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Abstract

We first give a review of the geometrical techniques (in particular G-structures and Generalized Complex Geometry) that are currently used in the study of supersymmetric $\mathcal{N}=1$ compactifications. Then we focus on the study of type IIB compactifications to four dimensional Anti de Sitter vacua with $\mathcal{N}=1$ supersymmetry. We give the general conditions that supersymmetry imposes on the solutions in particular, the internal manifold must have $SU(2)$-structure. Then we perform an exhaustive search of such vacua on cosets and group manifold. With some assumptions on the $SU(2)$ torsion classes and constant dilaton and warp factor, we find that such vacua are very rare : two on cosets and five on group manifolds. All of them are required to include intersecting O5 and O7-planes. This also means that there exist no sourceless AdS_4 vacua. We then study scales separation on the vacua in order to look give some insights for four dimensional effective theory and found that only two of them admit scales separation.

A long french summary of the thesis is given in appendix B.

Résumé court

Nous commençons par donner une revue des techniques géométriques (en particulier les G-structures et la Géometrie Complexe généralisée) qui sont couramment utilisées dans l'étude des compactifications supersymétriques $\mathcal{N}=1$. Ensuite nous nous concentrons sur l'étude des compactifications de type IIB vers des vides Anti de Sitter avec supersymétrie $\mathcal{N}=1$. Nous donnons les conditions générales que la supersymétrie impose, en particulier la variété interne doit avoir une structure $SU(2)$. Nous faisons une recherche exhaustive de tels vides sur les quotients et les groupes de Lie. Avec quelques hypothèses sur les classes de torsion $SU(2)$ et avec dilaton et facteur de warping constants, on trouve que de tels vides sont très rares : deux sur des quotients et cinq sur des groupes de Lie. Tous requierent des plans O5 et O7 qui s'intersectent. Cela veut également dire qu'il n'existe pas de vide AdS_4 sans sources. Nous étudions ensuite la séparation d'échelles sur ces vides afin de donner quelques intuitions sur les théories effectives à quatre dimensions. Nous avons trouvé que seulement deux d'entre elles admettent séparation d'échelles.

Un résumé long en français se trouve appendix B.

Contents

1 Introduction **1**

2 G-Structure and Torsion **5**
2.1 G-structures . . . 5
2.1.1 Structure group . . . 5
2.1.2 G-structure . . . 6
2.1.3 Almost complex structure and almost complex manifold . . . 7
2.1.4 Hermitian metric and almost Hermitian manifold . . . 8
2.1.5 Symplectic structures . . . 8
2.1.6 Product structure . . . 8
2.1.7 $SU(3)$ and $SU(2)$ structures in dimension 6 . . . 9
2.2 Holonomy group and torsion . . . 11
2.2.1 Examples of integrable structures . . . 11
2.2.2 Intrinsic torsion . . . 12
2.2.3 Special holonomy . . . 14
2.2.4 Torsion for $SU(3)$-structure in dimension 6 . . . 15
2.2.5 Torsion for $SU(2)$-structure in dimension 6 . . . 16
2.2.6 Calabi Yau manifold . . . 17

3 Generalized Complex Geometry **19**
3.1 The generalized tangent bundle . . . 19
3.2 Generalized complex structure . . . 21
3.2.1 Generalized complex structure and generalized complex manifolds . . . 21
3.2.2 The Courant Bracket . . . 22
3.3 Pure spinors . . . 24
3.3.1 $O(d,d)$ spinors . . . 24
3.3.2 Pure spinors . . . 25
3.3.3 Pure spinors and generalized complex structures . . . 26
3.3.4 Type of a pure spinor . . . 27
3.4 Generalized Calabi Yau manifolds . . . 27
3.4.1 Differential structure of the manifold . . . 28
3.5 Metric from pure spinors . . . 29

4 Supersymmetry and $\mathcal{N}=1$ flux compactifications **31**
4.1 Supersymmetric solutions . . . 32
4.1.1 Supersymmetry equations . . . 32
4.1.2 Bianchi identities and e.o.m for the forms . . . 33
4.2 Pure geometry: Calabi-Yau compactifications . . . 34

4.3 $\mathcal{N} = 1$ flux compactifications: Generalized Calabi-Yau manifolds . . . 36
4.3.1 Topological condition . . . 37
4.3.2 Differential condition . . . 39

5 Anti de Sitter vacua in type IIB SUGRA on cosets and group manifolds 43
5.1 Motivations . . . 43
5.2 $\mathcal{N} = 1$ SUSY AdS_4 vacua in type IIB SUGRA . . . 45
5.2.1 The SUSY equations for rigid SU(2) structure . . . 46
5.2.2 Link to the $SU(2)$ torsion classes . . . 48
5.3 Restriction to parallelizable manifolds . . . 49
5.3.1 Sources . . . 50
5.3.2 O5 and O7 planes . . . 51
5.4 Explicit examples 1: Cosets . . . 54
5.5 Explicit examples 2: Group manifolds . . . 57
5.5.1 Bianchi Identities . . . 59
5.5.2 Intermediate $SU(2)$ structures on group manifolds . . . 60
5.5.3 Explicit vacua on group manifolds . . . 61
5.6 Scales separation . . . 66
5.6.1 Separation of scales without sources? . . . 69
5.6.2 Study of scales separation on the explicit examples . . . 70

6 Conclusion 73

A De Sitter Compactification 77
A.1 Motivations . . . 77
A.2 The Manifold $\mathfrak{g}_{5,17}^{p,-p,\pm 1}$ . . . 78
A.3 Compactification . . . 79
A.3.1 The metric . . . 79
A.3.2 The fluxes . . . 80
A.3.3 The 10-dimensional action . . . 81
A.3.4 Bianchi Identities . . . 81
A.3.5 The 4-dimensional action . . . 81
A.4 Sources . . . 83
A.4.1 Structure . . . 84
A.4.2 SUSY source . . . 84
A.4.3 SUSY breaking sources . . . 84

B Résumé long en français 87
B.1 Structures sur les variétés . . . 87
B.1.1 Groupes de structures . . . 87
B.1.2 Les cas SU(2) et SU(3) . . . 87
B.2 Géométrie complexe généralisée . . . 88
B.2.1 Structure complexe généralisée . . . 89
B.2.2 Spineurs purs . . . 90
B.3 Supersymétrie . . . 91
B.3.1 Généralités . . . 91
B.3.2 Géométrie complexe généralisée et compactifications $\mathcal{N} = 1$ . . . 91
B.4 Etudes des vides AdS sur les variétés parallélisable . . . 92
B.4.1 Recherche sur les groupes de Lie . . . 92

B.4.2 Quotients 94
B.4.3 Séparation d'échelle 94

Chapter 1

Introduction

Physicists have always tried to describe the world under a formalism as general as possible and to unify all known interactions. During the 20$^{\text{th}}$ century, tremendous progress in this direction has been done thanks to two main theories. At the microscopic level, the Standard Model gives an accurate description of electro-weak and strong interactions using quantum field theory. It provides us with a quantum description of matter and with the recent discovery of the Higgs boson at the Large Hadron Collider, its achievements are widely acclaimed. At the other end of the spectrum, General Relativity based on Riemannian geometry gives a description of how spacetime interacts with matter. Both theories are nevertheless incomplete in the sense that they can't explain everything we observe.

Indeed, in the Standard Model, many parameters are not theoretically fixed but rather are chosen so that their values fit the experiments. Moreover, the origin of the gauge group of the Standard Model $SU(3)\times SU(2)\times U(1)$ is still unknown. Why is it this one that one encounters and not some other? The answer could be that this theory is only an effective theory and inherits its structure from a more fundamental description. Also the Standard Model doesn't include gravity. Similarly, General Relativity, despite the accuracy of its predictions, fails to give a correct description of all observations scientists make. Indeed, already at the classical level, it cannot completely describe singular objects such as black holes. Moreover obtaining a quantum field theory description of gravity is not easy since renormalization fails.

There is also another conceptual issue in the description of the universe via these two theories : the hierarchy problem. One of the typical energy scale of the Standard Model is the electroweak scale, which is the scale where electromagnetic and weak interaction have a unified description (around 246 GeV). One can also define a scale where quantum gravity effects should appear and this is what one calls the Planck scale (around 10^{19} GeV). The discrepancy between these two scales is not explained within the framework of these theories and so require a more fundamental one. Another scale that one has to consider is the cosmological constant. Precise cosmological measurements show that our universe is expanding and so should be described by a de Sitter spacetime with a positive but small cosmological constant of mass scale $M_\Lambda = \Lambda^{\frac{1}{4}} \sim 10^{-12}$ GeV. This is much below both the electroweak scale and the Planck scale! And once again, there are no theoretical explanations as to why it is the case.

All these reasons show that there must be something more. At the moment, a theory that seems to give answers to most of these questions is string theory. We will now give some of the main properties of this theory, the interested reader can find more details for example in [1, 2]. The basic idea of string theory is to replace the notion of point particle by a one-dimensional extended object (a string). Then particles are seen as oscillation modes of the strings. One can easily show that the spectrum of string theory always contains a spin two particle which can be identified with the graviton. That is to say that string theory automatically contains gravity and since string theory is a quantum theory, it should even contain quantum gravity. Note that there exists a characteristic length in string theory which is the length of the string l_s which is usually taken to be proportionnal to the Planck scale (this is because the quantum gravity effects should appear at this scale).

In order to also include fermions in the theory, one generally consider a string theory with supersymmetry (SUSY) to give superstring theory. Rigid SUSY is an extension of the Poincaré group that has the feature of relating bosons and fermions (the action of SUSY on a boson gives a fermion and vice-versa). There can be many supersymmetry generators and one denotes by $\mathcal{N}$ their number (for instance, in four dimensions $\mathcal{N}$ can vary from one to four). An important feature of supersymmetry is that it commutes with the momentum, so that fields can be arranged in multiplets of same mass.

We can make SUSY a local symmetry and include gravity: these are supergravity (SUGRA) theories. These theories are very constrained. In particular, in eleven dimensions, the theory is unique. Unfortunately, these theories are not renormalizable[1]. A non renormalizable theory is perfectly acceptable if taken as the low energy limit of a more complete and well defined theory. In the case of SUGRA theories, they emerge as the low energy limit of string theory. That's why even if the SUGRA theories are not finite, their study is important as effective theories of string theory.

Anomaly cancellation forces superstring theories to live in ten dimensions. Thanks to that and the strong constraints coming from SUSY, one can show that there exists only five different types of superstring theories : Type I, Type IIA/IIB, Heterotic SO(32) and Heterotic $E_8 \times E_8$. Each of these theories give SUGRA theories in ten dimensions with $\mathcal{N} = 1$ (heterotic and type I) or $\mathcal{N} = 2$ (type IIA and IIB) supersymmetry. It is believed that also the eleven dimensions SUGRA can be interpreted as the low energy limit of what is called M-theory where the fundamental objects are not strings but rather extended objects like membranes. All these theories are related by a web of dualities and this hints to the existence of a single theory which would encompass all these. In this thesis, we will focus on type II string theories.

As we said, superstring theories involve a ten dimensional spacetime. One can wonder how one recovers the usual four dimensional spacetime and how one should consider the extra six spacelike dimensions. The simplest mechanism is compactification, namely one assumes that the six extra dimensions are compact. This means that they are closed (ie not extended) on themselves and form a six dimensional compact manifold. One usually assumes that the typical scale of this manifold is very small which would explain why we are not able to probe the extra dimensions. We call this manifold the internal manifold. Then one has only to "integrate" over this internal manifold to recover a four dimensional theory.

Unfortunately, if the method is straightforward, applying it is less easy. Indeed, in order to have phenomenologically interesting theories, one has to select particular internal manifolds. The simplest examples of such manifolds are Calabi-Yau but theories based on this spaces give a large number of scalar fields that have no fixed vacuum expectation value, called moduli. These scalars are not physical, since, if they existed, they would produce a fifth force that we don't observe. String theory contains form potentials, the Neveu-Schwarz and Ramond-Ramond fields, which can be seen as higher-dimensional generalizations of the electro-magnetic potential. Considering more general backgrounds where some of these fields are non-trivial allows to get

[1]This statement is still under debate, in particular for $\mathcal{N} = 8$ in four dimensions but recent results tend to show that no SUGRA is finite.

rid, at least partially, of the unwanted moduli (see [3] for a comprehensive review on fluxes compactification). This is the only method we know to solve the moduli problem at the classical supergravity level. Higher order perturbative corrections or other stringy mechanism are of course viable alternatives. The presence of fluxes on a compact manifold must be compatible with Gauss law. There are indeed no-go theorems forbidding purely flux compactifications to Minkowksi space. These can be avoided by introducing negative tensions sources, the so called O-planes. One can also look at more complicated manifolds with fluxes, O-planes and D-branes. These are non perturbative objects that sources the RR and NS fields.

As we said, the universe appears to possess a positive cosmological constant and that's why one of the important fields of study in string theory is obtaining a stable de Sitter vacuum. While stable de Sitter solutions can be found in lower dimensional gauge supergravity theories, it is very hard to embed them in sting theory. de Sitter space-time is not compatible with supersymmetry. Breaking supersymmetry while having a positive cosmological constant is a difficult task in string theory. In particular this seems to require, beside background fluxes, stringy objects like orientifold, quantum and higher derivative correction and non/geometric fluxes.

One of the first step towards obtaining a de Sitter vacuum may be, oddly, an Anti de Sitter vacuum. Indeed one can devise a way to break SUSY in the AdS vacuum in order to lift it to a de Sitter vacuum. This is the KKLT proposal [4]: the idea is to start from a AdS_4 vacuum in type IIB with all moduli fixed by fluxes and non-perturbative effects, and then to lift it to de Sitter by adding a small portion of anti-D3 branes wrapped on the internal manifold.

A significant part of the work in this thesis is devoted to the study of AdS_4 vacua with $\mathcal{N} = 1$ supersymmetry in type II SUGRA. In type IIA, the literature is quite extensive and that's why we will consider type IIB. We will use powerful mathematical tools in order to achieve our goal namely G-structure, Generalized Complex Geometry and torsion. So we will devote a significant part of this thesis to a search for de Sitter vacua. The KKLT proposal is not the only option to achieve stable de Sitter compactifications in string theory. In [5] it was suggested to use an ansatz for a SUSY breaking calibrated source to find a classical de Sitter solution directly in ten dimensions on a given solvmanifold (For a definition, see for example [6]).

In appendix A of this thesis we will report on some partial attempt to recover such solutions from the point of view of the four-dimensional action obtained by compactifying on the solvmanifold.

Here is a brief outline of the thesis :

In Chapter 2 we give a review of some mathematical tools needed to do our analysis : G-structure and torsion. We describe the different topological structures one can put on manifolds. We also give how this structures can be incorporated in a differential setup.

In Chapter 3, we review Complex Generalized Geometry. It generalizes the notion of tangent bundle. We also introduce the Courant Bracket to generalize the concept of integrability of a structure. This framework will be an essential tool in our analysis.

In Chapter 4, we will use the formalism seen in the first chapters in order to rewrite the SUSY equations into differential equations on the internal manifold. We also present some results about $\mathcal{N} = 1$ compactifications.

In Chapter 5, we present our method of analysis for $\mathcal{N} = 1$, AdS_4 vacua on parallelizable manifolds. We use both the SU(2)-structure and the Generalized Complex Geometry in order to do this analysis. We also present orientifolds plane and their involution. Finally, we study scales separation on the vacua we found in order to have genuine four-dimensional vacua.

Appendix A contains some results about the search of stable de Sitter vacua in type IIA theory.

Appendix B is a summary of the thesis in french.

Chapter 2

G-Structure and Torsion

In this section we will review some useful definitions about G-structure, holonomy and torsion. These concepts are important when considering string compactifications because they permit to encode in an easy way topological and differential conditions on the compactification manifolds. Indeed, as we will see in chapter 4, the supersymmetric (SUSY) conditions will put constraints on the structure group and the holonomy of the manifold we compactify on. We will also review the intrinsinc torsion associated to a connection which will also be useful to parametrize the exterior derivative of the objects defined by the structure we put on the manifold. We will then focus on two examples $SU(3)$ and $SU(2)$-structures in six dimensions which will be of interest when compactifying.

2.1 G-structures

In this chapter we introduce the notion of G-structure. In fact, many of the topological properties characterizing a manifold are examples of G-structures.

2.1.1 Structure group

Consider a manifold M, of real dimension $d = 2n$. At each point p of the manifold one can define a vector, which is an element of the tangent space T_pM. The union of the T_pM for all points p of M is called the tangent bundle TM and its sections are called vector fields. The same construction for one-forms define the cotangent bundle T^*M on M, whose sections are one-form fields.

On each patch on the manifold, one can introduce a local frame $e_m^{(\alpha)}$, namely a set of d independent vectors spanning TM at each point. Then a vector v (or tensor) can be expanded in this basis, $v = v_{(\alpha)}^m e_m^{(\alpha)}$, and its components on two overlapping patches are related by a local change of coordinates

$$\left.\begin{array}{l} v_{(\alpha)} \in U_\alpha \\ v_{(\beta)} \in U_\beta \end{array}\right\} \Rightarrow v_{(\beta)}^m = (M_{\alpha\beta})^m_n v_{(\beta)}^n \quad \text{on} \quad U_\alpha \cap U_\beta \,, \tag{2.1}$$

where $M_{\alpha\beta} \in GL(2n, \mathbb{R})$, the group of general linear transformations. Since one can repeat this construction at every point on M, the matrices $M_{\alpha\beta}$ can be seen as functions from the manifold to $GL(2n, \mathbb{R})$

$$M_{\alpha\beta} \;:\; M \to GL(2n, \mathbb{R}) \tag{2.2}$$

$$x \to M_{\alpha\beta}(x)\,. \tag{2.3}$$

These are the transition functions and contain all the information about the non-trivial topology of the bundle TM. On a triple overlap $U_\alpha \cap U_\beta \cap U_\gamma$ they must satisfy the consistency condition

$$M_{\alpha\beta} M_{\beta\gamma} = M_{\alpha\gamma} \,. \tag{2.4}$$

and also

$$M_{\alpha\beta} M_{\beta\alpha} = 1 \,, \tag{2.5}$$

which gives the set of transition functions the properties of a group, G. The group G, in this case $GL(2n, \mathbb{R})$, is the structure group of the tangent bundle.

Alternatively, one can define the structure group of TM as the group of the transition functions of the frame bundle associated to the vector bundle TM[1]. Since the vector field $v = v^m_{(\alpha)} e^{(\alpha)}_m$ (no sum over α) is invariant across patches (and thus globally defined), it is clear that the frames must change with the same (inverse) transition functions

$$e^{(\alpha)}_m = (M^{-1}_{\alpha\beta})^n{}_m e^{(\beta)}_n \,, \qquad \text{on}\, U_\alpha \cap U_\beta \tag{2.6}$$

where the same transition functions $M_{\alpha\beta}$ as before.

Note that the fields one can define on the manifold carry representations of the structure group with transition functions taking values in different representations of the structure group. For instance

frame bundle	$\rightarrow$	adjoint
vectors	$\rightarrow$	fundamental

2.1.2 G-structure

If one can arrange all the transition functions of the frame bundle of a manifold M to take values in a subgroup $G \subset GL(2n, \mathbb{R})$, the structure group of the tangent bundle[2] is reducible to G. If it is possible to reduce the structure group of a manifold M to a subgroup $G \subset GL(2n, \mathbb{R})$, we say that the manifold M as a G-structure.

An alternative definition of a G-structure is given in terms of G-invariant tensors, or, if M is spin, G-invariant spinors : a manifold M has a G-structure if there exist globally defined (non-vanishing) G-invariant tensors or spinors

It is easy to prove that the two definitions are actually equivalent. Recall that tensors of a given type on M are in a certain representation of $GL(2n)$. Suppose now that the structure group of the frame bundle is reduced to G. Then the $GL(2n)$ representation of a given tensor can be decomposed into irreducible representations of G. It can happen that some components of the tensor field, under the decomposition, transform as singlets of G. This means that the corresponding bundle is trivial and thus admits a globally defined non-vanishing section ξ. In other words, we have a globally defined non-vanishing G-invariant tensor or spinor.

Let us now consider the converse. We have a non-vanishing, globally defined tensor or spinor ξ which is G-invariant. Since the invariant tensor ξ is globally defined, by considering the set of frames for which ξ takes the same fixed form, one can see that the structure group of the frame bundle must then reduce to G. Thus the existence of ξ implies we have a G-structure.

The relation between G-structures and globally defined tensors extends to other vector bundles with structure groups different from $\mathrm{GL}(d, \mathbb{R})$ and its subgroups. In particular it

[1] The frame bundle associated to TM is the bundle having as fibers the set of all frames.

[2] This is more generally true for the structure group G of any vector bundle

extends to spin bundles with spin groups as structure groups. In this case, a (reduced) G-structure can be usually associated to the existence of globally defined spinors.

The standard structures we are used to in differential geometry can be re-interpreted as different G-structures on M

Structure	Structure group
Riemannian manifold	$\mathrm{O}(2n)$
Orientable manifold	$\mathrm{SO}(2n)$
Almost Complex Structure	$\mathrm{GL}(n, \mathbb{C})$
Almost Hermitian manifold	$\mathrm{U}(n)$
Pre-symplectic	$\mathrm{Sp}(2n, \mathbb{R})$
Almost Product Structure	$\mathrm{GL}(l, \mathbb{R}) \times \mathrm{GL}(2n-l, \mathbb{R})$

where in the last line, l is the rank of T_T.

Below we discuss in more details some of the structure which are relevant for the discussion of Generalized Complex Geometry and supersymmetry.

2.1.3 Almost complex structure and almost complex manifold

Consider a manifold of real dimension $d = 2n$. An almost complex structure is a globally defined tensor (i.e. a tensor field) given by the map

$$I \ : \ T \to T \tag{2.7}$$

$$x^m \to I^m_n x^n \, , \tag{2.8}$$

such that

$$I_m{}^p I_p{}^n = -\delta_m{}^n. \tag{2.9}$$

Note that locally one can always define a tensor with such properties, but this is an almost complex structure only if it is globally defined, namely if it is a tensor field.

A manifold of real dimension $d = 2n$ is called almost complex if it admits an almost complex structure I. On any even-dimensional manifold, pointwise, one can introduce complex coordinates. The existence of an almost complex structure guaranties that the introduction of complex coordinates can be defined on the whole neighborhood and that the definition on different patches is consistent. To introduce explicitly the local set of complex coordinates, we can use (2.9). From this equation it follows that any almost complex structure has eigenvalues $\pm i$. Thus one can define the projection operators

$$(P_\pm)_m{}^n = \frac{1}{2}(\delta^n_m \mp i I_m{}^n) \, , \tag{2.10}$$

which project onto the $\pm i$-eigenspaces, and satisfy

$$P_\pm P_\pm = P_\pm \, , \qquad P_+ P_- = 0 \, . \tag{2.11}$$

Then locally we can split the tangent bundle in

$$T = T^{(1,0)} \oplus T^{(0,1)} \, . \tag{2.12}$$

On an almost complex manifold one can use the projectors (2.10) to decompose a real $(p+q)$-form ω^{p+q} into holomorphic and anti-holomorphic components

$$\omega^{p,q}_{m_1\ldots m_{p+q}} = (P^+)_{m_1}{}^{n_1}\ldots(P^+)_{m_p}{}^{n_p}(P^-)_{m_{p+1}}{}^{n_{p+1}}\ldots(P^-)_{m_{p+q}}{}^{n_{p+q}}\omega^{p+q}_{n_1\ldots n_{p+q}} \ . \tag{2.13}$$

In general we will denote the projections on the $+i$ eigenvalue subspace with an unbarred index i, and the projection on the $-i$ eigenvalue subspace with a barred index $\bar{\imath}$.

2.1.4 Hermitian metric and almost Hermitian manifold

A metric g_{mn} on an almost complex manifold is called Hermitian if it satisfies

$$I_m{}^p I_n{}^r g_{pr} = g_{mn} \ . \tag{2.14}$$

An almost Hermitian manifold is an almost complex manifold endowed with a Hermitian metric. On it, one can define a 2-form

$$J_{mn} = I_m{}^p g_{pn} \ , \tag{2.15}$$

$$J = \frac{1}{2} J_{mn} \mathrm{d}x^m \mathrm{d}x^n \ , \tag{2.16}$$

called the fundamental form. The relation (2.14) implies that J is a non-degenerate 2-form.

In local complex coordinates the hermitian metric is of type $(1,1)$ and has one barred and one unbarred index. Thus, raising and lowering indices with the hermitian metric converts holomorphic indices into anti-holomorphic ones and vice versa. Moreover the contraction of a holomorphic and an anti-holomorphic index vanishes. Also, on an almost hermitian manifold of real dimension $d = 2n$, forms of type $(p,0)$ vanish for $p > n$.

Because I is not constant, derivatives of a (p,q)-form give (extra pieces compared to complex manifolds see (2.35)) [7]

$$\mathrm{d}\omega^{(p,q)} = (\mathrm{d}\omega)^{(p-1,q+2)} + (\mathrm{d}\omega)^{(p,q+1)} + (\mathrm{d}\omega)^{(p+1,q)} + (\mathrm{d}\omega)^{(p+2,q-1)} \ . \tag{2.17}$$

2.1.5 Symplectic structures

A manifold M admit a pre-symplectic structure if there exists a globally defined skew-symmetric 2-form

$$\omega \in \Lambda^2 T^\star \ , \tag{2.18}$$

which can also be seen as a map

$$\omega \quad : \ T \to T^\star \tag{2.19}$$

$$x^m \to x^n \omega_n m = i_x \omega \ . \tag{2.20}$$

A manifold M admitting a pre-symplectic structure is called almost symplectic.

2.1.6 Product structure

An almost product structure is very similar to an almost complex structure. It is a globally defined tensor given by the map

$$R \ : \ T \to T \tag{2.21}$$

$$x^m \to R^m_n x^n \ , \tag{2.22}$$

such that

$$R_m{}^p R_p{}^n = \delta_m{}^n \ . \tag{2.23}$$

It has (+1) and (-1) eigenvalues and induces a split of T in two subbundles T_T and T_N.

2.1.7 $SU(3)$ and $SU(2)$ structures in dimension 6

In the rest of the thesis we will focus on compactifications to four-dimensions with with $\mathcal{N} = 1$ supersymmetry. In this case a very useful way to recast the conditions imposed by supersymmetry is in terms of $SU(3)$ and $SU(2)$ structures on the internal six-dimensional manifold. We will then discuss in some detail these two cases

2.1.7.1 $SU(3)$ structure

A six dimensional Riemannian manifold admits a spin bundle, so the structure group generically will be $SU(4) \sim SO(6)$. The irreducible spinor representation is the $\mathbf{4} \in SU(4)$. So a globally defined invariant spinor necessarily requires the reduction of the structure group. The simplest possibility is to consider a $SU(3)$ invariant spinor η_+

$$\begin{array}{ccc} SO(6) & \rightarrow & SU(3) \\ 4 & \rightarrow & 3+1 \\ \eta & \rightarrow & \quad \eta_+ \end{array}$$

We can also see the reduction of the structure group in terms of globally defined forms. On a 6-d manifold one can define 1,2,3,4,5,6-forms. Each of them is in a non-trivial representation of $SO(6)$. We can then look at the decomposition in $SU(3)$ representations and see whether there are $SU(3)$ singlets: these are the invariant tensors

	$SO(6)$	$\rightarrow$	$SU(3)$	
A_1	6	$\rightarrow$	$3 + \bar{3}$	
A_2	156	$\rightarrow$	$8 + 3 + \bar{3} + 1$	J
A_3	10_c	$\rightarrow$	$6 + 3 + 1$	Ω
A_4	15	$\rightarrow$	$8 + 3 + \bar{3} + 1$	$J \wedge J$
A_5	6	$\rightarrow$	$3 + \bar{3}$	$J \wedge \Omega = 0$
A_6	1	$\rightarrow$	1	$\Omega \wedge \bar{\Omega} \sim J \wedge J \wedge J$

Then an $SU(3)$ structure in 6 dimensions is equivalently defined by a real 2-form (fundamental form) and a complex 3-form (J,Ω), or a metric and a globally defined chiral spinor (g, η_+).

The two definitions are indeed equivalent since the forms can be defined as bilinears in the spinor

$$J_{mn} = -i\eta_+^\dagger \gamma_{mn}\eta_+ \tag{2.24}$$

$$\Omega_{mnp} = -i\eta_-^\dagger \gamma_{mnp}\eta_+ \tag{2.25}$$

Note that, unlike the $U(n)$ case, given J and Ω, the metric does not need to be specified in addition [8]. Essentially this is because, without the presence of a metric, Ω defines an almost complex structure, and J an almost symplectic structure. Treating J as the fundamental form, it is then a familiar result on almost hermitian manifolds that the existence of an almost complex structure and a fundamental form allow one to construct an hermitian metric.

2.1.7.2 $SU(2)$ structure

An SU(2)-structure is defined in terms of a globally defined, nowhere-vanishing complex vector z^m, a real two-form j and a complex two-form ω, satisfying

$$\omega \wedge j = \omega \wedge \omega = 0\,, \tag{2.26a}$$
$$\iota_z j = \iota_z \omega = 0\,, \tag{2.26b}$$
$$\omega \wedge \bar{\omega} = 2j^2\,, \tag{2.26c}$$

where z now denotes the 1-form $z_m \mathrm{d}x^m = g_{mn} z^n \mathrm{d}x^m$.

Alternatively, the SU(2)-structure can be characterized by the existence of *two* globally defined nowhere-vanishing and nowhere-parallel chiral spinors

$$\eta\,, \qquad \chi = \frac{1}{2} z^m \gamma_m \eta^*\,. \tag{2.27}$$

Here we have chosen χ to be orthogonal to η, which is always possible by subtracting the parallel part.

As for the SU(3) case, these tensors can be written as bilinears in the spinors (2.27)

$$z^m = \eta^T \gamma^m \chi\,, \tag{2.28a}$$
$$j_{mn} = (i/2)\eta^\dagger \gamma_{mn} \eta - (i/2)\chi^\dagger \gamma_{mn} \chi\,, \tag{2.28b}$$
$$\omega_{mn} = \chi^\dagger \gamma_{mn} \eta\,. \tag{2.28c}$$

An SU(2) structure can be seen as the intersection of two SU(3) structures, defined by the spinors η and χ, respectively. The corresponding fundamental forms are given by

$$J_+ = \frac{i}{2}\eta^\dagger \gamma_{mn} \eta\, \mathrm{d}x^m \wedge \mathrm{d}x^n \;=\; -\frac{i}{2} z \wedge \bar{z} + j\,,$$
$$J_- = \frac{i}{2}\chi^\dagger \gamma_{mn} \chi\, \mathrm{d}x^m \wedge \mathrm{d}x^n \;=\; -\frac{i}{2} z \wedge \bar{z} - j\,.$$

More generally, the SU(2)-structure determines an entire U(1) family of almost complex structures compatible with the metric. The corresponding (1,1)-forms are constructed as in (2.24) and (2.25) in terms of the normalized spinor

$$\psi = k_\| \, \eta + k_\perp \, \chi\,. \tag{2.29}$$

with $k_\|$ and $k_\perp$ positive and $k_\|^2 + k_\perp^2 = 1$. When $k_\| = 0$ and $k_\perp = 1$, one talks about a static $SU(2)$-structure whereas when $k_\| \neq 0$ and $k_\perp \neq 0$, one talks about an intermediate $SU(2)$-structure. When $k_\| = 1$ and $k_\perp = 0$, one recovers an $SU(3)$-structure.

The one-form z provides an almost product structure on M, defined locally by

$$R_m^n = z_m \bar{z}^n + \bar{z}_m z^n - \delta_m^n\,, \qquad\qquad m, n = 1, \ldots, 6\,, \tag{2.30}$$

which induces a (global) decomposition of the tangent space in

$$TM = T_2 M \oplus T_4 M\,. \tag{2.31}$$

The subbundle T_2M is spanned by the real and imaginary parts of the form z.

2.2 Holonomy group and torsion

The existence of a G-structure on a manifold M is related to its topological properties, namely the existence of certain bundles. In order to make contact with the supersymmetry conditions in string compactifications, we need to study the integrability properties of such structures. We will not give here a rigorous definition of integrability. Roughly speaking a given structure is integrable if it is possible to find a system of adapted coordinates on the manifold. What matters for us is that the integrability of a given structure can be rephrased in terms of intrinsic torsion. The structure is integrable is the connection is torsion free. We then recover special holonomy manifolds as special cases where all torsions vanishes.

2.2.1 Examples of integrable structures

Before defining the notion of intrinsic torsion we recall some examples of integrable structures.

2.2.1.1 Complex structure

An almost complex structure is integrable if there is an atlas of coordinates such that I can be put everywhere in the form

$$I = \begin{pmatrix} i\mathbb{I}_{n\times n} & 0 \\ 0 & -i\mathbb{I}_{n\times n} \end{pmatrix} . \tag{2.32}$$

An integrable almost complex structure is called a complex structure. There are two possible equivalent definition of integrability of an almost complex structure

1. the Nijenhuis tensor

$$N_{mn}{}^{p} = I_m{}^q \left(\partial_q I_n{}^p - \partial_n I_q{}^p\right) - I_n{}^q \left(\partial_q I_m{}^p - \partial_m I_q{}^p\right) \tag{2.33}$$

   must vanish.

2. the "(1, 0)" part of the complexified tangent bundle $T \otimes \mathbb{C}$ is integrable under the Lie bracket. This means that the Lie bracket of two (anti)-holomorphic vectors must be (anti)-holomorphic

$$P_{\mp}[P_{\pm}x, P_{\pm}y] = 0 \qquad \forall x, v \in T\,. \tag{2.34}$$

   One can see that both the real and imaginary part of the equation above are proportional to the Nijenhuis tensor.

An almost complex manifold admitting a complex structure is said to be a complex manifold. For an almost complex manifold we have seen that it is possible to define complex coordinates in a patch. For a complex manifold the transition functions between different patches are holomorphic functions of the complex coordinates. It is the holomorphicity of the transition functions that allows to put I in the diagonal form (2.32).

For a complex manifold the exterior derivative of a (p, q)-form takes the form

$$\mathrm{d}\omega^{(p,q)} = (\mathrm{d}\omega)^{(p,q+1)} + (\mathrm{d}\omega)^{(p+1,q)}\ . \tag{2.35}$$

2.2.1.2 Symplectic structures

A pre-symplectic structure ω on M is integrable if it is possible to find local coordinates on M such that the symplectic forms becomes

$$\omega = \mathrm{d}x_m \wedge \mathrm{d}y^m \tag{2.36}$$

and such that all transition functions are symplectic with respect to the standard symplectic structure. By Darboux theorem, the integrability condition is equivalent

$$\mathrm{d}\omega = 0 \tag{2.37}$$

A manifold M admitting a closed ω is called symplectic.

2.2.1.3 Product structure

The integrability if an almost product structure can also be expressed in terms of the Nijenhuis tensor becomes :

$$N_{mn}{}^p = R_m{}^q\left(\partial_q R_n{}^p - \partial_n R_q{}^p\right) - R_n{}^q\left(\partial_q R_m{}^p - \partial_m R_q{}^p\right) \tag{2.38}$$

The ± 1 subbundles T_T and T_N are integrable if the projections on T_T and T_N of the Nijenhuis tensor vanish, respectively. If both T_T and T_N are integrable, they define a product structure.

2.2.2 Intrinsic torsion

Let us first recall the definition of torsion and contorsion on a Riemannian manifold (M, g).

A connection D' on (M, g) is said to be a metric compatible connection D'_m on (M, g) if it satisfies

$$D'_m g_{np} = 0 \ . \tag{2.39}$$

From such a connection one can define the Riemann curvature tensor and the torsion tensor as follows

$$[D'_m, D'_n]v_p = -R_{mnp}{}^q v_q - 2T_{mn}{}^r D'_r v_p \ , \tag{2.40}$$

where v is an arbitrary vector field. The Levi-Civita connection is the unique connection without torsion compatible with the metric.

