigma(i)\,j} \tag{5.5.20}$$

for all $i, j \in I$. As S is unitary, its rows are linearly independent, and hence (5.5.20) implies that $\dot\sigma(i) = \dot\sigma^{-1}(i)$ for all i, i.e. that $\dot\sigma^2 = id$. Hence any σ that fulfills (5.3.6) and commutes with the T-matrix has order two. (Again, this result is just based on the property (5.5.2) of $\dot\sigma$, and therefore is valid independently of whether $\dot\sigma$ comes from a Galois group element σ or not.) As we will see in the next chapter, at least for WZW theories a kind of converse statement is also true, namely that any Galois group element of order two respects the T-matrix up to possibly minus signs.

Due to the presence of the signs ϵ_σ, the invariants (5.5.6) are generically *not* positive. However, at least for order $N = 2$ one sometimes gets invariants that are completely positive and moreover have a non-degenerate vacuum. The only required property of σ is that $\epsilon_\sigma(i)$ is universal for all length-two orbits, while the sign for fixed points is arbitrary. Fixed points with $\epsilon_\sigma(i) = -1$ simply get projected out; in fact, the latter are the only fields that can be directly projected out.

The kind of invariant that is defined by (5.5.6) depends on the vacuum orbit. If the identity is a fixed point, the signs $\epsilon(i)$ are all equal to the same overall sign ϵ, as shown in Section 5.4. Then, for $N = 2$, the choice $f_0 = 0$ and $f_1 = \epsilon$ in (5.5.6) immediately gives us a positive matrix Z that commutes with S and generates a fusion rule automorphism. If the vacuum is not fixed, the choice $f_0 = 0$, $f_1 = \epsilon(0)$ leads to an invariant with an extended chiral algebra in which at least the identity block is positive. It follows from unitarity of S that in such an invariant not all coefficients $f_l\epsilon_l(i)$ can be positive (otherwise $Z_{ij} \geq \delta_{ij}$, and hence $Z_{00} = \sum_{i,j\in I} S_{0i}Z_{ij}S_{0j} \geq \sum_{i\in I} S_{0i}S_{0i} = 1$, with equality only if $Z_{ij} = \delta_{ij}$; this is clearly a contradiction). The only way to get a positive invariant is then that the negative signs occur precisely for the fixed point orbits, which are then projected out. If $N = 2$ this is indeed possible. Note that T-invariance still remains to be checked in both cases.

For $N > 2$ it is much harder to get a physical invariant. First of all there must exist orbits that violate T-invariance, although such orbits might be projected out by the summation in (5.5.6). It is in fact easy to see that no positive integer invariant can be obtained from (5.5.6) if N is odd, for any choice of f_l. If N is odd, all coefficients except f_0 come in pairs f_l, f_{-l}. It follows that $Z_{jj} = f_0 \bmod 2$ for all $j \in I$, and since $Z_{00} = 1$ this means that none of the fields is projected out. Then the unitarity argument given above shows that a non-trivial positive invariant cannot exist. If $N > 2$ and even, hence not a prime, one has to distinguish various kinds of fixed point orbits. Positive modular invariants may then well exist, but we will not consider this more complicated case at this place.

Let us stress that even if the matrix (5.5.6) contains negative entries, or does not commute with T, it can still be relevant for the construction of physical invariants, because the prescription may be combined with other procedures in such a manner that the negative contributions cancel out. For example one may use simple currents to extend the chiral algebra before employing the Galois transformation, or it may happen that a certain linear combination with other known elements of the integer commutant of S is a physical invariant.

5.6 Discussion

There are several striking similarities between Galois symmetries and simple current symmetries. First of all both are related to general properties of fusion rings, and not to particular (e.g. WZW) models. Both imply equalities among certain matrix elements of S up to signs or phases. Both symmetries organize the fields of the theory into orbits, whose length is a divisor of the order N of the symmetry. In both cases one can give very simple generic formulas for S-invariants, and in both cases the phenomenon of 'fixed points', i.e. of orbits whose length is less than N, occurs. In both cases such fixed points can appear with multiplicities larger than 1 in certain modular invariants in which the chiral algebra is extended. Note that this kind of structure is empirically observed in nearly all exceptional (not simple current generated) invariants found thus far. However, we believe this is the first time that at least in some cases the apparent 'orbits' and 'fixed points' of exceptional invariants are actually related to an underlying discrete symmetry. This might in fact be of some help in the still open problem of resolving fixed points of exceptional invariants.

There is also an important difference between Galois and simple current symmetries. In the latter case one can give a general construction of invariants that are positive and are also T-invariant. For Galois invariants it may well be possible to find a general criterion for T-invariance (as we will see for WZW models), but positivity appears to be a much more difficult requirement. There is, however, one set of S-invariants that is always positive, namely those due to a $\mathbb{Z}_2$ Galois symmetry that fixes the vacuum. In WZW models such invariants (that also commute with T) are abundant: this includes all charge conjugation invariants and also at least some of the simple current automorphism invariants that were first constructed in [7]. Remarkably, very few exceptional ones are known.

Let us also note that formula (5.4.4) can be generalized to automorphisms $\dot\sigma$ which change the vacuum, i.e. obey $\dot\sigma(0) \neq 0$ (and hence are not automorphisms of the fusion ring as a *unital* ring). In this situation, (5.4.4) gets replaced by

$$\begin{aligned} \mathcal{N}_{ij}^{\ \ k} &= \sigma(\mathcal{N}_{ij}^{\ \ k}) = \sigma(\sum_{l\in I}\frac{S_{il}S_{jl}S^*_{kl}}{S_{0l}}) \\ &= \sum_{l\in I}\frac{\epsilon_\sigma(i)\epsilon_\sigma(j)\epsilon_\sigma(k)\, S_{\dot\sigma(i)\,l}S_{\dot\sigma(j)\,l}S^*_{\dot\sigma(k)\,l}}{\epsilon_\sigma(0)S_{\dot\sigma(0)\,l}} = \epsilon_\sigma(0)\epsilon_\sigma(i)\epsilon_\sigma(j)\epsilon_\sigma(k)\ {}_{\dot\sigma(0)}\mathcal{N}_{\dot\sigma(i)\,\dot\sigma(j)}^{\ \ \ \ \ \dot\sigma(k)}\,, \end{aligned} \tag{5.6.1}$$

where ${}_l\mathcal{N}_{ij}^{\ \ k} \equiv \sum_{m\in I} S_{im}S_{jm}S^*_{km}/S_{lm}$. Note that the numbers ${}_l\mathcal{N}_{ij}^{\ \ k}$ are well-defined only if $S_{lm} \neq 0$ for all $m \in I$, in which case according to (5.6.1) they are actually integers; in the present situation this condition is met because $S_{\dot\sigma(0)\,i} = \epsilon_\sigma(0)\epsilon_\sigma(i)\, S_{0\,\dot\sigma(i)} \neq 0$ for all $i \in I$. This result can be interpreted as follows. Allowing also for negative structure constants, we can introduce a second fusion product $\star_\sigma$, with structure constants ${}_{\dot\sigma(0)}\mathcal{N}_{ij}^{\ \ k}$, on the same ring $\mathbb{Z}^{|I|}$. Defining $\tilde\phi_i := \epsilon_\sigma(0)\epsilon_\sigma(i)\,\phi_{\dot\sigma(i)}$, it follows that $\tilde\phi_i \star_\sigma \tilde\phi_j = \sum_{k\in I}\mathcal{N}_{ij}^{\ \ k}\,\tilde\phi_k$, i.e. both fusion structures are isomorphic. Some special cases of this phenomenon have already been noticed in [34]. While our argument uses symmetries of number fields, in [34] the representation theory of the modular group is employed; thus our observation suggests a relation between number fields and modular forms.

In this chapter we have presented a procedure for constructing modular invariant partition functions directly from symmetries of the matrix S, without any explicit knowledge of its matrix elements. This method is valid for all rational conformal field theories, and not a priori restricted to WZW models and coset theories, unlike conformal embeddings or level-rank duality. Previously only two such methods were known, namely charge conjugation (actually an example of Galois symmetry) and simple currents, and usually the term 'exceptional invariant' was used to refer to anything else. By providing a third general procedure, the results of this chapter define a new degree of 'exceptionality' for modular invariant partition functions. Invariants satisfying this new definition of exceptionality do exist; this may be taken as an indication that still more interesting structure remains to be discovered. We will see in Chapter 7 that there is in fact a non-trivial generalization of Galois symmetry, at least in the case WZW theories, which we will call Quasi-Galois symmetries.

Chapter 6

Galois Modular Invariants of WZW Models

The set of modular invariants that can be obtained from Galois transformations is investigated systematically for WZW models. It is shown that a large subset of Galois modular invariants coincides with simple current invariants. For algebras of type B and D infinite series of previously unknown exceptional automorphism invariants are found.

6.1 Introduction

In the previous chapter [47], we introduced a novel method for the construction of modular invariants based on a Galois symmetry of the matrix S of a rational conformal field theory. The main purpose of this chapter is to study in more detail the application of this new method to WZW models.

We have seen that Galois symmetry organizes the fields of a CFT into orbits, and along these orbits the matrix elements of S are algebraically conjugate numbers. Based on this knowledge we can write down a number of integer-valued matrices P that commute with S, but do not necessarily commute with T and are not necessarily positive. These matrices span what we call the 'Galois-commutant' of S. This commutant can be constructed in a straightforward manner from the Galois orbits, which in turn can be obtained by scaling vectors in weight space by certain integers, and mapping them back into the fundamental affine Weyl chamber (for a more precise formulation we refer to Section 6.3 and the appendix). This is a simple algorithm that can be carried out easily with the help of a computer. The time required for this computation increases linearly with the number of primary fields, and for each primary the number of calculational steps is bounded from above by the order of the Weyl group. This should be compared with the computation of the modular matrix S, which grows quadratically with the number of primaries, and which requires a sum over the full Weyl group (although several shortcuts exist, for example simple currents and of course Galois symmetry).

Our second task is then to find the positive T-invariants within the Galois commutant. In some cases this can be done analytically. This class, which contains only simple current invariants, is discussed in Section 6.3. In general however one has to solve a set of equations for a number of integer coefficients. The number of unknowns can grow rather rapidly with increasing level of the underlying affine Kac-Moody algebra – Galois symmetry is a huge and very powerful symmetry – which is another limitation on the scope of our investigations.

In practice we have considered algebras with rank ≤ 8 and up to 2500 primary fields, but this range was extended when there was reason to expect something interesting. Although a lot of exploratory work has already been done on the classification of modular invariants, only fairly recently new invariants were found [124] for E_6 and E_7 at rather low levels (namely 4 and 3), showing that there are still chances for finding something new. Indeed, we did find new invariants, namely an infinite series of exceptional automorphism invariants for algebras of type B at level 2, starting at rank 7, as well as for algebras of type D at level 2. In addition we find for the same algebras some clearly unphysical extensions by spin-1 currents. This is explained in Section 6.4. Other exceptional invariants that can be explained in terms of Galois symmetry are presented in Section 6.5.

We have also considered the possibility of combining Galois orbits with simple current orbits. In Section 6.6 we discuss two ways of doing that, one of which is to apply Galois symmetry to simple current extensions of the chiral algebra.

To conclude this introduction we fix some notations. If $Z_{0i} = Z_{i0} = 0$ for all $i \neq 0$, the matrix Z defines a permutation of the fields in the theory that leaves the fusion rules invariant. We will refer to this as an *automorphism invariant.* Under multiplication such matrices form a group which is a subgroup of the group of *fusion rule automorphisms.* These are all permutations of the fields that leave the fusion rules invariant, but which do not necessarily commute with S or T. Finally there is a third group of automorphisms we will encounter, namely that of *Galois automorphisms.* They act as a permutation combined with sign flips, and may act non-trivially on the identity. It is important not to confuse these three kinds of automorphisms.

If a matrix Z does not have the form of an automorphism invariant, and if the partition function is a sum of squares of linear combinations of characters, we will refer to it as a (*chiral algebra*) *extension.* If it is not a sum of squares it can be viewed as an automorphism invariant of an extended algebra [110, 28] (at least if an associated CFT exists).

A matrix Z corresponding to a chiral algebra extension may contain squared terms appearing with a multiplicity higher than 1. Such terms will be referred to as 'fixed points', a terminology which up to now was appropriate only for extensions by simple currents. Galois automorphisms provide us with a second rationale for using this name. Usually such fixed points correspond to more than one field in the extended CFT, and they have to be 'resolved'. The procedure for doing this is available only in some cases, and then only for S, T, the fusion rules and in a few cases also the characters [129].

6.2 Galois Symmetry for WZW Models

Here we describe in detail how Galois scalings are implemented when the conformal field theory in question is a WZW theory based on an untwisted affine Lie algebra g at integral level k. Then the Galois group is a subgroup of $\mathbb{Z}^*_{M(k+g^\vee)}$, where $g^\vee$ is the dual Coxeter number of the horizontal subalgebra $\bar{\mathrm{g}}$ of g (i.e. the subalgebra generated by the zero modes of g) and M is the denominator of the metric on the weight space of $\bar{\mathrm{g}}$.

We label the primary fields by the shifted highest weight a with respect to the horizontal subalgebra $\bar{\mathrm{g}}$, which differs from the ordinary highest weight by addition of the Weyl vector ρ of $\bar{\mathrm{g}}$. Thus a is an integrable highest weight of g at level $k + g^\vee$, i.e. the components a^i of a in the Dynkin basis satisfy

$$a^i \in \mathbb{Z}_{\geq 0} \quad \text{for } i = 0, 1, \dots, \mathrm{rank}(\bar{\mathrm{g}}) \,, \tag{6.2.1}$$

where $a^0 \equiv k + g^\vee - \sum_{i=1}^{\mathrm{rank}(\bar{\mathrm{g}})} \theta_i a^i$ with θ_i the dual Coxeter labels of g. However, because of the shift not all such integrable weights belong to primary fields, but only the strictly dominant integral weights, i.e. the primary fields of the WZW theory correspond precisely to those weights a which obey

$$a^i \in \mathbb{Z}_{>0} \quad \text{for } i = 0, 1, \dots, \mathrm{rank}(\bar{\mathrm{g}}) \,. \tag{6.2.2}$$

A Galois transformation labelled by $\ell \in \mathbb{Z}^*_{M(k+g^\vee)}$ acts as the permutation [23]

$$\dot\sigma_{(\ell)}(a) = \hat w(\ell a)\,. \tag{6.2.3}$$

If we label the fields by the weights $\Lambda = a - \rho$ which are at level k, this is rewritten as

$$\dot\sigma_{(\ell)}(\Lambda) = \hat w(\ell\cdot(\Lambda+\rho)) - \rho\,, \tag{6.2.4}$$

That it is the shifted weight a rather than $a-\rho$ that is scaled is immediately clear from the formula (6.3.2) for the modular matrix S. In fact, it is possible to derive the formula (5.5.2) directly by scaling the row and column labels of S by ℓ and ℓ^{-1}, respectively, using (6.2.3). Galois symmetry is thus not required to derive this formula, nor is it required to show that (5.5.6) commutes with S. Galois symmetry has however a general validity and is not restricted to WZW models.

Substituting (6.2.3) into the formula (1.4.7) for WZW conformal weights one easily obtains a condition for T-invariance, namely $(\ell^2-1) = 0 \bmod 2M(k+g^\vee)$ (or mod $M(k+g^\vee)$ if all integers $M\,a\cdot a$ are even). Since ℓ has an inverse mod $M(k+g^\vee)$, it follows that $\ell = \ell^{-1} \bmod M(k+g^\vee)$, i.e. the order of the transformation must be 2, what is also true for arbitrary conformal field theories, as we have seen in Chapter 5.

Let us explain the prescription (6.2.3) in more detail. First one performs a dilatation of the shifted weight $a = (a^1, a^2, \ldots)$ by the factor $\ell \in \mathbb{Z}^*_{M(k+g^\vee)}$. Now the weight ℓa does not necessarily satisfy (6.2.2), i.e. does not necessarily correspond to a primary field. If it does not, then the dilatation has to be supplemented by the horizontal projection $\hat w \equiv \hat w_{(\ell;a)}$ of a suitable affine Weyl transformation. More precisely, to any arbitrary integral weight b one can associate an affine Weyl transformation $\hat w$ such that either $\hat w(b)$ satisfies (6.2.2), and in this case $\hat w$ is in fact unique, or else such that $\hat w(b)$ obeys $(\hat w(b))^i = 0$ for some $i \in \{0,1,\ldots,\mathrm{rank}(\bar{\mathrm g})\}$ (in the latter case $\hat w(b)$ lies on the boundary of the horizontal projection of the fundamental Weyl chamber of g at level $k+g^\vee$). To construct the relevant Weyl group element $\hat w$ for a given weight b as a product of fundamental Weyl reflections $w_{(l)}$ (i.e. reflections with respect to the lth simple root of g), one may use the following algorithm. Denote by $j_1 \in \{0,1,\ldots,\mathrm{rank}(\bar{\mathbf g})\}$ the smallest integer such that $b^{j_1} < 0$, and consider instead of b the Weyl-transformed weight $\hat w_1(b)$ with $\hat w_1 := \hat w_{(j_1)}$; next denote by j_2 the smallest integer such that $(\hat w_1(b))^{j_2} < 0$, and consider instead of $\hat w_1(b)$ the weight $\hat w_2 \hat w_1(b)$ with $\hat w_2 := \hat w_{(j_2)}$, and so on, until one ends up with a weight $\hat w_n \ldots \hat w_2 \hat w_1(b)$ obeying (6.2.2), and then $\hat w = \hat w_n \ldots \hat w_2 \hat w_1$ is the unique Weyl group element which does the job. (The presentation of an element $\hat w \in \hat W$ as a product of fundamental reflections is however not unique; the present algorithm provides one specific presentation of this type, which is not necessarily reduced in the sense that the number of fundamental reflections is minimal.)

It is worth noting that there is no guarantee that starting from an integral weight b one gets this way a weight satisfying (6.2.2), but in the case where b is of the form $b = \ell a$ with a integrable and ℓ coprime with $r(k+g^\vee)$, the algorithm does work. Here r denotes the maximal absolute value of the off-diagonal matrix elements of the Cartan matrix of $\bar{\mathbf g}$, i.e. $r=1$ if $\bar{\mathbf g}$ is simply laced, $r=2$ for the algebras of type B and C and for F_4, and $r=3$ for $\bar{\mathbf g} = G_2$. (The property that ℓ is coprime with $r(k+g^\vee)$ in particular holds whenever (6.2.3) corresponds to an element of the Galois group, and hence for Galois transformations the algorithm works simultaneously for all primary fields of the theory.) Namely, assume that for some choice of a there is no choice of $\hat w \in \hat W$ such that $\hat w(\ell a)$ obeys (6.2.2). This means that any $\hat w(\ell a)$ lies on the boundary of some affine Weyl chamber, and hence the same is already true for the weight ℓa. Then there must exist some non-trivial $\hat v \in \hat W$ which leaves ℓa fixed, $\hat{v}(\ell a) = \ell a$. Decomposing $\hat{v}$ into its finite Weyl group part $v \in W$ and its translation part $(k+g^\vee)t$ (with t an element of the coroot lattice of $\bar{\mathrm{g}}$), this means that we have $\ell\, v(a) + (k+g^\vee)t = \ell a$, or in other words,

$$\ell\,(a - v(a)) = (k+g^\vee)\,t\,. \tag{6.2.5}$$

Now assume that ℓ is coprime with $r(k+g^\vee)$. This implies that there exists integers m, n such that $m\ell = nr(k+g^\vee)+1$. Multiplying (6.2.5) with m then yields

$$a = v(a) + (k+g^\vee)[mt - nr\,(a - v(a))]\,. \tag{6.2.6}$$

Since for any integral weight a the weight $r(a-v(a))$ is an element of the coroot lattice, the same is also true for the expression in square brackets, and hence (6.2.6) states that the weight a stays fixed under some affine Weyl transformation. But a satisfies (6.2.2) and hence the fact that $\hat{W}$ acts freely on such weights implies that this Weyl transformation must be the identity. This implies that $\hat{v}$ must be the identity as well. Thus for ℓ coprime with $r(k+g^\vee)$ the assumption that $\hat{w}(\ell a)$ is not integrable leads to a contradiction.

In the general case where b is not of the form ℓa with a subject to (6.2.2) and ℓ coprime with $r(k+g^\vee)$, the algorithm described above still works unless at one of the intermediate steps one of the Dynkin labels becomes zero, which means that the weight lies on the boundary of the fundamental affine Weyl chamber. In the latter case any Weyl image of this weight lies on the boundary of some affine Weyl chamber as well, and hence we can never end up with a weight that satisfies (6.2.2), i.e. in the interior of the fundamental affine Weyl chamber. It may also be remarked that one can speed up the algorithm considerably using not the weight b itself as a starting point, but rather the weight $\tilde{b} = b + (k+g^\vee)t$ that is obtained from b by such a Weyl translation $(k+g^\vee)t$ for which the length of $\tilde{b}$ becomes minimal.

Finally, there is a general formula for the sign $\epsilon_{\sigma_{(\ell)}}$, namely

$$\epsilon_{\sigma_{(\ell)}}(a) = \eta_\ell\,\mathrm{sign}(w_{(\ell;a)})\,, \tag{6.2.7}$$

i.e. the sign is just given by that of the Weyl transformation $\hat{w}$, up to an overall sign η_ℓ that only depends on $\sigma_{(\ell)}$ [23], but not on the individual highest weight a. (Actually the cyclotomic field $\mathbb{Q}(\zeta_{M(k+g^\vee)})$ whose Galois group is $\mathbb{Z}^*_{M(k+g^\vee)}$ does not yet always contain the overall normalization $\mathcal{N}$ that appears in the formula (6.3.2) for S, but rather sometimes a slightly larger cyclotomic field must be used [23] However, the permutation $\dot{\sigma}$ of the primary fields that is induced by a Galois scaling can already be read off the generalized quantum dimensions, which do not depend on the normalization of S. The correct Galois treatment of the normalization of S just amounts to the overall sign factor η_ℓ, which is irrelevant for our purposes.)

