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The role of supersymmetry in the black hole/qubit correspondence

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Declaration

The work presented in this thesis was carried out in the Theoretical Physics Group at Imperial College London between August 2007 and July 2010 under the supervision of Professor Michael J. Duff. Except where otherwise stated, the work is original and has not been submitted before for a degree of this or any other university.

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September 2010

Abstract

This thesis explores the numerous relationships between the entropy of black hole solutions in supergravity and the entanglement of multipartite systems in quantum information theory: the so-called black hole/qubit correspondence.

We examine how, through the correspondence, the dyonic charges in the entropy of supersymmetric black hole solutions are directly matched to the state vector coefficients in the entanglement measures of their quantum information analogues. Moreover the U-duality invariance of the black hole entropy translates to the stochastic local operations and classical communication (SLOCC) invariance of the entanglement measures. Several examples are discussed, with the correspondence broadening when the supersymmetric classification of black holes is shown to match the entanglement classification of the qubit/qutrit analogues.

On the microscopic front, we study the interpretation of D-brane wrapping configurations as real qubits/qutrits, including the matching of generating solutions on black hole and qubit sides. Tentative generalisations to other dimensions and qubit systems are considered. This is almost eclipsed by more recent developments linking the nilpotent U-duality orbit classification of black holes to the nilpotent classification of complex qubits. We provide preliminary results on the corresponding covariant classification.

We explore the interesting parallel development of supersymmetric generalisations of qubits and entanglement, complete with two- and three-superqubit entanglement measures. Lastly, we briefly mention the supergravity technology of cubic Jordan algebras and Freudenthal triple systems (FTS), which are used to: 1) Relate FTS ranks to three-qubit entanglement and compute SLOCC orbits. 2) Define new black hole dualities distinct from U-duality and related by a 4D/5D lift. 3) Clarify the state of knowledge of integral U-duality orbits in maximally extended supergravity in four, five, and six dimensions.

For my parents

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Chapter 1

Introduction

1.1 An overview

The past four years have seen the development of a remarkable body of work linking what are, at first glance, two strikingly different areas of theoretical physics: string theory and the entropy of supersymmetric black hole solutions on the one hand, and quantum information theory and the entanglement of multipartite qubit (two-level) and qutrit (three-level) systems on the other.

The starting point is a 2006 paper by Mike Duff [1] in which he established the relationship between the entropy of black hole solutions of the so-called *STU* supergravity model [2] and an an invariant known as Cayley's hyperdeterminant [3]. In particular the fact that the hyperdeterminant, which was first written down in 1845, finds modern use as a measure of entanglement of three-qubit systems [4].

Having discovered that these disparate disciplines share the same mathematics the race was on to find evidence of a more substantive physical underpinning or duality; something to indicate that, underneath it all, the same physical phenomenon was being described. Kallosh and Linde [5] responded with several important results, one being a link between the classification of three qubits entanglement and whether *STU* black holes have vanishing horizon areas and whether they preserve a fraction of supersymmetry. In addition they showed that the most general black hole and black ring entropy in $\mathcal{N} = 8$ supergravity/M-theory, given by Cartan's quartic $E_{7(7)}$ invariant, can be written in a canonical basis that reduces it to Cayleys hyperdeterminant.

These ideas were developed further in a collaboration between Duff and Ferrara [6–8] in which the $\mathcal{N} = 8$ generalisation was fleshed out by providing a tripartite-entangled seven-qubit system with a proposed entanglement measure to correspond to the $\mathcal{N} = 8$

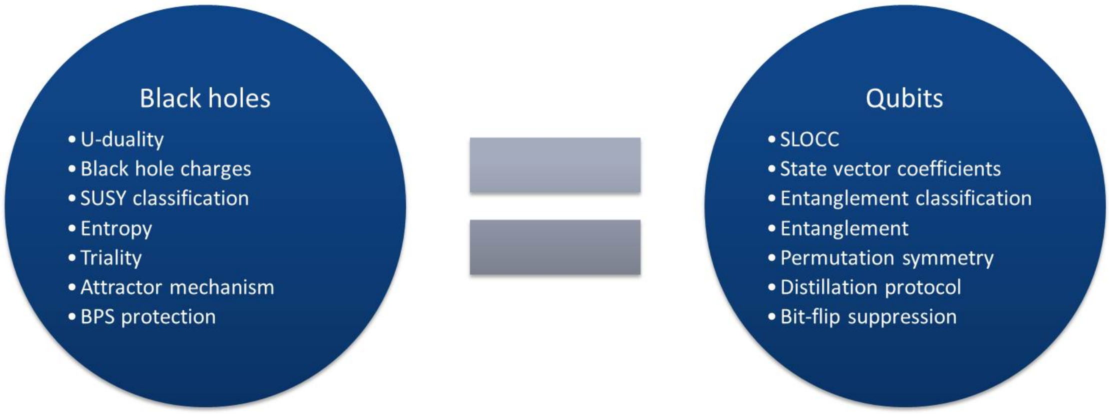


Figure 1.1: Some of the links between black holes and qubits

entropy. Moreover they put forward a $D = 5$ generalisation connecting black hole/string entropy to the bipartite entanglement of three qutrits, with Cartan's cubic $E_{6(6)}$ invariant playing the role of entropy and entanglement. Similar results in 4D and 5D were outlined for the "magic" $\mathcal{N} = 2$ supergravities. Meanwhile Levay [9–11] expanded the scope further, relating the attractor mechanism on the black hole side with optimal distillation protocols on the quantum information side, relating supersymmetric and non-supersymmetric black holes to the suppression or non-suppression of bit-flip errors, and providing an enlightening description of the seven-qubit tripartite entanglement in terms of the Fano plane, opening the door to the octonions, which would play several roles in future work within the correspondence.

This was the state of play at the outset of my involvement in the work done with L. Borsten, M. J. Duff, and W. Rubens, at times in collaboration with H. Ebrahim, A. Marrani, and S. Ferrara, and benefiting from helpful conversations with P. Levay. Since then further papers have enriched the correspondence [12–20], feeding the ever growing dictionary translating phenomena in one language to those in the other (see Figure 1.1). Meanwhile we have continued its expansion along the following avenues:

1. A microscopic interpretation of the correspondence based on wrapped D-branes, laying the foundations for the classification of entanglement based on intersecting

D-branes [21].

2. A set of dictionaries to transform between the duality frames of the $\mathcal{N} = 8$ entropy and seven-qubit tripartite entanglement, in effect providing confirmation of this correspondence. Broader links between entanglement classification and black hole classification. A new outlook on the symmetries of M-theory based on imaginary quaternions and octonions [22, 23].

3. A new classification of three qubit entanglement using supergravity technology [24].

4. The discovery of a new symmetry of black holes called Freudenthal duality [25].

5. The introduction of the superqubit, a supersymmetric generalisation of the qubit, with super-analogues of familiar entanglement measures [26].

6. An investigation of the integral U-duality orbits of black holes [27].

7. A derivation of the classification of four-qubit entanglement using string theory [28].

1.2 Motivation and outline

Clearly then, in a short time the black hole/qubit correspondence has become a sprawling, multifaceted body of work. Nevertheless, there are a few loosely unifying themes that can be picked out to lend more coherence to our review of recent developments. One path is the recurring topic of supersymmetry, which plays an obvious role in the superqubit construction, but is present throughout our work in the matching of entanglement classification to black hole classification, be it in the guise of our four qubit discoveries or proposed generalisations to our brane wrapping perspective. Another identifiable narrative is of course the use of the supergravity technology involving cubic Jordan algebras and Freudenthal triple systems to classify entanglement, construct new dualities, and inspect discrete U-duality orbits.

Since two of the co-authors of [21, 23–28] are submitting their theses simultaneously, let us make crystal clear that while both draw from [23], we here focus on the role of supersymmetry and the papers [21, 26, 28]. The technological, algebraic developments of the papers [24, 25, 27] are dealt with elsewhere, so we limit ourselves to a most rudimentary discussion: a mere statement of the developments, including such definitions as are necessitated by the highly interrelated nature of the research. In reviewing the role of supersymmetry our two main motivations are

- To survey the correspondences between the entanglement hierarchies of various multipartite systems and the SUSY classification of black holes in several theories.
- To study the generalisation of quantum systems experiencing entanglement to their superentangled, minimal supersymmetric extensions.

In beginning the exploration of the black hole/qubit correspondence, we require a certain amount of prerequisite material as pertains to black holes in supergravity and quantum information theory. In chapter 2 the necessary concepts such as U-duality, extremality, etc are introduced, followed by the theories of interest. We then review the basics of entanglement and entanglement classification in chapter 3, supported by some elementary concepts in classical invariant theory in section 3.3. This is rounded off by a discussion of the early correspondences in chapter 4, dealing with the STU model in section 4.1, the $\mathcal{N} = 8$ generalisation in section 4.2, and the $D = 5$ generalisation in section 4.3.

Moving on to the main topic, we begin our review of newer material in chapter 5, starting in section 5.1 with a brief outline of the explicit dictionaries through which we translated between different black hole perspectives and the qubit perspectives. This leads in to section 5.2 where we look at the alignment of black hole SUSY classification in various theories with qubit and qutrit entanglement classifications. The octonions are examined in section 5.3 as they play multiple distinct roles in the early correspondences.

Subsequently, we consider the wrapped brane perspectives in four and five dimensions in chapter 6. In chapter 7 we discuss the recent derivation of a four-qubit entanglement classification into 31 families by means of the SUSY distinguished U-duality orbits resulting from timelike reduction of string theory from $D = 4$ to $D = 3$. We then dive into full-on superlinear algebra with the supersymmetric generalisation of qubits and entanglement in chapter 8. This thesis then concludes in chapter 9 with a summary of the discoveries reviewed and avenues for future research. We have also provided in Appendix A supporting material concerning the algebraic tools of cubic Jordan algebras and Freudenthal triple systems.

As this is a review of aspects of our published body of work some chapters are closely based on our existing papers, however, previously unpublished material is incorporated, particularly in section 3.3.2, section 5.3, section 7.1, and section 9.2.

Chapter 2

Black holes

2.1 Black hole entropy

As discussed in chapter 1, the core of the black hole/qubit correspondence is an entropy/entanglement relationship. We begin then with a brief note on black hole entropy.

Work done by Bekenstein and Hawking [29, 30] in the 1970s established that black holes radiate energy and could be characterised as thermodynamic systems, with a set of relationships known as the laws of black hole mechanics describing geometrical properties of spacetime in a manner identical to the description of the properties of a statistical system by the laws of thermodynamics [31].

For example, the analogue of the first law of thermodynamics describes the relationship between the conserved quantities of a black hole when subject to small variations

$$\delta M = \frac{\kappa_S}{8\pi}\delta A + \mu\delta Q + \Omega\delta J, \tag{2.1}$$

where $M, \kappa_S, A, \mu, Q, \Omega$, and J are respectively mass, surface gravity, event horizon area, electric potential, charge, angular velocity and angular momentum. In particular, this law relates quantities defined at the event horizon such as area and surface gravity to quantities defined at infinity: mass, charge, and angular momentum.

Since the first law of thermodynamics contains a TdS term, one might hope to identify the geometrical analogue of entropy, provided the black-body temperature of radiation emitted by a black hole. This is of course the Hawking temperature T_{H}, fixed by treating the black hole in quantum field theory in a curved background, with gravity treated classically and matter fields quantum mechanically

$$T_{\rm H} = \frac{\hbar \kappa_S}{2\pi}. \tag{2.2}$$

Combining (2.1) and (2.2) suggests that the entropy of the black hole, the Bekenstein-hawking entropy, is proportional to its horizon area:

$$S_{\rm BH} = \frac{A}{4\hbar G_4}, \tag{2.3}$$

where G_4 is the 4D Newton constant. In light of its semiclassical gravity origins it is subject to higher order quantum corrections. The thermodynamic analogy is borne out by the second law of black hole mechanics which states $dA \geq 0$.

This is the macroscopic picture, but Boltzmann's entropy formula $S = k_{\rm B} \log W$ raises the question of the microscopic origin of the entropy, the understanding of which was provided only in 1996 by string theory [32] for the class of extremal black holes.

The supergravities of the various string theories admit black hole solutions [13,33,34], and one can compute their entropy. In the cases we consider it is expressed (solely, in the extremal case) in terms of electric and magnetic black hole charges. We now turn to the subject of extremality.

2.2 Extremal black holes

Even with no prior knowledge, it is reasonable to expect that quantum entanglement is expressed in terms of state vector coefficients. Our concern here is to clarify why black hole entropy should be expressed in terms of dyonic charges, knowing already that it is given by the horizon area.

Consider the line element for a static, spherically symmetric 4D black hole

$$ds^2 = -e^{2h(r)}dt^2 + e^{2k(r)}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{2.4}$$

In Einstein-Maxwell theory, the most general static black hole solution is the Reissner-Nordström solution

$$e^{2h(r)} = e^{-2k(r)} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{2.5}$$

which yields two horizons

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{2.6}$$

For $M \geq |Q|$ the singularity is hidden behind the horizon ($M < |Q|$ is excluded by the cosmic censorship hypothesis). In the special, *extremal* case $M = |Q|$ the horizons coincide at $r_+ = r_- = |Q|$ and the area of the horizon reduces to $4\pi Q^2$. As a consequence the entropy is fixed by (2.3) in terms of the charges. We note also that the surface gravity also vanishes at $M = |Q|$ and hence so does the Hawking temperature, leading one to conclude that the extremal black hole is stable. These considerations easily generalise to the case with nonzero magnetic charge P by replacing Q^2 with $Q^2 + P^2$.

The supergravity theories we concern ourselves with differ in having more than one photon (four for the STU model, 28 for $\mathcal{N} = 8, D = 4$, etc). Consequently there are more electric and magnetic charges. Extremal black holes in these cases obey generalised mass = charge relations, which also involve scalar moduli. This situation can be improved to one in which the moduli dependence drops out by fixing them (freezing them) at their horizon values - in this manner the entropy is ensured to be moduli independent [35]. As a simple example, consider the axion dilation extremal black hole [5, 36]. It has four charges p^0, p^1, q_0, and q_1, and its entropy is given by an $\mathrm{SL}_2(\mathbb{R})$ invariant, determinant-like expression:

$$S = \pi \left| p^0 q_1 - q_0 p^1 \right| . \tag{2.7}$$

An important point for our considerations is that a black hole preserving some unbroken supersymmetries is called *BPS* (after Bogomol'nyi-Prasad-Sommerfield). Being BPS implies extremality, but is not a necessary condition for it.

2.3 U-duality

The black hole entropies we consider are required to be U-duality invariants, with the charges transforming in representations of the U-duality group [37–40]. As a result of the correspondence, these invariants show up again on the qubit side. Consequently, a brief discussion of U-duality is in order.

One can view string theory as a worldsheet sigma model with background space-time as its target space [41, 42]. While different backgrounds can correspond to different quantum string theories, some can produce physically equivalent theories. Such backgrounds can be mapped into each other by discrete transformations coming from the symmetry groups of string dualities. The dualities that transform one theory to another are classified into T, S, and U-dualities where U-duality subsumes the others.

Consider superstring theory compactified on a circle of radius R. T-duality [43] sends the worldsheet theory with radius R to α'/R, where α' is the string tension. This generalises to toroidal compactification where the compact space is T^k, a k-dimensional torus. Fields of dimensionally reduced theory transform as representations of T-duality group. T-duality holds perturbatively order-by-order in the string coupling constant g_s. From the open string or D-brane perspective, T-duality transforms Dirichlet boundary conditions into Neumann boundary conditions and vice versa [44]. S-duality [45–51], on the other hand, is a generalisation of the electromagnetic duality of supersymmetric Yang-mills theories [52] and acts non-perturbatively in contrast to T-duality.

For $\mathcal{N} = 2$ compactifications the combined dualities give $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SO}_{\ell,2}(\mathbb{Z})$ where the low energy limit is $D = 4, \mathcal{N} = 2$ supergravity coupled to $\ell + 1$ vector multiplets [53]. For $\mathcal{N} = 4$ compactifications we have $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SO}_{6,m}(\mathbb{Z})$ where the low energy limit is $D = 4, \mathcal{N} = 4$ supergravity coupled to m vector multiplets [54].

Unifying S and T dualities yields U-duality. For M-theory on $\mathbb{R}^d \times T^k$ or string theory on $\mathbb{R}^d \times T^{k-1}$, where $d + k = 11$, the reduced theories are invariant under a global symmetry group called the U-duality group. The reduced d-dimensional low energy effective action, which is the d-dimensional supergravity theory, is invariant under a continuous symmetry group and the discrete subgroup of it, the U-duality group, is the symmetry of the full theory.

An analysis of the global symmetries of dimensionally reduced theories is accomplished through the consideration of the symmetries in their scalar sectors [55]. In essence, the symmetry extends to higher valence fields and acts linearly, with the non-linear realisation for the scalars only. Special care must be taken, since the scalars are not merely dilatons and axions, but also arise from the Hodge dualisation of $(D-1)$-form field strengths (i.e. $(D-2)$ form potentials) to 1-form field strengths of scalars. The dualisation generally entails a flip in the sign of the field's associated dilaton vector, and an interchange of Chern-Simons and transgression terms. It is the dilation vectors that hold the key to analysing the scalar manifold, specifically the vectors $\vec{b}_{i,i+1}$ and $\vec{a}_{123}$, with $i \in \{1, \ldots, 11-D\}$ for a D-valent Lagrangian form (in the case of a torus reduction from 11 dimensions). These are the simple roots of the global symmetry group G. A brief aside: the generators in the Lie algebra of a group can be partitioned into the Cartan subalgebra generators $\vec{H}$ and ladder operators $E_{\vec{\alpha}}$ satisfying:

$$\begin{aligned} \left[\vec{H}, E_{\vec{\alpha}}\right] &= \vec{\alpha} E_{\vec{\alpha}} \\ \left[E_{\vec{\alpha}}, E_{\vec{\beta}}\right] &= N(\alpha, \beta) E_{\vec{\alpha}+\vec{\beta}} \end{aligned} \tag{2.8}$$

The $\vec{\alpha}$ are root vectors, and these can be partitioned into positive and negative roots, where the prescription for allocating signs is to take the sign of their first non-zero elements. Simple roots are obtained as the minimal subset of the positive vectors that can

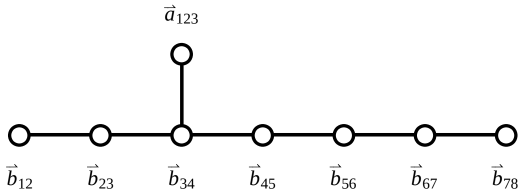


Figure 2.1: The Dynkin diagram for the global symmetry of a Lagrangian obtained via T^8 reduction of $D = 11$ SUGRA. The roots are labelled with the corresponding dilaton vectors. To obtain the diagram for T^n reduction, one only need restrict to the dilaton vectors that appear in the reduction in question.

generate all of the positive roots via linear combinations of themselves with nonnegative integer coefficients. The simple roots are sufficient to classify Lie algebras, and it is the relative root lengths and angles that get encoded in Dynkin diagrams. The general Dynkin diagram for a torus reduction from 11 dimensions is given in Figure 2.1 where one specialises to the dimensionality under consideration by amputating the appropriate root/dilaton vectors. Returning to the scalar manifold, one reads off G from the dilaton vectors, and by assembling the exponentiations of the positive-root and Cartan generators with dilatons and axions, it is possible to write the scalar Lagrangian in a form that makes the symmetries manifest. The procedure depends on whether dualisation of field strengths can add to the scalar content of the reduced theory, but in either case the object $\mathcal{V}$ can be constructed:

$$\mathcal{V} = e^{1/2\,\vec{\phi}\cdot\vec{H}} \left(\prod_{i<j} e^{\mathcal{A}^{ij}_{[0]} E^{ij}} \right) \exp\left(\sum_{i<j<k} A^{ijk}_{[0]} E^{ijk} \right) \tag{2.9}$$

where generators E^{ij} and E^{ijk} correspond to root/dilaton vectors $\vec{b}_{ij}$ and $\vec{a}_{ijk}$ respectively, and terms are ordered with indices appearing in reverse-canonical order. It is here that the transgression terms in the field strengths arises, since non-vanishing commutators between positive root generators contribute then via Baker-Campbell-Hausdorf. $\mathcal{V}$ constructed in this manner is in an upper triangular (or Borel) gauge. Defining $\mathcal{M} := \mathcal{V}^{\mathsf{T}}\mathcal{V}$ allows one to write the scalar Lagrangian as $\frac{1}{4}\,\mathrm{tr}\left(\partial\mathcal{M}^{-1}\partial\mathcal{M}\right)$. The beauty of this form of the Lagrangian is that a G transformation Λ on the scalars $\mathcal{V} \to \mathcal{V}\Lambda$ will result in the transformation $\mathcal{M} \to \Lambda^{\mathsf{T}}\mathcal{M}\Lambda$, which clearly leaves the Lagrangian invariant, thanks to trace cyclicity. In truth, the G transformation will generally break the Borel gauge and a compensating transformation $\mathcal{O}$ needs to be made so that $\mathcal{V} \to \mathcal{O}\mathcal{V}\Lambda$, which preserves the invariance. This $\mathcal{O}$ will lie in the maximal compact subgroup $K \subset G$ by virtue of the Iwasawa decomposition, which states that a general group element g can be written as $g_K g_H g_N$ with $g_K \in K, g_N \in G$ and g_H in the Cartan subalgebra. The fact that points on the scalar manifold are then specified via G transformations modulo K compensations indicates that the scalar manifold is the coset G/K. The trace form of the Lagrangian is for this reason referred to as a coset space Lagrangian, with $\mathcal{V}$ being a coset representative. The number of the scalar fields of the compactified theory is equal to the dimension of the coset space, $\dim G - \dim K$. A classification of symmetry groups of the supergravities with 32 supercharges in different dimensions has been given in Table 2.1 [56].

For reduction past dimension 6, the scalars resulting from dualisation complicate matters and a generalisation of the transpose is needed to replace the $\mathcal{M} = \mathcal{V}^{\mathsf{T}}\mathcal{V}$ relation since $\mathrm{USp}_8, \mathrm{SU}_8$, and SO_{16} are not orthogonal. The Cartan involution does the trick by operating at the generator level, flipping the signs of the noncompact ones; for example, the involution is simply dagger, † when acting on unitaries. The $\mathrm{E}_{n(n)}$ groups are the exceptional groups in their maximally noncompact form: whereas usually the ladder operator generators provide, through linear combinations, equal numbers of compact and noncompact generators, in this case the Cartan generators are also noncompact. (More generally, the bracketed term corresponds to the number of non-compact generators less the number of compact ones.) In fact, the general prescription for $3 \leq k \leq 8$ is $\mathrm{E}_{k(k)}$ [57].

Of special interest to us of course is the compactification to four dimensions where the U-duality group is the 133 dimensional exceptional Lie group $\mathrm{E}_{7(7)}$, with 63 compact and 70 non-compact generators. The resulting low-energy limit is $(D = 4, \mathcal{N} = 8)$ supergravity with 28 abelian vector fields. The 28 electric and 28 magnetic black hole charges transform as an irreducible **56** of $\mathrm{E}_{7(7)}$ as shown in Table 2.2, taken from [58,59]. We will later encounter the SL_2^7 subgroup in the context of a special 7-qubit system.

Table 2.1: The symmetry groups (G) of the low energy supergravity theories with 32 supercharges in different dimensions (D) and their maximal compact subgroups (K).

D	scalars	vectors	G	K
10A	1	1	$\mathrm{SO}_{1,1}(\mathbb{R})$	–
10B	2	0	$\mathrm{SL}_2(\mathbb{R})$	$\mathrm{SO}_2(\mathbb{R})$
9	3	3	$\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SO}_{1,1}(\mathbb{R})$	$\mathrm{SO}_2(\mathbb{R})$
8	7	6	$\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_3(\mathbb{R})$	$\mathrm{SO}_2(\mathbb{R}) \times \mathrm{SO}_3(\mathbb{R})$
7	14	10	$\mathrm{SL}_5(\mathbb{R})$	$\mathrm{SO}_5(\mathbb{R})$
6	25	16	$\mathrm{SO}_{5,5}(\mathbb{R})$	$\mathrm{SO}_5(\mathbb{R}) \times \mathrm{SO}_5(\mathbb{R})$
5	42	27	$\mathrm{E}_{6(6)}(\mathbb{R})$	USp_8
4	70	28	$\mathrm{E}_{7(7)}(\mathbb{R})$	SU_8
3	128	-	$\mathrm{E}_{8(8)}(\mathbb{R})$	$\mathrm{SO}_{16}(\mathbb{R})$

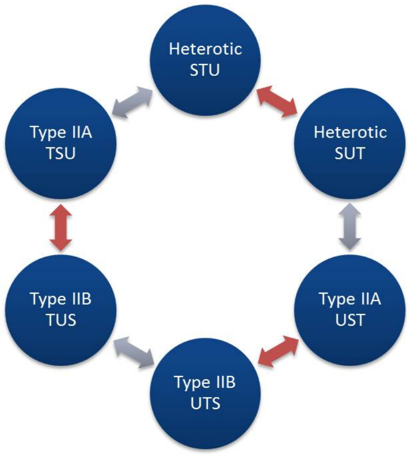


Figure 2.2: String/string/string triality. Red arrows are mirror transformations, grey arrows are string/string dualities.

