The signature of the quadratic generalized uncertainty principle on the newtonian gravity and galaxy mass profile

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Abstract. In this study, we discuss the corrections implies by the presence of the general uncertainty principle (GUP) on Newton's law of gravity by virtue of Verlinde's proposal. We argue here that GUP leads to twofold modification, namely on the equipartition theorem and the holographic relation (Bekenstein-Hawking formula). Hence, following Verlinde's proposal, we obtain quantum corrections term to the Newtonian gravity. In addition, we also report the quantum corrected mass profile of the galaxy. We restricted our derivation to first order in the GUP's free parameter and compared it analytically with the other relevant works.

1. Introduction

Statistical mechanics has given a satisfactory explanation for the aspects of thermodynamics. Given a particle distribution, one could describe the phase space of the particles and their distribution function to generally derived the equation of state [1,2]. Unlike the volume of space which could have value arbitrarily, the phase-space volume is bounded by the Heisenberg principle. Accordingly, any modifications on the commutation relations (or Poisson relation) should imply a modification of the invariant phase-space volume [3].

The standard quantum mechanics was initially being considered without any concern for the effects of gravity. Standard quantum mechanics have the canonical relationship between position and momentum operator, which eventually leads to the Heisenberg Uncertainty Principle (HUP) [4]. The expression of the HUP equation does not provide an upper or lower bound for either Δx or Δp. More precisely, there is no restriction to the uncertainty of the position and momentum of a particle. The effect is that the position of objects cannot be ascertained with certainty in space. In other words, space fuzziness $\Delta x_{min.} = 0$. Quantum gravity predicts modification on the quantum mechanical canonical relation [5,6,7]. The Generalized Uncertainty Principle (GUP) was introduced to include the space fuzziness (namely minimal length proposal) into the formulation of quantum mechanics. To achieve this, one must relax the Planck constant to be momentum-dependent [7,8]. Due to quantum gravity, the granular structure should be relevant at the Planckian scale, and so the parameter is on the Planckian scale. Therefore, just by taking the quadratic function of p, one should obtain $\Delta x_{min.} \approx lp \neq 0$, where lp is the Planck length $\approx 10^{-35}$ m. The Planck length containing the gravitational constant indicates the inclusion of gravitational effects in quantum mechanics.

Motivated by black hole thermodynamics, it is natural to think that there are deep connections between gravity and thermodynamics [9,10]. More recently, an interesting setting was proposed by Verlinde, which argues that gravity is instead an emergent phenomenon [11]. He argues that space is a storage space for information. Given a compact region of space, the information on the bulk has resided on the closed surface surrounding it, which he calls as the holographic screens. Unlike his predecessor

[9,10], this assumption does not restrict the closed surface to be a horizon, which raised several contradictions [12]. By assuming that these screens satisfy the Bekenstein-Hawking relation and the bits of information obey the equipartitions law, he obtained the Newtonian and Einstein gravity.

Since Verlinde's proposal relies on the Bekenstein-Hawking relation, it is interesting to include the quantum gravity corrections and observed the correction it generates. In reference [13] obtained the quantum correction to Newtonian gravity within the linear GUP approach, which is a resemblance with the result by the Randall-Sundrum II model with a different sign [14]. Afterwards, on reference [15], discuss about modified Newtonian gravity, which aims to show that by taking the subsequent leading correction in Bekenstein-Hawking, the terms containing $\hbar$ have appeared. They argue that as the concept of minimal length has been invoked in the Bekenstein entropic derivation, the GUP, which is a direct consequence of the minimal length, should be considered in the entropic interpretation of gravity. Indeed, based on GUP, it has been demonstrated that the black hole Bekenstein entropy area law has a logarithmic correction as the next leading term. When applying it to the entropic interpretation, the resulting gravity force law does include sub-leading order correction terms that depend on $\hbar$. Recently in [16], discussed modified Newtonian gravity with various types of GUP, namely linear, quadratic, and linear-quadratic GUP. Together with the modified equation of state (EoS) for the degenerate Fermi gas generated by GUP, they constraint the GUP's free parameter by comparing it with the recent data of white dwarf mass profile [16].