Any metric compatible torsionful connection can be written in terms of the Levi-Civita connection

$$D^{(T)} = D + \kappa \ , \tag{2.41}$$

where we denote by D and $D^{(T)}$ the covariant derivative with respect to the Levi-Civita connection and a connection with torsion, respectively. The tensor $\kappa_{mn}{}^p$ in (2.41) is the contorsion tensor. Metric compatibility implies

$$\kappa_{mnp} = -\kappa_{mpn} \ , \tag{2.42}$$

where $\kappa_{mnp} = \kappa_{mn}{}^r g_{rp}$. There is a one-to-one correspondence between the torsion and the contorsion[3]

$$T_{mn}{}^p = \frac{1}{2}(\kappa_{mn}{}^p - \kappa_{nm}{}^p) \equiv \kappa_{[mn]}{}^p \ , \tag{2.43}$$

$$\kappa_{mnp} = T_{mnp} + T_{pmn} + T_{pnm} \ . \tag{2.44}$$

[3] One can prove it inserting (2.42) into (2.40)

From these relations it follows that, given a torsion tensor T, there exist a unique connection $D^{(T)}$ whose torsion is precisely T.

We can now go back to our G-structure. In general the G-invariant tensors (or spinor) ξ is not constant under the Levi-Civita connection

$$D\xi \neq 0\,. \tag{2.45}$$

However, it possible to prove [9], that there always exist some connection with torsion $D^{(T)}$ compatible with the G-structure, that it is say

$$D^{(T)}\xi = 0\ . \tag{2.46}$$

In general there is more than one connection compatible with a given G-structure and they have different torsion. However, it is possible to identify a part of the torsion which is independent of the choice of the connection and only depends on the G-structure: the intrinsic torsion. This can be defined using the Levi-Civita connection

$$D^{(T)}\xi = D\xi + \kappa_0\xi = 0\ . \tag{2.47}$$

The tensor κ_0 is called intrinsic contorsion and it is the component of the contorsion that acts non-trivially on the invariant tensor. To see this we can look at the symmetry properties of the contorsion, (2.42): κ is an element of $\Lambda^1 \otimes \Lambda^2$ where Λ^n is the space of n-forms. Since the space of two-forms on the manifold is isomorphic to the algebra of $SO(2n)$ ($\Lambda^2 \cong so(2n)$), we can also consider the contorsion as a one form with values in the Lie-algebra $so(2n)$

$$\kappa \in \Lambda^1 \otimes so(2n)\,. \tag{2.48}$$

Given the existence of a G-structure, we can decompose $so(2n)$ into a part in the Lie algebra $\mathcal{G}$ of G and an orthogonal piece $\mathcal{G}^\perp = so(2n)/\mathcal{G}$. The same splitting can be enforced on the contorsion tensor

$$\kappa = \kappa_0 + \kappa_{\mathcal{G}}\ , \tag{2.49}$$

where κ_0 is the part in $\Lambda^1 \otimes \mathcal{G}^\perp$. Since ξ is an invariant tensor (or spinor), it does not transform under the action of $\mathcal{G}$ (the generators of G), so that

$$D^{(T)}\xi = (D + \kappa_0 + \kappa_{\mathcal{G}})\,\xi = (D + \kappa_0)\,\xi = 0\ . \tag{2.50}$$

In summary the intrinsic contorsion is independent of the choice of G-compatible connection. Basically it is a measure of the degree to which $D\xi$ fails to vanish and as such is a measure of the G-structure itself.

Using the isomorphism between torsion and contorsion (2.43), one can also define the intrinsic torsion as

$$T^{0}_{mn}{}^{p} = \kappa_{0\ [mn]}{}^{p} \in \Lambda^1 \otimes \mathcal{G}^\perp\ . \tag{2.51}$$

The intrinsic torsion also provide a classification of G-structures. The idea is that one can decompose κ_0 into irreducible representations of the group G. Then a G-structure will be specified in terms of the representations appearing in the decomposition. In particular, in the special case where κ_0 vanishes so that $D\xi = 0$, one says that the structure is torsion-free.

2.2.2.1 Example: Complex structure

An almost complex structure I defines a $\mathrm{GL}(d/2,\mathbb{C})$ structure on M. A compatible connection ∇ is such that $\nabla I = 0$. The integrability of an almost complex structure is equivalent to the vanishing of the Nijenhuis tensor, N_I. An almost complex structure on M corresponds to a $\mathrm{GL}(d/2,\mathbb{C})$ structure. The Nijenhuis tensor can be written in terms of the torsion of the $\mathrm{GL}(d/2,\mathbb{C})$ compatible torsion, and integrability becomes the condition of the structure being *torsion free*.

2.2.3 Special holonomy

The holonomy group H of a Riemannian manifold M consists of the matrices that realize parallel transport of a generic field ϕ on M around a close curve γ

$$\phi \to \phi' = U\phi \qquad\qquad U = P(e^{\int_\gamma \omega \mathrm{d}x}) \tag{2.52}$$

where ω is the spin connection, which is a $SO(2n)$ gauge field. The matrices U are element of $SO(2n)$ and in general H coincide with $SO(2n)$. If the holonomy group is smaller than $SO(2n)$, the manifold is said to be of special holonomy. Note that the holonomy is associated to the curvature tensor, and hence to the choice of a connection. When it is not specified otherwise, the holonomy is that of the Levi-Civita connection.

The reduction of the holonomy group is related to existence of covariantly constant tensors on M. Consider a tensor bundle on M and suppose that it admits a covariantly constant tensor (if the manifold is spin we can also have a covariantly constant spinor)

$$\nabla\xi = 0\,. \tag{2.53}$$

Such a tensor is invariant under parallel transport and hence under the holonomy group. This means that the holonomy group is reduced to the subgroup $G \in SO(d)$ that leaves the tensor ξ invariant. The opposite statement is also true: if the holonomy is reduced to a subgroup G, then there exist constant tensors. Therefore studying the holonomy of a connection or its constant tensors is equivalent.

Consider now a connection, ∇ of reduced holonomy. From (2.47), it follows that it is always possible to find a G-structure such Hol(∇)= G. If the connection is the Levi-Civita connection, then the corresponding G-structure is torsion free. For simply-connected manifolds M of dimension n, with a Riemannian metric g, that is irreducible and non-symmetric, the possible reduced holonomy groups are given in the Table below. In the same table we list the corresponding constant tensors and/or spinors (η_+ are chiral spinors) .

Holonomy	dim(M)	constant tensors	Type of manifold
SO(d)	d	g	Orientable
U(m)	$d = 2m$, $m \geq 2$	(g, J)	Kähler
SU(m)	$d = 2m$, $m \geq 2$	(J, Ω_m) or η_+	Calabi-Yau
Sp(m)	$d = 4m$, $m \geq 2$		Hyperkähler
Sp(m) Sp(1)	$d = 4m$, $m \geq 2$		Quaternionic Kähler
G_2	7	ϕ_3 or η	G_2
Spin(7)	8	Ω_4 or η_+	Spin(7)

2.2.4 Torsion for $SU(3)$-structure in dimension 6

As an example we can consider again the case of an $SU(3)$-structure in six dimensions. If we write explicitly the relation between the torsionful connection and the Levi-Civita :

$$D^{(T)}_m \eta_+ = D_m \eta_+ - \frac{1}{4}\kappa_{mnp}\gamma^{np}\eta_+ = 0 \ , \tag{2.54}$$

we see that the contorsion is indeed the obstruction to η_+ being covariantly constant with respect to the Levi-Civita connection. We can also write the analogous condition on J and Ω, and show that they are no longer covariantly constant

$$D^{(T)}_m J_{np} = D_m J_{np} - \kappa_{mn}{}^r J_{rp} - \kappa_{mp}{}^r J_{nr} = 0 \, , \tag{2.55}$$

$$D^{(T)}_m \Omega_{nmp} = D_m \Omega_{npq} - \kappa_{mn}{}^r \Omega_{rpq} - \kappa_{mp}{}^r \Omega_{nrq} - \kappa_{mq}{}^r \Omega_{npr} = 0 \, . \tag{2.56}$$

Again κ is measuring the obstruction to J and Ω being covariantly constant with respect to the Levi-Civita connection.

The connection with torsion $D^{(T)}$ preserves the $SU(3)$ structure in that η_+ or equivalently J and Ω are constant with respect to it.

As we have already seen in the general case, the obstruction to having a covariantly constant spinor (or equivalently J and Ω) is actually measured by not the full contorsion but by "intrinsic contorsion" part κ^0. Indeed κ_{mnp} takes values in $\Lambda^1 \otimes \Lambda^2$, and Λ^2 is isomorphic to the Lie algebra $so(6)$. We can further decompose it according to $SU(3)$

$$so(6) \cong su(3) \oplus su(3)^{\perp} \qquad \Rightarrow \qquad \kappa^{su(3)} + \kappa^0 \qquad \begin{cases} \kappa^{su(3)} \in \Lambda^1 \otimes su(3) \\ \kappa^0 \in \Lambda^1 \otimes su(3)^{\perp} \end{cases} \tag{2.57}$$

If we apply this decomposition to the covariant derivative of the spinor, since η_+ is an $SU(3)$ singlet, the action of $su(3)$ on η_+ vanishes, and we are left with

$$D_m \eta = \frac{1}{4}\kappa^0_{mnp}\gamma^{np}\eta_+ \ . \tag{2.58}$$

Similar expressions can be derived for the covariant derivatives of J and Ω

$$\mathrm{d}J_{mnp} = 6 T^0_{[mn}{}^r J_{r|p]} \ , \tag{2.59}$$

$$\mathrm{d}\Omega_{mnpq} = 12 T^0_{[mn}{}^r \Omega_{r|pq]} \ . \tag{2.60}$$

Because of the isomorphism between contorsion and torsion, the same definitions hold for the torsion T.

Both the intrinsic torsion and contorsion can be decomposed into irreducible representations of $SU(3)$, and, hence, different $SU(3)$ structures can be characterized by the non-trivial $SU(3)$ representations T^0 carries.

$$\begin{aligned} T^0 \in \Lambda_1 \otimes SU(3)^{\perp} &= (3 \oplus \bar{3}) \otimes (1 \oplus 3 \oplus \bar{3}) \\ &= (1 \oplus 1) \oplus (8 \oplus 8) \oplus (6 \oplus \bar{6}) \oplus (3 \oplus \bar{3}) \oplus (3 \oplus \bar{3})' \\ &= \quad \mathcal{W}_1 \qquad\quad \mathcal{W}_2 \qquad\quad \mathcal{W}_3 \qquad\quad \mathcal{W}_4 \qquad\quad \mathcal{W}_5 \end{aligned} \tag{2.61}$$

The $\mathcal{W}_i$ are called torsion classes, they correspond to form of different types

$\mathcal{W}_1 \rightarrow$ complex scalar

$\mathcal{W}_2 \rightarrow$ complex primitive $((W_2)_{mn}J^{mn} = 0)$ (1,1) form

$\mathcal{W}_3 \rightarrow$ real primitive (2,1) + (1,2) form

$\mathcal{W}_4 \rightarrow$ real vector

$\mathcal{W}_5 \rightarrow$ complex (1,0) form

For $SU(3)$ structure it is possible to express each torsion class in terms of a component in the $SU(3)$ decomposition of dJ and dΩ.

$$\begin{array}{rcl} \mathrm{d}J &=& \left[(\mathrm{d}J)^{3,0}+(\mathrm{d}J)^{0,3}\right]+\left[(\mathrm{d}J)^{2,1}_0+(\mathrm{d}J)^{1,2}_0\right]+\left[(\mathrm{d}J)^{1,0}+(\mathrm{d}J)^{0,1}\right] \\ 20 &=& (1\oplus 1)\oplus(6\oplus\bar{6})\oplus(3\oplus\bar{3}) \\ && \quad\mathcal{W}_1 \qquad \mathcal{W}_3 \qquad \mathcal{W}_4 \end{array} \tag{2.62}$$

$$\begin{array}{rcl} \mathrm{d}\Omega &=& (\mathrm{d}\Omega)^{3,1}+(\mathrm{d}\Omega)^{2,2}_0+(\mathrm{d}\Omega)^{0,0}\ , \\ 24 &=& (3\oplus\bar{3})'\oplus(8\oplus 8)\oplus(1\oplus 1) \\ && \quad\mathcal{W}_2 \qquad \mathcal{W}_5 \qquad \mathcal{W}_1 \end{array} \tag{2.63}$$

According to this decomposition we can write the exterior derivatives of J and Ω as

$$\mathrm{d}J = \frac{3}{2}Im(\bar{\mathcal{W}}_1\Omega)+\mathcal{W}_4\wedge J+\mathcal{W}_3 \tag{2.64}$$

$$\mathrm{d}\Omega = \mathcal{W}_1\, J\wedge J+\mathcal{W}_2\wedge J+\bar{\mathcal{W}}_5\wedge\Omega \tag{2.65}$$

Manifolds with SU(3) structure can then be classified depending on which torsion classes are non-zero. Here are some examples

Name	Torsion classes
Complex	$\mathcal{W}_1=\mathcal{W}_2=0$
Symplectic	$\mathcal{W}_1=\mathcal{W}_3=\mathcal{W}_4=0$
Nearly Kähler	$\mathcal{W}_2=\mathcal{W}_3=\mathcal{W}_4=\mathcal{W}_5=0$
Kähler	$\mathcal{W}_1=\mathcal{W}_2=\mathcal{W}_3=\mathcal{W}_4=0$
Half-flat	$\mathrm{Im}\,\mathcal{W}_1=\mathrm{Im}\,\mathcal{W}_2=\mathcal{W}_4=\mathcal{W}_5=0$
Nearly Calabi-Yau	$\mathcal{W}_1=\mathrm{Im}\,\mathcal{W}_2=\mathcal{W}_3=\mathcal{W}_4=\mathcal{W}_5=0$
Calabi-Yau	$\mathcal{W}_1=\mathcal{W}_2=\mathcal{W}_3=\mathcal{W}_4=\mathcal{W}_5=0$

Table 2.1: Classification of geometries according to the vanishing SU(3) torsion classes.

2.2.5 Torsion for $SU(2)$-structure in dimension 6

In the rest of this thesis we will be interested in the torsion classes for an SU(2)-structure in six dimensions. A simple way to obtain them is to decompose the SU(3) torsion classes given in the previous section according to SU(3) $\to$ SU(2) $\times$ U(1). As a result the intrinsic torsion (and contorsion) can be decomposed into 20 irreducible representations of SU(2)

$$\begin{aligned} T^0\in T^*M\otimes su(2)^\perp =&(2\cdot\mathbf{1}\oplus 2\cdot\mathbf{2})\otimes(4\cdot\mathbf{1}\oplus 4\cdot\mathbf{2}) \\ =&16\cdot\mathbf{1}\oplus 16\cdot\mathbf{2}\oplus 8\cdot\mathbf{3} \end{aligned} \tag{2.66}$$

That is to say that one has 8 complex scalars S_i, 8 holomorphic vectors V_i and 4 complex tensors T_i. One can define these torsion classes from the exterior differentials on the forms defining the SU(2) structure [10]

$$\begin{aligned}
\mathrm{d}z &= S_1\omega + S_2 j + S_3 z\wedge\bar z + S_4\bar\omega + z\wedge(V_1+\bar V_2) + \bar z\wedge(V_3+\bar V_4) + T_1\\
\mathrm{d}j &= S_5\bar z\wedge\omega + S_6 z\wedge\omega + \frac{1}{2}(S_7+\bar S_8)z\wedge j + j\wedge V_5 + z\wedge\bar z\wedge V_6 + z\wedge T_2 + \text{c.c.}\\
\mathrm{d}\omega &= S_7 z\wedge\omega + S_8\bar z\wedge\omega - 2\bar S_5 z\wedge j - 2\bar S_6\bar z\wedge j + i z\wedge\bar z\wedge(\bar V_6 \llcorner \omega) + j\wedge(V_7+\bar V_8)\\
&\quad + z\wedge T_3 + \bar z\wedge T_4\,.
\end{aligned} \tag{2.67}$$

where the relations between the representations in dj and dω are implied by the conditions

$$\mathrm{d}(j\wedge\omega) = \mathrm{d}(j\wedge j) = \mathrm{d}(\omega\wedge\omega) = 0\,. \tag{2.68}$$

In (2.67) we added two vector representations in dz that were missing in [10]

Notice that the doublet V_i are holomorphic vectors with respect to the complex structure defined by j,

$$\omega\wedge V_i = 0 \tag{2.69}$$

while the T_i are (1,1) and primitive

$$j\wedge T_i = \omega\wedge T_i = 0 \tag{2.70}$$

and can be decomposed on the basis of anti-self dual two-forms $\tilde j_1$, $\tilde j_2$, $\tilde j_3$ transforming in the **3** of SU(2)

$$T_i = \sum_{a=1}^{3} t^i_a \tilde j_a\,. \tag{2.71}$$

2.2.6 Calabi Yau manifold

When all torsion classes of an SU(3) structure manifold vanishes

$$\nabla^{\mathrm{LC}}_m\eta = 0 \qquad \Leftrightarrow \qquad \mathrm{d}J = 0 \qquad\qquad \mathrm{d}\Omega = 0\,, \tag{2.72}$$

the internal manifold is a Calabi-Yau three-fold. Calabi-Yau manifolds played a fundamental role in purely geometrical string compactifications, and the whole idea of applying Generalized Complex Geometry to to non zero flux backgrounds is to try to extend to this case some of the properties of Calabi-Yau compactifications.

There are several definitions of a Calabi-Yau manifold of dimension $d = 2n$

- Calabi-Yau manifold is a Riemannian manifold with closed fundamental form d$J = 0$ (Kähler manifold) with $SU(n)$ holonomy.
- A Calabi Yau manifold as a Kähler manifold with vanishing first Chern class $c_1(M) = 0$. The first Chern class is the second cohomology group of the manifold.
- A Calabi-Yau manifold has a globally defined closed real two-from and holomorphic n-form, such that

  $$\Omega\wedge\bar\Omega = cJ^n\,, \tag{2.73}$$

  where the proportionality factor must be a constant.

The interest of the definition is that it allows to express the geometrical properties of the manifold in terms of differential equations for some forms. It is in this formulation that we will generalize the idea of Calabi-Yau manifolds to the case of non-zero fluxes.

Chapter 3

Generalized Complex Geometry

Generalized Complex Geometry (GCG) was introduced by Hitchin and then further developed by Gualtieri [8, 11] with the aim of treating on the same ground complex and symplectic geometries. In doing so, GCG allows to geometrize the NS B-field and that's way it is of fundamental interest in string compactifications.

3.1 The generalized tangent bundle

Generalized complex geometry is the generalization of complex geometry to the sum of the tangent and cotangent bundle of a manifold. The idea is to combine vectors and one-forms into a single object. Given a $d = 2n$-dimensional manifold M, one defines the generalized tangent bundle E, which is en extension of TM by T^*M

$$0 \longrightarrow T^*M \longrightarrow E \xrightarrow{\pi} TM \longrightarrow 0\,. \tag{3.1}$$

Sections of E are called generalized vectors and locally can be written as

$$X = (v + \xi) \in TM \oplus T^*M \tag{3.2}$$

where $v \in TM$ and $\xi \in T^*M$. The projection operator $\pi : E \to TM$ in (3.1) is defined by

$$\pi \;:\; v + \xi \mapsto v\;. \tag{3.3}$$

On the overlap of two patches U_α and U_β the sections transform as

$$x_{(\alpha)} + \xi_{(\alpha)} = a_{(\alpha\beta)} x_{(\beta)} + \left[a^{-T}_{(\alpha\beta)} \xi_{(\beta)} - i_{a_{(\alpha\beta)} x_{(\beta)}} \omega_{(\alpha\beta)} \right]\,, \tag{3.4}$$

where $a_{(\alpha\beta)} \in GL(d, \mathbb{R})$ gives the usual patching of vectors and one-form, and the two-form $\omega_{(\alpha\beta)}$ gives the non trivial fibration of T^*M over TM[1]. From (3.2) we see that there is an isomorphism between E and $TM \oplus T^*M$ which is not canonical since it depends on the choice of the two-form ω

$$x + (\xi - \iota_v \omega) \in \Gamma(TM \oplus T^*M)\,, \tag{3.6}$$

[1]This is a gerbe structure. Indeed, if we set $\omega_{(\alpha\beta)} = -\mathrm{d}\Lambda_{(\alpha\beta)}$ with

$$\Lambda_{(\alpha\beta)} + \Lambda_{(\beta\gamma)} + \Lambda_{(\gamma\alpha)} = g_{(\alpha\beta\gamma)} \mathrm{d} g_{(\alpha\beta\gamma)} \tag{3.5}$$

on $U_\alpha \cap U_\beta \cap U_\gamma$ and $g_{\alpha\beta\gamma} := e^{i\alpha}$ is a $U(1)$ element. This is analogous to the patching of a $U(1)$ bundle, but here the transition "functions" are one-forms where Γ denote the sections of $TM \oplus T^*M$. Such a structure has a natural interpretation in string theory, where the two-form ω can be identified with the NS two-form B. Indeed while the field-strength, $H = \mathrm{d}B$ is globally defined, B itself generically is not. In the rest of this chapter, unless it is necessary to specify them, we will use E and $TM \oplus T^*M$ in an equivalent way.

Given the split into vectors and forms, there is a natural $O(d,d)$-invariant metric $\mathcal{I}$ on $TM \oplus T^*M$ given by the natural pairing of vectors and forms

$$\mathcal{I}(X_1, X_2) = (v_1 + \xi_1, v_2 + \xi_2) = \frac{1}{2}(\xi_1(v_2) + \xi_2(v_1)). \tag{3.7}$$

This is a non degenerate symmetric bilinear form of signature (d,d). Using a two-component notation to distinguish the vector and form parts of X

$$X = \begin{pmatrix} v \\ \xi \end{pmatrix} \tag{3.8}$$

we can write

$$\mathcal{I}(X,X) = X^T \mathcal{I} X\,, \qquad \mathcal{I} = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}\,. \tag{3.9}$$

The structure group of E can be reduced to $SO(d,d)$ by choosing a natural volume form :

$$\mathrm{vol} = \frac{1}{(n!)^2}\epsilon^{i_1 \dots i_n}\frac{\partial}{\partial x^{i_1}} \wedge \frac{\partial}{\partial x^{i_n}} \wedge \epsilon_{j_1 \dots j_n} dx^{j_1} \wedge \dots \wedge dx^{i_n}\,.$$

The metric is invariant under $O(d,d)$ transformations acting on the fibers of E. Explicitly a general element $O \in O(d,d)$ can be written in terms of $d \times d$ matrices a, b, c, and d as

$$O = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, \tag{3.10}$$

under which a generic element $X \in TM \oplus T^*M$ transforms by

$$X = \begin{pmatrix} x \\ \xi \end{pmatrix} \mapsto OX = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} v \\ \xi \end{pmatrix}. \tag{3.11}$$

The requirement that $\mathcal{I}(OX, OX) = \mathcal{I}(X,X)$ implies $a^T c + c^T a = 0$, $b^T d + d^T b = 0$ and $a^T d + c^T b = \mathbb{I}$. Note that $GL(d)$ action on the fibers of TM and T^*M embeds as a subgroup of $O(d,d)$, leaving $\mathcal{I}$ invariant. Concretely it maps

$$X \mapsto X' = \begin{pmatrix} a & 0 \\ 0 & a^{-T} \end{pmatrix}\begin{pmatrix} v \\ \xi \end{pmatrix}. \tag{3.12}$$

where $a \in GL(d)$ and $a^{-T} = (a^{-1})^T$. In what follows we will also discuss two other elements of $O(d,d)$. Given a two form ω, we define

$$e^{\omega} = \begin{pmatrix} \mathbb{I} & 0 \\ \omega & \mathbb{I} \end{pmatrix} \qquad \text{such that} \qquad X = v + \xi \mapsto X' = v + (\xi - i_v \omega). \tag{3.13}$$

This is usually referred to as a B-transform and forms an abelian subgroup $G_B \subset O(d,d)$. Similarly we can consider a bivector β and define the β-transform

$$e^{\beta} = \begin{pmatrix} \mathbb{I} & \beta \\ 0 & \mathbb{I} \end{pmatrix} \qquad \text{such that} \qquad X = v + \xi \mapsto X' = (v + i_\xi \beta) + \xi. \tag{3.14}$$

3.2 Generalized complex structure

A *generalized almost complex structure* is a map

$$\mathcal{J} \quad : \quad T \oplus T^* \to T \oplus T^* \tag{3.15}$$

such that

$$\mathcal{J}^2 = -\mathbb{I}_{2d}\,, \qquad \mathcal{J}^t \mathcal{I} \mathcal{J} = \eta\,. \tag{3.16}$$

The second condition is the hermiticity with respect to the natural metric, $\mathcal{I}$, on $TM \oplus T^*M$.

The existence of $\mathcal{J}$ reduces the structure group of $TM \oplus T^*M$ further, to $U(n,n)$.

As for an almost complex structures, it is possible to define locally a holomorphic and anti-holomorphic splitting of $T \oplus T^*$

$$T \oplus T^* = \mathcal{L}_{\mathcal{J}} + \mathcal{L}_{\bar{\mathcal{J}}} \tag{3.17}$$

by introducing the projectors on the $\pm i$ eigenbundles of $\mathcal{J}$

$$\begin{aligned} \Pi &= \frac{1}{2}(\mathbb{I}_{2n} - i\mathcal{J})\,, \\ \bar{\Pi} &= \frac{1}{2}(\mathbb{I}_{2n} + i\mathcal{J})\,. \end{aligned} \tag{3.18}$$

We also have to impose that $\Pi X = X$, where $X = v + \xi$ is a section of $T \oplus T^*$.

The $\pm i$-eigenbundles of the generalized almost complex structure are maximally isotropic subbundles of $T \oplus T^{*}$[2] Indeed they have dimension n and are null with respect to the metric $\mathcal{I}$ in (3.7), since for $X, Y \in L_{\mathcal{J}}$,

$$(X,Y) = X\,\mathcal{I}\,Y = X\mathcal{J}^t\,\mathcal{I}\,\mathcal{J}Y = (iX)\,\mathcal{I}\,(iY) = -X\,\mathcal{J}\,Y = -(X,Y)\,. \tag{3.19}$$

3.2.1 Generalized complex structure and generalized complex manifolds

Just as for an ordinary almost complex structure, it is possible to give an integrability condition for a generalized almost complex structure. We will define integrability as the requirement that the "holomorphic" part of the complexified $TM \oplus T^*M$ is integrable with respect to a bracket, the Courant bracket, that generalize the Lie bracket to$TM \oplus T^*M$. The definition of the Courant bracket will be given in the next section.

A generalized almost complex structure $\mathcal{J}$ is *integrable* if its i eigenbundle $L_{\mathcal{J}}$ is closed under the Courant bracket

$$\bar{\Pi}\,[\Pi(v+\xi), \Pi(w+\eta)]_{\mathrm{C}} = 0\,, \tag{3.20}$$

where Π is the projector on $L_{\mathcal{J}} \subset TM \oplus T^*M$. In this case, $\mathcal{J}$ is called a *generalized complex structure.*

A manifold on which such a tensor exists is called a *generalized complex manifold.*

[2]A *maximally isotropic subbundle* L is such that

- it is a null space: $(X,Y) = 0 \qquad \forall X, Y \in L$
- it has maximal dimension, which in signature (n,n) is n.

In summary

$$
\begin{array}{c|c|c}
 & T & T \oplus T^* \\
\hline
\begin{array}{c}\text{almost complex}\\ \text{structure}\end{array} & \begin{array}{c} J \\ J^2 = -1_d \end{array} & \begin{array}{c} \mathcal{J} \\ \mathcal{J}^2 = -1_{2d} \end{array} \\
\text{projectors} & \pi_\pm = (1_d \pm iJ)/2 & \Pi_\pm = (1_{2d} \pm i\mathcal{J})/2 \\
\text{integrability} & \pi_+[\pi_-(u), \pi_-(v)] = 0 & \Pi_+[\Pi_-(X), \Pi_-(Y)]_C = 0
\end{array}
\tag{3.21}
$$

The simplest examples of generalized complex structures are provided by the embedding in $TM \oplus T^*M$ of the standard complex and symplectic structures

$$
\mathcal{J}_I \equiv \begin{pmatrix} I & 0 \\ 0 & -I^t \end{pmatrix} \ , \qquad \mathcal{J}_J \equiv \begin{pmatrix} 0 & J \\ -J^{-1} & 0 \end{pmatrix} \tag{3.22}
$$

where $I = I_m{}^n$ obeys $I^2 = -\mathbb{I}_d$, i.e. it is a regular almost complex structure for the tangent bundle, and $J = J_{mn}$ is a non degenerate two–form J_{mn}, i.e. an almost symplectic structure for the tangent bundle.

In these two examples, the integrability of $\mathcal{J}$ turns into a condition on the building blocks, $I_m{}^n$ and J_{mn}. Integrability of $\mathcal{J}_I$ forces I to be an integrable almost complex structure on T and hence a complex structure. In other words the manifold is complex. For $\mathcal{J}_J$, integrability imposes $\mathrm{d}J = 0$, thus making J into a symplectic form, and the manifold a symplectic one.

We can construct explicitly $\mathcal{J}_{I,J}$ and their $\pm i$ eigenbundles for the very simple case of a two-torus. If we call e^1 and e^2 the vielbein on the two-torus, we can define the fundamental and the holomorphic forms as $J = e^1 \wedge e^2$ and $\Omega_1 = e^1 + ie^2$, respectively. Then

$$
\mathcal{J}_I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ , \qquad \mathcal{J}_J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \ . \tag{3.23}
$$

The holomorphic eigenbundles are

$$
\begin{aligned}
L_{\mathcal{J}_1} &= \left\langle \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix} \right\rangle = T^{1,0} \oplus (T^*)^{0,1} \ , \\
L_{\mathcal{J}_2} &= \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix} \right\rangle = \{v^m + iv^m J_{mn}\} \ .
\end{aligned} \tag{3.24}
$$

3.2.2 The Courant Bracket

To impose integrability of the almost complex structure we need a bracket on $TM \oplus T^*M$. This cannot be the Lie bracket since this is defined on vectors. So we must look for a generalization of Lie. This is provided by the Courant bracket, which is a skew-symmetric bracket defined on smooth sections of $TM \oplus T^*M$

$$[v+\xi, w+\eta]_C = [v,w] + \mathcal{L}_v\eta - \mathcal{L}_w\xi - \tfrac{1}{2}\mathrm{d}\left(\iota_v\eta - \iota_w\xi\right) \ , \tag{3.25}$$

where $[v,w]$ is the usual Lie bracket between vectors, and $\mathcal{L}_v$ is the Lie derivative[3] Note that Courant reduces to the ordinary Lie bracket when restricted to vectors, while it vanishes on one-forms.

The Courant bracket does not satisfy the Jacobi identity on $TM \oplus T^*M$ (actually there is no bracket on $TM \oplus T^*M$ satisfying it). However it does so when restricted on isotropic subbundles of $TM \oplus T^*M$. One can measure the violation of the Jacobi identity by a trilinear operator, the Jacobiator

$$\mathrm{Jac}(X,Y,Z) = [[X,Y],Z]_C + [[Y,Z],X]_C + [[Z,X],Y]_C \tag{3.30}$$

which in turns can be expressed as the derivative of the so-called Nijenhuis operator

$$\mathrm{Jac}(X,Y,Z) = \mathrm{dNij}(X,Y,Z)\,, \tag{3.31}$$

$$\mathrm{Nij}(X,Y,Z) = \frac{1}{3}\left(([X,Y]_C,Z) + ([Y,Z]_C,X) + ([Z,X]_C,Y)\right) \ . \tag{3.32}$$

There is an alternative definition of the Courant bracket as a particular case of so–called derived brackets (see for example [12]), which is more useful later when considering pure spinors. We first define the Lie bracket as a derived bracket

$$[\{\iota_v, d\}, \iota_w] = \iota_{[v,w]_{\mathrm{Lie}}} \ , \tag{3.33}$$

where $i_{[v,w]} = [\mathcal{L}_v, \iota_w]$ and all the variables are operators acting on differential forms. The brackets on the left hand side are commutators and anticommutators; the one on the right hand side is the Lie bracket. We can now define the Courant bracket analogously

$$[\{X\cdot, d\}, Y\cdot] \equiv [X,Y]_{\mathrm{Courant}}\cdot \ , \tag{3.34}$$

where X and Y are sections of $T \oplus T^*$. Here $X\cdot$ denotes the action of a section of E differential forms

$$X\cdot = \iota_v + \zeta\wedge \tag{3.35}$$

where vectors act by contraction and one–forms act by wedging. By computing it explicitly we obtain the definition (3.25).

[3]Given two vector fields

$$v(x) = v^i(x)\partial_i|_x \qquad w(x) = w^i(x)\partial_i|_x \tag{3.26}$$

the Lie bracket is defined as

$$[v,w](f) \;=\; v(w(f)) - w(v(f)) \qquad \text{basis independent} \tag{3.27}$$

$$[v,w]\,(x) \;=\; v^i\frac{\partial w^j}{\partial x^i}\frac{\partial}{\partial x^j}|_x - w^i\frac{\partial v^j}{\partial x^i}\frac{\partial}{\partial x^j}|_x \qquad \text{basis dependent} \tag{3.28}$$

The Lie derivative along a vector v is defined as

$$\mathcal{L}_v = \iota_v\mathrm{d} + d\iota_v. \tag{3.29}$$

3.2.2.1 Symmetries of the Courant Bracket

As the Lie bracket, the Courant bracket is invariant under diffeomorphisms of M, but has also an additional symmetry given by the B-field transformations with closed B

$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \tag{3.36}$$

where $v + \xi \mapsto v + \xi + \iota_v B$.

3.2.2.2 Twisted Courant bracket

The main feature of a derived bracket is that it contains a differential. For both Lie and Courant the differential is d, but one can generalize it to other differentials. A natural generalization is the inclusion a closed three–form H, to form the differential $\mathrm{d} - H\wedge$. The bracket will of course be modified in a way that is very natural in string theory. Indeed, if B is not a globally defined form, but is actually a field, like it is the case for the (e.g. the NS two-form, a B-transform is not be a symmetry of Courant. We can can use it to modify the definition of the Courant bracket, and introduce the *twisted Courant bracket*

$$[v + \xi, w + \eta]_H = [v, w] + \mathcal{L}_v\eta - \mathcal{L}_w\xi - \tfrac{1}{2}\mathrm{d}\left(\iota_v\eta - \iota_w\xi\right) + \iota_v\iota_w H\,. \tag{3.37}$$

The twisted bracket appears naturally in the supersymmetry equations for type II supergravity.

3.3 Pure spinors

On the tangent bundle TM there is a one-to-one correspondence between almost complex structures and Weyl spinors. An analogous property holds on $TM \oplus T^*M$ between generalized almost complex structures and pure spinors.

3.3.1 $O(d,d)$ spinors

Given the metric $\mathcal{I}$ one can define $Spin(d,d)$ spinors. The Clifford algebra, $Cliff(d,d)$,

$$\{\gamma^m, \gamma_n\} = \delta^m_n \tag{3.38}$$

$$\{\gamma^m, \gamma^n\} = \{\gamma_m, \gamma_n\} = 0 \tag{3.39}$$

has two irreducible representation, one of positive and one of negative chirality $S_\pm(TM \oplus T^*M)$. The spin representation splits into two chiral ones because, due to the signature (d,d) of the metric, the volume form (the chiral gamma) squares to 1, and thus has ± 1 eigenvalues.

