ON THE THEORY OF THE STOCHASTIC METHOD OF PARTICLE ACCELERATION AND BEAM STACKING

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(presented by A. A. Kolomenskij)

All modern methods of particle acceleration in cyclic accelerators may be classified into two groups —those in which the accelerating voltage is scheduled or programmed and those in which it is a random function of time. The first group includes all autophasing accelerators, whose theory is now well understood. The second type, on the other hand, has practically received no attention since it was first proposed in 1948 [1]. It should be noted, however, that stochastic schemes of acceleration, in addition to being theoretically interesting, may also prove to be practical for increasing the intensity of the beam of particles in existing phasotrons, thus eliminating one of their major shortcomings, namely the pulse nature of their operation. Stochastic schemes may also be of interest for use in intersecting-beam accelerators and stacking systems of the ring phasotron type [2], for overcoming transition energy and so on.

Let us consider some of the relationships for particle motion in a fixed magnetic field for the case when the voltage $V(t)$ across the accelerating element is a random function of time (" noise ") having a given spectral intensity $P(\omega)$.

Disregarding betatron oscillations, it is convenient to express the equations of motion in action variables by means of the Hamiltonian [3]

$$H = -e\int F(\theta,t)d\theta + E(z)\,. \qquad (1)$$

Here θ is the generalized azimuth, and the canonical momentum z is related to energy by the relation

$$dz = dE/\Omega(E) \qquad (2)$$

where $\Omega(E)$ is the revolution frequency of a particle having energy E. The quantity eF represents the work performed by the external emf during one revolution. For the case of one accelerating gap, F may be written in the following form:

$$F = V(t)\sum_K \delta(\int \Omega dt - 2\pi K) \qquad (3)$$

We shall assume that $V(t)$ represents a stationary process, i.e. its correlation function $B_v(\tau)$ is not explicitly dependent on time [4]

$$B_v(\tau) = \langle V(t)V(t-\tau)\rangle \qquad (4)$$

(the brackets denote time averaging).

Strictly speaking, $F(\theta(t), t)$ does not represent a stationary process, since the frequency Ω depends on the magnitude $z(t)$. We assume, however, that the maximum correlation distance of the $V(t)$ process, i.e. the value $\tau = \tau_c$ at which B_v decreases to zero, is much smaller than the time required for $\Omega(t)$ to change perceptibly. This condition is always fulfilled when $V(t)$ is sufficiently small, which is actually the case in practice. (It should be noted that the value $\omega_c = 2\pi/\tau_c$ characterizes the lower limit of the accelerating voltage spectrum). With this assumption, it is easy to obtain for the correlation function of $F(\theta, t)$ the expression:

$$B_F(\tau) = B_v(\tau)\sum_K \delta(\Omega\tau - 2\pi K) \qquad (5)$$

showing it to be likewise independent of time.

Now, using the equation of motion, we obtain due to the ergodicity of the process $\overline{dz/dt} = 0$;

$$2D = \frac{\overline{dz^2}}{dt} = 2e^2 \int_0^t \overline{F[\theta(t),t]F[\theta(\tau),\tau]}d\tau$$

$$= 2e^2 \int_0^t B_F(\tau)d\tau = 2e^2 \int_0^t B_v(\tau)\sum_{K=0}^{\infty} \delta(\Omega\tau - 2\pi K)d\tau \qquad (6)$$

where the bar denotes averaging over the assembly of particles.

From Eq. (6) it immediately follows that if the correlation distance is sufficiently small

$$\frac{\Omega\tau_c}{2\pi} \ll 1\,; \quad \frac{\Omega}{\omega_c} \ll 1 \tag{7}$$

i.e. if the accelerating voltage spectrum lies in the frequency range $\omega \gg \Omega$, the quantity D for $t > \tau_c$ is independent of time and is simply equal to

$$D(z) = \frac{\langle V^2 \rangle e^2}{2\Omega(z)}\,. \tag{8}$$

Actually it is precisely the case that is discussed in the paper by Burshtein et al.[1], where it was assumed that during passage through the accelerating gap the voltage across the gap may with equal probability have the values $\pm V_0$.

