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Black holes in massive conformal gravity

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ABSTRACT

We analyze the classical stability of Schwarzschild black hole in massive conformal gravity which was recently proposed for another massive gravity model. This model in the Jordan frame is conformally equivalent to the Einstein–Weyl gravity in the Einstein frame. The coupled linearized Einstein equation is decomposed into the traceless and trace equation when one chooses $6m^2\varphi = \delta R$. Solving the traceless equation exhibits unstable modes featuring the Gregory–Laflamme s-mode instability of five-dimensional black string, while we find no unstable modes when solving the trace equation. It is shown that the instability of the black hole in massive conformal gravity arises from the massiveness where the geometry of extra dimension trades for mass.

1. Introduction

Recently, massive conformal gravity was proposed as another massive gravity model [1]. This model is composed of a conformally coupled scalar to Einstein–Hilbert term (conformal relativity) and Weyl-squared term which are invariant under conformal transformations. Apparently, this model is related to the Einstein–Weyl gravity of $R + aC_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ [2], which is not manifestly invariant under conformal transformations. This model seems to be promising because the conformal symmetry restricts the number of counter-terms arising from the perturbative quantization of the metric tensor [3]. However, Stelle has shown that a definite combination of $aC_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + bR^2$ is necessary to improve the perturbative properties of Einstein gravity [4]. In this sense, massive conformal gravity including the Weyl-squared term only might not be a candidate for a proper quantum gravity model.

On the other hand, massive conformal gravity plays a role of being massive gravity model [5,6] because it includes Einstein–Hilbert term and Weyl-squared term in addition to a conformally coupled scalar. Actually, this action in the Jordan frame is conformally equivalent to the Einstein–Weyl action in the Einstein frame. It turned out that the Schwarzschild black hole is unstable against the Gregory–Laflamme (GL) $s(l=0)$-mode metric perturbation [7] in massive gravity models [8–10]. This is possible because the extra dimension in five-dimensional black string could be replaced by the mass [11]. That is, trading geometry for mass is a plausible argument for the instability of Schwarzschild black hole in the massive gravity. If one takes into account the number of degrees of freedom (DOF), it is easy to show why the Schwarzschild black hole is physically stable in the Einstein gravity, whereas the Schwarzschild black hole is unstable in massive conformal gravity. The number of DOF of the metric perturbation is 2 DOF in the Einstein gravity, while the number of DOF is $6 = 5 + 1$ in massive conformal gravity. The s-mode analysis is suitable for a massive graviton with 5 DOF, whereas 1 DOF is described by a conformally coupled scalar (linearized Ricci scalar) which satisfies a massive scalar equation.

In this work, we investigate the classical stability of Schwarzschild black hole in massive conformal gravity. The coupled linearized Einstein equation is decomposed into the traceless and trace equation when one chooses $6m^2\varphi = \delta R$. Solving the traceless equation exhibits unstable modes featuring the GL s-mode instability of five-dimensional black string, while we find no unstable modes from solving the trace equation. This implies that massive conformal gravity could not provide the Schwarzschild black hole solution.

2. Massive conformal gravity

We consider the action for massive conformal gravity which is composed of conformal relativity and Weyl-squared term [1]

$$S_{\mathrm{MCG}} = \frac{1}{32\pi}\int d^4x\sqrt{-g}\left[\alpha\left(\phi^2 R + 6\partial_\mu\phi\partial^\mu\phi\right) - \frac{1}{m^2}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}\right], \tag{1}$$

where the Weyl tensor squared is given by

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$$C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}=2\left(R^{\mu\nu}R_{\mu\nu}-\frac{1}{3}R^2\right)+\left(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}-4R^{\mu\nu}R_{\mu\nu}+R^2\right). \tag{2}$$

Here the last of Gauss–Bonnet term could be neglected because it does not contribute to equation of motion. Also, we use the Planck units of $c=\hbar=G=1$ and m is the mass of massive spin-2 graviton. The action (1) is invariant under the conformal transformations of

$$g_{\mu\nu}\to\Omega^2(x)g_{\mu\nu},\qquad \phi\to\Omega^{-1}\phi, \tag{3}$$

where $\Omega(x)$ is an arbitrary function of the spacetime coordinates.