2.4 The STU model

The STU model, so named for it's three complex scalar fields or moduli (the dilaton/axion, complex Kähler form, and complex structure fields), is the theory whose low energy limit is a four dimensional $\mathcal{N} = 2$ supergravity coupled to three vector multiplets interacting through the special Kähler manifold $[\mathrm{SL}_2 / \mathrm{SO}_2]^3$ [15, 60–62]. It can be obtained via dimensional reduction of type IIA supergravity on the Calabi-Yau $K3$ with $D0, D2, D4$, and $D6$ branes wrapping on the cycles. It can be usefully embedded in an $\mathcal{N} = 4$ theory obtained from compactification of the heterotic string on T^6 and exhibits an $\mathrm{SL}_2(\mathbb{Z})_S$ strong/weak coupling duality and an $\mathrm{SL}_2(\mathbb{Z})_T \times \mathrm{SL}_2(\mathbb{Z})_U$ target space duality. String/string duality furnishes an alternative view in which it is a truncation of an $\mathcal{N} = 2$ theory obtained by compactifying Type IIA on $K3 \times T^2$ where S and T exchange roles [48, 63, 64]. This is yet again equivalent through mirror symmetry to a Type IIB string on the mirror manifold. Indeed it seems all roads lead to STU, but this profusion is a symptom of $D = 4$ string/string/string triality [61] as shown in Figure 2.2. Combined, the theory exhibits a full S-T-U triality symmetry and SL_2^3 duality [61]. The

Table 2.2: The representations of the U-duality group G of all the forms of maximal supergravities in any dimension, omitting the scalars which parameterise the coset G/K. The $(D-2)$-forms dual to the scalars always belong to the adjoint representation.

D	G	Form Valence									
		1	2	3	4	5	6	7	8	9	10
10A	$\mathbb{R}^+$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$		$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$ $\mathbf{1}$
10B	$SL_2(\mathbb{R})$		$\mathbf{2}$		$\mathbf{1}$		$\mathbf{2}$		$\mathbf{3}$		$\mathbf{4}$ $\mathbf{2}$
9	$SL_2(\mathbb{R}) \times \mathbb{R}^+$	$\mathbf{2}$ $\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$ $\mathbf{1}$	$\mathbf{3}$ $\mathbf{1}$	$\mathbf{3}$ $\mathbf{2}$	$\mathbf{4}$ $\mathbf{2}$ $\mathbf{2}$	
8	$SL_3(\mathbb{R}) \times SL_2(\mathbb{R})$	$(\overline{\mathbf{3}},\mathbf{2})$	$(\mathbf{3},\mathbf{1})$	$(\mathbf{1},\mathbf{2})$	$(\overline{\mathbf{3}},\mathbf{1})$	$(\mathbf{3},\mathbf{2})$	$(\mathbf{8},\mathbf{1})$ $(\mathbf{1},\mathbf{3})$	$(\mathbf{6},\mathbf{2})$ $(\overline{\mathbf{3}},\mathbf{2})$	$(\mathbf{15},\mathbf{1})$ $(\mathbf{3},\mathbf{3})$ $(\mathbf{3},\mathbf{1})$ $(\mathbf{3},\mathbf{1})$		
7	$SL_5(\mathbb{R})$	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\mathbf{10}$	$\mathbf{24}$	$\overline{\mathbf{40}}$ $\overline{\mathbf{15}}$	$\mathbf{70}$ $\mathbf{45}$ $\mathbf{5}$			
6	$SO_{5,5}$	$\mathbf{16}$	$\mathbf{10}$	$\overline{\mathbf{16}}$	$\mathbf{45}$	$\mathbf{144}$	$\mathbf{320}$ $\overline{\mathbf{126}}$ $\mathbf{10}$				
5	$E_{6(+6)}$	$\mathbf{27}$	$\overline{\mathbf{27}}$	$\mathbf{78}$	$\mathbf{351}$	$\overline{\mathbf{1,728}}$ $\overline{\mathbf{27}}$					
4	$E_{7(+7)}$	$\mathbf{56}$	$\mathbf{133}$	$\mathbf{912}$	$\mathbf{8,645}$ $\mathbf{133}$						
3	$E_{8(+8)}$	$\mathbf{248}$	$\mathbf{3,875}$ $\mathbf{1}$	$\mathbf{147,250}$ $\mathbf{3,875}$ $\mathbf{248}$							

black hole solutions have 8 charges, p^i, q_i with $i \in \{0,1,2,3\}$. The action is as follows:

$$\begin{aligned}\mathcal{S}_{\rm STU} = \frac{1}{16\pi G}\int e^{-\eta}\Bigg[\Bigg(R + \frac{1}{4}\Big(&\text{tr}\left[\partial\mathcal{M}_T^{-1}\partial\mathcal{M}_T\right] \\ &+ \text{tr}\left[\partial\mathcal{M}_U^{-1}\partial\mathcal{M}_U\right]\Big)\Bigg)\star 1 \\ &+ \star d\eta\wedge d\eta - \frac{1}{2}\star H_{[3]}\wedge H_{[3]} \\ &- \frac{1}{2}\star F_{S[2]}^{\mathsf{T}}\wedge\left(\mathcal{M}_T\otimes\mathcal{M}_U\right)F_{S[2]}\Bigg]\end{aligned} \tag{2.10}$$

$$\mathcal{M}_S = \frac{1}{\Im(S)}\begin{pmatrix}1 & \Re(S)\\ \Re(S) & |S|^2\end{pmatrix} \quad \text{etc,} \tag{2.11}$$

where the object $F_{S[2]}$ is a vector of field strengths for the 4 gauge fields: the graviphoton and the three coupled vectors. The last term in (2.10) is understood as a typical kinetic-style term, with matrix multiplication operative on the non-Lorentz indices; the transpose also only operates on the non-Lorentz index on $F_{S[2]}$. The invariant trace terms $\text{tr}[\partial\mathcal{M}^{-1}\partial\mathcal{M}]$ were already discussed in section 2.3.

Crucially the model admits extremal black hole solutions carrying four electric and four magnetic charges [2, 61, 65] denoted q_0, q_1, q_2, q_3, p^0, p^1, p^2, p^3. These may be organised into a $2\times2\times2$ *hypermatrix* with an associated *hyperdeterminant*, first introduced by Cayley in 1845 [3]. However, the generating solution depends on just $8-3=5$ parameters [66, 67], after fixing the action of the isotropy subgroup SO_2^3. Whether the solution is embedded in $\mathcal{N}=4$ with symmetry $\text{SL}_2\times\text{SO}_{6,22}$ and charges transforming in a $(\mathbf{2},\mathbf{28})$ or $\mathcal{N}=8$ with symmetry $\text{E}_{7(7)}$ and charges transforming in the $\mathbf{56}$, in all cases, remarkably, the same five parameters suffice to describe these 56-charge black holes [66, 67].

Calculating the area (with a view to calculate the entropy) requires evaluating the moduli not asymptotically, but with their frozen horizon values which are fixed in terms of the charges [35]. This ensures that the entropy is moduli-independent, as it should be. This calculation for the model with the *STU* prepotential was carried out in [2] with charges denoted (p^0,q_0), (p^1,q_1), (p^2,q_2), (p^3,q_3) and $\text{O}_{2,2}$ scalar products

$$\begin{aligned} p^2 &= (p^0)^2 + (p^1)^2 - (p^2)^2 - (p^3)^2,\\ q^2 &= (q_0)^2 + (q_1)^2 - (q_2)^2 - (q_3)^2,\\ p\cdot q &= (p^0q_0) + (p^1q_1) + (p^2q_2) + (p^3q_3).\end{aligned} \tag{2.12}$$

In these variables, the entropy is given by the quartic polynomial expression

$$\begin{aligned}(S/\pi)^2 = &-(p\cdot q)^2\\ &+4\Big[(p^1q_1)(p^2q_2)+(p^1q_1)(p^3q_3)+(p^3q_3)(p^2q_2)\\ &+q_0p^1p^2p^2-p^0q_1q_2q_3\Big]\end{aligned} \tag{2.13}$$

which is $[\mathrm{SL}_2(\mathbb{Z})]^3$ invariant as required and symmetric under transformations: $p^1 \leftrightarrow p^2 \leftrightarrow p^3$ and $q_1 \leftrightarrow q_2 \leftrightarrow q_3$. It is in fact none other than the hyperdeterminant of the charge hypermatrix.

2.5 $\mathcal{N}=8$ black holes

The most general class of black hole solutions in $\mathcal{N}=8$ supergravity/M-theory are equipped with charges belong to the fundamental 56-dimensional representation of $\mathrm{E}_{7(7)}$, and the black hole entropy is a quartic polynomial in the 56 charges given by

$$S=\pi\sqrt{|I_4|}, \tag{2.14}$$

where I_4 is Cartan's quartic E_7 invariant [57, 68–70], the singlet in $\mathbf{56}\times\mathbf{56}\times\mathbf{56}\times\mathbf{56}$ given by

$$I_4=-\operatorname{tr}(xy)^2+\tfrac{1}{4}(\operatorname{tr}xy)^2-4\left(\operatorname{Pf}x+\operatorname{Pf}y\right), \tag{2.15}$$

where x^{IJ} and y_{IJ} are 8×8 antisymmetric matrices and Pf is the Pfaffian. An alternative expression has been provided by Cremmer and Julia [57]

$$I_4=\operatorname{tr}(\bar{Z}Z)^2-\tfrac{1}{4}(\operatorname{tr}\bar{Z}Z)^2+4\left(\operatorname{Pf}Z+\operatorname{Pf}\bar{Z}\right). \tag{2.16}$$

Here

$$Z_{AB}=-\tfrac{1}{4\sqrt{2}}(x^{IJ}+iy_{IJ})(\Gamma^{IJ})_{AB}, \tag{2.17}$$

and

$$x^{IJ}+iy_{IJ}=-\tfrac{\sqrt{2}}{4}Z_{AB}(\Gamma^{AB})_{IJ}. \tag{2.18}$$

The matrices of the SO_8 algebra are $(\Gamma^{IJ})_{AB}$ where (I,J) are the 8 vector indices and (A,B) are the 8 spinor indices. The $(\Gamma^{IJ})_{AB}$ matrices can be considered also as $(\Gamma^{AB})_{IJ}$ matrices due to equivalence of the vector and spinor representations of the SO_8 Lie algebra. The exact relation between the Cartan invariant in (2.15) and Cremmer-Julia invariant [57] in (2.16) was established in [71, 72]. Here Z_{AB} is the central charge matrix and (x, y) are the quantised charges of the black hole (28 electric and 28 magnetic). The relation between the entropy of stringy black holes and the Cartan-Cremmer-Julia $E_{7(7)}$ invariant was established in [69]. The Cartan basis of (2.15) only displays a manifest SO_8 symmetry, whereas it was proved in [68] and [57] that the sum of all terms in (2.15) is invariant under an SU_8 symmetry. To address this, the central charge matrix Z_{AB} can be brought through an SU_8 transformation to a canonical, antisymmetric matrix format:

$$Z_{AB} = \begin{pmatrix} z_1 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & 0 & z_3 & 0 \\ 0 & 0 & 0 & z_4 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.19}$$

where $z_i = \rho_i e^{i\varphi_i}$ are complex. In this way the parameter count is reduced from 56 to 8. The meaning of these parameters was clarified in a systematic treatment in [73]. Writing the four complex parameters as $z_i = \rho_i e^{i\varphi_i}$ one can remove a further three phases by an SU_8 rotation, but the overall phase cannot be removed as it is related to an extra parameter in the class of black hole solutions [66, 74]. In this basis, the quartic invariant takes the form [69]

$$\begin{aligned} I_4 = \sum_i |z_i|^4 - 2\sum_{i<j} |z_i|^2|z_j|^2 + 4\left(z_1z_2z_3z_4 + \bar{z}_1\bar{z}_2\bar{z}_3\bar{z}_4\right) \\ = \quad & (\rho_1 + \rho_2 + \rho_3 + \rho_4) \\ & \times (\rho_1 + \rho_2 - \rho_3 - \rho_4) \\ & \times (\rho_1 - \rho_2 + \rho_3 - \rho_4) \\ & \times (\rho_1 - \rho_2 - \rho_3 + \rho_4) \\ & + \; 8\rho_1\rho_2\rho_3\rho_4\left(\cos\varphi - 1\right). \end{aligned} \tag{2.20}$$

Therefore a 5-parameter solution is called a generating solution for other black holes in $\mathcal{N} = 8$ supergravity/M-theory [67, 75].

If the phase in (2.20) vanishes (which is the case if the configuration preserves at least 1/4 supersymmetry [73]), I_4 of (2.20) becomes

$$I_4 = s_1s_2s_3s_4, \tag{2.21}$$

where s_i are given by the ρ_i of (2.20)

$$\begin{aligned} s_1 &= \rho_1 + \rho_2 + \rho_3 + \rho_4, \\ s_2 &= \rho_1 + \rho_2 - \rho_3 - \rho_4, \\ s_3 &= \rho_1 - \rho_2 + \rho_3 - \rho_4, \\ s_4 &= \rho_1 - \rho_2 - \rho_3 + \rho_4, \end{aligned} \tag{2.22}$$

and we order the s_i so that $s_1 \geq s_2 \geq s_3 \geq |s_4|$. The charge orbits [70,73,76] for the black holes depend on the number of unbroken supersymmetries or the number of vanishing eigenvalues as in Table 2.3. "Large" and "small" black holes are classified by the sign of I_4:

$$I_4 > 0, \tag{2.23a}$$

$$I_4 = 0, \tag{2.23b}$$

$$I_4 < 0. \tag{2.23c}$$

Non-zero I_4 corresponds to large black holes, which are BPS for $I_4 > 0$ and non-BPS for $I_4 < 0$, and vanishing I_4 corresponds to small black holes. Case (2.23a) requires that only 1/8 of the supersymmetry is preserved, while we may have 1/8, 1/4 or 1/2 for case (2.23b). This situation is summarised in Table 2.3.

Table 2.3: Classification of $\mathcal{N} = 8, D = 4$ black holes. The distinct charge orbits are determined by the number of non-vanishing eigenvalues and I_4, as well as the number of preserved supersymmetries.

Orbit	s_1	s_2	s_3	s_4	I_4	Black hole	SUSY
$E_{7(7)} / (E_{6(6)} \ltimes \mathbb{R}^{27})$	> 0	0	0	0	0	small	1/2
$E_{7(7)} / (O_{5,6} \ltimes \mathbb{R}^{32} \times \mathbb{R})$	> 0	> 0	0	0	0	small	1/4
$E_{7(7)} / (F_{4(4)} \ltimes \mathbb{R}^{26})$	> 0	> 0	> 0	0	0	small	1/8
$E_{7(7)} / E_{6(2)}$	> 0	> 0	> 0	> 0	> 0	large	1/8
$E_{7(7)} / E_{6(2)}$	> 0	> 0	> 0	< 0	< 0	large	0
$E_{7(7)} / E_{6(6)}$	> 0	> 0	> 0	< 0	< 0	large	0

When the x^{IJ} and y_{IJ} charge matrices are themselves SU_8 transformed into the eight parameter canonical form of (2.19) we have

$$(x^{IJ} + y_{IJ})_{\text{can}} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.24}$$

The Cartan invariant can be computed in this basis and rewritten again in terms of x and y matrices to obtain

$$\begin{aligned} I_4 = \\ &-(x^{01}y_{01} + x^{23}y_{23} + x^{45}y_{45} + x^{67}y_{67})^2 \\ &+ 4(x^{01}x^{23}y_{01}y_{23} + x^{01}x^{45}y_{01}y_{45} + x^{23}x^{45}y_{23}y_{45} \\ &+ x^{01}x^{67}y_{01}y_{67} + x^{23}x^{67}y_{23}y_{67} + x^{45}x^{67}y_{45}y_{67}) \\ &- 4(x^{01}x^{23}x^{45}x^{67} + y_{01}y_{23}y_{45}y_{67}), \end{aligned} \tag{2.25}$$

which is simply Cayley's hyperdeterminant (2.13) under the identifications

$$\begin{aligned} \lambda_1 &= x^{01} + iy_{01} = q_0 + ip^0, \\ \lambda_2 &= x^{23} + iy_{23} = -p^3 + q_3, \\ \lambda_3 &= x^{45} + iy_{45} = p^2 - iq_2, \\ \lambda_4 &= x^{56} + iy_{56} = p^1 - iq_1. \end{aligned} \tag{2.26}$$

This result for $E_{7(7)}$ also applies, *mutatis mutandis*, to $E_7(\mathbb{C})$. In other words the Cartan invariant reduces to Cayley's hyperdeterminant in this basis, permitting *STU* black holes to be classified using the same eigenvalue notation as $\mathcal{N} = 8$ black holes, which we do in Table 2.4. In contrast to $\mathcal{N} = 8$, cases (2.23a) and (2.23b) both require preservation of 1/2 of the supersymmetry. The orbits for the large $\mathcal{N} = 2$ black holes were previously found in [70, 77] while those of the small black holes were more recently found in [24].

One can also consider $\mathcal{N} = 4$ supergravity coupled to m vector multiplets (symmetry $SL_2(\mathbb{Z}) \times SO_{6,m}(\mathbb{Z})$) and $\mathcal{N} = 2$ supergravity coupled to $\ell+1$ vector multiplets (symmetry $SL_2(\mathbb{Z}) \times SO_{\ell,2}(\mathbb{Z})$) and the entropies are again the square root of quartic invariants.

Table 2.4: Classification of $\mathcal{N}=2, D=4$ *STU* black holes, see section 2.4. The values of I_4 and the eigenvalues s_i distinguish the different charge orbits. Here, small black holes have a vanishing horizon.

Orbit	s_1	s_2	s_3	s_4	I_4	Black hole	SUSY
$\mathrm{SL}_2^3/(\mathrm{SO}_{1,1}^2 \ltimes \mathbb{R}^3)$	>0	0	0	0	0	small	1/2
$\mathrm{SL}_2^3/(\mathrm{O}_{2,1} \times \mathbb{R})$	>0	>0	0	0	0	small	1/2
$\mathrm{SL}_2^3/\mathbb{R}^2$	>0	>0	>0	0	0	small	1/2
$\mathrm{SL}_2^3/\mathrm{U}_1^2$	>0	>0	>0	>0	>0	large	1/2
$\mathrm{SL}_2^3/\mathrm{U}_1^2$	>0	>0	>0	<0	<0	large	0 ($Z=0$)
$\mathrm{SL}_2^3/\mathrm{SO}_{1,1}^2$	>0	>0	>0	<0	<0	large	0 ($Z\neq 0$)

2.6 $D = 5$ black holes

In five dimensions we might consider $\mathcal{N}=8$ supergravity where the symmetry is the non-compact exceptional group $\mathrm{E}_{6(6)}(\mathbb{Z})$ and the black holes carry charges belonging to the fundamental 27-dimensional representation (all electric) [73]. The electrically charged objects are point-like and the magnetic duals are one-dimensional, or string-like, transforming according to the contragredient representation. The black hole entropy is a cubic polynomial in the 27 charges given by

$$S = \pi\sqrt{|I_3(Q)|}, \tag{2.27}$$

where I_3 is Cartan's cubic $\mathrm{E}_{6(6)}$ invariant [68, 70, 73, 78, 79] which may be written

$$I_3(Q) = q_{ij}\Omega^{jl}q_{lm}\Omega^{mn}q_{np}\Omega^{pi}, \tag{2.28}$$

where q_{ij} is the charge vector transforming as a **27** which can be represented as traceless Sp_8 matrix.

In five dimensions the compact group K is USp_8 (with conventions chosen so that $\mathrm{USp}_2 = \mathrm{SU}_2$). This time the commutator of the supersymmetry generators yields a central charge matrix Z_{AB} which can be brought to a normal form by a USp_8 transformation. In the normal form the central charge matrix can be written as

$$Z_{AB} = \begin{pmatrix} s_1+s_2-s_3 & 0 & 0 & 0 \\ 0 & s_1+s_3-s_2 & 0 & 0 \\ 0 & 0 & s_2+s_3-s_1 & 0 \\ 0 & 0 & 0 & -(s_1+s_2+s_3) \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.29}$$

where the s_i can be ordered so that $|s_1| \geq |s_2| \geq |s_3|$. The cubic invariant, in this basis, becomes

$$I_3 = s_1 s_2 s_3, \tag{2.30}$$

Which furnishes three distinct possibilities as shown in Table 2.5. Unlike the four-

Table 2.5: Classification of $D = 5, \mathcal{N} = 8$ black holes. The distinct charge orbits are determined by the number of non-vanishing eigenvalues and I_3, as well as the number of unbroken supersymmetries.

Orbit	s_1	s_2	s_3	I_3	Black hole	SUSY
$\mathrm{E}_{6(6)}/(\mathrm{O}_{5,5} \ltimes \mathbb{R}^{16})$	>0	0	0	0	small	1/2
$\mathrm{E}_{6(6)}/(\mathrm{O}_{5,4} \ltimes \mathbb{R}^{16})$	>0	>0	0	0	small	1/4
$\mathrm{E}_{6(6)}/\mathrm{F}_{4(4)}$	>0	>0	>0	>0	large	1/8

dimensional case where flipping the sign of I_4 interchanges BPS and non-BPS black holes, the sign of the I_3 (2.30) has no effect since it changes under a CPT transformation. There are no non-BPS orbits in five dimensions.

One can also consider $\mathcal{N} = 4$ supergravity coupled to $m-1$ vector multiplets (symmetry $\mathrm{SO}_{1,1}(\mathbb{Z}) \times \mathrm{SO}_{m-1,5}(\mathbb{Z})$) and $\mathcal{N} = 2$ supergravity coupled to ℓ vector multiplets (symmetry $\mathrm{SO}_{1,1}(\mathbb{Z}) \times \mathrm{SO}_{\ell-1,1}(\mathbb{Z})$) where the entropies are again the square root of cubic invariants.

2.7 Magic supergravities

In both four and five dimensions one can consider $\mathcal{N} = 2, 4, 8$ supergravities. In four dimensions one has the maximal case, $\mathcal{N} = 4$ coupled to m vector multiplets, and $\mathcal{N} = 2$ coupled to $\ell + 1$ vector multiplets, where there exist quartic invariants giving the black hole entropy in each case. The $\mathcal{N} = 8$ case admits a quantum information theoretic interpretation, which descends to the $\mathcal{N} = 4, m = 6$ and $\mathcal{N} = 2, \ell = 2$ cases[1] as truncations. There is a similar story in five dimensions where one has cubic invariants instead.

However, as suggested by Levay [9], one might also consider the "magic" supergravities [80–84]. In four dimensions these correspond to the $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (real, complex, quaternionic and octonionic) $\mathcal{N} = 2$ supergravity coupled to $6, 9, 15$ and 27 vector multiplets with symmetries $\mathrm{Sp}_6(\mathbb{Z}), \mathrm{SU}_{3,3}, \mathrm{SO}^*_{12}$ and $\mathrm{E}_{7(-25)}$, respectively. Once again, it has been shown [85] in all cases that there are quartic invariants whose square root yields

[1]The $\ell = 2$ case is of course the one we already knew [1].

the corresponding black hole entropy. In five dimensions [80–82] the magic supergravities are the $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\,\mathcal{N} = 2$ supergravity coupled to 5, 8, 14 and 26 vector multiplets with symmetries $\mathrm{SL}_3(\mathbb{R}), \mathrm{SL}_3(\mathbb{C}), \mathrm{SU}_6^*$ and $\mathrm{E}_{6(-26)}$ respectively, all with cubic invariants whose square root yields the corresponding black hole entropy [85].

