Wahlquist metric revisited

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Abstract. Here we continue studying the Wahlquist metric. We know that the wave equation written for a zero mass scalar particle in the background of this metric gives Heun type solutions. To be able to use the existing literature on Heun functions, we try to put our wave equation to the standard form for these functions. Then we calculate the reflection coefficient of a wave coming from infinity and scattered at the center using this formalism.

1. Introduction

The work on Heun class solutions of wave equations written for particles in the background metric of different metrics is still popular. One of the most recent papers is given in [1]. Here we study a similar problem for a metric, which was given a while ago.

The Wahlquist metric [2, 3, 4] is an exact interior solution for a finite rotating body of perfect fluid to Einstein's field equations with equation of state corresponding to constant gravitational mass density. It is also given in the celebrated book *Exact Solutions to Einstein's Field Equations* [5]. This metric is axially symmetric, stationary and is type D. As stated in [6]", quoting Wahlquist [2], it can be "described as a superposition of a Kerr-NUT metric [7, 8] and is a rigidly rotation perfect fluid in the same space-time region". Equation of the state for the p perfect fluid is $\rho + 3p = \mu_0$, a constant.

The original metric written in [2], was slightly modified by Senovilla [9, 10], and put to new form by Mars [11], "to show that the Kerr-de Sitter and Kerr metrics are contained as sub cases". Mars states in [11] that Kramer [12] showed the vanishing of the Simon tensor [13] for this metric. Mars also states that the space-time admits a Killing tensor as shown in [14]".

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A negative point about this metric is the paper, [15], where it was shown that " the Wahlquist perfect fluid space-time can not be smoothly joined to an exterior asymptotically flat vacuum region since the conditions for matching the induced metrics and extrinsic curvatures are mutually contradictory". Thus, " the Wahlquist metric can not describe an isolated rotating body". Bradley et al., however, showed that this obstruction is removed for a limiting form of the Wahlquist solution [16], and also for every D-type metric [17] if the requirement of an asymptotically flatness is removed.
More recent work in this field exists, "where the existence of a rank-2 generalized closed conformal Killing-Yano tensor with a skew-symmetric torsion " [18], and "the separability of the Maxwell equation on the Wahlquist spacetime" are shown [19].
Here we start with the metric as given in [18] and try to write the wave equation, in the background of this metric, as given in [6], in the standard form given by [20]. How to perform this task is described in [21], as quoted by [22], and used meticulously by [23, 24]. The same method is recently used by Vieira et al in [25] . We need the standard form of the wave equation to be able to apply the information given in the existing literature to our work. We will first summarize our previous work, [6], then show how to convert the wave equation to the standard form. In the third section, we give the approximate reflection coefficient if a wave , coming from infinity is scattered at the origin. We conclude with a few remarks.

2. Summary of former work

Here, we will summarize the work in [6] below. First, using the metric given in [2], we try to calculate ϕ where it obeys the equation

$$\frac{1}{\sqrt{g}}\partial_\mu g^{\mu\nu}\sqrt{g}\partial_\nu\phi = 0. \tag{1}$$

Here g is the determinant of the metric coefficients $g_{\mu\nu}$.
We first write the metric as is given in [18], which is equivalent to the one giving in [11]. We follow some of the work in [18] in the first part, and [6] in the second part of this section.
Here "the comoving, pseudoconfocal, spatial coordinates are used, which are closely related to the oblate-spheroidal coordinates in Euclidean geometry" [2]. We set $4\pi G = c = 1$. Then, the metric reads

$$ds^2 = (v_1+v_2)\big(\frac{dz^2}{U}+\frac{dw^2}{V}\big)+\frac{U}{v_1+v_2}(d\tau+v_2d\sigma)^2-\frac{V}{v_1+v_2}(d\tau-v_1d\sigma)^2 \tag{2}$$

where

$$U = Q_0 + a_1\frac{sinh(2\beta z)}{2\beta} - \frac{\nu_0}{\beta^2}\frac{cosh(2\beta z)-1}{2\beta^2}$$
$$-\frac{\mu_0}{2\beta^2}\big[\frac{cosh(2\beta z)-1}{2\beta^2} - z\frac{sinh(2\beta z)}{2\beta}\big], \tag{3}$$