From (1), the Einstein equation is derived to be

$$\alpha m^2\left[\phi^2G_{\mu\nu}+g_{\mu\nu}\nabla^2(\phi^2)-\nabla_\mu\nabla_\nu(\phi^2)+6\partial_\mu\phi\partial_\nu\phi-3(\partial\phi)^2g_{\mu\nu}\right]-2W_{\mu\nu}=0, \tag{4}$$

where the Einstein tensor is given by

$$G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu} \tag{5}$$

and the Bach tensor $W_{\mu\nu}$ takes the form

$$W_{\mu\nu}=2\left(R_{\mu\rho\nu\sigma}R^{\rho\sigma}-\frac{1}{4}R^{\rho\sigma}R_{\rho\sigma}g_{\mu\nu}\right)-\frac{2}{3}R\left(R_{\mu\nu}-\frac{1}{4}Rg_{\mu\nu}\right)+\nabla^2R_{\mu\nu}-\frac{1}{6}\nabla^2Rg_{\mu\nu}-\frac{1}{3}\nabla_\mu\nabla_\nu R. \tag{6}$$

Its trace is zero ($W^\mu{}_\mu=0$).

The other scalar equation is given by

$$\nabla^2\phi-\frac{1}{6}R\phi=0, \tag{7}$$

which is conformally covariant. Taking the trace of (4) leads to

$$-\phi^2R+3\nabla^2(\phi^2)-6(\partial\phi)^2=0 \tag{8}$$

which vanishes when one uses the scalar equation (7).

Considering the background ansatz

$$\bar R_{\mu\nu}=0,\qquad \bar R=0,\qquad \bar\phi=\sqrt{\frac{2}{\alpha}}, \tag{9}$$

Eqs. (4) and (7) provide the Schwarzschild black hole solution

$$ds_S^2=\bar g_{\mu\nu}dx^\mu dx^\nu=-f(r)dt^2+\frac{dr^2}{f(r)}+r^2d\Omega_2^2 \tag{10}$$

with the metric function

$$f(r)=1-\frac{r_0}{r}. \tag{11}$$

It is easy to show that the Schwarzschild black hole (10) is also the solution to the Einstein equation of $G_{\mu\nu}=0$ in Einstein gravity.

We introduce the metric and scalar perturbations around the Schwarzschild black hole

$$g_{\mu\nu}=\bar g_{\mu\nu}+h_{\mu\nu},\qquad \phi=\bar\phi(1+\varphi)=\sqrt{\frac{2}{\alpha}}(1+\varphi). \tag{12}$$

Then, the linearized Einstein equation takes the form

$$m^2\left[\delta G_{\mu\nu}+2(\bar g_{\mu\nu}\bar\nabla^2-\bar\nabla_\mu\bar\nabla_\nu)\varphi\right]=\left[\bar\nabla^2\delta G_{\mu\nu}+2\bar R_{\rho\mu\sigma\nu}\delta G^{\rho\sigma}\right]+\frac{1}{3}\left[\bar g_{\mu\nu}\bar\nabla^2-\bar\nabla_\mu\bar\nabla_\nu\right]\delta R, \tag{13}$$

where the linearized Einstein tensor, Ricci tensor, and Ricci scalar are given by

$$\delta G_{\mu\nu}=\delta R_{\mu\nu}-\frac{1}{2}\delta R\bar g_{\mu\nu}, \tag{14}$$

$$\delta R_{\mu\nu}=\frac{1}{2}\left(\bar\nabla^\rho\bar\nabla_\mu h_{\nu\rho}+\bar\nabla^\rho\bar\nabla_\nu h_{\mu\rho}-\bar\nabla^2h_{\mu\nu}-\bar\nabla_\mu\bar\nabla_\nu h\right), \tag{15}$$

$$\delta R=\bar g^{\mu\nu}\delta R_{\mu\nu}=\bar\nabla^\mu\bar\nabla^\nu h_{\mu\nu}-\bar\nabla^2h \tag{16}$$

with $h=h^\rho{}_\rho$.