6.3 Infinite Series of Invariants

In this section we will discuss an infinite class of WZW modular invariants that can be obtained both by a Galois scaling as well as by means of simple currents. Both Galois transformations and simple currents organize the fields of a CFT into orbits. In general, the respective orbits are not identical. In the special case of WZW models these orbits are in fact never identical, except for a few theories with too few primary fields to make the difference noticeable. However, since the orbits are used in quite different ways to derive modular invariants, it can nevertheless happen that these invariants are the same.

The Galois scalings we consider are motivated by the following argument. As already mentioned, Galois automorphisms of the fusion rules arise if the field $L_{(0)}$ is strictly smaller than the field L. In the case of WZW theories L is contained in the cyclotomic field $\mathbb{Q}(\zeta_{M(k+g^\vee)})$, while the quantum Weyl formula [117]

$$\frac{S_{a,\rho}}{S_{\rho,\rho}} = \prod_{\alpha>0} \frac{\sin[\pi\, a\cdot\alpha^\vee/(k+g^\vee)]}{\sin[\pi\,\rho\cdot\alpha^\vee/(k+g^\vee)]} \tag{6.3.1}$$

shows that $L_{(0)}$ is already contained in $\mathbb{Q}(\zeta_{2(k+g^\vee)})$. Now as any element of $\mathcal{G}al(L/\mathbb{Q})$ can be described by at least one element of $\mathcal{G}al(\mathbb{Q}(\zeta_{M(k+g^\vee)})/\mathbb{Q})$, we do not loose anything by working with the latter Galois group. Any Galois automorphism of the fusion rules can now be described by at least one element of $\mathcal{G}al(\mathbb{Q}(\zeta_{M(k+g^\vee)})/L_{(0)})$. Unfortunately, $L_{(0)}$ is not explicitly known in practice; therefore we would like to replace $L_{(0)}$ by the field $\mathbb{Q}(\zeta_{2(k+g^\vee)})$ in which it is contained. However, M is not always even, and hence we consider instead of $\mathbb{Q}(\zeta_{2(k+g^\vee)})$ the smaller field $\mathbb{Q}(\zeta_{k+g^\vee})$ and the corresponding Galois group $\mathcal{G}al(\mathbb{Q}(\zeta_{M(k+g^\vee))}/\mathbb{Q}(\zeta_{k+g^\vee}))$. The elements of this group are precisely covered by scalings by a factor $m(k+g^\vee)+1$. This way we recover at least part of the automorphisms, but due to the difference between $\mathbb{Q}(\zeta_{2(k+g^\vee)})$ and $\mathbb{Q}(\zeta_{k+g^\vee})$, generically some of these scalings do not describe automorphisms, but rather correspond to an extension of the chiral algebra.

Consider now the Kac-Peterson (1.4.8) formula for the modular matrix S which reads in terms of the shifted weights

$$S_{ab} = \mathcal{N} \sum_w \varepsilon(w)\, \exp[-2\pi \mathrm{i}\, \frac{w(a)\cdot b}{k+g^\vee}] \ . \tag{6.3.2}$$

Here $\mathcal{N}$ is a normalization factor which follows by the unitarity of S and is irrelevant for our purposes, and the summation is over the Weyl group of the horizontal subalgebra of the relevant affine Lie algebra; a and b are integrable weights, shifted by adding the Weyl vector ρ. In the following we will denote such shifted weights by roman characters $a, b, \ldots$, while for the Lie algebra weights $a-\rho$, $b-\rho, \ldots$ we will use greek characters.

The scaling by a factor $\ell = m(k+g^\vee)+1$ is an allowed Galois scaling if the following condition is fulfilled (note that m is defined modulo M):

$$\text{(a)}\quad m(k+g^\vee)+1 \text{ is prime relative to } M(k+g^\vee)\,. \tag{6.3.3}$$

We will return to this condition later. (Let us mention that even if condition (a) is not met, the scaling by ℓ can still be used to define an S-invariant. We will describe the implications of such 'quasi-Galois' scalings in the next chapter.)

Under such a scaling one has

$$\begin{aligned} S_{ab} \mapsto \sigma S_{ab} &= \mathcal{N} \textstyle\sum_w \varepsilon(w)\, \exp[-2\pi \mathrm{i}\, \frac{w(a)\cdot b}{k+g^\vee}(m(k+g^\vee)+1)] \\ &= \mathrm{e}^{-2\pi \mathrm{i} m a\cdot b} S_{ab}\ , \end{aligned} \tag{6.3.4}$$

where the last equality holds if $mw(a)\cdot b = ma\cdot b \bmod 1$ for all Weyl group elements w. To analyze when this condition is fulfilled, first note that any Weyl transformation can be written as a product of reflections with respect to the planes orthogonal to the simple roots. For a Weyl reflection r_i with respect to a simple root α_i ($i \in \{1,2,\ldots,\text{rank}\}$) one has in general

$$\begin{aligned} r_i(a)\cdot b &= a\cdot b - (\tfrac{2}{\alpha_i\cdot\alpha_i})\ \alpha_i\cdot a\ \alpha_i\cdot b \\ &= a\cdot b - \tfrac{1}{2}\,\alpha_i\cdot\alpha_i\ a_i b_i\ , \end{aligned} \tag{6.3.5}$$

where a_i and b_i are Dynkin labels. Thus $r_i(a)\cdot b$ equals $a\cdot b$ modulo integers if and only if all simple roots have norm 2 (which is for all algebras our normalization of the longest root), i.e. iff the algebra is simply laced. However, the derivation depends on this relation with an extra factor m. This yields one more non-trivial solution, namely $m=2$ for B_n, n odd. Note that for B_n with n even, one has $M=2$ so that the only allowed scaling, $m=2$, yields a trivial solution. This is also true for all other non-simply laced algebras.

As is easily checked, the quantity $a\cdot b \bmod 1$ is closely related to the product of the simple current charges; we find:

$$\begin{array}{ll} A_n: & a\cdot b = -(n+1)\mathcal{Q}(a)\mathcal{Q}(b) \\ B_n: & 2a\cdot b = 2n\mathcal{Q}(a)\mathcal{Q}(b) \\ D_n\ (n\ \text{odd}): & a\cdot b = 4n\mathcal{Q}(a)\mathcal{Q}(b) \\ D_n\ (n\ \text{even}): & a\cdot b = 2\mathcal{Q}_s(a)\mathcal{Q}_s(b) + 2\mathcal{Q}_c(a)\mathcal{Q}_c(a) + (n-2)\mathcal{Q}_v(a)\mathcal{Q}_v(b) \\ E_6: & a\cdot b = 3\mathcal{Q}(a)\mathcal{Q}(b) \\ E_7: & a\cdot b = 2\mathcal{Q}(a)\mathcal{Q}(b)\ . \end{array} \tag{6.3.6}$$

Here $\mathcal{Q}(a)$ is the monodromy charge with respect to the simple current J of a WZW representation with highest weight a (which is at level $k+g^\vee$). This should not be confused with the simple current charge of the field labelled by a, which we denote by $Q(a)$. The relation between these two quantities is

$$Q(a) = \mathcal{Q}(a-\rho) = \mathcal{Q}(a) - \mathcal{Q}(\rho)\ , \tag{6.3.7}$$

since the field labelled by a has highest weight $a-\rho$ (which is at level k). The charge $\mathcal{Q}$ (as well as Q) depends only on the conjugacy class of the weight. The WZW theory with algebra D_n, n even, has a center $\mathbb{Z}_2\times\mathbb{Z}_2$ and simple currents J_s, J_v and $J_c = J_v\star J_s$. It has thus two independent charges, for which one may take Q_v and Q_s.

If ρ is on the root lattice, then $\mathcal{Q}(\rho)=0$ and the shift in (6.3.7) is irrelevant, i.e. $\mathcal{Q}=Q \bmod 1$. In general, either ρ is a vector on the root lattice, or it is a weight with the property that 2ρ is on the root lattice. In the cases of interest here, ρ is on the root lattice for A_n, n even, D_n with $n = 0 \bmod 4$ or $1 \bmod 4$, and for E_6. In all other cases $Q=\mathcal{Q}+\frac{1}{2} \bmod 1$ (if the algebra is $D_n, n=2 \bmod 4$, the charges affected by this shift are Q_s and Q_c).

Note that the left hand sides of the relations (6.3.6) are always of the form $lN\mathcal{Q}(a)\mathcal{Q}(b)$ or a sum of such terms, where N is the order of the simple current and l is an integer. The relation for B_n has an essential factor of 2 in the left hand side. Since the relations are defined modulo integers we cannot simply divide this factor out. The most convenient way to deal with it is to rewrite m in this case as $m=2\tilde{m}$ (we have already seen above that m has to be even for B_n). After substituting (6.3.6) into (6.3.4) we get generically

$$\sigma S_{ab} = \mathrm{e}^{-2\pi i lmN\mathcal{Q}(a)\mathcal{Q}(b)} S_{ab}\ . \tag{6.3.8}$$

This formula holds for B_n if one replaces m by $\tilde{m}$, and for D_n, n even, if one replaces the exponent by the appropriate sum, as in (6.3.6). We will postpone the discussion of the latter case until later, and consider for the moment only theories with a center $\mathbb{Z}_N$.

Now we wish to make use of the simple current relation

$$S_{J^n a,b} = \mathrm{e}^{2\pi i n Q(b)} S_{ab}\ . \tag{6.3.9}$$

This is simplest if we can replace $\mathcal{Q}$ by Q, and this is the case we consider first. This replacement is allowed if ρ is on the root lattice, but this is not a necessary condition because of the extra factor lmN. Suppose $\mathcal{Q} = Q + \frac{1}{2}$. Then we see from the foregoing that N is even and l odd. Replacing $\mathcal{Q}$ by Q in the exponent of (6.3.8) yields the extra terms

$$\tfrac{1}{2}lmNQ(a) + \tfrac{1}{2}lmNQ(b) + \frac{1}{4}lmN \;, \tag{6.3.10}$$

which should be an integer. Now $N\mathcal{Q}(a)$ (or $N\mathcal{Q}(b)$) is an integer, which as a function of a (or b) takes all values modulo N. Hence each of the three terms must separately be an integer. The first two terms are integers if and only if m is even. Then the last one is an integer as well, since N is even. Thus the condition that $\mathcal{Q}$ can be replaced by Q is equivalent to

$$\text{(b)} \quad m\rho \text{ is an element of the root lattice.} \tag{6.3.11}$$

We remind the reader that for B_n this is valid with m replaced by $\tilde{m} = \frac{1}{2}m$. Hence condition (b) is in fact not satisfied for B_n for any non-trivial value of m. In all remaining algebras M (the denominator of the inverse symmetrized Cartan matrix) is equal to N.

If conditions (a) and (b) hold we can derive

$$\sigma S_{ab} = S_{J^{-mlNQ(a)}a,b} = S_{a,J^{-mlNQ(b)}b} \;. \tag{6.3.12}$$

On the other hand according to (5.3.6) Galois invariance implies

$$\sigma S_{ab} = \epsilon_\sigma(a) S_{\dot\sigma a,b} = \epsilon_\sigma(b) S_{a,\dot\sigma b} \;. \tag{6.3.13}$$

Furthermore if $m\rho$ is an element of the root lattice, it is easy to see that the scale transformation fixes the identity field: the identity is labelled by ρ, and transforms into $\rho' = \rho + m(k+g^\vee)\rho$. The second term is a Weyl translation if $m\rho$ is on the root lattice. In these cases ρ' is mapped to ρ by the transformations described in the appendix, which implies that the identity primary field is fixed. Then it follows that $\epsilon \equiv 1$, and hence we find

$$S_{J^{-mlNQ(a)}a,b} = S_{\dot\sigma a,b} \;, \tag{6.3.14}$$

or

$$S_{a,b} = S_{\tau a,b} \;, \tag{6.3.15}$$

where $\tau a = J^{mlNQ(a)}\dot\sigma a$. Then unitarity of S implies $\delta_{a,\tau a} = \sum_b S_{\tau a,b} S^*_{ba} = \sum_b S_{ab} S^*_{ba} = 1$, so that $a = \tau a$, and hence $\dot\sigma a = J^{-mlNQ(a)}a$.

As described in Chapter 5, any Galois transform that fixes the identity generates an automorphism of the fusion rules, and in this case we see that it connects fields on the same simple current orbit. It is a positive S-invariant, but so far it was not required to respect T-invariance. Thus the last condition we will now impose is

$$\text{(c)} \quad T\text{-invariance}. \tag{6.3.16}$$

In general for simple currents of order N one has

$$h(J^n a) = h(a) + h(J^n) - nQ(a) \bmod 1 \tag{6.3.17}$$

and

$$h(J^n) = \frac{rn(N-n)}{2N} \bmod 1 \;, \tag{6.3.18}$$

where r is the monodromy parameter, which is equal to k for A_n at level k, to $3nk \bmod 8$ for D_n, n odd, to $2k$ for E_6, and to $3k$ for E_7. Condition (c) amounts to the requirement that the difference $h(J^{-mlNQ(a)}a) - h(a)$ of conformal weights be an integer. We have

$$\begin{aligned} h(J^{-mlNQ(a)}a) &= h(a) + h(J^{-mlNQ(a)}) + mlNQ(a)Q(a) \\ &= h(a) - \tfrac{r}{2}mlNQ(a) - (\tfrac{r}{2}(ml)^2 - ml)NQ(a)Q(a)\,. \end{aligned} \tag{6.3.19}$$

For algebras of type A or E, the second term on the right hand side is always an integer, or can be chosen integer: if N is odd, r is defined modulo N and hence can always be chosen even (provided one makes the same choice also in the third term), whereas if N is even by inspection one sees that m must be even as well in order for $m\rho$ to be an element of the root lattice, and hence $mr/2 \in \mathbb{Z}$. Then the only threat to T-invariance is the last term, $(\frac{r}{2}ml-1)mlNQ(a)Q(a)$. This is an integer for any a if and only if $(\frac{r}{2}ml-1)ml = 0 \bmod N$.

Now we will determine the solutions to the three conditions (a), (b) and (c) formulated above. Any solution to these conditions will be a positive modular invariant of automorphism type, that can be obtained both from Galois symmetry as well as from simple currents.

Consider first E_6. Condition (b) is trivial, so that m has to satisfy (a) $m(k+12)+1 \neq 0 \bmod 3$, i.e. $km+1 \neq 0 \bmod 3$, and (c) $(km-1)m = 0 \bmod 3$. We may assume that $m \neq 0$ to avoid the trivial Galois scaling. Then both conditions are satisfied if and only if $km = 1 \bmod 3$. There is always a solution for m, namely $m = k \bmod 3$, unless k is a multiple of 3.

Next consider E_7. Now m has to be even in order that $m\rho$ is a root, and this only allows the trivial solution $m=0$.

For A_n the problem is a bit more complicated. As T-invariance must hold for any monodromy charge $Q(a)$, it is clearly sufficient to consider $Q(a) = \frac{1}{N}$. Several cases have to be distinguished. We start with odd $N = n+1$. Then condition (b) is automatically satisfied. For even level $k = 2j$ the other two conditions read

$$\begin{aligned} &\text{(a)} \quad \mathrm{GCD}(2jm+1, N) = 1\,, \\ &\text{(c)} \quad (jm+1)m = 0 \bmod N\,. \end{aligned} \tag{6.3.20}$$

The solution of the second equation depends crucially on the common factors of j and N. It is easy to see that if j and N have a common factor p, then m is divisible by p as many times as N. In particular, if $N = p^\ell$ and j contains a factor p, then the only solution is the trivial one. To remove common factors, write $j = j'q_a$, $m = m'q_b$ and $N = N'q_b$, where q_a is the greatest common divisor of j and N, and q_b consists of all the prime factors of q_a to the power with which they appear in N. Now the second equation becomes

$$(j'q_a m'q_b + 1)m' = 0 \bmod N'\,. \tag{6.3.21}$$

Now we know that N' has no factors in common with j', q_a or q_b, and hence we can find a m' for which the first factor vanishes $\bmod N'$. This solution m' is non-trivial provided $N' \neq 1$; if $N' = 1$ the solution is $m' = 1$ (or 0), i.e. $m = 0 \bmod N$.

The solution m' has no factors in common with N'. Hence we may write $2jm+1 = jm + (jm+1) = jm \bmod N' = j'q_a m'q_b \bmod N'$, so that we see that $2jm+1$ and N' have no common factors. Furthermore $2jm+1$ and q_b have no common factors, since m has a factor q_b. Hence $2jm+1$ has no common factors with $N = N'q_b$, and therefore the first equation is also satisfied.

In addition to the solution described here, (6.3.21) may have additional solutions with m' and N' having a common factor. It is again easy to see that if m' contains any such prime factor, it must contain it with the same power with which it occurs in N'. Let us denote the total common factor as p_b, which is in general a product of several prime factors. Then the second equation reads

$$(j' q_a m'' p_b q_b + 1) m'' = 0 \bmod N'' \ , \tag{6.3.22}$$

where $m' = m'' p_b$ and $N' = N'' p_b$. We now look for solutions where m'' and N'' have no further common factors. Such a solution does indeed exist, since the coefficient of m'' has no factors in common with N''. To show that the first condition is also satisfied one proceeds exactly as in the foregoing paragraph.

When N is odd and k is also odd, we choose the even monodromy parameter $r = k + N$, and define $j = \frac{k+N}{2}$. The rest of the discussion is then exactly as before.

If N is even condition (b) implies that m must be even as well, and condition (c) becomes $(km/2 + 1)m = 0 \bmod N$, or, writing $m = 2t$, $N = 2p$, $(kt + 1)t = 0 \bmod p$. Condition (a) reads $\text{GCD}(km + 1, N) = 1$, which is equivalent to $\text{GCD}(2kt + 1, p) = 1$. Now we have succeeded in bringing the conditions in exactly the same form as (6.3.20), and we can read off the solutions almost directly. The only slight difference is that above N was odd, whereas here p can be odd or even. However, the value of N did not play any rôle anywhere in the discussion *following* (6.3.20) (it *was* used to derive (6.3.20), though), and hence everything does indeed go through.

If the algebra is D_n, n odd, then we have to distinguish two cases. If $n = 1 \bmod 4$, then condition (b) is trivially satisfied, and condition (a) reads

$$\text{(a)} \quad \text{GCD}(m(k + 2n - 2) + 1, 4) = 1 \ , \tag{6.3.23}$$

from which we conclude that mk (and hence $mr = 3mk$) must be even, so that just as for A_n and E_n the second term on the right hand side of (6.3.19) plays no rôle. Condition (c) thus reduces to

$$\text{(c)} \quad -\left(\frac{r}{2}(mn)^2 - mn\right) = 0 \bmod 4 \ , \tag{6.3.24}$$

with k satisfying $3nk = r \bmod 8$, or what is the same, $nk = 3r \bmod 8$. To substitute this we multiply the first argument of (a) with n, which does not affect this condition. Afterwards we use that $n = 1 \bmod 4$, and then the conditions simplify to

$$\begin{array}{ll} \text{(a)} & \text{GCD}(3mr + 1, 4) = 1 \ , \\ \text{(c)} & -\left(\frac{r}{2} m^2 - m\right) = 0 \bmod 4 \ . \end{array} \tag{6.3.25}$$

If r is even, $r = 2j$, condition (c) then reduces to $jm^2 - m = 0 \bmod 4$. This clearly has a non-trivial solution if j is odd (then m is odd), but only trivial solutions if j is even. If r is odd the only solution to both equations is $m = 2$.

If $n = 3 \bmod 4$ this argument goes through in much the same way, but now solutions for odd m are eliminated by condition (b).

6.3.1 Automorphisms from fractional spin simple currents

Nearly all these results can be summarized as follows. Define $\tilde{N} = N$ if N is odd, $\tilde{N} = N/2$ if N is even. Decompose $\tilde{N}$ into prime factors, $\tilde{N} = p_1^{n_1} \dots p_l^{n_l}$. Then the set of solutions m consists of all integers of the form $m = m'' \frac{N}{\tilde{N}} p_1^{k_1} \dots p_l^{k_l}$, where $k_i = n_i$ if the monodromy parameter r is divisible by p_i, and $k_i = 0$ or $k_i = n_i$ otherwise. The solutions are thus labelled by all combinations of distinct prime factors of N that are not factors of r. The parameter m'' for each solution in this set is the unique solution of the equation

$$\frac{1}{2}rlm''(p_1^{k_1}\dots p_l^{k_l}) = 1 \bmod N'' , \qquad (6.3.26)$$

where $N'' = \frac{\tilde{N}}{p_1^{k_1}\dots p_l^{k_l}}$, and r chosen even if N is odd. These automorphism invariants have both a Galois interpretation and a simple current interpretation: they can be generated by the Galois scaling $m(k+g^\vee)+1$ or alternatively by the fractional spin simple current J^m.

These are precisely all the pure automorphisms generated by single simple currents $K = J^m$ of fractional spin which have a "square root", i.e. for which there exists a simple current K' such that $(K')^2 = K$. Such a square root exists always if K has odd order, but if K has even order it must be an even power m of the basic simple current J. The condition on the common factors of r and N has a simple interpretation in terms of simple currents: If it is not satisfied, then there are integral spin currents on the orbit of J. If one constructs the simple current invariant associated with J these currents extend the chiral algebra, so that one does not get a pure automorphism invariant.

The condition that K must have a square root is a familiar one: in [126] the same condition appeared as a requirement that an invariant can be obtained by a simple left-right symmetric orbifold-like construction with "twist operator" $L\bar{L}^c$. If K does not have a square root and r is even, then there are additional invariants, which were described in [126] and derived in [127]. Recently in [100] it was observed that these invariants could be described as orbifolds with discrete torsion. It is quite interesting that precisely these discrete torsion invariants are missing from the list of Galois invariants.

There is one exception, namely the automorphism invariants of D_{4l+1} at level $2j$, which are Galois invariants even though they violate the foregoing empirical rule: In this case $\tilde{N} = 2$, which is a factor of r. Indeed, they are generated by the current J_s (or J_c) which does not have a square root. Technically the reason for the existence of this extra solution is that this is the only simply laced algebra with ρ lying on the root lattice but N even.