Chapter 3

Quantum information theory

3.1 Qubits and entanglement

Quantum entanglement is a phenomenon in which the quantum states of two or more objects must be described with reference to each other, even though the individual objects may be spatially separated [86–92]. This leads to correlations between observable physical properties of the systems that are classically forbidden. For example it is possible to prepare two particles in a single quantum state such that when one is observed to be spin-up, the other one will always be observed to be spin-down and vice versa, this despite the fact that it is impossible to predict, according to quantum mechanics, which set of measurements will be observed. As a result, measurements performed on one system seem to be instantaneously influencing other systems entangled with it. Note, however, that quantum entanglement does not enable the transmission of classical information faster than the speed of light.

On a more philosophical note, the correlations predicted by quantum mechanics, and observed in experiment, reject the principle of local realism, which is that information about the state of a system should only be mediated by interactions in its immediate surroundings and that the state of a system exists and is well-defined before any measurement.

While a physical property, indeed a physical resource, entanglement is not exactly an observable - much like the state itself. As a resource it is leveraged in the fields of quantum computing and quantum cryptography and has been used to implement quantum teleportation experimentally.

The prototypical physical systems experiencing entanglement are quantum bits or *qubits*: two-level quantum systems

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle \tag{3.1}$$

where $a_A \in \mathbb{C}$ and $\langle\psi|\psi\rangle = 1$ so that in marked contrast to classical bits the system can exist in a complex superposition of states $|0\rangle$ and $|1\rangle$. Entanglement presupposes multiple systems to compare with each other so we see no entanglement for a lone qubit. To describe two qubits, one forms the tensor product of two single qubit states

$$\begin{aligned} |\psi\rangle &= \sum_{A,B=0}^{1} a_{AB}|A\rangle \otimes |B\rangle =: \sum_{A,B=0}^{1} a_{AB}|AB\rangle \\ &= a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle. \end{aligned} \tag{3.2}$$

From the standpoint of this model, the physical phenomenon of entanglement is the property of the state corresponding to the failure of the sum of tensor products (3.2) to factorise: non-separability. When a two-level system doesn't suffice, one has recourse to *qutrits*: three-level quantum systems

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle + a_2|2\rangle, \tag{3.3}$$

and more generally to *qudits*: d-level systems

$$|\psi\rangle = \sum_{A=0}^{d-1} a_A|A\rangle. \tag{3.4}$$

Finally one can describe any number n of such systems as a linear aggregate of n-fold tensor products

$$|\psi\rangle = \sum_{A_1\cdots A_n} a_{A_1\cdots A_n}|A_1\rangle \otimes \cdots \otimes |A_n\rangle \equiv \sum_{A_1\cdots A_n} a_{A_1\cdots A_n}|A_1\cdots A_n\rangle, \tag{3.5}$$

where the sums can in general be of differing lengths to accommodate combinations of qubit, qutrits, etc. The 2^n coefficients $a_{A_1\cdots A_n}$ of an n qubit system can be arranged in n-dimensional hypercubes as in Figure 3.1. Of particular significance to us is the $n = 3$ cube, which can be interpreted as a $2 \times 2 \times 2$ *hypermatrix* sporting a generalisation of a matrix determinant known as Cayley's *hyperdeterminant*. These hypercubes famously make an appearance in classical information theory, where the minimum number of edges between nodes is the Hamming distance between them, a useful metric in error detection

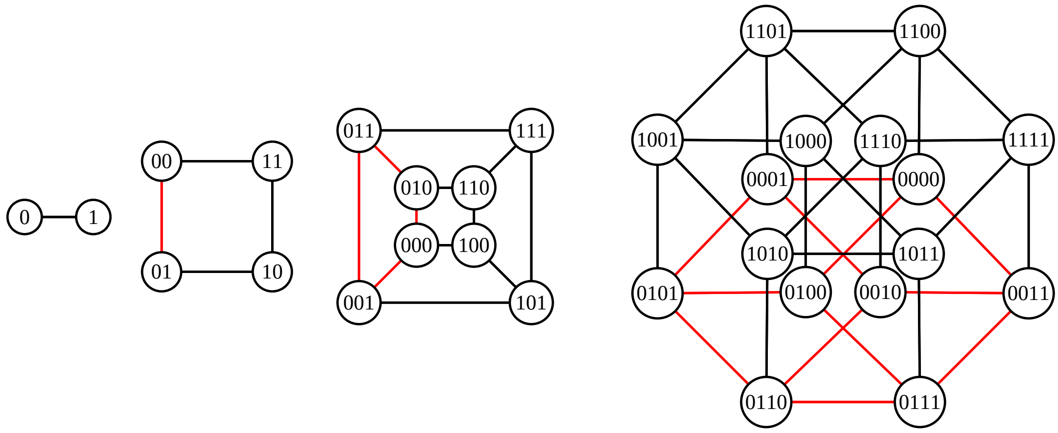


Figure 3.1: Two-dimensional projections of qubit coefficient hypercubes for one to four qubits. The red edges in a projection correspond to the preceding hypercube.

and correction [93].

A little more formally, qudits occupy a complex Hilbert space $\mathcal{H}$ equipped with a one-to-one map into its dual space $\mathcal{H}^\dagger$,

$$\begin{aligned} {}^\dagger : \mathcal{H} &\to \mathcal{H}^\dagger, \\ |\psi\rangle &\mapsto (|\psi\rangle)^\dagger := \langle\psi| \end{aligned} \tag{3.6}$$

which defines an inner product $\langle\psi|\phi\rangle$ and satisfies

$$\left.\begin{aligned} (\alpha|\psi\rangle)^\dagger &= \langle\psi|\alpha^*, \\ (|\psi\rangle + |\phi\rangle)^\dagger &= \langle\psi| + \langle\phi|, \end{aligned}\right\} \tag{3.7a}$$

$$\langle\psi|\phi\rangle^* = \langle\phi|\psi\rangle, \tag{3.7b}$$

$$\langle\psi|\psi\rangle \geq 0, \tag{3.7c}$$

for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}, \alpha \in \mathbb{C}$ with saturation of the inequality for null $|\psi\rangle$ only.

In particular a qubit inhabits the 2-dimensional complex Hilbert space $\mathbb{C}^2$. An arbitrary n-qubit system is then simply a vector in the n-fold tensor product Hilbert space $\mathbb{C}^2\otimes, \ldots \otimes \mathbb{C}^2 = [\mathbb{C}^2]^n$.

3.2 Entanglement classification

3.2.1 The (S)LOCC paradigm

Entanglement being a resource, one is compelled to quantify it and discriminate systems by means of it, just as one does for more familiar classical resources such as energy or entropy [94]. Non-separability is an easy to understand notion of entanglement, but it is the more physical description of classically forbidden inter-state correlations that furnishes a means of quantifying entanglement.

The general idea is to partition state space into orbits using classical-correlation respecting transformations. The appropriate transformations are supplied by the principle of *(stochastic) local operation and classical communication* or (S)LOCC. In the LOCC paradigm, a multipartite state may be split into its relevant subsystems which are distributed to different labs where they may be experimented upon, the results optionally being disseminated by classical means. Classical correlations can be established using LOCC, but since the protocol can only create separable states it is unable to establish genuine quantum correlations: LOCC cannot create entanglement [95]. This of course leads to the restriction that any putative entanglement measure be a monotonically decreasing function of *any* LOCC transformation: an *entanglement monotone*. The LOCC method of transforming states forms a hierarchy through some states not being interconvertible. States that can be LOCC related should however be physically equivalent with respect to their entanglement properties, a notion called LOCC equivalence. It was shown in [96] that two states of a composite system are LOCC equivalent if and only if they may be transformed into one another using the group of *local unitaries* (LU), unitary transformations which factorise into separate transformations on the component parts. In the case of n qudits, the LU group (up to a phase) is given by $[\mathrm{SU}_d]^n$. The LU orbits carve the Hilbert space into equivalence classes. For a n-qudit system the space of orbits is given by [97, 98]:

$$\frac{[\mathbb{C}^d]^n}{\mathrm{U}_1 \times [\mathrm{SU}_d]^n}. \tag{3.8}$$

As it happens however, this LU classification is very restrictive and fails to relate nominally equivalent states: even simple bipartite systems will not, in general, be related [99]. Furthermore, continuous parameters are required to describe the space of entanglement classes [97, 98, 100, 101]. Coarse graining this classification to one that is more physically acceptable is what puts the S in SLOCC: rather than require that states be deterministically related to each other by LOCC, instead we require only that they may be transformed into one another with some non-zero probability of success, hence stochastic

LOCC [96, 99].

It is proved in [99] that for n qudits, the SLOCC equivalence group is (up to an overall complex factor) $[\mathrm{SL}_d(\mathbb{C})]^n$. Essentially, we may identify two states if there is a non-zero probability that one can be converted into the other and vice-versa, which means we get $[\mathrm{SL}_d(\mathbb{C})]^n$ orbits rather than the $[\mathrm{SU}_d]^n$ kind of LOCC. This generalisation may be physically motivated by the fact that any set of SLOCC equivalent entangled states may be used to perform the very same non-classical, entanglement dependent, operations, only with varying likelihoods of success. For a n-qudit state the space of SLOCC equivalence classes is given by [99]:

$$\frac{[\mathbb{C}^d]^n}{[\mathrm{SL}_d(\mathbb{C})]^n}. \tag{3.9}$$

Invariants of the quotient group (which are polynomials in the state vector coefficients) may then act as discriminating polynomials - entanglement measures [95,96,98,101–113]. In the qubit case $d = 2$, the lower bound on the number of continuous variables needed to parameterise the space of orbits is $2(2^n - 1) - 6n$ so that for three qubits in particular the space of orbits is finite and discrete giving the concise classification of entanglement classes of [99].

3.2.2 Two qubits

The state norm, despite being fixed for normalised states, is generically a discriminating polynomial of the simplest variety, serving to distinguish the trivial null class from all others. This is all one has for one qubit (transforming as a $\mathbf{2}$ of SL_2) since all non-null states are separable. For two qubits transforming as a $(\mathbf{2}, \mathbf{2})$ of $[\mathrm{SL}_2]^2$ the bipartite measure is the concurrence C_{AB}

$$C_{AB} = 2\sqrt{\det \rho_A} = 2\sqrt{\det \rho_B} = 2|\det a_{AB}| = 2|a_{00}a_{11} - a_{01}a_{10}| \tag{3.10}$$

where ρ_A and ρ_B are reduced density operators formed as partial traces of the full pure state density operator $|\psi\rangle\langle\psi|$

$$\begin{aligned}(\rho_A)_{A_1A_2} &= \delta^{B_1B_2} a_{A_1B_1} a^*_{A_2B_2},\\ (\rho_B)_{B_1B_2} &= \delta^{A_1A_2} a_{A_1B_1} a^*_{A_2B_2}.\end{aligned} \tag{3.11}$$

This is to be compared with the von Neumann entropy $\mathcal{E}$ (an extension of the Shannon entropy):

$$\mathcal{E} = -\operatorname{tr}\left[\rho_A \log_2(\rho_A)\right] = -\operatorname{tr}\left[\rho_B \log_2(\rho_B)\right]. \tag{3.12}$$

C_{AB} is a nonlinear function of $\mathcal{E}$, but as Figure 3.2 shows, $\mathcal{E}$ increases monotonically from 0 to 1 as C goes from 0 to 1,

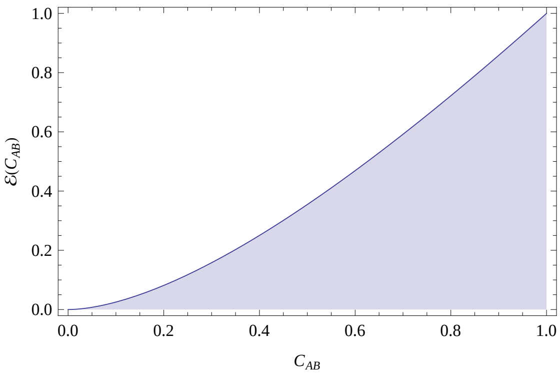


Figure 3.2: The von Neumann entropy as a function of concurrence

$$\begin{aligned}\mathcal{E}(C_{AB}) &= h\left(\tfrac{1}{2}\left[1+\sqrt{1-C_{AB}^2}\right]\right)\\ h(x) &:= -x\log_2(x) - (1-x)\log_2(1-x),\end{aligned} \tag{3.13}$$

making both acceptable as entanglement measures, though the concurrence becomes more complicated for mixed states. Typically the relevant discriminating polynomial is expressed as the two-qubit 2-tangle

$$\tau_{AB} := C_{AB}^2 = 2[(\operatorname{tr}\rho)^2 - \operatorname{tr}\rho^2] \tag{3.14}$$

where ρ is the total density operator. Exemplary states are

- Separable: $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$ for which $\tau_{AB} = 0$
- Bell state/EPR state [87, 89]: $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ for which $\tau_{AB} = 1$

3.2.3 Three qubits

For three qubits transforming as a $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ of $[\mathrm{SL}_2]^3$ there are six $\mathrm{SL}_2(\mathbb{C})^3$ invariant polynomials that measure the entanglement and discriminate between the orbits. There is as always the quadratic norm, followed at quartic order by three *local entropies* $S_{A,B,C}$

$$\begin{aligned} S_A &= 4\det\rho_A = 4\det\operatorname{tr}_{BC}|\psi\rangle\langle\psi|, \\ S_B &= 4\det\rho_B = 4\det\operatorname{tr}_{CA}|\psi\rangle\langle\psi|, \\ S_C &= 4\det\rho_C = 4\det\operatorname{tr}_{AB}|\psi\rangle\langle\psi|. \end{aligned} \tag{3.15}$$

otherwise known as three-qubit 2-tangles and denoted $\tau_{A(BC)}, \tau_{B(AC)}$, and $\tau_{C(AB)}$. These may be converted to an alternative set of three polynomials which are also referred to as three-qubit 2-tangles [114]

$$\begin{aligned} \tau_{AB} &= C_{AB}^2 = \tfrac{1}{2}\left(-\tau_{C(AB)} + \tau_{A(BC)} + \tau_{B(CA)} - \tau_{ABC}\right), \\ \tau_{BC} &= C_{BC}^2 = \tfrac{1}{2}\left(-\tau_{A(BC)} + \tau_{B(CA)} + \tau_{C(AB)} - \tau_{ABC}\right), \\ \tau_{CA} &= C_{CA}^2 = \tfrac{1}{2}\left(-\tau_{B(CA)} + \tau_{C(AB)} + \tau_{A(BC)} - \tau_{ABC}\right), \end{aligned} \tag{3.16}$$

where C_{AB},C_{BC} and C_{CA} are the corresponding concurrences. These 2-tangles give bipartite entanglements between pairs in 3-qubit system as shown in Figure 3.3.

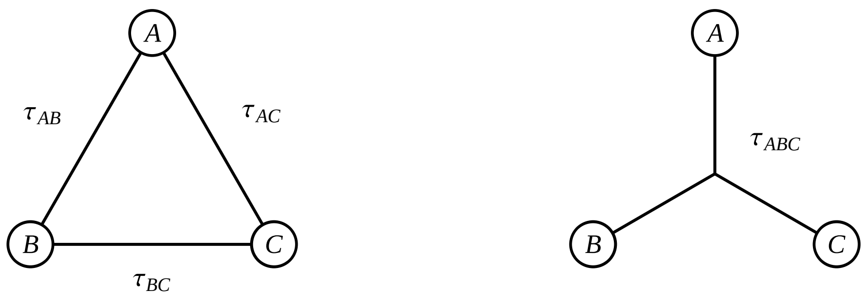


Figure 3.3: Tangles.

Next there is the sextic Kempe invariant [104]

$$\begin{aligned} K &:= a_{A_1B_1C_1}a_{A_2B_2C_2}a_{A_3B_3C_3}\bar{a}_{A_1B_2C_3}\bar{a}_{A_2B_3C_1}\bar{a}_{A_3B_1C_2} \\ &= \operatorname{tr}(\rho_A \otimes \rho_B\rho_{AB}) - \operatorname{tr}(\rho_A^3) - \operatorname{tr}(\rho_B^3) \\ &= \operatorname{tr}(\rho_B \otimes \rho_C\rho_{BC}) - \operatorname{tr}(\rho_B^3) - \operatorname{tr}(\rho_C^3) \\ &= \operatorname{tr}(\rho_C \otimes \rho_A\rho_{CA}) - \operatorname{tr}(\rho_C^3) - \operatorname{tr}(\rho_A^3). \end{aligned} \tag{3.17}$$

It happens that this invariant is redundant for the purposes of classification. Nevertheless, in [115] the Kempe invariant was shown to vary continuously for a fixed orbit while the other polynomials were held fixed, indicating that being an entanglement measure is more restrictive than simply being an invariant and a monotone.

Finally there is Cayley's hyperdeterminant $\operatorname{Det} a_{ABC}$ or 3-tangle τ_{ABC}

$$\tau_{ABC} = 4|\operatorname{Det} a_{ABC}|, \tag{3.18}$$

$$\begin{aligned}\operatorname{Det} a := -\tfrac{1}{2}\, & \varepsilon^{A_1A_2}\varepsilon^{B_1B_2}\varepsilon^{A_3A_4}\varepsilon^{B_3B_4}\varepsilon^{C_1C_4}\varepsilon^{C_2C_3} \\ & \times a_{A_1B_1C_1}a_{A_2B_2C_2}a_{A_3B_3C_3}a_{A_4B_4C_4}.\end{aligned} \tag{3.19}$$

The 3-tangle, in contrast to the 2-tangles, is a measure of the genuine 3-way entanglement as depicted in Figure 3.3. The hyperdeterminant is compared to the ordinary determinant in Figure 3.4. Unlike the two qubit array a_{AB}, contraction with epsilon

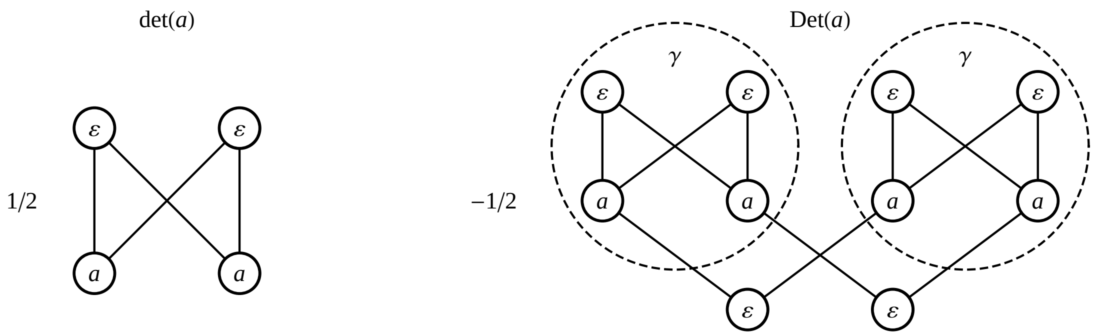


Figure 3.4: The hyperdeterminant (right) compared to the ordinary determinant (left) in diagrammatic tensor notation. The encircled letters are tensors and the lines denote index contractions. The hyperdeterminant can be viewed as the determinant of the determinant-like quantity γ.

tensors in a determinant-like manner fails to saturate the indices of the three-qubit array a_{ABC}. However the resulting object, which we call γ, does have two indices and can be saturated with a determinant style epsilon contraction. There are three ways to construct γ

$$\begin{aligned}(\gamma^A)_{A_1A_2} &= a_{A_1}{}^{BC}a_{A_2BC}, \\ (\gamma^B)_{B_1B_2} &= a^A{}_{B_1}{}^C a_{AB_2C}, \\ (\gamma^C)_{C_1C_2} &= a^{AB}{}_{C_1}a_{ABC_2},\end{aligned} \tag{3.20}$$

but they satisfy

$$\det\gamma^A = \det\gamma^B = \det\gamma^C = -\operatorname{Det} a. \tag{3.21}$$

The values of the five polynomials (recalling that K is redundant) carve the state space into seven classes, with the entanglement classification complete with representative states summarised in Table 3.1. The objects $\gamma^{A,B,C}$ satisfy a complementary

Table 3.1: The values of the local entropies S_A, S_B, and S_C and the hyperdeterminant $\operatorname{Det} a$ are used to partition three-qubit states into entanglement classes.

Class	Representative	Condition				
		ψ	S_A	S_B	S_C	$\operatorname{Det} a$
Null	0	$= 0$	$= 0$	$= 0$	$= 0$	$= 0$
A-B-C	$\|000\rangle$	$\neq 0$	$= 0$	$= 0$	$= 0$	$= 0$
A-BC	$\|010\rangle + \|001\rangle$	$\neq 0$	$= 0$	$\neq 0$	$\neq 0$	$= 0$
B-CA	$\|100\rangle + \|001\rangle$	$\neq 0$	$\neq 0$	$= 0$	$\neq 0$	$= 0$
C-AB	$\|010\rangle + \|100\rangle$	$\neq 0$	$\neq 0$	$\neq 0$	$= 0$	$= 0$
W	$\|100\rangle + \|010\rangle + \|001\rangle$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$= 0$
GHZ	$\|000\rangle + \|111\rangle$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$

relationship with the local entropies $S_{A,B,C}$: where in Table 3.1 $S_A = 0, S_{B,C} \neq 0$ we would have $\gamma^{B,C} = 0, \gamma^A \neq 0$, etc. A visual representation of the SLOCC classes is provided by the onion-like classification [4] of Figure 3.5.

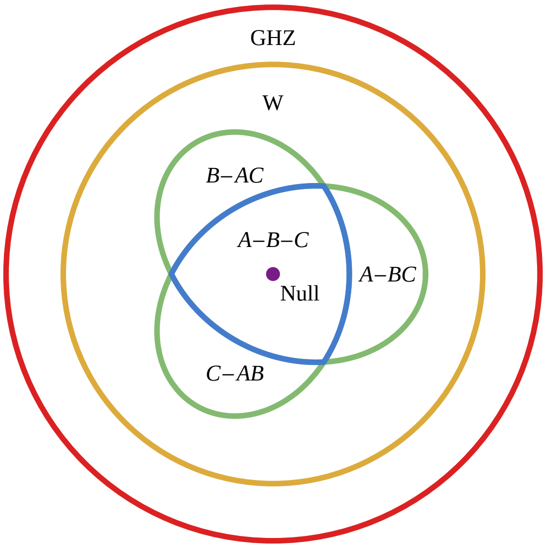


Figure 3.5: Onion-like classification of SLOCC orbits.

The classes and representatives

Null: The trivial zero entanglement orbit corresponding to vanishing states,

$$\text{Null}: \quad 0. \tag{3.22}$$

Separable: Another zero entanglement orbit for completely factorisable product states,

$$A\text{-}B\text{-}C: \quad |000\rangle. \tag{3.23}$$

Biseparable: Three classes of bipartite entanglement

$$\begin{aligned} A\text{-}BC: &\quad |010\rangle + |001\rangle, \\ B\text{-}CA: &\quad |100\rangle + |001\rangle, \\ C\text{-}AB: &\quad |010\rangle + |100\rangle. \end{aligned} \tag{3.24}$$

W: Three-way entangled states that do not maximally violate Bell-type inequalities in the same way as the GHZ class discussed below. However, they are robust in the sense that tracing out a subsystem generically results in a bipartite mixed state that is maximally entangled under a number of criteria [99],

$$\mathrm{W}: \quad |100\rangle + |010\rangle + |001\rangle. \tag{3.25}$$

GHZ: Genuinely tripartite entangled Greenberger-Horne-Zeilinger [116] states. These maximally violate Bell-type inequalities but, in contrast to class W, are fragile under the tracing out of a subsystem since the resultant state is completely unentangled,

$$\mathrm{GHZ}: \quad |000\rangle + |111\rangle. \tag{3.26}$$

These W and GHZ state definitions are readily generalised

$$|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\right) \tag{3.27}$$

$$|\mathrm{W}_N\rangle = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left[\bigotimes_{j=1}^{i-1}|0\rangle \otimes |1\rangle \otimes \bigotimes_{j=i+1}^{N}|0\rangle\right] \tag{3.28}$$

however, there are multiple alternative W and GHZ states which do not conform to these patterns, but respect the entanglement properties of the conventional definitions.

The seven listed entanglement classes are arranged in the entanglement hierarchy [99] shown in Figure 3.6. Note that no SLOCC operations (invertible or not) relate the GHZ and W classes; they are genuinely distinct classes of genuine tripartite entanglement. However, from either the GHZ class or W class one may use non-invertible SLOCC transformations to descend to one of the biseparable or separable classes and hence we have a hierarchical entanglement structure.