From (7), we derive the linearized scalar equation

$$\bar\nabla^2\varphi-\frac{1}{6}\delta R=0 \tag{17}$$

which is surely a coupled equation for φ and δR. Plugging (17) into (13), one finds a simpler linearized Einstein equation

$$m^2[\delta G_{\mu\nu}-2\bar\nabla_\mu\bar\nabla_\nu\varphi]=\left[\bar\nabla^2\delta G_{\mu\nu}+2\bar R_{\rho\mu\sigma\nu}\delta G^{\rho\sigma}\right]-\frac{1}{3}\bar\nabla_\mu\bar\nabla_\nu\delta R. \tag{18}$$

It might be difficult to solve (18) directly because it is a coupled second-order equation for $\delta G_{\mu\nu}$, δR, and φ. Taking the trace of (18) together with $\delta G^\mu{}_\mu=-\delta R$ leads to (17) too. In order to simplify the linearized equation (18), one way is to find a condition of non-propagating linearized Ricci scalar ($\delta R=0$). However, it is not justified to impose $\delta R=0$ because of conformal symmetry in massive conformal gravity. In Appendix A, we have $\delta R=0$ for the new massive conformal gravity where the conformal symmetry is broken due to the addition of the Einstein–Hilbert term.

The other way to resolve the coupling difficulty is to propose a relation between φ and δR because the massive conformal gravity implies 6 DOF of massive graviton (with 5 DOF) and scalar. If one requires the relation

$$\varphi=\frac{1}{6m^2}\delta R \tag{19}$$

the linearized equation (18) is simplified further as

$$\bar\nabla^2\delta G_{\mu\nu}+2\bar R_{\rho\mu\sigma\nu}\delta G^{\rho\sigma}=m^2\delta G_{\mu\nu}. \tag{20}$$

Before we proceed, we would like to mention that the relation (19) between φ and δR is taken specially for the massive conformal gravity. If we do not use this relation, we could not make a further progress on the stability analysis. The apparently two DOF of φ and δR becomes a single DOF due to the relation (19). Plugging (19) into the linearized scalar equation (17) leads to the massive scalar equation

$$(\bar\nabla^2-m^2)\varphi=0. \tag{21}$$

Also, the same equation is recovered when one takes the trace of (20)

$$(\bar\nabla^2-m^2)\delta R=0 \tag{22}$$

which is called the trace equation. However, Eq. (20) describes 6 DOF of a massive graviton and (Ricci) scalar wholly. Splitting $\delta R_{\mu\nu}$ into the traceless linearized Ricci tensor $\delta\tilde R_{\mu\nu}$ with $\bar g^{\mu\nu}\delta\tilde R_{\mu\nu}=0$ and the linearized Ricci scalar δR as

$$\delta R_{\mu\nu}=\delta\tilde R_{\mu\nu}+\frac{1}{4}\delta R\bar g_{\mu\nu}, \tag{23}$$

the linearized Einstein tensor is given by

$$\delta G_{\mu\nu}=\delta\tilde R_{\mu\nu}-\frac{1}{4}\delta R\bar g_{\mu\nu}. \tag{24}$$

Then, the linearized Einstein equation (20) takes the form

$$\bar{\nabla}^2\delta\tilde{R}_{\mu\nu}+2\bar{R}_{\rho\mu\sigma\nu}\delta\tilde{R}^{\rho\sigma}-m^2\delta\tilde{R}_{\mu\nu}-\frac{\bar{g}_{\mu\nu}}{4}(\bar{\nabla}^2-m^2)\delta R=0. \tag{25}$$

At first sight, Eq. (25) seems to be a coupled equation for $\delta\tilde{R}_{\mu\nu}$ and δR. Using the trace equation (22) to eliminate δR, Eq. (25) is reduced to the traceless linearized Ricci tensor equation

$$\bar{\nabla}^2\delta\tilde{R}_{\mu\nu}+2\bar{R}_{\rho\mu\sigma\nu}\delta\tilde{R}^{\rho\sigma}=m^2\delta\tilde{R}_{\mu\nu} \tag{26}$$

which is our main result.

On the other hand, we note that in the Einstein–Weyl gravity [10], the non-propagation of the linearized Ricci scalar ($\delta R=0$) is an essential requirement to arrive at the linearized massive Ricci tensor equation

$$\bar{\nabla}^2\delta R_{\mu\nu}+2\bar{R}_{\rho\mu\sigma\nu}\delta R^{\rho\sigma}-m^2\delta R_{\mu\nu}=0 \tag{27}$$

which describes a massive graviton with 5 DOF propagating around the Schwarzschild black hole.