6.3.2 Automorphisms from integer spin simple currents

Finally, we have to return to the case D_n, n even. Since $M = 2$ in this case, the only potentially interesting solution is $m = 1$. Hence $\mathcal{Q}$ is equivalent to Q if and only if ρ is on the root lattice, which is true if and only if $n = 0 \bmod 4$. It is straightforward to derive the analogue of (6.3.12):

$$\sigma S_{ab} = S_{J_s^{2mQ_s(a)} J_c^{2mQ_c(a)} J_v^{(n-2)mQ_v(a)} a, b} \; . \qquad (6.3.27)$$

(Since the three currents and charges are dependent this is a somewhat redundant notation.) The solution $m = 1$ satisfies condition (a) if and only if the level is even. This implies immediately that all three currents J_s, J_v and J_c have integer spin, and we can write the transformation of S in the following symmetric way:

$$\sigma S_{ab} = S_{J_s^{2Q_s(a)} J_c^{2Q_c(a)} J_v^{2Q_v(a)} a, b} \; . \qquad (6.3.28)$$

Since $Q_s + Q_c + Q_v = 0 \bmod 1$ for any weight a, at least one of the charges, say Q_v, must vanish. Then $Q_s = Q_c \bmod 1$, and the field a is transformed to $J_s^{2Q_s(a)} J_c^{2Q_s(a)} a = J_v^{2Q_s(a)} a$. Since J_v has integral spin and $Q_v(a) = 0$, this field has the same conformal weight as a, and hence T-invariance is respected. Due to the symmetry in s, c and v the same is true for any other field as well. Thus we do find an infinite series of modular invariant partition functions. These are automorphism invariants, again with both a Galois and a simple current interpretation, although this time they are due to simple currents of integer spin. Invariants of this type have been described before in [122].

6.3.3 Chiral Algebra Extensions

Now we will examine what happens if we relax condition (b), i.e. we will consider the case that the replacement of $\mathcal{Q}$ by Q leads to a different answer. This obviously requires that ρ is not on the root lattice, and that the extra terms (6.3.10) are non-integral for some values of Q. The latter is true if m is odd, or if the algebra is B_n, n odd, and $m = 2$ ($\tilde{m} = 1$). Now we can write (omitting for the moment the case D_n, n even)

$$\begin{aligned} \sigma S_{ab} &= \mathrm{e}^{-2\pi i lmN[Q(a)+\frac{1}{2}][Q(b)+\frac{1}{2}]} S_{ab} \\ &= \mathrm{e}^{-\pi i lmN[Q(a)+\frac{1}{2}]} S_{J^{-lmN[Q(a)+\frac{1}{2}]}a\,,\,b} \end{aligned} \tag{6.3.29}$$

instead of (6.3.8). As before, a similar formula holds also for B_n, n odd, with m replaced by $\tilde{m} = \frac{1}{2}m$.

Since $mlNQ(a)$ is always integral and N is even, the exponential prefactor is in fact a sign, and the result may be written as

$$\sigma S_{ab} = \eta(a)\, S_{J^{-lmN[Q(a)+\frac{1}{2}]}a\,,\,b}\ . \tag{6.3.30}$$

Comparing this with (6.3.13) we find now that $S_{a,b} = \omega(a) S_{\tau a,b}$, where ω is the product of the overall signs η and ϵ, and $\tau a = J^{-lmN[Q(a)+\frac{1}{2}]}\dot{\sigma} a$. Unitarity of S now gives $\delta_{a,\tau a} = \sum_b S_{\tau a,b} S^*_{ba} = \sum_b \omega(a) S_{ab} S^*_{ba} = \omega(a)$, which implies that $\omega = 1$, i.e. $\eta = \epsilon$, and that τ is the trivial map.

Also in this case the Galois transformation generates an automorphism that lies within simple current orbits, and hence if it generates a positive modular invariant, it must be a simple current invariant. The identity is not fixed in this case: it must thus be mapped to a simple current. The candidate modular invariant has the form $P = \mathbf{1} + \eta(0)\Pi$, where Π is the matrix representing the transformation (6.3.30).

Galois automorphisms of this type always have orbits with positive and negative signs. A positive invariant can only be obtained if the negative sign orbits are in fact fixed points of the Galois automorphism (these should not be confused with fixed points of the simple current!). One sees immediately from (6.3.29) that the sign $\eta(a)$ is opposite for fields of charge $Q(a) = 0$ and $Q(a) = \frac{1}{N}$. Since the former includes the identity we fix that sign to be positive. Hence the orbits of charge $\frac{1}{N}$ must be fixed points. This leads to the condition

$$-\,lmN[\frac{1}{N} + \frac{1}{2}] = 0 \bmod N\ , \tag{6.3.31}$$

or, writing $N = 2N'$, $lm(N'+1) = 0 \bmod 2N'$. From this we conclude that N' must be odd and lm must be a multiple of $N' = N/2$.

We are now in the familiar situation of an extension by a simple current of order 2, and clearly T-invariance will then require this current to have integral spin. The solutions can now easily be listed:

$$\begin{array}{lll} A_{4l+1}, & \text{level } 4j & (l, j \in \mathbb{Z})\,, \\ B_{2l+1}, & \text{level } 2j & (l, j \in \mathbb{Z})\,, \\ E_7, & \text{level } 4j & (j \in \mathbb{Z})\,. \end{array} \tag{6.3.32}$$

Now consider D_n for even n. Then ρ is not an element of the root lattice, but a vector weight if $n = 2 \bmod 4$. Hence $Q_s(\rho) = Q_c(\rho) = \frac{1}{2}$ and $Q_v(\rho) = 0$. The transformation of S is now

$$\sigma S_{ab} = e^{2\pi i[Q_s(a)+Q_c(a)]} S_{J_s^{2[Q_s(a)+\frac{1}{2}]} J_c^{2[Q_c(a)+\frac{1}{2}]} a, b} , \qquad (6.3.33)$$

where we set $m = 1$, the only acceptable value. It is not hard to see that the resulting S-invariant cannot be a positive one, since there do exist wrong-sign Galois orbits that are not fixed points.

There are several simple current extensions that cannot be obtained from Galois symmetry, at least not in the way described here. Since we considered here only a single Galois scaling, only Galois automorphisms of order 2 can give us a positive modular invariant [47] (this is also true for the automorphism invariants discussed earlier in this section, as one may verify explicitly). Hence there is *a priori* no chance to obtain extensions by more than one simple current. However, some simple currents of order 2 are missing as well, namely those generated by the current J^{2l} of A_{4l-1}, the current J of B_l, l even, and the currents J_v of D_l and J_s, J_c of D_{2l}, with levels chosen so that these currents have integer spin. Note that the existence of a modular invariant of order two implies the existence of a "Galois-like" automorphism. This may suggest the existence of some generalization of Galois symmetry that would also explain those invariants.

6.4 New Infinite Series

In this section we will describe several infinite series of exceptional invariants that we obtained from Galois symmetry. They occur for algebras of type B and D at level 2 and certain values of the rank. Let us start the discussion with type B, which is slightly simpler.

The new invariants occur for the algebras B_7, B_{10}, B_{16}, B_{17}, B_{19}, B_{22} etc., always at level 2. The pattern of the relevant ranks n becomes clear when we consider the number $2n + 1$, corresponding to the identity $B_n = \mathrm{so}(2n + 1)$; namely, $2n + 1$ must have at least two distinct prime factors. For example, for so(15) at level 2 we find the following three non-diagonal modular invariants:

$$\mathcal{P}_1 = |\mathcal{X}_0 + \mathcal{X}_1|^2 + 2\,(\,|\mathcal{X}_4|^2 + |\mathcal{X}_5|^2 + |\mathcal{X}_6|^2 + |\mathcal{X}_7|^2 + |\mathcal{X}_8|^2 + |\mathcal{X}_9|^2 + |\mathcal{X}_{10}|^2\,)\,, \qquad (6.4.1)$$

$$\mathcal{P}_2 = |\mathcal{X}_0|^2 + |\mathcal{X}_1|^2 + |\mathcal{X}_2|^2 + |\mathcal{X}_3|^2 + |\mathcal{X}_5|^2 + |\mathcal{X}_6|^2 + |\mathcal{X}_8|^2 + (\mathcal{X}_4\mathcal{X}_9^c + \mathcal{X}_7\mathcal{X}_{10}^c + \text{c.c.})\,, \quad (6.4.2)$$

$$\mathcal{P}_3 = |\mathcal{X}_0 + \mathcal{X}_1|^2 + |\mathcal{X}_4 + \mathcal{X}_9|^2 + |\mathcal{X}_7 + \mathcal{X}_{10}|^2 + 2\,(\,|\mathcal{X}_5|^2 + |\mathcal{X}_6|^2 + |\mathcal{X}_8|^2\,)\,. \qquad (6.4.3)$$

Here the labels $i = 1, 2 \ldots 10$ of $\mathcal{X}_i$ denote the following representations:

$$\begin{array}{llll} 0: & (0,0,0,0,0,0,0) & 6: & (0,0,0,0,1,0,0) \\ 1: & (2,0,0,0,0,0,0) & 7: & (0,0,0,1,0,0,0) \\ 2: & (0,0,0,0,0,0,1) & 8: & (0,0,1,0,0,0,0) \\ 3: & (1,0,0,0,0,0,1) & 9: & (0,1,0,0,0,0,0) \\ 4: & (0,0,0,0,0,0,2) & 10: & (1,0,0,0,0,0,0) \\ 5: & (0,0,0,0,0,1,0) & & \end{array} \qquad (6.4.4)$$

The first of these invariants is not new: it corresponds to the conformal embedding so(15) $\subset$ su(15). The fields $i = 4 \ldots 10$ are fixed points, each of which is resolved into two distinct complex conjugate fields in the extended algebra. In su(15) the two fields originating from the so(15) field i are the antisymmetric tensor representations $[4 + i]$ and $[11 - i]$. The invariant $\mathcal{P}_1$ is in fact an integer spin simple current invariant. The other two B_7 invariants are manifestly not simple current invariants.

The second B_7 invariant is new, as far as we know, and can be explained in the following way. The algebra A_{14} at level 1 has three distinct automorphism invariants which are generated by the simple currents J, J^3 and J^5. They read

$$\sum_{i=0}^{14} \mathcal{X}_i \mathcal{X}^*_{-i}\,, \qquad \sum_{i=0}^{14} \mathcal{X}_i \mathcal{X}^*_{-11i}\,, \qquad \sum_{i=0}^{14} \mathcal{X}_i \mathcal{X}^*_{-4i}\,, \tag{6.4.5}$$

respectively, where the labels are defined modulo 15. The first one is equal to the charge conjugation invariant, and the last one is the "product" of the first two. The existence of an $A_{14,1}$ automorphism implies relations among the matrix elements of the modular matrix S of that algebra. Owing to the existence of the conformal embedding $B_{7,2} \subset A_{14,1}$, these matrix elements are related to those of $B_{7,2}$. The precise relation is

$$\begin{aligned} & S_{00}[A_{14,1}] = 2S_{00}[B_{7,2}]\ , \\ & S_{0,4+i}[A_{14,1}] = S_{0,11-i}[A_{14,1}] = S_{0,i}[B_{7,2}]\ , \\ & S_{4+i,4+j}[A_{14,1}] = S_{11-i,11-j}[A_{14,1}] \\ = \ & S^*_{4+i,11-j}[A_{14,1}] = S^*_{11-i,4+j}[A_{14,1}] = \tfrac{1}{2}S_{ij}[B_{7,2}] + \mathrm{i}\Sigma_{ij}\ . \end{aligned} \tag{6.4.6}$$

Here Σ denotes the fixed point resolution matrix. The first automorphism, charge conjugation, just sends i to $-i$ and hence acts trivially on the $B_{7,2}$ fields. The other two su(15) automorphisms interchange the $B_{7,2}$ fields $(4,9)$ and $(7,10)$, leaving $5,6$ and 8 fixed (in addition one gets relations from the imaginary part on the matrix elements of Σ). This implies relations like $S_{0,4} = S_{0,9}$ and $S_{4,7} = S_{9,10}$ for the $B_{7,2}$ matrix elements. All these relations hold also if the label 0 is replaced by 1, but we do not get any relations for matrix elements involving the fields that are projected out, i.e. the fields 2 and 3. In the general case, the absence of relations involving fields that get projected out implies that the automorphisms of an algebra g do not lead to automorphisms for a conformal subalgebra h $\subset$ g. The present case is an exception, since all the fields on which the automorphism acts (and in fact all the fields with labels $4,\ldots,10$) are fixed points of the $B_{7,2}$ simple current that extends the algebra. Then the matrix elements $S_{2,i}$ and $S_{3,i}$ vanish for $i = 4,\ldots,10$ and we need no further relations among them.

This explains the presence of the second invariant listed above. The third one is a linear combination of the foregoing ones and the diagonal invariant: $\mathcal{P}_3 = \mathcal{P}_1 + \mathcal{P}_2 - \mathbf{1}$. This is a remarkable invariant: it looks like a normal extension by a spin 1 current, but it does not follow from any conformal embedding. The only conformal embedding of B_7 at level 2 is in su(15), and the corresponding invariant is $\mathcal{P}_1$, not $\mathcal{P}_3$. This implies in particular that there cannot exist any conformal field theory corresponding to the modular invariant $\mathcal{P}_3$! In fact, it is not even possible to write down a fusion algebra for this invariant, because there does not exist a fixed point resolution matrix. In [129] another example of this kind was described, although that theory was unphysical for a somewhat different reason.

The existence of $\mathcal{P}_3$ can also be seen as a consequence of the closure of the set of Galois automorphisms. Each Galois modular invariant, automorphism invariants as well as chiral algebra extensions, originates from a Galois symmetry of S, which acts on the fields as a permutation accompanied by sign flips. For the "chiral extension" $\mathcal{P}_3$ this Galois automorphism is represented by the matrix $\mathcal{P}_3 - \mathbb{1}$. This set of Galois automorphisms will always close as a group. Indeed, the automorphism underlying $\mathcal{P}_3$ is simply the product of that of $\mathcal{P}_1$ and $\mathcal{P}_2$.

By the same arguments there will be pure automorphism invariants for $B_{n,2}$ whenever $2n+1$ contains at least two different prime factors. The spin-1 extension always involves an identity block plus n fixed points that yield each two su$(2n+1)$ level 1 fields (this is true since all non-trivial representations of su$(2n+1)$ are complex). If there is only one prime factor the only automorphism is charge conjugation, which acts trivially. When there are K different prime factors there are 2^K distinct pure Galois automorphisms for su$(2n+1)$ at level 1, including the identity and the charge conjugation invariant. When "projected down" to $B_{n,2}$ these are related in pairs by charge conjugation, and we expect therefore 2^{K-1} distinct $B_{n,2}$ modular invariants of automorphism type. In addition there is of course the invariant corresponding to the conformal embedding in su$(2n+1)$ itself. In combination with the $2^{K-1}-1$ *non-trivial* automorphisms this extension gives rise to as many other invariants that look like conformal embeddings, but actually do not correspond to a consistent conformal field theory.

How does this come out in terms of Galois symmetry? First of all the spin-1 extension of the conformal embedding is in fact a simple current extension, and we have seen in the previous section that it follows from Galois symmetry only for B_n with n odd. If n is odd the Galois periodicity is $4(2n+1)$ for $B_{n,2}$ and $2(n+1)(2n+1)$ for $A_{2n,1}$. Hence the cyclotomic field of the former is contained in that of the latter, so that all Galois transformations of $A_{2n,1}$ have a well-defined action on the modular matrix S of $B_{n,2}$. In this case we may thus expect 2^K distinct Galois modular invariants, including the identity and the unphysical invariants described above. If n is even the Galois periodicities are respectively $2(2n+1)$ and $2(n+1)(2n+1)$, so that also in this case all Galois transformations are well-defined on B_n. But due to the fact that the simple current invariant is not a Galois invariant, we get only half the number of invariants now, namely 2^{K-1}.

For n odd the su$(2n+1)$ simple current automorphisms are mapped to two B_n modular invariants: one physical automorphism and one chiral extension, which (except for the one originating from the diagonal invariant, i.e. the conformal embedding invariant) is unphysical. For n even each su$(2n+1)$ automorphism is mapped to just one B_n invariant. The diagonal invariant is mapped to the diagonal one of B_n, but it turns out that the non-trivial automorphisms are mapped to either a pure automorphism or an unphysical chiral extension, in such a way that the closure of the set of Galois automorphisms is respected.

Now consider algebras of type D. Again the crucial ingredient is the conformal embedding so$(2n)_2 \subset$ su$(2n)_1$. In terms of D_n fields the su$(2n)$ characters are built as follows: The identity character is the combination $\mathcal{X}_0+\mathcal{X}_v$ and the antisymmetric tensor $[n]$ has a character equal to $\mathcal{X}_s+\mathcal{X}_c$. All other su$(2n)$ representations are complex, and each pair of complex conjugate representations arises from a resolved fixed point of the vector current of D_n. Even though D_n has complex representations itself for n odd, these get projected out, and all the non-real contributions to the su(n) modular matrix S arise from fixed point resolution.

The center of the su$(2n)$ WZW theory is $\mathbb{Z}_{2n}$, but the 'effective center' is $\mathbb{Z}_n$. This means that only the simple current J^2 of the su$(2n)$ theory yields non-trivial modular invariants, and that the order $2n$ current J may be ignored. It is easy to see that the field $[n]$ has zero charge with respect to J^2, so that it is mapped onto itself by any automorphism generated by powers of J^2. This implies that, just as before, all su$(2n)$ simple current automorphisms act non-trivially only on resolved fixed points, and hence can be 'projected down' to D_n. If n is prime, then the only automorphism is equivalent to charge conjugation, and hence it projects down to the trivial invariant. Hence just as before we will get non-trivial D_n automorphisms whenever n contains at least two distinct prime factors, where the prime is now allowed to be two. The counting of invariants is the same as for $B_{(n-1)/2}$ above. Again they come in pairs: an automorphism and an unphysical extension by a spin-1 current.

All these invariants exist, but not all of them follow from Galois theory. Just as for B_n, the automorphism invariants do, but the conformal embedding invariant does not always follow. In fact, it never comes out as a result of the scalings discussed in the previous section. However, if $n = 3 \bmod 4$ the simple current extension by the current J_v is an exceptional Galois invariant only at level 2 (see the Table 6.1). In that case all the expected invariants are Galois invariants. For all other values of n only half of the expected invariants are Galois invariants, and from each pair only one member appears, either the automorphism or the unphysical extension.

There is still one interesting observation to be made here. If there are just two distinct prime factors, and $n = 6 \bmod 8$, then the extra invariant is an unphysical extension. Remarkably, however, that extension is a simple current invariant. It is equal to the extension by J_v, but it has additional terms of the form $|\mathcal{X}_a + \mathcal{X}_b|^2$, where a and b are fields that appear diagonally, as fixed points of order 2, in the normal simple current invariant. The fields a and b are however on the same orbit with respect to the current J_s, which makes this a simple current invariant by definition. Nevertheless, it is not part of the classification presented in [63], because that classification was obtained under a specific regularity condition on the matrix S that is not satisfied here (indeed, D_{2n} at level 2 was explicitly mentioned as an exception in the appendix of [63]; the reason for it being an exception is that all orbits except for the identity field are fixed points of one or all currents). It also follows that this simple current invariant cannot be obtained using orbifolds with discrete torsion, unlike the simple current invariants within the classification of [100]. Hence the fact that it is unphysical is not in contradiction with the expectation that simple current invariants should normally be physical.

In the previous case the automorphism would be obtained by subtracting the normal spin-1 extension, and adding the identity matrix. Clearly the resulting automorphism is not really exceptional, but is simply the automorphism generated by the spinor simple current J_s (or J_c, which at level 2 gives the same result). The same happens if the rank is 2 mod 8, except that in that case the automorphism comes out directly as a Galois invariant. It is listed in Table 6.1. To get really new automorphisms that are not simple current invariants for $n = 2 \bmod 8$ or $n = 6 \bmod 8$ one has to consider cases where n contains three or more distinct prime factors. Finally, if the rank is divisible by 4 the spinor currents have integer spin, and do not interfere with the exceptional automorphisms discussed in this section.

6.5 Pure Galois Invariants

Here we list all the remaining Galois invariants of simple WZW models, i.e. not including those described in the previous sections. All these invariants are positive and result directly from a single Galois automorphism of order 2. Although the full Galois commutant was investigated, in all but one case there is only a single non-trivial orbit contributing (in terms of the formula (5.5.6) this means that f_0 is used to get $P_{00} = 1$, and apart from f_0 only one other coefficient f_σ is non-zero.) The exception is the E_8-type invariant of A_1 at level 28, which can also be interpreted as a combined simple current/Galois invariant, and which is therefore included in Table 6.2. The results are listed in Table 6.1. The notation is as follows:

- CE: Conformal embedding.

- $S(J)$: Simple current invariant. The argument of S is the simple current responsible for the invariant.

- RLD: Rank-Level Dual. The S-matrices of $\mathrm{su}(N)_k$, $\mathrm{so}(N)_k$ and $C_{n,k}$ are related to those of respectively $\mathrm{su}(k)_N$, $\mathrm{so}(k)_N$ and $C_{k,n}$ by level-rank duality. One might expect that Galois transformations of one matrix are mapped to similar transformations of the other. The relation is not quite that straightforward however, and we will not examine the details here. The results clearly respect this duality.

- EA: Exceptional Automorphism. These are modular invariants of pure automorphism type that are not due to simple currents. The only invariants of this type known so far were found in [139], and appear also in Table 6.1.

- HSE: Higher Spin Extension, an extension of the chiral algebra by currents of spin larger than 1 that are not simple currents. Some of these invariants can be predicted using level-rank duality; all other known ones are related to meromorphic $c = 24$ theories [124].

Note that there are some simple current invariants in this list. This is not in conflict with the results of Section 6.3, as we did not claim that the list given there was complete. The scales of the Galois transformations for which these simple current invariants are obtained are interesting. For A_{4m-1} and D_{4m+3} these scales are equal respectively to $(2m+1)(k+g^\vee)-1$ and $3(k+g^\vee)-1$. If the contribution -1 were replaced by $+1$, they would be of the kind discussed in Section 6.3. In fact we can write these scales as $(-1)[(2m-1)(k+g^\vee)+1] \bmod 4m(k+g^\vee)$ and $(-1)[(k+g^\vee)+1] \bmod 4(k+g^\vee)$, respectively, which shows that these Galois automorphisms are nothing but the product of a scaling of the type discussed in Section 6.3 and charge conjugation. It can be checked that without the charge conjugation one does not get a positive invariant: certain fields are transformed to their charge conjugate with a sign flip. After multiplying with the charge conjugation automorphism these fields become fixed points. The scale factor for C_{4m}, $4m+3$, is of the form $(k+g^\vee)+1$, but for C_n the arguments of Section 6.3 break down right from the start, so that no conclusions can be drawn for this case. For the other simple current invariants the scale factor does not have the right form, and hence the arguments of Section 6.3 simply do not apply.