$$V = Q_0 + a_2 \frac{sin(2\beta w)}{2\beta} + \frac{\nu_0}{\beta^2} \frac{-cos(2\beta w) + 1}{2\beta^2}$$
$$- \frac{\mu_0}{2\beta^2} [\frac{cos(2\beta w) - 1}{2\beta^2} + w \frac{sin(2\beta w)}{2\beta}] \tag{4}$$

and

$$v_1 = \frac{cosh(2\beta z) - 1}{2\beta^2}, v_2 = \frac{-cos(2\beta w) + 1}{2\beta^2}. \tag{5}$$

This metric has six real constants $Q_0, a_1, a_2, \nu_0, \mu_0, \beta$. Here a_1 is related to the NUT [26, 27] parameter; a_2 is related to the mass parameter. One writes the other variables to express the energy density, pressure and the fluid velocity of the perfect fluid. Here β is related to the scaling of z and w, both space coordinates, in the linear transformation of these variables given in the original paper by Wahlquist [2]. U and V are are related to h_2, h_1 in the original metric. All these parameters are scaled so that they do not vanish in the β going to zero limit.
When we use the ansatz

$$\phi = R(z)Y(w)T(\tau)S(\sigma) \tag{6}$$

for the solution, the wave equation written in this metric separates easily. We have two Killing vectors since the metric does not depend on τ and σ, related to t and θ in the original metric [2]. If we make a Wick rotation which changes w to $y = iw$, a_2 to $-ia_2$ [11] where i is the square root of minus unity, the metric becomes symmetrical

$$ds^2 = (v_1 + v_2)(\frac{dz^2}{U} - \frac{dw^2}{V}) + \frac{U}{v_1 + v_2}(d\tau - v_2 d\sigma)^2 - \frac{V}{v_1 + v_2}(d\tau + v_1 d\sigma)^2. \tag{7}$$

Then we have identical equations for z and y. The equation for z reads

$$\partial_z(U\partial_z)R(z) + (\frac{v_1^2}{U}\partial_\tau^2 + 2\frac{v_1}{U}\partial_\tau\partial_\sigma + \frac{1}{U}\partial_\sigma^2)R = 0. \tag{8}$$

We get exactly the same equation for the new variable y, with appropriate changes like U going to V and v_1 going to v_2.

$$\partial_y(V\partial_y)S(y) + (\frac{v_2^2}{V}\partial_\tau^2 + 2\frac{v_2}{V}\partial_\tau\partial_\sigma + \frac{1}{V}\partial_\sigma^2)S(y) = 0. \tag{9}$$

Since the functions v_1, U (similarly v_2, V) don't depend on τ and σ, the solutions for τ and σ are just exponential functions, giving us constants upon differentiation.

Here we will first try to write the differential equation for a zero mass scalar field for the radial coordinate z, in the background of this metric. Then, we make more independent variable changes to put our equation in the the standard form , as given in [20].
We find that this is not an easy task. The presence of hyperbolic sine and cosine functions in the wave equation prevents us from using standard analytical methods in solving differential equations. We change our variables to $x = exp(2\beta z)$ which makes it possible to write the hyperbolic sine and cosine functions in terms of powers. $sinh 2\beta z = 1/2[x - 1/x]$, $cosh 2\beta z - 1 = 1/2[x + 1/x - 2]$. Then, however, exists the relation $z = lnx/(2\beta)$.
The codes we have to solve differential equations analytically, does not recognize lnx. Thus, we can not give an exact solution for all values of the independent variable x. We can get solutions only in one patch, by using an algebraic expression approximating lnx in this region.
Upon the variable change from z to x, we get the new equation

$$\frac{d^2\psi(x)}{dx^2} + \big(\frac{\frac{dU}{dx}}{U} + \frac{1}{x}\big)\frac{d}{dx}\psi(x) - \frac{1}{x^2U^2}\big(\omega^2 v_1^2 + 2v_1\omega s + s^2\big)\psi(x) = 0. \quad (10)$$

