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DIVERGENCE CONDITIONS FOR VECTOR AND AXIAL VECTOR CURRENTS *

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ABSTRACT

Equations for the divergence of the vector and axial-vector currents follow from the assumptions of Lorentz invariance, locality, the chiral SU(3) × SU(3) algebra of current densities (time components), and the usual electromagnetic and weak Hamiltonians. The divergence conditions lead to derivations of the low-energy meson theorems which do not involve "Schwinger" terms.

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During the past few years, there have been several spectacular successes in the area of low-energy pion theorems brought about by the application of current algebra[1] and PCAC.[2] Some of these are the Adler-Weisberger sum rule for G_A/G_V,[3] the relations of the $K\ell_3$ and $K\ell_4$ decays to the $K\ell_2$ decays,[4,5] the pion-nucleon S-wave scattering lengths,[6] the relations between $K \rightarrow 3\pi$ and $K \rightarrow 2\pi$ decays,[7] the S-wave hyperon non-leptonic decay amplitudes,[8] etc. In all these examples, the derivations proceed by considering the S-matrix elements in which at least one pion has been "reduced" out, whereupon the application of the PCAC hypothesis and the zero energy limit leads to the equal time commutators (ETC) among current densities. In general, the ETC between current densities can involve more complicated structure than simple δ-functions, as pointed out by Schwinger.[9] The existence of these more complicated terms may nevertheless not play any role as to the validity of the low-energy theorems. This is because these theorems actually involve only the commutators of a charge with a current density, which is of a more restricted nature than the commutator for two densities.

In this note, we show that Lorentz invariance, locality and the SU(3) × SU(3) algebra of current densities (time components), coupled with PCAC and the usual electromagnetic and weak Hamiltonians, allow divergence equations for vector and axial-vector currents V_μ and A_μ of the form[10]

$$\text{(1a)} \quad \partial^\mu V_\mu^\alpha = -ieb^\alpha a^\mu V_\mu^\alpha + GL^{\mu\gamma} C_{\beta\gamma}^\alpha \left(V_\mu^\beta - A_\mu^\beta\right)$$

$$\text{(1b)} \quad \partial^\mu A_\mu^\alpha = a\pi^\alpha - ieb^\alpha a^\mu A_\mu^\alpha - GL^{\mu\gamma} C_{\beta\gamma}^\alpha \left(V_\mu^\beta - A_\mu^\beta\right)$$

where the indices α, β, γ refer to internal degrees of freedom (γ is a charge index; for α, β, see Ref. 10), a_μ and L_ν are the electromagnetic field and lepton current, respectively, b^α and $C_{\beta\gamma}^\alpha$ are numerical constants, and π^α the pion fields.

The low energy theorems involving soft pions follow simply from Eq. (1b). For example, consider the decay $K \longrightarrow \pi e \nu$, with the S-matrix element $_{out}\langle \pi e \nu | K \rangle_{in}$. Reducing the pion and replacing the interpolating pion field by

$$\pi^{\alpha} = \frac{1}{a}\left\{\partial^{\mu} A^{\alpha}_{\mu} + G L^{\mu\gamma} C^{\alpha}_{\beta\gamma}\left(V^{\beta}_{\mu} - A^{\beta}_{\mu}\right)\right\}$$

leads immediately to the results of Callan and Treiman[4] in the limit where the pion four-momentum vanishes. All low energy theorems follow in a similar manner.

Equations (1a) - (1b) were postulated by Veltman[11] as independent of current algebra.[12] We show here that those equations follow essentially from the algebra of the j^{α}_{o}. Thus ETC contain more information than do the "divergence conditions" (1a) - (1b), in case of commutators which yield Schwinger terms. The fact that the low energy theorems may be derived from (1a) - (1b) directly shows that the Schwinger terms in the ETC do not affect the low energy results.[13]