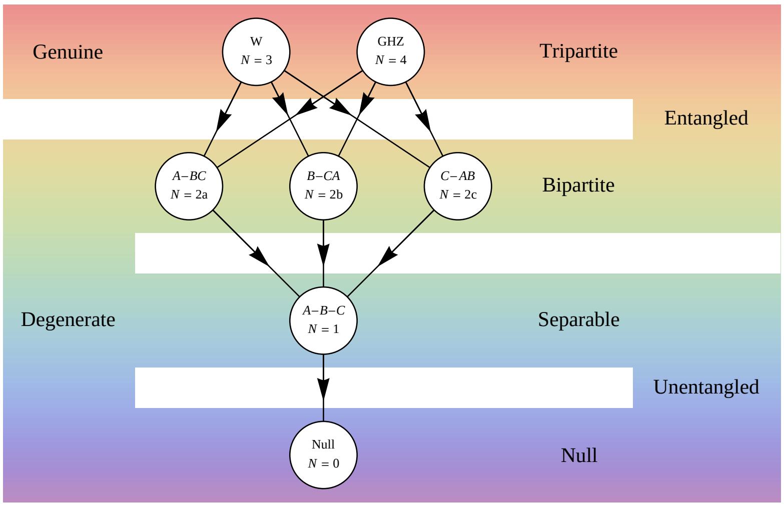


Figure 3.6: Three qubit entanglement hierarchy. The arrows are non-invertible SLOCC transformations between classes that generate the stratification.

In anticipation of the brane wrapping interpretation of section 6.1 we present an alternative parameterisation. For unnormalised three qubit states, the number of parameters [97] needed to describe inequivalent states is given by the dimension of the space of orbits

$$\frac{\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2}{\mathrm{U}_1 \times \mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SU}_2}, \tag{3.29}$$

namely $16 - 10 = 6$. For subsequent comparison with the STU black hole, however, we restrict our attention to states with *real* coefficients a_{ABC}. In this case one has

$$\frac{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2}{\mathrm{SO}_2 \times \mathrm{SO}_2 \times \mathrm{SO}_2}, \tag{3.30}$$

with dimension $8-3 = 5$. Hence, the most general real three-qubit state can be described by just five parameters [117], conveniently taken as four real numbers N_0, N_1, N_2, N_3 and an angle θ [1]:

$$\begin{aligned} |\Psi\rangle = &-N_3 \cos^2\theta|001\rangle - N_2|010\rangle \\ &+ N_3 \sin\theta\cos\theta|011\rangle - N_1|100\rangle \\ &- N_3 \sin\theta\cos\theta|101\rangle + (N_0 + N_3\sin^2\theta)|111\rangle. \end{aligned} \tag{3.31}$$

Representatives states in this parameterisation are provided in Table 3.2.

[1] This is obtained from the canonical form for real states, Eq. (11) of [117], by applying two different $SO(2)$ transformations on the second and third bits.

Table 3.2: Entanglement class representatives for the five parameter state (3.31).

Class	Entanglement	Representative
A-B-C	Separable	$N_0\|111\rangle$
A-BC, AC-B, AB-C	Biseparable	$N_0\|111\rangle - N_1\|100\rangle$
W	Full bipartite	$-N_1\|100\rangle - N_2\|010\rangle - N_3\|001\rangle$
GHZ	Tripartite	$N_0\|111\rangle - N_1\|100\rangle - N_2\|010\rangle - N_3\|001\rangle$

3.2.4 Two qutrits

The bipartite entanglement of A and B is given by the 2-tangle [118–122]

$$\tau_{AB} = 27 \det \rho_A = 27\,|\det a_{AB}|^2, \tag{3.32}$$

$$\begin{aligned}\det a_{AB} &= \tfrac{1}{3!}\varepsilon^{A_1A_2A_3}\varepsilon^{B_1B_2B_3}a_{A_1B_1}a_{A_2B_2}a_{A_3B_3}\\ &= a_{00}(a_{11}a_{22} - a_{12}a_{21})\\ &\quad - a_{01}(a_{10}a_{22} - a_{12}a_{20})\\ &\quad + a_{02}(a_{01}a_{21} - a_{11}a_{20}),\end{aligned} \tag{3.33}$$

where ρ_A is the reduced density matrix

$$\rho_A = \mathrm{Tr}_B|\Psi\rangle\langle\Psi|. \tag{3.34}$$

The determinant is invariant under $\mathrm{SL}_{3,A} \times \mathrm{SL}_{3,B}$, with a_{AB} transforming as a $(\mathbf{3}, \mathbf{3})$, and under a discrete duality that interchanges A and B. The only other discriminating polynomial we require is C_2, the sum of the principal minors of the density matrix:

$$\begin{aligned}C_2 = \ & |a_{00}a_{11} - a_{01}a_{10}|^2 + |a_{02}a_{10} - a_{00}a_{12}|^2 + |a_{01}a_{12} - a_{02}a_{11}|^2\\ &+ |a_{01}a_{20} - a_{00}a_{21}|^2 + |a_{00}a_{22} - a_{02}a_{20}|^2 + |a_{10}a_{21} - a_{11}a_{20}|^2\\ &+ |a_{12}a_{20} - a_{10}a_{22}|^2 + |a_{02}a_{21} - a_{01}a_{22}|^2 + |a_{11}a_{22} - a_{12}a_{21}|^2.\end{aligned} \tag{3.35}$$

The entanglement classification for two qutrits is then summarised in Table 5.2.

Table 3.3: Two qutrit entanglement classification

Class	ψ	C_2	τ_{AB}
Null	0	0	0
A-B	> 0	0	0
Rank 2 Bell	> 0	> 0	0
Rank 3 Bell	> 0	> 0	> 0

As was done in section 3.2.3, we present an alternative parameterisation in anticipation of comparison with the $D = 5$ black hole. This time restricting to unnormalised states with real coefficients results in the space of orbits

$$\frac{\mathbb{R}^3 \times \mathbb{R}^3}{\mathrm{SO}_3 \times \mathrm{SO}_3}, \tag{3.36}$$

with dimension namely $9 - 6 = 3$. Hence, the most general two-qutrit state can be described by just three parameters, which may be conveniently taken to be three real numbers N_0, N_1, N_2:

$$|\Psi\rangle = N_0|00\rangle + N_1|11\rangle + N_2|22\rangle. \tag{3.37}$$

3.3 Qubit covariants in classical invariant theory

3.3.1 Transvection and the Omega process

It is possible to systematically generate the discriminating polynomials of section 3.2.2 and section 3.2.3 (and more besides) using techniques from classical invariant theory [110, 123–128]. The basic method was introduced by Cayley in 1845 [3] when he introduced hyperdeterminant theory. Essentially, one operates on homogeneous polynomials (forms) with differential operators to produce *covariants*. Covariants of a form in this context are functions $f(\vec{a}, \vec{x})$ of the form's variables $\vec{x}$ and coefficients $\vec{a}$ that are unchanged under general linear transformations, modulo factors of the determinant Δ of the transformation

$$f(\vec{a}, \vec{x}) = \Delta^w f'(\vec{a}', \vec{x}'). \tag{3.38}$$

where w is the *weight* of the covariant. Invariants are covariants with no $\vec{x}$ dependence. When $w = 0$ the covariant or invariant is called *absolute*. The process by which covariants are constructed from a base form is called *transvection* which is itself based upon the *Omega process*. The mth order Omega process with respect to an $m \times m$ matrix of variables V is a differential operator defined as

$$\begin{aligned} \boldsymbol{\Omega} &:= \begin{vmatrix} \frac{\partial}{\partial V_{11}} & \cdots & \frac{\partial}{\partial V_{1m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial V_{m1}} & \cdots & \frac{\partial}{\partial V_{mm}} \end{vmatrix}, \\ &= \textstyle\sum_{\pi \in \mathcal{S}_m} (-)^\pi \prod_{i=1}^m \frac{\partial}{\partial V_{i\pi(i)}}. \end{aligned} \tag{3.39}$$

For example, a second order Omega process would be $\frac{\partial}{\partial V_{11}\partial V_{22}} - \frac{\partial}{\partial V_{12}\partial V_{21}}$. We will call the matrix of which $\mathbf{\Omega}$ is the determinant the Omega matrix Ω. The essential property of the Omega matrix that permits the construction of covariants when acting on base forms is the fact that it transforms with a factor of the inverse of the determinant of the transformation, making it an example of an invariant process.

Defining $P(\overrightarrow{V_i}) := P(V_{i1}, \dots, V_{ip})$, we use the tensor product notation $\bigotimes_{i=1}^p P_i$ to denote $P_1(\overrightarrow{V_1}), \dots, P_p(\overrightarrow{V_p})$. A *complete* n-transvectant is then defined by

$$(P_1, \dots, P_p)^{(n)} := \operatorname{tr} \mathbf{\Omega}^n \bigotimes_{i=1}^p P_i, \tag{3.40}$$

where tr sets all vectors of variables to be equal: $\overrightarrow{V_1} = \overrightarrow{V_2} = \cdots = \overrightarrow{V_p}$. Under the exchange of any of the P_i, the transvectant picks up a factor of $(-)^n$. For a complete n-transvectant, the dimension v of the $\overrightarrow{V_i}$ vectors satisfies $p = v = m$. However more generally, v can be any integer multiple of p: $p = v/d$. In such case, the polynomials are expected to be multiforms; that is, while the polynomials accept $p \times d$ arguments, they are homogeneous in each of the d sets of v variables. Separate Omega processes can then operate on the d sets of variables to form a complete $(n_1, \dots, n_d)$-transvectant:

$$(P_1, \dots, P_p)^{(n_1,\dots,n_d)} := \operatorname{tr} \mathbf{\Omega}_1^{n_1} \cdots \mathbf{\Omega}_d^{n_d} \bigotimes_{i=1}^p P_i. \tag{3.41}$$

When the P_i have multiweights $(w_{11}, \dots, w_{1d}), \dots, (w_{p1}, \dots, w_{pd})$ the resulting covariant has multiweight $(w_{11} + \cdots + w_{p1} + n_1, \dots, w_{1d} + \cdots + w_{pd} + n_d)$. In the multidegree case, transvectants satisfy $p = v/d = m$.

In the case of n qudits the base form is the state itself $|\psi\rangle = \sum_{i=0}^{d-1} a_{A_1\cdots A_n} |A_1 \cdots A_n\rangle$ where the coefficients $\vec{a}$ are $a_{A_1\cdots A_n}$ and the variables $\vec{x}$ are the basis kets $|A_1 \cdots A_n\rangle$. This trivial covariant is a multiform of multidegree $(\underbrace{1, \dots, 1}_{n})$ which we will denote $A_{\underbrace{1 \dots 1}_{n}}$. Following [125] we then transvect in the following particular manner

$$(A_{1\cdots 1}, \cdots (A_{1\cdots 1}, (A_{1\cdots 1}, A_{1\cdots 1})^{s_{11}\cdots s_{1n}})^{s_{21}\cdots s_{2n}} \dots)^{s_{q1}\cdots s_{qn}} \tag{3.42}$$

so that we have a $q \times n$ transvectant specification matrix s. We restrict to $\sum_{j=1}^n s_{ij} > 0$ (where $s_{ij} \geq 0$ by definition) to avoid transvectants that are merely products of previously generated ones. The resulting transvectant can be systematically named

$$X_{(q+1-2\sum_{i=1}^q s_{i1})\cdots(q+1-2\sum_{i=1}^q s_{in})} \tag{3.43}$$

where X is the $(q+1)$th letter of the alphabet. For example if we don't transvect at all we have a $0 \times n$ matrix s so that X is the first letter of the alphabet A and $q+1-2\sum_{i=1}^{q} s_{ij} = 1$ for all $1 \leq j \leq n$. In other words this notation is consistent with calling the state $A_{1\cdots 1}$. Transvecting once we have

$$(A_{1\cdots 1}, A_{1\cdots 1})^{s_{11}\cdots s_{1n}} = B_{(2-2s_{11})\cdots(2-2s_{1n})}. \tag{3.44}$$

As an example, for two qubits the two nonvanishing covariants are A_{11} and $B_{00} = (A_{11}, A_{11})^{11}$. This notation allows one to read off the form of the monomials of a covariant from its name or transvection specification:

$$a^{(q+1)}_{A_1\cdots A_n} |A_1\rangle^{\otimes(q+1-2\sum_{i=1}^{q} s_{i1})} \otimes |A_n\rangle^{\otimes(q+1-2\sum_{i=1}^{q} s_{in})}. \tag{3.45}$$

Naturally the s_{ij} must also be such that $q+1-2\sum_{i=1}^{q} s_{ij} \geq 0$ for all $1 \leq j \leq n$ to prevent negative tensor powers. Most acceptable covariant specifications s will actually result in vanishing transvectants. In the case when all tensor powers are empty the covariant is actually an invariant, such as the determinant B_{00}. Though informative, this notation will still assign the same name to different covariants with identical monomial structure so, depending on the task at hand, one must keep track of the underlying polynomials.

For three qubits one obtains the degree 2 covariants $B_{200}, B_{020}, B_{002}$ which are in fact $\gamma^A, \gamma^B, \gamma^C$. At degree 3 one finds C_{111} which is related to the Kempe invariant K,and at degree 4 there is the hyperdeterminant D_{000}. If one is insensitive to permutations of qubits, covariants collapse into multiweight classes: $B_{200}, B_{020}, B_{002} \to 3B_{200}$. While Δ and $\langle\psi|\psi\rangle$ correspond directly to their LOCC class counterparts modulo numerical factors, the local entropies S_i and Kempe invariant obey the following

$$\begin{aligned} S_i &= 16\Big[\textstyle\sum_j \langle B^j|B^j\rangle - \langle B^i|B^i\rangle\Big] \\ \langle B^i|B^i\rangle &= \tfrac{1}{32}\big[\textstyle\sum_j S_j - 2S_i\big] \end{aligned} \tag{3.46}$$

$$\textstyle\sum_i \langle B^i|B^i\rangle = \frac{1}{32}\sum_i S_i \tag{3.47}$$

$$K = \langle A_{111}|A_{111}\rangle^3 + \tfrac{3}{2}[\langle C_{111}|C_{111}\rangle - \tfrac{1}{16}\langle A_{111}|A_{111}\rangle \textstyle\sum_i S_i] \tag{3.48}$$

where $B^1 = B_{200}, B^2 = B_{020}, B^3 = B_{002}$. There are however some additional three-qubit covariants at order 4: $3D_{220}$ as shown in Figure 3.7. When one attempts a covariant based entanglement classification including this added set of covariants it turns out that they are redundant (i.e. generate no new classes), just as C_{111} already was. We note that the trilinear form C_{111} has the same multiweight structure as the state itself, and can be considered a transformed state. This perspective is employed in section A.2.2 in

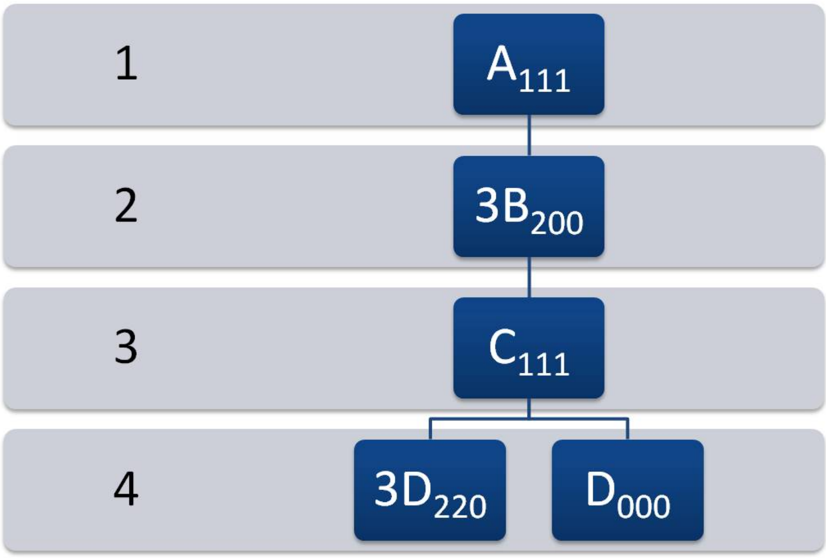


Figure 3.7: Three-qubit covariant tree down to degree 4 (numbers on the left indicate degree).

which C_{111} is proportional to the triple product T.

3.3.2 Molecular notation

A rather instructive molecular notation for transvection was developed by Clifford, Sylvester, Kempe, Olver, and others [123] which can provide some insight to the structure of the qubit covariants. We will explore the notation for the three qubit case, but the definitions extend trivially to n qubits. First note that second order omega processes can be denoted

$$\Omega_{\alpha\beta} := \det \begin{pmatrix} \partial_{x_\alpha} & \partial_{y_\alpha} \\ \partial_{x_\beta} & \partial_{y_\beta} \end{pmatrix} =: [\alpha\beta], \tag{3.49}$$

Where the *bracket factor* $[\alpha\beta]$ is antisymmetric. When written in this form,transvectants are referred to as bracket polynomials. The three-qubit covariants can then be obtained as

$$\begin{aligned} B_{200} &= \tfrac{1}{2} \operatorname{tr} (\quad)([12])([12])\, A_{111} \otimes A_{111}, \\ B_{020} &= \tfrac{1}{2} \operatorname{tr} ([12])(\quad)([12])\, A_{111} \otimes A_{111}, \\ B_{002} &= \tfrac{1}{2} \operatorname{tr} ([12])([12])(\quad)\, A_{111} \otimes A_{111}, \end{aligned} \tag{3.50}$$

$$C_{111} = \operatorname{tr} ([12])([23])([23])\, A_{111}^{\otimes 3}, \tag{3.51}$$

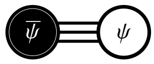
$$D_{000} = \tfrac{1}{2} \operatorname{tr} ([12][34])([14][23])([14][23])\, A_{111}^{\otimes 4}. \tag{3.52}$$

The notation essentially keeps track of the antisymmetrisations between constituent copies of the base form A_{111}. With that in mind, one can view the basic triform as a trivalent atom (though we distinguish between $|A_{111}\rangle$ and $\langle A_{111}|$):

$$|A_{111}\rangle = |\psi\rangle =$$ ,

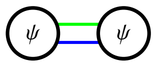
$$\text{and } \langle A_{111}| = \langle\psi| =$$ . (3.53)

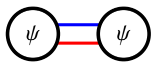
One can introduce two types of bond between these atoms. The first represents a simple overlap between bras and kets. For example, the norm is given by:

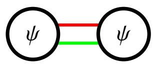
$$\langle\psi|\psi\rangle =$$ . (3.54)

The overlap bond is strictly between atoms of distinct type. In contrast, the second type of bond represents the action of an Omega process. The bonds are colour-coded to distinguish between the partial transvection slots.

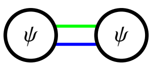
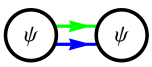
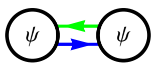
For example, the B^i are given by dimers:

$$2B_{200} = \mathrm{tr}\,(\quad)([12])([12])\,\psi\otimes\psi =$$ ,

$$2B_{020} = \mathrm{tr}\,([12])(\quad)([12])\,\psi\otimes\psi =$$ ,

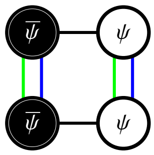
$$2B_{002} = \mathrm{tr}\,([12])([12])(\quad)\,\psi\otimes\psi =$$ . (3.55)

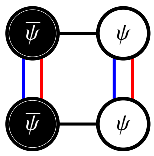
Strictly speaking, these bonds are directed since $[\alpha\beta]$ is antisymmetric

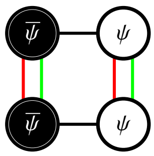
 $\equiv$  $\equiv -$ . (3.56)

The fact that the B^i dimers are ionic corresponds to the fact that they are covariants.

To form invariants, we may neutralise them with overlap bonds.

$4\langle B_{200}|B_{200}\rangle =$ 

$4\langle B_{020}|B_{020}\rangle =$  (3.57)

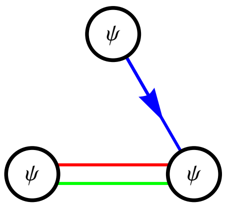
$4\langle B_{002}|B_{002}\rangle =$ 

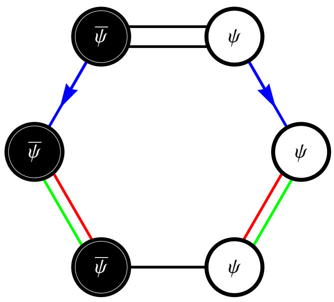
Attempting to neutralise with omega bonds causes them to vanish.

Next there is the trimer C_{111}

$$C_{111} = \mathrm{tr}\ ([12])([23])([23])\ \psi^3 \tag{3.58}$$

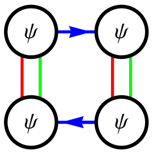
Equality still holds under interchange of colours. Again C_{111} is ionic and we neutralise with overlap bonds.

$C_{111} =$ 

$\langle C_{111}|C_{111}\rangle =$  (3.59)

Finally, there is the hyperdeterminant, which is neutral without needing overlaps

$$D_{000} = \frac{1}{2}\ \mathrm{tr}\ ([12][34])([14][23])([14][23])\ \psi^{\otimes 4}, \tag{3.60}$$

$D_{000} =$  . (3.61)

By establishing rules to determine whether a molecule vanishes or is equivalent to another, the generation and examination of covariants by hand is greatly simplified.

Chapter 4

Preliminary analogies

4.1 Three-qubits and $\mathcal{N} = 2, D = 4$ black holes

The black hole/qubit correspondence begins with the recognition that the entropy S of STU black holes is simply the three-qubit entanglement measure τ_{ABC} in a different basis. That is,

$$S = \frac{\pi}{2}\sqrt{\tau_{ABC}} \tag{4.1}$$

with the "dictionary" between bases given by:

$$\begin{aligned} &\big(\; p^0\;,\quad p^1\;,\quad p^2\;,\quad p^3\;,\; q_0\;,\; q_1\;,\; q_2\;,\; q_3\;\big) \\ =\; &\big(a_{000},\; -a_{001},\; -a_{010},\; -a_{100},\; a_{111},\; a_{110},\; a_{101},\; a_{011}\big), \end{aligned} \tag{4.2}$$

where this is the convention of [5] and differs from [1] in that the signs of $a_{000}, a_{011}, a_{100}, a_{111}$ are flipped, though of course both satisfy (4.1). As a side note, in [5] a simpler example of such a correspondence is made between the entropy of the axion dilaton black hole and the two-qubit 2-tangle $S = \pi \det(a_{AB}) = \frac{\pi}{2}\sqrt{\tau_{AB}}$.

Two important distinctions between the black hole and quantum information contexts are

- Qudit wavefunctions may be normalised but there is no analogous restriction on black hole charges.
- The quantised charges of STU black holes are integers transforming under $\mathrm{SL}_2(\mathbb{Z})^3$ rather than $\mathrm{SL}_2(\mathbb{C})^3$ [1, 2, 5, 61]

The second point may be addressed[1] by restricting to real qubits or *rebits* in which the

[1] More recently, the Kostant-Sekiguchi correspondence allows us to map real orbits of black holes to

state vector coefficients are real. This results in three cases to consider

$$\operatorname{Det} a < 0, \tag{4.3a}$$

$$\operatorname{Det} a = 0, \tag{4.3b}$$

$$\operatorname{Det} a > 0. \tag{4.3c}$$

Case (4.3a) corresponds to the non-separable or GHZ class [116], for example,

$$|\Psi\rangle = \tfrac{1}{2}(-|000\rangle + |011\rangle + |101\rangle + |110\rangle). \tag{4.4}$$

Case (4.3b) corresponds to the separable (A-B-C, A-BC, B-CA, C-AB) and W classes, for example

$$|\Psi\rangle = \tfrac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle). \tag{4.5}$$

Case (4.3c) is also GHZ, for example the state (4.4) with a sign flip

$$|\Psi\rangle = \tfrac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle), \tag{4.6}$$

however in this case we could just as well use the canonical GHZ state.

In [1], cases (4.3a) and (4.3b) were shown to correspond to BPS black holes, for which half of the supersymmetry is preserved. Case (4.3a) has non-zero horizon area and entropy ("large" black holes), and case (4.3b) has vanishing horizon area and entropy ("small" black holes), at least at the semi-classical level[2]. This alignment of the classification of $\mathcal{N} = 2$ supersymmetric STU black holes with the entanglement classification of three qubits is summarised in Table 4.1.

4.1.1 Higher order corrections

The small black holes have a singular horizon with vanishing area and entropy at the classical level, but may acquire nonvanishing area and entropy due to quantum corrections, characterised by higher derivatives in the supergravity Lagrangian. One can interpret this as consequence of the quantum stretching of the horizon conjectured by Susskind [129] and Sen [130, 131]. See also [132–137].