There is an isomorphism between the spinor bundle and the exterior algebra of T^*M, $\Lambda^\bullet T^*M$

$$S_+ \cong \Lambda^{\mathrm{even}}T^*M \qquad S_- \cong \Lambda^{\mathrm{odd}}T^*M \tag{3.40}$$

so that positive and negative chirality spinors are even and odd forms, respectively. To see this, consider the action of the Clifford algebra on forms: the gamma matrices of the $Cliff(d,d)$ algebra are vectors v and one–forms ζ (acting by $\zeta\wedge$). As a basis, we can consider

$$\gamma^m = \mathrm{d}x^m, \qquad \gamma_m = \iota_m\,, \tag{3.41}$$

that satisfy

$$\{dx^m\wedge\,,\, dx^n\wedge\} = 0\ , \quad \{dx^m\wedge\,,\, \iota_{\partial_n}\} = \delta^m{}_n\ , \quad \{\iota_{\partial_m}\,,\, \iota_{\partial_n}\} = 0\ . \tag{3.42}$$

Then the action of an element of the Clifford algebra on forms is

$$(v + \xi) \cdot \Phi = \iota_v \Phi + \xi \wedge \Phi \,, \tag{3.43}$$

and it is easy to check that this is exactly the Clifford algebra with metric (3.7)

$$X^2 = (X, X) \qquad \forall X \in TM \oplus T^*M \,. \tag{3.44}$$

Indeed

$$\begin{aligned} ((v + \xi)(v + \xi)) \cdot \Phi &= (v + \xi)(\iota_v \Phi + \xi \wedge \Phi) && (3.45) \\ &= \iota_v(\iota_v \Phi + \xi \wedge \Phi) + \xi \wedge (\iota_v \Phi + \xi \wedge \Phi) && (3.46) \\ &= (\iota_v \xi)\Phi = (v + \xi, v + \xi)\Phi && (3.47) \end{aligned}$$

On the spinor representation it is possible to define a symmetric bilinear form. Because of the isomorphism between spinors and forms, this translates into an inner product on the space of forms, the *Mukai pairing*. This is defined as

$$\langle A, B \rangle \equiv (A \wedge \lambda(B))_d \;, \qquad \lambda(A_p) = (-1)^{\mathrm{Int}[p/2]} A_p \,, \tag{3.48}$$

where the subindices d and p denote the degree of the form. More precisely, the Mukai pairing selects the component of the wedge product of highest degree. In $d = 6$ this pairing is antisymmetric[4]

The Mukai pairing can be used to define the *norm* of a spinor. Consider the inner product of a spinor and his conjugate $\langle \Phi, \bar{\Phi} \rangle$. In this case the component of highest degree is a top form. Since the top form is proportional to the volume form "vol", one can define the norm as the constant of proportionality between the Mukai pairing and the volume form

$$\langle \Phi, \bar{\Phi} \rangle = -i||\Phi||^2 \mathrm{vol} \,. \tag{3.49}$$

3.3.2 Pure spinors

From the action of the Clifford algebra (3.43), one defines the annihilator of a spinor as the subspace

$$L_\Phi = \{v + \zeta \in TM \oplus T^*M \,|\, (v + \zeta) \cdot \Phi = 0\} \,. \tag{3.50}$$

From (3.44), it follows that the annihilator space L_Φ of any spinor Φ is isotropic. It can have at most dimension d, in which case it is maximally isotropic.

A spinor is said to be *pure* if its annihilator is maximally isotropic. This is equivalent to say that a pure spinor is a vacuum of the Clifford algebra since it is annihilated by half of the gamma matrices. A single pure spinor reduces the structure group of $TM \oplus T^*M$ to $SU(n, n)$.

3.3.2.1 Examples

- The holomorphic-three form Ω in six dimensions is a pure spinor. One can define holomorphic and anti-holomorphic gammas: $\gamma^i, \gamma_i, \gamma^{\bar{i}}, \gamma_{\bar{i}}$. We choose γ^i and $\gamma^{\bar{i}}$ to be the creators and γ_i and $\gamma_{\bar{i}}$ the annihilators. Then Ω is annihilated by

$$\gamma^i \Omega = \gamma_{\bar{i}} \Omega = 0 \,. \tag{3.51}$$

[4]It is symmetric in dimension $4k$ k=1,2,... and skew symmetric in dimension $4k + 2$.

- The exponential of the fundamental form is a pure spinor

$$(\gamma_m + iJ_{mn}\gamma^n)e^{iJ} = 0\,. \qquad (3.52)$$

- The identity is a pure spinor

$$(v+\xi)\mathbb{I} = \xi \qquad (3.53)$$

since the contraction with any vector is zero. Then the annihilator of the identity is the tangent bundle itself

$$\mathcal{L}_{\mathbb{I}} = \{v + \xi s.t.\xi = 0\} = TM\,, \qquad (3.54)$$

which is maximally isotropic.

3.3.3 Pure spinors and generalized complex structures

There is a one-to-one correspondence between generalized complex structures and pure spinors. The isomorphism holds both at the topological and differential level.

At the topological level, the correspondence is based on the fact that, given an almost complex structure, $\mathcal{J}$, one can always built a pure spinor that has the $+i$ eigenspace of $\mathcal{J}$ as annihilator

$$\text{annihilator of}\,\Phi = i - \text{eigenspace of}\,\mathcal{J}\,. \qquad (3.55)$$

For instance, for an $SU(3)$ structure we have

$$\mathcal{J}_1 \longleftrightarrow \Omega \qquad\qquad \mathcal{J}_2 \longleftrightarrow e^{-iJ}\,. \qquad (3.56)$$

Given a pure spinor Φ, we can also define the corresponding generalized almost complex structure $\mathcal{J}$ as

$$\mathcal{J}_{\pm\,\Lambda\Sigma} = \langle \mathrm{Re}(\Phi_\pm)), \Gamma_{\Lambda\Sigma}\mathrm{Re}(\Phi_\pm)\rangle\,, \qquad (3.57)$$

where Λ, Σ are indices on $TM \oplus T^*M$, and Γ_Λ are Cliff(d,d) gamma matrices.

Since rescaling the pure spinor Φ does not change its annihilator L_Φ, to each almost complex structure we can associate a line bundle of pure spinors. In general this line bundle does not have a global section. When this is the case, the structure group on $TM \oplus T^*M$ is further reduced from $U(n,n)$ to $SU(n,n)$.

At the differential level, the correspondence is a relation between the integrability of $\mathcal{J}$ and some differential properties of the associated pure spinor.

$$\mathcal{J}\,\text{integrable} \qquad \Leftrightarrow \qquad \mathrm{d}\Phi = (\iota_v + \xi\wedge)\Phi\,, \qquad (3.58)$$

for some v and a ξ. To see this, let us consider the $+i$ eigenspace of $\mathcal{J}$ and use the definition of the Courant bracket as a derived bracket, (3.34). For $X, Y \in L_{\mathcal{J}}$,

$$[X,Y]_{\mathrm{C}}\Phi = (XY - YX)\mathrm{d}\Phi = 0\,. \qquad (3.59)$$

I $\mathcal{J}$ is integrable, $[X,Y]_{\mathrm{C}} = 0$ and $\mathrm{d}\Phi = 0$. The converse is also true: imposing $\mathrm{d}\Phi = 0$ implies that $[X,Y]_{\mathrm{C}} \in L_\Phi = L_{\mathcal{J}}$, and hence by definition $\mathcal{J}$ is integrable. Notice that the condition $\mathrm{d}\Phi = 0$ is actually too strong. For $\mathrm{d}\Phi$ to be annihilated by the two gamma matrices X, Y it is enough that it is at most at level one starting from the Clifford vacuum Φ

$$\mathrm{d}\Phi = (\iota_v + \zeta\wedge)\Phi \qquad (3.60)$$

for some v and ζ.

Let us consider again the two-dimensional examples of generalized complex structures of (3.23) and let us construct the corresponding pure spinors and their annihilators. Let us take first the generalized complex structure $\mathcal{J}_{\mathcal{I}}$. Then Φ_I must be such that $L_{\Phi_I} = L_{\mathcal{J}_{\mathcal{I}}}$ annihilates it

$$(\iota_{\partial_1} + i\,\iota_{\partial_2})\Phi_I = 0\ , \qquad (e^1 + i\,e^2) \wedge \Phi_I = 0\ . \tag{3.61}$$

This gives

$$\Phi_I = c_-\,(e^1 + i\,e^2) = c_-\,\Omega_1\ , \tag{3.62}$$

where the complex number c_- can be freely rescaled and gives the normalization of Φ_I. The same procedure for $L_{\mathcal{J}_{\mathcal{J}}}$ gives

$$\left.\begin{array}{l}(\iota_{\partial_1} + i\,e^2\wedge\,)\Phi_J = 0\\ (\iota_{\partial_2} - i\,e^1\wedge\,)\Phi_J = 0\end{array}\right\} \Rightarrow \Phi_J = c_+\,(1 - ie^1 \wedge e^2) = c_+\,e^{-iJ}\ . \tag{3.63}$$

3.3.4 Type of a pure spinor

Another characterization of a pure spinor is the type. In any dimension $d = 2n$ a pure spinor can be put in the form

$$\Phi = e^{B+ij} \wedge \omega_k \tag{3.64}$$

where B and j real two-forms and ω_k a holomorphic k-form $(0 \leq k \leq n)$[5]. The degree k of the holomorphic form is called type of Φ and corresponds to the dimension of the intersection of the annihilator $L_{\mathcal{J}}$ with the tangent bundle TM.

From (3.64) we see that the most general pure spinor is a hybrid of the two examples we discussed above. Notice also that in general the type of a pure spinor can vary over the manifold M: it is as low as allowed by parity, and can jump in steps of two at some special loci. More precisely, an even pure spinor Φ_+ will be of type 0, and jump to type 2 at some locus on the manifold, while an odd one, Φ_-, will be of type 1, and jump to type 3 at some loci.

3.4 Generalized Calabi Yau manifolds

A manifold M admitting a closed pure spinor with non vanishing norm

$$\mathrm{d}\Phi = 0\,. \tag{3.65}$$

is called a generalized Calabi-Yau (GCY).

Note that the requirement of non-vanishing norm is essential, since otherwise any manifold would be a generalized Calabi-Yau. To see this consider the identity operator, which, as we have already seen, $\mathbb{I}$ is a pure spinor. However it has zero norm, since $\mathbb{I} \wedge \mathbb{I}$ has no top–form part, (see (3.48)).

The generalized Calabi-Yau condition is the parallel on the generalized tangent bundle of the well-known Calabi-Yau condition. In summary

[5] The degree of ω cannot be bigger than N because in that case the spinor Φ will have zero norm.

	T	$T \oplus T^*$
Clifford algebra	$Cliff(6)$ $\{\gamma^m, \gamma^n\} = 2g^{mn}$	$Cliff(6,6)$ $\{\gamma^m, \gamma^n\} = 0, \{\gamma_m, \gamma_n\} = 0, \{\gamma^m, \gamma_n\} = \delta^m_n$
spinors	$(0,q)$ forms	(p,q) forms
pure spinor	η_0 vacuum of Cliff(6)	Φ vacuum of Cliff(6,6)
	$D_m \eta_0 = 0$ Calabi Yau	$\mathrm{d}\Phi = 0$ Generalized Calabi Yau

(3.66)

For applications to string theory, it is useful to define the notion of twisted generalized Calabi-Yau manifold. Notice that given a closed spinor Φ, we can build other closed two-spinors with non-zero norm by acting on Φ with a closed B-transform

$$\Phi_B = e^B \wedge \Phi\,, \qquad\qquad e^B = 1 + B\wedge + \frac{1}{2} B \wedge B \wedge + \ldots\,. \tag{3.67}$$

where B is given in (3.36). Notice that the new spinor corresponds to the generalized complex structure

$$\mathcal{J}_B = \mathcal{B}\mathcal{J}\mathcal{B}^{-1}\,. \tag{3.68}$$

So in general a GCY does not admit a unique closed Φ.

What is more interesting is what happens for when we act on a closed Φ with non-closed B–field. Using the twisted Courant bracket (3.37) we can state a correspondence between twisted generalized complex structures (3.68) and twisted pure spinors (3.67)

$$(d - H\wedge)\Phi = (\iota_v + \zeta\wedge)\Phi \qquad \Leftrightarrow \qquad \mathcal{J} \text{ twisted integrable}\,. \tag{3.69}$$

We can then introduce the notion of twisted generalized Calabi–Yau: it is a manifold on which there exists a pure spinor Φ which is closed under $d - H\wedge$

$$\mathrm{d}\Phi + H \wedge \Phi = 0\,.$$

3.4.1 Differential structure of the manifold

The existence of an integrable pure spinor allows to determine the local geometry of the manifold. If the integrable pure spinor has type k, the generalized complex manifold is locally equivalent to a product

$$\mathbb{C}^k \times (\mathbb{R}^{d-2k}, J) \qquad \left\{ \begin{array}{ll} z^1, \ldots, z^k & \text{holomorphic} \\ x^{2k+1}, \ldots, x^6 & \text{real} \end{array} \right. \,, \tag{3.70}$$

where

$$J = dx^{2k+1} \wedge dx^{2k+2} + \ldots + dx^{d-1} \wedge dx^d \tag{3.71}$$

is the standard symplectic structure and k is again the type. This is a complex–symplectic "hybrid". Some examples that will be relevant in type II string compactifications are

SU(3) structure	IIA	Φ_+ is type 0	$\Rightarrow$	Y_6 is symplectic
	IIB	Φ_- is type 3	$\Rightarrow$	Y_6 is complex
SU(2) structure	IIA	Φ_+ is type 2	$\Rightarrow$	Y_6 is hybrid 1 symplectic -2 complex
	IIB	Φ_- is type 1	$\Rightarrow$	Y_6 is hybrid 1 complex - 2 symplectic

3.5 Metric from pure spinors

Two pure spinors are said to be *compatible* if they have three common annihilators. Alternatively, they must have equal norm

$$\langle \bar{\Phi}^-, \Phi^- \rangle = \langle \bar{\Phi}^+, \Phi^+ \rangle \tag{3.72}$$

and must satisfy

$$\langle \Phi^+, (v+\zeta)\cdot \Phi^- \rangle = \langle \Phi^+, (v+\zeta)\cdot \bar{\Phi}^- \rangle = 0 \; , \qquad \forall\, v+\zeta \in TM \oplus T^*M \, . \tag{3.73}$$

Two compatible pure spinors reduce the structure to $SU(n)\times SU(n)$.

For instance, on a manifold of $SU(3)$ structure there exist two compatible natural pure spinors

$$\Phi_+ = e^{-iJ} = 1 - iJ - \frac{1}{2}J^2 + \frac{i}{6}J^3 \tag{3.74}$$

which is annihilated by $\gamma_m + iJ_{mn}\gamma^n$ and

$$\Phi_- = \Omega \tag{3.75}$$

with annihilators $\gamma^i, \gamma_{\bar{i}}$.

Using the isomorphism between pure spinors and generalized complex structures, we also have a pair of compatible generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$: they commute and are such that

$$\mathcal{H} = -\mathcal{I}\mathcal{J}_1\mathcal{J}_2 \tag{3.76}$$

is a positive definite metric on $TM \oplus T^*M$. The metric $\mathcal{H}$ can be constructed also introducing a split of E into two orthogonal (with respect to $\mathcal{I}$) n-dimensional sub-bundles[6]

$$E = C_+ \oplus C_- \tag{3.79}$$

such that the metric $\mathcal{I}$ decomposes into a positive-definite metric on C_+ and a negative-definite metric on C_-. Then, the generalized metric $\mathcal{H}$ is defined by

$$\mathcal{H} = \mathcal{I}|_{C_+} - \mathcal{I}\,|_{C_-} \; . \tag{3.80}$$

It is easy to see that $\mathcal{H}^2 = \mathbb{1}$ $\mathcal{I}(\mathcal{H}X, \mathcal{H}Y) = \mathcal{I}(X,Y)$.

The explicit form of $\mathcal{H}$ can be derived from (3.77) and (3.80), by noticing that $(\mathbb{1} \pm \mathcal{H})$ are projectors into $C_\pm$. This gives

$$\mathcal{H} = \frac{1}{2}\begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \, . \tag{3.81}$$

[6]The two subbundles $C_\pm$ can be defined as the graphs

$$C_\pm = \{X \in E \;:\; X_\pm = v + (B \pm g)v\} \, , \tag{3.77}$$

where g is a Riemannian metric on M and B is a two-form. Both g and B are seen as maps from TM to T^*M

$$(B \pm g)v \equiv (B \pm g)_{mn} v^n \mathrm{d}x^m \, . \tag{3.78}$$

From this expression we see that a $U(n) \times U(n)$ structure implies the existence of a metric g and B–field. Notice that B appears indeed as in a B–transform, (3.67)

$$\mathcal{H} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} . \tag{3.82}$$

Note also that the metric $M = \mathcal{I} G$ appeared in T–duality (see for example [13]) as a combination that transforms by conjugation under $\mathrm{Sl}(2, \mathbb{R})$.

In the two-dimensional examples (3.23), it is easy to prove that the two complex structures are compatible and define a metric $\mathcal{H}$

$$\mathcal{H} = -\mathcal{I}\mathcal{J}_I\mathcal{J}_J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.83}$$

Here there is no B-field and the metric g is just the 2×2 identity matrix. In this example it is easy to verify that the relative sign of the two generalized complex structures is important for $\mathcal{H}$ to be a metric. Indeed, even if change the sign of one of the $\mathcal{J}$'s, they would still commute, but the metric g would not be positive definite.

Chapter 4

Supersymmetry and $\mathcal{N} = 1$ flux compactifications

In this thesis we will focus on type II compactifications to four dimensions and we look for classical vacua. For this reason we will always work in the context of the low energy supergravity actions in ten dimensions.

Type II supergravities are ten-dimensional theories with local $\mathcal{N} = 2$ supersymmetry. The bosonic sector consists of the Neveu-Schwarz and Ramond-Ramond fields. The NS sector is the same for both theories: the metric, the dilaton and the NS two-form. The RR sector depends on the theory. For IIA, It consists of odd p-forms for IIA, and even for IIB. We will use the democratic formulation [14], which considers all RR potentials, C_n with $n = 1, \dots, 9$ for IIA and C_n with $n = 0, \dots, 0$, for IIB. These potential are not all independent: for instance C_2 is equivalent to C_6 because of Hodge duality. Therefore to reduce to the independent degrees of freedom we will impose a self-duality constraint on the field strengths

$$F_n = (-1)^{Int[n/2]} \star_{(10)} F_{10-n} \,. \tag{4.1}$$

The fermionic sector consist of two Majorana-Weyl spin 3/2 spinors, the gravitinos ψ^i_M, and two Majorana-Weyl spin1/2 spinors, the dilatinos λ_i. Gravitinos and dilatinos have opposite chirality. In IIA the gravitinos (hence the dilatinos) have opposite chirality, while in two IIB the gravitinos have the same chirality, which we choose positive. Correspondingly the dilatinos will have negative chirality.

We are interested in compactifications to four dimensions, where the four dimensional space are maximally symmetric: Minkowski, Anti de Sitter or de Sitter spaces. To this extent we can make some hypothesis on the form of the 10-dimensional solutions we are looking for:

- the space time is a (warped) product of 4-dimensional space-time and a compact internal 6-dimensional manifold

  $$M_{10} = X_4 \times_w M \,. \tag{4.2}$$

  The corresponding metric has the form

  $$\mathrm{d}s^2_{10} = e^{2A(y)} \mathrm{d}s^2_4 + \mathrm{d}s^2_6 \,, \tag{4.3}$$

  where the warp factor A can be a function of the coordinates on the internal manifold M. The $4d$ metric will have Poincaré, $SO(1,4)$ or $SO(2,3)$ symmetry for M_4, AdS_4 or dS_4, respectively.

- some of the RR and/or NS fields can have non-zero background values. In order to preserve maximal symmetry in four dimensions the fluxes can be non trivial only on the internal manifold

$$F_p^{(10)} = \mathrm{vol}_4 \wedge \hat{F}_{p-4} + F_p \tag{4.4}$$

$$= F_p + \mathrm{vol}_4 \wedge \lambda(*F_{6-p}) \qquad \text{with} \quad \lambda(F_n) = (-1)^{\mathrm{Int}[n/2]} F_n \tag{4.5}$$

  Then the Hodge duality (4.1) implies for the following relation among internal and external components of the RR fields

$$\tilde{F}_p = *_6 \lambda(F_{6-p}) \quad \Leftrightarrow \quad \tilde{F} = *_6 \lambda(F) \; . \tag{4.6}$$

- imposing maximal symmetry sets all the vacuum expectation values of the fermionic fields to zero. So we will look for purely bosonic solutions.

4.1 Supersymmetric solutions

We will actually restrict even more the form of the solutions we are after, by imposing that they have minimal supersymmetry, namely $\mathcal{N} = 1$ in four dimensions.

From a technical point of view, looking for supersymmetric solutions simplifies life a lot. Indeed, it can be shown that, under some hypothesis[1] that are verified for the type of solutions we are interested in [15], instead of solving the equations of motion, which are second order differential equations, one can solve a set of first order equations

- vanishing of supersymmetry variations for the gravitino and the dilatino
- Bianchi identities and equations of motion for the fluxes

Minimal supersymmetry is a phenomenological requirement, since extended supersymmetries do not admit chiral fermions and thus do not give rise to a physically relevant spectrum of particles in the low energy actions obtained compactifying around one such vacuum. We do not address here the issue of how supersymmetry is broken. We simply assume that it is spontaneously broken at low enough energies.

4.1.1 Supersymmetry equations

For a purely bosonic background, the conditions for unbroken supersymmetry is that the variations of the fermionic fields vanish. Indeed, an unbroken supersymmetry is such that its generator (the conserved supercharge) annihilates the vacuum state

$$Q|0>= 0 \, , \tag{4.7}$$

which is equivalent to the condition that for all operators O in the theory

$$< 0|Q, O|0 >= 0 \, . \tag{4.8}$$

[1]It has been proven by [15] that whenever in the Einstein tensor there are no mixed time-internal components, the SUSY variations plus the flux Bianchi identities imply the equations of motion for the dilaton and the metric. This is indeed the case for backgrounds corresponding to warped product. More recently, [16] extended the result to the e.o.m for the NS field H.

For all bosonic operators this is true since the anticommutator $\{Q, O\}$ is fermionic, and thus it must vanish to preserve Lorentz invariance. So the only non trivial condition from (4.8) is when the operator O is fermionic. The constraint is actually quite simple since for a fermionic operator

$$< 0|Q, O|0 >=< 0|\delta O|0 >\sim \delta O \quad \text{classically}\,. \tag{4.9}$$

Then, in type II supergravity we have to set to zero the bosonic part of the gravitino and dilatino variations

$$\delta\psi_M = 0 \qquad\qquad \delta\lambda = 0\,. \tag{4.10}$$

We will always work in the string frame and we use the democratic formulation. If we write the two gravitino and the two dilatino as doublets $\psi_M = (\psi^1_M, \psi^2_M)$ and $\lambda = (\lambda^1, \lambda^2)$, then their supersymmetry variations read

$$\delta\psi_M = (D_M\epsilon + \frac{1}{4}H_M\mathcal{P})\epsilon + \frac{1}{16}e^{\phi}\sum_n \not{F}^{(2n)}\,\Gamma_M\mathcal{P}_n\,\epsilon\,, \tag{4.11}$$

$$\delta\lambda = (\not{\partial}\phi + \frac{1}{2}\not{H}\mathcal{P})\epsilon + \frac{1}{8}e^{\phi}\sum_n(-1)^{2n}(5-2n)\not{F}^{(2n)}\,\mathcal{P}_n\epsilon\,, \tag{4.12}$$

where the supersymmetry parameter $\epsilon = (\epsilon^1, \epsilon^2)$ is also doublet of Majorana-Weyl spinors. The matrices $\mathcal{P}$ and $\mathcal{P}_n$ are different in IIA and IIB. For IIA $\mathcal{P} = \Gamma_{11}$ and $\mathcal{P}_n = \Gamma_{11}\sigma_1$, while in IIB $\mathcal{P} = -\sigma_3$, $\mathcal{P}_n = \sigma_1$ for $n + 1/2$ even and $i\sigma_2$ for $n + 1/2$ odd.

Because of the product structure of the space-time, the 10-d Lorentz group reduces accordingly

$$SO(1,9) \to SO(1,3) \times SO(6)\,. \tag{4.13}$$

The ten-dimensional gamma matrices Γ^M split accordingly in terms of four- and six-dimensional gamma matrices $\hat{\gamma}^\mu$ (associated with the unwarped X_4 metric) and γ^m in the following way

$$\Gamma^\mu \;=\; e^{-A}\hat{\gamma}^\mu \otimes \mathbb{1} \qquad\quad \Gamma^m \;=\; \gamma_{(4)} \otimes \gamma^m \tag{4.14}$$

where $\gamma_{(4)} = i\hat{\gamma}^{0123}$ is the standard four-dimensional chiral operator. The six-dimensional chiral operator is in turn $\gamma_{(6)} = -i\gamma^{123456}$ and so we have that $\Gamma_{(10)} = \gamma_{(4)} \otimes \gamma_{(6)}$.

Then supersymmetry parameters decompose as

$$\begin{aligned}\epsilon_1 &= \zeta_+ \otimes \eta^1_+ + \zeta_- \otimes \eta^1_- \\ \epsilon_2 &= \zeta_+ \otimes \eta^2_- + \zeta_- \otimes \eta^2_+\,,\end{aligned} \tag{4.15}$$

in IIA while for IIB

$$\epsilon_i = \zeta_+ \otimes \eta^i_+ + \zeta_- \otimes \eta^i_- \qquad\qquad i = 1, 2\,, \tag{4.16}$$

where $\zeta_\pm$ is a $4d$ chiral spinor ($\zeta^*_+ = \zeta_-$) and $\eta^{(i)}_\pm$ is a $6d$ chiral spinor ($\eta^{i*}_+ = \eta^i_-$).

4.1.2 Bianchi identities and e.o.m for the forms

The distinction between Bianchi identities and equations of motion depends on what we take for the electric and magnetic components of the fluxes. Following the splitting (4.6), it is natural to choose F, the component entirely in the internal directions, as electric, and $\tilde{F}$ as the magnetic one.

Then Bianchi identities for the NS and the RR fluxes can be rewritten in terms of the internal fluxes. If we define the sum of the internal fluxes

$$\text{IIA} \qquad F = F_0 + F_2 + F_4 + F_6 \qquad\qquad \text{IIB} \qquad F = F_1 + F_3 + F_5\,, \tag{4.17}$$

the Bianchi identities become

$$(\mathrm{d} - H\wedge)F = \delta(source) \tag{4.18}$$
$$\mathrm{d}H = 0\,. \tag{4.19}$$

where $\delta(source)$ is the contribution from brane or orientifold sources. Similarly the equations of motion read

$$(\mathrm{d} + H\wedge)(e^{4A} * F) = \delta_s \tag{4.20}$$
$$\mathrm{d}(e^{4A-2\phi} * H) = \pm e^{4A} F_n \wedge *\lambda(F_{n+2})\,, \tag{4.21}$$

where the upper/lower sign corresponds to IIA/IIB. Note that the Hodge star is the six-dimensional one. As we will see later the equations of motion, (4.20), follow from the supersymmetry conditions. So the only extra conditions to impose for supersymmetric vacua are the Bianchi identities. In fact, these latter are highly nontrivial constraints on the solutions. We will see that they enforce the no-go theorem about flux compactification in the special case of supersymmetric flux compactification.

4.2 Pure geometry: Calabi-Yau compactifications

We start by considering the case of purely geometric compactifications, where the only non trivial field is the metric. Since all the fluxes are set to zero, in order to find a solution it is enough to solve the supersymmetry variations.

When all fluxes are set to zero, using the metric ansazt (4.3) and the splitting (4.15) and (4.16) for the spinors, the dilatino variation (4.11) reduces to the six-dimensional equations[2]

$$\partial\!\!\!/\phi\, \eta_{1,2} = 0\ , \tag{4.23}$$

where $\partial\!\!\!/\phi = \gamma^m \partial_m \phi$. Since $||\partial\!\!\!/\phi\, \eta_{1,2}||^2 = (\partial\phi)^2 ||\eta_{1,2}||^2$, it follows that the dilaton must be constant.

The gravitino variations reduce to the requirement that the supersymmetry parameters must be covariantly constant

$$\delta\psi^1_M = \nabla_M \epsilon_1 = 0\,, \qquad \delta\psi^2_M = \nabla_M \epsilon_2 = 0\,, \tag{4.24}$$

where $\nabla_M = \partial_M + \frac{1}{4}\omega_M{}^{AB}\Gamma_{AB}$ is the standard spinorial covariant derivative.

Using the metric ansatz (4.3) and 4 plus 6 splitting of the gamma matrices and spinors, (4.14)-(4.16), the space-time component of equation (4.24) becomes (an identical equation holds for η_2)

$$D_\mu \zeta \otimes \eta_1 - \frac{1}{2} e^A (\gamma_\mu \zeta^* \otimes \partial\!\!\!/ A\, \eta_1^*) + \text{c.c.} = 0\,, \tag{4.25}$$

[2]In order to be able to describe at the same time type IIA and type IIB, in the spinorial SUSY variation $\eta_{1,2}$ denote six-dimensional chiral spinors, of opposite chirality in IIA and same chirality in IIB

$$\gamma_7 \eta_1 = \eta_1 \qquad\qquad \gamma_7 \eta_2 = \mp \eta_2 \quad \text{in IIA/IIB}\,. \tag{4.22}$$

where D_m is the standard covariant derivative with respect to the four-dimensional unwarped metric. This equation requires the warp factor to be constant[3] and

$$D_\mu \zeta = 0 \,. \tag{4.26}$$

This leads to the integrability condition

$$[D_\mu, D_\nu]\epsilon = \frac{1}{4} R_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\epsilon = \frac{\Lambda}{6}\gamma_{\mu\nu}\epsilon \,, \tag{4.27}$$

where we used the expression for the curvature tensor for a maximally symmetric (unwarped) four-dimensional metric: $R_{\mu\nu\rho\sigma} = \frac{1}{3}\Lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$. Combining (4.26) and (4.27), we obtain that $\Lambda\gamma_{\mu\nu}\zeta = 0$, which implies the vanishing of the cosmological constant

$$\Lambda = 0 \,. \tag{4.28}$$

Then, from the external gravitino equations it follows that the warp factor must be constant and the four-dimensional space must be Minkowski.

We are left with the internal gravitino equations, (4.53), which reduce to

$$D_m \eta_i = 0 \,, \qquad i = 1, 2 \,, \tag{4.29}$$

where D_m is the covariant derivatives with respect to the internal six-dimensional metric. This equation is highly non trivial and determines many of the properties of the solution. More precisely it implies

- Ricci flatness. Applying the integrability condition on the internal manifold, we see that the internal metric must be Ricci flat $R_{mn} = 0$.
- SU(3) holonomy. As already mentioned, a covariantly constant spinor implies a reduction of the holonomy group of a Riemannian manifold. From (4.29) it follows that the internal metric must have at most holonomy SU(3). The internal metric has exactly SU(3) holonomy SU(3) if the two internal spinors η^1 and η^2 are proportional[4].

A Ricci flat manifold of SU(3) holonomy is a Calabi-Yau.

An alternative definition of a Calabi-Yau manifold is via closed forms. Without loss of generality we can set $\eta^1_+ = \eta^2_+ = \eta$ with $\eta^\dagger\eta = 1$ and define the forms

$$\begin{aligned} J_{mn} &= -i\,\eta_+^\dagger \gamma_{mn}\,\eta_+ \,, & (4.32)\\ \Omega_{mnp} &= -i\,\eta_-^\dagger \gamma_{mnp}\,\eta_+ & (4.33)\end{aligned}$$

[3]Alternatively we should impose that $\not{\partial} A\,\eta^{i*}$ is proportional to $\eta_{1,2}$, but this is impossible since $\eta^\dagger_{1,2}\gamma_m\eta_{1,2} = 0$.

[4] Notice that we could have started from a more general spinorial ansatz than (4.15) and (4.16)

$$\epsilon_1 = \zeta_1 \otimes \eta_1 + \text{c.c.} \,, \qquad \epsilon_2 = \zeta_2 \otimes \eta_2 + \text{c.c.} \,, \tag{4.30}$$

where ζ_1 and ζ_2 are completely independent. This ansatz gives exactly the same supersymmetry conditions (with both ζ_1 and ζ_2 constant) as for the case $\zeta_1 = \zeta_2$ considered above. If the internal space has strict SU(3)-holonomy (η^1_+ and η^2_+ are proportional), ζ_1 and ζ_2 give eight independent real supercharges and the background preserves $\mathcal{N} = 2$ four-dimensional supersymmetry. If the internal space has a holonomy group smaller then SU(3), the equation $\nabla_m\eta = 0$ has two (for SU(2)-holonomy) or four (for flat-space) independent solutions η_a and we can set

$$\eta^1_+ = \sum_a c^1_a \eta_a \,, \qquad \eta^2_+ = \sum_a c^2_a \eta_a \,, \tag{4.31}$$

with arbitrary constant $c^{1,2}_a$. Thus we have $\mathcal{N} = 4(8)$ four-dimensional supersymmetry.

which define an SU(3) structure as in 2.24 and 2.25. They allow to rewrite the Calabi-Yau condition in an alternative form

$$D_m \eta = 0 \ \Rightarrow \left\{ \begin{array}{l} D_m J_{np} = 0 \\ D_m \Omega_{npq} = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \mathrm{d}J = 0 \\ \mathrm{d}\Omega = 0 \end{array} \right. \tag{4.34}$$

J and Ω are then the Kähler form and the holomorphic $(3,0)$-form of the Calabi-Yau.

The interest of this definition is that it allows to express the geometrical properties of the manifold in terms of differential equations for some forms. It is in this formulation that we will generalize the idea of Calabi-Yau manifold to the case of non-zero fluxes.

In view of the generalization to flux backgrounds it is worth stressing that the supersymmetry condition $D_m \eta^i_+ = 0$ (see (4.34)) actually splits into two parts: a topological condition about the existence of globally defined spinors on the manifold, which is equivalent to the existence of a SU(3)-structure, and one on the differential properties of the spinor (and tensors), which is the SU(3) holonomy condition

$$D_m \eta_+ = 0$$

$$\Downarrow$$

2 types of constraints

$\swarrow \quad \searrow$

topological	differential
$\exists$ a globally defined non-vanishing invariant spinor η_+	the spinor is covariantly constant $D_m \eta_+ = 0$
$\Updownarrow$	$\Updownarrow$
reduction of the structure group (G-Structure) $SO(6) \rightarrow SU(3)$	reduced holonomy (special holonomy) $SO(6) \rightarrow SU(3)$
$\Updownarrow$	$\Updownarrow$
$\exists$ globally defined invariant forms $J_{mn} = -i\eta^\dagger_+ \gamma_{mn} \eta_+$, $\Omega_{mnp} = -i\eta^\dagger_- \gamma_{mnp} \eta_+$	the forms are closed $\mathrm{d}J = 0$, $\mathrm{d}\Omega = 0$

Note that the topological condition, namely the existence of a globally defined spinor, is necessary in order to perform the KK-reduction of the action and to have supersymmetry, here $\mathcal{N} = 2$ in the effective 4-d low energy theory. The differential condition, that the Levi-Civita connection has $SU(3)$ holonomy, tells that the effective theory has a vacuum corresponding to 4-d Minkowski.

4.3 $\mathcal{N} = 1$ flux compactifications: Generalized Calabi-Yau manifolds

We now turn to more general solutions of type II supergravity where some of the fluxes have non-zero vacuum expectation values. The presence of such fluxes drastically changes the properties of the solutions. This can be seen both from the equations of motion and the supersymmetry variations. Indeed, from the Einstein equation, which reads schematically

$$R_{MN} \sim H_{MPQ} H_N{}^{PQ} + \sum_p F_{MQ_1 \ldots Q_p} F_N{}^{Q_1 \ldots Q_p} \,, \tag{4.35}$$

we see that the fluxes back-react on the metric, which generically cannot be Ricci-flat (and thus Calabi-Yau) anymore. Another generic feature is a non trivial warp factor in the ten-dimensional metric (4.3).

The supersymmetry variations are also modified. For example, from (4.11) and (4.12) one can see that in the presence of RR fluxes the supersymmetry conditions relate ϵ^1 and ϵ^2 so that the four-dimensional components $\zeta_{1,2}$ cannot be chosen independently anymore, as in (4.30). Therefore, in the presence of RR-fluxes one generically obtains $\mathcal{N} = 1$ in four dimensions.