In this work, we applied the modification due to the presence of quadratic GUP on the phase space volume to derive the equipartitions law and equation of state for ideal gas. Later, we obtain quantum corrections term following the Verlinde's procedure for the Newtonian gravity and quantum corrected mass profile of the object in the galaxy. Finally, the results of this modification of gravitational force are compared with the other relevant studies [13,15,16]. The differences between our results and the three references above, i.e., in the resulting Newtonian gravity expression, has a distinct form although with quantum correction. The given correction term sign, the power r of the correction term, and the behaviour of the correction to the parameter sign differ from references [13,15] but are somewhat similar to reference [16]. This present paper is organized as follows. In section 2, we discuss the used formalism briefly. Next, in section 3, we provide the result and discussion, while section 4 is for the conclusion.

2. Formalism

In this section, we briefly discuss the main idea of the generalized uncertainty principle, entropic gravity, and galaxy mass profile. After reviewing the key aspects, we will apply the formalism in the next section to observed the quantum corrections generates by introducing GUP.

2.1. The Quadratic Generalized Uncertainty Principle (GUP) Equations

The GUP approach used is limited only to reviewing the modified phase-space volume and the standard Heisenberg commutation relation that changes become to GUP. This GUP is a modification of the standard uncertainty principle of the Heisenberg Uncertainty Principle / (HUP), with the expression HUP [4]

$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{1}$$

with $\hbar = \frac{h}{2\pi}$, h denote Planck constant. The modified HUP relation in quadratic GUP becomes [7]

$$\Delta x \Delta p \geq \frac{\hbar}{2}(1+\beta p^2) \tag{2}$$

Where β can be scale as $\beta = \frac{\beta_0 l_p^2}{\hbar^2}$, with β_0 is free parameter dimensionless of GUP, and Planck length $l_p = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-35}\, m$. The minimum volume of modified phase-space from $\Delta V = \hbar^3$ to $\Delta V_{GUP} = \hbar(\text{p})^3$ which for quadratic model $\Delta V_{GUP} = \hbar^3(1+\beta p^2)^3$. As a result, the volume invariance of the phase-space become modified [3]

$$\int \frac{d^3xd^3p}{\hbar^3} \rightarrow \int \frac{d^3xd^3p}{\hbar^3(1+\beta p^2)^3} \quad (3)$$

Hence, states with high momentum are suppressed. This modification affects the number of microstates in the thermodynamic system and affects the Equation of State (EoS) ideal gas.

2.2. The Entropic Gravity

The entropic gravity or emergent gravity is also called the entropic force. The idea of entropic force is used by emergent gravity to show that gravity is an emergent phenomenon.

Based on Verlinde's work [11], it is reviewed quantitatively if the rotating polymer chain is placed in a hot reservoir with temperature T (so that the temperature is constant throughout the process) is pulled by the external force F as far as Δx. Then an entropic force will appear, which opposes the external force. Through the definition of thermodynamics $\frac{1}{T} = \frac{\partial S}{\partial E}$ and force $F = \frac{\partial E}{\partial x}$, where E is the energy in the system, combining equations $\frac{1}{T} = \frac{\partial S}{\partial E}$ and force were obtained

$$\frac{F}{T} = \frac{\partial S}{\partial x} \quad (4)$$

Then from [11], the number of information or number of bits on the holographic surface is assumed to be proportional to the surface area (Bekenstein-Hawking formula)

$$N = \frac{Ac^3}{G\hbar} = \frac{4S_{BH}}{k_B} \quad (5)$$

$$S_{BH} = \frac{k_B A}{4l_p^2} \quad (6)$$

this system of bits on the holographic surface satisfies the ideal gas distribution, so the ideal gas equation is used

$$E = \frac{Nk_BT}{2} \quad (7)$$

Verlinde then used Bekenstein's idea that if a test object is brought near along Δx is the test body Compton wavelength towards the surface, then the related surface's entropy will increase $2\pi k_B$. So that equations (4) and (7) can be obtained

$$F = T\left(\frac{2\pi k_B mc}{\hbar}\right) = \left(\frac{2E}{Nk_B}\right)\frac{2\pi k_B mc}{\hbar} \quad (8)$$

energy E is the total energy of the mass system or mass source M, so $E = Mc^2$. By substituting E and N through equation (5) and the surface area of the holographic surface $A = 4\pi R^2$, it will be obtained

$$F = \frac{GMm}{R^2} \quad (9)$$

Amazingly, Newton's equations of gravity emerge.