Kallosh and Linde [5] have noted that this quantum entropy also admits an interpretation in terms of qubit entanglement measures. They propose a general formula that

---

the complex orbits of qubits and bypass rebits.

[2]However, small black holes may acquire a non-zero entropy through higher order quantum effects.

Table 4.1: The values of the local entropies S_A, S_B, and S_C and the hyperdeterminant $\mathrm{Det}\, a$ (defined in section 3.2.3) are used to partition three-qubit states into entanglement classes. The black hole/qubit correspondence relates these to $D = 4, \mathcal{N} = 2, STU$ model black holes). Specifically, the to absence/presence of a horizon (small/large) and the extent of supersymmetry.

Class	S_A	S_B	S_C	$\mathrm{Det}\, a$	Black hole	SUSY
A-B-C	0	0	0	0	small	1/2
A-BC	0	>0	>0	0	small	1/2
B-CA	>0	0	>0	0	small	1/2
C-AB	>0	>0	0	0	small	1/2
W	>0	>0	>0	0	small	1/2
GHZ	>0	>0	>0	<0	large	1/2
GHZ	>0	>0	>0	>0	large	0

correctly reduces to the known special cases. It is given by

$$S_{total} = \frac{\pi}{2}\sqrt{\tau_{ABC} + \frac{4c_2}{3}(C_{AB} + C_{BC} + C_{CA}) + \frac{8K^2}{3}|\Psi|}, \tag{4.7}$$

where c_2 and K are constants that depend upon the compactification. Clearly this now involves the bipartite entanglement of three qubits.

For completely separable states with only one nonzero charge, this reduces to

$$S = K\sqrt{\tfrac{2}{3}|\Psi|} = \pi K\sqrt{\tfrac{2}{3}|q_0|}. \tag{4.8}$$

For the bipartite and W-states at large values of the charges, the concurrences are much greater than $|\Psi|$ and the formula reduces to

$$S_{total} = \pi\sqrt{\frac{c_2}{3}(C_{AB} + C_{BC} + C_{CA})} \tag{4.9}$$

$$= 4\pi\sqrt{|q_0(p^1 + p^2)|}. \tag{4.10}$$

Finally, for the GHZ states the (unnormalised) 3-tangle is much greater than the concurrences and we regain

$$S = \tfrac{\pi}{2}\sqrt{\tau_{ABC}}. \tag{4.11}$$

4.2 The $\mathcal{N} = 8$ generalisation

4.2.1 Decomposing $\mathrm{E}_{7(7)} \supset \mathrm{SL}_2^7$

Just as the entropy of the STU model possesses an $\mathrm{SL}_2(\mathbb{Z})^3$ symmetry which permits a correspondence with the $\mathrm{SL}_2(\mathbb{C})^3$ invariant three-way entanglement measure of three qubits we seek a system whose entanglement corresponds to the $\mathrm{E}_{7(7)}(\mathbb{Z})$ invariant $\mathcal{N} = 8$ entropy. This despite the fact that we can only expect an $\mathrm{SL}_d(\mathbb{C})^n$ symmetry from n qudits. Nevertheless $\mathrm{E}_{7(7)}(\mathbb{Z})$ contains $\mathrm{SL}_2(\mathbb{Z})^7$ as a subgroup:

$$\begin{aligned} \mathrm{E}_{7(7)}(\mathbb{Z}) &\supset \mathrm{SL}_2(\mathbb{Z})^7, \\ \mathrm{E}_{7(7)}(\mathbb{C}) &\supset \mathrm{SL}_2(\mathbb{C})^7. \end{aligned} \tag{4.12}$$

This initially suggests a simple analogy with a seven qubit system, but this perception is altered by the details of the decomposition of the fundamental $\mathbf{56}$. One begins by decomposing under the maximal subgroup $\mathrm{SL}_2 \times \mathrm{SO}_{6,6}$

$$\begin{array}{ccccc} \mathrm{E}_{7(7)} & \supset & \mathrm{SL}_{2,A} & \times & \mathrm{SO}_{6,6}, \\ \mathbf{56} & \to & (\mathbf{2}, \mathbf{12}) & + & (\mathbf{1}, \mathbf{32}), \end{array} \tag{4.13}$$

where the SL_2 is labelled with an A in anticipation of a seven qubit interpretation. The decomposition then proceeds as shown in Figure 4.1 down to

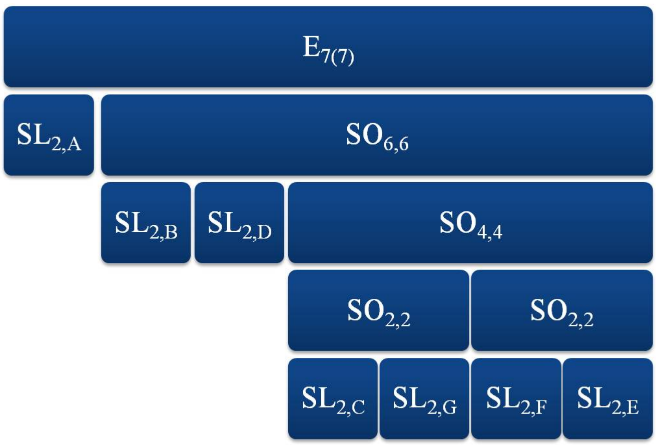


Figure 4.1: $\mathrm{E}_{7(7)}(\mathbb{Z}) \supset \mathrm{SL}_2(\mathbb{Z})^7$ decomposition

$$\mathrm{E}_{7(7)} \supset \mathrm{SL}_{2,A} \times \mathrm{SL}_{2,B} \times \mathrm{SL}_{2,D} \times \mathrm{SL}_{2,C} \times \mathrm{SL}_{2,G} \times \mathrm{SL}_{2,F} \times \mathrm{SL}_{2,E}, \tag{4.14}$$

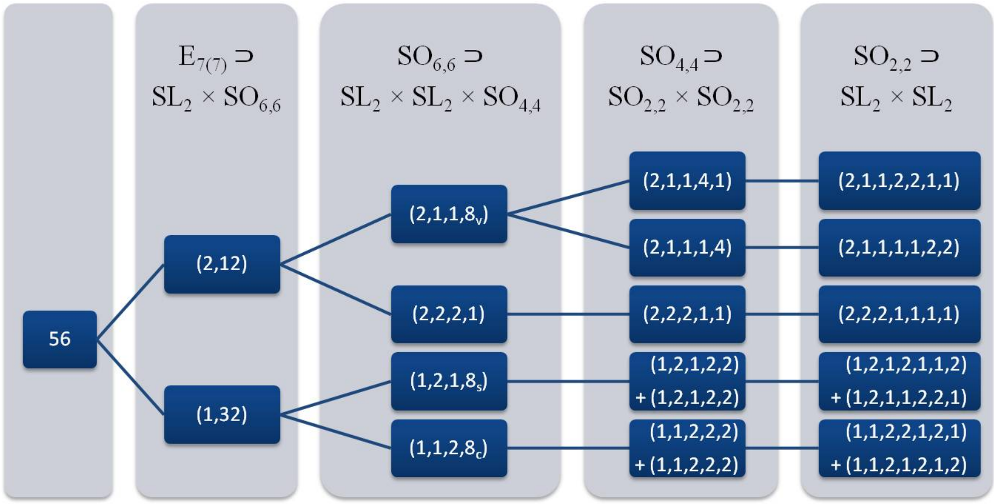


Figure 4.2: Detailed decomposition of the **56** of $E_{7(7)}(\mathbb{Z})$ under $E_7 \supset SL_2^7$

with the detailed decomposition of the **56** shown in Figure 4.2 where the ordering of the SL_2 factors is the same as in Figure 4.1. This particular ordering of factors is chosen to admit an interesting decomposition pattern upon the permutation (34)(57)

$$E_{7(7)} \supset SL_{2,A} \times SL_{2,B} \times SL_{2,C} \times SL_{2,D} \times SL_{2,E} \times SL_{2,F} \times SL_{2,G}, \tag{4.15}$$

so that

$$\begin{aligned} \mathbf{56} \to \quad & (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1}) \\ & + (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1}) \\ & + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) \\ & + (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2}) \\ & + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) \\ & + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) \\ & + (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2}). \end{aligned} \tag{4.16}$$

An analogous decomposition holds for

$$E_7(\mathbb{C}) \supset SL_2(\mathbb{C})^7. \tag{4.17}$$

4.2.2 Interpretation

Instead of a straightforward seven qubits, which would have the wrong counting $\mathbf{56} \not\to (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})$, we find the direct sum of seven copies of the $(2^3 = 8)$-dimensional three qubit $(\mathbf{2},\mathbf{2},\mathbf{2})$s. That is, rather than

$$a_{ABCDEFG}|ABCDEFG\rangle, \tag{4.18}$$

we have

$$\begin{aligned} |\Psi\rangle_{56} = \quad & a_{AB\bullet D\bullet\bullet\bullet}|AB\bullet D\bullet\bullet\bullet\rangle \\ +\ & b_{\bullet BC\bullet E\bullet\bullet}|\bullet BC\bullet E\bullet\bullet\rangle \\ +\ & c_{\bullet\bullet CD\bullet F\bullet}|\bullet\bullet CD\bullet F\bullet\rangle \\ +\ & d_{\bullet\bullet\bullet DE\bullet G}|\bullet\bullet\bullet DE\bullet G\rangle \\ +\ & e_{A\bullet\bullet\bullet EF\bullet}|A\bullet\bullet\bullet EF\bullet\rangle \\ +\ & f_{\bullet B\bullet\bullet\bullet FG}|\bullet B\bullet\bullet\bullet FG\rangle \\ +\ & g_{A\bullet C\bullet\bullet\bullet G}|A\bullet C\bullet\bullet\bullet G\rangle, \end{aligned} \tag{4.19}$$

which we abbreviate to

$$\begin{aligned} |\Psi\rangle_{56} = \quad & a_{ABD}|ABD\rangle \\ +\ & b_{BCE}|BCE\rangle \\ +\ & c_{CDF}|CDF\rangle \\ +\ & d_{DEG}|DEG\rangle \\ +\ & e_{EFA}|EFA\rangle \\ +\ & f_{FGB}|FGB\rangle \\ +\ & g_{GAC}|GAC\rangle. \end{aligned} \tag{4.20}$$

This state has the following properties

- Any pair of states has a qubit in common
- Each qubit is excluded from four out of the seven states
- Two given qubits are excluded from two out of the seven states
- Three given qubits are never excluded.

Despite not being a subsector of the seven qubit space there are seven qubits coupled to each other in groups of three as shown in Figure 4.3. Consequently we refer to this situation as the *tripartite entanglement of seven qubits*, though this phrase needs to be carefully interpreted. One could in fact consider (4.20) to be a special 56-dimensional

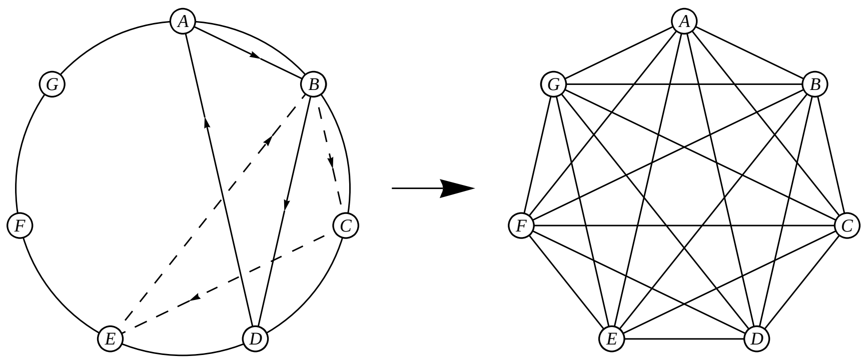


Figure 4.3: E_7 entanglement diagram corresponding to the decomposition (4.16) and the state (4.20). Each of the seven vertices $A, B, \ldots, G$ represents a qubit and each of the seven triangles $ABD, BCE, \ldots, GAC$ describes a tripartite entanglement.

subspace of a $3^7 = 2,187$-dimensional seven qutrit space, where we decompose

$$\mathrm{SL}_3^7 \supset \mathrm{SL}_2^7, \tag{4.21}$$

$$\begin{aligned}
(\mathbf{3},\mathbf{3},\mathbf{3},\mathbf{3},\mathbf{3},\mathbf{3},\mathbf{3}) \to \quad & 1 \text{ term like } (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2}) \\
+ & \ 7 \text{ terms like } (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1}) \\
+ & \ 21 \text{ terms like } (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) \\
+ & \ 35 \text{ terms like } (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1}) \\
+ & \ 35 \text{ terms like } (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) \\
+ & \ 21 \text{ terms like } (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) \\
+ & \ 7 \text{ terms like } (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) \\
+ & \ 1 \text{ term like } (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) \quad ,
\end{aligned} \tag{4.22}$$

so that the singlets originate from $\mathbf{3} \to \mathbf{2} + \mathbf{1}$. Thus, our state (4.20) is indeed a seven qubit subsector. Nevertheless the main point of interest is the hidden $E_{7(7)}$ symmetry which must be respected by the entanglement measure. If we're in the market for a quartic $E_{7(7)}$ invariant, the sole possibility is Cartan's I_4. To express I_4 in the basis of the amplitudes $a, b, \ldots, g$ one writes the decomposition (4.16) in a more terse form with singlets discarded and doublets appropriately labelled

$$\mathbf{56} = (ABD) + (BCE) + (CDF) + (DEG) + (EFA) + (FGB) + (GAC), \tag{4.23}$$

or schematically

$$\mathbf{56} = a + b + c + d + e + f + g. \tag{4.24}$$

The invariant is then the singlet in $\mathbf{56} \times \mathbf{56} \times \mathbf{56} \times \mathbf{56}$:

$$\begin{aligned} I_4 = a^4 + b^4 + c^4 + d^4 + e^4 + f^4 + g^4 \\ + 2\Big[a^2b^2 + a^2c^2 + a^2d^2 + a^2e^2 + a^2f^2 + a^2g^2 \\ + b^2c^2 + b^2d^2 + b^2e^2 + b^2f^2 + b^2g^2 \\ + c^2d^2 + c^2e^2 + c^2f^2 + c^2g^2 \\ + d^2e^2 + d^2f^2 + d^2g^2 \\ + e^2f^2 + e^2g^2 \\ + f^2g^2\Big] \\ + 8\left[abce + bcdf + cdeg + defa + efgb + fgac + gabd\right]. \end{aligned} \tag{4.25}$$

The terms like

$$\begin{aligned} a^4 &= (ABD)(ABD)(ABD)(ABD) \\ &= \tfrac{1}{2}\varepsilon^{A_1A_2}\varepsilon^{B_1B_2}\varepsilon^{D_1D_4}\varepsilon^{A_3A_4}\varepsilon^{B_3B_4}\varepsilon^{D_2D_3} \\ &\quad \times a_{A_1B_1D_1}a_{A_2B_2D_2}a_{A_3B_3D_3}a_{A_4B_4D_4}, \end{aligned} \tag{4.26}$$

are just seven hyperdeterminants (modulo sign), each excluding four qubits. Products like

$$\begin{aligned} a^2b^2 &= (ABD)(ABD)(BCE)(BCE) \\ &= \tfrac{1}{2}\varepsilon^{A_1A_2}\varepsilon^{B_1B_3}\varepsilon^{D_1D_2}\varepsilon^{B_2B_4}\varepsilon^{C_3C_4}\varepsilon^{E_3E_4} \\ &\quad \times a_{A_1B_1D_1}a_{A_2B_2D_2}b_{B_3C_3E_3}b_{B_4C_4E_4}, \end{aligned} \tag{4.27}$$

are $3 \times 7 = 21$ cross entanglements excluding two qubits. A more compact definition is afforded via the γ decomposition of the hyperdeterminant

$$-\tfrac{1}{2}\,\mathrm{tr}[(\gamma^2(a)\cdot\varepsilon)^{\mathsf{T}}\cdot\varepsilon\cdot\gamma^1(b)]. \tag{4.28}$$

Finally, products like

$$\begin{aligned} abce &= (ABD)(BCE)(CDF)(EFA) \\ &= \tfrac{1}{2}\varepsilon^{A_1A_4}\varepsilon^{B_1B_2}\varepsilon^{C_2C_3}\varepsilon^{D_1D_3}\varepsilon^{E_2E_4}\varepsilon^{F_3F_4} \\ &\quad \times a_{A_1B_1D_1}b_{B_2C_2E_2}c_{C_3D_3F_3}e_{E_4F_4A_4}, \end{aligned} \tag{4.29}$$

are seven cross entanglements excluding one qubit. The full entanglement can then be expressed as

$$I_4 = \sum_{i=1}^{7} a_i^4 + 2\sum_{i=1}^{7}\sum_{j=i+1}^{7} a_i^2 a_j^2 + 8\sum_{i=1}^{7} a_i a_{|i+1|_7} a_{|i+2|_7} a_{|i+4|_7} \tag{4.30}$$

These results may be verified using the dictionary between amplitudes and the Cartan basis charges discussed in section 5.1.

4.2.3 The Fano plane

The particular pattern of the decomposition (4.16) and the state (4.20) is encapsulated in the Fano plane as shown in Figure 4.4. The Fano plane is a projective plane with seven points and seven lines (the circle counts as a line). We may associate it to the state (4.20) by interpreting the points as the seven qubits *A*-*G* and the lines as the seven tripartite entanglements. This is consistent as there are three points on every line and three lines through every point. We refer to the amplitudes of the state (4.20) as the *Fano basis* of the Cartan invariant. If the plane is oriented like ours, one may use the directed lines to read off a multiplication table for imaginary octonions. This

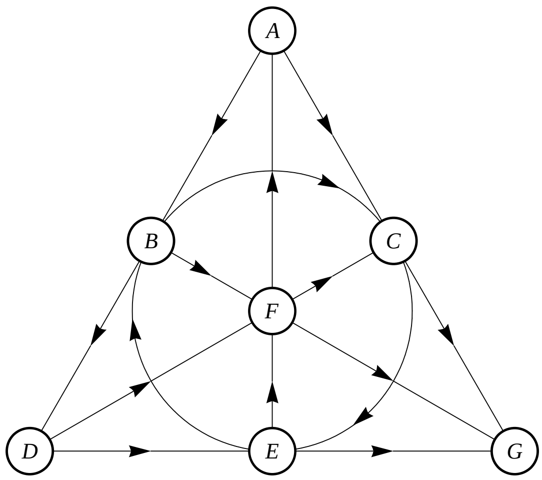


Figure 4.4: The Fano plane with nodes labelled with qubits $A, \ldots, G$ in a pattern that matches the decomposition (4.16). Edges are oriented in a way that allows one to read off a multiplication table for imaginary octonions.

is not the end of the story since the decomposition (4.16) can be tabulated against the corresponding amplitudes $a, \ldots, g$ and qubits $A, \ldots, G$ as in Table 4.2 to reveal a dual description in which the Fano nodes and edges swap roles, resulting in a dual state

Table 4.2: The seven terms in decomposition (4.16) may be written in a grid such that Fano lines and vertices are rows and columns. This permits easy identification of the dual lines and vertices, which are simply given by columns and rows.

	A	B	C	D	E	F	G
a	2	2	1	2	1	1	1
b	1	2	2	1	2	1	1
c	1	1	2	2	1	2	1
d	1	1	1	2	2	1	2
e	2	1	1	1	2	2	1
f	1	2	1	1	1	2	2
g	2	1	2	1	1	1	2

$$
\begin{aligned}
|\widetilde{\Psi}\rangle_{56} = \quad & A_{aeg}|a\,e\,g\rangle \\
&+ B_{bfa}|b\,f\,a\rangle \\
&+ C_{cgb}|c\,g\,b\rangle \\
&+ D_{dac}|d\,a\,c\rangle \\
&+ E_{ebd}|e\,b\,d\rangle \\
&+ F_{fce}|f\,c\,e\rangle \\
&+ G_{gdf}|g\,d\,f\rangle,
\end{aligned}
\tag{4.31}
$$

and a dual Fano plane shown in Figure 4.5 which also furnishes a multiplication table for the imaginary octonions.

4.2.4 Subsectors

Having discussed the correspondence between $\mathcal{N} = 8$ black holes and the tripartite entanglement of seven qubits using $E_{7(7)}$, we can obtain analogous correspondences in the $\mathcal{N} = 4$ and $\mathcal{N} = 2$ cases by taking them as subsectors of the full $\mathcal{N} = 8$ theory, examining the $SL_2 \times SO_{6,6}$ subgroup for $\mathcal{N} = 4$ and an $SL_2 \times SO_{2,2}$ subgroup for $\mathcal{N} = 2$.

In the first case we restrict to $\mathcal{N} = 4$ by retaining just the 24 NS-NS charges belonging to the $(\mathbf{2}, \mathbf{12})$ part of the decomposition of the fundamental $\mathbf{56}$ under its maximal subgroup. These can be read off of Figure 4.2 to give

$$
|\Psi\rangle = a_{ABD}|ABD\rangle + e_{EFA}|EFA\rangle + g_{GAC}|GAC\rangle. \tag{4.32}
$$

So only qubit A is shared. This corresponds to the three lines passing through A in

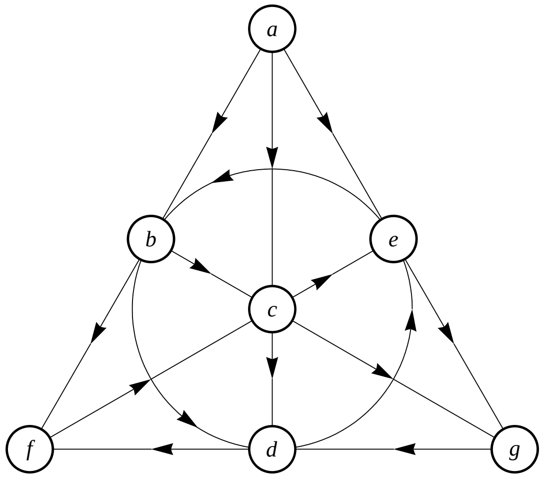


Figure 4.5: The dual Fano plane has nodes and edges swapped with the Fano plane of Figure 4.4 and is associated with the dual seven qubit state (4.31). With the above orientation applied to the edges the dual plane also yields an imaginary octonion multiplication table.

the Fano plane or the *aeg* line in the dual Fano plane, meaning that we reduce from imaginary octonions to imaginary quaternions. Then the equation analogous to (4.23) is

$$(\mathbf{2}, \mathbf{12}) = (ABD) + (EFA) + (GAC) = a + e + g, \tag{4.33}$$

and the corresponding quartic invariant, I_4, reduces to the singlet in $(\mathbf{2}, \mathbf{12}) \times (\mathbf{2}, \mathbf{12}) \times (\mathbf{2}, \mathbf{12}) \times (\mathbf{2}, \mathbf{12})$

$$I_4 \sim a^4 + e^4 + g^4 + 2[e^2g^2 + g^2a^2 + a^2e^2]. \tag{4.34}$$

The 24 numbers $(a_{ABD}, e_{EFA}, g_{GAC})$ can be identified with (P^μ, Q_ν) with $\mu, \nu = 0, \ldots, 11$ to yield the $\mathrm{SL}_2 \times \mathrm{SO}_{6,6}$ invariant [61, 66, 74]

$$I_4 = P^2Q^2 - (P \cdot Q)^2. \tag{4.35}$$

So

$$I_4 = I_{aeg} \equiv \det(\gamma^1(a) + \gamma^2(g) + \gamma^3(e)). \tag{4.36}$$

A different subsector which excludes qubit A is obtained by keeping just the R-R charges: the $\mathbf{1}, \mathbf{32}$ in (4.13), which can again be read off Figure 4.2

$$(\mathbf{1}, \mathbf{32}) = (BCE) + (CDF) + (DEG) + (FGB) = b + c + d + f, \tag{4.37}$$

and the corresponding quartic invariant, I_4, reduces to the singlet in $(\mathbf{1},\mathbf{32})\times(\mathbf{1},\mathbf{32})\times(\mathbf{1},\mathbf{32})\times(\mathbf{1},\mathbf{32})$

$$I_4 \sim b^4 + c^4 + d^4 + f^4 + 2[b^2c^2 + c^2d^2 + d^2e^2 + d^2f^2 + c^2f^2 + f^2b^2] + 8bcdf. \quad (4.38)$$

This is does not correspond to any $\mathcal{N}=4$ black hole but rather to an $\mathcal{N}=8$ black hole with only the R-R charges switched on.