However, the massive conformal gravity implies that the linearized Einstein tensor with 6 DOF ($\delta\tilde{R}_{\mu\nu}$ and δR) propagating the Schwarzschild black hole satisfies the traceless equation (26) and the trace equation (22). The traceless condition of $\delta G^{\mu}{}_{\mu}=-\delta R=0$ is an important requirement to show the GL instability of the Schwarzschild black hole and it could be achieved only in the Einstein–Weyl gravity. On the contrary, the trace equation (22) plays an important role of obtaining the traceless equation (26) in massive conformal gravity. In the next section, we will prove that the conformally invariant action (1) in the Jordan frame is conformally equivalent to the Einstein–Weyl action in the Einstein frame.

3. Massive conformal gravity in Einstein frame

In this section, we transform the conformally invariant action (1) into the corresponding action in the Einstein frame. First of all, it would be better to show that the conformally invariant action (1) is nothing but the $\omega=-3/2$ Brans–Dicke theory plus Weyl-squared term for $\alpha=1/6$ when one chooses [12]

$$\frac{1}{12}\phi^2=e^{-\Phi}. \tag{28}$$

Then, (1) is given by

$$\tilde{S}_{\rm MCG}=\frac{1}{16\pi}\int d^4x\sqrt{-g}\left[e^{-\Phi}\left(R+\frac{3}{2}\partial_\mu\Phi\partial^\mu\Phi\right)-\frac{1}{2m^2}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}\right]. \tag{29}$$

On the other hand, the Brans–Dicke theory plus Weyl-squared term is described by

$$S^{\omega}_{\rm BDW}=\frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\phi_{\rm BD}R-\frac{\omega}{\phi_{\rm BD}}\partial_\mu\phi_{\rm BD}\partial^\mu\phi_{\rm BD}-\frac{1}{2m^2}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}\right]. \tag{30}$$

Choosing $\phi_{\rm BD}=e^{-\Phi}$, (30) could be rewritten as

$$\tilde{S}^{\omega}_{\rm BDW}=\frac{1}{16\pi}\int d^4x\sqrt{-g}\left[e^{-\Phi}\left(R-\omega\partial_\mu\Phi\partial^\mu\Phi\right)-\frac{1}{2m^2}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}\right]. \tag{31}$$

We note that $\tilde{S}^{\omega=-3/2}_{\rm BDW}=\tilde{S}_{\rm MCG}$ (29), indicating that the conformal relativity is just the Brans–Dicke theory with $\omega=-3/2$ in the Jordan frame.

Now we make conformal transformation of the conformally invariant action (1) with $\alpha=1/6$ only by choosing [13,14]

$$\hat{g}_{\mu\nu}=\Omega^2g_{\mu\nu},\qquad \hat{\phi}=\phi-\phi=0,\qquad \Omega=\frac{\phi}{2\sqrt{3}}. \tag{32}$$

Then, the transformed action takes the form

$$\hat{S}_{\rm MCG}=\frac{1}{16\pi}\int d^4x\sqrt{-\hat{g}}\left[\hat{R}-\frac{1}{2m^2}\hat{C}^{\mu\nu\rho\sigma}\hat{C}_{\mu\nu\rho\sigma}\right] \tag{33}$$

which is noting but the Einstein–Weyl gravity in the Einstein frame. Hence it is clear that the conformally invariant action (1) ($\omega=-3/2$ Brans–Dicke theory plus Weyl-squared term) in the Jordan frame is conformally equivalent to the Einstein–Weyl action (33) in the Einstein frame. The Schwarzschild black hole (10) is also obtained as the solution to the Einstein equation. Its linearized Einstein equation is given by [10]

$$\bar{\nabla}^2\delta\hat{R}_{\mu\nu}+2\bar{R}_{\rho\mu\sigma\nu}\delta\hat{R}^{\rho\sigma}-m^2\delta\hat{R}_{\mu\nu}=0 \tag{34}$$

together with transverse-traceless condition of $\bar{\nabla}^\mu\delta\hat{R}_{\mu\nu}=0$ and $\delta\hat{R}=0$. This implies that even though a conformally coupled scalar ϕ provides a different linearized Einstein equation (20) with (22) in the Jordan frame, it disappears in the Einstein frame.