In the general case, $D(z, t)$ is a step function of time, assuming for time $t \gtrsim \tau_c$ the stationary value

$$D(z) = \frac{e^2}{\Omega}\left[\sum_0^\infty B_v\left(\frac{2K\pi}{\Omega}\right) - \frac{\langle V^2 \rangle}{2}\right] \tag{9}$$

or, taking into account the fact that $B_v(\tau)$ and $P(\omega)$ are related by a Fourier Transformation,

$$D = \frac{e^2}{2}\sum_{K=1}^\infty P(K\Omega)\,. \tag{10}$$

Eq. (10) has a simple physical meaning, namely the particle " selects ", from the $V(t)$ spectrum, the harmonics which are multiples of its revolution frequency and act in an incoherent manner. It should be noted that if the maximum of the spectrum corresponds to high values of K, then, by substituting in Eq. (10) integration for summation, we again obtain Eq. (8).

If $V(t)$ is sufficiently small, the acceleration process may be considered as a diffusion along the z axis with coefficient $D(z, t)$. The particle distribution function $\psi(z, t)$ obeys the Fokker-Planck equation

$$\frac{\partial\psi}{\partial t} - \frac{\partial}{\partial z}\left[D(z,t)\frac{\partial\psi}{\partial z}\right] = J\delta(z-z_i) - \frac{\psi}{\tau(z)} \tag{11}$$

where the terms on the right-hand side represent the appearance of new particles from the injector (J is the injector current), and the losses of particles in the injector mechanism, these losses are due to scattering on residual gas, and other losses ($\tau(z)$ is the mean lifetime). The boundary conditions for ψ are

$$\psi = 0 \Big|_{z = z_{\min}}^{z = z_{\max}} \tag{12}$$

where $z_{\max}$ and $z_{\min}$ represent, respectively, the maximum and minimum particle energy in the accelerator. To obtain a complete solution of Eq. (11), it is necessary to know the actual dependence of $P(\omega)$ and $\tau(z)$. By way of example, let us consider an idealized case of stochastic stacking. Assume that particles with an initial energy corresponding to $z = z_i$ are continuously introduced into the chamber, where they are subjected to the action of an accelerating voltage $V(t)$. In order to qualitatively visualize what happens, let us consider the distribution for the case when $D(z)$ and $\tau(z)$ are slowly varying functions. Eq. (11) may then be solved by the WBK method and easily obtained in the following form:

$$\psi = J\left(\frac{\tau}{D}\right)_i^{\frac{1}{4}}\left(\frac{\tau}{D}\right)^{\frac{1}{4}} \mathrm{sh}\int_{z_{\min}}^{z_i}\frac{dz}{\sqrt{D\tau}}\cdot\left(\mathrm{sh}\int_{z_{\min}}^{z_{\max}}\frac{dz}{\sqrt{D\tau}}\right)^{-1}\mathrm{sh}\int_{z}^{z_{\max}}\frac{dz}{\sqrt{D\tau}}$$

$$z > z_i \tag{13}$$

from which it follows that $\psi \sim D^{-\frac{1}{4}}$. Qualitatively, this conclusion should also remain valid for the case when $D \to 0$ at some points, although the method of solution near these points is, generally speaking, inapplicable. In other words, if, at some z_0, D becomes zero, it may be expected that at $z \simeq z_0$ the distribution will have a sharp maximum, i.e. the particles will accumulate near this point, for example on the maximum radius of machine. Indeed, let us assume, for example, that at z_0 D has a zero of the second order, and τ, as before, is a slowly varying function. Then it can be easily shown that at point z_0 the distribution function has a pole of the order

$$\frac{1}{2}\left\{1+\left[1+\left(\tau\frac{d^2D}{dz^2}\right)_0\right]^{-1}\right\}. \tag{14}$$

The pole itself, naturally, has no physical meaning and is associated with the adopted idealization. Its presence, however, indicates that the particles indeed accumulate at points $D = 0$. In order to obtain an exact evaluation of the distribution function, transient time, etc., it is necessary to know, in the first place, the function $\tau(z)$ and, secondly, the actual dependence of $P(\omega)$.