For $\mathcal{N}=2$, examining a $\mathrm{SL}_2\times\mathrm{SO}_{2,2}$ subgroup simply reproduces the $\mathrm{SL}_2)^3$ of the *STU* model which we have already considered

$$(\mathbf{2},\mathbf{2},\mathbf{2}) = (ABD) = a, \quad (4.39)$$

so the corresponding quartic invariant

$$I_4 \sim a^4, \quad (4.40)$$

is just Cayley's hyperdeterminant

$$I_4 = -\operatorname{Det} a. \quad (4.41)$$

4.3 The $D=5$ generalisation

4.3.1 Decomposing $\mathrm{E}_{6(6)} \supset \mathrm{SL}_3^3$

We saw in section 4.2.1 and section 4.2.2 that the $\mathrm{E}_{7(7)}$ invariant 4D $\mathcal{N}=8$ entropy could be decomposed under $\mathrm{SL}_2(\mathbb{C})^7$ to yield an analogy with a tripartite entangled seven-qubit system. Similarly, we seek an $\mathrm{E}_{6(6)}$ 5D black hole/string entropy decomposition to $\mathrm{SL}_d(\mathbb{C})^n$. It so happens that E_6 admits the following decompositions

$$\mathrm{E}_{6(6)} \supset \mathrm{SL}_3(\mathbb{R})^3, \quad (4.42)$$

and

$$\mathrm{E}_6(\mathbb{C}) \supset \mathrm{SL}_3(\mathbb{C})^3. \quad (4.43)$$

This suggests a correspondence to a simple three qutrit system [126], but just as the details of the decomposition of the **56** in 4D led to a more specialised system, so too does the detailed decomposition of $\mathrm{E}_{6(6)}$. Specifically, under

$$\mathrm{E}_{6(6)} \supset \mathrm{SL}_{3,A}\times\mathrm{SL}_{3,B}\times\mathrm{SL}_{3,C}, \quad (4.44)$$

the **27** decomposes as

$$\mathbf{27} \to (\mathbf{3}',\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3}',\mathbf{3}') + (\mathbf{3},\mathbf{1},\mathbf{3}). \quad (4.45)$$

An analogous decomposition holds for

$$E_6(\mathbb{C}) \supset \mathrm{SL}_3(\mathbb{C})^3. \tag{4.46}$$

Note the new feature of this decomposition: in contrast to (4.16) we see primed representations alongside unprimed ones. This is not to be ignored and suitable allowances must be made in the interpretation of any embedded qutrit system.

4.3.2 Interpretation

Instead of a straightforward three qutrits (which would this time have the correct counting), we see from (4.45) that rather than

$$a_{ABC}|ABC\rangle, \tag{4.47}$$

we have

$$|\Psi\rangle_{27} = a_{A'B\bullet}|A'B\bullet\rangle + b_{\bullet BC}|\bullet BC\rangle + c_{A'\bullet C'}|A' \bullet C'\rangle, \tag{4.48}$$

which we abbreviate to

$$|\Psi\rangle_{27} = a_{A'B}|A'B\rangle + b_{B'C'}|B'C'\rangle + c_{CA}|CA\rangle, \tag{4.49}$$

where $A, A' = 0, 1, 2$. Note that:

1. Any pair of states has a qutrit in common

2. Each qutrit is excluded from one out of the three states

Despite not being a subsector of the three qutrit space $(\mathbf{3}, \mathbf{3}, \mathbf{3})$, there are three qutrits coupled to each other pairwise as shown in Figure 4.6. The appearance of both primed and unprimed representations is respected by having the upper and lower indices distinguished

$$a^A = \tfrac{1}{2}\varepsilon^{AB_1B_2}a_{[B_1B_2]} \tag{4.50}$$

in contrast to the qubit case where the ε symbol simply related the two $a^A = \varepsilon^{AB}a_B$. The antisymmetry of $a_{[B_1B_2]}$ allows the interpretation of the $3'$ as a pair of indistinguishable "fermions". Just as we referred to the 4D situation as the tripartite entanglement of seven qubits, we can refer to the present situation as the *bipartite entanglement of three qutrits*, and once again this terminology requires some interpretation.

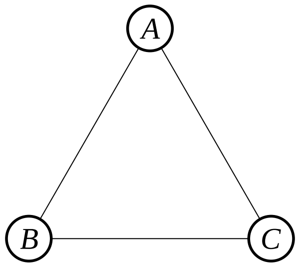


Figure 4.6: The $D = 5$ analogue of Figure 4.3 is the three qutrit entanglement diagram corresponding to the decomposition (4.45) and the state (4.49). It is a triangle with vertices A, B, C representing the qutrits and the lines AB, BC and CA representing the entanglements.

This time, one must explain the primed representations as well as the singlets. This is achievable under the embedding

$$\mathrm{SL}_7 \supset \mathrm{SL}_3 \tag{4.51}$$

under which

$$\mathbf{7} \to \mathbf{3} + \mathbf{3}' + \mathbf{1}, \tag{4.52}$$

so that the qutrits are embedded in 7-dits. Under

$$\mathrm{SL}_{7,A} \times \mathrm{SL}_{7,B} \times \mathrm{SL}_{7,C} \supset \mathrm{SL}_{3,A} \times \mathrm{SL}_{3,B} \times \mathrm{SL}_{3,C} \tag{4.53}$$

we have

$$\begin{aligned}(\mathbf{7},\mathbf{7},\mathbf{7}) \to \quad & (\mathbf{3}',\mathbf{3}',\mathbf{3}') + (\mathbf{3},\mathbf{3},\mathbf{3}) + (\mathbf{1},\mathbf{1},\mathbf{1}) \\ & + (\mathbf{3}',\mathbf{3}',\mathbf{3}) + (\mathbf{3}',\mathbf{3},\mathbf{3}') + (\mathbf{3},\mathbf{3}',\mathbf{3}') \\ & + (\mathbf{3}',\mathbf{3},\mathbf{3}) + (\mathbf{3},\mathbf{3}',\mathbf{3}) + (\mathbf{3},\mathbf{3},\mathbf{3}') \\ & + (\mathbf{3}',\mathbf{3}',\mathbf{1}) + (\mathbf{3}',\mathbf{1},\mathbf{3}') + (\mathbf{1},\mathbf{3}',\mathbf{3}') \\ & + (\mathbf{3},\mathbf{3},\mathbf{1}) + (\mathbf{3},\mathbf{1},\mathbf{3}) + (\mathbf{1},\mathbf{3},\mathbf{3}) \\ & + (\mathbf{3}',\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3}',\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{3}') \\ & + (\mathbf{3},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{3}) \\ & + (\mathbf{3}',\mathbf{1},\mathbf{3}) + (\mathbf{3}',\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3},\mathbf{3}') \\ & + (\mathbf{3},\mathbf{1},\mathbf{3}') + (\mathbf{3},\mathbf{3}',\mathbf{1}) + (\mathbf{1},\mathbf{3}',\mathbf{3}),\end{aligned} \tag{4.54}$$

so that the state (4.49) is indeed a special 27-dimensional subspace of a $7^3 = 343$-dimensional three 7-dit space. The hidden $\mathrm{E}_{6(6)}$ symmetry must now be respected by the entanglement measure, which must be given by the cubic Cartan invariant (2.28)

$$\tau(ABC) = 27|I_3|^2. \tag{4.55}$$

To express I_3 in the basis of amplitudes a, b, c on starts by rewriting (4.45) with singlets suppressed and triplets labelled

$$\mathbf{27} = (AB) + (BC) + (CA), \tag{4.56}$$

or symbolically

$$\mathbf{27} = a + b + c. \tag{4.57}$$

The invariant I_3 is then the singlet in $\mathbf{27} \times \mathbf{27} \times \mathbf{27}$:

$$I_3 = a^3 + b^3 + c^3 + 6abc. \tag{4.58}$$

Terms like

$$\begin{aligned} a^3 &= (AB)(AB)(AB) \\ &= \tfrac{1}{6}\varepsilon_{A_1A_2A_3}\varepsilon^{B_1B_2B_3}a^{A_1}{}_{B_1}a^{A_2}{}_{B_2}a^{A_3}{}_{B_3}, \end{aligned} \tag{4.59a}$$

$$\begin{aligned} b^3 &= (BC)(BC)(BC) \\ &= \tfrac{1}{6}\varepsilon_{B_1B_2B_3}\varepsilon_{C_1C_2C_3}b^{B_1C_1}b^{B_2C_2}b^{B_3C_3}, \end{aligned} \tag{4.59b}$$

$$\begin{aligned} c^3 &= (CA)(CA)(CA) \\ &= \tfrac{1}{6}\varepsilon^{C_1C_2C_3}\varepsilon^{A_1A_2A_3}c_{C_1A_1}c_{C_2A_2}c_{C_3A_3}, \end{aligned} \tag{4.59c}$$

exclude one qutrit (C, A, and B respectively), and the products

$$\begin{aligned} abc &= (AB)(BC)(CA) \\ &= \tfrac{1}{6}a^{A}{}_{B}b^{BC}c_{CA}, \end{aligned} \tag{4.60}$$

exclude none. One can truncate to just the $(\mathbf{3}, \mathbf{3})$ in (4.45) which excludes B

$$|\Psi\rangle = c_{CA}|CA\rangle, \tag{4.61}$$

which is described by just that line not passing through B in the ABC triangle,

$$(\mathbf{3}, \mathbf{3}) = (CA) = c, \tag{4.62}$$

and the corresponding cubic invariant, I_3, reduces to the singlet in $(\mathbf{3},\mathbf{3}) \times (\mathbf{3},\mathbf{3}) \times (\mathbf{3},\mathbf{3})$

$$I_3 = \det c_{CA} \sim c^3. \tag{4.63}$$

4.4 Magic supergravities

The black hole/qubit correspondence continues to hold for the magic supergravities. The essential point is that although the $\mathcal{N} = 8$ black hole charges are real and the entropy is invariant under $\mathrm{E}_{7(7)}(\mathbb{Z})$, the amplitudes of (4.20) are complex, with entanglement measure invariant under $E_7(\mathbb{C})$ which contains both $E_{7(7)}(\mathbb{Z})$ and $E_{7(-25)}(\mathbb{Z})$ as subgroups. Thus, one could equally well have chosen the magic octonionic $\mathcal{N} = 2$ supergravity rather than the conventional $\mathcal{N} = 8$ supergravity. It is true that

$$\mathrm{E}_{7(7)}(\mathbb{Z}) \supset \mathrm{SL}_2(\mathbb{Z})^7, \tag{4.64}$$

but

$$\mathrm{E}_{7(-25)}(\mathbb{Z}) \not\supset \mathrm{SL}_2(\mathbb{Z})^7 \tag{4.65}$$

but this is irrelevant, since all that matters is

$$\mathrm{E}_7(\mathbb{C}) \supset \mathrm{SL}_2(\mathbb{C})^7. \tag{4.66}$$

This line of reasoning also holds for the magic real, complex and quaternionic $\mathcal{N} = 2$ supergravities which are, in any case truncations of $\mathcal{N} = 8$ (in contrast to the octonionic). With this in mind one may revisit the ordinary $\mathcal{N} = 2$ and $\mathcal{N} = 4$ cases. Conventionally we give the seven qubit subsector an $\mathcal{N} = 4$ supergravity interpretation with symmetry $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SO}_{6,6}$ [6], but we could equally have given an interpretation in terms of $\mathcal{N} = 2$ supergravity coupled to 11 vector multiplets with symmetry $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SO}_{10,2}$. Since $\mathrm{SO}_{\ell-1,2}$ is contained in $\mathrm{SO}_{\ell+1}(\mathbb{C})$ and $\mathrm{SO}_{6,m}$ is contained in $\mathrm{SO}_{12+m}(\mathbb{C})$ one can give a qubit interpretation to more vector multiplets for both $\mathcal{N} = 2$ and $\mathcal{N} = 4$, at least in the case of $\mathrm{SO}_{4n}(\mathbb{C})$ which contains $\mathrm{SL}_2(\mathbb{C})^{\otimes 2n}$.

The same argument applies for the $D = 5$ magic supergravities. This time the crucial observation is that the $\mathrm{E}_6(\mathbb{C})$ invariance of the entanglement (4.55) contains both $\mathrm{E}_{6(6)}(\mathbb{Z})$ and $\mathrm{E}_{6(-26)}(\mathbb{Z})$ as subgroups, so that magic octonionic $\mathcal{N} = 2$ supergravity is an equally good black hole correspondence. Although

$$\mathrm{E}_{6(6)}(\mathbb{Z}) \supset \mathrm{SL}_3(\mathbb{Z})^3, \tag{4.67}$$

but

$$\mathrm{E}_{6(-26)}(\mathbb{Z}) \not\supset \mathrm{SL}_3(\mathbb{Z})^3 \tag{4.68}$$

the relevant fact is still

$$\mathrm{E}_6(\mathbb{C}) \supset \mathrm{SL}_3(\mathbb{C})^3. \tag{4.69}$$

Again, this holds for the magic real, complex and quaternionic $\mathcal{N} = 2$ supergravities which are, in any case truncations of $\mathcal{N} = 8$ (in contrast to the octonionic). When revisiting the ordinary $\mathcal{N} = 2$ and $\mathcal{N} = 4$ cases one notes that $\mathrm{SO}_{\ell,1}$ is contained in $\mathrm{SO}_{\ell+1}(\mathbb{C})$ and $\mathrm{SO}_{m,5}$ is contained in $\mathrm{SO}_{5+m}(\mathbb{C})$, so we can give a qutrit interpretation to more vector multiplets for both $\mathcal{N} = 2$ and $\mathcal{N} = 4$, at least in the case of $\mathrm{SO}_{6n}(\mathbb{C})$ which contains $\mathrm{SL}_3(\mathbb{C})^n$.

Chapter 5

Deepening correspondences

5.1 The Cartan invariant in various duality frames

Cartan's quartic invariant I_4 can be written in several bases that manifestly display the symmetries of subgroups of E_7, with the three most useful subgroups for our considerations being SO_8, SL_2^7, and E_6. Black holes are more conveniently described in either the SO_8 or E_6 bases, whereas the SL_2^7 basis is tailored to the qubits. Hence the computation of explicit dictionaries between these bases was an important milestone in the black hole/qubit correspondence as it cemented the validity of the SL_2^7 decomposition of I_4 and permitted easier comparisons between the very different perspectives. The SO_8 basis is the first one we encountered, namely the Cartan basis with its electric-magnetic split

$$\begin{gathered} E_7 \supset \mathrm{SO}_8, \\ \mathbf{56} \to \mathbf{28} + \mathbf{28}, \\ I_4(x, y) = -\operatorname{tr}(xy)^2 + \tfrac{1}{4}(\operatorname{tr} xy)^2 - 4(\operatorname{Pf} x + \operatorname{Pf} y). \end{gathered} \tag{5.1}$$

We recall that this can be SU_8 transformed to a canonical form that reduces to the hyperdeterminant. Secondly we have the Fano basis, in which is written in terms of the Fano amplitudes $a, \ldots, g$

$$\mathrm{E}_7 \supset \mathrm{SL}_2^7, \tag{5.2}$$

$$\begin{aligned}\mathbf{56} \to &(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})\\ &+(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})\\ &+(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1})\\ &+(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2})\\ &+(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1})\\ &+(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})\\ &+(\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2}),\end{aligned} \tag{5.3}$$

$$\begin{aligned}I_4 = {}& a^4+b^4+c^4+d^4+e^4+f^4+g^4\\ &+2\Big[a^2b^2+a^2c^2+a^2d^2+a^2e^2+a^2f^2+a^2g^2\\ &\quad+b^2c^2+b^2d^2+b^2e^2+b^2f^2+b^2g^2\\ &\quad+c^2d^2+c^2e^2+c^2f^2+c^2g^2\\ &\quad+d^2e^2+d^2f^2+d^2g^2\\ &\quad+e^2f^2+e^2g^2\\ &\quad+f^2g^2\Big]\\ &+8\left[abce+bcdf+cdeg+defa+efgb+fgac+gabd\right].\end{aligned} \tag{5.4}$$

The dictionary between the Fano basis and the Cartan basis charges is

$$x^{ij} = \eta_{il}C^l_{jk}a^k_{\phi(i,j)}, \tag{5.5}$$

$$y_{ij} = \eta_{il}C^l_{jk}a^k_{\tilde{\phi}(i,j)}. \tag{5.6}$$

Where $i,j,k,l \in \{0,\dots,7\}$, C is the dual Fano plane structure constant array (i.e. the structure constants of the multiplication table defined by the dual Fano plane generalised so that $C^i_{jk}=0$ for $j=0$ or $k=0$), η is the 8-dimensional Minkowski matrix (negative signature), $(a^0,a^1,\dots,a^7)=(0,a,\dots,g)$, and ϕ and $\tilde{\phi}$ are given by

$$\phi(i,j) := \begin{cases}7 & i=0 \text{ or } j=0\\ |(i-j)^2|_7 & \text{else}\end{cases} \tag{5.7}$$

$$\tilde{\phi}(i,j) := 7-\phi(i,j). \tag{5.8}$$

The $\phi(i,j)$ are known as (shifted) quadratic residues modulo seven. It can be helpful to regard the eight components of each a^i as a pair of quaternions.

Finally there is the Freudenthal/Jordan basis in which the E_7 invariant is given as a quartic form on an object called a Freudenthal Triple system (FTS). See section A.2.1 for details. The FTS uses the Springer construction of cubic Jordan algebras, which here means the Jordan algebra elements are represented by hermitian octonionic matrices (actually split octonions)

$$\begin{aligned} & E_7 \supset E_6, \\ & \mathbf{56} \to \mathbf{1} + \mathbf{27} + \mathbf{1} + \mathbf{27}', \\ I_4(p^0, P; q_0, Q) = & -\Big[p^0 q_0 + \operatorname{tr}(J_3(P) \circ J_3(Q))\Big]^2 \\ & + 4\Big[-p^0 J_3(Q) + q_0 J_3(P) + \operatorname{tr}(J_3{}^{\#}(P) \circ J_3{}^{\#}(Q))\Big]. \end{aligned} \tag{5.9}$$

One can insist that the the a amplitudes correspond to the eight STU subsector charges to permit a seamless reduction to the STU model correspondence once the other charges are switched off. In essence, we decompose $\mathrm{E}_{7(7)} \to \mathrm{E}_{6(6)} \to \mathrm{SO}_{4,4}$ to obtain

$$\mathbf{56} \to \mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{8_s}+\mathbf{8_c}+\mathbf{8_v}+\mathbf{8_s}+\mathbf{8_c}+\mathbf{8_v}, \tag{5.10}$$

and recognise the a amplitudes as the singlets. To properly assign the remaining amplitudes we instead decompose, for example

$$\mathrm{E}_{7(7)} \supset \mathrm{SL}_{2,A} \times \mathrm{SL}_{2,B} \times \mathrm{SL}_{2,D} \times \mathrm{SO}_{4,4}, \tag{5.11}$$

under which,

$$\mathbf{56} \to (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{8_v}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{8_s}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{8_c}). \tag{5.12}$$

to see that the A qubit transforms as a doublet with the $\mathbf{8_v}$, so that the e and g amplitudes may be associated to it (the a amplitudes already being accounted for). Repeating the process for the remaining qubits reveals that b and f correspond to the $\mathbf{8_s}$'s and c and d correspond to the $\mathbf{8_c}$'s. Retaining only $\mathbf{8}$s of a given type in (5.9) and comparing with (5.4) eventually allows one to tease out the explicit transformation. The Fano-Freudenthal dictionary can be written concisely if we make the following definition.

$$[i,j,k,l]_{mn} := \begin{cases} -\begin{pmatrix} P_i^n - P_k^n & -P_j^n + P_l^n \\ P_j^n + P_l^n & P_i^n + P_k^n \end{pmatrix} & m = 0 \\ -\begin{pmatrix} Q_n^i - Q_n^k & -Q_n^j + Q_n^l \\ Q_n^j + Q_n^l & Q_n^i + Q_n^k \end{pmatrix} & m = 1 \end{cases} \tag{5.13}$$

With $i,j,k,l \in \{0,\ldots,7\}$, $m \in \{0,1\}$, and $n \in \{c,s,v\}$. The dictionary is (with $i \in$

$\{0, 1\}$, and σ^i being Pauli matrices):

$$\begin{array}{c|c} \text{Fano} & \text{Freudenthal} \\ \hline d_{i\bullet\bullet} & -\sigma^3 \cdot [0,1,4,5]_{ic} \cdot \sigma^3 \\ c_{\bullet i\bullet} & \sigma^1 \cdot [2,3,6,7]_{ic} \cdot \sigma^1 \\ f_{\bullet\bullet i} & \sigma^1 \cdot [2,3,6,7]^{\top}_{is} \cdot \sigma^3 \\ b_{i\bullet\bullet} & \sigma^3 \cdot [0,1,4,5]_{is} \cdot \sigma^1 \\ g_{\bullet i\bullet} & -\sigma^1 \cdot [0,1,4,5]_{iv} \cdot \sigma^1 \\ e_{\bullet\bullet i} & -\sigma^1 \cdot [2,3,6,7]_{iv} \cdot \sigma^1 \end{array} \tag{5.14}$$

Take note of the transpose for f. The $a_{\bullet\bullet\bullet}$ amplitudes are given by the Kallosh-Linde dictionary, which in this notation is

$$a_{\bullet\bullet 0} = \sigma^3 \cdot [2,3,0,1]_0 \cdot \sigma^3, \tag{5.15}$$

$$a_{\bullet\bullet 1} = \sigma^1 \cdot [2,3,0,1]_1 \cdot \sigma^1. \tag{5.16}$$

The remaining dictionary and the inverse dictionaries are then trivial to compute, allowing one to navigate the three descriptions with ease.

5.2 SUSY classification meets entanglement classification

Using the canonical basis (2.24) to convert the Cartan invariant to the Cayley's hyperdeterminant permits the entanglement classification of the seven qubit system to be derived in the same manner as for three qubits as shown in Table 5.1. The A-B-C, A-BC, W and GHZ states are just those of the STU model. For $\mathcal{N} = 8$, as for $\mathcal{N} = 2$, the large black holes correspond to the two classes of GHZ-type (entangled) states and small black holes to the separable or W class. Note, however, that we obtain a finer supersymmetry and charge orbit correspondences than for the $\mathcal{N} = 2$ STU theory of Table 2.4.

Similarly, the classification of the bipartite entanglements of three qutrits is related to the classification of $\mathcal{N} = 8, D = 5$ supersymmetric black holes [21] shown in Table 5.2.

The classification correspondence for the STU model is enriched by the perspective in which a three-qubit state is mapped to a Freudenthal triple system. The details are provided in section A.2.2, but in essence the state is grouped into permutation insensitive

Table 5.1: As in Table 4.1 entanglement measures are used to classify states, but this time concerning the tripartite entanglement of seven qubit states. The correspondence relates these to the $D = 4, \mathcal{N} = 8$ black holes discussed in section 2.5.

Class	S_A	S_B	S_C	$\operatorname{Det} a$	Black hole	SUSY
A-B-C	0	0	0	0	small	1/2
A-BC	0	> 0	> 0	0	small	1/4
B-CA	> 0	0	> 0	0	small	1/4
C-AB	> 0	> 0	0	0	small	1/4
W	> 0	> 0	> 0	0	small	1/8
GHZ	> 0	> 0	> 0	< 0	large	1/8
GHZ	> 0	> 0	> 0	> 0	large	0

Table 5.2: The $D = 5$ analogue of Table 4.1 relates two-qutrit entanglements and their corresponding $D = 5, \mathcal{N} = 8$ black holes.