Equations (1a) - (1b) are derived in the following manner: Given a current $j^{\alpha}_{\mu}(\vec{x}t)$, we construct its charge $Q^{\alpha}(t)$ as

$$Q^{\alpha}(t) = \int d^3\vec{x}\, j^{\alpha}_{o}(\vec{x}t) \quad . \tag{2}$$

Let $H(\vec{x}t)$ be the energy density. Then

$$\partial^{\mu} j^{\alpha}_{\mu}(\vec{x}t) = i\left[H(\vec{x}t), Q^{\alpha}(t)\right] + \sum_{\substack{n \geq 2 \\ m}} \partial^{r_1} \cdots \partial^{r_n} R^{\alpha}_{r_1 \cdots r_n \underbrace{0 \ldots 0}_{m}}(\vec{x}t) \tag{3}$$

where r_i are 1, 2, 3 (space) indices. This form follows from space-rotation invariance and the requirement that

$$\partial^{o} Q^{\alpha}(t) = i\left[H, Q^{\alpha}(t)\right] \tag{4}$$

where H is the total Hamiltonian.[14] The fact that the summation on the right-hand side of Eq. (3) starts from n = 2 follows from the vector transformation property of $j_\mu^\alpha(\vec{x}t)$ under Lorentz transformations (the generator of a Lorentz transformation in the K direction is $M_{oK} = tP_K - \int d^3\vec{x}\, x_K\, H(\vec{x}t)$). The summation is usually over a finite number of terms.[15]

Suppose now that

(5) $$H(\vec{x}t) = H_o(\vec{x}t) + H_1(\vec{x}t)$$

such that

(6) $$\left[H_o(t),\, Q^\alpha(t)\right] = 0$$

where $H_o(t) = \int d^3\vec{x}\, H_o(\vec{x}t)$. Then, with Eq. (3),

(7) $$\partial^\mu j_\mu^\alpha(\vec{x}t) = i\left[H_1(\vec{x}t), Q^\alpha(t)\right] + \sum_{\substack{m \\ n \geq 1}} \partial^{r_1} \ldots \partial^{r_n} \tilde{R}^\alpha_{r_1 \ldots r_n \underbrace{0 \ldots 0}_{m}}(\vec{x}t)$$

We now show how to get the "divergence equations" essentially from the algebra of the j_o^α. Let us start with the case where

(8) $$H_1(\vec{x}t) = H^{e.m.}(\vec{x}t) = e\, j_\mu^{e.m.}(\vec{x}t)\, a^\mu(\vec{x}t)$$

and let $Q^\alpha(t)$ be an axial charge. We neglect for the moment other contributions to H_1. We want to calculate the commutator $\left[H^{e.m.}(\vec{x}t), Q_A^\alpha(t)\right]$. To this end we note that

(9) $$\left[Q_A^\alpha(t),\, a_\mu(\vec{x}t)\right] = 0$$

and

(10) $$\left[Q_A^\alpha(t),\, j_\mu^{e.m.}(\vec{x}t)\right] = b^\alpha A_\mu^\alpha(\vec{x}t) + g_\mu^k \sum_{\substack{n \\ m}} \partial^{r_1} \ldots \partial^{r_n} N_{k\, r_1 \ldots r_n \underbrace{0 \ldots 0}_{m}}(\vec{x}t)$$

This form is dictated by space rotation invariance and by

$$\left[A_o^\alpha(\vec{x}t),\ Q^{e.m.}(t)\right] = b^\alpha A_o^\alpha(\vec{x}t) \tag{11}$$

From Eqs. (7) - (10), we get

$$\begin{aligned}\partial^\mu A_\mu^\alpha(\vec{x}t) &= -ieb^\alpha a^\mu A_\mu^\alpha(\vec{x}t)\\ &+\left\{-iea^k(\vec{x}t)\sum_{\substack{n\\m}}\partial^{r_1}\ldots\partial^{r_n}N^\alpha_{kr_1\ldots r_n\underbrace{0\ldots0}_{m}}(\vec{x}t)\right.\\ &\left.+\sum_{\substack{n\geq1\\m}}\partial^{r_1}\ldots\partial^{r_n}\tilde{R}^\alpha_{r_1\ldots r_n\underbrace{0\ldots0}_{m}}(\vec{x}t)\right\}\end{aligned} \tag{12}$$