In conclusion, it should be noted that this method may be used to determine the output current of a stochastic accelerator in accordance with the formula

$$J_{ac} = -\left[D\frac{\partial\psi}{\partial z}\right]_{z_{max}} \tag{15}$$

Let us assume for this purpose that accelerated particles are lost in the working energy range only if they collide with the injector. If, moreover, there are no frequencies in the spectrum of the accelerating voltage $P(\omega)$ that are multiples of the revolution frequency, we obtain for the efficiency of the accelerator for not very large $P(\omega)$

$$\eta = \frac{\left[\frac{\pi\tau}{Te^2P(\Omega)\cdot\Omega}\right]_i^{\frac{1}{2}}}{\int_{E_i}^{E_{max}} \frac{dE}{e^2\Omega P(\Omega)}} \exp\left[-\frac{1}{\pi\sqrt{\frac{2\tau}{T_i}}}\frac{\Delta E}{[e^2P(\Omega)\cdot\Omega]_i^{\frac{1}{2}}}\right] \tag{16}$$

where T is the period of revolution and the subscript i refers to the value at injection. The quantity ΔE represents the energy that a particle must gain after injection in order to clear the danger zone from the standpoint of collision with the injector. From Eq. (15), it is seen that the efficiency of acceleration depends, in the main, on the spectrum density at injection. In particular, for the parameters of the electron ring phasotron [5], the losses at the injector become sufficiently small for $P_i > 1$ [V^2/Hz] and the acceleration efficiency is about 5×10^{-3}%.

In order to secure a regime of acceleration that is analogous to the stochastic, we can provide a law of frequency change for the accelerating field that is deliberately made to violate the auto-phasing condition ($\cos\phi_s \gg 1$). In this case, after each cycle of frequency change—in spite of the fact that the average energy of the beam remains practically unchanged—the energy spread of the beam increases. The efficiency of such an accelerating method is roughly equal to the one considered above, but from the engineering point of view this method may prove to be preferable.

In conclusion, let us briefly consider a regime of acceleration where in fixed field machines there are several accelerating voltages V_1, $V_2 \ldots\ldots V_s$, changing with various frequencies ω_1, $\omega_2 \ldots\ldots \omega_s$. These frequencies may be selected, for example, in the interval $q_{min}\,\Omega_{min}$ — $q_{max}\,\Omega_{max}$, where Ω_{min} and Ω_{max} are the minimum and maximum values of the revolution frequency in the magnet system and q is the harmonic number. The regime of such type had already been considered in our Institute in 1948-1949 for some special cases. But at that time this method did not find application. For cases of practical interest, the mean time of acceleration corresponds to many periods of change for the voltages V_k. Therefore, even with good frequency stability, the coherent nature (phasing) of the different voltages will probably be disturbed. The particle will be subjected to the action of voltages V_k with frequencies ω_k, whose phase may be considered as a random quantity. Therefore, under certain conditions, the action of a system with several fixed frequencies may be described by means of the stochastic theory of acceleration developed here.

LIST OF REFERENCES

1. Burshtein, E. L., Veksler, V. I. and Kolomenskij, A. A. A stochastic method of particle acceleration. AERE (*) Lib/Trans 623. October 1955. (Original Russian appeared *In* : Some problems in the theory of cyclic accelerators. Moscow, USSR Academy of Sciences, 1955, p. 3-6).

2a. Kolomenskij, A. A. "Simmetrichnyj" Kol'tsevoj fazotron s vstrechnymi puchkami. Zh. eksper. teor. Fiz. SSSR, *33*, p. 298-9, 1957. (*in* Russian); A "*symmetric*" *circular synchro-cyclotron with oppositely directed beams.* Soviet physics, JETP, *6*, p. 231-3, 1958. (in English).

b. Ohkawa, T. Two-beam fixed field alternating gradient accelerator. Rev. sci. Instrum., *29*, p. 108-17, 1958.