It is natural to ask whether it is possible to have a geometric characterization of $\mathcal{N} = 1$ flux backgrounds generalizing the Calabi-Yau condition. This is what Generalized Complex Geometry provides. The ten-dimensional supersymmetry conditions for flux vacua can be rephrased in terms of differential conditions on a pair of pure spinors on the internal manifold [17–19]. As in the case of Calabi-Yau compactifications this condition splits in two parts, a topological condition and a differential condition. The topological condition is that there exists a pair of compatible pure spinors. This means that we will have to introduce pure spinors in supergravity. The differential condition states that one of the two spinor is closed. This amounts in deriving the closure of Φ from the SUSY variations.

4.3.1 Topological condition

The way pure spinors on $TM \oplus T^*M$ appear in supergravity is via bispinors, namely tensor products of $Cliff(d)$ spinors.

The idea is that if a manifold M admit an SU($d/2$)×SU($d/2$) structure on the generalized tangent bundle, it is possible to write the two pure spinors $\Phi_\pm$ as Spin(d) bispinors. This is because the pure spinors can be seen as Spin(d)×Spin(d) objects and then as bispinors of Spin(d). We define a basis for the gamma matrices adapted to the splitting of the generalized tangent bundle $E = C_+ \oplus C_-$ induced by $\mathcal{H}$ (see section 3.5)

$$\Gamma^\pm_m = \Gamma_m \pm (g \mp B)_{mn}\Gamma^n \,. \tag{4.36}$$

With this choice of basis, the Cliff(d,d) algebra factorizes into two independent Cliff(d) algebrae on C_+ and C_-, respectively

$$\{\Gamma^\pm_m, \Gamma^\pm_n\} = \pm\, 2 g_{mn}, \quad \{\Gamma^+_m, \Gamma^-_n\} = 0\,. \tag{4.37}$$

$\Gamma^\pm_m$ generate two independent Spin(d) groups. For each of the two Spin(d) groups, we can consider a pair of chiral spinors $\eta^{1,2}_+$, that are annihilated by half of the Spin(d) gamma matrices. The spinors η^i_+ are associated to a pair of SU($d/2$) structures on M (J_1, Ω_1) and (J_2, Ω_2) and their annihilators are given by

$$(\mathbb{1} + iI_1)^n{}_m\, \Gamma^+_n \qquad\qquad (\mathbb{1} + iI_2)^n{}_m\, \Gamma^-_n \,, \tag{4.38}$$

where $I_{1,2} = g^{-1}J_{1,2}$ are the two almost complex structures associated to the two SU($d/2$) structures.

Given our pure chiral spinors η_1 and η_2, we can build two Spin(d,d) spinors as tensor products[5]

$$\Phi_+ \sim \eta^1_+ \otimes \eta^{2\,\dagger}_+ \quad , \quad \Phi_- \sim \eta^1_+ \otimes \eta^{2\,\dagger}_- \,, \tag{4.40}$$

[5] We recall that the Fierz identity we are using

$$\eta^1_+ \otimes \eta^{2\,\dagger}_\pm = \frac{1}{8}\sum_{k=0}^{6} \frac{1}{k!} \left(\eta^{2\,\dagger}_\pm \gamma_{m_k \ldots m_1} \eta^1_+\right) \gamma^{m_1 \ldots m_k} \,. \tag{4.39}$$

where the subscripts $\pm$ denote an even/odd polyforms. Written this way, $\Phi_\pm$ can be seen as bispinor of Spin(d).

We thus have two different representation of Spin(d,d) spinors, as polyforms and as Spin(d) bispinors, which are related by the so-called *Clifford map*:

$$\omega \equiv \sum_p \frac{1}{p!}\omega_{m_1\ldots m_p} \mathrm{d}x^{m_i}\wedge\ldots\wedge \mathrm{d}x^{m_p} \quad \longleftrightarrow \quad \not{\omega} \equiv \sum_p \frac{1}{p!}\omega_{m_1\ldots m_p}\gamma^{m_i\ldots m_p} \ , \tag{4.41}$$

where γ_m are ordinary Spin(d) matrices. Under this map one can identify the Clifford action of $\Gamma^\pm_m$ on forms and bispinors as

$$\begin{aligned} \Gamma^+_m\cdot\omega &\equiv \iota_m\omega + g_{mn}\mathrm{d}x^n\wedge\omega & \longleftrightarrow & \qquad \gamma_m\not{\omega} \ , \\ \Gamma^-_m\cdot\omega &\equiv \iota_m\omega - g_{mn}\mathrm{d}x^n\wedge\omega & \longleftrightarrow & \qquad (-)^{|\omega|+1}\not{\omega}\,\gamma_m \ , \end{aligned} \tag{4.42}$$

where $|\omega|$ is the degree of ω and in the action of $\Gamma^\pm_m$ (we still have $B=0$).

In type II supergravity, we have two natural globally defined pure spinors, the SUSY parameters η^1 and η^2. By tensoring them, we can construct a pair of Spin(d,d) spinors that are automatically pure and compatible. Hence they define an SU(3) $\times$ SU(3) structure on $TM\oplus T^*M$. We have

$$\Phi_+ = \eta^1_+\otimes\eta^{2\,\dagger}_+ \qquad \Phi_- = \eta^1_+\otimes\eta^{2\,\dagger}_- \,. \tag{4.43}$$

It is easy to prove that the two $Cliff(6,6)$ spinors defined above are by construction pure and compatible. In six dimensions a pure spinor must have six annihilators. These are the 3 annihilators of η^1 and the 3 annihilators of η^2, acting from the left and the right, respectively. By virtue of (4.42), we can translate these $3+3$ annihilators into 6 annihilators in $Cliff(6,6)$

$$\begin{aligned} &(\delta+iJ_1)_m^n\gamma_n\eta^1_+\otimes\eta^{2\,\dagger}_\pm = 0 \qquad && (J_{1\,mn} = -i\eta^{1\,\dagger}_+\gamma_{mn}\eta^1_+) \\ &\eta^1_+\otimes\eta^{2\,\dagger}_\pm\gamma_n(\delta\mp iJ_2)^n_m = 0 \qquad && (J_{2\,mn} = -i\eta^{2\,\dagger}_+\gamma_{mn}\eta^2_+) \end{aligned}$$

This means $\Phi_\pm$ are pure. By the same token we see that the two pure spinors are also compatible since they share three annihilators: the three gamma matrices $(\delta+iJ_1)_m{}^n\,\overrightarrow{\gamma^n}$.

The explicit expressions of the pure spinors (4.43) depends on the relation between the two supersymmetry parameters η^i or, in other words, on the G-structure on TM. More precisely we have

- SU(3) structure on TM

$$\Phi_+ = \frac{a\bar{b}}{8}e^{-iJ}\,, \tag{4.44}$$

$$\Phi_- = -i\frac{ab}{8}\Omega\,, \tag{4.45}$$

  where J and Ω are defined as in (2.24).

- static SU(2) structure on TM

$$\Phi_+ = -i\frac{a\bar{b}}{8}\omega\wedge e^{z\wedge\bar{z}/2}\,, \tag{4.46}$$

$$\Phi_- = -\frac{ab}{8}e^{-ij}\wedge z\,, \tag{4.47}$$

  where again the forms are defined as in (2.28a).

- intermediate $SU(2)$-structure

$$\Phi_- = -\frac{ab}{8} z \wedge (k_\perp e^{-ij} + ik_\| \omega) \,, \tag{4.48}$$

$$\Phi_+ = \frac{a\bar{b}}{8} e^{z\bar{z}/2} (k_\| e^{-ij} - ik_\perp \omega) \,, \tag{4.49}$$

4.3.2 Differential condition

As for the Calabi-Yau case, we decompose the ten dimensional supersymmetry conditions according to the 4 plus 6 splitting (4.3) and (4.15) and (4.16).

From the external part of the gravitino variations, (4.12) it follows that the four-dimensional chiral spinor ζ in (4.15) must satisfy

$$D_\mu \zeta^* = \frac{1}{2} \mu \gamma_\mu \zeta \ , \tag{4.50}$$

where the constant μ is related to the four-dimensional cosmological constant

$$\Lambda = -3 \, |\mu|^2 \ . \tag{4.51}$$

Thus, supersymmetry backgrounds are possible only when the four-dimensional space is either Minkowski ($\mu = 0$) or AdS ($\Lambda < 0$). The remaining conditions coming from the external gravitino equations give constraints on the internal manifold

$$\begin{aligned} \partial\!\!\!/ A \, \eta_1 + \frac{1}{4} e^\phi \gamma_7 F\!\!\!/ \eta_2 + e^{-A} \mu \eta_1^* = 0 \ , \\ \partial\!\!\!/ A \, \eta_2 - \frac{1}{4} e^\phi \gamma_7 F\!\!\!/^\dagger \eta_1 + e^{-A} \mu \eta_2^* = 0 \ . \end{aligned} \tag{4.52}$$

The internal gravitino equations $\delta\psi_m^{1,2} = 0$ give the six-dimensional conditions

$$\begin{aligned} (D_m + \frac{1}{4} H\!\!\!/_m) \eta_1 + \frac{1}{8} e^\phi F\!\!\!/ \gamma_m \gamma_7 \eta_2 = 0 \ , \\ (D_m - \frac{1}{4} H\!\!\!/_m) \eta_2 - \frac{1}{8} e^\phi F\!\!\!/^\dagger \gamma_m \gamma_7 \eta_1 = 0 \ . \end{aligned} \tag{4.53}$$

Finally, a combination of the dilatino equations and the trace of the gravitino ones give a condition relating the NS fields and the spinors $\eta_{1,2}$ on the internal manifold

$$\begin{aligned} (D\!\!\!\!/ - \partial\!\!\!/ \Phi + 2\partial\!\!\!/ A + \frac{1}{4} H\!\!\!/) \eta_1 + 2e^{-A} \mu \eta_1^* = 0 \,, \\ (D\!\!\!\!/ - \partial\!\!\!/ \Phi + 2\partial\!\!\!/ A - \frac{1}{4} H\!\!\!/) \eta_2 + 2e^{-A} \mu \eta_2^* = 0 \ . \end{aligned} \tag{4.54}$$

Notice that the integrability of equations (4.53) we can directly see that the internal manifold is no longer Ricci flat and hence no longer Calabi-Yau, since

$$[D_m, D_n] \eta_{1,2} = \frac{1}{4} R_{mn}{}^{pq} \gamma_{pq} \eta_{1,2} \neq 0 \,. \tag{4.55}$$

In [17] it was shown that it is possible to derive a set of differential conditions on the pure spinors (4.43) that are equivalent to the SUSY variations. Since the pure spinors are tensor product of the SUSY parameters we expect these conditions to be obtained by suitable manipulations of the the SUSY variations[6]. The result consists of three differential equations involving the two pure spinors plus two conditions relating the norms of the internal spinors to the warping. The content of the differential constraints is different for compactifications to four-dimensional Minkowski or Anti de Sitter. The equations for M_4 vacua have a nice geometrical interpretation, while this is less clear for for AdS_4 vacua. In this section we will only discuss the SUSY conditions for M_4, and we will postpone to the next chapter the AdS_4 case.

For compactifications to four-dimensional Minkowski space the ten-dimensional supersymmetry variations are equivalent to [17]

$$(\mathrm{d} - H\wedge)(e^{2A-\phi}\Phi_1) = 0, \tag{4.60}$$

$$(\mathrm{d} - H\wedge)(e^{A-\phi}\mathrm{Re}\,\Phi_2) = 0\,, \tag{4.61}$$

$$(\mathrm{d} - H\wedge)(e^{3A-\phi}\mathrm{Im}\,\Phi_2) = -\frac{1}{8}e^{4A} * \lambda(F)\,, \tag{4.62}$$

where ϕ is the dilaton, A the warp factor, F is the sum of the RR field strength on M, (4.17), and λ is the transposition operator defined in (3.48). The equations have the same structure in type IIA and type IIB, with

IIA	$\Phi_1 = \Phi_+$	$\Phi_2 = \Phi_-$
IIB	$\Phi_1 = \Phi_-$	$\Phi_2 = \Phi_+$

The equation for Φ_1 tells us that the pure spinor with the same parity as the RR fluxes is closed and has a nice interpretation in terms of Generalized Geometry: a necessary condition for a manifold to allow for an $\mathcal{N} = 1$ vacuum is to be a (twisted) generalized Calabi-Yau. This means that a supersymmetric compactification to flat space naturally posses an integrable generalized complex structure $\mathcal{J}_1$. The other equation, (4.62), shows that the RR fluxes act as torsion creating an obstruction to the integrability of the second generalized complex structure associated to Φ_2. We have then a very straightforward parallel with the case of Calabi-Yau compactifications

	T	$T \oplus T^*$
spinors	$(0, q)$ forms	(p, q) forms
pure spinor	η_0 vacuum of Cliff(6)	Φ vacuum of Cliff(6,6)
	$D_m\eta_0 = 0$	$\mathrm{d_H}\Phi = 0$
	Calabi Yau	twisted Gen. Calabi Yau

[6] The basic point is to express their exterior derivative in terms of the covariant derivatives of η^1 and η^2

$$\mathrm{d}\Phi_\pm = \mathrm{d}x^m D_m \Phi_\pm\,. \tag{4.56}$$

Using the relation between the action of the gamma matrices on bispinors and on forms (4.42), the action of $\mathrm{d}x^m$ translates into

$$\mathrm{d}x^m = \gamma^m \not{C}_k + (-)^k \not{C}_k\,\gamma^m\,. \tag{4.57}$$

Then we can rewrite the exterior derivative of $\Phi_\pm$ in terms of bispinors as

$$\mathrm{d}\Phi_+ = \{\gamma^m, D_m(\eta^1_+ \otimes \eta^{2\,\dagger}_+)\}\,, \tag{4.58}$$

$$\mathrm{d}\Phi_- = [\gamma^m, D_m(\eta^1_+ \otimes \eta^{2\,\dagger}_+)]\,. \tag{4.59}$$

As shown in [19], it is easy to see that the equation of motion for F is implied by the supersymmetry conditions (4.60)-(4.62). Acting with the operator λ on (4.62) we obtain

$$\lambda[(\mathrm{d} - H\wedge)(e^{3A-\phi}\mathrm{Im}\Phi_2)] = \mp(\mathrm{d} + H\wedge)[e^{3A-\phi}\lambda(\mathrm{Im}\Phi_2)] = \mp\frac{1}{8}e^{4A} * F \qquad (4.63)$$

where the upper (lower) signs correspond to IIA (IIB) and come from commuting λ with the Hodge star and and $(\mathrm{d} - H\wedge)$. From (4.63) it follows $e^{4A} * F$ is $\mathrm{d} + H\wedge$ exact, and hence also $\mathrm{d} + H$ closed.

Chapter 5

Anti de Sitter vacua in type IIB SUGRA on cosets and group manifolds

In this chapter, we will apply the formalism developed in the previous chapters to the study of four-dimensional $\mathcal{N} = 1$ Anti de Sitter vacua in type IIB theory. We will focus on parallelizable manifolds, on cosets and group manifolds, and we will show how the pure spinor approach allows to reduce the supersymmetry constraints to a set of algebraic equations and, in some cases, to fully scan for possible vacua in a large family of manifolds (namely group manifolds). It is also easy to see if such vacua are admitted on a given parallelizable manifold (recently progress as been made on non homogeneous manifolds [20, 21]). We first give some motivations to the search for AdS vacua.

5.1 Motivations

Even if of no direct phenomenological interest, compactifications to four-dimensional Anti de Sitter space are worth to study for several reasons. For instance, they are relevant for the $\text{CFT}_3/\text{AdS}_4$ correspondence and, also, they might be the first step in the construction of de Sitter vacua in string theory.

In type IIA the literature on SUSY AdS_4 flux vacua is plentiful: examples have been found both with [22–26] or without sources [27, 28]. Among the vacua with sources some (see for instance [29, 30] and their T-duals [19, 31, 32]) contain fully localized sources. However most examples involve intersecting sources, D-branes or O-planes, which are then smeared in the transverse directions. This raises the question of what is the meaning of a smeared orientifold plane and how such solutions can lift to full string theory [33, 34]. The way the solution is supposed to change for fully localized sources is still an open question, even if some interesting progress was made in [35]. For compactifications with sources that are *parallel* (or have an F-theory interpretation), such as for the no-scale orientifold compactifications of [29, 30] and their T-duals [19, 31, 32], it is known how to treat fully localized sources. For these cases the backreaction does not invalidate the existence of the solutions, but it is expected to be very relevant when computing fluctuations around the vacuum (see for instance [36–38]).

There is also a more stringy issue that troubles these vacua and concerns the proper definition of string theory with O6 planes when there is non-zero Romans mass [33, 39]. Since there is no conventional lift of massive IIA supergravity to 11 dimensions[1] it is not clear how the

[1]See however the intriguing proposal of [40], or the alternative suggestion that a lift is unnecessary since massive IIA cannot be strongly coupled at weak curvature [41].

orientifold singularities can be resolved and whether the background makes sense[2].

Another open question is whether the known AdS_4 vacua in type IIA can give rise to truly four-dimensional effective actions. Before the original KKLT proposal [4] (and [43]), none of the AdS vacua in string theory were truly lower-dimensional in the sense that the AdS scale was not parametrically larger than the length scale of the extra dimensions. The AdS solutions that are used for holography typically do not have scales separation and it is important to understand how holography works for AdS vacua with scales separation [44]. The KKLT construction and its descendants are not entirely explicit from a 10-dimensional point of view, which complicates a possible holographic understanding. For that reason, and for reasons of elegance and simplicity, it would be desirable to have solutions of classical supergravity in ten dimensions. This was first claimed in a series of papers constructing such vacua in massive IIA supergravity with intersecting O6 planes [22–24] (see also [25, 26] for later work on these solutions). In massive IIA many solutions without sources are also known [27, 28], but they cannot achieve scales separation [45]. Only for solutions with O6 planes is this possible, although no no-go theorem excluding other possibilities has been found. And, unfortunately, the AdS compactifications in IIA with scales separation involve *intersecting* O6 planes and, apart from the partial results in [35], not much is known.

For the reasons just named it is relevant to find other classical AdS solutions with scales separation in a different context. In this chapter we study AdS_4 vacua in type IIB theory. These are less studied than their IIA counterparts. It is commonly claimed that this theory cannot achieve moduli stabilisation at the classical level, but this statement can readily be violated by considering non-geometric fluxes or by moving beyond the usual O3/O7 compactifications and instead relying on O5/O7 orientifolds. A first attempt at finding such vacua has been done in [46], where the authors considered four dimensional effective theories obtained by consistent truncations on specific $SU(2)$ structure manifolds (built from coset space coverings) with smeared O5/O7 intersections. While some of the models considered allow for full moduli stabilisation, it is not clear whether they admit a limit in which the solution is at large volume, weak coupling and with scales separation.

One of the aim of this chapter is to further study O5/O7 compactifications of IIB supergravity to four-dimensional, unwarped AdS space, the absence of warping being a necessary outcome of the approximation of smeared sources. We construct the solutions directly in ten dimensions using the pure spinors approach of [17–19] that we described in the previous chapter. Our results have partial overlap with an earlier investigation on SUSY AdS vacua in IIA/IIB SUGRA [47].

Another question we aim at answering is the existence of AdS_4 SUSY vacua without sources. These clearly avoid all the aforementioned problems about the possible validity of the solutions and are clearly important in the context of AdS/CFT. While supersymmetric AdS_4 vacua without sources are known in type IIA , there is only one example in type IIB [48]. Using $SU(2)$ structure techniques, the authors of [47] showed that, for constant warp factor and a specific choice of $SU(2)$ torsions, only non supersymmetric sourceless vacua can be found. In this work we will extend the analysis to a larger class of manifolds. However, the most general form of the supersymmetry equations is too complicated to give general results. For this reason we will focus on group manifolds admitting an SU(2) structure[3] and look for solutions with constant warp factor. We will see that under these assumption it is not possible to have sourceless solutions.

[2] See reference [42] for more radical doubts about the use of orientifold planes.

[3] As shown in [25, 49] $\mathcal{N} = 1$ susy vacua in type IIB supergravity only exist on manifolds with SU(2) structure.

From simple manipulations of the traced Einstein equations it is easy to see that in absence of sources the internal manifold must have positive curvature [49]. This condition already rules out all nilmanifolds as candidates for sourceless vacua in type IIB. In this thesis we will prove a stronger result: under some restriction on the SU(2) torsions (namely the vectors in the torsion classes are set to zero), it is not possible to have sourceless solutions. We will leave the analysis of the warped case for future work.

For compactifications to four dimensions, this formalism allows to reduce the study of ten-dimensional supersymmetric backgrounds to the analysis of a set of equations involving only the components of the fields on the internal manifold. In this case, it is easy to show that the O-plane projections and supersymmetry require the internal manifold to admit a rigid $SU(2)$ structure. It is then possible to write down a general solution for the fields on the compactification manifold. By general solution we mean a set of constraints on the six-dimensional fields that are applicable to a whole class of manifolds instead of a specific example. This is typically achieved through writing the solution in terms of the $SU(2)$ invariant forms on the manifolds. To go from this general form to a concrete example one only has to compute the canonical forms for a given manifold. This is clearly beneficial and more insightful than minimizing F and D terms for a given manifold. When the compactification manifolds allow for consistent truncations, which is the case for homogeneous manifolds with smeared sources, then the minima of the scalar potential must lift consistently to solutions of the equations of motion in ten dimensions. Reference [50] explicitly analyzed how the IIA vacua in 4D lift to 10 IIA SUGRA solutions with smeared sources.

5.2 $\mathcal{N}=1$ SUSY AdS$_4$ vacua in type IIB SUGRA

We are interested in $\mathcal{N}=1$ SUSY AdS$_4$ vacua in type IIB theories. As already mentioned in the previous chapter, in order to study $\mathcal{N}=1$ vacua with non trivial fluxes, it is convenient to use the language of Generalized Complex Geometry [8, 51]. This formalism allows to reduce the study of ten-dimensional supersymmetric backgrounds to the analysis of a set of equations involving only the pure spinors $\Phi_\pm$ (see section 4.3)

$$\Phi_\pm = \eta^1_+ \otimes \eta^{2\dagger}_\pm \,, \tag{5.1}$$

and the components of the RR and NS fields on the internal manifold. As already discussed in section (4.3.1), the explicit form of $\Phi_\pm$ depend on the choice of structure on M. In the most general case of dynamical SU(2) structure they are

$$\Phi_- = -\frac{ab}{8} z \wedge (k_\perp e^{-ij} + i k_\parallel \omega) \,, \tag{5.2}$$

$$\Phi_+ = \frac{a\bar{b}}{8}\, e^{z\bar{z}/2} (k_\parallel e^{-ij} - i k_\perp \omega) \,, \tag{5.3}$$

where z, j and ω are the forms defining the $SU(2)$ structure (see section 2.1.7.2). The norm of the pure spinors $\Phi_\pm$ is related to the norm of the spinors η^i by

$$\langle \Phi_\pm, \bar{\Phi}_\pm \rangle = -i \|\Phi_\pm\|^2 \mathrm{vol}_6 = -\frac{i}{8} |a|^2 |b|^2 \mathrm{vol}_6 \,, \tag{5.4}$$

where vol$_6$ is the volume of the internal manifold and the product $\langle \Phi_\pm, \bar{\Phi}_\pm \rangle$ is the Mukai pairing defined in (3.48).

As shown in [17], for type IIB compactifications to AdS_4 the ten-dimensional supersymmetry variations are equivalent to the following set of equations on the pure spinors $\Phi_\pm$

$$
\begin{aligned}
&(\mathrm{d} - H\wedge)(e^{2A-\phi}\Phi_-) = -2\mu e^{A-\phi}\mathrm{Re}\Phi_+, && (5.5)\\
&(\mathrm{d} - H\wedge)(e^{A-\phi}\mathrm{Re}\,\Phi_+) = 0\,, && (5.6)\\
&(\mathrm{d} - H\wedge)(e^{3A-\phi}\mathrm{Im}\,\Phi_+) = -3e^{A-\phi}\mathrm{Im}\,(\bar\mu\Phi_-) - \frac{1}{8}e^{4A} * \lambda(F)\,, && (5.7)
\end{aligned}
$$

where ϕ is the dilaton, A the warp factor in (4.3) and F is the sum of the RR field strength on Y, $F = F_1 + F_3 + F_5$. The complex number μ determines the size of the AdS_4 cosmological constant

$$\Lambda = -|\mu|^2\,. \tag{5.8}$$

Notice also that, for AdS vacua, supersymmetry constraints the norms of the two six-dimensional spinors to be equal [19]

$$|a|^2 = |b|^2 = e^A\,. \tag{5.9}$$

Only the relative scale between the spinor being relevant, we can always rescale η_+ in such a way that

$$\bar b = a\,, \qquad \frac{b}{a} = e^{-i\theta}\,. \tag{5.10}$$

It is convenient to introduce the rescaled forms

$$\hat\omega = e^{i\theta}\omega \tag{5.11}$$

$$\hat z = \frac{\bar\mu}{|\mu|}z\,, \tag{5.12}$$

but for simplicity of notation, we will drop the ˆ symbols in the rest of the thesis.

5.2.1 The SUSY equations for rigid SU(2) structure

From equation (5.5) it is immediate to see that it is not possible to have AdS_4 vacua with $SU(3)$ structure, as found in [52][4]. Let us consider then the most general pure spinors defined in (5.2) and (5.3), and first expand (5.5). The zero-form component gives

$$\mu\, k_\| \cos\theta = 0\,, \tag{5.14}$$

which implies

$$k_\| = 0 \qquad \text{or} \qquad \cos\theta = 0\,. \tag{5.15}$$

The first choice corresponds to a rigid $SU(2)$ structure, while the second fixes the relative phase of a and b. In the rest of this thesis, we will focus on the $k_\| = 0$ case. Indeed, when we will look at cosets, we will require the presence of O5 and O7 which require $k_\| = 0$ (see [16] for more

[4] Indeed, in this case the Φ_- only contains a three-form term, so that one has to zero the zero- and two-form terms in $\mathrm{Re}\Phi_+$, which for $k_\perp = 0$, give

$$
\begin{aligned}
&\cos\theta = 0\,,\\
&\sin\theta(j + \frac{i}{2}z\wedge\bar z) = 0\,. && (5.13)
\end{aligned}
$$

Clearly these two equations cannot be solve at the same time.

details). As for group manifolds, as we will see, the case $\cos(\theta) = 0$ provide no AdS vacua. This means that the pure spinors take the following forms :

$$\Phi_+ = -\frac{ie^A}{8}\omega \wedge e^{\frac{z\wedge\bar{z}}{2}} \tag{5.16}$$

$$\Phi_- = -\frac{e^A\mu}{8|\mu|} z \wedge e^{-ij} \tag{5.17}$$

With the redefinition made above, the two-, four- and six-form components of (5.5) give

$$\mathrm{d}(e^{3A-\phi}z) = 2|\mu|e^{2A-\phi}\,\omega_I\,, \tag{5.18}$$

$$\mathrm{d}(e^{3A-\phi}z\wedge j) = ie^{2A-\phi}H\wedge z + e^{2A-\phi}|\mu|\, z\wedge\bar{z}\wedge\omega_R\,, \tag{5.19}$$

$$\mathrm{d}(e^{3A-\phi}z\wedge j\wedge j) = 2ie^{3A-\phi}H\wedge z\wedge j\,. \tag{5.20}$$

Plugging (5.18) in (5.19) and recalling that $\omega_I \wedge j = 0$ for an $SU(2)$ structure we obtain

$$z\wedge(\mathrm{d}j - iH + |\mu|e^{-A}\,\bar{z}\wedge\omega_R) = 0 \tag{5.21}$$

It is also straightforward to show that (5.20) is implied by (5.19). Indeed substituting (5.19) in (5.20) gives

$$z\wedge j\wedge(\mathrm{d}j - iH) = 0 \tag{5.22}$$

which is a consequence of (5.21).
Let us now consider the second equation, (5.6),

$$(\mathrm{d} - H\wedge)(e^{A-\phi}\mathrm{Re}\,\Phi_+) = 0\,. \tag{5.23}$$

Expanded in forms it gives a three- and five-form equation

$$\mathrm{d}(e^{2A-\phi}\omega_I) = 0\,, \tag{5.24}$$

$$\mathrm{d}(e^{2A-\phi}z\wedge\bar{z}\wedge\omega_R) = 2ie^{2A-\phi}H\wedge\omega_I \tag{5.25}$$

Finally we have to expand (5.7)

$$(\mathrm{d} - H\wedge)(e^{3A-\phi}\mathrm{Im}\,\Phi_+) = -3e^{A-\phi}\mathrm{Im}\,(\bar{\mu}\Phi_-) - \frac{1}{8}e^{4A} * \lambda(F) \tag{5.26}$$

This gives

$$*F_5 = 3e^{-A-\phi}\,|\mu|z_I\,, \tag{5.27}$$

$$*F_3 = -e^{-4A}\,\mathrm{d}(e^{4A-\phi}\omega_R) + 3e^{-A-\phi}|\mu|z_R\wedge j\,, \tag{5.28}$$

$$\begin{aligned} *F_1 = &-i\,\mathrm{d}(2A-\phi)z\wedge\bar{z}\wedge\omega_I - e^{-\phi}H\wedge\omega_R \\ &+\frac{1}{2}e^{-A-\phi}|\mu|z_I\wedge j\wedge j \end{aligned} \tag{5.29}$$

where in the last equation we used (5.24). In summary the non trivial susy conditions are

$$\mathrm{d}(e^{3A-\phi}z) = 2|\mu|e^{2A-\phi}\omega_I\,, \tag{5.30}$$

$$z\wedge(\mathrm{d}j - iH + |\mu|e^{-A}\,\bar{z}\wedge\omega_R) = 0 \tag{5.31}$$

$$\mathrm{d}(e^{2A-\phi}\omega_I) = 0\,, \tag{5.32}$$

$$\mathrm{d}(e^{2A-\phi}z\wedge\bar{z}\omega_R) = 2ie^{2A-\phi}H\wedge\omega_I \tag{5.33}$$

plus equations (5.27)-(5.29) for the fluxes. Note that in order to have a full solution of the ten-dimensional equations of motion, one must also check that the RR fluxes determined this way satisfy the Bianchi identities and that the NS three form is closed (we put no NS five-brane) :

$$\mathrm{d}H = 0\,, \tag{5.34}$$

$$\mathrm{d}F - H \wedge F = \delta(sources)\,, \tag{5.35}$$

where $\delta(sources)$ denotes the charge density of the space-filling sources. But we will come back to these in section 5.3.1.

5.2.2 Link to the $SU(2)$ torsion classes

To make contact with previous literature, we can express the equations above using the $SU(2)$ intrinsic torsions (2.67). The idea is to decompose all the objects in the equations in representations of $SU(2)$ and then obtain a set of conditions for the fields in the various representations. To decompose the exterior derivatives we use the torsion classes defined in (2.67), while for the fluxes we have

$$H = h_1 z \wedge \hat{\omega} + h_2 \bar{z} \wedge \hat{\omega} + h_3 z \wedge j + z \wedge \bar{z} \wedge h_1^{(2)} + h_2^{(2)} \wedge j + z \wedge h^{(3)} + \text{c.c.}\,, \tag{5.36}$$

$$F_1 = f_1 z + f_1^{(2)} + \text{c.c.}\,, \tag{5.37}$$

$$F_3 = f_2 z \ \wedge \hat{\omega} + f_3 \bar{z} \wedge \hat{\omega} + f_4 z \wedge j + z \wedge \bar{z} \wedge f_2^{(2)} + f_3^{(2)} \wedge j + z \wedge f^{(3)} + \text{c.c.}\,, \tag{5.38}$$

$$F_5 = f_5 z \wedge j \wedge j + z \wedge \bar{z} \wedge j \wedge f_4^{(2)} + \text{c.c.}\,, \tag{5.39}$$

where h_i and f_i are complex scalars in the singlet representation of $SU(2)$, $h_i^{(2)}$ and $f_i^{(2)}$ are holomorphic vectors in the **2** and $h^{(3)}$ and $f^{(3)}$ are complex two forms in the triplet representation, which are (1,1) and primitive with respect to j.

For completeness we also give the decomposition of Hodge dual fluxes

$$\begin{aligned} *H =& - ih_1 z \wedge \hat{\omega} + ih_2 \bar{z} \wedge \hat{\omega} - ih_3 z \wedge j - iz \wedge *_4 h^{(3)} + 2i *_4 h_1^{(2)} \\ & - \frac{i}{2} z \wedge \bar{z}(h_2^{(2)} \llcorner j) + \text{c.c.}\,, \end{aligned} \tag{5.40}$$

$$*F_1 = - \frac{i}{2} f_1 z \wedge j \wedge j - \frac{i}{2} z \wedge \bar{z} \wedge *_4 f_1^{(2)} + \text{c.c.}\,, \tag{5.41}$$

$$\begin{aligned} *F_3 =& - if_2 z \wedge \hat{\omega} + if_3 \bar{z} \wedge \hat{\omega} - if_4 z \wedge j - iz \wedge *_4 f^{(3)} + 2i *_4 f_2^{(2)} \\ & - \frac{i}{2} z \wedge \bar{z} \wedge (f_3^{(2)} \llcorner j) + \text{c.c.}\,, \end{aligned} \tag{5.42}$$

$$*F_5 = - 2if_5 z + 2if_4^{(2)} \llcorner j + \text{c.c.}\,, \tag{5.43}$$

where we used the fact that a product structure allows to split $*_6 = *_2 *_4$ and $*_4(v \wedge \xi) = (-1)^{\deg \xi} v \llcorner *_4 \xi$ and the following convention for the Hodge star :

$$*_2 1 = - z_R \wedge z_I \qquad *_2 z_R = z_I \qquad *_2 z_I = - z_R \qquad *_2 z_R \wedge z_I = - 1 \tag{5.44}$$

$$*_4 1 = \frac{1}{2} j \wedge j \qquad *_4 j, \omega_R, \omega_I = j, \omega_R, \omega_I \qquad *_4 \tilde{j}_i = - \tilde{j}_i \qquad *_4 j \wedge j = 2 \tag{5.45}$$

We can now look at the SUSY variations. Let us first consider (5.30) - (5.33). We find that the singlets in the torsions must satisfy

$$\begin{aligned} S_2 &= 0\,, & S_1 &= -S_4 = -i|\mu| e^{-A}\,, \\ S_3 &= \tfrac{1}{2} \partial_{\bar{z}} (3A - \phi)\,, & S_5 &= \bar{S}_6 = i\bar{h}_1 - \tfrac{1}{2} e^{-A} |\mu|\,, \\ & & S_7 &= \bar{S}_8 = -\tfrac{1}{2} \partial_z (2A - \phi)\,. \end{aligned} \tag{5.46}$$

Similarly, there are conditions on the vectors

$$\begin{array}{ll} V_3 = V_4 = V_6 = 0\,, & V_7 = i(\bar{\partial}_4 A + \bar{h}_1^{(2)})\llcorner\omega\,, \\ V_5 = ih_2^{(2)}\,, & V_8 = i[\bar{\partial}_4(3A-\phi) + \bar{h}_1^{(2)}]\llcorner\omega\,, \\ V_1 = V_2 = \partial_4(3A-\phi)\,, & \end{array} \tag{5.47}$$

and the two-forms

$$T_1 = 0\,, \qquad T_2 = -ih^{(3)}\,, \qquad T_3 = \bar{T}_4\,, \tag{5.48}$$

and the NS flux singlets

$$h_1 = \bar{h}_2\,, \qquad h_3 = -\frac{i}{2}\partial_z(2A-\phi)\,. \tag{5.49}$$

Finally the equations (5.27) - (5.29) for the RR fluxes give

$$\begin{array}{ll} f_1 = -e^{-\phi}(4ih_1 - \frac{1}{2}|\mu|e^{-A})\,, & f_1^{(2)} = ie^{-\phi}\omega\llcorner[\bar{\partial}_4(2A-\phi) + \bar{h}_1^{(2)}]\,, \\ f_2 = \bar{f}_3 = -\frac{i}{2}e^{-\phi}\partial_z A\,, & f_2^{(2)} = \frac{i}{2}e^{-\phi}\omega\llcorner[\bar{\partial}_4(4A-\phi) - \bar{h}_1^{(2)}]\,, \\ f_4 = \frac{1}{2}e^{-\phi}(4h_1 + i|\mu|e^{-A}) & f_3^{(2)} = f_4^{(2)} = 0\,, \\ f_5 = \frac{3}{4}e^{-A-\phi}|\mu| & f^{(3)} = ie^{-\phi}T_3\,. \end{array} \tag{5.50}$$

5.3 Restriction to parallelizable manifolds

A general analysis of the $SU(2)$ structure constraints derived in the previous section is very involved, due to the large number of torsion classes. In order to proceed we have to make some simplifying hypothesis.