6.6 Combination of Galois and Simple Current Symmetries

In Section 6.3 we have discussed a large set of invariants for which the Galois and simple current methods overlap. If they do not overlap, it may be fruitful to combine them. To do so we first have to understand how the orbit structures of both symmetries are interfering with each other. This can be seen by computing $\sigma S_{Ja,b}$. On the one hand, this is equal to

$$\sigma S_{Ja,b} = \epsilon_\sigma(Ja)\, S_{\dot\sigma Ja,b}\ . \qquad (6.6.1)$$

On the other hand, it is equal to

$$\begin{aligned}\sigma[\mathrm{e}^{2\pi\mathrm{i}Q(b)}\, S_{ab}] &= \mathrm{e}^{2\pi\mathrm{i}lQ(b)}\epsilon_\sigma(a)\, S_{\dot\sigma a,b} \\ &= \epsilon_\sigma(a)\, S_{J^l\dot\sigma a,b}\ .\end{aligned} \qquad (6.6.2)$$

Algebra	level	Galois scaling	Type	Interpretation
A_2	5	19	Extension	CE $\subset A_5$
A_{4m-1}	2	$8m^2+8m+1$	Extension	$S(J^{2m})$; RLD of $A_{1,4m}$
A_4	3	11	Extension	CE $\subset A_9$
A_9	2	31	Extension	RLD of $A_{1,10}$
C_{4m}	1	$4m+3$	Extension	$S(J)$; RLD of $C_{1,4m}=A_{1,4m}$
D_{8m+2}	2	$8m+1$	Automorphism	$S(J_s)$
D_{4m+3}	2	$24m+17$	Extension	$S(J_v)$
D_7	3	49	Extension	HSE; RLD of $\mathrm{so}(3)_{14}=A_{1,28}$
G_2	3	8	Extension	CE $\subset E_6$
G_2	4	5	Automorphism	EA
G_2	4	11	Extension	CE $\subset D_7$
F_4	3	5	Extension	CE $\subset D_{13}$
F_4	3	11	Automorphism	EA
E_6	4	7	Extension	HSE
E_7	3	13	Extension	HSE

Table 6.1: Pure Galois modular invariants for WZW models.

Here l is the power to which σ raises the generator of the cyclotomic field. In the first step we used that the simple current phase factor is contained in the field M, which follows from $\mathrm{e}^{2\pi\mathrm{i}Q(b)}=S_{Ja,b}/S_{ab}\in M$. Using unitarity of S we then find that

$$\begin{aligned}\epsilon_\sigma(Ja) &= \epsilon_\sigma(a)\,,\\ \dot\sigma J &= J^l\dot\sigma\ .\end{aligned} \tag{6.6.3}$$

Here J denotes the permutation of the fields that is generated by the simple current J. Since l is prime with respect to the order of the cyclotomic field, it is – at least in the case of WZW models – also prime with respect to the order N of the simple current. If $N=2$ this means that l must be odd so that $J^l=J$, and hence we conclude that $\dot\sigma$ and J commute. For all other values of N they do not commute unless $l=1$ mod N, but at least it is true that $\dot\sigma$ maps simple current orbits to simple current orbits, and furthermore it respects the orbit length.

If $N=2$ the simple currents yield the relation

$$S_{Ja,Jb}=\mathrm{e}^{2\pi\mathrm{i}(Q(a)+Q(b)+\frac{r}{2})}S_{ab} \tag{6.6.4}$$

among matrix elements of S, where r is the monodromy parameter. If r is even (which is the case for simple currents of integer or half-integer spin) this relation takes the form

$$S_{ab}=\epsilon(a)\epsilon(b)S_{Ja,Jb}\ , \tag{6.6.5}$$

since the phase factors are in fact signs. This is precisely the form of a Galois symmetry, as expressed in (5.5.2). We can represent this symmetry in matrix notation as

$$\Pi_J S\Pi_J=S\ , \tag{6.6.6}$$

where $\Pi_J=(\Pi_J)^{-1}$ is an orthogonal matrix that commutes with the analogous matrices representing the Galois group. Hence we can extend the Galois group by this transformation as explained in Chapter 5. Furthermore if $r=2$ mod 4 the simple current invariant produced by J is a fusion rule automorphism that can also be used to extend the Galois group.

We have not examined these extended Galois-like symmetries systematically, but we will illustrate that new invariants can be found by giving one example. Consider A_1 at level 10. One of the Galois invariants (invariant under S as well as T) is

$$\mathcal{P}_1 = |\mathcal{X}_0 + \mathcal{X}_6|^2 + |\mathcal{X}_4 + \mathcal{X}_{10}|^2 + |\mathcal{X}_1 - \mathcal{X}_9|^2 + 2|\mathcal{X}_3|^2 + 2|\mathcal{X}_7|^2 \ , \tag{6.6.7}$$

where the indices are the highest weights (in the Dynkin basis). The only problem with this invariant is that it is not positive. However, at level 10 we also have the D-type invariant

$$\mathcal{P}_2 = |\mathcal{X}_0|^2 + (\mathcal{X}_1\mathcal{X}_9^* + \mathcal{X}_3\mathcal{X}_7^* + \text{c.c.}) + |\mathcal{X}_2|^2 + |\mathcal{X}_4|^2 + |\mathcal{X}_5|^2 + |\mathcal{X}_6|^2 + |\mathcal{X}_8|^2 + |\mathcal{X}_{10}|^2 \ , \tag{6.6.8}$$

which is a simple current automorphism. If we now take the linear combination

$$\mathcal{P}_1 + \mathcal{P}_2 - \mathbb{1} \,, \tag{6.6.9}$$

we get a positive modular invariant which is in fact the well-known E_6-type invariant.

There is a second way of combining simple currents and Galois symmetries. One can extend the chiral algebra of the WZW model by integer spin simple currents. This projects out some of the fields, so that the negative sign Galois orbits of some Galois invariants are removed. It is essential that the Galois automorphisms respect the simple current orbits, and that the matrix elements of S are constant on these orbits for the fields that are not projected out. The simple current extension has its own S-matrix which can be derived partly from that of the original theory. This matrix has the form (1.5.7)

$$\tilde{S}_{a_i,b_j} = \frac{N_a N_b}{N} S_{ab} + \Sigma_{ab} P_{ij} \ . \tag{6.6.10}$$

All general considerations regarding Galois transformations can be applied directly to this new S-matrix. Clearly the matrix elements S_{ab} which correspond to the primary fields of the original theory that are not projected out belong to a number field M' which is contained in the number field M of the original theory. While $P_{ij} = \delta_{ij} - \frac{1}{N}$ and $N_a N_b/N$ are both rational and hence transform trivially under $\mathcal{Gal}(M'/\mathbb{Q})$, the presence of the matrix Σ_{ab} in (6.6.10) may require this number field to be extended to a field $\tilde{M}' \supset M'$ (a simple example is provided by the $A_{1,4}$ WZW theory, which has a real matrix S, whereas the S-matrix of the extended algebra $A_{2,1}$ is complex). Now because of the projections $\tilde{M}'$ does not necessarily contain the original number field M; however, at the possible price of redundancies we can consider an even larger number field $\tilde{M}$ that contains both $\tilde{M}'$ and M. When working with $\tilde{M}$, we do not loose any of the Galois transformations that act non-trivially on the surviving matrix elements S_{ab}. Note that any element of $\mathcal{Gal}(\tilde{M}/M)$ acts trivially on S_{ab} and hence induces a permutation which leaves non-fixed points invariant and acts completely within the set of primary fields into which a fixed point gets resolved. Further, for any element of $\mathcal{Gal}(\tilde{M}/\mathbb{Q})$ the associated permutation must act on the labels a, b in the same way in both terms on the right hand side of (6.6.10). In particular, for any matrix element involving only non-fixed points the action of a Galois transformation on S already determines its action on $\tilde{S}$, since the two matrix elements are equal up to a rational factor. The same is true for all matrix elements between fixed points and full orbits, since in that case Σ is absent, too. This is often already enough information to determine the Galois orbits of the extended theory completely. The transformations of the fixed point - fixed point elements of $\tilde{S}$ are more subtle, and in principle would require

Algebra	level	Galois scaling	Simple current	Type	Interpretation
A_1	10	7	J (†)	Extension	CE $\subset B_2$
A_1	28	11	J	Extension	CE $\subset G_2$
A_2	9	17	J	Extension	CE $\subset E_6$
A_2	21	35×53 (*)	J	Extension	CE $\subset E_7$
A_3	8	7	J	Extension	CE $\subset D_{10}$
A_7	4	7	J^2	Extension	HSE; RLD of $A_{3,8}$
A_7	$4m+2$	$4m+11$	J^4	Aut $\times$ Ext	$S(J^2)$
A_{27}	2	71	J^{14}	Extension	HSE; RLD of $A_{1,28}$
C_3	4	7	J	Extension	CE $\subset B_{10}$
C_4	3	7	J	Extension	RLD of $C_{3,4}$
D_{4m+2}	$4l$	$8m+4l+3$	J_s	Extension	$S(J_v) \star S(J_s)$
D_4	6	5	J_s, J_v	Extension	CE $\subset D_{14}$

Table 6.2: Modular invariants of WZW models obtained by combination of Galois and simple current symmetries.
(†) This is a simple current of half-integer spin; see the main text for details.
(*) Invariant originating from a non-cyclic subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ of the Galois group.

knowledge of the matrix Σ. However, as already pointed out any element of the Galois group must act on Σ exactly as it does on S. Although this still leaves undetermined the action within the set of primary fields into which the relevant fixed point is resolved, this limited information nevertheless can provide useful additional information on the matrix Σ, whose determination in general is a problem that is far from being solved.

Fortunately, as long as we are only interested in modular invariants of the original theory, we may in fact ignore fixed point resolution completely. By definition that issue is determined solely by S (and T), and the precise form of Σ should not matter.

We have performed a computer search for invariants of the type described above, and obtained the results shown in Table 6.2. Note that this table contains a few infinite series of simple current invariants. Since they were inferred from a finite computer scan, the statement that the series continues is a conjecture. Presumably these series can also be derived by arguments similar to those in Section 6.3, but we have not pursued this.

We have in principle just looked for invariants originating from single orbits, but there is one exception, namely the modular invariant of A_2 at level 21. This invariant is obtained as a sum over a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of the Galois group that is generated by the two scalings indicated in Table 6.2. Separately each of these scalings yields an S, T invariant with a few minus signs.

6.7 Conclusions

To conclude, let us make a rough comparison between the various methods for constructing modular invariants that were mentioned in the introduction. We will compare them on the basis of the following aspects.

- **Generality**
  A common property of simple currents and Galois symmetry is that neither is *a priori* restricted to WZW models, unlike all other methods. (In practice this is less

important than it may seem, since essentially all RCFT's we know are WZW models or WZW-related coset theories.)

- **Positivity**
  Most methods do not directly imply the existence of positive modular invariants, but rather they yield generating elements of the commutant of S and T that have to be linearly combined to get a positive invariant; the exceptions are simple currents, conformal embeddings and level-rank duality.

- **Existence of a CFT**
  It should be emphasized that a positive modular invariant partition function is only a necessary condition for a consistent conformal field theory. Most methods do not guarantee that a conformal field theory exists. Exceptions are conformal embeddings (the new CFT is itself a WZW model) and probably simple current invariants, since the construction of the new theory can be rephrased in orbifold language. Clearly any construction that may yield negative invariants cannot guarantee existence of the theory, and this includes Galois invariants. Indeed, we found examples of positive Galois modular invariants that cannot correspond to any sensible CFT.

- **Explicit construction**
  Simple current invariants can be constructed easily and straightforwardly. On the other hand, the explicit construction of an invariant corresponding to a conformal embedding is usually extremely tedious. Indeed, many of these invariants are not known explicitly. The other methods fall somewhere between these two extremes. The explicit construction of a Galois invariant is straightforward but requires long excursions through the Weyl group, as explained in the appendix.

- **Classification**
  All simple current invariants have been classified in [62, 63] and [100], under a mild regularity assumption for S, which, as we have seen in Section 6.4, is not always satisfied. The simple currents of WZW models were classified in [43]. All conformal embeddings have been classified in [9,125] All cases of level-rank duality are presumably known, but all other methods mentioned in the introduction have only been applied to a limited number of cases, without claims of completeness. Our results on Galois invariants are based partly on computer searches (inevitably restricted to low levels) and partly on rigorous derivations (Section 6.3). For the pure Galois invariants we expect our results to be complete, but we have no proof.

To summarize, we find that the Galois construction does not yield all solutions, but also that it is not contained in any of the previously known methods. It generates invariants of all known types. Most of the partition functions we found were already known in the literature, but we did find several new infinite series of pure automorphism invariants not due to simple currents.

In the course of this investigation we realized that the restriction that the scaling be prime with respect to $M(k + g^{\vee})$ can in fact be dropped, at least for WZW models. This yields even more relations among elements of S, which take the form of sum rules, and hence even more information about modular invariants. These transformations, which we call 'Quasi-Galois' symmetries, will be discussed in the next chapter.

Chapter 7

Quasi-Galois Symmetries of the Modular S-Matrix

The Galois symmetries of rational conformal field theory introduced in Chapter 5 are generalized, for the case of WZW theories, to 'quasi-Galois symmetries'. These symmetries can be used to derive a large number of equalities and sum rules for entries of the modular matrix S, including some that previously had been observed empirically. In addition, quasi-Galois symmetries allow to construct modular invariants and to relate S-matrices as well as modular invariants at different levels. They also lead us to a convenient closed expression for the branching rules of the conformal embeddings $\mathsf{g} \hookrightarrow \widehat{\mathrm{so}}(\dim \bar{\mathsf{g}})$.

7.1 Introduction

In the study of rational conformal field theories, modular transformations play an essential role. They turn the set of the characters of all primary fields into a unitary module of $SL(2,\mathbb{Z})$, the twofold covering of the modular group of the torus. Via the Verlinde formula, they are also closely related to the fusion rules.

In all cases where the modular matrix S is explicitly known, one observes that it contains surprisingly few different numbers, and that among the distinct numbers there are linear relations. While it has been known for a long time that simple currents lead to relations between individual S-matrix elements [126,130,86], many other relations, in particular sum rules, have remained so far somewhat mysterious. Recently it has become clear that Galois symmetries [26,23] are an independent source for relations between individual elements of S [47,51]. Both simple current and Galois symmetries exist for arbitrary rational conformal field theories, independent of the structure of the chiral algebra.

In this chapter we will show that in the special case of WZW theories, Galois symmetries can be generalized to what we will call *quasi-Galois symmetries*. A crucial ingredient of our construction (which is not available for other conformal field theories than WZW theories) is the Kac–Peterson formula for the S-matrix. These new symmetries turn out to be rather powerful and allow to derive three new types of relations between the entries of S: first, a sum rule which relates signed sums of S-matrix elements, see (7.3.4); second, the equality, modulo signs, of certain specific S-matrix elements, see (7.4.1); third, a new systematic reason for S-matrix elements to vanish, see the remarks after (7.2.9).

Just as in the case of Galois symmetries, the relations we find can be employed to construct elements of the commutant of S, and therefore to generate modular invariants. Moreover, they can be used to obtain relations between invariants at different values of the level, i.e. between different WZW theories. Finally, we show that our results allow to determine the branching rules of certain conformal embeddings.

The rest of this chapter is organized as follows. In Section 7.2 we recall the basic facts about Galois symmetries of rational conformal field theories, and of WZW theories in particular, and show how in the WZW case they can be generalized to quasi-Galois symmetries. Also, as a first application, we describe how these symmetries force certain S-matrix elements to vanish. In Section 7.3 we construct integral-valued matrices that commute with the S-matrix; as a by-product we obtain an interesting sum rule (7.3.4) for the entries of S. In Section 7.4 we obtain another symmetry, (7.4.1), of S as well as relations (see (7.4.8), (7.4.9)) between the S-matrices for WZW theories at different heights h_1, h_2, where h_1 is a multiple of h_2. Again, these results lead to a prescription for constructing S-matrix invariants, now both at the smaller and at the larger height (see (7.4.16) and (7.4.20), respectively). Finally, in Section 7.5 we consider a special case of the latter invariants, which leads us to a closed formula for the branching rules of the conformal embeddings $\mathbf{g} \hookrightarrow \widehat{\mathrm{so}}(\dim \bar{\mathbf{g}})$ which can easily be evaluated explicitly.

7.2 Quasi-Galois scalings

When analyzing the mathematical structure of a WZW theory, we are dealing with integrable highest weight representations of an untwisted affine Lie algebra g at a fixed integral level $k^\vee$. As the level is fixed, the g-weights are already fully determined by their horizontal part, i.e. by the weight with respect to the horizontal subalgebra $\bar{\mathbf{g}}$ of $\mathbf{g}$. In the following it will be convenient to shift all weights according to $a \mathrel{\hat=} \lambda_a + \rho$ by the Weyl vector ρ. Note that if the non-shifted weight λ_a is at level $k^\vee$, the shifted weight a is at level h, where

$$h := k^\vee + g^\vee \tag{7.2.1}$$

with $g^\vee$ the dual Coxeter number of $\bar{\mathbf{g}}$; we will call h the *height* of the weight a. The set of (shifted) integrable weights of the affine Lie algebra $\mathbf{g}$ at height h is

$$P_h := \{a \in L^{\mathrm{w}} \mid 0 < a^i \le k^\vee + 1 \text{ for } i = 0, 1, \dots, r\}\,. \tag{7.2.2}$$

Here L^{w} denotes the weight lattice, i.e. the $\mathbb{Z}$-span of the fundamental weights. In other words, the weights (7.2.2) are precisely the integral weights in the interior of the dominant affine Weyl chamber at level $k^\vee + g^\vee$.

An important tool for studying the modular properties of WZW theories is the Kac–Peterson formula (1.4.8) for the modular matrix S which reads in terms of shifted weights and the height h as

$$S_{a,b} = \mathcal{N} \sum_{w\in W} \mathrm{sign}(w)\, \exp[-\tfrac{2\pi \mathrm{i}}{h}\,(w(a), b)]\,. \tag{7.2.3}$$

Here the summation is over the Weyl group W of the finite-dimensional horizontal subalgebra $\bar{\mathbf{g}}$ of $\mathbf{g}$. Some immediate consequences of this formula are the following. First, the fact that according to (7.2.3) $S_{a,b}$ depends on a and b only via the inner products $(w(a), b)$ and the identity $(w(\ell a), b) = \ell\,(w(a), b) = (w(a), \ell b)$ imply that

$$S_{\ell a,b} = S_{a,\ell b}\,; \tag{7.2.4}$$

and second, for any element $\hat{w}$ of the affine Weyl group $\hat{W}$ (i.e. the horizontal projection of the Weyl group of the affine algebra $\mathbf{g}$), one has

$$S_{\hat{w}(a),b} = \mathrm{sign}\,(\hat{w})\, S_{a,b}\,. \tag{7.2.5}$$

This implies in particular that $S_{a,b} = 0$ whenever a or b lies on the boundary of an affine Weyl chamber. Note that in (7.2.4) and (7.2.5) it is implicit that the quantity $S_{a,b}$ given by (7.2.3) can be considered also for weights which are not integrable. This is possible because we are free to take the formula (7.2.3) (which for integrable weights yields the entries of the actual S-matrix, i.e. of the matrix which realizes the modular transformation $\tau \mapsto -1/\tau$ on the characters) for arbitrary weights a, b as the definition of $S_{a,b}$. Analogously, these weights need not even be integral, and hence (7.2.4) is valid for arbitrary numbers ℓ, not just for integers.

Recall from Chapter 5 that for WZW theories a Galois transformation labeled by $\ell \in \mathbb{Z}^*_{Mh}$ [1] and induces the permutation $\Lambda \mapsto \hat{w}(\ell(\Lambda+\rho))-\rho$ of the highest weights carried by the primary WZW fields, or equivalently, the permutation

$$\dot{\sigma} \equiv \dot{\sigma}_{(\ell)}: \quad a \mapsto \dot{\sigma}a := \hat{w}_a(\ell a) \tag{7.2.6}$$

of shifted highest weights. Here $\hat{w}_a$ is an element of the affine Weyl group at level h, i.e.

$$\hat{w}_a(b) = w_a(b) + h\, t_a\,, \tag{7.2.7}$$

where w_a is some element of the finite Weyl group W and t_a some weight which belongs to the coroot lattice $L^\vee$ of $\bar{\mathfrak{g}}$. They are defined by the condition that $\hat{w}_a(\ell a) \in P_h$, which determines w_a and t_a uniquely. Substituting (7.2.6) into the formula for WZW conformal dimensions one easily obtains a condition for T-invariance, namely $\ell^2 = 1 \bmod 2Mh$ (or $\bmod\ Mh$ if all integers $M(a,a)$ are even).