If one starts with a non-conformally invariant action, there exists a scalar kinetic term of $-\frac{\lambda}{2}\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ which could be reduced to a canonical form of $-\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi$ in terms of a minimally coupled scalar $\Psi=\sqrt{\lambda}\phi$. The non-conformally invariant action ($\omega>-3/2$ Brans–Dicke theory plus Weyl-squared term) in the Jordan frame is conformally equivalent to the scalar-Einstein–Weyl gravity in the Einstein frame [15]

$$\hat{S}_{\rm MNCG}=\frac{1}{16\pi}\int d^4x\sqrt{-\hat{g}}\left[\hat{R}-\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi-\frac{1}{2m^2}\hat{C}^{\mu\nu\rho\sigma}\hat{C}_{\mu\nu\rho\sigma}\right]. \tag{35}$$

It was proposed that the stability of black holes does not depend on the frame [16], even though there exists an apparent difference between (20) and (34). The difference is the trace equation (22) which becomes the conformal scalar equation (21). We will check the above proposal.

4. Instability of Schwarzschild black hole in massive conformal gravity

Considering the number of DOF, it is helpful to show why the Schwarzschild black hole is physically stable in the Einstein gravity [17–19], whereas the Schwarzschild black hole is unstable in massive conformal gravity. From Eq. (20) together with the linearized Bianchi identity ($\bar{\nabla}^\mu\delta G_{\mu\nu}=0$), the number of DOF for massive spin-2 graviton is $10-4=6$ in massive conformal gravity. On the other hand, the number of DOF for the massless spin-2 graviton is 2 in Einstein gravity since one requires -4 further for a residual diffeomorphism after a gauge-fixing and the traceless condition. The s-mode analysis is relevant to the massive graviton in massive conformal gravity, but not to the massless graviton in the Einstein gravity. In general, the s-mode analysis of the massive graviton with 5 DOF shows the GL-instability which never appears in the massless spin-2 analysis.

To perform the stability of Schwarzschild black hole in massive conformal gravity completely, we have to solve two linearized equations: the trace equation (22) and traceless equation (26) with the same mass-squared m^2. These are different from those arising from the forth-order gravity of $R-\alpha R^2-\beta R_{\mu\nu}R^{\mu\nu}$ [10] because the latter provides different masses $m_0^2=-1/2(3\alpha+\beta)$ and $m_2^2=1/\beta$. If $\alpha=-\beta/3$ (Weyl-squared term), the linearized Ricci scalar is decoupled from the theory because its mass m_0^2 blows up.

First of all, we wish to solve the massive scalar equation (21) [equivalently, Ricci scalar equation (22)] around the Schwarzschild black hole. It turned out that the scalar mode does not have any unstable modes if $m^2\geqslant 0$ [20,21]. Explicitly, considering the scalar perturbation

$$\varphi(t,r,\theta,\phi)=e^{i\omega t}\frac{\psi(r)}{r}Y_{lm}(\theta,\phi) \tag{36}$$

and introducing the tortoise coordinate

$$r^*=r+r_0\ln\left[\frac{r}{r_0}-1\right] \tag{37}$$

the linearized equation (21) reduces to the Schrödinger-type equation as

$$\frac{d^2\psi}{dr^{*2}}+\left(\omega^2-V_\psi\right)\psi=0 \tag{38}$$

with the potential

$$V_\psi=\left(1-\frac{r_0}{r}\right)\left[\frac{l(l+1)}{r^2}+\frac{r_0}{r^3}+m^2\right]. \tag{39}$$

The potential V_ψ is always positive exterior the event horizon $r=r_0$ for $l\geqslant 0$ and $m^2\geqslant 0$, implying that the black hole is stable against the scalar [Ricci scalar] perturbation.