The first assumption we will take is that the warp factor and the dilaton will both be constant. The second one is that all vectors (ie the **2** of $SU(2)$) in (2.67) will be put to zero by hand. This simplifies dramatically the expressions and still provides us with a large class of interesting vacua.

Then we will restrict to manifolds for which we can compute the $SU(2)$ structure explicitly. These are typically homogeneous manifolds (groups and cosets). These manifolds admit left-invariant forms, which can be used to build the SU(2) structure. All the examples we will present are parallelizable manifolds: they possess a basis of globally defined one-forms e^i (the frame bundle is trivial) which are left-invariant. Then, up to redefinition of the e^i, one can always put the $SU(2)$ structure forms under the following form :

$$\begin{aligned} z &= z_1 e^1 + i z_2 e^2\,, \\ j &= j_1 e^{36} + j_2 e^{45}\,, \\ \omega_R &= \frac{j_1 j_2}{\omega_1} e^{34} + \omega_1 e^{56}\,, \\ \omega_I &= -\frac{j_1 j_2}{\omega_2} e^{35} + \omega_2 e^{46}\,, \end{aligned} \tag{5.51}$$

with all the coefficient being real. Moreover, one can also define a basis $\tilde{j}_i$ of two-forms spanning the **3** of $SU(2)$ namely :

$$\begin{aligned} \tilde{j}_1 &= j_1 e^{36} - j_2 e^{45}\,, \\ \tilde{j}_2 &= -\frac{j_1 j_2}{\omega_1} e^{34} + \omega_1 e^{56}\,, \\ \tilde{j}_3 &= -\frac{j_1 j_2}{\omega_2} e^{35} - \omega_2 e^{46}\,. \end{aligned} \tag{5.52}$$

In particular, the T_i in (2.67) can be decomposed as follows : $T_i = t_i^a \tilde{j}_a$ with t_i^a being complex.

Moreover we expect to have O-planes in our solutions as they are required to achieve a hierarchy of scales. The possible $SU(2)$ structures one can define out of left-invariant forms, consistent with the orientifold involutions, is restricted. Even if we use homogeneous spaces to justify the specific choice of $SU(2)$ torsions, the general solutions we derive could be applicable to more general manifolds.

5.3.1 Sources

In general we expect our solutions to require intersecting sources, D-branes or O-planes. Since we do not know how to find exact solutions that describe generic intersecting branes or O-planes, we smear them over the internal manifold, and we write the source terms as invariant decomposable forms on the internal manifold, dual to the cycle wrapped by the brane[5]

$$\begin{aligned}\delta(D7/O7) &= \sum N_{(D7/O7)ij} e^i \wedge e^j \\ \delta(D5/O5) &= \sum N_{(D5/O5)ijkl} e^i \wedge e^j \wedge e^k \wedge e^l \,.\end{aligned}$$

We can also define vol^{ij} as the decomposable form proportional to $e^i \wedge e^j$ and normalized such that $< \mathrm{vol}^{ij}, \mathrm{Im}\,\Phi_+ >= -\mathrm{vol}_6 = -8i < \Phi_+, \overline{\Phi}_+ >$. Similarly, one defines vol^{ijkl}. They represent the volume forms dual to the cycles wrapped by the sources. Then the sources can be written as

$$\delta(D7/O7) = \sum n_{(D7/O7)ij} \mathrm{vol}^{ij} \tag{5.53}$$

$$\delta(D5/O5) = \sum n_{(D5/O5)ijkl} \mathrm{vol}^{ijkl} \tag{5.54}$$

The constant n and N have different interpretations. $n_{(D7/O7)ij}$ and $n_{(D5/O5)ijkl}$ can be seen as the densities of charges on the cycle wrapped by the D-branes or orientifolds and they give the sign of the charges. On the other hand, $N_{(D7/O7)ij}$ and $N_{(D5/O5)ijkl}$ are more similar to total charges and these should be generally of order one on a solution since they are directly related to the number of D-branes or orientifolds.

For compactifications to Minkowski space there is a no–go theorem that rules out vacua in which the internal compact manifold has non-zero background fluxes and no sources. The no-go theorem is based on the simple observation that background fluxes contribute a positive energy momentum tensor that it is not compensated in four dimensions. It is derived using the external components of the modified Einstein equations of motion. However, for supersymmetric compactifications, it should be possible to obtain the above no-go theorems directly from the supersymmetry conditions and the Bianchi identity [19, 54]. The idea is that one can use the susy equation (4.62) to show that the Bianchi identity has a definite sign. This can be done by taking the Mukai pairing of (4.62) with $e^{3A-\phi}\mathrm{Im}\Phi_2$, which is the calibration form of the cycle wrapped by a spacetime–filling brane or an orientifold, as derived in the context of generalized complex geometry in [55]. Using the adjunction property

$$\int \langle A, (\mathrm{d} - H\wedge)B \rangle = \int \langle (\mathrm{d} - H\wedge)A, B \rangle \tag{5.55}$$

[5] We refer to Appendix C of [26] for an explanation on smeared source terms and the corresponding microscopic interpretation in terms of orientifolds and their involutions, whereas Appendix D of [53] contains some first attempts for charge and flux quantisation.

for the differential $d - H$, one derives

$$\int \langle (\mathrm{d} - H\wedge)F,\ e^{3A-\phi}\mathrm{Im}\Phi_2\rangle = \int \langle F,\ (\mathrm{d} - H\wedge)e^{3A-\phi}\mathrm{Im}\Phi_2\rangle = -\frac{1}{8}\int e^{4A}\,\langle F,\ *\lambda(F)\rangle\,, \quad (5.56)$$

from which we see that there must be sources of negative charge.

If we apply this condition to smeared sources, we find

$$\begin{aligned} \int \sum_p |F_p|^2\, \mathrm{vol}_6 &= \int \langle F, *\lambda(F)\rangle \\ &= \int \sum_i n_i \langle \mathrm{vol}^i, \mathrm{Im}\,\Phi_+\rangle = -\int \sum_i n_i\, \mathrm{vol}_6\,, \end{aligned} \quad (5.57)$$

where, for simplicity, we denoted by n_i the charge density of a generic source and by vol^i the corresponding transverse volume. We also take

$$\int vol^6 = 1\,. \quad (5.58)$$

This means that we can associate orientifolds to a negative charge density n_i and branes to a positive one.

In AdS compactifications, the presence of a negative cosmological constant, allows to avoid such no-go theorem, and solutions with trivial fluxes and no sources can be found (see for instance [27, 28] in type IIA). We can use the same convention to compute the charge and determine their signs: if $n_{(D7/O7)ij}$ and/or $n_{(D5/O5)ijkl}$ are negative we have an overall O-plane charge and viceversa.

5.3.2 O5 and O7 planes

The last ingredient we need before discussing concrete examples are the orientifold projections. Indeed, all of our examples will require both O5 and O7 either because we put them by hand (on the cosets) or because the SUSY equations require them (for group manifolds).

The O5 and O7 projections are given by

$$\Pi_{\mathrm{O3/O7}} = \Omega_{WS}(-1)^F\sigma \qquad\qquad \Pi_{\mathrm{O5/09}} = \Omega_{WS}\sigma\,, \quad (5.59)$$

where Ω_{WS} is the world-sheet reflection, F is the left-mover fermion number and σ is the target space involution. The action of the target space involution on the pure spinors is given by [16, 19]

$$\sigma(\Phi_+) = \pm\lambda(\bar{\Phi}_+) \qquad\qquad \sigma(\Phi_-) = \mp\lambda(\Phi_-) \quad (5.60)$$

where σ is the orientifold involution, λ is the transposition operator (3.48) and the upper and lower signs correspond to O5 and O7 planes, respectively. From this we can deduce how the orientifold involution acts on the $SU(2)$ structure forms. An intermediate $SU(2)$ structure is not compatible with both O5 and O7 projections, since, when $k_{\|} \neq 0$ the phases of the spinors have to be different for O5 and O7 planes [16]

$$\mathrm{O5}\,:\, a = \pm b \qquad\qquad \mathrm{O7}\,:\, a = \pm ib\,. \quad (5.61)$$

Therefore, all we need is the orientifold action on the rigid $SU(2)$ structure forms

$$\begin{aligned}\sigma(z) &= \mp z\,, \\ \sigma(j) &= -j\,, \\ \sigma(\omega) &= \pm\bar{\omega}\,. \end{aligned} \tag{5.62}$$

From the equations above we see that the one-form z must be orthogonal to the O5-planes and parallel to the O7's. The triplet $\tilde{j}_i$ of anti self-dual two-forms (5.52) have the same transformation properties as j, ω_R and ω_I

$$\sigma(\tilde{j}_1) = -\tilde{j}_1\,, \qquad \sigma(\tilde{j}_2) = \pm\tilde{j}_2 \qquad \sigma(\tilde{j}_3) = \mp\tilde{j}_3\,, \tag{5.63}$$

where, as before, upper and lower signs correspond to O5 and O7-planes. It is also useful to remind how the NS and RR fluxes transform under the orientifold involutions

$$\begin{aligned}\sigma(H) &= -H\,, \\ \sigma(F_1) &= \mp F_1\,, \\ \sigma(F_3) &= \pm F_3\,, \\ \sigma(F_5) &= \mp F_5\,, \end{aligned} \tag{5.64}$$

where, as before, upper and lower signs correspond to O5 and O7-planes.

Given the ansatz (5.51), it easy to see that the most general configuration of O5 and O7 compatible with $\mathcal{N} = 1$ supersymmetry is

plane	1	2	3	4	5	6
O5			x	x		
O5					x	x
O7	x	x	x		x	
O7	x	x		x		x

Table 5.1: O5- and O7-planes

These orientifolds are intersecting , that's why we will take them to be smeared on the internal manifold. Notice that these orientifold projections can be also should be better seen as asymmetric orbifold of $T^6/(\mathbb{Z}_2 \times (-1)^{F_L}\mathbb{Z}_2)$ with one single O-plane. As pointed out in [46], asymmetric orbifold of this type should still have a valid supergravity description.

5.3.2.1 SUSY and Bianchi identities

The orientifold projections in table 5.1 put further constraints on the SUSY equations. First of all notice that, even if we hadn't put the torsion classes in the **2** of $SU(2)$ to zero by hand, the presence of orientifolds would have forced us to do so. The same thing can be said about constant dilaton and warp factor, because of the smearing of the orientifolds.

As a result the supersymmetry conditions (5.46) - (5.50) reduce to

$$\begin{aligned} \mathrm{d}z &= 2|\mu|e^{-A}\hat{\omega}_I\,, \\ \mathrm{d}j &= (2i\bar{h}_1 - |\mu|e^{-A})\bar{z}\wedge\hat{\omega}_R - iz\wedge h^{(3)} + \text{c.c.}\,, \\ \mathrm{d}\hat{\omega}_R &= (2ih_1 + |\mu|e^{-A})z\wedge j + z\wedge T_3 + \text{c.c.}\,, \\ \mathrm{d}\hat{\omega}_I &= 0\,, \end{aligned} \tag{5.65}$$

while the fluxes become

$$
\begin{aligned}
H &= 2h_1 z\wedge\hat\omega_R + z\wedge h^{(3)} + \text{c.c.}\,,\\
F_1 &= e^{-\phi}(4ih_1 - \frac{1}{2}|\mu|e^{-A})z + \text{c.c.}\,,\\
F_3 &= -\frac{1}{2}e^{-\phi}(i|\mu|e^{-A} + 4h_1)z\wedge j + \frac{i}{2}e^{-\phi}z\wedge T_3 + \text{c.c.}\,,\\
F_5 &= e^{-\phi}\, f_5 z\wedge j\wedge j + \text{c.c.}\,.
\end{aligned}
\tag{5.66}
$$

Comparing (5.64) and (5.63) we can see that only one of the three components survive for each T_i and $h^{(3)}$

$$
T_3 = t_3\tilde{j}_1\,, \qquad\qquad h^{(3)} = h_4\tilde{j}_2\,. \tag{5.67}
$$

What remains to be solved are the Bianchi identities (5.34) and (5.35). To do so we need the derivatives of T_3 and $h^{(3)}$, which can be easily determined from (5.67) and

$$
\begin{aligned}
\mathrm{d}\tilde{j}_1 &= t_3 z\wedge\hat\omega_R - a_2 z\wedge\tilde{j}_2 + \text{c.c}\,,\\
\mathrm{d}\tilde{j}_2 &= -ih_4 z\wedge j + a_2 z\wedge\tilde{j}_1 + \text{c.c}\,,
\end{aligned}
\tag{5.68}
$$

where the equations above can be obtained expanding the $\mathrm{d}\tilde{j}_i$ as in (2.67) and imposing the orientifold projections. Note that a_2 is a complex number.

Let us start with the BI identities for NS three-form. Using (5.65), (5.66) and (5.68) we obtain

$$
|h_4|^2 - 4|h_1|^2 + 2\,\mathrm{Im}(e^{-A}|\mu|h_1) = 0\,, \qquad\qquad \mathrm{Im}(2h_1\bar{t}_3 + h_4\bar{a}_2) = 0\,. \tag{5.69}
$$

The equation for the five-form flux is trivially satisfied. We are left with the BI involving sources (by abuse of notation we also denote by a δ the contribution of smeared sources)

$$
\mathrm{d}F_1 = \delta(D7/O7)\,, \tag{5.70}
$$

$$
\mathrm{d}F_3 = H\wedge F_1 + \delta(D5/O5)\,. \tag{5.71}
$$

Using again (5.65) and (5.66), they give

$$
\delta(D7/O7) = -2e^{-\phi}(|e^{-A}\mu|^2 + 8\,\mathrm{Im}(e^{-A}|\mu|h_1))\hat\omega_I\,, \tag{5.72}
$$

$$
\begin{aligned}
\delta(D5/O5) = &-2ie^{-\phi}(\mathrm{Re}\,(a_2\bar{t}_3) - \mathrm{Im}\,(e^{-A}|\mu|h_4) - 6\mathrm{Re}\,(\bar{h}_1 h_4))z\wedge\bar{z}\wedge\tilde{j}_2\\
&+ie^{-\phi}(2|t_3|^2 + 24|h_1|^2 - |e^{-A}\mu|^2)z\wedge\bar{z}\wedge\hat\omega_R\,.
\end{aligned}
$$

Notice that the parameters in the previous equations have to satisfy further consistency conditions, namely $\mathrm{d}^2 j = \mathrm{d}^2\hat\omega_R = 0$ and $\mathrm{d}^2\tilde{j}_i = 0$. More precisely, taking the exterior derivative of (5.65) and (5.68) we obtain (the consistency conditions on $\tilde{j}_1$ and $\tilde{j}_2$ give the same equations)

$$
\mathrm{Re}\,(h_4\bar{a}_2 + 2h_1\bar{t}_3) - \mathrm{Im}\,(e^{-A}|\mu|t_3) = 0\,, \tag{5.73}
$$

$$
\mathrm{Re}\,(e^{-A}|\mu|h_4) + \mathrm{Im}\,(2\bar{h}_1 h_4 + a_2\bar{t}_3) = 0\,. \tag{5.74}
$$

In summary, in order to find a generic $\mathcal{N} = 1$ AdS$_4$ vacua with the choice of O-plane of Table 5.1, one has to solve (5.69), (5.73) and (5.74). The fluxes and the geometry are then given by (5.65) and (5.66). The general solutions to these equations are easy to obtain but, since the expressions are not very illuminating, we do not give them in this thesis.

5.4 Explicit examples 1: Cosets

Coset manifolds are very good candidates for string compactifications since the existence of globally defined left-invariant forms makes it possible to perform a Kaluza-Klein reduction even in presence of fluxes. Coset manifolds in string compactification have been studied in [26, 56]. In [46] a search for type IIB AdS_4 vacua has been performed looking for minima of the four-dimensional effective action obtained by reduction on a generic SU(2) structure manifolds with the orientifold projections of table (5.1). Our aim is to compute solutions directly in 10 dimensions when they exist and to demonstrate when solutions cannot exist. This was an open issue in [46], while with the pure spinor technology this is quite straightforward to settle.

We refer to [57] for a thorough discussion of coset manifolds. Here we simply recall some simple facts that help making our derivation clearer.

A coset manifold $M = G/H$ where G is a Lie group and H is a closed subgroup of G, is completely determined by the corresponding algebrae, g and h. We denote by $\{H_a\}$, with $a = 1, \ldots \dim H$, a basis of generators of h and by $\{K_i\}$, with $i = 1, \ldots, \dim G - \dim H$ a basis for the complement of h in g. Then the structure constants are given by

$$\begin{aligned}
[H_a, H_b] &= f^c_{ab} H_c \,, \\
[H_a, K_i] &= f^j_{ai} K_j + f^c_{ai} H_c \,, \\
[K_i, K_j] &= f^i_{jk} K_i + f^a_{jk} H_a \,.
\end{aligned} \tag{5.75}$$

The coframe $e^i(y)$ on G/H is defined by

$$L^{-1} \mathrm{d}L = e^i K_i + \omega^a H_a \,, \tag{5.76}$$

where $L(y)$ is a coset representative and y^i are local coordinates on G/H. A p-form

$$\phi = \phi_{i_i \ldots i_p} e^{i_1} \wedge \ldots \wedge e^{i_p} \tag{5.77}$$

is then said to be left-invariant under the action of G if and only if its coefficients $\phi_{i_i \ldots i_p}$ are constant and

$$f^j_{a[i_1} \phi_{i_2 \ldots i_p]j} = 0 \,. \tag{5.78}$$

From the algebra (5.75) we have

$$\mathrm{d}e^i = -\frac{1}{2} f^i_{jk} e^j \wedge e^k - f^i_{aj} \omega^a \wedge e^j \,. \tag{5.79}$$

It is then easy to show that (5.78) guarantees that the exterior derivative preserve the property of left-invariance.

As mentioned before, we want the $SU(2)$ structure to be also left-invariant. As shown in [57] this requires that $H \subset SU(2)$. The list of reductive coset manifold that satisfy this property is [46]

$$\frac{SU(3) \times U(1)}{SU(2)} \qquad \frac{SU(2)^2}{U(1)} \times U(1) \qquad SU(2) \times SU(2) \qquad SU(2) \times U(1)^3 \,. \tag{5.80}$$

The rest of this section is devoted to the study of $\mathcal{N} = 1$ SUSY AdS_4 on such manifolds.

5.4.0.2 $\frac{SU(3)\times U(1)}{SU(2)}$

Of the 9 generators of $SU(3)\times U(1)$ we denote by T_2 and T_7, T_8, T_9 the generators of $U(1)$ and $SU(2)$, respectively. The algebra is given by

$$\begin{aligned} &f^1_{46} = -\frac{\sqrt{3}}{2} \quad (\text{and cyclic})\,, \qquad f^1_{35} = \frac{\sqrt{3}}{2} \quad (\text{and cyclic})\,,\\ &f^9_{78} = 1 \quad (\text{and cyclic})\,,\\ &f^7_{65} = f^7_{34} = f^8_{63} = f^8_{45} = f^9_{64} = f^9_{53} = \frac{1}{2} \quad (\text{and cyclic})\,. \end{aligned} \tag{5.81}$$

The left-invariant forms compatible with the O5 and O7 projections are

$$\begin{array}{ll} \text{1 forms} & e^1,\, e^2 \quad , \\ \text{2 forms} & e^{36}+e^{45},\, e^{34}+e^{56},\, e^{35}-e^{46}\,, \end{array}$$

Since the $SU(2)$ structure forms must also be left-invariant, in the ansatz (5.51) and (5.52) we set

$$j_1 = j_2 \qquad \omega_1 = \epsilon_1 j_2 \qquad \omega_2 = \epsilon_2 j_2\,, \tag{5.82}$$

with $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 1$. It is easy to see that none of the forms $\tilde{j}_i$ is left-invariant, which implies $t_3 = h_4 = 0$. Solving the constraints (5.69), (5.73) and (5.74) gives the solution

$$\begin{aligned} h_1 =&0\,,\\ z_1 =& -\epsilon_1 \frac{\sqrt{3}}{2|\mu|}\,,\\ j_2 =& -\epsilon_1\epsilon_2 \frac{3}{8|\mu|^2}\,,\\ a_2 =&0\,, \end{aligned} \tag{5.83}$$

where

$$\begin{aligned} \rho^3 =& -\epsilon_1 \frac{9\sqrt{3}}{128}\frac{z_2}{|\mu|^5}\,,\\ R_6 =&4|\mu|^2\,, \end{aligned} \tag{5.84}$$

and the orientifold charges are

$$\begin{aligned} n_{(D7/O7)35} =& n_{(D7/O7)46} = -n_{(D5/O5)1234} = -n_{(D5/O5)1234} = -e^{-\phi}\frac{|\mu|^2}{4}\,,\\ N_{(D7/O7)35} =& -N_{(D7/O7)46} = \frac{3e^{-\phi}\epsilon_1}{4}\,,\\ N_{(D5/O5)1234} =& N_{(D5/O5)1256} = \frac{3\sqrt{3}e^{-\phi}\epsilon_1\epsilon_2 z_2}{8|\mu|}\,. \end{aligned} \tag{5.85}$$

Note that by consistency the orientifold planes should wrap directions whose dual source forms should be left-invariant. The source forms that are Poincare dual to the surfaces wrapped by the O5 planes are not left-invariant, although we have said O planes have to be consistent with the left-invariant forms. This problem is however cured since the sum of the two O5 forms, $e^{1234}+e^{1256}$ is left-invariant. If we interpret each of these two terms as a separate orientifold source with its own involution then consistency requires the two O-planes to have exactly the same charge, such that the source term in the Bianchi identity for F_3 is given by the sum of the two forms, and hence left-invariant. This is anyhow a necessary requirement from the point of view of charge quantisation. Orientifolds, unlike D-branes, cannot be stacked. So for each involution we should have a single unit of orientifold charge.

5.4.0.3 $\frac{SU(2)^2}{U(1)} \times U(1)$

For this coset, out of the 7 generators of $SU(2)^2 \times U(1)$ we denote by T_7 the generator of the $U(1)$. The algebra is given by :

$$\begin{aligned} &f^1_{35} = 1 \text{ (and cyclic)}, \quad f^7_{46} = 1 \text{ (and cyclic)} \\ &f^1_{46} = f^5_{37} = -f^3_{57} = -1\,. \end{aligned} \tag{5.86}$$

As in the previous case, the $SU(2)$ structure must be left-invariant. This means that in the ansatz (5.51) and (5.52) we set

$$j_1 = j_2 \qquad \omega_1 = \epsilon_1 j_2\,, \tag{5.87}$$

with $\epsilon_1 = \pm 1$. As before, the requirement of left-invariant implies $h_4 = 0$ and $t_3 = 0$. The solution is

$$\begin{aligned} z_1 =& -\epsilon_1 \frac{1}{2|\mu|}\,, \\ \omega_2 =& -\epsilon_1 \frac{1}{4|\mu|^2}\,, \\ j_2 =& -\epsilon_2 \frac{1}{4|\mu|^2}\,, \\ h_1 =& 0\,, \\ h_4 =& t_3 = 0\,, \\ a_2 =& |\mu|\,, \end{aligned} \tag{5.88}$$

with $\epsilon_2 = \pm 1$. The volume and curvature are

$$\begin{aligned} \rho^3 =& -\frac{\epsilon_1 z_2}{32|\mu|^5}\,, \\ R_6 =& 4|\mu|^2\,, \end{aligned} \tag{5.89}$$

and the orientifold charges are

$$n_{(D7/O7)35} = n_{(D7/O7)46} = -n_{(D5/O5)1234} = -n_{(D5/O5)1234} = -e^{-\phi}\frac{|\mu|^2}{4} \tag{5.90}$$

$$\begin{aligned} N_{(D7/O7)35} =& -N_{(D7/O7)46} = e^{-\phi}\frac{\epsilon_1}{2}\,, \\ N_{(D5/O5)1234} =& N_{(D5/O5)1256} = e^{-\phi}\frac{\epsilon_2 z_2}{4|\mu|}\,. \end{aligned} \tag{5.91}$$

5.4.0.4 $SU(2) \times SU(2)$ and $SU(2) \times U(1)^3$

There are no SUSY solutions on these two manifolds. This is most easily seen for $SU(2)\times SU(2)$ since the SUSY equations (see (5.65)) require the one-form $\mathrm{Im}\,(z)$ to be closed whereas there are no closed (left-invariant) one-forms on $SU(2) \times SU(2)$.

5.5 Explicit examples 2: Group manifolds

A second class of examples we are interested in are AdS_4 vacua on group manifolds. With the assumption that dilaton and warp factor are constant, and that there no vectors in the SU(2) torsion classes, we are able to perform a complete scan of group manifolds admitting $\mathcal{N}=1$ AdS_4 vacua. In this case we do not impose any orientifold projection since we want to be able to look for sourceless solutions as well. We will see that, under our assumption, there are no sourceless solutions and that we are always forced to have O5 and O7 planes.

The strategy is the following. We consider a generic six-dimensional homogeneous group manifold. A homogeneous group manifold is specified by a basis of globally defined one forms, e^i, satisfying the Maurer-Cartan equations

$$\mathrm{d}e^i = -\frac{1}{2} f^i_{jk} e^j \wedge e^k \tag{5.92}$$

with f^i_{jk} constant. In this case imposing $\mathrm{d}^2 e^i = 0$ gives the Jacobi identities

$$f^i_{[jk} f^k_{l]i} = 0\,. \tag{5.93}$$

We take the SU(2) structure (5.51) and (5.52) and we impose the SUSY conditions of section 5.2.2. Under the hypothesis discussed above this gives

$$\begin{aligned}
S_2 =& S_3 = S_7 = S_8 = 0 & S_1 =& - S_4 = -i|\mu| & S_5 =& \bar{S}_6 = i\bar{h}_1 - \frac{1}{2}|\mu| \\
V_i =& 0 & T_1 =& 0 & T_3 =& \bar{T}_4 \\
h^{(3)} =& iT_2 & h_2 =& \bar{h}_1 & h_3 =& 0
\end{aligned}$$

Or equivalently :

$$\begin{aligned}
\mathrm{d}z &= 2|\mu| e^{-A} \hat{\omega}_I \,, \\
\mathrm{d}j &= (2i\bar{h}_1 - |\mu| e^{-A}) \bar{z} \wedge \hat{\omega}_R + z \wedge t^a_2 \tilde{j}_a + \text{c.c.}\,, \\
\mathrm{d}\hat{\omega}_R &= (2ih_1 + |\mu| e^{-A}) z \wedge j + z \wedge t^a_3 \tilde{j}_a + \text{c.c.}\,, \\
\mathrm{d}\hat{\omega}_I &= 0\,,
\end{aligned} \tag{5.94}$$

In order to solve the Bianchi identities, we need to parametrize the exterior derivatives of the anti-self dual two-forms $\tilde{j}_a$. Taking already into account the SUSY constraints we can write them as

$$\begin{aligned}
d\tilde{j}_1 =& z_R \wedge (t^1_{2R} j + t^1_{3R} \omega_R + \tilde{t}^5_{11} \tilde{j}_2 + \tilde{t}^6_{11} \tilde{j}_3) + z_I \wedge (-t^1_{2I} j - t^1_{3I} \omega_R + \tilde{t}^5_{12} \tilde{j}_2 + \tilde{t}^6_{12} \tilde{j}_3) \\
d\tilde{j}_2 =& z_R \wedge (t^2_{2R} j + t^2_{3R} \omega_R - \tilde{t}^5_{11} \tilde{j}_1 + \tilde{t}^6_{21} \tilde{j}_3) + z_I \wedge (-t^2_{2I} j - t^2_{3I} \omega_R - \tilde{t}^5_{12} \tilde{j}_1 + \tilde{t}^6_{22} \tilde{j}_3) \\
d\tilde{j}_3 =& z_R \wedge (t^3_{2R} j + t^3_{3R} \omega_R - \tilde{t}^6_{11} \tilde{j}_1 - \tilde{t}^6_{21} \tilde{j}_2) + z_I \wedge (-t^3_{2I} j - t^3_{3I} \omega_R - \tilde{t}^6_{12} \tilde{j}_1 - \tilde{t}^6_{22} \tilde{j}_2)
\end{aligned} \tag{5.95}$$

where the $\tilde{t}^a_{bj}$ are new free real parameters. We remind the reader that the t^a_b are defined by $T_b = t^a_b \tilde{j}_a$ where T_b is defined in (2.67).

In (2.67) and (5.95), we haven't enforced the whole $SU(2)$ structure. Indeed, the equations of the type $d(d(j)) = 0$ are not automatically verified. Unfortunately these equations are quadratic in the torsion coefficients and the flux parameters. We can bypass this problem by using the structure of the group manifolds. Once we use the expressions (5.51) and (5.52) for z, j, ω and $\tilde{j}_a$, the quadratic constraints $\mathrm{d}^2() = 0$ are clearly satisfied since we know that the one-forms e_i must satisfy the Jacobi identity $\mathrm{d}^2 e^i = 0$. Clearly we can't use the same technique on the cosets because the one forms e^i are not necessarily left-invariant so d^2 is not necessarily 0 on them.

The idea is then to express the structure constants in terms of the torsion parameters and the coefficients of H, and then solve explicitly the Jacobi identities in terms of such parameters. Plugging (5.51) and (5.52) in the torsion equations (5.94) and (5.95) we find the following non-zero structure constants

$$f^1_{jk} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2|\mu|j_1 j_2}{z_1\omega_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{2|\mu|\omega_2}{z_1} \\ 0 & 0 & -\frac{2|\mu|j_1 j_2}{z_1\omega_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2|\mu|\omega_2}{z_1} & 0 & 0 \end{pmatrix} \qquad f^2_{jk} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f^3_{j1} = \begin{pmatrix} 0 \\ 0 \\ -(t^2_{3R} - t^1_{2R})z_1 \\ -\frac{(2t^3_{2R}+\tilde{t}^6_{11})z_1\omega_2}{2j_1} \\ -\frac{\left(-4h_{1I}-2t^2_{2R}-2t^1_{3R}-\tilde{t}^5_{11}+2|\mu|\right)z_1\omega_1}{2j_1} \\ -\frac{(\tilde{t}^6_{21}-2t^3_{3R})z_1\omega_2\omega_1}{2j_1 j_2} \end{pmatrix} \qquad f^3_{j2} = \begin{pmatrix} 0 \\ 0 \\ -(t^1_{2I} - t^2_{3I})z_2 \\ -\frac{(\tilde{t}^6_{12}-2t^3_{2I})z_2\omega_2}{2j_1} \\ \frac{(4h_{1R}-2(t^2_{2I}+t^1_{3I})+\tilde{t}^5_{12})z_2\omega_1}{2j_1} \\ -\frac{(2t^3_{3I}+\tilde{t}^6_{22})z_2\omega_2\omega_1}{2j_1 j_2} \end{pmatrix}$$

$$f^4_{j1} = \begin{pmatrix} 0 \\ 0 \\ -\frac{(2t^3_{2R}-\tilde{t}^6_{11})j_1 z_1}{2\omega_2} \\ -(t^1_{2R} + t^2_{3R})z_1 \\ -\frac{(2t^3_{3R}-\tilde{t}^6_{21})z_1\omega_1}{2\omega_2} \\ \frac{\left(-4h_{1I}-2t^2_{2R}+2t^1_{3R}+\tilde{t}^5_{11}+2|\mu|\right)z_1\omega_1}{2j_2} \end{pmatrix} \qquad f^4_{j2} = \begin{pmatrix} 0 \\ 0 \\ \frac{(2t^3_{2I}+\tilde{t}^6_{12})j_1 z_2}{2\omega_2} \\ (t^1_{2I} + t^2_{3I})z_2 \\ \frac{(2t^3_{3I}+\tilde{t}^6_{22})z_2\omega_1}{2\omega_2} \\ -\frac{(4h_{1R}-2t^2_{2I}+2t^1_{3I}-\tilde{t}^5_{12})z_2\omega_1}{2j_2} \end{pmatrix}$$

$$f^5_{j1} = \begin{pmatrix} 0 \\ 0 \\ -\frac{\left(4h_{1I}-2t^2_{2R}-2t^1_{3R}+\tilde{t}^5_{11}-2|\mu|\right)j_1 z_1}{2\omega_1} \\ -\frac{(2t^3_{3R}+\tilde{t}^6_{21})z_1\omega_2}{2\omega_1} \\ -(t^1_{2R} - t^2_{3R})z_1 \\ -\frac{(2t^3_{2R}-\tilde{t}^6_{11})z_1\omega_2}{2j_2} \end{pmatrix} \qquad f^5_{j2} = \begin{pmatrix} 0 \\ 0 \\ -\frac{(4h_{1R}+2(t^2_{2I}+t^1_{3I})+\tilde{t}^5_{12})j_1 z_2}{2\omega_1} \\ -\frac{(\tilde{t}^6_{22}-2t^3_{3I})z_2\omega_2}{2\omega_1} \\ -(t^2_{3I} - t^1_{2I})z_2 \\ \frac{(2t^3_{2I}+\tilde{t}^6_{12})z_2\omega_2}{2j_2} \end{pmatrix}$$

$$f^6_{j1} = \begin{pmatrix} 0 \\ 0 \\ \frac{(2t^3_{3R}+\tilde{t}^6_{21})j_1 j_2 z_1}{2\omega_2\omega_1} \\ -\frac{\left(-4h_{1I}+2t^2_{2R}-2t^1_{3R}+\tilde{t}^5_{11}+2|\mu|\right)j_2 z_1}{2\omega_1} \\ -\frac{(2t^3_{2R}+\tilde{t}^6_{11})j_2 z_1}{2\omega_2} \\ (t^1_{2R} + t^2_{3R})z_1 \end{pmatrix} \qquad f^6_{j2} = \begin{pmatrix} 0 \\ 0 \\ -\frac{(2t^3_{3I}-\tilde{t}^6_{22})j_1 j_2 z_2}{2\omega_2\omega_1} \\ -\frac{(-4h_{1R}-2t^2_{2I}+2t^1_{3I}+\tilde{t}^5_{12})j_2 z_2}{2\omega_1} \\ -\frac{(\tilde{t}^6_{12}-2t^3_{2I})j_2 z_2}{2\omega_2} \\ -(t^1_{2I} + t^2_{3I})z_2 \end{pmatrix}$$

Then we can impose the Jacobi identities $d^2 e^i = 0$. Fortunately some of these equations are linear in t^a_{ib} and the H coefficients and we can simply solve them. The result is quite simple: $f^3_{j1}, f^4_{j1}, f^5_{j1}, f^6_{j1}$ are put to zero and the other ones are left unchanged. This can also be translated in conditions on the torsion classes and coefficients of the NS three form H :

$$\begin{aligned} t^1_{2R} &=0 & t^2_{2R} &=0 & t^3_{2R} &=0 & t^1_{3R} &=0 & t^2_{3R} &=0 \\ t^3_{3R} &=0 & h_{1I} &=\frac{|\mu|}{2} & \tilde{t}^5_{11} &=0 & \tilde{t}^6_{11} &=0 & \tilde{t}^6_{21} &=0 \end{aligned}$$

5.5.1 Bianchi Identities

Only one thing is left for us to claim that we have a genuine 10D vacuum and that is to solve the Bianchi identities.