Interestingly, quantum gravity models have a common prediction for the Bekenstein-Hawking relation (6). Various approaches [17, 18, 19, 20] predict that the next leading term is in the logarithmic form. As can be seen in [18] that GUP, as a low energy consequence of quantum gravity is also generated a similar correction term which is in the form

$$S = S_{BH} + c_0 ln(S_{BH}) + \sum_{n=1}^{\infty} c_n (S_{BH})^{-n} + const. \tag{10}$$

For GUP to first order has an expression [17, 18]

$$S = \frac{k_B A}{4l_p^2} - \frac{k_B \pi \beta_0}{16} ln\left(\frac{A}{4l_p^2}\right) \tag{11}$$

Whereas can be seen from the equation above that the GUP parameter is taking the place as the expansion coefficients (here, we only show to first order). We will obtain the correction generates by such a modification on Newtonian gravity in the next section.

2.3. Standard Galaxy Mass Profile

In relaxed galaxy clusters, to estimate the mass of the cluster is to place the Newtonian hydrostatic equilibrium equation for spherical symmetry [21]

$$\frac{dP}{dr} = \frac{-GM\rho}{r^2} \tag{12}$$

This equation can also be used to determine the structure of a star. In the ideal gas state equation, it can be directed to determine the mass of the hydrostatic equilibrium *M (r)*, which depends on the temperature of the gas *T (r)* and its density *ρ (r)*

$$M(r) = -\frac{k_B T r}{G\mu m_p}\left(\frac{dln\rho(r)}{dlnr} + \frac{dlnT}{dlnr}\right) \tag{13}$$

Equation (13) is a combination of the Newtonian equation with the equation of the ideal gas state.

3. Result and Discussion

In this section, we elaborate on our result. We will explain concerning the utilization of GUP on Newtonian gravity, phase-space volume, and galaxy mass profile.

3.1. GUP Correction on The Ideal Gas

Using formalism before, the results will be determined. We begin use equation (3) to the closed phase-space with the energy surface equation [1]

$$\sum(E) = \frac{1}{\hbar^{3N}} \int_{H(q,p)\,=\,E} \frac{d^3p_1 \ldots d^3p_N d^3q_1 \ldots d^3q_N}{(1+\beta p^2)^{3N}} \tag{14}$$

Momentum $p^2 = 2.M_b.E$, where M_b is the mass of an individual atom of the ideal gas so that we can write

$$\sum(E) = \frac{V^N}{\hbar^{3N}(1+2\beta M_b E)^{3N}} \int_{H(q,p)=E} d^3p \tag{15}$$

Which after the evaluation yields

$$\sum(E) = \frac{\pi^{3N/2} V^N}{N!\frac{3N}{2}\Gamma\left(\frac{3N}{2}\right)\hbar^{3N}} \frac{\partial B(E)}{\partial E} \tag{16}$$

where $B(E) = \left(\frac{\sqrt{2M_b E}}{1+2\boldsymbol{\beta} M_b \boldsymbol{E}}\right)^{3N}$. Since the entropy is defined as S = $k_B.\ln \sum(E)$, then it modified expression become

$$S = k_B \ln\left[\left(\frac{\pi^{3N/2} V^N (2M_b E)^{3N/2}}{N!\,\Gamma\left(\frac{3N}{2}\right)\hbar^{3N}}\right)\frac{1-2\beta M_b E}{(1+2\beta M_b E)^{3N+1}}\right] \tag{17}$$