Class	C_2	τ_{AB}	Black hole	SUSY
A-B	0	0	small	1/2
Rank 2 Bell	> 0	0	small	1/4
Rank 3 Bell	> 0	> 0	large	1/8

subsets

$$|\psi\rangle = a_{ABC}|ABC\rangle \quad \leftrightarrow \quad \Psi = (a_{111}, a_{000}, (a_{100}, a_{010}, a_{001}), (a_{011}, a_{101}, a_{110})) \tag{5.17}$$

and operated upon by a system of rank polynomials arising the in the formalism of triple systems. The associated ranks 0,1,2,3,4 correspond exactly to the familiar null, separable, biseparable, W and GHZ classes as shown in Figure 5.1. The rank 0, 1, and 4 conditions are trivially related to their corresponding entanglement classes since the rank polynomials involve the state norm and hyperdeterminant. To complete the analogy one identifies some of the remaining rank polynomials with the $3B_{200}$ and C_{111} covariants encountered in section 3.3.1 and proves the constraint

$$\text{two } B^i = 0 \Leftrightarrow C_{111} = 0, \tag{5.18}$$

making C_{111} redundant for classification purposes as promised. The result is that the class conditions and rank conditions can be seen as equivalent using the three-qubit covariants. In this manner we provided a manifestly SLOCC invariant three-qubit classification while providing new insights about the SLOCC orbits and the coset character-

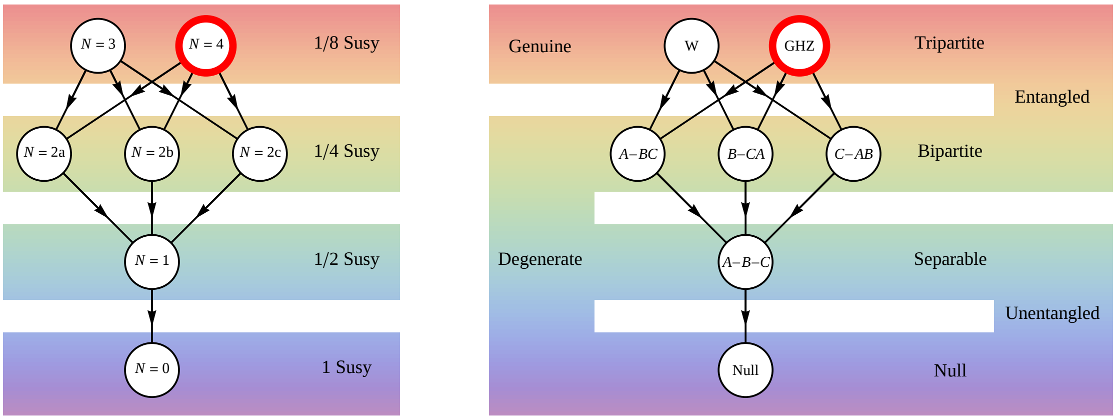


Figure 5.1: Exact matching of three-qubit SLOCC orbits and STU black hole orbits. N corresponds to FTS rank.

isation of the small STU black hole orbits, displayed in Table 5.3.

Table 5.3: Coset spaces of the orbits of the 3-qubit state space $\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2$ under the action of the SLOCC group $\mathrm{SL}_2(\mathbb{C})^3$.

Class	FTS Rank	Orbits	dim	Projective orbits	dim
Separable	1	$\frac{\mathrm{SL}_2(\mathbb{C})^3}{\mathrm{SO}_2(\mathbb{C})^2 \ltimes \mathbb{C}^3}$	4	$\frac{\mathrm{SL}_2(\mathbb{C})^3}{[\mathrm{SO}_2(\mathbb{C}) \ltimes \mathbb{C}]^3}$	3
Biseparable	2	$\frac{\mathrm{SL}_2(\mathbb{C})^3}{\mathrm{O}_3(\mathbb{C}) \times \mathbb{C}}$	5	$\frac{\mathrm{SL}_2(\mathbb{C})^3}{\mathrm{O}_3(\mathbb{C}) \times [\mathrm{SO}_2(\mathbb{C}) \ltimes \mathbb{C}]}$	4
W	3	$\frac{\mathrm{SL}_2(\mathbb{C})^3}{\mathbb{C}^2}$	7	$\frac{\mathrm{SL}_2(\mathbb{C})^3}{\mathrm{SO}_2(\mathbb{C}) \ltimes \mathbb{C}^2}$	6
GHZ	4	$\frac{\mathrm{SL}_2(\mathbb{C})^3}{\mathrm{SO}_2(\mathbb{C})^2}$	7	$\frac{\mathrm{SL}_2(\mathbb{C})^3}{\mathrm{SO}_2(\mathbb{C})^2}$	7

5.3 The octonions

The octonions serve as somewhat of a crossroads in the early black hole/qubit correspondence, arising in the description of the $\mathcal{N} = 8$ charge vector, in $\mathcal{N} = 4$ and $\mathcal{N} = 2$ subsectors (via their quaternionic subsector), and in the Cartan dictionary. As an algebra, the octonions $\mathbb{O}$ (with product denoted by juxtaposition) possess numerous interesting properties, some of which are notable absences of familiar properties. Historical notes and other extensive details can be found in [138], and some interesting physical applications may be found in [139–148]. Although the octonions do appear in a quantum information context [149,150], this seems to be unrelated to their role in the black hole/qubit correspondence.

Typical octonions $a, b, c \in \mathbb{O}$ are:

- 8-tuples of real numbers: $a, b, c \in \mathbb{R}^8$ so that they form an 8-dimensional vector space, with basis elements $e_0, \ldots, e_7$.
- non-real: $a \neq a^*$, like the complexes. The conjugate $\bullet^* : \mathbb{O} \to \mathbb{O}$ trivially extends the conjugate for $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$ so that basis element e_μ is mapped to $\eta_{\mu\nu} e_\nu$ (with $\eta \equiv \text{diag}(1, -\mathbb{1}_7)$). Under the familiar partition $\mu = (0, i)$, scalar multiples of e_0 are real octonions, and scalar multiples of e_i are imaginary octonions.
- non-commutative: $ab \neq ba$, like the quaternions.
- non-associative: $a(bc) \neq (ab)c$, a new property not present in $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.
- alternative - meaning that the subalgebra generated by any two elements is associative, or equivalently, the associator $[\bullet, \bullet, \bullet] : \mathbb{O}^3 \to \mathbb{O}$, $(a, b, c) \mapsto a(bc) - (ab)c$ is alternating: $[a_1, a_2, a_3] = (-)^\pi [a_{\pi(1)}, a_{\pi(2)}, a_{\pi(3)}]$ with $\pi \in S_3$.
- a division algebra, so that when a product of octonions is zero one of the multiplied octonions must have been zero: $ab = 0 \Rightarrow a = 0 \wedge b = 0$. They share this property with $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and no other algebras.
- normed: $|ab| = |a||b|$, which implies the division algebra property. Like the conjugate, the norm $|\bullet| : \mathbb{O} \to \mathbb{R}$ is also a trivial extension of the norm for the other division algebras: $a \mapsto a^* a$.

Clearly the octonions are closely related to the other division algebras, and indeed in this context it is instructive to classify $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ as $*$-algebras (star algebras). Such algebras are characterised by the possession of a real-linear conjugation map that is involutive ($(a^*)^* \equiv a$) and an anti-automorphism ($(ab)^* \equiv b^* a^*$). The salient point is that as an $*$-algebra, the octonions can be constructed using quaternions, which themselves can be obtained via complexes (a.k.a. binarions). This is an iterative procedure that extends down to the reals called the Cayley-Dickson process. It produces a 2^n dimensional $*$-algebra after n steps and is implemented via a multiplication rule and a conjugation rule:

$$\begin{aligned} (a_1, a_2) \cdot (b_1, b_2) &:= ((a_1 \cdot b_1 - b_2^* \cdot a_2), (b_2 \cdot a_1 + a_2 \cdot b_1^*)) \\ (a_1, a_2)^* &:= (a_1^*, -a_2) \end{aligned} \tag{5.19}$$

the result of which, at $n = 3$ is shown in Table 5.4. This table does not exactly reproduce the Fano plane multiplication table, but is an automorphism of it. As (5.19) is recursively

Table 5.4: The octonion multiplication table resulting from the Cayley-Dickson process. Due to the generality of (5.19) as a method for generating $*$-algebras this table is considered canonical. The top-left quadrant is actually the quaternion multiplication table, and the top left quadrant of *that* is the complex (binarion) table. Apart from the first row and column, and disregarding the -1's on the diagonal, the table is antisymmetric.

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

applied the new algebras lose more properties. Initially we have the reals at step zero, which have the property of being their own conjugates. The complexes lose this property, but are commutative. The quaternions lose that but retain associativity. The octonions lose that and retain the division algebra property, and at step 4 - the sedenions, we lose that as well. From then on, the procedure "stabilises" so that no more properties are lost. All of the algebras generated in this way are nicely normed, which means $a + a^* \in \mathbb{R}$ and $a^*a = aa^* > 0$.

In a similar fashion, with a change of sign [151] one obtains the split-octonions of Table 5.5).

$$(a_1, a_2) \cdot (b_1, b_2) := ((a_1 \cdot b_1 + b_2^* \cdot a_2), (b_2 \cdot a_1 + a_2 \cdot b_1^*)) \tag{5.20}$$

The sign change only takes effect at the top level in the process and doesn't affect lower dimensional multiplication, though further generalisations permit this. At any rate, it is the split octonion multiplication table that is used in the Freudenthal construction to generate I_4. An alternative method of generating hypercomplex numbers involves appending a new basis element to the vector space and generating new elements until the algebra closes. In this regime, the octonions are generated by the three basis elements

Table 5.5: The canonical split-octonion multiplication table results from a single sign change in (5.19) to give (5.20). This time, the last four elements on the main diagonal are +1 rather than -1. Since the sign change in (5.19) is only operative at the top level and doesn't percolate down to the quaternions and complexes, we see that the top-left quadrant still gives the lower dimensional tables.

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	e_0	$-e_1$	$-e_2$	$-e_3$
e_5	e_5	e_4	$-e_7$	e_6	e_1	e_0	e_3	$-e_2$
e_6	e_6	e_7	e_4	$-e_5$	e_2	$-e_3$	e_0	e_1
e_7	e_7	$-e_6$	e_5	e_4	e_3	e_2	$-e_1$	e_0

i, j, and k:

$$\begin{aligned} &\mathbb{R} : 1 \\ &\mathbb{C} : 1, i \\ &\mathbb{H} : 1, i, j, ij \\ &\mathbb{O} : 1, i, j, ij, k, ik, jk, (ij)k, \end{aligned} \tag{5.21}$$

which obviously should not be confused with quaternionic basis elements. This has a certain transparency, but doesn't capture the full generality of the Cayley-Dickson process. An alternative characterisation of the split vs. non split "hypernumbers" is to refer to them as circular vs. hyperbolic [152, 153]. Circular and hyperbolic here refers to the norms of these numbers, which in the split/hyperbolic case has a split signature. A succinct multiplication rule based on the Cayley-Dickson process is provided by the binary or "dyadic" formalism [154]

$$e_x e_y = (-)^{x_3(y_1y_2+y_3)+x_2(y_2+y_3+y_1y_3)+x_1(y_1+y_2+y_3+y_2y_3)} e_{x\oplus y} \tag{5.22}$$

where $x, y \in \{0, \ldots, 7\}$, $x_i, y_i \in \{0, 1\}$, and $\oplus$ is the bitwise OR operation. Note that we differ in conventions from [154] by reversing the order of the digits. The dyadic formalism is general enough to encompass not only $*$-algebras, but Clifford algebras (and hence Grassmann algebras) too. These algebras share a factorisation property in their multiplication rule

$$e_x e_y = V_{xy} e_{x\oplus y}, \tag{5.23}$$

where V_{xy} is referred to as a scalar factor, which itself factorises

$$V_{xy} = (-)^{k_{xy}} \prod_{i=1}^{n} \gamma_i^{x_i y_i}. \tag{5.24}$$

The k_{xy} matrix characterises the algebra along with an anti-involution vector f_x

$$e_x \mapsto e_x^* = (-)^{f_x} e_x, \tag{5.25}$$

both of which can be made to obey recursion relations that reproduce the Cayley-Dickson process. The γ_i also help characterise the algebra: in the $*$-algebra case they are either 1 or -1 and distinguish ordinary algebras from their various split cousins, in the Clifford case they are the eigenvalues of the associated quadratic form matrix, and consequently they vanish in the Grassmann case (though the factorisation (5.24) as written would require the interpretation $0^0 = 1$).

Graphically representing the structure contents of $*$-algebras as in Figure 5.2 reveals an interesting fractal structure. The non-zero elements clearly lie in tetrahedral patterns (reminding us that a regular tetrahedron can be embedded in a cube,sharing vertices); specifically these appear to be Sierpinski tetrahedra (a.k.a Sierpinski sponge, or "tetrix"). It seems that the objects would be truly fractal in the limit of infinite $*$-algebra dimension. Despite appearances, it would not possess a truly three-dimensional character, instead having a (capacity) fractal dimension of two. Viewing the i-j plane shows the signs of i-j products, while the height in k specifies the basis element resulting from an i-j product. Switching to a Grassmann algebra collapses the tetrahedron down to an ordinary Sierpinski triangle.

Restricting to a quaternionic cycle (a line in the Fano plane) permits a striking result: that the $\mathcal{N} = 4$ Cartan invariant can be written as the hyperdeterminant over the imaginary octonions. Writing the imaginary quaternion multiplication as

$$e_i e_j = -\delta_{ij} + \varepsilon_{ijk} e_k \tag{5.26}$$

one forms

$$e_i e_j e_k e_l + e_i e_k e_l e_j = 2\delta_{ij}\delta kl + \dots, \tag{5.27}$$

where the dots signify terms vanishing when contracted with

$$-\tfrac{1}{2}\varepsilon^{A_1A_2}\varepsilon^{B_1B_2}\varepsilon^{A_3A_4}\varepsilon^{B_3B_4}\varepsilon^{D_1D_4}\varepsilon^{D_2D_3} a^i_{A_1B_1D_1} a^j_{A_2B_2D_2} a^k_{A_3B_3D_3} a^l_{A_4B_4D_4}. \tag{5.28}$$

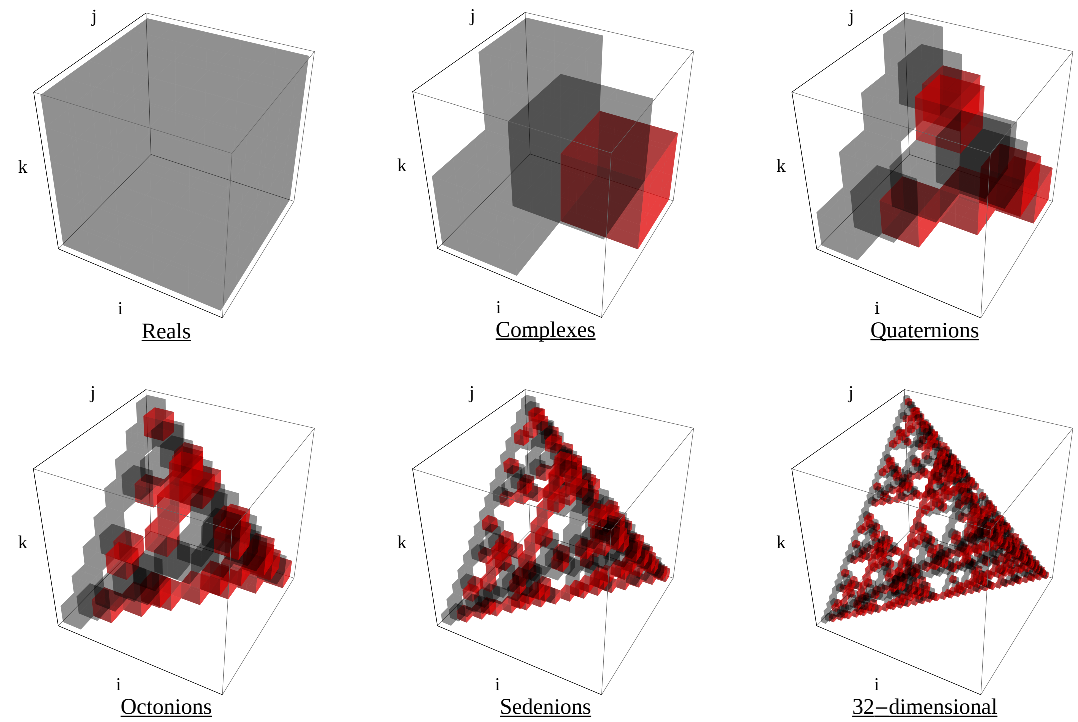


Figure 5.2: Structure constant diagrams: the axes i, j, k in each 3D plot correspond to the indices of the structure constant array of the plot's $*$-algebra. Each i, j, k position is filled with a cube coloured according to the value of the structure constant with that i, j, k specification. Values are depicted as follows: $1 \to$ black, $-1 \to$ red, and $0 \to$ empty.

Choosing the imaginary quaternions to be

$$q = ae_1 + ee_5 + ge_7 \tag{5.29}$$

then results in the aeg $\mathcal{N} = 4$ subsector of I_4.

Chapter 6

The brane wrapping perspective

6.1 Three-qubits and D3 branes

With the correspondence between the tripartite entanglement measure of three qubits and the macroscopic entropy of the four-dimensional $\mathcal{N} = 2, D = 4$ *STU* black holes established, one can ask whether there is a true physical underpinning, or if it is simply a group theoretic analogy. By examining the microscopic string-theoretic origin of the eight *STU* charges we emerge with a brane wrapping perspective that suggests a more substantive link.

As we have seen, there are many ways of embedding the *STU* model in string/M-theory, but the most useful for the present purpose is Type IIB compactified on T^6. Dp-branes wrapping around the six compact dimensions provide the string-theoretic interpretation of the black holes. A Dp-brane wrapped around a p-dimensional cycle of the compact directions $(x^4, x^5, x^6, x^7, x^8, x^9)$ looks like a D0-brane from the four-dimensional (x^0, x^1, x^2, x^3) perspective. In this case we have four D3-branes wrapping the $(579), (568), (478), (469)$ cycles with wrapping numbers N_0, N_1, N_2, N_3 and intersecting over a string [155]. This picture is consistent with the interpretation of the 4-charge black hole as a bound state at threshold of four 1-charge black holes [61, 156, 157]. A fifth parameter θ is obtained [158, 159] by allowing the N_3 brane to intersect at an angle which induces additional effective charges on the $(579), (569), (479)$ cycles. The microscopic calculation of the entropy consists of taking the logarithm of the number of microstates and, to leading order, yields the same result as the macroscopic analysis [160].

The wrapped circles are denoted by crosses x and the unwrapped circles by noughts o as shown in Table 6.1. We associate the three-qubit basis vectors $|ABC\rangle$, with wrapping configurations of these intersecting D3-branes.

Table 6.1: Three-qubit interpretation of the 8-charge $D = 4$ black hole from four D3-branes wrapping around the lower four cycles of T^6 with wrapping numbers N_0, N_1, N_2, N_3 and then allowing N_3 to intersect at an angle θ.

dimension						charges		$\|ABC\rangle$
4	5	6	7	8	9	macro	micro	
x	o	x	o	x	o	p^0	0	$\|000\rangle$
o	x	o	x	x	o	q_1	0	$\|110\rangle$
o	x	x	o	o	x	q_2	$-N_3 \sin\theta \cos\theta$	$\|101\rangle$
x	o	o	x	o	x	q_3	$N_3 \sin\theta \cos\theta$	$\|011\rangle$
o	x	o	x	o	x	q_0	$N_0 + N_3 \sin^2\theta$	$\|111\rangle$
x	o	x	o	o	x	$-p^1$	$-N_3 \cos^2\theta$	$\|001\rangle$
x	o	o	x	x	o	$-p^2$	$-N_2$	$\|010\rangle$
o	x	x	o	x	o	$-p^3$	$-N_1$	$\|100\rangle$

To wrap or not to wrap; that is the qubit.
-M. J. Duff

Specifically, we associate the three T^2 with the $SL_{2,A} \times SL_{2,B} \times SL_{2,C}$ of the three qubits A, B, and C. The 8 different cycles then yield 8 different basis vectors $|ABC\rangle$ as in the last column of the Table, where $|0\rangle$ corresponds to xo and $|1\rangle$ to ox. It is then immediately apparent that the GHZ state of Table 3.2 is described by four D3-branes intersecting over a string, or groups of four wrapping cycles with just one cross in common. Performing a T-duality transformation, one obtains a Type IIA interpretation with N_0 D0-branes and N_1, N_2, N_3 D4-branes where $|0\rangle$ now corresponds to xx and $|1\rangle$ to oo.

This interpretation manages to relate a well-known fact of quantum information theory, that the most general real three qubit state can be parameterised by four real numbers and an angle, to a well-known fact of string theory, that the most general STU black hole can be described by four D3-branes intersecting at an angle. This analysis also provided an explanation for the appearance of the qubit two-valuedness (0 or 1) that was lacking in previous treatments: the brane can wrap one circle or the other in each T^2. Further, the number of qubits is three because of the number of extra dimensions is six.

6.2 Two-qutrits and M2 branes

The brane wrapping perspective employed for three-qubits and four D3-branes intersecting on a string in the STU model suggests that the correspondence between $D = 5$ black holes and qutrits should admit an analogous interpretation. This expectation bears out. In this case 9-charge $\mathcal{N} = 2$, $D = 5$ black hole is most conveniently embedded in the $\mathcal{N} = 8$ theory by way of three M2-branes [155, 161] wrapping the (58), (69), (710) cycles of the T^6 compactification of $D = 11$ M-theory and intersecting over a point with wrapping numbers N_0, N_1, N_2 as shown in Table 6.2. This time we associate the two T^3

Table 6.2: Two qutrit interpretation of the 9-charge $D = 5$ black hole from M2-branes in $D = 11$ wrapping around the upper three cycles of T^6 with wrapping numbers N_0, N_1, N_2. Note that they intersect over a point.

dimension						charges		$\|AB\rangle$
5	6	7	8	9	10	macro	micro	
x	o	o	x	o	o	p^0	N_0	$\|00\rangle$
o	x	o	o	x	o	p^1	N_1	$\|11\rangle$
o	o	x	o	o	x	p^2	N_2	$\|22\rangle$
x	o	o	o	x	o	p^3	0	$\|01\rangle$
o	x	o	o	o	x	p^4	0	$\|12\rangle$
o	o	x	x	o	o	p^5	0	$\|20\rangle$
x	o	o	o	o	x	p^6	0	$\|02\rangle$
o	x	o	x	o	o	p^7	0	$\|10\rangle$
o	o	x	o	x	o	p^8	0	$\|21\rangle$

with the $\mathrm{SL}_{3,A} \times \mathrm{SL}_{3,B}$ of the two qutrits A and B. The 9 different cycles then yield the 9 different basis vectors $|AB\rangle$ as in the last column of Table 6.2, where $|0\rangle$ corresponds to xoo, $|1\rangle$ to oxo, and $|2\rangle$ to oox. It is once again obvious that we reproduce the three parameter two-qutrit state $|\Psi\rangle$ of (3.37). This time the wrapped branes interpretation provides an explanation for the appearance of the qutrit three-valuedness: the brane can wrap one of the three circles in each T^3 and the number of qutrits is two because of the number of extra dimensions is six.

Chapter 7

Four qubit entanglement

7.1 Attempts to classify four-qubit entanglement

Having discussed the matching of classifications for three qubit entanglement and extremal black holes in the *STU* model, it is only natural to attempt to extend this to four qubits. Indeed, since experimentalists now control entanglement with four qubits [162] the classification of this entanglement is an active area of research. In contrast to two- and three-qubit entanglement, which are well understood as we have seen, the situation for four qubits seems more confused, with apparently contradictory claims made in the literature, as illustrated in Table 7.1.

Table 7.1: Various results on four-qubit entanglement.