The key idea in the present chapter is to allow in the transformation (7.2.6) for arbitrary integers ℓ rather than only elements of $\mathbb{Z}^*_{Mh}$. As we will show, these generalized transformations lead to interesting new information. Note that if $\ell \notin \mathbb{Z}^*_{Mh}$, then in order for the map (7.2.6) of the integrable weights to be still well-defined, we must slightly extend the prescription for the Weyl group element $\hat{w}_a$. Namely, $\hat{w}_a$ is now determined by the condition that either ℓa lies on the boundary of some affine Weyl chamber (in which case $\hat{w}_a$ can simply be taken to be the identity), or else that $\hat{w}_a(\ell a) \in P_h$. In the latter case, $\hat{w}_a$ is the unique element of $\hat{W}$ with this property, and we write

$$\mathrm{sign}\,(\hat{w}_a) = \mathrm{sign}\,(w_a) =: \epsilon_\ell(a)\,, \tag{7.2.8}$$

while in the former case we put $\epsilon_\ell(a) = 0$. While the map (7.2.6) is thus still well-defined for $\ell \notin \mathbb{Z}^*_{Mh}$, it can no longer be induced by a mapping $\zeta_{Mh} \mapsto (\zeta_{Mh})^\ell$ of the number field, and hence in particular it does no longer correspond to a Galois transformation. Nevertheless the similarity with Galois transformations is still so close that we call the map $a \mapsto \ell a$, with ℓ not coprime with Mh, a *quasi-Galois scaling* and the associated map $\dot{\sigma}$ (7.2.6) a *quasi-Galois transformation.*

For a quasi-Galois scaling there will in general exist some $a \in P_h$ for which ℓa lies on the boundary of an affine Weyl chamber, so that $\dot{\sigma}$ is not even an endomorphism of the set of integrable weights. However, in terms of WZW primary fields the latter situation corresponds to mapping the primary field with highest weight a to zero, so that $\dot{\sigma}$ can still be interpreted as a linear map on the fusion ring that is spanned by the primary fields. Moreover, this can also be translated back to the language of weights by adding to the set P_h a single element $\mathcal{B}$ which stands for the union of all boundaries of affine Weyl chambers. In this setting, the map (7.2.6) supplemented by $\dot{\sigma}(\mathcal{B}) = \mathcal{B}$ is an endomorphism of the set $P_h \cup \{\mathcal{B}\}$, though it is not any more a permutation.

[1] Actually the cyclotomic field $\mathbb{Q}(\zeta_{Mh})$ does not yet always contain the normalization $\mathcal{N}$ appearing in (7.2.3); rather, sometimes a slightly larger cyclotomic field must be used [23]. However, the permutation $\dot{\sigma}$ can already be determined from the generalized quantum dimensions, which do not depend on $\mathcal{N}$. Accordingly, the correct Galois treatment of $\mathcal{N}$ just amounts to an overall sign factor which is irrelevant for our purposes.

Consider now an arbitrary scaling $a \mapsto \ell a$, $\ell \in \mathbb{Z} \setminus \{0\}$, with associated (quasi-) Galois transformation given by (7.2.6). As follows immediately by applying the identities (7.2.4) and (7.2.5) to $S_{\dot\sigma a,b}$, we then have the identity

$$\epsilon_\ell(a)\, S_{\dot\sigma a,b} = \epsilon_\ell(b)\, S_{a,\dot\sigma b}\,. \tag{7.2.9}$$

For genuine Galois scalings, this result was already obtained in [23]. In the quasi-Galois case, the two sides of (7.2.9) are not necessarily non-vanishing, and this provides us with an explanation for the vanishing of certain S-matrix elements. Namely, if for the quasi-Galois scaling ℓ the weights b and $c := \dot\sigma a$ are contained in P_h, but $\dot\sigma b$ is not (i.e. ℓb lies on the boundary of an affine Weyl chamber), then (7.2.9) tells us that $S_{c,b} = 0$. (Another systematic reason for S-matrix elements to be zero is provided by simple current symmetries: $S_{a,b} = 0$ if a is a fixed point of the simple current J and b has non-vanishing monodromy charge [130] with respect to J.)

7.3 Quasi-Galois modular invariants

Consider for a given quasi-Galois scaling ℓ the matrix Π with entries in $\{0, \pm 1\}$ that describes the mapping induced by the scaling on the primary fields, i.e.

$$\Pi_{a,b} \equiv \Pi^{(\ell)}_{a,b} := \epsilon_\ell(a)\, \delta_{b,\dot\sigma a}\,. \tag{7.3.1}$$

Equation (7.2.9) can then be written as

$$(\Pi S)_{a,b} = \epsilon_\ell(a)\, S_{\dot\sigma a,b} = \epsilon_\ell(b)\, S_{a,\dot\sigma b} = (S \Pi^t)_{a,b}\,. \tag{7.3.2}$$

Multiplying this equation from both the left and the right with S^+, the hermitean conjugate of S, using the unitarity of S and taking the hermitean conjugate of this equation, we see that

$$(\Pi^t S)_{a,b} = (S \Pi)_{a,b}\,. \tag{7.3.3}$$

This relation describes in fact a rather remarkable sum rule for S-matrix elements: writing the matrix multiplication in (7.3.3) explicitly, it reads

$$\sum_{c \in P_h} \epsilon_\ell(c)\, \delta_{a,\dot\sigma c} S_{c,b} = \sum_{c \in P_h} \epsilon_\ell(c)\, \delta_{b,\dot\sigma c} S_{a,c}\,. \tag{7.3.4}$$

Generically the sums appearing in (7.3.4) contain more than one non-vanishing term; to our knowledge it is the first time that a relation of this type between S-matrix elements has been established in a general framework.

By introducing the pre-images of a quasi-Galois transformation,

$$\Sigma^{-1}(a) := \{c \in P_h \mid \dot\sigma(c) = a\} \tag{7.3.5}$$

for any $a \in P_h$, we may rewrite the sum rule (7.3.4) in the more suggestive manner

$$\sum_{c \in \Sigma^{-1}(a)} \epsilon_\ell(c)\, S_{c,b} = \sum_{c \in \Sigma^{-1}(b)} \epsilon_\ell(c)\, S_{a,c}\,. \tag{7.3.6}$$

If the map (7.2.6) is invertible, then (7.3.6) reduces to the relation

$$\epsilon_\ell(\dot\sigma^{-1} a)\, S_{\dot\sigma^{-1} a,b} = \epsilon_\ell(\dot\sigma^{-1} b)\, S_{a,\dot\sigma^{-1} b}\,, \tag{7.3.7}$$

which is equivalent to the identity (7.2.9) applied to the map $\dot{\sigma}^{-1}$.

Combining the two relations (7.3.2) and (7.3.3), it follows that the matrix

$$Z^{(\ell)} := \Pi + \Pi^t \tag{7.3.8}$$

commutes with the modular matrix S,

$$[Z^{(\ell)}, S] = 0\,. \tag{7.3.9}$$

Typically the S-matrix invariant $Z^{(\ell)}$ obtained this way is not positive, nor does it commute with T. This pattern already arises for ordinary Galois scalings. However, just as in the Galois case in Chapter 5, it is still possible to construct physical modular invariants, because one can get rid of the minus signs and achieve T-invariance by suitably adding up various invariants of the type above and possibly combining with other methods such as simple currents. Note that in the invariant (7.3.8) typically some of the fields are projected out, and hence when using quasi-Galois transformations it is in fact easier to obtain T-invariance than in the Galois case.

To give an example for a matrix that commutes with the S-matrix and that is obtained by the above prescription, let us consider the scaling $\ell = 3$ for the A_1 WZW theory at height $h = 6$. In terms of non-shifted highest weights, this scaling maps $\Lambda = 0$ and $\Lambda = 4$ with a positive sign ϵ_ℓ on $\Lambda = 2$, the weight $\Lambda = 2$ with a negative sign on itself, and the weights $\Lambda = 1$, 3 on the boundary $\mathcal{B}$. Thus the matrix $Z^{(3)}$ defined by (7.3.8) reads

$$Z^{(3)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{7.3.10}$$

While this matrix has negative entries and is hence unphysical, the combination

$$\hat{Z} = (Z^{(3)})^2 + 2Z^{(3)} \tag{7.3.11}$$

is a physical invariant, namely the D-type invariant of the height 6 A_1 theory. As the number of primary fields is rapidly increasing with the rank and level, most applications of our prescription which lead to physical invariants involve rather complex expressions; therefore we will not display more complicated examples explicitly.

Actually the invariant (7.3.11) can also be obtained from genuine Galois transformations. An example for a physical modular invariant which cannot be explained that way, but which is obtainable as a linear combination of quasi-Galois invariants is the exceptional E_7-type invariant of A_1 at level 16. However, the concrete expression is rather lengthy so that we refrain from presenting it here. As we shall see later, also for the E_7-type invariant there exists a close relation to the matrix $Z^{(3)}$ displayed in (7.3.10) even though they are invariants at different heights.

7.4 S-matrix invariants: increasing and lowering the height

In this section we consider the special case where the scaling factor $\ell \in \mathbb{Z}_{>0}$ is a divisor of the height; to simplify notation, we will make this explicit by denoting the height of the theory to which the scaling is applied by ℓh. As we will see, in this situation there exist intimate relations between the WZW theories at height ℓh and at height h. As we are now dealing with weights at two distinct heights, we find it convenient to denote the elements of P_h by lower case and the elements of $P_{\ell h}$ by upper case roman letters, respectively. Similarly, we use the capital letter 'S' for the S-matrix of the height ℓh theory and the symbol 's' for the S-matrix of the height h theory.

Before describing the relationship between height h and height ℓh theories, let us first prove another new symmetry property of the S-matrix: if the height is divisible by ℓ, then for any $B \in P_{\ell h}$ the signed S-matrix elements

$$\epsilon_\ell(C) \cdot \mathrm{S}_{\ell a,C} \tag{7.4.1}$$

are identical for all $C \in \Sigma^{-1}(B)$. To check this statement, take any fixed $B \in P_{\ell h}$ and any $C \in \Sigma^{-1}(B)$. Then considering weights of the form $A = \ell a$ with $a \in P_h$, and using the fact that $\dot\sigma C = w_C(\ell C) + \ell h\, t_C$ with $w_C \in W$ and $t_C \in L^\vee$, as well as $\epsilon_\ell(C) = \mathrm{sign}(w_C)$, we find

$$\begin{aligned} \mathrm{S}_{\ell a,C} &= \mathcal{N} \sum_{w\in W} \mathrm{sign}(w) \exp[-\tfrac{2\pi \mathrm{i}}{\ell h}\,(w(\ell a), \ell^{-1} w_C^{-1}(B) + h t'_C)] \\ &= \mathcal{N} \sum_{w\in W} \mathrm{sign}(w) \exp[-\tfrac{2\pi \mathrm{i}}{h}\,(w_C w(a), \ell^{-1} B)] \\ &= \mathrm{sign}(w_C) \cdot \mathcal{N} \sum_{w\in W} \mathrm{sign}(w) \exp[-\tfrac{2\pi \mathrm{i}}{h}\,(w(a), \ell^{-1} B)]\,. \end{aligned} \tag{7.4.2}$$

The only dependence of the right hand side on the weight C is thus via the sign $\epsilon_\ell(C) \equiv \mathrm{sign}(w_C)$, and hence we have established the symmetry (7.4.1).

The primary WZW fields φ_a and ϕ_A which are associated to the weights in P_h and in $P_{\ell h}$, respectively, can be viewed as the generators of the fusion rings $\mathcal{R}_h$ and $\mathcal{R}_{\ell h}$ of the height h and height ℓh WZW theories, respectively. Let us introduce the mappings

$$\begin{aligned} P:\ \ & \mathcal{R}_{\ell h} \to \mathcal{R}_h \\ & \phi_A \mapsto P(\phi_A) = \sum_{b\in P_h} P_{A,b}\, \varphi_b\,, \qquad P_{A,b} := \epsilon_\ell(A)\, \delta_{\dot\sigma A,\ell b} \end{aligned} \tag{7.4.3}$$

and

$$\begin{aligned} D:\ \ & \mathcal{R}_h \to \mathcal{R}_{\ell h} \\ & \varphi_a \mapsto D(\varphi_a) = \sum_{B\in P_{\ell h}} D_{a,B}\, \phi_B\,, \qquad D_{a,B} := \delta_{\ell a,B} \end{aligned} \tag{7.4.4}$$

between these two fusion rings. Note that because of

$$\ell^{-1}\dot\sigma A = \ell^{-1}\,(w_A(\ell A) + \ell h\, t_A) = w_A(A) + h\, t_A \tag{7.4.5}$$

with $w_A \in W$ and $t_A \in L^\vee$ for any $A \in P_{\ell h}$, the weight $\ell^{-1}\dot\sigma A$ is integral and either an element of P_h or else on the boundary of an affine Weyl chamber at height h. Also, $P_{b,b} = 1$ (here the first label b is to be considered as an element of $P_{\ell h}$) which shows that the map P is always non-zero.

The relation (7.4.5) implies that there is a close connection, which will prove to be useful later on, between the conformal dimensions Δ mod $\mathbb{Z}$ of all those fields which belong to the same pre-image under the map $\dot\sigma$. Namely, from the definition $\Delta_a = [(a,a) - (\rho,\rho)]/2h$ of the conformal dimensions at height h (and the fact that any Weyl group element $w \in W$ is an isometry), it follows that

$$\begin{aligned} \ell\,(\Delta_b - \Delta_c) &= (2h\ell)^{-1}\,[(a + ht_b, a + ht_b) - (a + ht_c, a + ht_c)] \\ &= \ell^{-1}\,(a, t_b - t_c) + \tfrac{1}{2}\, h\ell^{-1}[(t_b,t_b) - (t_c,t_c)] \end{aligned} \tag{7.4.6}$$

for $b,c \in \Sigma^{-1}(a)$; we will use this equation only modulo $\mathbb{Z}$. Since $t_b,\ t_c \in L^\vee$, we have $(a,t_b) \in \mathbb{Z}$, $(t_b,t_b) \in 2\mathbb{Z}$, and analogously for t_c, and hence the right hand sight of (7.4.6) is an integral multiple of ℓ^{-1}. If in addition the height is divisible by ℓ, then according to (7.4.5) this is also true for the Dynkin components of any a for which $\Sigma^{-1}(a)$ is non-empty, and hence in this case the right hand side is in fact an integer, so that $\Delta_b - \Delta_c \in \ell^{-1}\mathbb{Z}$ for $h = \ell h'$ and $b,c \in \Sigma^{-1}(a)$. In the notation appropriate to the height ℓh theory we thus have, for all $A \in P_{\ell h}$,

$$\Delta_B - \Delta_C \in \ell^{-1}\,\mathbb{Z} \qquad \text{for} \quad B,C \in \Sigma^{-1}(A)\,. \tag{7.4.7}$$

The relevance of the maps P and D that we introduced in (7.4.3) and (7.4.4) comes from the fact that they provide direct relations between the two modular matrices S and s. Namely, denoting the rank of $\bar{\mathbf{g}}$ by r, we find

$$\mathrm{S}\,D^t = \ell^{-r/2}\,P\,\mathrm{s} \tag{7.4.8}$$

$$P^t\,\mathrm{S} = \ell^{r/2}\,\mathrm{s}\,D\,. \tag{7.4.9}$$

Equivalently, by taking the transpose, we can write these identities as

$$D\,\mathrm{S} = \ell^{-r/2}\,\mathrm{s}\,P^t \tag{7.4.10}$$

$$\mathrm{S}\,P = \ell^{r/2}\,D^t\,\mathrm{s}\,. \tag{7.4.11}$$

To prove (7.4.8), we first separate the height-independent part of the normalization factor $\mathcal{N}$ in the Kac–Peterson formula (7.2.3) from the rest,

$$\mathcal{N} \equiv \mathcal{N}_{(h)} = \mathrm{i}^{(d-r)/2}\,|L^{\mathrm{w}}/L^\vee|^{-1/2}\,h^{-r/2} =: h^{-r/2}\,\overline{\mathcal{N}}\,, \tag{7.4.12}$$

where d is the dimension of $\bar{\mathbf{g}}$. Then we compute

$$\begin{aligned}(\mathrm{S}\,D^t)_{A,b} = \mathrm{S}_{A,\ell b} &= (\ell h)^{-r/2}\,\overline{\mathcal{N}}\sum_{w\in W}\mathrm{sign}(w)\exp[-\tfrac{2\pi\mathrm{i}}{\ell h}\,(w(A),\ell b)] \\ &= (\ell h)^{-r/2}\,\overline{\mathcal{N}}\sum_{w\in W}\mathrm{sign}(w)\exp[-\tfrac{2\pi\mathrm{i}}{h}\,(w(A),b)]\end{aligned} \tag{7.4.13}$$

and, once again making use of $\dot\sigma A = w_A(\ell A) + \ell h\,t_A$ with $w_A \in W$ and $t_A \in L^\vee$, and of $\epsilon_\ell(A) = \mathrm{sign}(w_A)$,

$$\begin{aligned}(P\,\mathrm{s})_{A,b} = \epsilon_\ell(A)\,\mathrm{s}_{\ell^{-1}\dot\sigma A,b} &= h^{-r/2}\,\overline{\mathcal{N}}\,\mathrm{sign}(w_A)\sum_{w\in W}\mathrm{sign}(w)\exp[-\tfrac{2\pi\mathrm{i}}{h}\,(w(w_A(A)+ht_A),b)] \\ &= h^{-r/2}\,\overline{\mathcal{N}}\sum_{w\in W}\mathrm{sign}(w)\exp[-\tfrac{2\pi\mathrm{i}}{h}\,(w(A),b)]\,.\end{aligned} \tag{7.4.14}$$

Comparing (7.4.13) and (7.4.14), we obtain (7.4.8).

The relation (7.4.9) can now be proven by multiplying (7.4.8) from the left with the hermitean conjugate S^+ of S and from the right with s^+. Using the unitarity of S and s and taking the hermitean conjugate yields (7.4.9).

We can now apply the results just proven to the construction of S-matrix invariants, both at height h and at height ℓh. Namely, assume first that the matrix Z belongs to the commutant of the S-matrix of the height ℓh theory, i.e. that

$$[Z,\mathrm{S}] = 0\,. \tag{7.4.15}$$

Further, define

$$\tilde{z} := P^t\, Z\, D^t + D\, Z\, P\,. \tag{7.4.16}$$

Explicitly, we have

$$\tilde{z}_{a,b} = \sum_{A\in\Sigma^{-1}(\ell a)} \epsilon_\ell(A)\, Z_{A,\ell b} + \sum_{B\in\Sigma^{-1}(\ell b)} \epsilon_\ell(B)\, Z_{\ell a,B}\,. \tag{7.4.17}$$

Using (7.4.15) as well as the relations (7.4.8) – (7.4.11) proven above, we can then derive that

$$\begin{aligned}\tilde{z}\,\mathrm{s} &= P^t\, Z\, D^t\, \mathrm{s} + D\, Z\, P\, \mathrm{s} = \ell^{-r/2}\, P^t\, Z\, \mathrm{S}\, P + \ell^{r/2} D\, Z\, \mathrm{S}\, D^t \\ &= \ell^{-r/2}\, P^t\, \mathrm{S}\, Z\, P + \ell^{r/2} D\, \mathrm{S}\, Z\, D^t = \mathrm{s}\, D\, Z\, P + \mathrm{s}\, P^t\, Z\, D^t = \mathrm{s}\, \tilde{z}\,.\end{aligned} \tag{7.4.18}$$

Similarly, let z be an S-matrix invariant of the height h theory,

$$[z, \mathrm{s}] = 0\,, \tag{7.4.19}$$

and define

$$\tilde{Z} := D^t\, z\, P^t + P\, z\, D\,. \tag{7.4.20}$$

Using the convention that $z_{a,b} = 0$ whenever a or b is not in P_h, the matrix elements of $\tilde{Z}$ read

$$\tilde{Z}_{A,B} = \epsilon_\ell(A)\, z_{\ell^{-1}\dot\sigma A,\ell^{-1}B} + \epsilon_\ell(B)\, z_{\ell^{-1}A,\ell^{-1}\dot\sigma B}\,. \tag{7.4.21}$$

By employing (7.4.19) and again (7.4.8) – (7.4.11), we obtain

$$\begin{aligned}\tilde{Z}\,\mathrm{S} &= D^t\, z\, P^t\, \mathrm{S} + P\, z\, D\, \mathrm{S} = \ell^{r/2}\, D^t\, z\, \mathrm{s}\, D + \ell^{-r/2}\, P\, z\, \mathrm{s}\, P^t \\ &= \ell^{r/2}\, D^t\, \mathrm{s}\, z\, D + \ell^{-r/2}\, P\, \mathrm{s}\, z\, P^t = \mathrm{S}\, P\, z\, D + \mathrm{S}\, D^t\, z\, P^t = \mathrm{S}\, \tilde{Z}\,.\end{aligned} \tag{7.4.22}$$

We have thus proven the following remarkable facts: Given an S-matrix invariant Z at height ℓh, the formula (7.4.16) provides us with an S-matrix invariant $\tilde{z}$ at height h,

$$[\tilde{z}, \mathrm{s}] = 0\,; \tag{7.4.23}$$

and conversely, given an S-matrix invariant z at height h, the formula (7.4.20) defines an S-matrix invariant $\tilde{Z}$ at height ℓh,

$$[\tilde{Z}, \mathrm{S}] = 0\,. \tag{7.4.24}$$

Not surprisingly, the prescriptions (7.4.16) and (7.4.20) do not respect positivity, i.e. even if Z (respectively z) is a positive invariant, this needs not hold for $\tilde{z}$ ($\tilde{Z}$).

As an example, let us take for Z the exceptional invariants of A_1 which occur all at heights a multiple of 6, namely for $h = 12, 18, 30$, and obtain from them by (7.4.16) invariants of A_1 at height 6. For $h = 12$ and $h = 30$ the prescription (7.4.16) yields the zero matrix. More interesting is the E_7-type invariant at $h = 18$; in this case $\tilde{z}$ is precisely the quasi-Galois invariant (7.3.10) obtained in the previous section.