However, the s-mode analysis is responsible for detecting an instability of a massive graviton propagating on the Schwarzschild black hole in massive gravity. The even-parity metric perturbation is designed for a $s(l=0)$-mode analysis in the massive gravity and whose form is given by H_{tt}, H_{tr}, H_{rr}, and K as [7]

$$h^{(m)}_{\mu\nu}=e^{\Omega t}\begin{pmatrix} H_{tt}(r) & H_{tr}(r) & 0 & 0\\ H_{tr}(r) & H_{rr}(r) & 0 & 0\\ 0 & 0 & K(r) & 0\\ 0 & 0 & 0 & \sin^2\theta K(r)\end{pmatrix}. \tag{40}$$

Even though one starts with 4 DOF, they are related to each other when one uses the transverse-traceless gauge of $\bar\nabla^\mu h^{(m)}_{\mu\nu}=0$ and $h^{(m)}=0$. Hence, we have one decoupled equation for H_{tr} from the massive graviton equation

$$\bar\nabla^2 h^{(m)}_{\mu\nu}+2\bar R_{\rho\mu\sigma\nu}h^{(m)\rho\sigma}=m^2h^{(m)}_{\mu\nu}. \tag{41}$$

Since Eq. (41) is the same linearized equation for four-dimensional metric perturbation around five-dimensional black string, we use the GL instability analysis in asymptotically flat spacetimes [7]. Eliminating all but H_{tr}, Eq. (41) reduces to a second-order radial equation for H_{tr}

$$AH''_{tr}+BH'_{tr}+CH_{tr}=0, \tag{42}$$

where A, B and C are given by

$$A=-m^2f-\Omega^2+\frac{f'^2}{4}-\frac{ff''}{2}-\frac{ff'}{r}, \tag{43}$$

$$B=-2m^2f'-\frac{3f'f''}{2}-\frac{3\Omega^2f'}{f}+\frac{3f'^3}{4f}+\frac{2m^2f}{r}+\frac{2\Omega^2}{r}+\frac{3f'^2}{2r}+\frac{ff''}{r}-\frac{2ff'}{r^2}, \tag{44}$$

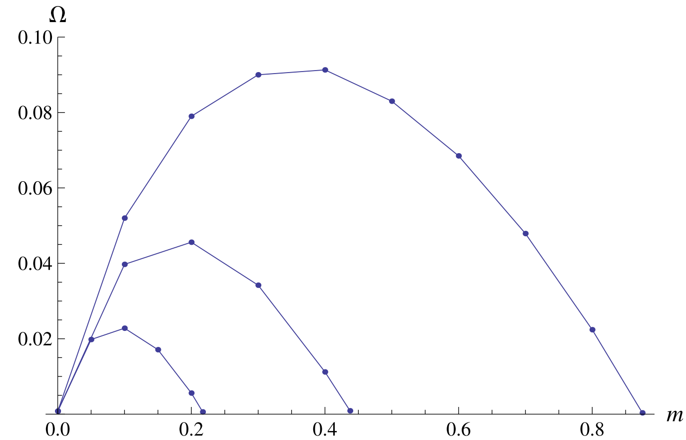

Fig. 1. Plots of unstable modes on three curves with $r_0=1,2,4$. The $y(x)$-axis denote $\Omega(m)$. The smallest curve represents $r_0=4$, the medium denotes $r_0=2$, and the largest one shows $r_0=1$.

$$C=m^4+\frac{\Omega^4}{f^2}+\frac{2m^2\Omega^2}{f}-\frac{5\Omega^2f'^2}{4f^2}+\frac{m^2f'^2}{4f}+\frac{f'^4}{4f^2}-\frac{m^2f''}{2}-\frac{\Omega^2f''}{2f}-\frac{f'^2f''}{4f}-\frac{f''^2}{2}-\frac{2m^2f'}{r}-\frac{\Omega^2f'}{rf}+\frac{f'^3}{rf}-\frac{3f'f''}{r}+\frac{2\Omega^2}{r^2}+\frac{2m^2f}{r^2}-\frac{5f'^2}{2r^2}+\frac{ff''}{r^2}+\frac{2ff'}{r^3} \tag{45}$$

with the metric function $f=1-r_0/r$ (11).

It is worth noting that the s-mode perturbation is described by single DOF but not 5 DOF. We solve (42) numerically and find unstable modes. See Fig. 1 that is generated from the numerical analysis. From the observation of Fig. 1 with $\mathcal{O}(1)\simeq 0.86$, we find unstable modes [8] for

$$0<m<\frac{\mathcal{O}(1)}{r_0} \tag{46}$$

with mass m. As the horizon size r_0 increases, the instability becomes weak as in the Schwarzschild black hole.