Here we used the solution to the τ equation in the form $exp(-i\omega\tau)$ and the solution for the $i\sigma$ equation a solution in the form $exp(-i\sigma s)$. Note that our metric had Killing vectors for both τ and σ. Here

$$U = Q_0 + a_1\frac{(x - 1/x)}{4\beta} - \big[\frac{\mu_0}{4\beta^4} + \frac{\nu_0}{4\beta^3}\big]\frac{(x-1)^2}{x} + \frac{\mu_0}{8\beta^4}\ln x\frac{(x^2-1)}{x}\big], \quad (11)$$

$$v_1 = \frac{(x-1)^2}{4\beta^2 x} \quad (12)$$

,

$$\frac{dU}{dx} = \big[\frac{a_1}{\beta}(1+\frac{1}{x^2}) - (\frac{\nu_0}{\beta^2} + \frac{\mu_0}{\beta^4}(1-\frac{1}{x^2}) + \frac{\mu_0}{8\beta^4 x}(x-\frac{1}{x}) + \frac{\mu_0}{8\beta^4}\ln x(1+\frac{1}{x^2})\big]. \quad (13)$$

From here on, we will define $\mu_0' = \frac{\mu_0}{(8\beta^4)}$ and $\nu_0' = \frac{\nu_0}{(4\beta^3)}$, $a_1' = a_1/\beta$ and use these new constants in our equations.
By studying the equation written in terms of the variable x, we know that there is a singularity near $x = 1$. To study the solution around this singularity, we expand lnx in the neighborhood of this point and use $x - 1$ instead of lnx in the wave equation we use in our calculation, after we differentiate U with respect to x.
We find that, if we keep all our constants non zero, we get [6] the exact solution in terms the general Heun function [20, 28, 29, 30] up to an exponential and terms (powers of polynomials) multiplying this function. The regular singularities are at zero and at three other finite points.

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Then we try the next possible choice. We try to find the solution when we keep all the terms in U aside from Q_0, which we set equal to zero. Then, our equation, again, has four regular singularities at $0, 1$ and the points

$$x_a = -\frac{1}{2\mu_0'}(M - [N]^{1/2}), \tag{14}$$

$$x_b = -\frac{1}{2\mu_0'}(M + [N]^{1/2}), \tag{15}$$

where

$$M = (a_1' - \nu_0' - 2\mu_0'), \tag{16}$$

$$N^2 = M^2 - 4\mu_0'(a_1' + \nu_0' + \mu_0'). \tag{17}$$

The $HeunG$, General Heun form solution is retained when we set Q_0 to zero. To further simplify our expressions, we define $a_1'' = \frac{a_1'}{\mu_0'}, \nu_0'' = \frac{\nu_0'}{\mu_0'}$.

3. Reduction to the standard form

How to reduce an equation with four regular singular points at finite values to an equation with singularities at zero, unity, a finite point and infinity is described in [22]. If the coefficients of the first derivatives satisfy a linear relation, one just has to make a homographic substitution to bring three of the original singularities b_1, b_2, b_4 to zero, unity and infinity. This transformation is

$$u = \frac{x - b_1}{x - b_4}\frac{b_2 - b_4}{b_2 - b_4} \tag{18}$$

In our case $b_1 = 0, b_1 = 1, b_3 = x_a, b_4 = x_b$, which gives $u = \frac{x(1-x_b)}{x-x_b}$. The singularity at x_b is moved to infinity, the singularity at x_a is moved to $\frac{-x_a(1-x_b)}{x_b-x_a}$. Then our equation reads

$$\frac{d^2\phi(u)}{du^2} + (\frac{1}{x_a x_b} - \frac{1}{u-1} + \frac{(1+x_b) - x_a x_b \frac{1}{x_a}}{(1-x_a)(u + \frac{x_a(1-x_b)}{x_b-x_a})})\frac{d\phi(u)}{du} + \frac{J}{2Tu}\phi(u)$$
$$+\frac{1}{4}(\frac{D}{T^2} + \frac{2s\omega}{EuT} + \frac{2s^2}{F(u-1)T} + \frac{4s^2}{H(u-1)^2} + \frac{\omega^2}{4(x_a x_b u)^2})\phi(u) = 0. \tag{19}$$