Note that the maps (7.4.3) and (7.4.4) are related to the map Π introduced in (7.3.1) by $\Pi = PD$:

$$\Pi_{A,B} = \epsilon_\ell(A)\, \delta_{B,\dot\sigma A} \equiv \sum_{c\in P_h} \epsilon_\ell(A)\, \delta_{\ell c,\dot\sigma A} \delta_{B,\ell c} = \sum_{c\in P_h} P_{A,c} D_{c,B}\,. \tag{7.4.25}$$

The prescription (7.4.20) actually provides a generalization of the quasi-Galois S-matrix invariant (7.3.8). Namely, according to (7.4.25), when considering the diagonal invariant $z = \mathbb{1}$, (7.4.20) yields

$$\tilde{Z} = P\, D + D^t\, P^t = \Pi + \Pi^t\,, \tag{7.4.26}$$

i.e. reproduces the invariant (7.3.8). A still more special case is obtained by performing the scaling by the factor ℓ at height $\ell g^\vee$. Then the smaller level is in fact zero, so that there is a single primary field with shifted weight $a=\rho$, and hence a single nontrivial invariant $z_{a,b}=\delta_{a,\rho}\delta_{b,\rho}$. In this situation, (7.4.21) reads

$$\tilde Z_{A,B} = \delta_{A,\ell\rho} \sum_{C\in\Sigma^{-1}(\ell\rho)} \epsilon_\ell(C)\,\delta_{B,C} + \delta_{B,\ell\rho} \sum_{C\in\Sigma^{-1}(\ell\rho)} \epsilon_\ell(C)\,\delta_{A,C}\,. \tag{7.4.27}$$

In applications (see in particular Section 7.5 below) it is often not the matrix (7.4.27) that is directly relevant, but rather the combination

$$\hat Z := \tilde Z^2 - 2\epsilon_\ell(\ell\rho)\,\tilde Z \tag{7.4.28}$$

(compare the similar formula (7.3.11)). The entries of (7.4.28) read

$$\hat Z_{A,B} = |\tilde\Sigma^{-1}(\ell\rho)|\,\delta_{A,\ell\rho}\delta_{B,\ell\rho} + \sum_{C,D\in\tilde\Sigma^{-1}(\ell\rho)} \epsilon_\ell(C)\epsilon_\ell(D)\,\delta_{A,C}\,\delta_{B,D}\,, \tag{7.4.29}$$

where

$$\tilde\Sigma^{-1}(\ell\rho) := \Sigma^{-1}(\ell\rho)\ \setminus \{\ell\rho\}\,. \tag{7.4.30}$$

Note that in the invariant $\hat Z$ only fields belonging to $\Sigma^{-1}(\ell\rho)$ get mixed; by (7.4.7) this implies that $\hat Z$ is not only S-invariant, but also invariant under T^ℓ. It is also easily checked that $\hat Z^2 = |\tilde\Sigma^{-1}(\ell\rho)|\,\hat Z$, so that by taking powers of $\hat Z$ we cannot produce any new invariants.

We can also apply the constructions (7.4.20) and (7.4.16) consecutively to a height h S-matrix invariant, or in the opposite order to a height ℓh invariant. The computation then involves the identities $PD=\Pi$, $DD^t=\mathbb{1}$, $P^tP=\ell^r\mathbb{1}$, as well as $DP=\pi$ and $D^tD=Q$ with

$$\pi_{a,b} := \epsilon_\ell(\ell a)\,\delta_{\ell b,\dot\sigma(\ell a)} \tag{7.4.31}$$

and

$$Q_{A,B} := \delta_{A,B}\cdot \sum_{b\in P_h} \delta_{A,\ell b}\,. \tag{7.4.32}$$

We find

$$\tilde{\tilde z} = 2\ell^r\, z + \pi z \pi + \pi^t z \pi^t \tag{7.4.33}$$

and a similar formula for $\tilde{\tilde Z}$. The result (7.4.33) means that whenever z commutes with s, then so does the matrix $\pi z\pi + \pi^t z\pi^t$. Also note that in (7.4.31) the map $\dot\sigma$ is the quasi-Galois transformation with scale factor ℓ at height ℓh. This implies that $\dot\sigma(\ell a) = \ell\,(w_{\ell a}(\ell a) + h t_{\ell a})$, and hence the δ-symbol in (7.4.31) imposes the constraint that the weight b is related to a by a quasi-Galois transformation with the same scale factor ℓ, but now at height h. In other words, as already anticipated in the notation, the map $\pi = DP$ implements the same quasi-Galois scaling for the height h theory as the map $\Pi = PD$ (7.4.25) implements for the height ℓh theory.

7.5 Conformal embeddings

Conformal embeddings are embeddings $\mathsf{g} \hookrightarrow \mathsf{h}$ of untwisted affine Lie algebras for which the irreducible highest weight modules possess finite branching rules. The explicit form of these branching rules has been determined for various cases (see e.g. [93, 94, 141, 21, 80, 5, 140]), but a general formula is not known, and there are still many conformal embeddings for which all known methods are inapplicable.

The list of conformal embeddings [125, 9] contains several infinite series. Here we are interested in a particular infinite series, namely the embedding $\mathbf{g}_{g^\vee} \hookrightarrow \widehat{\mathrm{so}}(d)_1$, i.e. of $\mathbf{g}$ at level $g^\vee$ (with $\mathbf{g}$ an arbitrary untwisted affine Lie algebra) into $\widehat{\mathrm{so}}(d)$, with $d \equiv \dim \bar{\mathbf{g}}$, at level one. In terms of the horizontal algebras, the embedding is the one for which the vector representation of so(d) branches to the adjoint representation of the smaller algebra $\bar{\mathbf{g}}$. Such embeddings are of particular interest because they are connected with the 'fermionization' [8,74,41] of WZW models with level $g^\vee$, which is due to the fact that $\widehat{\mathrm{so}}(d)$ can be written in terms of free fermions. This will play a rôle in the following.

The diagonal level one $\widehat{\mathrm{so}}(d)$ partition function is

$$\mathcal{Z}_{\mathrm{so}(d)}(\tau,\bar\tau) = |\mathcal{X}_\mathrm{o}|^2 + |\mathcal{X}_\mathrm{v}|^2 + |\mathcal{X}_\mathrm{s}|^2 + |\mathcal{X}_\mathrm{c}|^2 \qquad \text{for } d \text{ even} \tag{7.5.1}$$

and

$$\mathcal{Z}_{\mathrm{so}(d)}(\tau,\bar\tau) = |\mathcal{X}_\mathrm{o}|^2 + |\mathcal{X}_\mathrm{v}|^2 + |\mathcal{X}_\mathrm{s}|^2 \qquad \text{for } d \text{ odd,} \tag{7.5.2}$$

where o, v, s and c refer to the singlet, vector, spinor, and conjugate spinor representation of so(d), respectively. Our objective is to write each of these characters in terms of the characters χ_Λ of $\mathbf{g}$ at level $g^\vee$.

The branching rule for the $\widehat{\mathrm{so}}(d)$ spinor(s) is already known explicitly ([91], see also [75, 94,42]). Up to a multiplicity, they branch to a single irreducible representation, namely the one whose (unshifted) highest weight is the Weyl vector ρ. We will denote this irreducible representation by L_ρ. The dimension of the analogous irreducible representation of the horizontal algebra $\bar{\mathbf{g}}$ is 2^{N_+}, where $N_+ = (d-r)/2$ is the number of positive roots (and r is the rank of $\bar{\mathbf{g}}$); hence the multiplicity with which L_ρ is contained in the $\widehat{\mathrm{so}}(d)$ spinors is $2^{r/2-1}$ if d is even, and $2^{(r-1)/2}$ if d is odd. A closed formula for the branching rules of the $\widehat{\mathrm{so}}(d)$ singlet and vector is also known [94], but (see (7.5.20) below) it involves the image $\hat{W}(\rho)$ of the Weyl vector under the affine Weyl group and hence is not convenient for explicit calculations. (As a matter of fact, only in very few cases, such as for $\bar{\mathbf{g}} = G_2$ [21], the branching has already been determined explicitly.) Accordingly, we will not employ this formula, but rather prove an equivalent formula which allows for an immediate evaluation on a computer. To start, we make the following general ansatz for the relation between level one $\widehat{\mathrm{so}}(d)$ and $\mathbf{g}_{g^\vee}$ characters:

$$\mathcal{X}_\mathrm{o} = \sum_{\Lambda\in P_{g^\vee}} \mathrm{m}_\mathrm{o}^\Lambda\, \chi_\Lambda\,, \qquad \mathcal{X}_\mathrm{v} = \sum_{\Lambda\in P_{g^\vee}} \mathrm{m}_\mathrm{v}^\Lambda\, \chi_\Lambda\,, \qquad \mathcal{X}_\mathrm{s} = \mathcal{X}_\mathrm{c} = 2^{r/2-1}\chi_\rho \tag{7.5.3}$$

for d even, and

$$\mathcal{X}_\mathrm{o} = \sum_{\Lambda\in P_{g^\vee}} \mathrm{m}_\mathrm{o}^\Lambda\, \chi_\Lambda\,, \qquad \mathcal{X}_\mathrm{v} = \sum_{\Lambda\in P_{g^\vee}} \mathrm{m}_\mathrm{v}^\Lambda\, \chi_\Lambda\,, \qquad \mathcal{X}_\mathrm{s} = 2^{(r-1)/2}\chi_\rho \tag{7.5.4}$$

for d odd. Here and below we label the integrable $\mathbf{g}_{g^\vee}$ representations by their *unshifted* highest weights (in particular we will use $\Lambda = \rho$ in place of $a = 2\rho$); accordingly, the summations in (7.5.3) and (7.5.4) are over the unshifted fundamental chamber $P_{g^\vee}(\mathbf{g})$; also, m_o and m_v are non-negative integral vectors in the space of all characters. The equality of the decomposition of the two $\widehat{\mathrm{so}}(d)$ spinor characters for even d implies that these representations will appear as a fixed point of order 2 in the $\mathbf{g}_{g^\vee}$ modular invariant. Hence the invariant will have the form

$$\mathcal{Z}_{\mathrm{c.e.}} = |\sum_{\Lambda\in P_{g^\vee}} \mathrm{m}_\mathrm{o}^\Lambda\, \chi_\Lambda|^2 + |\sum_{\Lambda\in P_{g^\vee}} \mathrm{m}_\mathrm{v}^\Lambda\, \chi_\Lambda|^2 + 2\cdot|2^{r/2-1}\chi_\rho|^2 \tag{7.5.5}$$

for d even, and

$$\mathcal{Z}_{\text{c.e.}} = |\sum_{\Lambda\in P_{g^\vee}} \mathrm{m}_{\mathrm{o}}^\Lambda \chi_\Lambda|^2 + |\sum_{\Lambda\in P_{g^\vee}} \mathrm{m}_{\mathrm{v}}^\Lambda \chi_\Lambda|^2 + |2^{(r-1)/2}\chi_\rho|^2 \tag{7.5.6}$$

for d odd.

The identity and vector characters of $\widehat{\mathrm{so}}(d)$ branch to distinct $\mathbf{g}_{g^\vee}$ characters, since the difference of conformal dimensions of identity and vector is non-integral. Thus the vectors m_{o} and m_{v} are orthogonal. We will focus first on the cases where also the spinor(s) have different conformal weights modulo integers than identity and vector, which holds if $d \neq 0 \bmod 8$, compare (1.4.9). Then by the same argument the spinor(s) branch to different $\mathbf{g}_{g^\vee}$ characters than identity and vector characters, and hence we have $\mathrm{m}_{\mathrm{o}}^\rho = \mathrm{m}_{\mathrm{v}}^\rho = 0$. This situation is covered by the following simple theorem. Consider any S-invariant (such as (7.5.5), (7.5.6)) that is a sum of squares, i.e. of the form

$$\mathcal{M} = \sum_p N_p |\sum_{\Lambda\in P_{g^\vee}} \mathrm{m}_p^\Lambda \chi_\Lambda|^2 \,. \tag{7.5.7}$$

This can be written as $\sum_{\Lambda,\Lambda'\in P_{g^\vee}} \chi_\Lambda M_{\Lambda,\Lambda'} \chi^*_{\Lambda'}$, where M is the matrix with entries

$$M_{\Lambda,\Lambda'} = \sum_p N_p\, \mathrm{m}_p^\Lambda \mathrm{m}_p^{\Lambda'} \,. \tag{7.5.8}$$

Further, suppose that the vectors m_p are orthogonal,

$$\sum_{\Lambda\in P_{g^\vee}} \mathrm{m}_p^\Lambda\, \mathrm{m}_{p'}^\Lambda = R_p \delta_{pp'} \,. \tag{7.5.9}$$

Let us also impose the physical requirement that there is a unique vacuum, i.e. that M satisfies $M_{00} = 1$; then among the vectors m_p there must be precisely one, conventionally labeled by $p = 0$, which contains the identity character, i.e. we must have $N_0 = 1$ and $\mathrm{m}_0^0 = 1$. Next consider the matrix M^2; it has entries $(M^2)_{\Lambda,\Lambda'} = \sum_p N_p^2 R_p \mathrm{m}_p^\Lambda\, \mathrm{m}_p^{\Lambda'}$; in particular, $(M^2)_{00} = R_0$. Thus the matrix $M^2 - R_0 M$ has entries $(M^2 - R_0 M)_{\Lambda,\Lambda'} = \sum_p (N_p^2 R_p - N_p R_0)\, \mathrm{m}_p^\Lambda\, \mathrm{m}_p^{\Lambda'}$. Finally, the square Z of the latter matrix has entries

$$Z_{\Lambda,\Lambda'} \equiv ([M^2 - R_0 M]^2)_{\Lambda,\Lambda'} = \sum_p (N_p R_p - R_0)^2 N_p R_p\, \mathrm{m}_p^\Lambda\, \mathrm{m}_p^{\Lambda'} \,. \tag{7.5.10}$$

This is a manifestly non-negative matrix, it obeys $Z_{00} = 0$, and because it is a polynomial in M it commutes with S. Thus $0 = Z_{00} = \sum_{\Lambda,\Lambda'\in P_{g^\vee}} S_{0\Lambda} Z_{\Lambda,\Lambda'} S_{0\Lambda'} \geq 0$, with equality only if $Z_{\Lambda,\Lambda'} = 0$ for all $\Lambda, \Lambda' \in P_{g^\vee}$; i.e., any such matrix must vanish. By (7.5.10), the vanishing of Z implies that for any p the sum rule

$$N_p \sum_{\Lambda\in P_{g^\vee}} (\mathrm{m}_p^\Lambda)^2 \equiv N_p R_p = R_0 \tag{7.5.11}$$

holds. This is equivalent to the property $M^2 = R_0 M$, so that M is idempotent up to a normalization.

In the situation of our interest, these sum rules give useful information because we know N_p and m_p for the spinor characters. For even d, the spinors have $N = 2$, and hence (7.5.11) tells us that

$$R_{\mathrm{o}} = N_{\mathrm{v}} R_{\mathrm{v}} = 2 \cdot (2^{r/2-1})^2 = 2^{r-1} \,, \tag{7.5.12}$$

and for d odd we get

$$R_{\rm o} = N_{\rm v} R_{\rm v} = (2^{(r-1)/2})^2 = 2^{r-1} \,. \tag{7.5.13}$$

Since for $d \neq 8 \bmod 16$ the vector representation of level one $\widehat{\rm so}(d)$ has different conformal dimension modulo integers than the other representations, we have $N_{\rm v} = 1$. As we will see below, the matrix M has all entries except the spinor entries equal to 0 or 1, and in that case the sum rule (7.5.11) tells us that the identity and the vector of $\widehat{\rm so}(d)$ each branch to 2^{r-1} different irreducible representations of the conformal subalgebra $\mathbf{g}$.

For the following argument it is convenient to summarize the spinor branching rules in (7.5.3) and (7.5.4) as $\tilde{\mathcal{X}}_{\rm s} = 2^{[r/2]}\chi_\rho$, where $[n]$ stands for the integer part of n, and where $\tilde{\mathcal{X}}_{\rm s} = \mathcal{X}_{\rm s}$ for odd d and $\tilde{\mathcal{X}}_{\rm s} = (\mathcal{X}_{\rm s} + \mathcal{X}_{\rm c})/2$ for even d. Then by performing the modular transformation $\tau \mapsto -1/\tau$ and using the explicit form of the S-matrix of the $\widehat{\rm so}(d)$ theory, we have

$$2^{[r/2]-r/2}\,(\mathcal{X}_{\rm o} - \mathcal{X}_{\rm v})(\tau) = \tilde{\mathcal{X}}_{\rm s}(-\frac{1}{\tau}) = 2^{[r/2]}\chi_\rho(-\frac{1}{\tau}) = 2^{[r/2]} \sum_{\Lambda\in P_{g^\vee}} (S_{\mathbf{g}})_{\rho,\Lambda}\chi_\Lambda(\tau) \,. \tag{7.5.14}$$

This formula holds in fact for the full characters, not just for the Virasoro specialized ones. Since the full characters form a basis of the relevant module of $SL(2,\mathbb{Z})$, and since in the expansions of $\mathcal{X}_{\rm o}$ and $\mathcal{X}_{\rm v}$ into powers of $q = \exp(2\pi i\tau)$ the fractional powers of q are different, it follows that (7.5.14) already determines the branching rules of the singlet and vector characters uniquely. In particular the knowledge that χ_0 must appear with multiplicity one in the branching rule for $\mathcal{X}_{\rm o}$ implies that $(S_{\mathbf{g}})_{\rho,0} = 2^{-r/2}$, and that for any $\Lambda \in P_{g^\vee}$, $(S_{\mathbf{g}})_{\rho,\Lambda}$ must be an integral multiple of this number.

All the properties of the conformal embedding invariants that were obtained above follow by rather general arguments. We will now discuss how one can obtain these invariants (i.e. the form of the vectors $\mathrm{m}_{\rm o}$ and $\mathrm{m}_{\rm v}$) in a much more explicit manner by employing a quasi-Galois scaling by a factor 2. Thus consider $\mathbf{g}$ at height $h = 2g^\vee$, and the quasi-Galois scaling $\ell = 2$. Applying the prescription (7.4.20), we obtain the special case $\ell = 2$ of the S-matrix invariant (7.4.29). In terms of unshifted weights, (7.4.29) reads

$$\hat{Z}_{\Lambda,\Lambda'} = |\tilde{\Sigma}^{-1}(\rho)|\,\delta_{\Lambda,\rho}\delta_{\Lambda',\rho} + \sum_{\mu,\mu'\in\tilde{\Sigma}^{-1}(\rho)} \epsilon(\mu)\epsilon(\mu')\,\delta_{\Lambda,\mu}\,\delta_{\Lambda',\mu'} \,. \tag{7.5.15}$$

As it turns out, the sign ϵ is not constant on $\Sigma^{-1}(\rho)$, so that (unlike in the, otherwise similar, situation of (7.3.10)) the invariant $\hat{Z}$ (7.5.15) is not positive. By the remark after (7.4.30) it follows, however, that it does commute with T^2.

Furthermore, according to (7.2.9) we have

$$\epsilon(0)\,(S_{\mathbf{g}})_{\rho,\Lambda} \equiv \epsilon(0)\,(S_{\mathbf{g}})_{\dot\sigma 0,\Lambda} = \epsilon(\Lambda)\,(S_{\mathbf{g}})_{0,\dot\sigma\Lambda} \tag{7.5.16}$$

for any $\Lambda \in P_{g^\vee}$, and hence the observation after (7.5.14) implies that $\epsilon(0) = 1$ and

$$(S_{\mathbf{g}})_{\rho,\Lambda} = \epsilon(\Lambda)\cdot 2^{-r/2} \tag{7.5.17}$$

for all $\Lambda \in P_{g^\vee}$. Combining this information with (7.5.14) and the fact that the full characters form a basis, we learn that

$$\mathcal{X}_{\rm o} = \sum_{\substack{\Lambda\in\Sigma^{-1}(\rho)\\ \epsilon(\Lambda)=1}} \chi_\Lambda \,, \qquad \mathcal{X}_{\rm v} = \sum_{\substack{\Lambda\in\Sigma^{-1}(\rho)\\ \epsilon(\Lambda)=-1}} \chi_\Lambda \,. \tag{7.5.18}$$

This is the announced closed formula for the branching rules of the embedding $\mathbf{g} \hookrightarrow \widehat{\rm so}(\dim \mathbf{g})$. Note that in terms of unshifted weights the explicit form of the quasi-Galois transformation reads $2\rho = \rho + \rho = \dot\sigma\Lambda + \rho \equiv \hat w_\Lambda(2(\Lambda+\rho)) = 2\,w_\Lambda(\Lambda+\rho) + 2g^\vee\beta_\Lambda$ with $w_\Lambda \in W$ and $\beta_\Lambda \in L^\vee$, which can be rewritten as

$$\Lambda = w_\Lambda^{-1}(\rho) - \rho - g^\vee\, w_\Lambda^{-1}(\beta_\Lambda) = \hat u(\rho) - \rho\,, \tag{7.5.19}$$

where the last equality defines a unique element $\hat u$ of the affine Weyl group $\hat W$ at level $g^\vee$. Thus our result (7.5.18) can be rewritten as

$$\mathcal{X}_{\rm o} = \sum_{\Lambda\in P_{g^\vee}\cap {\rm R}_+} \chi_\Lambda\,, \qquad \mathcal{X}_{\rm v} = \sum_{\Lambda\in P_{g^\vee}\cap {\rm R}_-} \chi_\Lambda \tag{7.5.20}$$

with

$${\rm R}_\pm := \{\hat w(\rho) - \rho \mid \hat w \in \hat W,\ {\rm sign}(w) = \pm 1\}\,. \tag{7.5.21}$$

The formula (7.5.20) has already been obtained in [94]. It is equivalent to (7.5.18), but for explicit calculations has the disadvantage that it involves the sets ${\rm R}_\pm$; these sets are infinite due to the fact that all elements of the affine Weyl group must be taken into account.

Let us describe some aspects of the formula (7.5.18) in more detail. First, for all simple $\bar{\mathbf{g}}$ except $\bar{\mathbf{g}} = A_r$ with r even, we observe the following. A certain number K of representations with integer conformal weight is mapped via the quasi-Galois transformation to L_ρ with a positive sign; an equal number of representations with half-integer conformal weight flows to L_ρ with a negative sign; all other representations as well as L_ρ itself flow to the boundary. (This has been checked explicitly for rank less than 9; the continuation of this specific result to higher rank is only a conjecture.) For A_r with r even, there are two differences with respect to the foregoing. First of all the numbers K and K' of fields with integral and half-integral conformal weight, respectively, that flow to L_ρ are different, and secondly L_ρ does not flow to the boundary, but to itself. In this case $d = r(r+2)$, which is a multiple of 8, implying that the $\widehat{\rm so}(d)$ spinor has integral or half-integral conformal weight. The sign associated with the flow of L_ρ to itself is plus or minus for these two cases respectively.