For a massive gravity theory in the Minkowski background, there is correspondence between linearized Ricci tensor $\delta R_{\mu\nu}$ and Ricci spinor Φ_{ABCD} when one uses the Newman–Penrose formalism [22]. Here the massive gravity requires null complex tetrad to specify six polarization modes [23,24]. This implies that in massive conformal gravity, one takes the linearized Ricci tensor $\delta R_{\mu\nu}$ (15) with 5 DOF as physical observables [10] by requiring the transversality condition of $\bar\nabla^\mu\delta R_{\mu\nu}=0$ from the contracted Bianchi identity and the traceless condition of $\delta R=0$. That is, the traceless linearized Ricci tensor $\delta\tilde R_{\mu\nu}$ has the same 5 DOF as the metric perturbation $h_{\mu\nu}$ does have in massive gravity theory. Actually, Eq. (26) is considered as a boosted-up version of (41) [25]. Similarly, we find Eq. (41) when we replace $\delta\tilde R_{\mu\nu}$ by $h^{(m)}_{\mu\nu}$ in (26). Hence, a relevant equation for $\delta\tilde R_{tr}$ takes the same form

$$A\delta\tilde R''_{tr}+B\delta\tilde R'_{tr}+C\delta\tilde R_{tr}=0 \tag{47}$$

which shows the same unstable modes appeared in Fig. 1.

Consequently, we have found unstable s-mode from the traceless equation (26), but have not found unstable modes from the trace equation (22) [scalar equation (21)] in the Jordan frame. If one uses the linearized equation (34) arisen from the Einstein–Weyl gravity in the Einstein frame, one finds the same unstable modes. This implies that the instability of black holes in massive gravity does not depend on the frame.

5. Discussions

We discuss on the following issues.

- Ghosts and linearized Ricci tensor.

Since the linearized equation (13) is a fourth-order derivative equation, it involves the linearized ghosts [25]. The ghost appears surely when one introduces an auxiliary tensor $f_{\mu\nu}$ to reduce fourth-order gravity theory to second-order theory [26]. This implies that if one uses the massive spin-2 equation (41) to analyze the instability of Schwarzschild black hole in the massive conformal gravity, its instability might not be legitimate. If one uses the linearized Ricci tensor $\delta\bar{R}_{\mu\nu}$ instead of the metric perturbation $h_{\mu\nu}$ [10], its linearized equation is a second-order equation (26) which is free from any ghosts.

- Renormalizability and conformal symmetry.

It was suggested that the conformal invariant action (1) enhances the renormalizability because the conformal symmetry restricts the number of counter-terms arising from the perturbative quantization of the metric tensor [1]. However, Stelle [4] has shown that the quadratic curvature gravity of $a(R^2_{\mu\nu}-R^2/3)+bR^2$ in addition to the Einstein–Hilbert term (R) is necessary to improve the perturbative properties of Einstein gravity. If $ab\neq 0$, the renormalizability was achieved but the unitarity was violated, indicating that the renormalizability and unitarity exclude to each other. Although the a-term of providing the massive graviton improves the ultraviolet divergence, it induces ghost excitations which spoil the unitarity simultaneously. The price one has to pay for making the theory renormalizable in this way is the loss of unitarity. If one excludes bR^2, there is no massive spin-0 corrections. In this sense, the conformal invariant action (1) is unhealthy and it might not enhance the renormalizability without unitarity.

- Massive conformal gravity and black hole.

As was shown in most massive gravity theories [8–10], it is difficult for massive conformal gravity to accommodate the static black hole solution because the GL $s(l=0)$-mode instability [7] was found. It could be understood that the instability of the black hole in massive conformal gravity arises from the massiveness of $m^2\neq 0$, where the geometry of extra dimension in five-dimensional black string is replaced by the mass [11].

- Role of a conformally coupled scalar φ.