First of all, we will look at the Bianchi identity for the RR 1-form. Then we have :

$$n_{(D7/O7)35} = n_{(D7/O7)46} = -\frac{5e^{-3A-\phi}}{4}|\mu|^2 < 0\,, \tag{5.96}$$

That is to say that in order to have an AdS SUSY solution on a group manifold, one must always have two intersecting O7-planes. Since we have intersecting orientifolds, we will only consider smeared sources. The compatibility of the algebrae with the orientifolds involution forces to set :

$$t^3_{2I} =0 \qquad t^3_{3I} =0 \qquad \tilde{t}^6_{12} =0 \qquad \tilde{t}^6_{22} =0.$$

The expression for f^i_{jk} we gave above are not very manageable. Moreover, in looking for solutions, one usually proceed in the opposite way: given an internal manifold, one would like to find the coefficients of the torsions and H flux in terms of the structure constants. By inverting these relation we find the following expressions for the parameters in the $SU(2)$

$$\omega_2 = -\frac{f^1_{46} z_1}{2e^{-A}|\mu|} \qquad j_2 = -\frac{D}{4j_1}\,, \tag{5.97}$$

with $D = \frac{f^1_{35} f^1_{46} z_1^2}{4|e^{-A}\mu|^2}$, the torsions and H flux coefficients :

$$\begin{aligned} h_{1R} &= -\frac{1}{8j_1 z_2 \omega_1}\left(f^4_{26}D + f^3_{25}j_1^2 - f^5_{23}\omega_1^2 - \frac{f^6_{24}j_1^2\omega_1^2}{D}\right) \\ t^2_{2I} &= -\frac{1}{4j_1 z_2 \omega_1}\left(-f^4_{26}D - f^3_{25}j_1^2 - f^5_{23}\omega_1^2 - \frac{f^6_{24}j_1^2\omega_1^2}{D}\right) \\ t^1_{3I} &= -\frac{1}{4j_1 z_2 \omega_1}\left(f^4_{26}D - f^3_{25}j_1^2 - f^5_{23}\omega_1^2 + \frac{f^6_{24}j_1^2\omega_1^2}{D}\right) \\ \tilde{t}^5_{12} &= -\frac{1}{2j_1 z_2 \omega_1}\left(-f^4_{26}D + f^3_{25}j_1^2 - f^5_{23}\omega_1^2 + \frac{f^6_{24}j_1^2\omega_1^2}{D}\right) \end{aligned} \tag{5.98}$$

We also find that

$$f^5_{25} = -f^3_{23} \qquad f^6_{26} = -f^4_{24}\,. \tag{5.99}$$

The non-zero free parameters are now : j_1, z_1,z_2, ω_1, f^1_{35}, f^1_{46} and f^3_{23}, f^3_{25}, f^4_{24}, f^4_{26}, f^5_{23}, f^6_{24}.

The NS Bianchi Identity can be rewritten as :

$$\begin{aligned} dH =&4(4h_{1R}^2-(t_{2I}^1)^2-(t_{2I}^2)^2)z_R\wedge z_I\wedge j+2(4h_{1R}t_{3I}^1-t_{2I}^2\tilde{t}_{12}^5)z_R\wedge z_I\wedge\tilde{j}_1\\ &+2(4h_{1R}t_{3I}^2+t_{2I}^1\tilde{t}_{12}^5)z_R\wedge z_I\wedge\tilde{j}_2-4(t_{2I}^1t_{3I}^1+t_{2I}^2t_{3I}^2))z_R\wedge z_I\wedge\omega_R\\ &=0 \end{aligned} \quad (5.100)$$

One can solve this equation. Indeed there are sufficiently few solutions that one can look at them all one by one.

We then study the O5 source equations in each of the three solutions of the NS Bianchi identity. If the charge densities $n_{(D5/O5)}$ are negative, there are additional constraints on the structure constants due to the presence of an O5-plane in the directions 34 and 56

$$f_{23}^3=f_{24}^4=0\,. \quad (5.101)$$

5.5.2 Intermediate $SU(2)$ structures on group manifolds

Up to know we restricted our analysis to the case of rigid $SU(2)$ structure $k_\parallel=0$. In this section we will briefly discuss the case of intermediate $SU(2)$ structure when restricted on group manifolds (we reserve the term dynamical $SU(2)$ for the case where $k_\perp$ and $k_\parallel$ are non constant, see [58] for more details). As we saw in Section 5.2.1, having both $k_\parallel\neq 0$ and $k_\perp\neq 0$ forces to set $\theta=-\frac{\pi}{2}$. Then, for constant dilaton and warp factor, (5.5) and (5.6) give

$$\mathrm{d}z_R=-\frac{2e^{-A}|\mu|}{k_\perp}\left[k_\perp\omega_R+k_\parallel(j+z_R\wedge z_I)\right]\,, \quad (5.102)$$

$$\mathrm{d}z_I=0\,, \quad (5.103)$$

$$\mathrm{d}\omega_R=\frac{k_\parallel}{k_\perp^2}\left[-k_\perp\mathrm{d}j+2|\mu|e^{-A}z_I\wedge(k_\parallel j+k_\perp\omega_R)\right] \quad (5.104)$$

$$z\wedge\left[\mathrm{d}j-ik_\perp^2H-ik_\perp k_\parallel\mathrm{d}\omega_I-|\mu|e^{-A}\bar{z}\wedge\big(\frac{k_\parallel}{k_\perp}(1-2k_\perp^2)2ij-\omega_I+2ik_\parallel^2\omega_R\big)\right]=0 \quad (5.105)$$

$$ik_\perp z\wedge\bar{z}\wedge\mathrm{d}\omega_I-2k_\parallel j\wedge H-2k_\perp\omega_R\wedge H-ik_\parallel z\wedge\bar{z}\wedge H=0\,, \quad (5.106)$$

while (5.7) can be used, as always, to determine the RR fluxes

$$e^\phi * F_5=3z_I|\mu e^{-A}|k_\perp\,, \quad (5.107)$$

$$e^\phi * F_3=k_\parallel H-k_\perp d\omega_I+3|\mu e^{-A}|\left(k_\perp z_R\wedge j+k_\parallel(z_I\wedge\omega_I-z_R\wedge\omega_R)\right)\,, \quad (5.108)$$

$$e^\phi * F_1=\frac{-1}{2k_\perp}\left(2k_\parallel k_\perp j\wedge dj+2k_\perp^2\omega_I\wedge H-|\mu e^{-A}|z_I\wedge j\wedge j(1+3k_\parallel^2)\right)\,. \quad (5.109)$$

One can straightforwardly repeat the analysis of the SUSY conditions and Jacobi identities. Then we look at the Bianchi identities as in section 5.5.1. In particular we find that :

$$n_{(D7/O7)36}=n_{(D7/O7)12}=-\frac{e^{-3A-\phi}|\mu|^2k_\parallel^2(-8+3k_\perp^2)}{4k_\perp^2}<0 \quad (5.110)$$

This means that we must have at least an orientifold in each of the orthogonal direction. Then the orientifold involution on z (namely $\sigma(z)=z$) can't be satisfied. It means that in our framework, there can't be any AdS$_4$ vacua on manifolds with intermediate $SU(2)$-structure.

5.5.3 Explicit vacua on group manifolds

After having imposed the SUSY variations and the Bianchi identities as described in the previous section, we find that $\mathcal{N}=1$ AdS$_4$ solutions are very rare and have at least one O5-plane besides the O7-planes we already mentioned. We give the list of solutions in Table 5.2 where

algebra	$O5$	type of manifold	name
$(35+\epsilon 46, 0, 0, 0, 23, 24)$	✓	nilmanifold	$n3.13,\ n3.14$
$(35+46, 0, 0, 0, 23, 0)$	✓	nilmanifold	$n4.1$
$(35+46, 0, 0, 0, 0, 0)$	✓	nilmanifold	$n5.1$
$(35+\epsilon 46, 0, 25, -\epsilon 26, -\epsilon 23, 24)$	✓	solvmanifold	$\mathfrak{g}6.88$

Table 5.2: list of group manifolds admitting an AdS SUSY solution ($\epsilon = \pm 1$).

the O5 column tells us if there is presence or absence of O5 plane.

We will now give the explicit vacua on these manifolds (omitting n3.13 since it is very close to n3.14). Since O5 and O7 are necessarily present, we use the notations of section 5.3.2. Moreover since the warp factor A is constant, one can, without loss of generality assume that it is zero.

5.5.3.1 Nilmanifold 3.14

The solution is given by

$$\begin{aligned} z_1 =& -2|\mu|\epsilon_1\sqrt{j_1 j_2}\,, \\ \omega_2 =& \epsilon_1\sqrt{j_1 j_2}\,, \\ h_1 =& \frac{i|\mu|}{2} + \frac{(j_1+j_2)\omega_1}{8 j_1 j_2 z_2}\,, \\ h_4 =& -\frac{(j_1+j_2)\omega_1}{4 j_1 j_2 z_2}\,, \\ t_3 =& -a_2 = -\frac{i(j_1-j_2)\omega_1}{4 j_1 j_2 z_2}\,. \end{aligned} \tag{5.111}$$

where the volume and curvature read

$$\begin{aligned} \rho^3 =& -2\epsilon_1|\mu| z_2 (j_1 j_2)^{\frac{3}{2}}\,, \\ R_6 =& -4|\mu|^2 - \frac{(j_1^2+j_2^2)\omega_1^2}{2 j_1^2 j_2^2 z_2^2}\,. \end{aligned} \tag{5.112}$$

The orientifold charges are :

$$\begin{aligned} N_{O7}^{(1)} =& N_{O7}^{(2)} = -10\epsilon_1 e^{-\phi}|\mu|^2\sqrt{j_1 j_2}\,, \\ N_{O5}^{(1)} =& \frac{4e^{-\phi}|\mu|\epsilon_1}{\sqrt{j_1 j_2} z_2 \omega_1}\left(5|\mu|^2 j_1^2 j_2^2 z_2^2 + (j_1^2 + j_1 j_2 + j_2^2)\omega_1^2\right)\,, \\ N_{O5}^{(2)} =& 20 e^{-\phi}\epsilon_1|\mu|^3\sqrt{j_1 j_2} z_2 \omega_1\,, \end{aligned} \tag{5.113}$$

$$\begin{aligned} n_{O7}^{(1)} =& n_{O7}^{(2)} = n_{O5}^{(2)} = -\frac{5e^{-\phi}}{4}|\mu|^2\,, \\ n_{O5}^{(1)} =& -\frac{5e^{-\phi}}{4}|\mu|^2 - \frac{e^{-\phi}}{4 j_1^2 j_2^2 z_2^2}(j_1^2 + j_1 j_2 + j_2^2)\omega_1^2\,. \end{aligned} \tag{5.114}$$

with $\epsilon_1 = \pm 1$

5.5.3.2 Nilmanifold 4.1

The solution is

$$
\begin{aligned}
z_1 =& -2|\mu|\epsilon_1\sqrt{-j_1 j_2}\,,\\
\omega_2 =& -\epsilon_1\sqrt{-j_1 j_2}\,,\\
h_1 =& \frac{i|\mu|}{2} + \frac{j_1}{8 z_2 \omega_1}\,,\\
h_4 =& -i a_2 = i t_3 = \frac{j_1}{4 z_2 \omega_1}\,.
\end{aligned}
\tag{5.115}
$$

where

$$
\begin{aligned}
\rho^3 =& 2\epsilon_1|\mu| z_2(-j_1 j_2)^{\frac{3}{2}}\,,\\
R_6 =& -4|\mu|^2 - \frac{j_1^2}{2 z_2^2 \omega_1^2}\,,
\end{aligned}
\tag{5.116}
$$

and the charges are

$$
\begin{aligned}
N_{O7}^{(1)} =& -N_{O7}^{(2)} = -10\epsilon_1 e^{-\phi}|\mu|^2\sqrt{-j_1 j_2}\,,\\
N_{O5}^{(1)} =& -20\epsilon_1 e^{-\phi}|\mu|^3(-j_1 j_2)^{\frac{3}{2}}\frac{z_2}{\omega_1}\,,\\
N_{O5}^{(2)} =& 4\epsilon_1 e^{-\phi}|\mu|\sqrt{-j_1 j_2}\left(\frac{j_1^2 + 5|\mu|^2 z_2^2 \omega_1^2}{z_2 \omega_1}\right)\,,
\end{aligned}
\tag{5.117}
$$

$$
\begin{aligned}
n_{O7}^{(1)} =& n_{O7}^{(2)} = n_{O5}^{(1)} = -\frac{5e^{-\phi}}{4}|\mu|^2\,,\\
n_{O5}^{(2)} =& -\frac{5e^{-\phi}}{4}|\mu|^2 - \frac{e^{-\phi} j_1^2}{4 z_2^2 \omega_1^2}\,.
\end{aligned}
\tag{5.118}
$$

with $\epsilon_1 = \pm 1$

5.5.3.3 Nilmanifold 5.1

The solution is :

$$
\begin{aligned}
z_1 =& 2|\mu|\epsilon_1\sqrt{-j_1 j_2}\,,\\
\omega_2 =& -\epsilon_1\sqrt{-j_1 j_2}\,,\\
h_1 =& \frac{i|\mu|}{2}\,,\\
h_4 =& a_2 = t_3 = 0\,,
\end{aligned}
\tag{5.119}
$$

where

$$
\begin{aligned}
\rho^3 =& 2\epsilon_1|\mu|(-j_1 j_2)^{\frac{3}{2}} z_2\,,\\
R_6 =& -4|\mu|^2\,,
\end{aligned}
\tag{5.120}
$$

and the charges are

$$\begin{aligned}
N_{O7}^{(1)} =& - N_{O7}^{(2)} = -10\epsilon_1 e^{-\phi}|\mu|^2\sqrt{-j_1j_2}\,,\\
N_{O5}^{(1)} =& - 20\epsilon_1 e^{-\phi}|\mu|^3(-j_1j_2)^{\frac{3}{2}}\frac{z_2}{\omega_1}\,,\\
N_{O5}^{(2)} =& 20\epsilon_1 e^{-\phi}|\mu|32\sqrt{-j_1j_2}z_2\omega_1\,, \qquad (5.121)\\
n_{O7}^{(1)} =& n_{O7}^{(2)} = n_{O5}^{(1)} = n_{O5}^{(2)} = -\frac{5e^{-\phi}}{4}|\mu|^2\,. \qquad (5.122)
\end{aligned}$$

with $\epsilon_1 = \pm 1$

This solution can be obtained from a T-duality of the O6 toroidal orientifold in massive IIA SUGRA.

5.5.3.4 Solvmanifold g6.88

The solution is :

$$\begin{aligned}
z_1 =& 2|\mu|\epsilon_1 j_1\,,\\
j_2 =& - \epsilon j_1\,,\\
\omega_2 =& \epsilon_1\epsilon j_1\,,\\
h_1 =& \frac{i|\mu|}{2}\,,\\
h_4 =& 0\,,\\
t_3 =& - \frac{i(j_1^2 - \epsilon\omega_1^2)}{2j_1z_2\omega_1}\,,\\
a_2 =& \frac{i(j_1^2 + \epsilon\omega_1^2)}{2j_1z_2\omega_1} \qquad (5.123)
\end{aligned}$$

where

$$\begin{aligned}
\rho^3 =& - 2\epsilon_1\epsilon|\mu|j_1^3z_2\,,\\
R_6 =& - 4|\mu|^2 - \frac{(j_1^2 - \epsilon\omega_1^2)^2}{j_1^2z_2^2\omega_1^2}\,, \qquad (5.124)
\end{aligned}$$

and the charges are

$$\begin{aligned}
N_{O7}^{(1)} =& - \epsilon N_{O7}^{(2)} = 10\epsilon_1 e^{-\phi}|\mu|^2 j_1\,,\\
N_{O5}^{(1)} =& \frac{4e^{-\phi}\epsilon\epsilon_1|\mu|j_1}{z_2\omega_1}\left(j_1^2(-\epsilon + 5|\mu|^2z_2^2) + \omega_1^2\right)\,,\\
N_{O5}^{(2)} =& - \frac{4e^{-\phi}\epsilon_1|\mu|j_1}{z_2\omega_1}\left(j_1^2 + (-\epsilon + 5|\mu|^2z_2^2)\omega_1^2\right)\,, \qquad (5.125)\\
n_{O7}^{(1)} =& n_{O7}^{(2)} = -\frac{5e^{-\phi}}{4}|\mu|^2\,,\\
n_{O5}^{(1)} =& \frac{e^{-\phi}}{4}\left(-5|e^{-A}\mu|^2 + \frac{\epsilon - \frac{\omega_1^2}{j_1^2}}{z_2^2}\right)\,, \qquad (5.126)\\
n_{O5}^{(2)} =& \frac{e^{-\phi}}{4}\left(-5|e^{-A}\mu|^2 + \frac{\epsilon - \frac{j_1^2}{\omega_1^2}}{z_2^2}\right)\,. \qquad (5.127)
\end{aligned}$$

with $\epsilon_1 = \pm 1$. Notice that we still have three free parameters (j_1, z_2 and ω_1). Note also that

$$n_{(D5/O5)1234} + n_{(D5/O5)1256} = -\frac{5e^{-\phi}}{2}|e^{-A}\mu|^2 - e^{-\phi}\left(\frac{j_1}{2z_2\omega_1} - \frac{\epsilon\omega_1}{2j_1z_2}\right)^2 < 0\,, \qquad (5.128)$$

which means that at least one of them is negative and so there is, as we said, at least one O5-plane.

Differently from the previous cases, the algebra in this solution is solvable. It is straightforward to check that it has nilradical of dimension 5. Solvable algebrae with five-dimensional nilradicals have been classified [6]. One can easily see that our manifold $(35+\epsilon 46, 0, 25, -\epsilon 26, -\epsilon 23, 24)$ has nilradical $\mathfrak{g}_{5.4}$. We rewrite our solution in Bock's notation in order to make the comparisons easier :

$$\begin{aligned} [X_2, X_4] &= X_1 & [X_3, X_5] &= X_1 \\ [X_2, X_6] &= X_4 & [X_3, X_6] &= -\,\epsilon X_5 \\ [X_4, X_6] &= -\,\epsilon X_2 & [X_5, X_6] &= X_3 \end{aligned} \qquad (5.129)$$

We consider manifolds that one can put under the following form :

$$\begin{aligned} [X_2, X_4] &= X_1 & [X_3, X_5] &= X_1 \\ [X_2, X_6] &= \epsilon_2 X_4 & [X_3, X_6] &= \epsilon_3 X_5 \\ [X_4, X_6] &= \epsilon_4 X_2 & [X_5, X_6] &= \epsilon_5 X_3 \end{aligned} \qquad (5.130)$$

with $\epsilon_i = \pm 1$. One can see that by redefining X_6 in $-X_6$, one can change all the signs of the ϵ_i. So without loss of generality, one can assume that $\epsilon_2 = 1$. These manifolds are all unimodular (ie $\forall X$, Tr(Ad$_X$)= 0) solvmanifolds of dimension six with nilradical $\mathfrak{g}_{5.4}$. One can see that up to real redefinitions of the X's, the different cases are :

$(\epsilon_3, \epsilon_4, \epsilon_5)$	Name
$(1, 1, 1)$	$\mathfrak{g}_{6.88}$
$(-1, -1, 1)$	$\times$
$(-1, 1, -1)$	$\times$
$(1, -1, -1)$	$\mathfrak{g}^*_{6.92}$
$(1, 1, -1)$	$\mathfrak{g}_{6.89}^{0,-1,1} = \mathfrak{g}_{6.91}$
$(-1, 1, 1)$	$\mathfrak{g}_{6.89}^{0,1,1} = \mathfrak{g}_{6.90}^{0,-1}$

The first row of the table corresponds to our solution with $\epsilon = -1$ whereas the second row corresponds to $\epsilon = 1$. As one can see, we weren't able to find it in the classification and called it solv$_1$ in the main text. Note that if one permits complex redefinitions of the X's, the first four rows are equivalent and the last two are also equivalent.

There is also another good reason to believe that such algebra and the corresponding solution make sense: it is easy to show that this solution can be obtained via T-duality from a Lust-Tsimpis type solution in IIA found in [19]. This is a IIA solution with an AdS$_4$-spacetime with O_6 planes on the solvmanifold (0,0,25,-26,-23,24). We will first rewrite it in our notation and then do a T-duality in order to recover our solution. The solution of [19] can be written as

$$
\begin{aligned}
J =& e^1 \wedge e^2 - e^4 \wedge e^5 + e^3 \wedge e^6 && (5.131)\\
\Omega =& -i(e^1 - ie^2) \wedge (e^3 + ie^6) \wedge (e^4 - ie^5) && (5.132)\\
H =& 2|\mu|\Omega_R && (5.133)\\
F_0 =& 5|\mu| && (5.134)\\
F_2 =& 0 && (5.135)\\
F_4 =& 3/2|\mu| J \wedge J && (5.136)\\
F_6 =& 0 && (5.137)
\end{aligned}
$$

and verifies the susy conditions for an AdS_4 vacuum in IIA

$$
\begin{aligned}
&(\mathrm{d} + H\wedge)(e^{2A-\phi}\Phi_+) = 2\mu e^{A-\phi}\mathrm{Re}\Phi_-, && (5.138)\\
&(\mathrm{d} + H\wedge)(e^{A-\phi}\mathrm{Re}\,\Phi_-) = 0\,, && (5.139)\\
&(\mathrm{d} + H\wedge)(e^{3A-\phi}\mathrm{Im}\,\Phi_-) = 3e^{A-\phi}\mathrm{Im}\,(\bar{\mu}\Phi_-) + \frac{1}{8}e^{4A} * F\,, && (5.140)
\end{aligned}
$$

where ϕ is the dilaton, A the warp factor, $\Phi_- = \Omega$ and $\Phi_+ = e^{-iJ}$. Since this will be relevant for T-duality, we define the flux part of H

$$
\begin{aligned}
B =& -2|\mu|(-1+\alpha)e^4 \wedge e^5 && (5.141)\\
H_{fl} =& 2|\mu| \left(-e^1 \wedge e^3 \wedge e^5 - e^1 \wedge e^4 \wedge e^6 + 2\alpha e^2 \wedge e^3 \wedge e^4 - (2-\alpha)e^2 \wedge e^5 \wedge e^6\right) && (5.142)
\end{aligned}
$$

with α an arbitrary real parameter.

Topologically the manifold (0,0,25,-26,-23,24) is

$$S^1_{\{1\}} \times M_5\,, \tag{5.143}$$

where M^5 is a $T^2_{\{3,5\}} \times T^2_{\{4,6\}}$–fibration over $S^1_{\{2\}}$. So the only possibility is to T-dualize along the direction 1 . We will note all the T-dual quantities with tilde. Since the metric is the identity and the B field is only along the base, these two quantities don't change under T-duality, and we have :

$$\mathrm{d}\tilde{s}^2 = \mathbb{I}_6 \qquad \tilde{B} = -2|\mu|(-1+\alpha)\tilde{e}^4 \wedge \tilde{e}^5 \tag{5.144}$$

Next we will do the T-duality on the pure spinors, following the rules in [19]. We first define the twisted pure spinors

$$\Phi_{+B} = e^B\Phi_+ \qquad \Phi_{-B} = e^B\Phi_-\,, \tag{5.145}$$

which are those transforming naturally under T-duality. T-duality in an element of the O(d,d) group of symmetries of the generalized tangent bundle, that act on the spinor representation by wedges and contractions (see section 3.3). In this case, T-duality in the direction 1 gives

$$
\begin{aligned}
\tilde{\Phi}_{+B} =& -(\iota_1 - e^1\wedge)\,\Phi_{-B}) && (5.146)\\
\tilde{\Phi}_{-B} =& (\iota_1 - e^1\wedge)\Phi_{+B} && (5.147)
\end{aligned}
$$

where in $\tilde{\Phi}_{\pm B}$ we replace e by $\tilde{e}$. Similarly, the T-dual of the fluxes is :

$$\tilde{F} = e^{-2\tilde{B}} \wedge (\iota_1 - e^1\wedge)(e^{2B} \wedge F) \tag{5.148}$$

with e replaced by $\tilde{e}$ as above.

Finally, according to Buscher rules [59], the components of the H-flux with one leg in the T-duality direction give off-diagonal elements in the metric that appear as new "geometric fluxes", namely new structure constants

$$f^1_{ab} \leftrightarrow H_{fl\ 1ab}\,. \tag{5.149}$$

This give the new algebra

$$(2|\mu|35 + 2|\mu|46, 0, 25, -26, -23, 24) \tag{5.150}$$

and a new $\tilde{H}_{fl}$

$$\tilde{H}_{fl} = 2|\mu|(\alpha\tilde{e}^2\wedge\tilde{e}^3\wedge\tilde{e}^4 - (2-\alpha)\tilde{e}^2\wedge\tilde{e}^5\wedge\tilde{e}^6) \tag{5.151}$$

In order to reproduce the algebra of our solution (with $\epsilon = 1$) : $(35+46, 0, 25, -26, -23, 24)$, we we rescale the vielbein as $\tilde{e}^1 \to 2|\mu|\tilde{e}^1$.

Now we can give explicitly some of the T-dual quantities :

$$z = 2|\mu|\tilde{e}^1 + i\tilde{e}^2 \qquad j = \tilde{e}^3\wedge\tilde{e}^6 - \tilde{e}^4\wedge\tilde{e}^5 \tag{5.152}$$

$$\omega_R = -\tilde{e}^3\wedge\tilde{e}^4 + \tilde{e}^5\wedge\tilde{e}^6 \qquad \omega_I = \tilde{e}^3\wedge\tilde{e}^5 + \tilde{e}^4\wedge\tilde{e}^6 \tag{5.153}$$

$$H = 2|\mu|(\tilde{e}^2\wedge\tilde{e}^3\wedge\tilde{e}^4 - \tilde{e}^2\wedge\tilde{e}^5\wedge\tilde{e}^6) \qquad F_1 = 10|\mu|^2\tilde{e}^1 \tag{5.154}$$

$$F_3 = 3|\mu|(\tilde{e}^2\wedge\tilde{e}^4\wedge\tilde{e}^5 - \tilde{e}^2\wedge\tilde{e}^3\wedge\tilde{e}^6) \qquad F_5 = -6|\mu|^2\tilde{e}^1\wedge\tilde{e}^3\wedge\tilde{e}^4\wedge\tilde{e}^5\wedge\tilde{e}^6 \tag{5.155}$$

which is the solution we had with $j_1 = 1$, $\epsilon_1 = 1$, $\omega_1 = 1$.

5.6 Scales separation

A question relevant for both compactifications and holography is whether genuine 4-dimensional vacua exist within $10d$ supergravity. To this extent some conditions have to be fulfilled: the string coupling constant e^ϕ needs to be tunably small in order to suppress string loop corrections, for α' corrections to be small the internal volume needs to be tunably large (in string units) and the AdS scale Λ_{AdS} needs to be tunably small. Moreover to be able to decouple the massive KK modes and to reduce to a fully 4-dimensional theory, all of these three conditions must combine in such a way that the AdS length scale, L_{AdS}, is parametrically larger than the length scale set by the compact dimensions L_{KK}

$$\frac{L_{KK}}{L_{AdS}} << 1\,. \tag{5.156}$$

Let us first discuss how to define L_{KK} and L_{AdS}. The four dimensional length scale is set by the inverse of the AdS_4 cosmological constant in the four-dimensional Einstein frame. We follow the notation of [60]. Since all solutions we will consider have constant warp factor, we will set $e^A = 1$. Then we rewrite the 10-dimensional string frame metric as

$$\mathrm{d}s^2_{10} = \tau_0^2\tau^{-2}\mathrm{d}s^2_4 + \rho\,\mathrm{d}\tilde{s}^2_6\,, \tag{5.157}$$

where $\mathrm{d}s^2_4$ is the 4-dimensional Einstein frame metric. We have rescaled the internal metric

$$\mathrm{d}s^2_6 = \rho\,\mathrm{d}\tilde{s}^2_6\,, \tag{5.158}$$

in such a way that the modulus $\rho = (\det g_6)^{1/6}$ measures the string frame volume of the internal manifold and

$$\int_6 \sqrt{\tilde{g}_6} = \mathcal{O}(1)\,. \tag{5.159}$$

The variable τ is the 4-dimensional dilaton and is given by

$$\tau^2 = \mathrm{e}^{-2\phi}\rho^3\,. \tag{5.160}$$

With τ_0 we denote the VEV of τ, such that in equation (5.157) only the dynamical part of τ is used to obtain $4d$ Einstein frame.

Then direct dimensional reduction of the 10-dimensional string frame action gives the $4d$ Planck mass in terms of the string mass scale M_s^2

$$m_p^2 = \tau_0^2 M_s^2\,. \tag{5.161}$$

We define the dimensionally reduced action as

$$S = \int \sqrt{g}(m_p^2 R - V)\,, \tag{5.162}$$

such that the scalar potential is a dimension four operator. The AdS cosmological constant is then defined as

$$\Lambda_{AdS} = \frac{V}{m_p^2}\,. \tag{5.163}$$

The number $|\mu|^2$ that appears in the supersymmetry equations of the previous section is related to the cosmological constant in the following way

$$|\mu|^2 = -6\Lambda_{AdS}\,. \tag{5.164}$$

We define the AdS length scale as $L_{AdS}^2 = -\Lambda_{AdS}^{-1}$, such that it determines the $4d$ curvature as follows

$$2R_{\mu\nu}^{(4)} = L_{AdS}^{-2} g_{\mu\nu}^{(4)}\,. \tag{5.165}$$

The size of the internal manifold is less straightforward to define. A natural guess for the KK scale, which we will adopt in this thesis, is

$$L_{KK}^2 = \rho\,. \tag{5.166}$$

The proper way to check the condition (5.156) would be to compute the Kaluza-Klein spectrum and see that the masses are indeed much larger than the AdS scale. Since this is often not easy to perform, one has to rely on some simpler estimates this ratio.

A way to determine under which conditions scales separation can be achieved is to study the dependence of the effective four-dimensional potential on two moduli, namely the volume of the compact manifold, ρ, and the dilaton, ϕ. As an example we briefly recall how the this can be applied to the model of [24], which is one of the first constructions of a type IIA vacuum admitting full moduli stabilisation and scales separation. The model of [24] corresponds to a compactification on an orbifold of T^6 with non-zero F_0, F_4, H-fluxes and O6 sources. The scalar potential depends on the moduli τ and ρ schematically as

$$V(\rho,\tau)/m_p^4 = |H|^2\rho^{-3}\tau^{-2} + T_{\mathrm{O6}}\tau^{-3} + |F_0|^2\tau^{-4}\rho^3 + |F_4|^2\tau^{-4}\rho^{-1}\,, \tag{5.167}$$

where T_{O6} is the O6 tension and is the only negative term in the potential. The coefficients in the potential are a priori functions of the other moduli. In the particular example of [24], it can be shown that, while the H and F_0 flux are constrained by the tadpole condition to be order one, the F_4 flux is an unbounded flux quantum. In what follows we assume that $|H|^2, T_{O6}, |F_0|^2$ are all order one in proper units and $|F_4|^2$ scales as N^2, where N is unbounded. From the detailed balance condition (all terms are of the same order in the potential) we can derive the N-dependence for ρ and τ at a critical point

$$\rho \sim N^{\frac{1}{2}}\,, \qquad \tau \sim N^{\frac{3}{2}} \qquad (\mathrm{e}^{\phi} \sim N^{-\frac{3}{4}})\,, \tag{5.168}$$

from which we see that large N implies large volume and weak coupling. Secondly, we find that the AdS scale becomes tunably small in the same limit. Using the scaling of the potential and the $4d$ Planck mass we find

$$V/m_p^4 \sim N^{-\frac{9}{2}} \qquad m_p^2 \sim M_s^2 \mathrm{e}^{-2\phi}\rho^3 \sim N^3 \qquad L_{AdS}^{-2} \sim N^{-3/2}\,. \tag{5.169}$$

Since $\rho = L_{KK}^2$ we indeed find scales separation

$$\frac{L_{KK}}{L_{AdS}} \to 0\,. \tag{5.170}$$

A similar argument can be also given for IIB solutions. On the type IIB side, an explicit example with the right properties of tunably large volume, small coupling and small AdS scale was found in [26] by T-dualising the type IIA torus example. A systematic study, from a $4d$ point of view, of IIB solutions was initiated in [46]. Typically we have models with F_1, F_3, F_5 flux, O5, O7 sources on some curved internal manifold. The scalar potential can be written as

$$V(\rho,\tau)/m_p^4 = \tau^{-4}\Big(|F_5|^2\rho^{-2} + |F_3|^2 + |F_1|^2\rho^2\Big) + \tau^{-3}\Big(T_{O7}\rho^{\frac{1}{2}} + T_{O5}\rho^{-\frac{1}{2}}\Big) + R_6\tau^{-2}\rho^{-1}\,. \tag{5.171}$$

where, as in the type IIA case, the coefficients are functions of all other moduli. The models of [26,46] are characterized by two unbounded flux quanta: F_5 and a component of F_1, whereas another component is determined by the O7 planes and not tunable. So let us scale both fluxes

$$|F_5|^2 \sim N^2 \qquad |F_1|^2 \sim N^C\,, \tag{5.172}$$

where C is some positive number. If we then balance F_5^2 against F_1^2 and T_{O7} we find the following N-dependence

$$\rho \sim N^{\frac{1}{2}-\frac{C}{4}}\,, \qquad \mathrm{e}^{\phi} \sim N^{-\frac{C}{4}}\,. \tag{5.173}$$

If $0 < C < 2$ the solution is indeed at large volume and weak coupling for large N. The F_5, F_1, T_{O7} contributions, which set the size of the AdS solution, scale as

$$V/m_p^4 \sim N^{-2-2C}\,, \tag{5.174}$$

and go to zero at large N. If we compute L_{AdS}^2 we again find a separation of scales. This argument relied on a detailed balance condition for the F_5, F_1 and T_{O7} contributions. We have not discussed the other contributions to the scalar potential. In the explicit solutions we derive in this thesis all terms in the potential will be of the same order of magnitude.

The Ricci scalar of the internal space scales as the inverse metric/ ρ^{-1}. Therefore one would be tempted to conclude that in the large volume limit the two definitions of scales separation

$$\text{scales separation (1)}: \qquad \frac{L_{AdS}^2}{L_{KK}^2} \to \infty\,, \tag{5.175}$$

$$\text{scales separation (2)}: \qquad \frac{R_6}{R_4} \to \infty\,, \tag{5.176}$$

are equivalent. However, these two definitions do not need to coincide if the normalized curvature $\tilde{R}$

$$R_6 = \rho^{-1}\tilde{R}(\phi_I)\,, \tag{5.177}$$

is not kept constant when the limit of large ρ is taken. The other moduli ϕ_I that appear inside the normalized curvature could also introduce an extra scaling. Below we find an explicit examples for which there is scales separation according to the first definition, but not according to the second definition. Nonetheless notice that, as we will show in the next sections, the ratio of the Ricci scalars can be very useful in setting general conditions on the torsion classes of the internal manifold in order to achieve separation of scales.

5.6.1 Separation of scales without sources?

The most trustworthy solutions are those without any orientifold or D-brane sources since there is no reason to worry about the smearing approximation or charge quantisation. Even in the case one knows the localized solutions one could have rightful worries about the use of supergravity in the presence of singular sources. A priori, sourceless AdS SUSY vacua can exist both for $SU(3)$ structure manifolds IIA and $SU(2)$ structure ones in IIB. When SUSY is broken many more solutions can exist, see for instance reference [28] for the IIA case. It is not clear whether solutions without sources allow for separation of scales [45]. The above scaling arguments do not obviously use the presence of a source term. We did include it in the analysis, but we could have equally discarded it. It turns out that it depends on the details of the manifold whether there exists the specific large flux limits that achieve scales separation. While no no-go theorem has been found so far, there is no example known of a solution in $10d$ SUGRA, whether SUSY or not, that achieves scales separation without sources.