Put in Stirling approximation and limit thermodynamic

$$S = Nk_B\left[\frac{5}{2} + \ln\left(\frac{V}{N\hbar^3(1+2\beta M_b E)^3}\left(\frac{4\pi M_b E}{3N}\right)^{3/2}\right)\right] \tag{18}$$

By defining

$$f(E) \equiv \frac{E}{(1+2\beta M_b E)^2} = \frac{3\hbar^2 N^{5/3}}{4\pi M_b V^{2/3}} \exp\left[\frac{2S}{3Nk_B} - \frac{5}{3}\right] \tag{19}$$

the temperature $T = \frac{2E}{3k_B N}$ can be written as

$$T_{mod.} = \frac{2}{3k_B N}\frac{(1+2\beta M_b E)^3}{1-2\beta M_b E} f(E) = \left(\frac{1+2\beta M_b E}{1-2\beta M_b E}\right)\frac{2E}{3k_B N} \tag{20}$$

Equation (20) shows that quadratic GUP gives a modified relation between temperature and the energy. For the pressure

$$P_{mod.} = -\frac{\partial \mathrm{E}}{\partial \mathrm{V}} = \frac{2E}{3V}\left(\frac{1+2\beta M_b E}{1-2\beta M_b E}\right) \tag{21}$$

From equations (20) and (21), we obtained the relation between pressure and temperature. Hence EoS results from the GUP effect do not change, which is given by $PV = Nk_BT$.

Other results related to GUP on EoS ideal gas equation have also been resolved in the following references [22, 23]. They develop statistical mechanics in the GUP framework and their implications in an ideal gas's statistical mechanics.

3.2. Quantum Correction to Newtonian Gravity and Galaxy Mass Profile

Next, we calculate for modified entropy from commutation relation in equation (2). Following ref. [17] after expanding Taylor, equation (2) and from standard entropy of Bekenstein-Hawking equation (6), obtained its modified entropy and used approximation to first order in the GUP's free parameter

$$S_{mod.} = S_{BH} + S_{correct.term} = \frac{k_B A}{4l_p^2} - \frac{k_B \pi \beta_0}{16} ln\left(\frac{A}{4l_p^2}\right) \tag{22}$$

Substitute (22) to (5) to get N modified $N_{mod.} = \frac{4S_{mod.}}{k_B}$

$$N_{mod.} = \frac{A}{l_p^2} - \frac{\pi\beta_0}{4} ln\left(\frac{A}{4l_p^2}\right) \tag{23}$$

Remember, energy E is $E = Mc^2$ namely the energy of bulk matter. Based on equation (6), gas equipartition on equation (7) and using equation (23), so equation (19) becomes

$$T_{mod.} = \frac{2E}{k_B\left[\frac{A}{l_p^2} - \frac{\pi\beta_0}{4} ln\left(\frac{A}{4l_p^2}\right)\right]}\left(\frac{1+2\beta M_b Mc^2}{1-2\beta M_b Mc^2}\right) \tag{24}$$

According to Verlinde, gravity is an entropic force, so that it can be derived from the relation $F_{mod.} = T_{mod.}\frac{\Delta S}{\Delta x}$. By using this and notice that, $M = N.\ M_b$, we obtain

$$F_{mod.} = \frac{4\pi Mmc^3}{\hbar}\left(\frac{1}{\left[\frac{A}{l_p^2} - \frac{\pi\beta_0}{4}ln\left(\frac{A}{4l_p^2}\right)\right]}\right)\left(\frac{1+2\beta N{M_b}^2c^2}{1-2\beta N{M_b}^2c^2}\right) \tag{25}$$

Since A = $4\pi R^2$, $\beta_0 = \frac{\beta\hbar^2}{l_p^2}$ and $l_p^2 = \frac{G\hbar}{c^3}$

$$F_{mod.} = \frac{GMm}{R^2}\left(\frac{1}{\left[1 - \frac{\beta\hbar^2\pi}{16\pi R^2}ln\left(\frac{A}{4l_p^2}\right)\right]}\right)\left(\frac{1+2\beta N{M_b}^2c^2}{1-2\beta N{M_b}^2c^2}\right) \tag{26}$$

and by approximating to first order

$$F_{mod.} = \frac{GMm}{R^2}\left(1+\beta\left[\frac{M^2G\hbar}{\pi R^2c} + \frac{\hbar^2}{16R^2}ln\left(\frac{\pi R^2}{l_p^2}\right)\right]\right) \tag{27}$$