Since the expression in curly brackets cannot be a Lorentz-scalar field, it has to vanish. Thus

$$\partial^\mu A_\mu^\alpha(\vec{x}t) = -ieb^\alpha a^\mu A_\mu^\alpha(\vec{x}t) \tag{13}$$

as the contribution of electromagnetism to the divergence of the A_μ^α current. Similarly, we can calculate the contribution from the weak Hamiltonian, adding to $H_1(\vec{x}t)$ of Eq. (8) a term $\frac{G}{\sqrt{2}}$ (lepton current) $\times$ (hadron current), with the hadron current given as in Ref. 16, and assuming that the hadron charges commute with the lepton current (analogous to Eq. (9) for electromagnetism).[17] The term $a\pi^\alpha$ in the expression for $\partial^\mu A_\mu^\alpha$ is due to the PCAC hypothesis.[2]

Finally, we may further note that each of the two terms in the curly brackets in Eq. (12) must vanish, due to the fact that one is a total divergence, while the other is not. This in turn implies that the Schwinger terms in the commutator $\left[A_o^\alpha(\vec{x}t),\ j_K^{e.m.}(\vec{y}t)\right]$ vanish after the $\vec{x}$ integration, as follows from Eq. (10), and that $\partial^\mu j_\mu^\alpha(\vec{x}t) = i\left[H_1(\vec{x}t),\ Q^\alpha(t)\right]$, as follows from Eq. (7).

REFERENCES AND FOOTNOTES

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4. C. G. Callan and S. B. Treiman, Phys. Rev. Letters 16, 153 (1966).
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8. H. Sugawara, Phys. Rev. Letters 15, 870, 997 (1965); M. Suzuki, Phys. Rev. Letters 15, 986 (1965).
9. J. Schwinger, Phys. Rev. Letters 3, 296 (1959).

10. We consider the divergences of strangeness conserving ($\Delta s = 0$) currents only. That means that α in Eqs. (1a) - (1b) corresponds to $\Delta s = 0$ only. The index β on the right-hand side, however, includes $\Delta s \neq 0$. If one wants to consider the divergence of a strangeness changing current, one has to include, in the right-hand side, contributions due to the medium-strong Hamiltonian. We also do not consider contributions due to non-leptonic weak decays.

11. M. Veltman, Phys. Rev. Letters 17, 553 (1966).

12. In Ref. 11, these equations were also written with W-meson fields replacing the lepton current. It is not clear from that reference what was the formalism employed for the vector fields. This is of importance in applying the reduction technique. See D. G. Boulware and L. S. Brown, Phys. Rev. 156, 1724 (1967).

13. M. Nauenberg (Phys. Rev. 154, 1455 (1967)) showed that the electromagnetic contributions to the divergence equations imply commutation relations of the vector charge density with vector and axial currents, with certain Schwinger terms. D. G. Boulware and L. S. Brown (Phys. Rev. 156, 1724 (1967)) showed that the weak contributions to the divergence equations, with W-meson fields replacing the lepton currents, lead to commutation relations between vector and axial-vector charge densities with all currents, with certain Schwinger terms.

14. We assume that surface terms at spacial infinity may be neglected, when forming matrix elements.

15. When an infinite number appears, certain relations among the various terms have to hold, in order not to spoil causality.

16. N. Cabibbo, Phys. Rev. Letters 10, 531 (1963).

17. When W-meson fields are used instead of lepton currents, this no longer holds in general. For example, in a non-abelian gauge formalism (T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters 18, 1029 (1967)), we have

$$\left[j_0^{\alpha}(\vec{x}t),\ W_0^{\beta}(\vec{y}t)\right] = i\,C^{\alpha\beta\gamma}\,\frac{G}{m_0^2}\,j_0^{\gamma}(\vec{x}t)\,\delta^3(\vec{x}-\vec{y}),$$

where m_0 is the bare mass of the W-meson. However, this introduces G^2 terms in the divergence equations (1a) and (1b), and does not affect lowest order results.