In the following we give a simple argument that seems to suggest that AdS without sources do not allow for scales separation. The argument below holds under two assumptions: 1) there is no warping and 2) the size of the internal manifold cannot be decoupled from its curvature radius.

Consider a general compactification with RR fluxes F_p, H flux and no sources. The scalar potential can be written as

$$V(\rho,\tau)/m_p^4 = -R_6\tau^{-2}\rho^{-1} + |H|^2\rho^{-3}\tau^{-2} + \sum_p |F_p|^2\rho^{3-p}\tau^{-4}\,. \tag{5.178}$$

The vacua of the theory must be extrema of the scalar potential. One can easily verify that the equations

$$\partial_\rho V = 0\,, \qquad \partial_\tau V = 0 \tag{5.179}$$

are specific linear combinations of the dilaton equation of motion in 10 dimensions, the trace over the internal indices of the (trace reversed) Einstein equation, and the external Einstein equation [61]. Upon eliminating $|H|^2$ in terms of the RR field strength densities, these are equivalent to

$$R_4 = 2V = -2\sum_p |F_p|^2 < 0\,, \tag{5.180}$$

$$R_6 = \sum_p \frac{9-p}{2}|F_p|^2 > 0\,, \tag{5.181}$$

where we did not write down the explicit ρ, τ dependence anymore. The first condition is an alternative derivation of the Maldacena–Nunez no-go theorem [62] in the simple case of no warping. The second condition was found before [34]. With the above equations we can compute the ratio

$$r = |\frac{R_6}{R_4}|\,, \tag{5.182}$$

and define scales separation as the possibility to have $r >> 1$ (5.176). However, our equations imply that r is bounded from above by a number r_{max}, since $p < 9$. We compute r_{max} by rewriting the inequality:

$$\sum_p (\frac{9-p}{2} - 2r_{max})|F_p|^2 < 0\,. \tag{5.183}$$

From this one deduces that

$$r_{max} = \frac{9-p_{max}}{4}\,, \tag{5.184}$$

where p_{max} is the highest rank field strength that is turned on in the vacuum solution. We then conclude that, under the assumptions we discussed above, AdS vacua not supported by sources cannot achieve scales separation. Note that indeed Freund–Rubin vacua [63] have r of order one. It would be most interesting to see whether the same argument holds when allowing for warping.

5.6.2 Study of scales separation on the explicit examples

We can now check whether the solutions we found on cosets and group manifolds admits small string coupling, large volume and scales separation. We proceed as follows. We look at possible scalings where, taking the limit of small cosmological constant

$$|\mu|^2 \to 0\,, \tag{5.185}$$

we can have small coupling and large volume. We also check whether separation of scales is possible according according to both definitions (5.175, 5.176) . Our results are summarized in Table 5.3

Only the solutions on Nil 4.1 and 5.1 can be tuned into a trustworthy regime. The solution on Nil 3.14 cannot be achieved for large volume and furthermore suffers from having a singular limit (vanishing volume) if scales separation is required. We also notice that for several examples there is no match between the two criteria for scales separation.

In finding the appropriate limits we have taken a conservative and safe viewpoint where each source term was taken to have finite prefactors as a consequence of charge quantisation (Appendix D of [53] contains some first attempts for charge and flux quantisation). Since not all source terms are represented by forms in the cohomology of the internal space, it is possible that charge quantisation is less restrictive on such forms and that certain numbers do not need

Manifold	weak coupling	Large volume	scales separation (1)	scales separation (2)
$\frac{SU(3)}{SU(2)} \times U(1)$	$\times$	✓	✓	$\times$
$\frac{SU(2)^2}{U(1)} \times U(1)$	$\times$	✓	✓	$\times$
Nil 3.14	✓	$\times$	✓	✓
Nil 4.1	✓	✓	✓	✓
Nil 5.1	✓	✓	✓	$\times$
$\mathfrak{g}_{92}^{6*}$	✓	$\times$	✓	✓

Table 5.3: The scaling regimes for the various manifolds with SUSY AdS vacua.

to take fixed values. In that case it is possible that certain solutions do allow weak coupling and scales separation although the above table indicates otherwise. We leave such subtle issues for further investigation.

Chapter 6

Conclusion

String compactification are the standard approach in string theory to make contact with the four-dimensional universe we observe. Among the solutions of string equations of motions, one selects those whose geometry is the product of the four-dimensional space we observe times a tiny compact six-dimensional space, the internal manifold. The geometry of such manifold determines the physical properties of the low energy effective actions. Since strings come with supersymmetry it is natural to look for supersymmetric solutions. It turns out that supersymmetry strongly constrains the geometry of the internal manifold. For example, if one looks for the simplest solutions, where only the metric is non trivial, requiring minimal supersymmetry forces the internal manifold to be a Calabi-Yau. In spite of their nice properties, Calabi-Yau manifolds are not suitable to give phenomenologically viable models, because of the moduli problem. In type II string theories, if one considers more general string backgrounds where also some of the Neveu-Schwarz and Ramond-Ramond fluxes are non trivial, it is possible to lift some (even all) moduli, and also to have $\mathcal{N} = 1$ supersymmetry (this is necessary to allow for chiral fermions). The backreacion of the fluxes changes the internal metric, which is no longer Calabi-Yau. We then face two problems: one is merely technical, namely finding explicitly such vacua, the second is more fundamental and consists in trying to see whether one can understand the geometry of these new internal manifolds.

As we review in the first three chapters of this thesis, for Calabi-Yau compactifications supersymmetry tells us the internal manifold must have reduced holonomy (SU(3) holonomy in six dimensions). The idea is then to see what remains of this condition when we add fluxes. This leads to the formalism of G-structures. Indeed, it is possible to see that the manifold still admits globally defined spinors (or equivalently tensors), as it is the case for reduced holonomy, which define a G-structure in the manifold. But now they are not covariantly constant (or closed). Their failure to be constant is due to the fluxes and it is measured by the intrinsic torsion of the Levi-Civita connection. The SUSY conditions can be rewritten as conditions on the torsion classes. For $\mathcal{N} = 1$ compactifications on six-dimensional manifolds we can have SU(3) or SU(2)-structure.

Generalized Complex Geometry is a further generalization of these ideas that allows to treat all $\mathcal{N} = 1$ compactifications in an unified way. The basic principle of Generalized Complex Geometry is quite simple, instead of considering the tangent bundle alone, one considers the sum of the tangent and the cotangent bundle, the generalized tangent bundle, whose sections are generalized vectors composed of a vector and a one-form. Then one can define the analogue of the SU(3) or SU(2)-structures and relate them to the existence of globally defined polyforms, which are pure spinors on the generalized tangent bundle. We see that Generalized Complex Geometry naturally encodes complex and symplectic structures and can even interpolate between them. One defines the Courant bracket in order to properly define the integrability of these generalized structures. The integrability of one such pure spinors define a generalized Calabi-Yau. This generalize the notion of Calabi-Yau manifolds.

What matters for string compactifications is that is possible to rewrite the supersymmetry variation of ten-dimensional type II supergravities and differential conditions on two compatible pure spinors. With a suitable ansatz for the metric and the fluxes, these equations contain only the components of all fluxes on the internal manifold. In this way we have reduced the study of supersymmetry to a purely six-dimensional problem. Let us also remind, that for the solutions we are interested in, solving the SUSY equations and the Bianchi identity automatically solves the full ten dimensional equations of motion.

In this thesis we have applied Generalized Complex Geometry (and G-structures) to study $\mathcal{N} = 1$ compactifications to AdS_4 in type II supergravity.

We first study in full generality the SUSY equations in terms of pure spinors for compactification to AdS_4. This allows us to reproduce very elegantly the result that the internal manifold must have SU(2)-structure. Solving the SUSY equations for vacua is very hard. So we make some assumptions in order to simplify our task: we consider constant dilaton and warp factor and put the torsion classes in the **2** of SU(2) to zero. We also restrict ourselves to parallelizable manifolds, cosets and group manifolds. The idea is to find an exhaustive list of vacua and also see whether they can give rise to truly four dimensional theories, meaning that they admit large volume, small curvature and scales separation. We also wanted to see whether sourceless solutions are possible in IIB (we know that some exist in IIA). The cosets admitting SU(2)-structure were known and it was easy with our technique to see which one admits AdS_4 vacua, completing this way the analysis started in [46]. For the group manifolds, the analysis is a bit more involved, but as described in Chapter 5 it is possible to perform a complete scan for AdS_4. In particular we find that there are no AdS_4 vacua on manifolds with intermediate SU(2)-structure and that all vacua must have both O5 and O7 planes present. In particular this means that with our assumptions no sourceless AdS_4 vacua exist. Even allowing for sources, we find that, under our assumptions, AdS_4 vacua are pretty rare: we only find only five of them. Four of are nilmanifolds which were already known for admitting Minkowski vacua (and they are the only nilmanifolds that do). The fifth solution is a totally new solvmanifold. In order to further argument its validity we showed that it is T-dual to a known solution in IIA.

Finally we check whether the seven solutions we found satisfy the requirements of large volume, small curvature and scale separation. We find that the cosets can't even have small coupling and so are not good candidates for a four-dimensional effective theory. For the group manifolds, two of them (n4.1 and n5.1) appear to verify all the conditions for scales separation and one should be able to define a genuine effective four-dimensional theory on them. In order to rightfully claim to have such an effective theory, one should of course compute the Kaluza-Klein spectrum but it is nonetheless promising that some of the vacua survive this first analysis.

There are various issues that call for further research.

First of all, there still is a lot to be done in the search for AdS_4 vacua. Indeed, our method is very general and can be adapted to much more complicated situations that those we looked at. I think that by relaxing some of our assumptions, one can obtain new interesting AdS_4 vacua. Indeed, the first step would be to take non constant dilaton and warp factor, and see whether we can find the known example of sourceless solutions in IIB [64] and also find new ones.

It would also be interesting to compute the four dimensional effective theory and study for example moduli stabilization in these vacua. Indeed in our papers we looked at ten dimensional vacua without doing the four dimensional analysis which could give exciting new results. The hope is to find a four dimensional theory with scales separation and all the moduli stabilized, the ten dimensional analysis providing a list of manifold on which this is possible.

Other important open questions concern the source we often encounter in flux vacua. A first problem is understanding charge quantization in these backgrounds, the difficulty being that in these solutions the sources wrap submanifolds that are not cycles (Appendix D of [53] contains some first attempts for charge and flux quantisation).

Besides that, it is essential to understand how to localize the orientifold planes. Indeed most of the solutions of parallelizable manifolds contain intersecting O-planes and branes, which then we have to take smeared. What it is the meaning of a smeared orientifold and whether it makes sense is a very debated question. Likely this question is more tractable using the pure spinor formalism, as was attempted for localized O6 solutions in massive IIA [35]. If an analogy can be made with solutions that feature parallel orientifolds [19, 31] then one can expect that localization will change the geometry but that the very existence of the solution is not invalidated.

Ultimately, it is perhaps even more relevant to break supersymmetry and to look for non-SUSY AdS vacua in this context or even de Sitter vacua. Notice that sourceless non-susy vacua are relatively simpler to obtain [15]. In IIA meta-stable non-SUSY AdS vacua have been found using Ansatze that are close to that of the SUSY AdS solutions [28]. Interestingly, the same has been done for dS solutions [5, 53, 61, 65–67][1], although none of the latter examples turned out meta-stable. Clearly more examples are required and the results in this thesis could offer a first step towards achieving this. Already a (unstable) de Sitter critical point was numerically found in [46]. It would be worthwhile to verify whether this numerical solution lifts to a simple 10-dimensional solution.

Non supersymmetric solutions are hard to study because we have to solve the full ten-dimensional equations of motion. Also, given a solution checking its stability is a non trivial problem. Continuing in the line of [70], it would be interesting to see whether using Generalized Complex Geometry it is possible to give a set of first order equations that are equivalent to the equations of motion, at least for a class of non-SUSY backgrounds. These fake BPS equations should correspond SUSY breaking calibrated sources. If they existed we could obtain a general mechanism of controlled SUSY breaking and perform a systematic study of the stability of such solutions.

Clearly the same method could be applied to the search of (meta)stable de Sitter vacua. This approach is along the same lines of what is proposed in [58] where a (possibly stable) de Sitter solution on a given solvmanifold was found, using an ansatz for a SUSY breaking calibrated source. This also relates with an ongoing project during the Ph.D. where we tried to see whether we could reproduce such solution in the four-dimensional effective action. We explicitly computed the four dimensional action coming from the Kaluza-Klein reduction. One can show that it admits a Minkowski $\mathcal{N} = 1$ vacuum when there is no supersymmetry breaking sources. In doing so, we recovered a known SUSY solution that had be obtained by a ten dimensional approach. We then tried to include supersymmetry breaking sources. We found a way to obtain a de Sitter vacuum but were not able to easily determine its stability. The analysis is still in progress and we hope to have some result in the near future.

[1]These de Sitter constructions are in part inspired on the proposals in [68, 69].

Appendix A

De Sitter Compactification

The first part of my PhD work was devoted to the construction of (meta)stable de Sitter solutions in type IIA theories. The project is not finished yet, because of technical and conceptual issues we still have to address. In this Appendix I will give a very schematic account of the work done up to know and of some of the open questions.

A.1 Motivations

Embedding (meta)stable de Sitter vacua in string theory is a highly non trivial issue. While vacua of this kind can be found, at least in principle, in gauged supergravities, their lift to the full ten dimensional theory is far for being clear. Even if there are no no-go theorems, no examples of (meta)stable de Sitter vacua have been found in purely classical supergravity. It seems that other stringy ingredients like sources, higher order corrections and non perturbative corrections are needed.

The difficulty partly resides in the need for breaking supersymmetry and having a positive cosmological constant. The requirement of having a positive cosmological constant restricts the choice of internal manifolds. Without supersymmetry, one is forced to solve the second order equations of motion, which can be in general very hard. Particularly hard to handle is the computation of the variation of the DBI action for the sources, which are necessarily present due to no-go theorems about flux compactifications.

For SUSY-preserving sources, using κ-symmetry one can write the DBI action as the pull-back of the non-integrable pure spinor (see the expression of the SUSY variations (4.60)-(4.62)).

$$\left(i^*[\mathrm{Im}\Phi_2] \wedge e^{\mathcal{F}}\right) = \frac{|a|^2}{8}\sqrt{|i^*[g] + \mathcal{F}|}\mathrm{d}^\Sigma x\,, \tag{A.1}$$

where i denotes the embedding of the world-volume into the internal manifold M, g is the internal metric and $\mathcal{F}$ is the gauge invariant combination of the field strength of the world-volume gauge field and the pullback of B. Then for sources preserving the bulk SUSY, one can replace the DBI action by the left-hand side of (A.1). The equations of motions derived this way are the same as those derived form DBI since the corrections are would be linear in the κ-symmetry condition, and thus vanish in the supersymmetric case.

With this approach one arrives at the following conditions for the four- and six-dimensional

Ricci scalars

$$R_4 = \frac{2}{3}(g_s^2|F_0|^2 - |H|^2) \ , \tag{A.2}$$

$$R_6 + \frac{1}{2}g_s^2|F_2|^2 + \frac{3}{2}(g_s^2|F_0|^2 - |H|^2) = 0 \, , \tag{A.3}$$

together with

$$g_s\frac{T_0}{p+1} = \frac{1}{3}\left[-2R_6 + |H|^2 + 2g_s^2(|F_0|^2 + |F_2|^2)\right] \ , \tag{A.4}$$

where T_0 denotes the source contribution. These equations show that two necessary requirements to have de Sitter solutions are $F_0 \neq 0$ and $R_6 < 0$.

The authors of [5] proposed to treat in a similar way also sources that do not preserve the bulk supersymmetry. The idea is to modify (A.1) in order to account for SUSY-breaking sources :

$$\left(i^*[\mathrm{Im}\, X_-] \wedge e^{\mathcal{F}}\right) = \sqrt{|i^*[g] + \mathcal{F}|}\mathrm{d}^\Sigma x \, , \tag{A.5}$$

where X_- is an odd polyform which, in general, is not a pure spinor. This proposal is inspired by what was done in [70] where the SUSY equations (4.60)-(4.62) were modified in this fashion. It permits to encode in a controlled way the SUSY breaking term by considering a fake BPS source. In particular, in the context of four dimensional compactification, the trace of the Einstein equation gives :

$$R_4 = \frac{2}{3}\left(\frac{g_s}{2}(T_0 - T) + g_s^2|F_0|^2 - |H|^2\right) \ , \tag{A.6}$$

where T_0 is different from the trace of the energy momentum tensor T (it is equal in the supersymmetric case). This suggest that this SUSY breaking source could help finding a stable de Sitter vacuum by providing a new term in the expression of the four dimensional Ricci scalar.

Using the "fake" calibration (A.5), in [5], the authors found a de Sitter solutions on the solvmanifold called $\mathfrak{g}_{5,17}^{p,-p,\pm 1}$ (we will define it in the next section) and checked its stability against the two universal moduli, namely the four-dimensional dilaton and the volume of the internal manifold.

Our goal is to find further evidence for the validity of the "fake" calibration. The idea is to compute the four-dimensional effective action by compactifying on the internal manifold, check whether it is in the class of theories that might admit a (meta)-stable de Sitter solution (see for instance [71, 72]) and compare with the ten-dimensional solution of [5].

This project is still work in progress. That's why in what follows, we only give a schematic outline of what we did.

A.2 The Manifold $\mathfrak{g}_{5,17}^{p,-p,\pm 1}$

Let's first study in a bit more detail the manifold $\mathfrak{g}_{5,17}^{p,-p,\pm 1}$ and define on it some quantities that will be useful in what follows. Its algebra is

$$(q_1(pe^{25} + e^{35}), q_2(pe^{15} + e^{45}), q_2(pe^{45} - e^{15}), q_1(pe^{35} - e^{25}), 0, 0) \, , \tag{A.7}$$

with q_1 and q_2 two positive real numbers. As discussed in [5], it is necessary to have two intersecting O6 planes, filling the four non-compact directions and wrapping the space-time and the directions (146) and (236) respectively of the internal manifold.

We can then define odd and even forms under the O6 involutions (the underlined forms are closed $\forall\, p$) :

	even (+)	odd (-)
1 forms	$\underline{6}$	$\underline{5}$
2 forms	14, 23	12, 13, 24, 34, $\underline{56}$
3 forms	$\underline{125}$, $\underline{135}$, $\underline{245}$, $\underline{345}$, 146, 236	$\underline{145}$, $\underline{235}$, 126, 136, 246, 346
4 forms	$\underline{3456}$, $\underline{2456}$, $\underline{1356}$, $\underline{1234}$, $\underline{1256}$	$\underline{1456}$, $\underline{2356}$
5 forms	$\underline{12346}$	$\underline{12345}$
6 forms	-	$\underline{123456}$

Table A.1: Parity of the forms under the orientifold involutions

For future convenience, we take the following basis $\omega^{I\pm}$ for the two-forms :

$$\omega^1 = e^{14} \qquad \omega^2 = e^{23} \qquad \omega^3 = e^{12} \qquad \omega^4 = e^{13} \qquad \omega^5 = e^{24} \qquad \omega^6 = e^{34}$$

where $\pm$ gives the parity under the orientifold involution (eg. an index summation $\alpha_{I+}\omega^{I+}$ means a summation on ω^1 and ω^2).

Next we define $\tilde{T}^I_{iJ}$ as $\mathrm{d}\omega^I = \tilde{T}^I_{iJ} e^i \wedge \omega^J$. One can show that only one $\tilde{T}^I_{iJ}$ is non zero, namely $\tilde{T}_5$ and :

$$\tilde{T}_5 = \begin{pmatrix} 0 & 0 & q_1 & -pq_1 & -pq_1 & -q_1 \\ 0 & 0 & -q_2 & -pq_2 & -pq_2 & q_2 \\ -q_2 & q_1 & 0 & 0 & 0 & 0 \\ -pq_2 & -pq_1 & 0 & 0 & 0 & 0 \\ -pq_2 & -pq_1 & 0 & 0 & 0 & 0 \\ q_2 & -q_1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.8}$$

We also define :

$$\omega^I \wedge \omega^J = \eta^{IJ} e^{\rho}\mathrm{vol}_4 \quad \text{with}$$

$$\eta^{IJ} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad e^{\rho}\mathrm{vol}_4 = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \tag{A.9}$$

which will be useful in order to have a compact formulation thereafter.

A.3 Compactification

In this section we briefly outline the compactification to four-dimensions. The results of this section are incomplete, since we still have to recast our results in a proper $\mathcal{N} = 2$ form.

A.3.1 The metric

We will consider the following ten dimensional metric :

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + g_{ij}\mathcal{E}^i\mathcal{E}^j + g_{ab} e^a e^b \tag{A.10}$$

where i, j denotes the indices $5, 6$ of the internal manifold, a, b, the indices $1, 2, 3, 4$ of the internal manifold and μ, ν the indices of the external spacetime. In fact, we will specify a bit more this metric by taking $\mathcal{E}^5 = e^5$ and $\mathcal{E}^6 = e^6 - G^6$ with G^6 an external one-form ($G^6 = G^6_\mu dx^\mu$). Note that due to the orientifold involutions, the only non zero terms in g_{ab} are $\{g_{11},\ g_{22},\ g_{33},\ g_{44},\ g_{14},\ g_{23}\}$ and g_{ij} is diagonal. We will take the following parametrization for these two metrics :

$$g_{ab} = \begin{pmatrix} \frac{e^{-\rho_1}}{\tau_{i1}} & 0 & 0 & \frac{e^{-\rho_1}\tau_{r1}}{\tau_{i1}} \\ 0 & \frac{e^{-\rho_2}}{\tau_{i2}} & \frac{e^{-\rho_2}\tau_{r2}}{\tau_{i2}} & 0 \\ 0 & \frac{e^{-\rho_2}\tau_{r2}}{\tau_{i2}} & \frac{e^{-\rho_2}(\tau_{r2}^2+\tau_{i2}^2)}{\tau_{i2}} & 0 \\ \frac{e^{-\rho_1}\tau_{r1}}{\tau_{i1}} & 0 & 0 & \frac{e^{-\rho_1}(\tau_{r1}^2+\tau_{i1}^2)}{\tau_{i1}} \end{pmatrix} \tag{A.11}$$

$$g_{ij} = \begin{pmatrix} \frac{e^{-\eta}}{\tau_{i3}} & 0 \\ 0 & e^{-\eta}\tau_{i3} \end{pmatrix} \tag{A.12}$$

note that ρ defined in (A.9) is such that $\rho = \rho_1 + \rho_2$. η (resp. ρ_1, ρ_2) can be seen as the volume in direction 56 (resp. 14, 23).

We also define :

$$H^{IJ} e^{\rho} \mathrm{vol}_4 = \omega^I \wedge *\omega^J \tag{A.13}$$

$$\text{ie } H^{IJ} = \frac{1}{\tau_{i1}\tau_{i2}} \begin{pmatrix} \tau_{i1}\tau_{i2}e^{\rho_1-\rho_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau_{i1}\tau_{i2}e^{-\rho_1+\rho_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & |\tau_1\tau_2|^2 & -|\tau_1|^2\tau_{r2} & \tau_{r1}|\tau_2|^2 & -\tau_{r1}\tau_{r2} \\ 0 & 0 & -|\tau_1|^2\tau_{r2} & |\tau_1|^2 & -\tau_{r1}\tau_{r2} & \tau_{r1} \\ 0 & 0 & \tau_{r1}|\tau_2|^2 & -\tau_{r1}\tau_{r2} & |\tau_2|^2 & -\tau_{r2} \\ 0 & 0 & -\tau_{r1}\tau_{r2} & \tau_{r1} & -\tau_{r2} & 1 \end{pmatrix} \tag{A.14}$$

A.3.2 The fluxes

The fluxes are constrained by the orientifold involutions. Indeed, F_0 and F_4 must be even (so A_3 also) whereas H and F_2 must be odd (so B and A_1 also). Using the table A.1, we can use the following general ansatz for the fluxes by decomposing them on the left invariant forms:

$$\begin{aligned} B &= B_{5\mu}dx^\mu e^5 + b_{12}e^5(e^6 - G^6) + b_{I-}\omega^{I-} \\ A_1 &= a_5 e^5 \\ A_3 &= C_3 + C_2(e^6 - G^6) + C_{1I+}\omega^{I+} + c_{6I+}(e^6 - G^6)\omega^{I+} + c_{5I-}e^5\omega^{I-} \\ H &= dB + H_{fl} \\ F_2 &= dA_1 + F_0 B + F_{2fl} \\ F_4 &= dA_3 + F_{4fl} - HA_1 + BF_{2fl} + 1/2F_0B^2 \end{aligned} \tag{A.15}$$

with

$$\begin{aligned} H_{fl} &= h_{mI-}\omega^{I-}e^6 + h_{pI+}\omega^{I+}e^5 \\ F_{2fl} &= f_2 e^5 e^6 + f_{mI-}\omega^{I-} \\ F_{4fl} &= f_4 e^{1234} + f_{m12I-}\omega^{I-}e^5e^6 \end{aligned} \tag{A.16}$$

A.3.3 The 10-dimensional action

We will consider the following action which is the type IIA supergravity action in IIA :

$$S = S_{\text{kin}} + S_{\text{top}} + S_{\text{source}}$$

with

$$\begin{aligned} S_{\text{kin}} =& \frac{1}{2}\int e^{-2\Phi}(R + 4d\Phi\wedge *d\Phi - \frac{1}{2}H\wedge *H) - \frac{1}{4}\int F_0^2 + F_2\wedge *F_2 + F_4\wedge *F_4 && \text{(A.17)} \\ S_{\text{top}} =& -\frac{1}{4}\int (A_3 H_{fl} + B(dA_3 + F_{4fl}) + B^2 F_{2fl} + \frac{1}{3}F_0 B^3)(dA_3 + F_{4fl}) && \text{(A.18)} \\ &+ \frac{1}{3}B^3 F_{2fl}F_{2fl} + \frac{1}{4}F_0 B^4 F_{2fl} + \frac{1}{20}F_0^2 B^5 \end{aligned}$$

and S_{sources} undefined for the moment. Note that we used the democratic formulation of [14] for the fluxes so that only the internal fluxes appear in the action.

A.3.4 Bianchi Identities

To the action A.17, we must supplement the Bianchi identities :

$$\begin{aligned} dF_0 =& 0 & dH =& 0 \\ dF_2 - HF_0 =& 2\kappa^2 T_p j_{source} & dF_4 - H\wedge F_2 =& 0 \end{aligned} \quad \text{(A.19)}$$

with j_{source} the source current and T_p the tension of the p-brane that gave rise to such a current. In order to verify these Bianchi identities, we take :

$$F_0 = \text{constant} \qquad \forall I-,\ h_{mI-} = 0 \qquad \text{(A.20)}$$

A.3.5 The 4-dimensional action

We now follow the steps of [73] in order to establish the four dimensional effective action. We will not explicitly put all the details but we refer the interested reader to [73] to have the whole scheme.

For the topological term in the action, we first do the following integration by part :

$$S_{\text{top}} = S_{\text{top}} + \frac{1}{4}\int \Big(d(h_{pI}c_6^I C_3) + d(dB_5 c_6^I C_I) - d(c_{6I}C^I dB_5) + \frac{1}{2}d(c_{6I}dC^I B_5) - d(h_{pI}C^I C2)\Big)$$

Then we compactify the action in string frame after dualization of the three-form C_3 which gives us $S = S_{\text{VTkin}} + S_{\text{top}} + S_{\text{Skin}} + S_{\text{pot}}$ with :

$$\begin{aligned} S_{\text{VTkin}} = -\frac{1}{4\kappa^2}\int \Big[& e^{-2\phi-\eta}\tau_{i3}dG^6\wedge *dG^6 + e^{-2\phi+\eta}\tau_{i3}(dB_5 + b_{12}dG^6)\wedge *(dB_5 + b_{12}dG^6) \\ &+ e^{-\eta}H^{IJ}(dC_I - c_{6I}dG^6)\wedge *(dC_J - c_{6J}dG^6) + \frac{e^{-\rho_1-\rho_2}}{\tau_{i3}}dC_2\wedge *dC_2\Big] \end{aligned}$$

$$\begin{aligned} S_{\text{top}} = -\frac{1}{4\kappa^2}\int & dC^{(2)}(2\mathcal{D}c_{5I}b^I + b_I b^I(f_2 G^6 - F_0 B_5) + 2C_I h_p^I - 2B_5 f_4) \\ & - 2(dB_5 + b_{12}dG^6)c_{6I}(dC^I - \frac{1}{2}c_6^I dG^6) + b_{12}dC^I dC_I \end{aligned}$$

$$S_{\rm Skin} = \frac{1}{2\kappa^2}\int \Big[e^{-2\phi}R*1 + 4e^{-2\phi}d\phi\wedge *d\phi - \frac{1}{2}e^{-2\phi}d\eta\wedge *d\eta - \frac{e^{-2\phi}}{2\tau_{i3}^2}d\tau_{i3}\wedge *d\tau_{i3}$$
$$-\frac{1}{2}e^{-2\phi}d(\rho_1)\wedge *d(\rho_1) - \frac{1}{2}e^{-2\phi}d(\rho_2)\wedge *d(\rho_2) - \frac{e^{-2\phi}}{2\tau_{i1}^2}d\tau_{i1}\wedge *d\tau_{i1}$$
$$-\frac{e^{-2\phi}}{2\tau_{i1}^2}d\tau_{r1}\wedge *d\tau_{r1} - \frac{e^{-2\phi}}{2\tau_{i2}^2}d\tau_{i2}\wedge *d\tau_{i2} - \frac{e^{-2\phi}}{2\tau_{i2}^2}d\tau_{r2}\wedge *d\tau_{r2}$$
$$-\frac{1}{2}e^{-\rho_1-\rho_2}\tau_{i3}(da_5 + F_0B_5 - f_2G^6)\wedge *(da_5 + F_0B_5 - f_2G^6)$$
$$-\frac{1}{2}e^{-2\phi+2\eta}db_{12}\wedge *db_{12} - \frac{1}{2}e^{-2\phi+\rho_1+\rho_2}H^{I-J-}db_{I-}\wedge *db_{J-}$$
$$-\frac{1}{2}\tau_{i3}H^{I-J-}(\mathcal{D}c_{5I} - a_5db_I)\wedge *(\mathcal{D}c_{5J} - a_5db_J)$$
$$-\frac{1}{2\tau_{i3}}H^{I+J+}dc_{6I+}\wedge *dc_{6J+}\Big]$$

$$S_{\rm pot} = -\frac{1}{4\kappa^2}\int \big[e^{-2\phi+\eta+\rho_1+\rho_2}\tau_{i3}H^{I+J+}(T^K_{5I}b_K + h_{pI})(T^K_{5J}b_K + h_{pJ}) + e^{\eta-\rho_1-\rho_2}(f_2 + F_0b_{12})^2$$
$$+e^{-\eta}H^{I-J-}(f_{mI} + F_0b_I)(f_{mJ} + F_0b_J) + e^{-\eta+\rho_1+\rho_2}\left(f_4 + b_{I-}(f_m^{I-} + \frac{1}{2}F_0b^{I-})\right)^2$$
$$+e^{\eta}H^{I-J-}(-\eta_{IP}T^P_{5K}c^K + b_If_2 + b_{12}(f_{mI} + b_IF_0) + f_{m12I})$$
$$(-\eta_{JP}T^P_{5K}c^K_6 + b_Jf_2 + b_{12}(f_{mJ} + b_JF_0) + f_{m12J})$$
$$-\frac{1}{4}e^{-2\phi+\eta}\tau_{i3}[H,T_5]^I{}_J[H,T5]^J{}_I + e^{-\eta-\rho_1-\rho_2}F_0^2$$
$$+e^{\eta+\rho_1+\rho_2}\left(b_I\left(f^I_{m12} + b_{12}f^I_m - T^I_{5J}c^J_6 + \frac{1}{2}b^I(f_2 + F_0b_{12})\right) + b_{12}f_4 - h_{pI}c^I_6\right)^2\Bigg] * 1$$

We defined :

$$\mathcal{D}c_{5I} = dc_{5I} + \eta_{KI}T^K_{5L}C^L - (f_2b_I + f_{m12I})G^6 + (f_{mI} + F_0b_I)B_5$$

and $\phi = \Phi + \frac{1}{2}(\eta + \rho)$. Note also that the H in $[H,T_5]$ is $H^I{}_J = H^{IK}\eta_{KJ}$.

Note that we have 21 scalars : $(\eta, \phi, \tau_{i3}, c_{6I}, \rho_1, \rho_2, b_{12}, a_5, \tau_{r1}, \tau_{r2}, \tau_{i1}, \tau_{i2}, b_I, c_{5I})$, 4 vectors : (B_5, G^6, C_I) and 1 tensor (C_2).

Since we are in four dimensions, we can dualize the tensor C_2 into a scalar γ_5. We do this dualization and go to Einstein frame in order to obtain :

$$S_{\rm Vkin} = -\frac{1}{4\kappa^2}\int \big[e^{-2\phi-\eta}\tau_{i3}dG^6\wedge *dG^6 + e^{-2\phi+\eta}\tau_{i3}(dB_5 + b_{12}dG^6)\wedge *(dB_5 + b_{12}dG^6)$$
$$+e^{-\eta}H^{IJ}(dC_I - c_{6I}dG^6)\wedge *(dC_J - c_{6J}dG^6)\big]$$

$$S_{\rm top} = -\frac{1}{4\kappa^2}\int -2(dB_5 + b_{12}dG^6)c_{6I}(dC^I - \frac{1}{2}c^I_6dG^6) + b_{12}dC^IdC_I$$

$$
\begin{aligned}
S_{\text{Skin}} = \frac{1}{2\kappa^2}\int \Big[& R*1 - 2d\phi\wedge *d\phi - \frac{1}{2}d\eta\wedge *d\eta - \frac{1}{2\tau_{i3}^2}d\tau_{i3}\wedge *d\tau_{i3} \\
& -\frac{1}{2}d(\rho_1)\wedge *d(\rho_1) - \frac{1}{2}d(\rho_2)\wedge *d(\rho_2) - \frac{1}{2\tau_{i1}^2}d\tau_{i1}\wedge *d\tau_{i1} \\
& -\frac{1}{2\tau_{i1}^2}d\tau_{r1}\wedge *d\tau_{r1} - \frac{1}{2\tau_{i2}^2}d\tau_{i2}\wedge *d\tau_{i2} - \frac{1}{2\tau_{i2}^2}d\tau_{r2}\wedge *d\tau_{r2} \\
& -\frac{1}{2}e^{2\phi-\rho_1-\rho_2}\tau_{i3}(da_5 + F_0B_5 - f_2G^6)\wedge *(da_5 + F_0B_5 - f_2G^6) \\
& -\frac{1}{2}e^{2\eta}db_{12}\wedge *db_{12} - \frac{1}{2}e^{\rho_1+\rho_2}H^{I-J-}db_{I-}\wedge *db_{J-} \\
& -\frac{e^{2\phi}}{2}\tau_{i3}H^{I-J-}(\mathcal{D}c_{5I} - a_5db_I)\wedge *(\mathcal{D}c_{5J} - a_5db_J) \\
& -\frac{1}{2}\tau_{i3}e^{2\phi+\rho_1+\rho_2}(\mathcal{D}\gamma_5 + b^{I-}\mathcal{D}c_{5I-}).\wedge *(\mathcal{D}\gamma_5 + b^{J-}\mathcal{D}c_{5J-}) \\
& -\frac{e^{2\phi}}{2\tau_{i3}}H^{I+J+}dc_{6I+}\wedge *dc_{6J+}\Big]
\end{aligned}
$$

$$
\begin{aligned}
S_{\text{pot}} = -\frac{1}{4\kappa^2}\int \big[& e^{2\phi+\eta+\rho_1+\rho_2}\tau_{i3}H^{I+J+}(T_{5I}^{K}b_K + h_{pI})(T_{5J}^{K}b_K + h_{pJ}) + e^{4\phi+\eta-\rho_1-\rho_2}(f_2 + F_0b_{12})^2 \\
& + e^{4\phi-\eta}H^{I-J-}(f_{mI} + F_0b_I)(f_{mJ} + F_0b_J) + e^{4\phi-\eta+\rho_1+\rho_2}\left(f_4 + b_{I-}(f_m^{I-} + \frac{1}{2}F_0b^{I-})\right)^2 \\
& + e^{4\phi+\eta}H^{I-J-}(-\eta_{IP}T_{5K}^{P}c^K + b_If_2 + b_{12}(f_{mI} + b_IF_0) + f_{m12I}) \\
& \qquad (-\eta_{JP}T_{5K}^{P}c_6^K + b_Jf_2 + b_{12}(f_{mJ} + b_JF_0) + f_{m12J}) \\
& -\frac{1}{4}e^{2\phi+\eta}\tau_{i3}[H,T_5]^I{}_J[H,T5]^J{}_I + e^{4\phi-\eta-\rho_1-\rho_2}F_0^2 \\
& + e^{4\phi+\eta+\rho_1+\rho_2}\left(b_I\left(f_{m12}^I + b_{12}f_m^I - T_{5J}^{I}c_6^J + \frac{1}{2}b^I(f_2 + F_0b_{12})\right) + b_{12}f_4 - h_{pI}c_6^I\right)^2\Big]*1
\end{aligned}
$$

with :

$$
\mathcal{D}\gamma_5 = d\gamma_5 + \frac{1}{2}b_Ib^I(f_2G^6 - F_0B_5) + 2C_Ih_p^I - 2B_5f_4
$$

We now have 22 scalars : $(\eta, \phi, \tau_{i3}, c_{6I}, \rho_1, \rho_2, b_{12}, a_5, \gamma_5, \tau_{r1}, \tau_{r2}, \tau_{i1}, \tau_{i2}, b_I, c_{5I})$ and 4 vectors : (B_5, G^6, C_I).