$$T = (u + \frac{x_a(1-x_b)}{x_b - x_a}), \tag{20}$$

$$D = \frac{1}{(x_b - x_a)^2}(\frac{\omega(1-x_a)}{2\beta^2 x_a} - \frac{2s}{1-x_a})^2, \tag{21}$$

$$E = 4x_a x_b(x_b - x_a), \tag{22}$$

$$F = (1-x_a)^2(1-x_b)(x_b - x_a) \tag{23}$$

$$H = (1 - x_a)^2(1 - x_b)^2, \tag{24}$$

$$J = -\frac{\omega^2 x_b(1 - x_a)}{\beta^4 x_a^2 x_b^2(x_b - x_a)} \tag{25}$$

The new equation is not in the standard General Heun form, though. One has to remove the terms where the singularities are in quadratic form from the new differential equation. One makes a F-homotopic transformation on the dependent variable $\phi(u)$ in the form

$$\phi(u) = (u)^\lambda(u - 1)^\xi(u + \frac{x_a(1 - x_b)}{x_b - x_a})^\kappa g(u), \tag{26}$$

which gives rather complicated expressions for these indices.

$$\lambda = \frac{1}{2}[1 \pm \sqrt{1 - \frac{w^2}{4\beta^4(x_a x_b)^2}}], \tag{27}$$

$$\xi = 1 \pm \sqrt{1 - \frac{4s^2}{(1 - x_a)^2(1 - x_b)2}}, \tag{28}$$

$$\kappa = -\frac{1}{2}[\frac{x_a}{x_a - x_b} - 1 \pm \sqrt{(\frac{x_a}{x_a - x_b})^2 - (\frac{w(x_a - 1)}{x_a\beta^2} - \frac{4s}{(x_a - 1)})^2}]. \tag{29}$$

Now we can write the new wave equation as

$$\frac{d^2g(u)}{du^2}+(\frac{c}{u}+\frac{d}{u-1}+\frac{e}{(u+\frac{x_a(1-x_b)}{x_b-x_a})})\frac{d}{du}g(u)+\frac{ABu+Q}{u(u-1)(u+\frac{x_a(1-x_b)}{x_b-x_a})}g(u)=0. \tag{30}$$

Here

$$c = [1 - \sqrt{1 - \frac{w^2}{4\beta^4(x_a x_b)^2}}], \tag{31}$$

$$d = 1 - 2\sqrt{1 - \frac{s^2}{(1 - x_a)^2(1 - x_b)2}}, \tag{32}$$

$$e = 1 - \sqrt{(\frac{x_a}{x_a - x_b})^2 - (\frac{w(x_a - 1)}{x_a\beta^2} - \frac{4s}{(x_a - 1)})^2} \tag{33}$$

Using the relation

$$c + d + e = A + B + 1, \tag{34}$$

which is valid when our equation has Heun form, we can also calculate A and B.

$$A = \frac{1}{2}(c + d + e - 1 + \sqrt{(c + d + e - 1)^2 - 4S}), \tag{35}$$

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$$B = \frac{1}{2}(c+d+e-1-\sqrt{(c+d+e-1)^2-4S}). \tag{36}$$

The expression inside the square root above simplifies

$$(c+d+e-1)^2-4S = [\frac{2}{x_a x_b} + (-1\frac{x_a x_b - 1}{x_b(x_b-x_a)})^2 \\ -[\frac{\omega(1-x_b)}{2\beta^2 x_b(x_b-x_a)} - \frac{2s}{(1-x_b)(x_b-x_a)}]^2]. \tag{37}$$

If we write $Q = Q_1 + Q_2$, we get

$$Q_1 = -\frac{1}{2}[ec + \frac{x_a(1-x_b)dc}{(x_b-x_a)} + \frac{x_a(1-x_b)}{x_a x_b(x_b-x_a)} + \frac{1+x_b-x_a x_b-\frac{1}{x_a}}{(x_b-x_a)(1-x_a)x_a x_b}], \tag{38}$$

$$Q_2 = \frac{\omega^2 x_b(1-x_a)}{2\beta^4 x_a^2 x_b^2(x_b-x_a)} - \frac{s\omega}{2\beta^2 x_a x_b(x_b-x_a)}. \tag{39}$$