3. Symon, K. R. and Sessler, A. M. Methods of radio frequency acceleration in fixed field accelerators with applications to high current and intersecting beam accelerators. CERN Symp. 1956. *1*, p. 44-58.

4. Kharkevich, A. A. Spectra and spectral analysis. Moscow, State Applied Sciences Publishing House, 1957.

5. Kanunnikov, V. N., Kolomenskij, A. A., Lebedev, A. N., Ovchinnikov, E. P., Stolov, A. M., Titov, V. A., Fateev, A. P., Yablokov, B. N. Investigations connected with the design of accelerators of the ring phasotron type. See p. 89.

(*) See note on reports, p. 696.

DISCUSSION

SYMON : I think one consideration with regard to the stochastic method would be the effect of a space charge limit near the injector where you have a much larger current than eventually reaches the target.

KOLOMENSKIJ : Symon's thoughts are correct, I think, because actually in the injection region there is, as a rule, a large space charge and because of this there are large particle losses. It is for this reason that we must have a sufficiently intensive $P(\omega)$ in the region of injection in order to keep the particles in this region only a small time. In order to take into account the injection problems we must choose their spectrum very carefully, and because the injection problems are very complicated we have only some evaluations in this direction; I think we may discuss this with Symon and with any other people who want to.

O'NEILL : Has Kolomenskij estimated the radio-frequency power required to make a voltage of several kilovolts over a wide frequency spectrum like this?

KOLOMENSKIJ : I think that by appropriate choice of the voltage spectrum the power in our case may be reduced to about 10 kilowatts.

BARBIER : I would like to know the mean acceleration time to get to 30 MeV and the number of accelerating sections, V_1, V_2, V_3.

KOLOMENSKIJ : The average time that corresponds to reaching the maximum energy is approximately

$$t_{\text{accel}} \simeq \left(\frac{E_{\max}}{\Delta E}\right)^2 t_{\text{revol}}$$

and here we must take into account the time of revolution. It is the order of magnitude but in various cases there may be different functions $P(\omega)$.

EXPERIMENTS ON STOCHASTIC ACCELERATION

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I. PRINCIPLES OF STOCHASTIC ACCELERATION

We have constructed a cyclic accelerator producing a continuous extracted beam of protons of 4.4 MeV and an internal beam of the order of 1 μA. This accelerator, which is in some respect similar to a synchro-cyclotron, has a diameter of 50 cm. The magnetic field of 14 000 G decreases by 8% from the centre to the radius of extraction. Its single electrode in a shape of a D is fed by a high frequency generator producing 5 kW and 2 000 V_{rms}. The voltage given by this generator varies at random and its Fourier analysis shows a continuous spectrum in the 21-23 Mc/s band. The idea of stochastic acceleration was put forward for the first time by Burshtein, Veksler and Kolomenskij [1]. We announced the practical application of this idea in an earlier paper [2].

To study the motion of the protons, reference should be made to the phase diagram on Fig. 1 (see a paper by the author [3]).

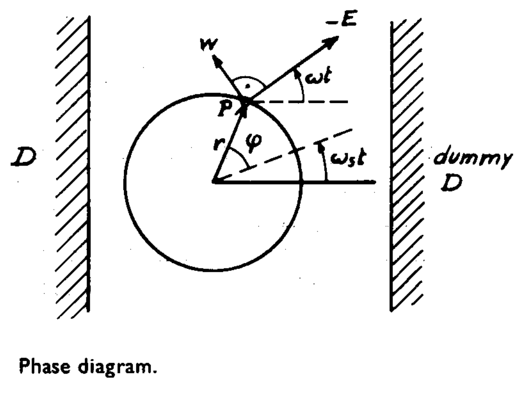

Fig. 1 Phase diagram.