In matrix notation, we thus have $\tilde Z = \Pi + \Pi^t$, with

$$\Pi = \begin{pmatrix} 0 & 0 & \vec e & 0 \\ 0 & 0 & -\vec e & 0 \\ 0 & 0 & \epsilon(\rho) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,, \tag{7.5.22}$$

for the matrix (7.4.20) that underlies (7.4.28), and hence

$$\hat Z = \begin{pmatrix} {\rm E} & -{\rm E} & 0 & 0 \\ -{\rm E} & {\rm E} & 0 & 0 \\ 0 & 0 & K+K' & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,. \tag{7.5.23}$$

Here the third column/row corresponds to L_ρ, the first one to all K fields with integral conformal weight which flow to L_ρ under the quasi-Galois transformation, the second to the K' fields with half-integral weight flowing to L_ρ, and the fourth to all remaining fields. The symbol $\vec e$ stands for a K, respectively K', component vector with all entries equal to

1, and $\mathrm{E} \equiv \vec{e} \otimes \vec{e}^t$ denotes the matrix of appropriate size (i.e., $K \times K$, $K \times K'$, $K' \times K$, and $K' \times K'$, respectively) each of whose entries is equal to 1; the 0's indicate matrices of zeroes of the proper size. Thus in particular for all cases except A_r with even rank, (7.5.23) can also be written as

$$\hat{Z} = \begin{pmatrix} \mathrm{E} & -\mathrm{E} & 0 & 0 \\ -\mathrm{E} & \mathrm{E} & 0 & 0 \\ 0 & 0 & 2K & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.5.24}$$

with all matrices E of size $K \times K$. Also recall that if L_ρ flows to the boundary, then $\epsilon(\rho) = 0$ so that the entry $\Pi_{\rho,\rho}$ of the matrix (7.5.22) vanishes. Further, if d is a multiple of 8, then not only the matrix (7.5.23), but also

$$\hat{Z}' := \hat{Z} + \epsilon(\rho)\,\tilde{Z} = \begin{pmatrix} \mathrm{E} & -\mathrm{E} & \epsilon(\rho)\,\vec{e} & 0 \\ -\mathrm{E} & \mathrm{E} & -\epsilon(\rho)\,\vec{e} & 0 \\ \epsilon(\rho)\,\vec{e}^t & -\epsilon(\rho)\,\vec{e}^t & K+K'+2\epsilon^2(\rho) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.5.25}$$

commutes with both S and T^2.

These results can be related to the conformal embedding invariant in the following way. Consider first the case of even d. The diagonal $\widehat{\mathrm{so}}(d)$ invariant can be written in terms of Jacobi theta functions and the Dedekind eta function, using

$$\begin{aligned} \mathcal{X}_\mathrm{o} &= \tfrac{1}{2}\,\eta^{-d/2}(\theta_3^{d/2} + \theta_4^{d/2})\,, \qquad & \mathcal{X}_\mathrm{v} &= \tfrac{1}{2}\,\eta^{-d/2}(\theta_3^{d/2} - \theta_4^{d/2}) \\ \mathcal{X}_\mathrm{s} &= \tfrac{1}{2}\,\eta^{-d/2}(\theta_2^{d/2} + \mathrm{i}^{d/2}\theta_1^{d/2})\,, \qquad & \mathcal{X}_\mathrm{c} &= \tfrac{1}{2}\,\eta^{-d/2}(\theta_2^{d/2} - \mathrm{i}^{d/2}\theta_1^{d/2})\,, \end{aligned} \tag{7.5.26}$$

where the arguments τ and z are suppressed ((7.5.26) reflects the possible description of the $\widehat{\mathrm{so}}(d)$ theory by free fermions). We are only considering Virasoro specialized characters here, i.e. these functions are in fact $\theta_i(\mathrm{z} = 0, \tau)$. Since $\theta_1(\mathrm{z} = 0, \tau) = 0$, in this setting the partition function (7.5.1) reads $\mathcal{Z}_{\mathrm{so}(d)} = \frac{1}{2}\,|\eta|^{-d}\left[\,|\theta_3|^d + |\theta_4|^d + |\theta_2|^d\,\right]$. This is modular invariant because S interchanges θ_4 and θ_3, while T interchanges θ_4 and θ_2, and all overall factors cancel.

This diagonal partition function is however not the one we obtain from quasi-Galois transformations. Using the modular transformation properties of the θ-functions one can write down another partition function that is only invariant under S and T^2, namely (fixing the normalization such as to make the square of the identity character appear exactly once) $\hat{\mathcal{Z}}_{\mathrm{so}(d)} = |\eta|^{-d/2}\left[\,|\theta_4|^d + |\theta_2|^d\,\right]$, or, re-expressed in terms of the $\widehat{\mathrm{so}}(d)$ characters (7.5.26),

$$\hat{\mathcal{Z}}_{\mathrm{so}(d)} = |\mathcal{X}_\mathrm{o} - \mathcal{X}_\mathrm{v}|^2 + |\mathcal{X}_\mathrm{s} + \mathcal{X}_\mathrm{c}|^2\,. \tag{7.5.27}$$

Both the diagonal modular invariant (7.5.1) and the partition function (7.5.27) contain more information than one strictly gets from specialized characters; one may check explicitly that both are S-invariant if the spinor characters are distributed symmetrically, as indicated.

If we write the matrix M corresponding to (7.5.27) in terms of **g**-representations we get

$$\begin{pmatrix} \mathrm{E}_{\mathrm{oo}} & -\mathrm{E}_{\mathrm{ov}} & 0 & 0 \\ -\mathrm{E}_{\mathrm{vo}} & \mathrm{E}_{\mathrm{vv}} & 0 & 0 \\ 0 & 0 & 2^r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,, \tag{7.5.28}$$

where $(\mathrm{E}_{pp'})_{\Lambda,\Lambda'} = \mathrm{m}_p^\Lambda\, \mathrm{m}_{p'}^{\Lambda'}$. The result (7.5.18) implies that $\mathrm{E}_{\mathrm{oo}} = \mathrm{E}_{\mathrm{ov}} = \mathrm{E}_{\mathrm{vo}} = \mathrm{E}_{\mathrm{vv}} = \mathrm{E}$, or in other words, that $\vec{\mathrm{m}}_\mathrm{o} = \vec{\mathrm{m}}_\mathrm{v} = \vec{e}$. Thus (7.5.28) can be identified with (7.5.24). There is also an independent consistency check of this identification. Namely, we find that $K = 2^{r-1}$, so that both m_o and m_v have 2^{r-1} components, each equal to 1. Hence they do satisfy the sum rule (7.5.12), so this rather nontrivial requirement for the matrix

$$Z_{\text{c.e.}} := \begin{pmatrix} \mathrm{E} & 0 & 0 & 0 \\ 0 & \mathrm{E} & 0 & 0 \\ 0 & 0 & 2^{r-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.5.29}$$

to commute with S is fulfilled. The matrix (7.5.29) is the modular invariant that corresponds to the branching rules (7.5.18). Note that the quasi-Galois symmetries imply that (7.5.24) commutes with S and T^2, while the step from (7.5.24) to (7.5.29) does not follow from any symmetry we know.

If d is a multiple of 8, then the above argument has to be slightly extended. Since in this case both (7.5.23) and (7.5.25) are S-T^2-invariants, we have in addition to (7.5.29) another matrix $Z'_{\text{c.e.}}$, and hence any physical linear combination $Z(u,v) := u\, Z_{\text{c.e.}} + v\, Z'_{\text{c.e.}}$, as candidates for the conformal embedding invariant. Explicitly, the matrix $Z'_{\text{c.e.}}$ reads

$$Z'_{\text{c.e.}} := \begin{pmatrix} \mathrm{E} & 0 & \vec{e} & 0 \\ 0 & \mathrm{E} & 0 & 0 \\ \vec{e}^{\,t} & 0 & 2^{r-1} + \epsilon^2(\rho) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.5.30}$$

for $d = 0 \bmod 16$ and

$$Z'_{\text{c.e.}} := \begin{pmatrix} \mathrm{E} & 0 & 0 & 0 \\ 0 & \mathrm{E} & \vec{e} & 0 \\ 0 & \vec{e}^{\,t} & 2^{r-1} + \epsilon^2(\rho) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.5.31}$$

for $d = 8 \bmod 16$, respectively. Fortunately, it is easy to eliminate all but one of the candidates, namely by imposing the 'quantum dimension' sum rule

$$\tfrac{1}{2} = \left(S_{\mathrm{so}(d)}\right)_{0,0} = \sum_{\Lambda \in P_{g^\vee}} (S_{\mathbf{g}})_{0,\Lambda} \tag{7.5.32}$$

(here the summation is over all fields that are combined with the identity field). Inserting the ansatz $Z(u,v)$, we find that for the case of A_r with even r, this yields the unique solution $u = 0$, $v = 1$, so that (7.5.30), respectively (7.5.31), is the correct solution (and we also have $\epsilon^2(\rho) = 1$). In contrast, for all other cases where d is a multiple of 8 (such as $\bar{\mathbf{g}}{=}E_8$), the unique solution is given by $u = 1$, $v = 0$, i.e. only (7.5.29) survives the constraint (7.5.32). Thus in all cases except A_r with r even the situation is the same as in the general case where d is not divisible by 8.

For odd d the use of theta functions is somewhat awkward, but it suffices to observe that the matrix

$$M = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{7.5.33}$$

commutes with the S-matrix

$$S_{\mathrm{so}(d)} = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} \tag{7.5.34}$$

Written in terms of $\mathbf{g}$-characters, (7.5.33) becomes identical to (7.5.28), and the rest of the argument is as before.

In the notation of (7.5.15), the conformal embedding invariant (7.5.29) reads

$$(Z_{\mathrm{c.e.}})_{\Lambda,\Lambda'} = 2^{r-1}\, \delta_{\Lambda,\rho}\delta_{\Lambda',\rho} + \sum_{\substack{\mu,\mu' \in \Sigma^{-1}(\rho) \\ \epsilon(\mu)=\epsilon(\mu')=1}} \delta_{\Lambda,\mu}\delta_{\Lambda',\mu'} + \sum_{\substack{\mu,\mu' \in \Sigma^{-1}(\rho) \\ \epsilon(\mu)=\epsilon(\mu')=-1}} \delta_{\Lambda,\mu}\delta_{\Lambda',\mu'}\,, \tag{7.5.35}$$

while (7.5.30) and (7.5.31) with $\epsilon(\rho) = \pm 1$ can be summarized as

$$(Z'_{\mathrm{c.e.}})_{\Lambda,\Lambda'} = (2^{r-1}+1)\, \delta_{\Lambda,\rho}\delta_{\Lambda',\rho} + \sum_{\substack{\mu,\mu' \in \Sigma^{-1}(\rho) \\ \epsilon(\mu)=\epsilon(\mu')=1}} \delta_{\Lambda,\mu}\delta_{\Lambda',\mu'} + \sum_{\substack{\mu,\mu' \in \Sigma^{-1}(\rho) \\ \epsilon(\mu)=\epsilon(\mu')=-1}} \delta_{\Lambda,\mu}\delta_{\Lambda',\mu'}\,. \tag{7.5.36}$$

(By inspection one easily verifies that these matrices commute with T, that the correct number $\dim(\mathrm{so}(d)) - \dim(\bar{\mathbf{g}}) = d(d-3)/2$ of spin one currents are combined with the identity field, and that the 'quantum dimension' sum rule (7.5.32) is satisfied also for d not a multiple of 8.) Note that in the summations in (7.5.35) and (7.5.36) (and also in those for the branching rules (7.5.18) of $\mathcal{X}_{\mathrm{o}}$ and $\mathcal{X}_{\mathrm{v}}$) the weight $\mu = \rho$ does not contribute, except for A_r with even r, in which case it contributes to $\mathcal{X}_{\mathrm{o}}$ (if $d \equiv r(r+2) = 0 \bmod 16$) and to $\mathcal{X}_{\mathrm{v}}$ (if $d = 8 \bmod 16$), respectively.

Let us finally present some examples for the explicit form of the conformal embedding invariants. The most interesting cases are those with exceptional $\bar{\mathbf{g}}$. The primary fields are again labeled by their unshifted highest weights. We find

$$\begin{aligned} \mathcal{Z}_{\mathrm{c.e.}}(F_{4,9}) = \;& |\,(0,0,0,0) + (0,0,1,6) + (0,0,2,1) + (0,1,0,0) \\ & \quad + (0,1,1,2) + (0,3,0,0) + (1,0,0,5) + (1,1,0,4)\,|^2 \\ & + |\,(0,0,0,7) + (0,0,2,0) + (0,0,3,0) + (0,1,0,3) \\ & \quad + (0,1,0,6) + (0,2,0,2) + (1,0,0,0) + (1,0,1,4)\,|^2 \\ & + 2 \cdot |\,2\,(1,1,1,1)\,|^2 \end{aligned} \tag{7.5.37}$$

and

$$
\begin{aligned}
\mathcal{Z}_{\mathrm{c.e.}}(E_{6,12}) = \;& |\,(0,0,0,0,0,0) + (0,0,0,0,12,0) + (0,0,1,0,0,0) + (0,0,1,0,9,0)\\
&\quad + (0,0,2,0,3,0) + (0,1,0,0,5,2) + (0,1,0,2,1,0) + (0,2,0,0,1,0)\\
&\quad + (0,2,0,0,7,0) + (1,0,0,0,7,2) + (1,0,0,2,0,0) + (1,0,3,0,1,0)\\
&\quad + (1,1,1,0,3,1) + (1,1,1,1,1,0) + (1,2,0,0,5,1) + (1,2,0,1,0,0)\\
&\quad + (2,0,0,1,3,1) + (2,0,1,0,2,0) + (2,0,1,0,5,0) + (3,0,2,0,0,0)\\
&\quad + (3,0,2,0,3,0) + (3,0,1,1,1,1) + (3,1,0,0,2,1) + (3,1,0,1,3,0)\\
&\quad + (4,0,0,0,4,0) + (5,0,0,2,1,1) + (5,0,0,1,0,2) + (5,0,1,0,2,0)\\
&\quad + (7,0,0,2,0,0) + (7,0,0,0,1,2) + (9,0,1,0,0,0) + (12,0,0,0,0,0)\,|^2\\
& + |\,(0,0,0,0,0,1) + (0,0,0,0,6,3) + (0,0,0,1,10,0) + (0,0,0,3,0,0)\\
&\quad + (0,0,4,0,0,0) + (0,1,0,0,8,1) + (0,1,0,1,0,0) + (0,1,2,0,2,0)\\
&\quad + (0,2,0,0,4,2) + (0,2,0,2,0,0) + (0,3,0,0,0,0) + (0,3,0,0,6,0)\\
&\quad + (1,0,1,0,4,1) + (1,0,1,1,2,0) + (1,1,0,0,6,1) + (1,1,0,1,1,0)\\
&\quad + (2,0,2,0,2,1) + (2,0,2,1,0,0) + (2,1,0,1,2,1) + (2,1,1,0,1,0)\\
&\quad + (2,1,1,0,4,0) + (3,0,0,0,3,1) + (3,0,0,1,4,0) + (4,0,0,2,0,2)\\
&\quad + (4,0,1,0,1,1) + (4,0,1,1,2,0) + (4,1,0,0,3,0) + (6,0,0,0,0,3)\\
&\quad + (6,0,0,1,1,1) + (6,0,0,3,0,0) + (8,0,0,1,0,1) + (10,1,0,0,0,0)\,|^2\\
& + 2\cdot|\,4\,(1,1,1,1,1,1)\,|^2
\end{aligned}
\tag{7.5.38}
$$

$$
\begin{aligned}
\mathcal{Z}_{\text{c.e.}}(E_{7,18}) = \; & |\,(0,0,0,0,0,0,0) + (0,0,0,0,0,10,4) + (0,0,0,0,0,1,3) + (1,0,0,0,0,12,2) \\
& + (0,0,0,0,1,16,0) + (0,0,0,0,4,0,2) + (4,0,0,0,0,7,1) + (0,0,0,0,5,0,0) \\
& + (0,0,0,1,0,0,1) + (0,0,0,1,0,0,3) + (0,1,0,0,0,10,2) + (0,1,0,0,0,0,0) \\
& + (0,0,0,1,1,0,1) + (0,2,0,0,0,12,0) + (0,0,0,3,0,4,1) + (0,3,0,0,0,4,0) \\
& + (0,0,0,6,0,0,0) + (0,0,1,0,0,14,0) + (1,0,1,0,0,8,2) + (0,0,1,0,1,1,1) \\
& + (2,0,1,0,0,10,0) + (3,0,1,0,0,5,1) + (0,0,1,0,3,2,0) + (0,3,1,0,0,2,0) \\
& + (0,0,2,0,0,1,1) + (0,0,2,0,0,8,0) + (0,0,2,0,3,0,0) + (0,0,3,0,0,6,0) \\
& + (0,1,0,0,2,0,2) + (2,0,0,1,0,8,1) + (0,1,0,0,3,0,0) + (0,1,0,1,2,2,1) \\
& + (2,1,0,1,0,5,0) + (0,2,0,1,0,2,1) + (0,1,0,2,0,6,0) + (0,1,0,4,0,2,0) \\
& + (1,0,1,1,0,6,1) + (0,1,1,0,1,2,0) + (2,1,1,1,0,3,0) + (0,1,1,2,0,4,0) \\
& + (0,1,2,0,1,0,0) + (2,0,0,2,0,3,1) + (0,2,0,0,2,4,0) + (0,2,0,2,2,0,0) \\
& + (0,3,0,2,0,0,0) + (0,2,1,0,2,2,0) + (2,1,0,3,0,1,0) + (0,4,0,0,3,0,0) \\
& + (0,5,0,0,1,0,0) + (3,0,0,0,1,6,2) + (1,0,0,0,3,1,1) + (4,0,0,0,1,8,0) \\
& + (1,2,0,0,1,3,1) + (1,0,0,2,1,4,0) + (1,0,1,0,2,1,1) + (2,0,1,0,1,6,0) \\
& + (1,0,1,2,1,2,0) + (2,0,2,0,1,4,0) + (1,1,0,1,1,4,1) + (1,1,0,1,1,3,0) \\
& + (1,3,0,1,1,1,0) + (1,1,1,1,1,1,0) + (1,2,0,2,1,2,0) + (2,0,0,4,1,0,0)\,|^2 \\
+ & |\,(0,0,0,0,0,18,0) + (0,0,0,0,0,0,4) + (0,0,0,0,0,11,3) + (0,0,0,0,1,0,2) \\
& + (1,0,0,0,0,0,0) + (4,0,0,0,0,6,2) + (0,0,0,0,4,1,1) + (5,0,0,0,0,8,0) \\
& + (0,1,0,0,0,13,1) + (0,1,0,0,0,9,3) + (0,0,0,1,0,1,2) + (0,0,0,1,0,15,0) \\
& + (1,1,0,0,0,11,1) + (0,0,0,2,0,0,0) + (0,3,0,0,0,3,1) + (0,0,0,3,0,5,0) \\
& + (0,6,0,0,0,0,0) + (0,0,1,0,0,0,0) + (0,0,1,0,1,0,2) + (1,0,1,0,0,9,1) \\
& + (0,0,1,0,2,0,0) + (0,0,1,0,3,1,1) + (3,0,1,0,0,6,0) + (0,0,1,3,0,3,0) \\
& + (0,0,2,0,0,7,1) + (0,0,2,0,0,2,0) + (3,0,2,0,0,4,0) + (0,0,3,0,0,0,0) \\
& + (2,0,0,1,0,7,2) + (0,1,0,0,2,1,1) + (3,0,0,1,0,9,0) + (2,1,0,1,0,4,1) \\
& + (0,1,0,1,2,3,0) + (0,1,0,2,0,5,1) + (0,2,0,1,0,3,0) + (0,4,0,1,0,1,0) \\
& + (0,1,1,0,1,1,1) + (1,0,1,1,0,7,0) + (0,1,1,1,2,1,0) + (0,2,1,1,0,1,0) \\
& + (1,0,2,1,0,5,0) + (0,2,0,0,2,3,1) + (2,0,0,2,0,4,0) + (2,2,0,2,0,2,0) \\
& + (0,2,0,3,0,3,0) + (2,0,1,2,0,2,0) + (0,3,0,1,2,1,0) + (3,0,0,4,0,0,0) \\
& + (1,0,0,5,0,1,0) + (1,0,0,0,3,0,2) + (3,0,0,0,1,7,1) + (1,0,0,0,4,0,0) \\
& + (1,0,0,2,1,3,1) + (1,2,0,0,1,4,0) + (2,0,1,0,1,5,1) + (1,0,1,0,2,2,0) \\
& + (1,2,1,0,1,2,0) + (1,0,2,0,2,0,0) + (1,1,0,1,1,2,1) + (1,1,0,1,1,5,0) \\
& + (1,1,0,3,1,1,0) + (1,1,1,1,1,3,0) + (1,2,0,2,1,0,0) + (1,4,0,0,2,0,0)\,|^2 \\
+ & 2 \cdot |\,8\,(1,1,1,1,1,1,1)\,|^2
\end{aligned}
\tag{7.5.39}
$$