4. Scattering at the origin

While changing our independent variable from z to x, we overlooked one point. Now x corresponds to minus infinity for z. To scatter from origin , we need to take $x = 1$. We, therefore, change our variables to $u - 1 = -y$. We know that $u = 1$ is the same point as $x = 1$. Then our equation reads

$$\frac{d^2 g(y)}{dy^2} + (\frac{c}{y-1} + \frac{d}{y} + \frac{e}{(y - \frac{x_b(1-x_a)}{x_b-x_a})})\frac{d}{dy}g(y) + \frac{AB(y-1)-Q}{y(y-1)(y-\frac{x_b(1-x_a)}{x_b-x_a})}g(y) = 0, \tag{40}$$

which gives as solution

$$g(y) = H_G(\frac{x_b(1-x_a)}{x_b-x_a}, -Q-AB; A, B, d, c, e; y = 1-u), \tag{41}$$

where H_G is the generalized Heun function, in the standard form[20]. We had a singular point at infinity. We want to have our wave to come from infinity and scatter at the origin. The other singular point at x_a can be arranged to be at a complex value.

For the scattering process, we will use the formula given by Dekar et al [31]. This formula is between two finite points. We bring the the point infinity to unity by using the transformation $t = \frac{y}{y-1}$. Now the point where y and t are equal to zero coincide, and y going to infinity is given by $t = 1$.

The new equation is not of the Heun form. To bring it back to the Heun form, we multiply the solution $g((t)$ by a power

$$g(t) = (t-1)^A h(t) \tag{42}$$

and try to use $h(t)$ in our further calculations. Our new dependent variable $h(t)$ satisfies the equation

$$\frac{d^2h(t)}{dt^2}+\Big(\frac{A+1-B}{t-1}+\frac{d}{t}+\frac{e}{t-\frac{x_b(1-x_a)}{x_a(1-x_b)}}\Big)\frac{d}{dt}h(t)+\Big(\frac{(A(d+e-B)t+Q_1}{t(t-1)(t-\frac{x_b(1-x_a)}{x_a(1-x_b)})}\Big)h(t)=0. \tag{43}$$

At this point, we make a change in our notation. We want to use the standard Heun notation. This means we will redefine our parameters as $A+1-B=\delta, d=\gamma, d+e-B=\beta$. The other parameters are defined as $e=\epsilon, A=\alpha$ will remain as they are. Now, we can use the standard notation, since our equations will be of the Heun form in the further part of this work, and we will be able to use known relations for this function.
Now, the differential equation is written as

$$\frac{d^2h(t)}{dt^2}+\Big(\frac{\gamma}{t}+\frac{\delta}{t-1}+\frac{\epsilon}{t-\frac{x_b(1-x_a)}{x_a(1-x_b)}}\Big)\frac{d}{dt}h(t)+\Big(\frac{(\alpha\beta)t+Q_1}{t(t-1)(t-\frac{-x_b(1-x_a)}{x_a(1-x_b)})}\Big)h(t)=0. \tag{44}$$

$$Q_1=\frac{(x_b-x_a)(\alpha(\gamma+\epsilon-\beta)-Q)-\gamma\alpha(x_b(1-x_a)}{x_a(1-x_b)}. \tag{45}$$

From now on, we will rename $h(t)$ as y_1. Now we have to write y_1 in terms of the two linearly independent solutions, which we name $y_3(1-t)$ and $y_4(1-t)$, the two linearly independent solutions one gets when one expands around $1-t=v$.
In other words, we want to write

$$y_1(t)=C_1y_3(1-t)+C_2y_4(1-t). \tag{46}$$

The right hand side gives us the two Heun solutions when the same equation is expanded as a function of $1-t$. This expansion is easily done

$$y_3=H_G\Big(\frac{(x_a-x_b)}{x_a(1-x_b)},Q_3;\alpha,\beta;\delta,\gamma,\epsilon;v=1-t\Big), \tag{47}$$

where $Q_3=Q_1-\alpha\beta$, and which is the solution of the equation

$$\frac{d^2y_3(v)}{dv^2}+\Big(\frac{\gamma}{v-1}+\frac{\delta}{v}+\frac{\epsilon}{v-\frac{x_a-x_a}{x_a(1-x_b)}}\Big)\frac{d}{dv}y_3(v)+\Big(\frac{(\alpha\beta)v+Q_3}{v(v-1)(v-\frac{(x_a-x_b)}{x_a(1-x_b)})}\Big)y_3(v)=0. \tag{48}$$