We obtain a new form of modified Newtonian gravity, which gives the result of attractive force on the correction term. As can be seen, the corrections have a quantum origin due to the presence of $\hbar$. The resulting equation (27) is a resemblance to the result in [15]. Since, in addition to the modified Bekenstein-Hawking relation, we also incorporate the modification on the temperature-energy relation, something that has not been considered in [16].

Next, we evaluate from (27) the gradient of the pressure $\frac{dP}{dr}$. We know that $dP.dA = F$, then $dA = \frac{dV}{dr}$. Where V is the variable of volume. So we can write $\frac{dP}{dr} = \frac{F_{mod.}}{dV}$

$$\frac{dP}{dr} = \frac{GM\rho}{r^2}\left(1+\beta\left[\frac{M^2G\hbar}{\pi r^2c} + \frac{\hbar^2}{16r^2}ln\left(\frac{\pi r^2}{l_p^2}\right)\right]\right) \tag{28}$$

Match it with equation (12)

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}\left(1+\beta\left[\frac{M^2G\hbar}{\pi r^2c} + \frac{\hbar^2}{16r^2}ln\left(\frac{\pi r^2}{l_p^2}\right)\right]\right) \tag{29}$$

$$M(r)_{Newt.} = -\frac{r^2}{G\rho}\left(\frac{dP}{dr}\right)\left\{1+\beta\left[\frac{M^2G\hbar}{\pi r^2c} + \frac{\hbar^2}{16r^2}ln\left(\frac{\pi r^2}{l_p^2}\right)\right]\right\}^{-1} \tag{30}$$

Combine with $\frac{dP}{dr}$ ideal gas [21]

$$\frac{dP}{dr} = \frac{k_B}{\mu m_p}\left(T\frac{d\rho}{dr} + \rho\frac{dT}{dr}\right) \tag{31}$$

Substitute equation (31) to equation (30) to obtain galaxy mass profile with quantum correction

$$M(r)_{mod.} = -\frac{k_BTr}{G\mu m_p}\left(\frac{dln\rho(r)}{dlnr} + \frac{dlnT}{dlnr}\right)\left\{1+\beta\left[\frac{M^2G\hbar}{\pi r^2c} + \frac{\hbar^2}{16r^2}ln\left(\frac{\pi r^2}{l_p^2}\right)\right]\right\}^{-1} \tag{32}$$

The mass of object galaxy clusters was corrected in equation (32). This mass also depends on the GUP's free parameter, which contains minimal length. So, GUP impacts the mass of objects in the galaxy that can modify the equation mass of the galaxy. Further, by matching the galaxy's mass data, the parameters range in the GUP's free parameter can be determined.

4. Conclusion

We can conclude that we have found a new form of Newton's law of gravity. The gravitational force of Newton has a quantum correction term which contains Planck length, which has origin from the deformation of the equipartition theorem and the holographic relation with entropy Bekenstein-Hawking formula. Then this can be implemented in the mass of objects in galaxy clusters because the mass of these objects depends on Newtonian masses. If Newton's gravitational force changes, it is clear that the mass of the galaxy will also be corrected. The result of this modification of Newton's law is compared to other relevant works on ref. [13,15,16]. The structure is similar to the expansion form of [16]. That is, it contains two correction terms in the first-order correction. By applying the GUP deformation on gravity and ideal gas EoS, we arrived at the modified expression for the galaxy mass profile. For future work, this corrected galaxy mass equation can be applied to the existing galaxy mass data, then determine the free parameter constraint on GUP that fits in what range.

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Acknowledgments

We acknowledge the Indonesian Endowment Fund for Education (LPDP), Ministry of Finance, the Republic of Indonesia for providing the funding. We also want to thank our colleague IH from Sampoerna University for precious discussions and many insights into our formalism.