$$
\begin{aligned}
\mathcal{Z}_{\text{c.e.}}(E_{8,30}) = \; & |(0,0,0,0,0,0,0,0) + (0,0,0,0,0,0,0,3) + (0,0,0,0,0,0,0,10) + (0,0,0,0,0,0,7,0) + (0,0,0,0,0,1,0,1) \\
& +(0,0,0,0,0,1,1,1) + (0,0,0,0,1,0,1,1) + (0,0,0,0,1,0,5,0) + (0,0,0,0,4,0,0,0) + (0,0,0,1,0,0,0,0) \\
& +(0,0,0,1,0,0,3,0) + (0,0,0,1,0,4,0,0) + (0,0,0,1,1,0,1,0) + (0,0,0,2,0,0,4,0) + (0,0,0,3,0,0,0,0) \\
& +(0,0,0,3,0,0,0,4) + (0,0,0,6,0,0,0,0) + (0,0,1,0,0,0,3,1) + (0,0,1,0,0,2,3,0) + (0,0,1,1,0,1,1,0) \\
& +(0,0,2,0,0,3,0,1) + (0,0,2,0,1,0,3,0) + (0,0,2,0,3,0,0,0) + (0,0,2,1,0,1,0,0) + (0,0,3,0,0,0,0,6) \\
& +(0,0,4,0,0,0,0,1) + (0,0,4,0,0,0,3,0) + (0,1,0,0,0,0,0,0) + (0,1,0,0,0,0,5,0) + (0,1,0,0,1,0,3,0) \\
& +(0,1,0,0,2,2,0,0) + (0,1,0,1,0,2,2,0) + (0,1,0,1,2,0,0,2) + (0,1,0,2,0,0,2,0) + (0,1,0,4,0,0,0,2) \\
& +(0,1,1,0,0,2,1,0) + (0,1,1,2,0,0,0,4) + (0,1,2,0,0,1,2,1) + (0,1,2,0,1,0,1,0) + (0,1,2,0,1,2,0,0) \\
& +(0,1,4,0,0,0,1,0) + (0,2,0,0,2,0,2,0) + (0,2,0,1,0,2,0,0) + (0,2,0,1,0,2,0,2) + (0,2,0,2,2,0,0,0) \\
& +(0,2,1,0,2,0,0,2) + (0,2,2,0,0,1,0,1) + (0,2,2,0,1,0,2,0) + (0,3,0,0,2,0,0,0) + (0,3,0,1,0,0,2,2) \\
& +(0,3,0,2,0,2,0,0) + (0,3,1,0,0,2,0,2) + (0,3,2,0,1,0,0,0) + (0,4,0,0,3,0,0,0) + (0,4,0,1,0,0,0,2) \\
& +(0,4,0,2,0,0,2,0) + (0,4,1,0,0,0,2,2) + (0,5,0,0,1,2,0,0) + (0,5,0,2,0,0,0,0) + (0,5,1,0,0,0,0,2) \\
& +(0,6,0,0,1,0,2,0) + (0,7,0,0,1,0,0,0) + (0,8,0,0,0,0,3,0) + (0,9,0,0,0,0,1,0) + (1,0,0,0,0,0,5,1) \\
& +(1,0,0,0,0,4,1,0) + (1,0,0,1,0,1,3,0) + (1,0,1,0,0,0,0,8) + (1,0,1,0,1,2,1,0) + (1,0,1,0,2,1,0,1) \\
& +(1,0,1,1,0,0,0,6) + (1,0,1,1,0,1,2,0) + (1,0,1,2,1,0,0,2) + (1,0,3,0,0,0,2,1) + (1,0,3,0,0,2,1,0) \\
& +(1,1,0,0,0,3,1,0) + (1,1,0,1,1,0,0,4) + (1,1,1,0,1,1,1,0) + (1,1,1,0,1,1,1,1) + (1,1,1,1,1,1,0,1) \\
& +(1,1,3,0,0,1,1,0) + (1,2,0,0,1,1,0,3) + (1,2,0,2,1,0,0,2) + (1,2,1,0,1,0,1,1) + (1,2,1,1,0,1,1,1) \\
& +(1,3,0,0,0,1,1,3) + (1,3,0,1,1,1,0,1) + (1,3,1,1,0,0,1,1) + (1,4,0,0,0,0,1,3) + (1,4,0,1,0,1,1,1) \\
& +(1,5,0,1,0,0,1,1) + (1,6,0,0,0,2,1,0) + (1,7,0,0,0,1,1,0) + (2,0,0,0,3,0,1,0) + (2,0,0,1,0,3,0,0) \\
& +(2,0,0,2,0,1,0,3) + (2,0,0,4,1,0,0,0) + (2,0,2,0,0,2,0,1) + (2,0,2,0,1,0,0,4) + (2,0,2,0,2,0,1,0) \\
& +(2,1,0,0,2,1,0,0) + (2,1,0,1,1,0,1,2) + (2,1,0,3,0,1,0,1) + (2,1,1,1,0,1,0,3) + (2,1,2,0,1,1,0,0) \\
& +(2,2,0,1,0,1,0,2) + (2,2,0,2,1,0,1,0) + (2,2,1,0,1,0,1,2) + (2,3,0,2,0,1,0,0) + (2,3,1,0,0,1,0,2) \\
& +(2,4,0,0,2,0,1,0) + (2,5,0,0,1,1,0,0) + (3,0,0,0,1,0,0,6) + (3,0,1,0,0,1,0,5) + (3,0,1,0,2,0,0,1) \\
& +(3,0,1,2,0,0,1,2) + (3,1,0,1,0,0,1,4) + (3,1,1,1,1,0,0,1) + (3,2,0,0,1,0,0,3) + (3,2,0,2,0,0,1,2) \\
& +(3,3,0,1,1,0,0,1) + (4,0,0,2,0,0,0,3) + (4,0,0,4,0,0,1,0) + (4,0,2,0,0,0,1,4) + (4,1,0,3,0,0,0,1) \\
& +(4,1,1,1,0,0,0,3) + (5,0,0,0,0,0,1,6) + (5,0,1,0,0,0,0,5)|^2 \\
& + |(0,0,0,0,0,0,1,2) + (0,0,0,0,0,0,6,1) + (0,0,0,0,0,1,0,2) + (0,0,0,0,0,2,0,0) + (0,0,0,0,0,5,0,0) \\
& +(0,0,0,0,1,0,0,0) + (0,0,0,0,1,0,2,0) + (0,0,0,0,2,0,0,0) + (0,0,0,1,0,0,2,1) + (0,0,0,1,0,1,4,0) \\
& +(0,0,0,2,0,1,0,0) + (0,0,1,0,0,0,0,0) + (0,0,1,0,0,0,4,0) + (0,0,1,0,1,0,2,0) + (0,0,1,0,1,3,0,0) \\
& +(0,0,1,0,3,0,0,1) + (0,0,1,1,0,1,3,0) + (0,0,1,2,0,0,1,0) + (0,0,1,3,0,0,0,3) + (0,0,2,0,0,0,0,7) \\
& +(0,0,2,0,0,2,0,0) + (0,0,3,0,0,0,3,1) + (0,0,3,0,0,3,0,0) + (0,0,3,0,1,0,0,0) + (0,0,5,0,0,0,0,0) \\
& +(0,1,0,0,0,0,0,9) + (0,1,0,0,0,0,4,1) + (0,1,0,0,0,3,2,0) + (0,1,0,1,0,1,2,0) + (0,1,0,2,0,0,0,5) \\
& +(0,1,1,0,1,1,2,0) + (0,1,1,0,1,2,0,1) + (0,1,1,1,0,1,1,0) + (0,1,1,1,2,0,0,1) + (0,1,3,0,0,0,1,1) \\
& +(0,1,3,0,0,1,2,0) + (0,2,0,0,0,3,0,0) + (0,2,0,0,2,0,0,3) + (0,2,0,3,0,0,0,3) + (0,2,1,0,1,0,2,1) \\
& +(0,2,1,0,1,1,0,0) + (0,2,1,1,0,2,0,1) + (0,2,3,0,0,1,0,0) + (0,3,0,0,0,2,0,3) + (0,3,0,1,2,0,0,1) \\
& +(0,3,1,0,1,0,0,1) + (0,3,1,1,0,0,2,1) + (0,4,0,0,0,0,2,3) + (0,4,0,1,0,2,0,1) + (0,4,1,1,0,0,0,1) \\
& +(0,5,0,0,0,0,0,3) + (0,5,0,1,0,0,2,1) + (0,6,0,0,0,3,0,0) + (0,6,0,1,0,0,0,1) + (0,7,0,0,0,1,2,0) \\
& +(0,8,0,0,0,1,0,0) + (0,10,0,0,0,0,0,0) + (1,0,0,0,0,0,0,0) + (1,0,0,0,0,0,6,0) + (1,0,0,0,1,0,4,0) \\
& +(1,0,0,0,3,1,0,0) + (1,0,0,1,0,3,1,0) + (1,0,0,2,0,0,3,0) + (1,0,0,2,1,0,0,3) + (1,0,0,5,0,0,0,1) \\
& +(1,0,1,0,0,2,2,0) + (1,0,2,0,0,2,1,1) + (1,0,2,0,1,0,2,0) + (1,0,2,0,2,1,0,0) + (1,0,2,1,0,0,0,5) \\
& +(1,0,4,0,0,0,2,0) + (1,1,0,0,2,1,1,0) + (1,1,0,1,0,2,1,0) + (1,1,0,1,1,1,0,2) + (1,1,0,3,1,0,0,1) \\
& +(1,1,1,1,1,0,0,3) + (1,1,2,0,0,1,1,1) + (1,1,2,0,1,1,1,0) + (1,2,0,0,2,0,1,0) + (1,2,0,1,0,1,1,2) \\
& +(1,2,0,2,1,1,0,0) + (1,2,1,0,1,1,0,2) + (1,2,2,0,1,0,1,0) + (1,3,0,1,0,0,1,2) + (1,3,0,2,0,1,1,0) \\
& +(1,3,1,0,0,1,1,2) + (1,4,0,0,2,1,0,0) + (1,4,0,2,0,0,1,0) + (1,4,1,0,0,0,1,2) + (1,5,0,0,1,1,1,0) \\
& +(1,6,0,0,1,0,1,0) + (1,8,0,0,0,0,2,0) + (2,0,0,0,0,4,0,0) + (2,0,0,1,0,0,0,7) + (2,0,1,0,1,0,0,5) \\
& +(2,0,1,0,1,2,0,0) + (2,0,1,0,2,0,1,1) + (2,0,1,2,0,1,0,2) + (2,0,3,0,0,2,0,0) + (2,1,0,1,0,1,0,4) \\
& +(2,1,1,0,1,1,0,1) + (2,1,1,1,1,0,1,1) + (2,2,0,0,1,0,1,3) + (2,2,0,2,0,1,0,2) + (2,2,1,1,0,1,0,1) \\
& +(2,3,0,0,0,1,0,3) + (2,3,0,1,1,0,1,1) + (2,4,0,1,0,1,0,1) + (2,6,0,0,0,2,0,0) + (3,0,0,0,3,0,0,0) \\
& +(3,0,0,2,0,0,1,3) + (3,0,0,4,0,1,0,0) + (3,0,2,0,0,1,0,4) + (3,0,2,0,2,0,0,0) + (3,1,0,1,1,0,0,2) \\
& +(3,1,0,3,0,0,1,1) + (3,1,1,1,0,0,1,3) + (3,2,0,2,1,0,0,0) + (3,2,1,0,1,0,0,2) + (3,4,0,0,2,0,0,0) \\
& +(4,0,0,0,0,1,0,6) + (4,0,1,0,0,0,1,5) + (4,0,1,2,0,0,0,2) + (4,1,0,1,0,0,0,4) + (4,2,0,2,0,0,0,2) \\
& +(5,0,0,4,0,0,0,0) + (5,0,2,0,0,0,0,4) + (6,0,0,0,0,0,0,6)|^2 \\
& + 2 \cdot |8\,(1,1,1,1,1,1,1,1)|^2
\end{aligned}
$$

Summary

Quantum field theory has proven to be a fundamental concept in modern physics and has given surprising new insight into many mathematical structures. Its applications in physics range from the standard model of elementary particle physics to the description of excitations in solid state physics. In mathematics, ideas motivated by quantum field theory have lead to new conjectures and novel proofs in many areas, e.g. the theory of modular forms and number theory, algebraic geometry or the theory of low-dimensional manifolds.

In two dimensions the structure of quantum field theories is particularly rich: while for theories in four or more dimensions statistics is governed by the permutation group, which restricts particles to have either bosonic or fermionic statistics, it is described in two dimensions by the braid group which allows for particles with more general statistics, e.g. anyons.

Another special feature of two dimensions is that in this case the conformal algebra is infinite-dimensional. As a consequence there are particularly powerful algebraic tools for the study of those two-dimensional quantum field theories which are covariant under the conformal algebra. These conformal field theories are also of considerable interest in several physical applications: they arise naturally in the description of critical behaviour of two-dimensional systems in statistical mechanics and of 'vacuum configurations' in string theory.

One aspect of two-dimensional quantum field theories which makes a large subset of them accessible to explicit calculations is that in two dimensions the number of superselection sectors can be finite, in which case the theory is called rational. For certain rational theories descriptions are known in which one can perform exact, and hence in particular fully non-perturbative, calculations. One such description is the so-called coset construction. It allows to describe conformal field theories in the mathematical framework of affine Lie algebras.

In this thesis various aspects of rational field theories are studied. In Part I we construct explicitly examples for a particularly interesting subclass of conformal field theories: $N = 2$ superconformal field theories. These theories are, in addition to their invariance under the conformal algebra, also invariant under an extended ($N = 2$) supersymmetry algebra. In these models the operator product of certain fields gives rise to a nilpotent finite-dimensional ring, the chiral ring. This ring structure allows to make contact to other theories like topological field theories. Another important aspect of these models is that they can be used as the inner sector of a space-time supersymmetric vacuum configuration in heterotic string compactifications.

In Chapter 2 we construct explicitly many examples for these models using the coset construction; we classify all coset conformal field theories which have $N = 2$ supersymmetry and derive several general properties of these models, e.g. that the set of Ramond ground state in these theories is invariant under charge conjugation. To obtain a fully consistent coset conformal field theory, several non-trivial constructions are required, in particular field identification fixed points, if they occur, have to be resolved. In Chapter

2 we obtain a complete proof that the models under consideration in this thesis are fully consistent conformal field theories.

As it turns out, distinct cosets can describe the same conformal field theory. In Chapter 3 we show that certain series of cosets for which rank and level (or simple functions thereof) are interchanged describe in fact one and the same conformal field theory. In the proof we use level-rank dualities for WZW theories which are also described in this chapter.

In Chapter 4 we use the models introduced in Chapter 2 to construct string vacua. To use a conformal field theory in a compactification of the heterotic string one has to implement several projections on it. It is explained how this can be done with the help of simple currents. This prescription leads to the definition of the so-called extended Poincaré polynomial. We use this polynomial to compute the massless spectra of the string compactifications based on the $N=2$ coset models constructed in Chapter 2.

In the investigations leading to the results presented in Part I of this thesis field identification fixed points had to be resolved. In a closer study of the theories which describe the resolution procedure we found that the matrix S, which implements in these theories the modular transformation $\tau \mapsto -1/\tau$ on the space of characters, possesses several surprising symmetries. Closer examination revealed that such symmetries are in fact present in any rational conformal field theory.

This new type of symmetry is the subject of Part II of this thesis. It is induced by the Galois group of the algebraic number field which contains the entries of S. Galois symmetry turns out to be extremely powerful: in Chapter 5 we show that it provides novel methods for the study of fusion rings. The Galois symmetries induce automorphisms of the underlying fusion rings; they can also be used to construct modular invariant partition functions.

The tools developed in Chapter 5 are applied in Chapter 6 to the fusion rings of WZW theories. It is shown that Galois symmetries can explain in a uniform way modular invariants that had previously been constructed by various other methods, e.g. simple currents, level-rank dualities or conformal embeddings. Moreover, in a systematic search, we discovered several infinite series of previously unknown exceptional invariants for WZW theories based on algebras of type B and D at level 2.

As it turns out, not all known modular invariants of WZW models can be explained by Galois symmetries. However, in the special case of WZW models one can generalize the method by allowing for more general mappings on weight space than Galois scalings. These quasi-Galois symmetries are the subject of Chapter 7 of the present thesis. They have various applications: they lead to sum rules for the elements of the modular matrix S which can be used for the construction of modular invariants. Moreover, they relate WZW theories at different levels and provide a powerful algorithm for the computation of the branching rules of conformal embeddings.

Samenvatting (summary in Dutch)

Quantumveldentheorie speelt een belangrijke en fundamentele rol in de moderne natuurkunde en heeft geleid tot verrassende nieuwe inzichten in diverse mathematische structuren. De toepassingen ervan in de natuurkunde lopen uiteen van het standaardmodel van de elementaire deeltjes tot de beschrijving van excitaties in vaste-stof fysica. In de wiskunde hebben ideeën uit quantumveldentheorie in velerlei gebieden geleid tot nieuwe vermoedens en bewijzen; voorbeelden daarvan zijn de theorie van modulaire vormen en getaltheorie, algebraische meetkunde en de theorie van laag-dimensionale variëteiten.

De structuur van quantumveldentheorieën is in het bijzonder in twee dimensies erg interessant: daar waar in vier of meer dimensies de statistiek van deeltjes wordt bepaald door de permutatiegroep, waardoor er slechts fermionen en bosonen mogelijk zijn, is het in twee dimensies de vlechtgroep die deze rol speelt en zorgt voor de mogelijkheid van deeltjes met andere statistiek, bv. anyonen.

Een andere bijzondere eigenschap in twee dimensies is dat de conforme algebra oneindig dimensionaal is. Als gevolg hiervan kan er gebruik gemaakt worden van krachtige algebraische methoden om conform invariante quantumveldentheorieën in twee dimensies te bestuderen.

Omdat in twee dimensies het aantal superselectiesectoren eindig kan zijn (de zogenaamde 'rationele' theorieën) staan veel tweedimensionale theorieën open voor expliciete berekeningen. Voor sommige van die theorieën zijn er formuleringen waarin het mogelijk is om exacte, en dus in het bijzonder niet-perturbatieve, berekeningen te doen. Eén van die formuleringen is de zogeheten 'coset' constructie. De beschrijving van conforme veldentheorieën op deze manier valt in het wiskundige kader van affiene Lie algebra's.

In dit proefschrift worden verschillende aspecten van rationele veldentheorieën bestudeerd. In het eerste gedeelte construeren we expliciete voorbeelden uit een bijzonder interessante subverzameling van conforme veldentheorieën, namelijk de $N = 2$ superconforme veldentheorieën. Deze zijn niet alleen invariant onder de conforme algebra, maar ook onder een uitgebreide ($N = 2$) supersymmetrie algebra. In deze modellen leiden de operatorproducten van bepaalde velden tot een nilpotente, eindigdimensionale ring, de chirale ring. Deze ringstructuur maakt het mogelijk contact te leggen met andere theorieën, zoals topologische veldentheorieën. Een ander belangrijk aspect van deze modellen is dat ze gebruikt kunnen worden voor de beschrijving van de interne sector van ruimte-tijd supersymmetrische vacuumtoestanden in compactificaties van heterotische strings.

In hoofdstuk 2 werken we expliciet een groot aantal voorbeelden van deze modellen uit door gebruik te maken van de coset constructie. De classificatie van alle coset conforme veldentheorieën met $N = 2$ supersymmetrie wordt uitgevoerd en diverse algemene eigenschappen van deze modellen, bv. dat de verzameling van Ramond grondtoestanden in deze theorieën invariant is onder ladingsconjugatie, worden afgeleid. Om te komen tot een volledig consistente coset conforme veldentheorie is het nodig om een aantal niet-triviale problemen op te lossen; in het bijzonder is het noodzakelijk dat de vaste punten die ontstaan bij de identificatie van velden opgelost worden. In hoofdstuk 2 wordt het volledige bewijs gegeven dat de modellen die in dit proefschrift worden behandeld inderdaad consistent zijn.

Het is bekend dat verschillende cosets verschillende beschrijvingen kunnen geven van één en dezelfde conforme veldentheorie. In hoofdstuk 3 wordt duidelijk gemaakt dat dit verschijnsel optreedt voor bepaalde reeksen cosets die aan elkaar gerelateerd zijn door verwisseling van de rang en de 'level' (of eenvoudige functies daarvan). In het bewijs daarvan wordt gebruik gemaakt van level-rang dualiteiten voor WZW theorieën, die ook in dit hoofdstuk worden beschreven.

In hoofdstuk 4 worden vervolgens de modellen uit hoofdstuk 2 gebruikt voor de constructie van stringvacua. Om in de compactificatie van de heterotische string gebruik te kunnen maken van conforme veldentheorieën is het noodzakelijk om een aantal projecties uit te voeren. Uitgelegd wordt hoe dit mogelijk is met behulp van 'simple currents'. Deze beschrijving leidt tot de definitie van het zogeheten uitgebreide Poincaré polynoom. Dit polynoom wordt daarna gebruikt om het spectrum van massaloze toestanden van de op $N = 2$ coset modellen gebaseerde stringcompactificaties uit hoofdstuk 2 uit te rekenen.

In het onderzoek uit het eerste gedeelte van dit proefschrift was het nodig om vaste punten op te lossen. Bij nadere bestudering van de theorieën die dit proces beschrijven hebben we gevonden dat de matrix S, die in deze theorieën de modulaire transformatie $\tau \mapsto -1/\tau$ op de ruimte van karakters beschrijft, verschillende verrassende symmetrieën heeft. Verder onderzoek laat zien dat deze symmetrieën ook aanwezig zijn in alle andere rationele conforme veldentheorieën.

Deze nieuwe symmetrie is het onderwerp van het tweede gedeelte van dit proefschrift. Ze wordt geïnduceerd door de Galois groep van het algebraische getallenlichaam die die componenten van S bevat. Galois symmetrie blijkt bijzonder krachtig te zijn: in hoofdstuk 5 laten we zien dat ze leidt tot nieuwe methoden voor de bestudering van fusieringen. De Galois symmetrieën induceren automorfismen van de onderliggende fusieringen; ze kunnen ook gebruikt worden voor de constructie van modulair invariante partitiefuncties.

De gereedschappen die in 5 zijn ontwikkeld worden vervolgens in hoofdstuk 6 toegepast op de fusieringen van WZW theorieën. We laten zien dat met behulp van Galois symmetrieën het optreden van bepaalde modulaire invarianten die eerder met andere methoden, zoals simple currents, level-rang dualiteiten en conforme inbeddingen zijn gevonden, kunnen worden begrepen. Daarnaast hebben we in een systematische studie verschillende oneindige reeksen van tot nog toe onbekende exceptionele invarianten van WZW theorieën, gebaseerd op algebras van het type B en D op level 2, geconstrueerd.

Het blijkt dat niet alle modulaire invarianten van WZW modellen kunnen worden gevonden met behulp van Galois symmetrieën. In het bijzondere geval van WZW modellen is het echter mogelijk om deze methode te generaliseren naar meer algemene afbeeldingen op de gewichtsruimte dan de Galois schalingen. Deze quasi-Galois symmetrieën zijn het onderwerp van hoofdstuk 7. Ze hebben verschillende toepassingen: ze leiden tot somregels voor de elementen van de modulaire matrix S, die gebruikt kunnen worden voor de constructie van modulaire invarianten. Bovendien relateren ze WZW theorieën op verschillende levels en leiden ze tot een krachtig algoritme voor de berekening van splitsingsregels van conforme inbeddingen.

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