Then, we make the transformation

$$y_4(v)=v^{(1-\delta)}j(v), \tag{49}$$

where $j(v)$ satisfies the equation

$$\frac{d^2j(v)}{dv^2}+(\frac{\gamma}{v-1}+\frac{2-\delta}{v}+\frac{\epsilon}{v-\frac{x_a-x_a}{x_a(1-x_b)}})\frac{d}{dv}j(v)+(\frac{(\gamma+\epsilon-\alpha)(\gamma+\epsilon-\beta)v+Q_4}{v(v-1)(v-\frac{(x_a-x_b)}{x_a(1-x_b)})})j(v)=0. \quad (50)$$

to give

$$y_4=(1-t)^{(1-\delta)}(H_G(\frac{x_a-x_b)}{x_a(1-x_b)},Q_4;\gamma+\epsilon-\alpha),\gamma+\epsilon-\beta;2-\delta,\gamma,\epsilon;v=1-t), \quad (51)$$

where $Q_4=Q_3-(1-\delta)(\gamma\frac{x_a-x_b}{x_a(1-x_b)}+\epsilon)$. Note that this last transformation is one of the transformations which does not change the Heun form of the differential equation. Furthermore,the relation $1+\alpha+\beta=\gamma+\delta+\epsilon$ is valid in this part of our calculations.
At this point we use the formula given in [31] for expanding $H_G(t)$ in terms of two linearly independent solutions of the equation written for $H_G(1-t=v)$, our equation (46).

$$y_1(t)=C_1y_3(1-t)+C_2y_4(1-t). \quad (52)$$

We find

$$C_1=H_G(\frac{x_b(1-x_a)}{x_a(1-x_b)},Q_3;\alpha,\beta;\gamma,\epsilon;1), \quad (53)$$

$$C_2=H_G\big(\frac{x_b(1-x_a)}{x_a(1-x_b)},Q_3-(\frac{x_b(1-x_a)}{x_a(1-x_b)})\gamma(1-\epsilon);\gamma+\epsilon-\alpha,\gamma+\epsilon-\beta;\gamma,\epsilon;1\big). \quad (54)$$

Note that the factor multiplying $j(1-t)$ in y_4 can be written as

$$\exp[(2i\sqrt{-1+\frac{s^2}{(1-x_a)^2(1-x_b)2}})\ln 1-t]. \quad (55)$$

Then we can write our eq. (46,52) as

$$\exp -i(V\ln(1-t))y_1(t)$$
$$=C_1\exp -i(V\ln(1-t))y_3(1-t)+C_2\exp(iV\ln(1-t))j(1-t), \quad (56)$$

where

$$V=\sqrt{-1+\frac{s^2}{(1-x_a)^2(1-x_b)2}}. \quad (57)$$

Then our reflection coefficient is given by

$$R=|\frac{C_2}{C_1}|^2. \quad (58)$$

This calculation is a formal one. We have to make sure that the Heun function, found by expansion around zero, is also finite at the second singular point for appropriate values of the parameters of our wave equation. A proper Leaver analysis [32] shows that for $\frac{x_a(1-x_b)}{x_b(1-x_a)}<1$, these terms are convergent at unity.

5. Conclusion

Here we put the wave equation for a zero mass scalar particle coupled to the Wahlquist metric to a form to give the Heun solutions in the standard notation and calculated the reflection coefficient for a particle coming from infinity at the origin.

Note that this calculation has weak points. We made an approximation by taking $lnx = x - 1$. This is valid only when x is close to unity. We made few transformations on the independent variable. We can not guarantee that this approximation is valid in the range we work. We have no choice, though, since we can not solve differential equations whose coefficients are transcendental functions. An interesting point is that we can satisfy an important algebraic constraint, given in [21, 20], in reducing the first equation to the standard form only if we make this approximation after we calculate $\frac{dU}{dx}$ exactly, and apply this approximation only after the differentiation is done.

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