Even the scalar is conformally coupled to Einstein–Hilbert action to give a conformally invariant action, its role in testing the black hole stability is trivial because the conformally invariant action (1) ($\omega=-3/2$ Brans–Dicke theory plus Weyl-squared term) in the Jordan frame is conformally equivalent to the Einstein–Weyl action (33) in the Einstein frame. The instability of Schwarzschild black hole is determined definitely by the massive linearized Ricci tensor equations (26) and (34) which are the same equation in both theories. The scalar field equation (21) [Ricci scalar equation (22) using $\varphi=\delta R/6m^2$] did not show any unstable modes for $m^2\geqslant 0$. This implies that the instability of Schwarzschild black hole is independent of choosing a frame.

- $f(R)$-gravity and massive conformal gravity.

A simple model of $f(R)=R+\alpha R^2$ provides a ghost-free massless graviton and massive spin-0 graviton [21], while massive conformal gravity shows a massless graviton, scalar, and massive spin-2 graviton with ghosts in terms of metric tensor. A similarity between two gravity theories is that both have a propagating linearized Ricci scalar (δR). A difference is that $f(R)$ gravity does not provide a propagating Ricci tensor ($\delta R_{\mu\nu}$), while massive conformal gravity have a propagating Ricci tensor.

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Appendix A. New massive conformal gravity

Adding the Einstein–Hilbert term is an easy way to break conformal symmetry in massive conformal gravity [27]. Then, the new massive conformal gravity action is proposed by

$$S_{\rm NMCG}=\frac{1}{32\pi}\int d^4x\sqrt{-g}\Big[-R+\alpha\big(\phi^2R+6\partial_\mu\phi\partial^\mu\phi\big)-\frac{1}{m^2}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}\Big]. \tag{48}$$

The Einstein equation is changed to be

$$G_{\mu\nu}=\alpha\big[\phi^2G_{\mu\nu}+g_{\mu\nu}\nabla^2(\phi^2)-\nabla_\mu\nabla_\nu(\phi^2)+6\partial_\mu\phi\partial_\nu\phi-3(\partial\phi)^2g_{\mu\nu}\big]-\frac{2}{m^2}W_{\mu\nu}. \tag{49}$$

However, the scalar equation remains unchanged as

$$\nabla^2\phi-\frac{1}{6}R\phi=0. \tag{50}$$

Taking the trace of (49) leads to

$$R=0 \tag{51}$$

which simplifies the scalar equation (50) as the uncoupled massless scalar equation

$$\nabla^2\phi=0. \tag{52}$$

The linearized Einstein equation around the Schwarzschild black hole is modified into

$$m^2\Big[\frac{1}{2}\delta G_{\mu\nu}+2\bar{g}_{\mu\nu}\bar{\nabla}^2\varphi-2\bar{\nabla}_\mu\bar{\nabla}_\nu\varphi\Big]=\big[\bar{\nabla}^2\delta G_{\mu\nu}+2\bar{R}_{\rho\mu\sigma\nu}\delta G^{\rho\sigma}\big]-\frac{1}{3}\big[\bar{\nabla}_\mu\bar{\nabla}_\nu-\bar{g}_{\mu\nu}\bar{\nabla}^2\big]\delta R. \tag{53}$$

The linearized scalar equation is

$$\bar{\nabla}^2\varphi=0. \tag{54}$$

Taking the trace of the linearized Einstein equation and using (54), one has

$$-\frac{m^2}{2}\delta R=0 \tag{55}$$

which implies the non-propagation of linearized Ricci scalar

$$\delta R=0 \tag{56}$$

unless $m^2=0$. We note that $\delta R=0$ is confirmed from linearizing $R=0$ (51). The choice of $\delta R=0$ reflects why we consider not the massive conformal gravity (1) but the new massive conformal gravity (48) as a starting action. If one does not break conformal symmetry, one could not achieve the non-propagation of the Ricci scalar. Plugging $\delta R = 0$ and (54) into Eq. (53) leads to the massive equation for the linearized Ricci tensor [10]

$$\bar{\nabla}^2 \delta R_{\mu\nu} + 2\bar{R}_{\rho\mu\sigma\nu}\delta R^{\rho\sigma} = m^2\left[\frac{1}{2}\delta R_{\mu\nu} - 2\bar{\nabla}_\mu \bar{\nabla}_\nu \varphi\right], \tag{57}$$

which is still difficult to be solved because of coupling $\delta R_{\mu\nu}$ and φ.

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