A.4 Sources

In this section we perform the reduction of the source terms as they appear in [5]. We perform the reduction both in the supersymmetric and non-supersymmetric case. In both cases we look for solutions, and in the susy case we can see that the solution we find correspond with the ten-dimensional counterpart found in [5].

A.4.1 Structure

We take the following $SU(2)$ structure :

$$
\begin{aligned}
z =& \sqrt{\frac{e^{-\eta}}{\tau_{i3}}} e^5 - i\sqrt{e^{-\eta}\tau_{i3}} e^6 \\
j =& e^{-\rho_1} e^{14} + e^{-\rho_2} e^{23} \\
\omega =& \frac{e^{-\frac{1}{2}(\rho_1+\rho_2)}}{\sqrt{\tau_{i1}\tau_{i2}}} (e^{12} - (\tau_{r1} + i\tau_{i1}) e^{24} + (\tau_{r2} + i\tau_{i2})(e^{13} + (\tau_{r1} + i\tau_{i1}) e^{34}))
\end{aligned}
$$

And construct a $SU(3)$ structure thanks to it :

$$
\Omega = (ij + \mathrm{Im}(\omega)) \wedge z \qquad J = \mathrm{Re}(\omega) + i\frac{z \wedge \bar{z}}{2} \tag{A.21}
$$

Then for the corresponding pure spinors we take :

$$
\Phi_+ = \frac{1}{8} e^{-iJ} \qquad \Phi_- = -\frac{i}{8}\Omega \tag{A.22}
$$

A.4.2 SUSY source

We will now consider the source term of the action. For a BPS source, we have :

$$
S_{\text{source}} = \int e^{-\Phi} < j_s, 8\mathrm{Im}\,\Phi_- > *1 \tag{A.23}
$$

with j_s being the source form defined by : $j_s = dF_2 - H \wedge F_0$. After compactification, it becomes :

$$
S_{source} = -4 \int e^{-\phi + \frac{1}{2}(\eta+\rho_1+\rho_2)} < j_s, \mathrm{Im}(\Phi_-) > *1
$$

We now have the entire action. We checked the equivalency of the 10d EOM and of the 4d EOM. Using this 4d action, we also recover the SUSY solution of [5] by setting :

$$
\begin{aligned}
g_{\mu\nu} =& \mathrm{diag}(-1, 1, 1, 1) \\
g_{ij} =& \mathrm{diag}(t_1, t_2\tau_3^2, t_2\tau_3^2, t_1, t_3, t_3\tau_6^2) \\
\varphi =& 0 \\
H =& 0 \\
F_0 =& 0 \\
F_2 =& \frac{1}{\sqrt{t_3}} \left((q_1 t_1 - q_2 t_2 \tau_3^2)(e^{34} - e^{12}) + p(q_1 t_1 - q_2 t_2 \tau_3^2)(e^{24} + e^{13}) \right) \\
F_4 =& 0
\end{aligned}
$$

A.4.3 SUSY breaking sources

We now modify the source term to :

$$
\begin{aligned}
S_{\text{source}} =& -\frac{1}{2\kappa^2} \int e^{-\varphi} < 2\kappa^2 T_p j_{\text{source}}, \mathrm{Im}(X) > \text{ with} \\
X =& \sqrt{|g_4|} d^4x \wedge X_6 \\
X_6 =& \alpha_0 \Phi_- + \tilde{\alpha}_0 \bar{\Phi}_- + \alpha_{mn} \gamma^m \Phi_- \gamma^n + \tilde{\alpha}_{mn} \gamma^m \bar{\Phi}_- \gamma^n
\end{aligned}
$$

Moreover, we will take α_{mn} and $\tilde{\alpha}_{mn}$ symmetric. We define

$$a_0^r = \mathrm{Re}\,(\alpha_0 - \tilde{\alpha}_0) \qquad a_{mn}^r = \mathrm{Re}\,(\alpha_{mn} - \tilde{\alpha}_{mn})$$
$$a_0^i = \mathrm{Im}\,(\alpha_0 + \tilde{\alpha}_0) \qquad a_{mn}^i = \mathrm{Im}\,(\alpha_{mn} + \tilde{\alpha}_{mn})$$

so that $\mathrm{Im}\,(X_6) = a_0^i \mathrm{Re}\,(\Phi_-) + a_0^r \mathrm{Im}\,(\Phi_-) + a_{mn}^r \gamma^m \mathrm{Im}\,(\Phi_-)\gamma^n + a_{mn}^i \gamma^m \mathrm{Re}\,(\Phi_-)\gamma^n$.

Then one can show that One can show that setting :

$$
\begin{aligned}
g_{ij} &= \mathrm{diag}(t_1, t_2\tau_3^2, t_2\tau_3^2, t_1, t_3, t_3\tau_6^2) & F_0 &= 0 & F_4 &= 0\\
g_{\mu\nu} &= \mathrm{diag}(-1 + \frac{1}{3}\Lambda x_2^2, \frac{1}{1-\frac{1}{3}\Lambda x_2^2}, x_2^2, \sin[x_3]^2 x_2^2) & \varphi &= 0 & H &= 0\\
F_2 &= \frac{2pq_2t_2\gamma\tau_3^2}{\sqrt{t_3}}(e^{13}+e^{24}) & a_{34}^r &= 0 & a_{24}^r &= 0\\
q_1 &= \frac{q_2t_2\tau_3^2}{t_1} & a_{12}^r &= 0 & a_{13}^r &= 0\\
b_i^- &= 0 & a_{13}^i &= 0 & a_{34}^i &= 0\\
a_{66}^r &= -\frac{\tau_6^4(t_3^2(1-\gamma^2) + 3k_{\|}\gamma a_{55}^r)}{3k_{\|}\gamma} & a_{24}^i &= 0 & a_{12}^i &= 0\\
a_{56}^i &= \frac{\tau_6(5t_3^2(-1+\gamma^2) - 12k_{\|}\gamma a_{55}^r)}{12k_{\|}\gamma} & a_0^r &= \frac{3+\gamma^2}{4k_{\|}\gamma} & &\\
\Lambda &= \frac{2p^2q_2^2t_2(1-\gamma^2)\tau_3^2}{3t_1t_3} & a_{44}^r &= -a_{11}^r & a_{22}^r &= -a_{33}^r
\end{aligned}
$$

is a solution of the 10D and 4D equations of motion. If one takes $\gamma^2 < 1$ then we have a dS vacuum. The study of the stability of this solution is still in progress.

There are still some issues to adress. Indeed, we would like to better comprehend the compactified action with the modified source X_- and in particular embed it in a $\mathcal{N} = 1$ or $\mathcal{N} = 2$ SUSY framework similar to those in [71, 72]. We would also like to give a physical interpretation to the modified source X_- and what object could give rise to it. Last, we would like to verify the stability of the de Sitter solution in order to be able to claim that we have a genuine de Sitter compactification.

Appendix B

Résumé long en français

B.1 Structures sur les variétés

B.1.1 Groupes de structures

On dit qu'une variété M de dimension $d = 2n$ admet un groupe de structure G si l'ensemble des fonctions de transitions est dans le groupe G. Pour une variété Riemanienne (ie munie d'une métrique), ce groupe est a priori $GL(d)$. Dans certains cas, ce groupe peut être strictement inclus dans $GL(d)$ on parle alors de structure de groupe réduite. Nous allons maintenant en voir quelques examples.

B.1.1.1 Structure complexe

Soit une variété M de dimension $d = 2n$. Une structure presque complexe est un tenseur globalement defini :

$$I \ : \ T \to T \tag{B.1}$$

$$x^m \to I^m_n x^n \, , \tag{B.2}$$

tel que

$$I_m{}^p I_p{}^n = -\delta_m{}^n. \tag{B.3}$$

Le groupe de structure est alors réduit à $GL(\frac{d}{2}, \mathbb{C})$. On dit que cette structure presque complexe est intégrable si le tenseur de Nijenhuis :

$$N_{mn}{}^p = I_m{}^q \left(\partial_q I_n{}^p - \partial_n I_q{}^p\right) - I_n{}^q \left(\partial_q I_m{}^p - \partial_m I_q{}^p\right) \tag{B.4}$$

est nul. On parle alors de structure complexe.

B.1.1.2 Structure Symplectique

Soit une variété M de dimension $d = 2n$. Elle admet une structure presque symplectique s'il existe une deux-forme globalement définie. Le groupe de structure est alors réduit à $Sp(d, \mathbb{R})$. Si de plus, cette deux-forme est fermée, on dit qu'elle définie une structure symplectique.

B.1.2 Les cas SU(2) et SU(3)

Nous allons maintenant étudier plus en détail deux structures de groupes qui vont particulièrement nous interesser : les structures $SU(3)$ et $SU(2)$ sur des variétés des dimension 6.

B.1.2.1 Structure SU(3)

On peut définir une structure $SU(3)$ sur une variété de dimension 6 à l'aide de deux objets : une deux-forme réelle J et une trois-forme complexe Ω vérifiant :

$$J \wedge \Omega = 0 \qquad J \wedge J \wedge J = \frac{3i}{4} \Omega \wedge \bar{\Omega}$$

On peut ensuite utiliser la théorie des groupes pour montrer que leur dérivée extérieure peut se mettre sous la forme :

$$\mathrm{d}J = \frac{3}{2} Im(\bar{W}_1 \Omega) + W_4 \wedge J + W_3 \tag{B.5}$$

$$\mathrm{d}\Omega = W_1 \, J \wedge J + W_2 \wedge J + \bar{W}_5 \wedge \Omega \tag{B.6}$$

Où $\mathcal{W}_1$ est un scalaire complexe, $\mathcal{W}_2$ est une forme complexe $(1,1)$ primitive (ie $J \wedge J \wedge \mathcal{W}_2 = 0$), $\mathcal{W}_3$ est une forme réelle $(2,1)+(1,2)$ primitive (ie $J \wedge \mathcal{W}_3 = 0$), $\mathcal{W}_4$ est un vecteur réel et $\mathcal{W}_5$ est une une-forme complexe. Les $\mathcal{W}$ sont appelés classes de torsion. Si elles sont toutes nulles, on dit que la variété est une variété de Calabi-Yau.

B.1.2.2 Structure SU(2)

On peut définir une structure $SU(2)$ sur une variété de dimension 6 à l'aide de troix objets : une une-forme complexe z, une deux-forme réelle j et une deux-forme complexe ω vérifiant :

$$\omega \wedge j = \omega \wedge \omega = 0 \, , \tag{B.7a}$$

$$\iota_z j = \iota_z \omega = 0 \, , \tag{B.7b}$$

$$\omega \wedge \bar{\omega} = 2 j^2 \, , \tag{B.7c}$$

Comme pour le cas $SU(3)$, on peut décomposer les dérivées extérieures comme suit :

$$\begin{aligned} \mathrm{d}z &= S_1 \omega + S_2 j + S_3 z \wedge \bar{z} + S_4 \bar{\omega} + z \wedge (V_1 + \bar{V}_2) + \bar{z} \wedge (V_3 + \bar{V}_4) + T_1 \, , \\ \mathrm{d}j &= S_5 \bar{z} \wedge \omega + S_6 z \wedge \omega + \frac{1}{2}(S_7 + \bar{S}_8) z \wedge j + j \wedge V_5 + z \wedge \bar{z} \wedge V_6 + z \wedge T_2 + \text{c.c.} \, , \\ \mathrm{d}\omega &= S_7 z \wedge \omega + S_8 \bar{z} \wedge \omega - 2\bar{S}_5 z \wedge j - 2\bar{S}_6 \bar{z} \wedge j + i z \wedge \bar{z} \wedge (\bar{V}_6 \llcorner \omega) + j \wedge (V_7 + \bar{V}_8) \\ &\quad + z \wedge T_3 + \bar{z} \wedge T_4 \, , \end{aligned} \tag{B.8}$$

où S_i est un scalaire complexe, V_i est un vecteur holomorphe ($\omega \wedge V_i = 0$) et T_i est un forme $(1,1)$ primitive (ie $j \wedge T_i = 0$).

B.2 Géométrie complexe généralisée

Afin d'étudier des compactifications, nous allons utiliser le formalisme de la géométrie complexe généralisée (GCG) qui permer de simplifier les équations. Le principe de base de la GCG est de considérer la somme de l'espace tangent et de l'espace cotangent et de généraliser toutes les structures que l'on définit habituellement sur l'espace tangent.

B.2.1 Structure complexe généralisée

Dans la géométrie complexe généralisée, on traite de la même manière les vecteurs et les une-forme. Les sections sont appelés vecteurs généralisés X :

$$X = (v + \xi) \in T(M_6) \oplus T^*(M_6) \tag{B.9}$$

Il existe alors une métrique naturelle invariante η sous $O(d, d)$ donnée par :

$$\eta(X_1, X_2) = (v_1 + \xi_1, v_2 + \xi_2) = \frac{1}{2}(\xi_1(v_2) + \xi_2(v_1)). \tag{B.10}$$

qui réduit le groupe de structure à $O(d, d)$.
Une structure presque complexe généralisée est une fonction $\mathcal{J}$:

$$\mathcal{J} \quad : \quad \mathcal{T} \oplus \mathcal{T}^* \to \mathcal{T} \oplus \mathcal{T}^* \tag{B.11}$$

telle que

$$\mathcal{J}^2 = -\mathbb{I}_{2d}\,, \tag{B.12}$$
$$\mathcal{J}^t \eta \mathcal{J} = \eta\,. \tag{B.13}$$

Celle-ci réduit le groupe de structure total à $U(n, n)$. On peut montrer que chaque sous espace propre de $\mathcal{J}$ est de dimension n et on note Π (reps. $\bar{\Pi}$) le projecteur sur le sous espace de valeur propre i (resp. $-i$). On peut également montrer que ces sous espaces propres $\mathcal{L}_{\mathcal{J}}$ et $\mathcal{L}_{\bar{\mathcal{J}}}$ sont maximalement isotropiques. On dit qu'un espace $\mathcal{L}$ est isotropique si $\forall X, Y \in \mathcal{L}, (X, Y) = 0$. On peut également montrer que sa dimension maximale est d et que dans ce cas, on dit que $\mathcal{L}$ est maximalement isotropique.

On veut maintenant définir l'analogue de l'intégrabilité sur cet espace. Pour cela, nous devons d'abord définir le crochet de Courant qui est une généralisation du crochet de Lie (qui agit sur les vecteurs) à $T(M_6) \oplus T^*(M_6)$:

$$[v + \xi, w + \eta]_C = [v, w] + \mathcal{L}_v \eta - \mathcal{L}_w \xi - \tfrac{1}{2}\mathrm{d}\left(\iota_v \eta - \iota_w \xi\right)\,, \tag{B.14}$$

où $[v, w]$ est le crochet de Lie habituelle et $\mathcal{L}_v$ est la dérivée de Lie définie par $\mathcal{L}_v = \iota_v \mathrm{d} + \mathrm{d}\iota_v$. On peut remarquer que ce crochet de Courant ne vérifie pas l'identité de Jacobi sur $T(M_6) \oplus T^*(M_6)$ (en fait, il n'existe pas de crochet sur cet espace vérifiant Jacobi) mais il la vérifie sur chacun des sous espaces propres de $\mathcal{J}$.

On définit alors l'intégrabilité de la structure presque complexe généralisée par :

$$\bar{\Pi}\left[\Pi(v + \xi), \Pi(w + \eta)\right]_{\mathrm{C}} = 0\,, \tag{B.15}$$

Les deux examples les plus simples de géométrie complexe généralisée sont :

$$\mathcal{J}_I \equiv \begin{pmatrix} I & 0 \\ 0 & -I^t \end{pmatrix}\,, \qquad \mathcal{J}_J \equiv \begin{pmatrix} 0 & J \\ -J^{-1} & 0 \end{pmatrix} \tag{B.16}$$

où $I = I_m^{\ n}$ vérifie $I^2 = -\mathbb{I}_d$ ie c'est une structure presque complexe standard et où $J = J_{mn}$ est une deux-forme non dégénérée ie c'est une structure presque symplectique standard. On peut facilement montrer que la condition d'intégrabilité de la structure presque complexe généralisée correspond aux conditions d'intégrabilité de la structure presque complexe standard et de la structure presque symplectique standard respectivement. On voit donc que la GCG est une génréralisation de ces deux structures et permet de les regrouper sous un même formalisme.

B.2.2 Spineurs purs

Sur T il y a une bijection entre les structure presque complexe et les spineurs de Weyl. Une propriété analogue existe sur $T \oplus T^*$ entre les structure presque complexe généralisée et les spineurs purs.

Etant donné la métrique η, on peut définir des spineurs de $Spin(d,d)$. L'algèbre de Clifford $Cliff(d,d)$,

$$\{\gamma^m, \gamma_n\} = \delta^m_n \tag{B.17}$$

$$\{\gamma^m, \gamma^n\} = \{\gamma_m, \gamma_n\} = 0 \tag{B.18}$$

a 2 représentation irréductible, une avec une chiralité positive et l'autre avec une chiralité négative$S_\pm(T \oplus T^*)$. Il y a un isomorphisme entre le fibré spinoriel et l'algèbre extérieure de T^*, $\Lambda^\bullet T^*$

$$S_+ \cong \Lambda^{\text{even}} T^* \qquad S_- \cong \Lambda^{\text{odd}} T^* \tag{B.19}$$

telle que la chiralité positive (resp. négative) correspond aux formes paires (resp. impaire). On peut le voir facilement en prenant comme base

$$\gamma^m = \mathrm{d}x^m, \qquad \gamma_m = \iota_m \,. \tag{B.20}$$

et alors l'action d'un élément de l'algèbre de Clifford sur les formes est :

$$(v + \xi) \cdot \Phi = \iota_v \Phi + \xi \wedge \Phi \tag{B.21}$$

pour la représentation spinorielle, il est possible de définir une forme bilinéaire symmétrique. En considérant l'isomorphisme ci-dessus, on peut définir un produit sur l'espace des formes, la paire de Mukai :

$$\langle A, B \rangle \equiv (A \wedge \lambda(B))_d \;, \qquad \lambda(A_p) = (-1)^{\text{Int}[p/2]} A_p \,, \tag{B.22}$$

où les indices d et p dénotent le degré de la forme. En dimension 6, ce produit est antisymétrique. On peut se servir de cette paire de Mukai pour définir la norme d'un spineur par :

$$\langle \Phi, \bar{\Phi} \rangle = -i ||\Phi||^2 \text{vol} \,. \tag{B.23}$$

On définit l'annihilateur d'un spineur Φ comme :

$$L_\Phi = \{v + \zeta \in T \oplus T^* \,|\, (v + \zeta) \cdot \Phi = 0\} \,. \tag{B.24}$$

qui est automatiquement isotropique. On dit qu'un spineur est pur si son annihilateur est maximalement isotropique. La présence d'un spineur pur réduit le groupe de structure à $SU(n,n)$.

Il est alors facile de voir la correspondance entre spineur pur et structure presque complexe généralisé. En effet, étant donné une structure presque complexe $\mathcal{J}$, on peut toujours construire un spineur pur ayant pour annihilateur le sous espace propre de valeur propre i de $\mathcal{J}$. Inversement, étant donné un spineur pur, il est facile de construire une structure presque complexe en la définissant sur ses sous espaces propres, le sous espace propre de valeur propre i correspondant à l'annihilateur du spineur pur. Au niveau de l'intégrabilité, on peut montrer que $\mathcal{J}$ est intégrable si et seulement s'il existe un vecteur v et une une-forme ξ telle que $\mathrm{d}\Phi = (\iota_v + \xi\wedge)\Phi$ où Φ est le pur spineur correspondant à $\mathcal{J}$.

B.3 Supersymétrie

B.3.1 Généralités

Nous allons nous interesser a des compactifications de type II et donc étudier les supergravités de type II qui sont des théories $\mathcal{N}=2$ en dimension 10. Il existe deux types de théories, la IIA et la IIB. Chaque secteur bosonique de ces théories est divisé :

- le secteur Neveu-Schwarz (NS) composé de la métrique g, du dilaton ϕ et de la deux-forme NS H
- le secteur Ramond-Ramond (RR) composé de de p-formes F_p paires pour le type IIA et impaires pour le type IIB

Nous ne nous interesserons pas au secteur fermionique. Nous considérons un espace temps à 10 dimensions qui est un produit d'un espace-temps à quatre dimensions et d'un espace interne à six dimensions avec une constante de "warping" non nulle :

$$\mathrm{d}s_{10}^2 = e^{2A(y)}\mathrm{d}s_4^2 + \mathrm{d}s_6^2 \,. \tag{B.25}$$

Dans la plupart des cas, l'espace-temps sera AdS_4. On peut montrer qu'alors les seuls flux non triviaux sont ceux présents sur la variété interne.

En fait, nous allons encore restreindre les solutions que nous cherchons en nous limitant à celles qui admettent $\mathcal{N}=1$ SUSY en 4 dimensions. Nous pouvons alors utiliser les résultats de [17] pour exprimer les équations de supersymétrie comme des équations différentielles du premier ordre. De plus, [15, 16] ont montré que les équations SUSY et les identités de Bianchi impliquait les équations du mouvement pour les différents champs considérés. Ainsi, au lieu de devoir résoudre des équations différentielles du deuxième ordre à 10 dimensions, on doit résoudre des équations différentielles du premier ordre à 6 dimensions. Nous verrons dans les sections suivantes que, dans notre formalisme, ces mêmes équations deviendrons des équations algébriques.

B.3.2 Géométrie complexe généralisée et compactifications $\mathcal{N}=1$

On peut réexprimer les équations de supersymmétrie en termes de spineurs purs et donc utiliser la géométrie complexe généralisée pour simplifier les équations. Cela se décompose en deux conditions : une condition topologique et une condition différentielle.

La condition topologique est l'existence de deux spineurs purs compatibles (ie les deux structures presque complexe généralisées associées commutent). Le groupe de structure sur $T \oplus T^*$ est alors $SU(3) \times SU(3)$. On peut donner l'expression de ces deux spineurs purs en fonction de formes différentielle définie en (B.7a) :

$$\Phi_- = -\frac{e^A}{8} z \wedge (k_\perp e^{-ij} + ik_\parallel \omega)\,, \tag{B.26}$$

$$\Phi_+ = \frac{e^A e^{i\theta}}{8} e^{z\bar{z}/2}(k_\parallel e^{-ij} - ik_\perp \omega)\,, \tag{B.27}$$

avec $k_\perp^2 + k_\parallel^2 = 1$ et θ un réel. On parle alors de structure $SU(2)$ intermédiaire sur T. Les deux limites $k_\parallel = 0$ et $k_\parallel = 1$ sont intéressantes et correspondent aux structures $SU(2)$ strictes et $SU(3)$ respectivement. Dans ces cas, on a :

- Structure $SU(3)$ sur T

$$\Phi_+ = \frac{e^A e^{i\theta}}{8} e^{-iJ}\,, \tag{B.28}$$
$$\Phi_- = -i\frac{e^A}{8}\Omega\,, \tag{B.29}$$

- Structure $SU(2)$ stricte sur T

$$\Phi_+ = -i\frac{e^A e^{i\theta}}{8}\omega \wedge e^{z\wedge\bar{z}/2}\,, \tag{B.30}$$
$$\Phi_- = -\frac{e^A}{8}e^{-ij} \wedge z\,, \tag{B.31}$$

La condition différentielle consiste en une équation sur chacun des spineurs purs. Pour le type IIB, on a

$$(\mathrm{d} - H\wedge)(e^{2A-\phi}\Phi_-) = -2\mu e^{A-\phi}\mathrm{Re}\Phi_+, \tag{B.32}$$
$$(\mathrm{d} - H\wedge)(e^{A-\phi}\mathrm{Re}\,\Phi_+) = 0\,, \tag{B.33}$$
$$(\mathrm{d} - H\wedge)(e^{3A-\phi}\mathrm{Im}\,\Phi_+) = -3e^{2A-\phi}\mathrm{Im}\,(\bar{\mu}\Phi_-) - \frac{1}{8}e^{4A} * \lambda(F)\,, \tag{B.34}$$

où F est la somme des champs de jauge, $F = F_1 + F_3 + F_5$. μ lié à la constante cosmologique par $\Lambda = -|\mu|^2$. λ a été défini par (B.22). En type IIA les equations de SUSY prennent à peu près la même forme que (B.32)-(B.34) avec Φ_+ et Φ_- échangé :

$$(\mathrm{d} + H\wedge)(e^{2A-\phi}\Phi_+) = 2\mu e^{A-\phi}\mathrm{Re}\Phi_-, \tag{B.35}$$
$$(\mathrm{d} + H\wedge)(e^{A-\phi}\mathrm{Re}\,\Phi_-) = 0\,, \tag{B.36}$$
$$(\mathrm{d} + H\wedge)(e^{3A-\phi}\mathrm{Im}\,\Phi_-) = 3e^{2A-\phi}\mathrm{Im}\,(\bar{\mu}\Phi_+) + \frac{1}{8}e^{4A} * F\,. \tag{B.37}$$

et ici $F = F_0 + F_2 + F_4 + F_6$.

B.4 Etudes des vides AdS sur les variétés parallélisable

B.4.1 Recherche sur les groupes de Lie

On va maintenant utiliser le formalisme que l'on a vu dans les sections précédentes afin de chercher des vides AdS sur les variétés parallélisable.

Pour cela, la première étape est très simple : il suffit d'utiliser B.30 et B.31 dans B.35,B.36 et B.37 (on peut montrer que dans le cas qui va nous intéresser, seul une structure SU(2) stricte donne des résultats) pour obtenir :

$$\mathrm{d}(e^{3A-\phi}z) = 2|\mu|e^{2A-\phi}\omega_I\,, \tag{B.38}$$
$$z \wedge (\mathrm{d}j - iH + |\mu|e^{-A}\,\bar{z} \wedge \omega_R) = 0 \tag{B.39}$$
$$\mathrm{d}(e^{2A-\phi}\omega_I) = 0\,, \tag{B.40}$$
$$\mathrm{d}(e^{2A-\phi}z \wedge \bar{z}\omega_R) = 2ie^{2A-\phi}H \wedge \omega_I \tag{B.41}$$

et

$$
\begin{aligned}
*F_5 &= 3e^{-A-\phi}\,|\mu| z_I\,, &\text{(B.42)}\\
*F_3 &= -e^{-4A}\,\mathrm{d}(e^{4A-\phi}\omega_R) + 3e^{-A-\phi}|\mu| z_R \wedge j\,, &\text{(B.43)}\\
*F_1 &= -i\,\mathrm{d}(2A-\phi) z\wedge \bar{z} \wedge \omega_I - e^{-\phi} H \wedge \omega_R &\\
&\quad + \frac{1}{2} e^{-A-\phi}|\mu| z_I \wedge j \wedge j &\text{(B.44)}
\end{aligned}
$$

Ces équations étant compliquées, on va utiliser quelques hypothèses supplémentaires. En effet, dans la suite nous allons supposer que nous avons un dilaton ϕ constant, un facteur de warping A constant (et donc on peut le prendre nul en le réabsorbant dans AdS) et enfin nous supposerons que toutes les classes de torsion dans le **2** de SU(2) sont nulles (ie $V_i = 0$ dans B.8). Nous nous restreignons également aux variétés parallélisables et même dans un premier temps aux groupes de Lie. Notre but est de faire un scan complet de ces variétés dans notre formalisme pour voir lesquelles admettent un vide AdS. Comme nous considérons des variétés parallelisables, on peut prendre l'ansatz suivant pour les formes de structure :

$$
\begin{aligned}
z &= z_1 e^1 + i z_2 e^2\,,\\
j &= j_1 e^{36} + j_2 e^{45}\,,\\
\omega_R &= \frac{j_1 j_2}{\omega_1} e^{34} + \omega_1 e^{56}\,,\\
\omega_I &= -\frac{j_1 j_2}{\omega_2} e^{35} + \omega_2 e^{46}\,,
\end{aligned}
\tag{B.45}
$$

B.4.1.1 Sources

Avant d'effectuer le scan proprement dit, nous allons étudier plus en détail les identités de Bianchi du secteur RR. En effet, on a :

$$
\begin{aligned}
\delta(D_{8-n}/O_{8-n}) = dF_n - H\wedge F_{n-3} &\hat{=} \sum N_{i_1\ldots i_{n+1}} e^{i_1}\wedge \ldots \wedge e^{i_{n+1}}\\
&\hat{=} \sum n_{i_1\ldots i_{n+1}} \mathrm{vol}^{i_1,\ldots,i_{n+1}}
\end{aligned}
$$

où l'on définit $\mathrm{vol}^{i_1,..,i_n}$ comme la forme proportionnelle à $e^{i_1}\wedge \ldots \wedge e^{i_n}$ et normalisée telle que $< \mathrm{vol}^{i_1,\ldots,i_n}, \mathrm{Im}\,\Phi_+ >= -8i < \Phi_+, \bar{\Phi}_+ >$.

$N_{i_1\ldots i_{n+1}}$ est l'analogue de la charge totale et doit être d'ordre 1. $n_{i_1\ldots i_{n+1}}$ est l'analogue de la densité de charge et son signe nous dit qu'elle sorte de source existe dans la direction perpendiculaire. Si $n_{i_1\ldots i_{n+1}}$ est positive alors il y a au moins une D-brane et si elle est négative, il y a au moins un orientifold.

B.4.1.2 Méthode

On commence par résoudre les équations SUSY qui, avec nos hypothèses sont simples à résoudre car linéaires en les classes de torsion. Ensuite nous imposons $d^2 e^i = 0$ sur la base des une-formes. L'ensemble de ces équations est divisé en deux ensembles : les équations qui sont linéaires et celles qui sont quadratiques. Heureusement, les équations linéaires permettent automatiquement de résoudre les quadratiques. C'est pour cette raisons que nous nous somme restreints dans un premier temps aux groupes de Lie.

Ensuite nous regardons ce que nous donne l'identité de Bianchi pour la une-forme F_1 :

$$n_{(D7/O7)35} = n_{(D7/O7)46} = -\frac{5e^{-3A-\phi}}{4}|\mu|^2 < 0\,, \tag{B.46}$$

On a donc nécéssairement présence de deux orientifold O7. En particulier, on ne peut pas avoir de solutions sans source. De plus, comme ils s'intersectent, nous sommes obligés de les prendre "smeared" sur tout l'espace interne. Mais alors l'involution des orientifold nous impose plus de contraintes sur nos equations en réduisant l'espace des solutions.

C'est cette réduction qui nous permet de continuer dans cette voie. En effet, on s'interesse maintenant à l'identité de Bianchi NS $dH = 0$. Celle-ci est quadratique mais il existe maintenant suffisament peu de solutions qu'on peut les considérer une par une. Dans chacun des cas on regarde également l'identité de Bianchi pour la tois-forme F_3 et on s'apercoit que l'on doit nécéssairement avoir au moins un orientifold O5 qui impose de nouvelles contraintes qui réduisent encore plus l'espace des solutions.

Ces réductions successives de l'espace des solutions nous permettent d'obtenir une liste exhaustive des groupes de Lie admettant un vide SUSY AdS :

Nom	Type de variété	algèbre
Nil 3.13	nilpotente	$(-35-46,0,0,0,23,-24)$
Nil 3.14	nilpotente	$(-35+46,0,0,0,23,-24)$
Nil 4.1	nilpotente	$(-35-46,0,-25,0,0,0)$
Nil 5.1	nilpotente	$(35+46,0,0,0,0,0)$
$\mathfrak{g}_{92}^{6*}$	résoluble	$(35+\epsilon 46,0,25,-\epsilon 26,-\epsilon 23,24)$

B.4.2 Quotients

On peut effectuer la même étude sur les quotients qui admettent une structure SU(2). C'est même plus facile car il existe une liste de tous les quotients de ce type (donnée par exempe dans [46]). Il suffit alors de vérifier si ces variétés admettent un vide SUSY AdS et on obtient :

Quotient G/H	SUSY AdS$_4$ solution
$\frac{SU(3)}{SU(2)} \times U(1)$	✓
$\frac{SU(2)^2}{U(1)} \times U(1)$	✓
$SU(2) \times SU(2)$	×
$SU(2) \times U(1)^3$	×

B.4.3 Séparation d'échelle

Nous voulons également vérifier si nous pouvons obtenir un vrai vide en quatre dimensions sur ces variétés. Nous devons donc vérifier en particulier que les approximations que nous utilisons sont correctes. Mais d'abord, comme nous l'avons dit section B.4.1.1, on doit vérifier que les charges totales des sources sont d'ordre 1. Nous devons également vérifier que les corrections à une boucle des cordes sont petites c'est à dire que le dilaton doit être petit devant 1. De même, il faut que les corrections en α' soient petites c'est à dire que le volume de la variété interne (nous prenons $\text{vol}_6 = \sqrt{\det(g_6)} = L_{int}^6$) doit être grand et que la constante cosmologique doit être petite.

Enfin, nous voulons que les modes massifs de Kaluza-Klein découplent. Pour cela, il faut que l'échelle typique de Kaluza-Klein soit petite devant la longueur typique d'AdS. Il y a alors deux choix naturels pour la longueur de Kaluza-Klein L_{KK}. Le premier (que nous noterons définition (1)) est de supposer qu'elle est égale à la longueur typique de la variété interne L_{int}. Le deuxième choix (que nous noterons définition (2)) est de supposer que $L_{KK}^2 = \frac{1}{R_6}$. On trouve les deux définitions dans la littérature et nous allons étudier les deux. Bien sûr, pour pouvoir réellemenent prétendre avoir une telle séparation d'échelle, on devrait calculer le spectre de Kaluza-Klein à partir de la théorie quatre dimensionnelle, ce qui requiert une autre étude que nous ne ferons pas ici. Dans le tableau suivant, nous donnons les résultats pour les variétés étudiées plus haut (sauf pour n3.13 dont létude est similaire à n3.14) dans la limite où la constante cosmologique est petite :

variété	couplage faible	volume large	séparation (1)	séparation (2)
$\frac{SU(3)}{SU(2)} \times U(1)$	×	✓	✓	×
$\frac{SU(2)^2}{U(1)} \times U(1)$	×	✓	✓	×
Nil 3.14	✓	×	✓	✓
Nil 4.1	✓	✓	✓	✓
Nil 5.1	✓	✓	✓	×
Solv	✓	×	✓	✓

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