Paradigm	Author	Year	Ref	result mod perms	result incl. perms
classes	Lamata et al,	2006	[163]	8 genuine,5 degenerate	16 genuine,18 degenerate
	Cao et al	2007	[164]	8 genuine,4 degenerate	8 genuine,15 degenerate
	Li et al	2007	[165]	?	$\geq$ 31 genuine,18 degenerate
	Akhtarshenas et al	2010	[166]	?	11 genuine, 6 degenerate
families	Verstraete et al	2002	[167]	9	?
	Chretrentahl et al	2007	[168]	9	?
	String theory	2010	[28]	9	31

While some of the disagreements are calculational, the main disparity is that the authors of Table 7.1 are actually employing distinct classification paradigms: "classes" vs. "families" (the two are in principle consistent and complementary perspectives). On the classes side is the covariant based approach we encountered in chapter 3 which uses discriminating polynomials to distinguish the SLOCC orbits. This approach was attempted by Briand et al. [125] with interesting, but partial results. They found a complete set of 170 independent generating covariants for degrees up to and including 12, and used them to distinguish representative states provided by the family approach. The family paradigm is a "normal form" approach which considers families of orbits where any given state may be transformed into a unique normal form. If the normal form depends on some of the algebraically independent SLOCC invariants it constitutes a family of orbits parameterised by these invariants. On the other hand a parameter-independent family contains a single orbit. This philosophy is adopted for the four-qubit case in [167, 168]. Up to permutation of the four qubits, these authors found 6 parameter-dependent families called G_{abcd}, L_{abc_2}, $L_{a_2b_2}$, $L_{a_2 0_{3\oplus\bar{1}}}$, L_{ab_3}, L_{a_4} and 3 parameter-independent families called $L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$, $L_{0_{5\oplus\bar{3}}}$, $L_{0_{7\oplus\bar{1}}}$:

$$\begin{aligned}
G_{abcd} &= \frac{a+d}{2}(|0000\rangle + |1111\rangle) + \frac{a-d}{2}(|0011\rangle + |1100\rangle) \\
&\quad + \frac{b+c}{2}(|0101\rangle + |1010\rangle) + \frac{b-c}{2}(|0110\rangle + |1001\rangle) \\
L_{abc_2} &= \frac{a+b}{2}(|0000\rangle + |1111\rangle) + \frac{a-b}{2}(|0011\rangle + |1100\rangle) \\
&\quad + c(|0101\rangle + |1010\rangle) + |0110\rangle \\
L_{a_2b_2} &= a(|0000\rangle + |1111\rangle) + b(|0101\rangle + |1010\rangle) \\
&\quad + |0110\rangle + |0011\rangle \\
L_{ab_3} &= a(|0000\rangle + |1111\rangle) + \frac{a+b}{2}(|0101\rangle + |1010\rangle) \\
&\quad + \frac{a-b}{2}(|0110\rangle + |1001\rangle) \\
&\quad + \frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle - |0111\rangle - |1011\rangle) \\
L_{a_4} &= a(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle) \\
&\quad + (i|0001\rangle + |0110\rangle - i|1011\rangle) \\
L_{a_2 0_{3\oplus\bar{1}}} &= a(|0000\rangle + |1111\rangle) + (|0011\rangle + |0101\rangle + |0110\rangle) \\
L_{0_{5\oplus\bar{3}}} &= |0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle \\
L_{0_{7\oplus\bar{1}}} &= |0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle \\
L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}} &= |0000\rangle + |0111\rangle,
\end{aligned}$$

with parameters $a, b, c, d \in \mathbb{C}$. Here the subscript notation refers to the eigenvalues of the Jordan normal form of the representative states once they are transformed into a particular matrix format, see [168] for full details. Note that L_{ab_3} should really be written L_{a_3b} to correspond to the given state, but we follow the literature convention of

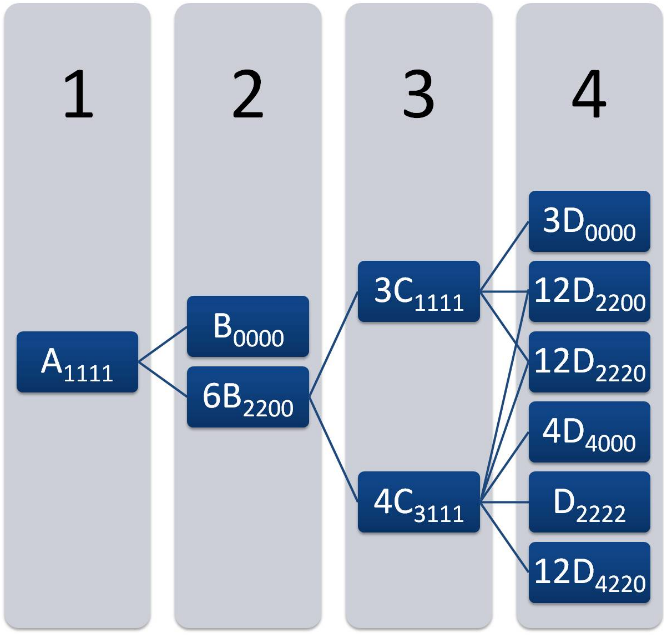


Figure 7.1: Four-qubit covariant lattice down to degree 4 (the numbers along the top correspond to degrees). C_{1111} contributes six D_{2200} and three D_{2220}, with the rest provided by C_{3111}; an ambiguity which did not arise in the three-qubit case.

mislabelling it.

To illustrate the difference between these two approaches, consider the separable EPR-EPR state $(|00\rangle+|11\rangle)\otimes(|00\rangle+|11\rangle)$. Since this is obtained by setting $b = c = d = 0$ in G_{abcd} it belongs to the G_{abcd} family, whereas in the covariant approach it forms its own class. Similarly, a totally totally separable A-B-C-D state, such as $|0000\rangle$, for which all (save one) covariants vanish, belongs to the family L_{abc_2}, which also contains genuine four-way entangled states. These interpretational differences were also noted in [163].

To begin with, we make our own attempt at a covariant classification, but in contrast to Briand et al. we consider the full web of covariants down to a given degree rather than restricting to the independent covariants, see for example Figure 7.2 which is already considerably more complicated than the three qubit case, being a lattice rather than a simple tree. On the other hand, we are only able to do so up to degree six, whereas Briand et al. terminate at the maximal degree which is 12. By considering the full lattice, we are able to group covariants into permutation insensitive subsets as we did in section 3.3.1. This affords us a simplification in the classification since representative states can have a permutation invariant set of amplitudes SLOCC-transformed to zero without affecting the classification. We develop the necessary formalism below.

Permutation invariant subsets For n-qubits, split the 2^n state vector coefficients $\{a_i\}_{i=0}^{2^n-1}$ into permutation-invariant subsets by placing them into $n+1$ totally symmetric tensors $A_{[0]}, A_{[1]}, \ldots, A_{[n]}$ (where the subscript is the tensorial valence), with zero entries at positions with any duplicate indices. The $A_{[p]}$ thus have $\binom{n}{p}$ independent components, correctly yielding 2^n components for all $n+1$ tensors. One also has invariant tensors $d_{[n]}$ and $d^{[n]}$ with entries proportional to $|\epsilon_{i_1\cdots i_n}|$ and $|\epsilon^{i_1\cdots i_n}|$ which may be used to dualise the $A_{[p]}$ with $p > n/2$ to $A^{[n-p]}$. When $n = 2m$ the tensor $A_{[m]}$ will be self-dual. The explicit entries of the $A_{[p]}$ may be defined as

$$A_{i_1\cdots i_p} := d_{i_1\cdots i_p} a_{\sum_{k=1}^{p} 2^{n-i_k}}, \tag{7.1}$$

so that the entries of $A^{[n-p]}$ are precisely the bit-flipped entries of $A_{[p]}$. For example, the tensors for $n = 1, 2, 3, 4$ qubits are:

1 qubit:

$$\{A_{[0]}, A^{[0]}\} = \{a_0, a_1\} \tag{7.2}$$

2 qubits:

$$\{A_{[0]}, A_{[1]}, A^{[0]}\} = \{a_{00}, \begin{pmatrix} a_{10} \\ a_{01} \end{pmatrix}, a_{11}\} \tag{7.3}$$

3 qubits:

$$\{A_{[0]}, A_{[1]}, A^{[1]}, A^{[0]}\} = \{a_{000}, \begin{pmatrix} a_{100} \\ a_{010} \\ a_{001} \end{pmatrix}, \begin{pmatrix} a_{011} \\ a_{101} \\ a_{110} \end{pmatrix}, a_{111}\} \tag{7.4}$$

4 qubits:

$$\begin{gathered}\{A_{[0]}, A_{[1]}, A_{[2]}, A^{[1]}, A^{[0]}\} = \\ \{a_{0000}, \begin{pmatrix} a_{1000} \\ a_{0100} \\ a_{0010} \\ a_{0001} \end{pmatrix}, \begin{pmatrix} 0 & a_{1100} & a_{1010} & a_{1001} \\ a_{1100} & 0 & a_{0110} & a_{0101} \\ a_{1010} & a_{0110} & 0 & a_{0011} \\ a_{1001} & a_{0101} & a_{0011} & 0 \end{pmatrix}, \begin{pmatrix} a_{0111} \\ a_{1011} \\ a_{1101} \\ a_{1110} \end{pmatrix}, a_{1111}\}\end{gathered} \tag{7.5}$$

Comparing with section A.2.2 we see that this formalism attempts to generalise the successful FTS approach.

SLOCC transformations To enable conversion of representative states to simpler forms we develop SLOCC transformations adapted to the permutation insensitive formalism. The $\mathfrak{sl}_2$ generators are

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{7.6}$$

A subset of the SLOCC generating transformations will then be given by the maps $\phi(C_{[1]})$ and it's dual under $p \to n-p$, $\psi(D^{[1]})$:

$$\begin{aligned} \phi(C_{[1]}) : A^{[p]} &\mapsto \sum_{k=p}^{n} C_{[1]}^{k-p} A^{[k]}, \\ \psi(D^{[1]}) : A_{[p]} &\mapsto \sum_{k=p}^{n} {D^{[1]}}^{k-p} A_{[k]}. \end{aligned} \tag{7.7a}$$

Taking care to treat the contractions correctly, one can rewrite these in the dual form

$$\begin{aligned} \phi(C_{[1]}) : A_{[p]} &\mapsto \sum_{k=n-p}^{n} d_{[n]}[C_{[1]}^{k+p-n} A^{[k]}] = \sum_{k=0}^{p} d_{[n]}[d^{[n]} C_{[1]}^{p-k} A_{[k]}], \\ \psi(D^{[1]}) : A^{[p]} &\mapsto \sum_{k=n-p}^{n} d^{[n]}[{D^{[1]}}^{k+p-n} A_{[k]}] = \sum_{k=0}^{p} d^{[n]}[d_{[n]} {D^{[1]}}^{p-k} A^{[k]}]. \end{aligned} \tag{7.7b}$$

Finally, we can rewrite (7.7) entirely in terms of the minimal tensors $A_{[p]}$ and $A^{[p]}$ with $p \leq n/2$:

$$\begin{aligned} \phi(C_{[1]}) &: \begin{cases} A_{[p]} \mapsto \sum_{k=0}^{p} d_{[n]}[d^{[n]} C_{[1]}^{p-k} A_{[k]}] \\ A^{[p]} \mapsto \sum_{k=0}^{\lfloor n/2 \rfloor} d^{[n]} C_{[1]}^{n-p-k} A_{[k]} + \sum_{k=p}^{\lceil n/2 \rceil -1} C_{[1]}^{k-p} A^{[k]} \end{cases}, \\ \psi(D^{[1]}) &: \begin{cases} A_{[p]} \mapsto \sum_{k=0}^{\lfloor n/2 \rfloor -1} d_{[n]} {D^{[1]}}^{n-p-k} A^{[k]} + \sum_{k=p}^{\lceil n/2 \rceil} {D^{[1]}}^{k-p} A_{[k]} \\ A^{[p]} \mapsto \sum_{k=0}^{p} d^{[n]}[d_{[n]} {D^{[1]}}^{p-k} A^{[k]}] \end{cases}. \end{aligned} \tag{7.8}$$

Let's examine what that means for the case $n = 3$. The tensors in this case are $\{A_{[0]}, A_{[1]}, A_{[2]}, A_{[3]}\}$, or in minimal form $\{A_{[0]}, A_{[1]}, A^{[1]}, A^{[0]}\}$. The explicit transformations on this set are, after simplification:

$$\begin{aligned}\phi(C_{[1]}) &: \begin{pmatrix}A_{[0]}\\A_{[1]}\\A^{[1]}\\A^{[0]}\end{pmatrix} \mapsto \begin{pmatrix}A_{[0]} & & & & & & \\ C_{[1]}A_{[0]} & + & A_{[1]} & & & & \\ d^{[3]}C_{[1]}^2A_{[0]} & + & d^{[3]}C_{[1]}A_{[1]} & + & A^{[1]} & & \\ d^{[3]}C_{[1]}^3A_{[0]} & + & d^{[3]}C_{[1]}^2A_{[1]} & + & C_{[1]}A^{[1]} & + & A^{[0]}\end{pmatrix}\\ \psi(D^{[1]}) &: \begin{pmatrix}A_{[0]}\\A_{[1]}\\A^{[1]}\\A^{[0]}\end{pmatrix} \mapsto \begin{pmatrix}d_{[3]}D^{[1]^3}A^{[0]} & + & d_{[3]}D^{[1]^2}A^{[1]} & + & D^{[1]}A_{[1]} & + & A_{[0]}\\ d_{[3]}D^{[1]^2}A^{[0]} & + & d_{[3]}D^{[1]}A^{[1]} & + & A_{[1]} & & \\ D^{[1]}A^{[0]} & + & A^{[1]} & & & & \\ A^{[0]} & & & & & & \end{pmatrix}.\end{aligned} \tag{7.9}$$

Note that there is no ambiguity in the d contractions. Under some obvious relabellings these reduce to the familiar expressions (A.28) of section A.2.1:

$$\begin{aligned}\phi(C) &: \begin{pmatrix}\beta\\A\\B\\\alpha\end{pmatrix} \mapsto \begin{pmatrix}\beta & & & & & & \\ C\beta & + & A & & & & \\ C^{\#}\beta & + & C\times A & + & B & & \\ N(C)\beta & + & (C^{\#},A) & + & (C,B) & + & \alpha\end{pmatrix}\\ \psi(D) &: \begin{pmatrix}\beta\\A\\B\\\alpha\end{pmatrix} \mapsto \begin{pmatrix}N(D)\alpha & + & (D^{\#},B) & + & (D,A) & + & \beta\\ D^{\#}\alpha & + & D\times B & + & A & & \\ D\alpha & + & B & & & & \\ \alpha & & & & & & \end{pmatrix},\end{aligned} \tag{7.10}$$

though other relabellings can achieve the same result. Let us also consider $n = 1$, $n = 2$ and $n = 4$:

$$\begin{aligned}\phi(C_{[1]}) &: \begin{pmatrix}A_{[0]}\\A^{[0]}\end{pmatrix} \mapsto \begin{pmatrix}A_{[0]} & & \\ d^{[1]}C_{[1]}A_{[0]} & + & A^{[0]}\end{pmatrix}\\ \psi(D^{[1]}) &: \begin{pmatrix}A_{[0]}\\A^{[0]}\end{pmatrix} \mapsto \begin{pmatrix}d_{[1]}D^{[1]}A^{[0]} & + & A_{[0]}\\ A^{[0]} & & \end{pmatrix}.\end{aligned} \tag{7.11}$$

$$
\begin{aligned}
\phi(C_{[1]}) : \begin{pmatrix} A_{[0]} \\ A_{[1]} \\ A^{[0]} \end{pmatrix} &\mapsto \begin{pmatrix} A_{[0]} & & & & \\ C_{[1]} A_{[0]} & + & A_{[1]} & & \\ d^{[2]} C_{[1]}^2 A_{[0]} & + & d^{[2]} C_{[1]} A_{[1]} & + & A^{[0]} \end{pmatrix} \\
\psi(D^{[1]}) : \begin{pmatrix} A_{[0]} \\ A_{[1]} \\ A^{[0]} \end{pmatrix} &\mapsto \begin{pmatrix} d_{[2]} {D^{[1]}}^2 A^{[0]} & + & D^{[1]} A_{[1]} & + & A_{[0]} \\ d_{[2]} D^{[1]} A^{[0]} & + & A_{[1]} & & \\ A^{[0]} & & & & \end{pmatrix}.
\end{aligned}
\tag{7.12}
$$

$$
\begin{aligned}
\phi(C_{[1]}) : \begin{pmatrix} A_{[0]} \\ A_{[1]} \\ A_{[2]} \\ A^{[1]} \\ A^{[0]} \end{pmatrix} &\mapsto \begin{pmatrix} A_{[0]} & & & & & & & & \\ C_{[1]} A_{[0]} & + & A_{[1]} & & & & & & \\ d_{[4]}[d^{[4]} C_{[1]}^2] A_{[0]} & + & d_{[4]}[d^{[4]} C_{[1]} A_{[1]}] & + & A_{[2]} & & & & \\ d^{[4]} C_{[1]}^3 A_{[0]} & + & d^{[4]} C_{[1]}^2 A_{[1]} & + & d^{[4]} C_{[1]} A_{[2]} & + & A^{[1]} & & \\ d^{[4]} C_{[1]}^4 A_{[0]} & + & d^{[4]} C_{[1]}^3 A_{[1]} & + & d^{[4]} C_{[1]}^2 A_{[2]} & + & C_{[1]} A^{[1]} & + & A^{[0]} \end{pmatrix} \\
\psi(D^{[1]}) : \begin{pmatrix} A_{[0]} \\ A_{[1]} \\ A_{[2]} \\ A^{[1]} \\ A^{[0]} \end{pmatrix} &\mapsto \begin{pmatrix} d_{[4]} {D^{[1]}}^4 A^{[0]} & + & d_{[4]} {D^{[1]}}^3 A^{[1]} & + & {D^{[1]}}^2 A_{[2]} & + & D^{[1]} A_{[1]} & + & A_{[0]} \\ d_{[4]} {D^{[1]}}^3 A^{[0]} & + & d_{[4]} {D^{[1]}}^2 A^{[1]} & + & D^{[1]} A_{[2]} & + & A_{[1]} & & \\ d_{[4]} {D^{[1]}}^2 A^{[0]} & + & d_{[4]} D^{[1]} A^{[1]} & + & A_{[2]} & & & & \\ D^{[1]} A^{[0]} & + & A^{[1]} & & & & & & \\ A^{[0]} & & & & & & & & \end{pmatrix}.
\end{aligned}
\tag{7.13}
$$

The remaining SLOCC transformations are generated by the scaling transformation $T(\lambda_{[1]})$:

$$
T(\lambda_{[1]}) : A_{i_1 \cdots i_p} \mapsto \prod_{k=1}^{n} \exp\left[\lambda_{n+1-k} \prod_{m=1}^{p} (-)^{\delta_{k,i_m}}\right] A_{i_1 \cdots i_p}. \tag{7.14}
$$

Generalised Jordan ranks We generalise the notion of Jordan rank in section A.1.2 by identifying the contractions $d^{[n]} C_{[1]}^k$, $k \in \{1, \ldots, n\}$ as the relevant rank tensors for a degree n "Jordan" algebra. The $n+1$ ranks are then given by

$$
\operatorname{rank} C_{[1]} \leq k-1 \Leftrightarrow d^{[n]} C_{[1]}^k = 0, \quad k \in \{1, \ldots, n\}, \tag{7.15}
$$

so that in particular

$$
\begin{aligned}
\operatorname{rank} C_{[1]} = 0 &\Leftrightarrow \quad C_{[1]} = 0, \\
\operatorname{rank} C_{[1]} = n &\Leftrightarrow \quad d^{[n]} C_{[1]}^n \neq 0.
\end{aligned}
\tag{7.16}
$$

For the familiar case of $n = 3$ this means

$$
\begin{aligned}
\operatorname{rank} C_{[1]} = 0 &\Leftrightarrow \quad C_{[1]} = 0, \\
\operatorname{rank} C_{[1]} \leq 1 &\Leftrightarrow \quad d^{[3]} C_{[1]}^2 = C_{[1]}^{\#} = 0, \\
\operatorname{rank} C_{[1]} \leq 2 &\Leftrightarrow \quad d^{[3]} C_{[1]}^3 = N(C_{[1]}) = 0, \\
\operatorname{rank} C_{[1]} = 3 &\Leftrightarrow \quad d^{[3]} C_{[1]}^3 = N(C_{[1]}) \neq 0.
\end{aligned}
\tag{7.17}
$$

Canonical forms We may attempt to use the SLOCC transformations (7.7) to convert states to canonical forms with fewer variables. For a generic n-qubit state, the best we have been able to do so far is a reduction by $n+1$ variables, short of the more sweeping results of Krutelevich [169, 170]. Start by assuming the state is not null and proceed as follows:

1. Ensure that $A^{[0]} \neq 0$ using a $\phi(C_{[1]})$ transformation with $C_{[1]}$ of sufficiently high rank to involve the first nonzero A tensor:

$$\phi(C_{[1]}) : A^{[0]} \mapsto \sum_{k=0}^{\lfloor n/2 \rfloor} d^{[n]} C_{[1]}^{n-k} A_{[k]} + \sum_{k=0}^{\lceil n/2 \rceil - 1} C_{[1]}^{k} A^{[k]}. \tag{7.18}$$

   Provided the rank tensors are non-degenerate this will work whenever the state isn't null.

2. Scale $A^{[0]}$ to 1 using a T transformation.

3. The state is now in the form $\{A_{[0]}, A_{[1]}, \ldots, A^{[1]}, 1^{[0]}\}$. If $A^{[1]} = 0$ we are finished, else perform a $\psi(-A^{[1]})$ transformation to yield the form $\{A_{[0]}, A_{[1]}, \ldots, A^{[2]}, 0^{[1]}, 1^{[0]}\}$.

For our four qubit case this means representative states possess no more that 11 kets, rather than a potential full 16.

Covariant classification The full covariant lattice up to degree six is shown in Figure 7.2, however, the large number of shared descendants at higher degrees prevents one from inferring the individual contributions of a given covariant to its descendants and one must keep track while generating them. The situation is ameliorated by listing the adjacency matrix in Table 7.2. While this affords us a full understanding of the relationships between covariants of two adjacent degrees, if one wants to tease out the detailed relationship across multiple degrees a more fine-grained description is required. Such a fine-graining within multiweight classes is possible with our present methods, but remains to be attempted.

The next step is to scan through these covariants with the reduced permutation invariant state $\{A_{[0]}, A_{[1]}, A^{[2]}, 0^{[1]}, 1^{[0]}\}$, switching off amplitudes in all possible ways modulo permutations. When this is done it is possible to eliminate redundant covariants, though there is some ambiguity surrounding the particular choice of retained covariants. In Table 7.3 we make the particular choice of $4F_{4000}$ and $4F_{0000}$ when we could just as well have well picked any two out of $68E_{3111}$, $10E_{1111}$, $76F_{4222}$, $300F_{4220}$, $93F_{2222}$, $160F_{2220}$, and $114F_{2200}$. We have filled in the degenerate classes and four-qubit GHZ

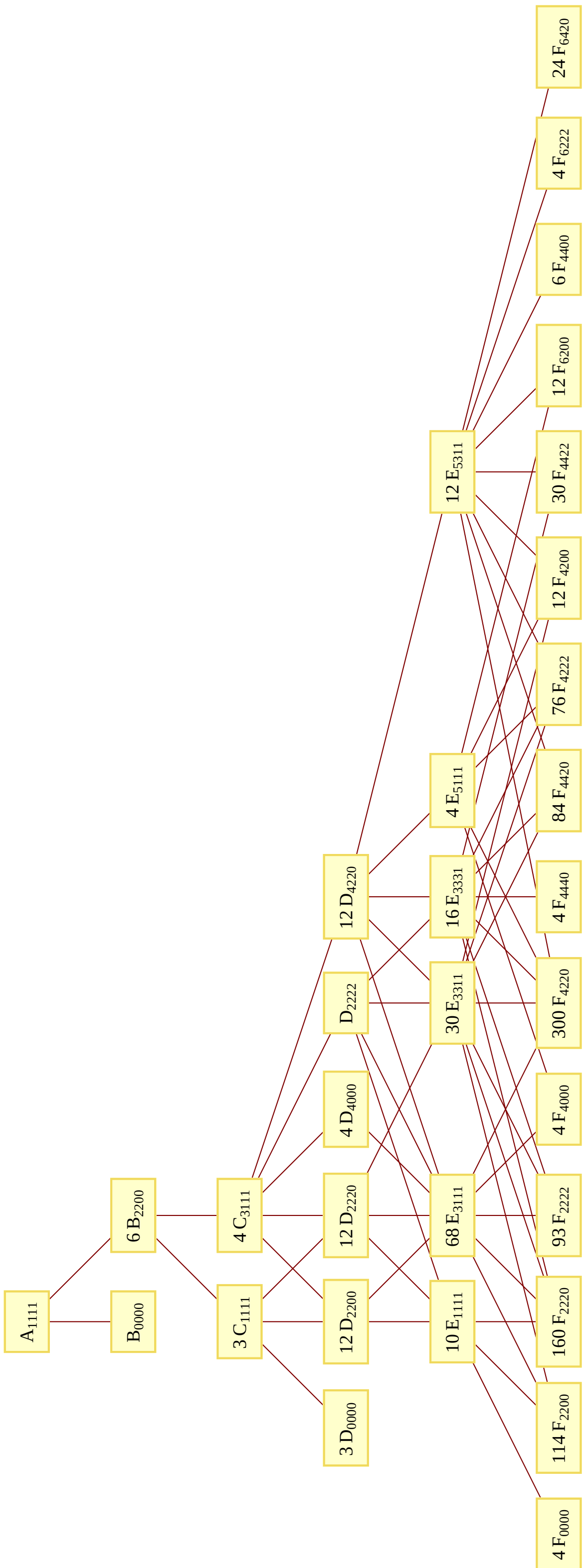


Figure 7.2: Four-qubit covariant lattice down to degree 6. The contributions of a given covariant to its descendants can be read off Table 7.2.

class unambiguously, and have indicated the families we believe correspond to a couple of the remaining classes. One does after all expect that each class corresponds to a single family, but not vice versa. This remains work in progress, and we have only been able to scan through covariants up to degree six, whereas in principle we would have to reach degree 12. Nevertheless our results so far include the primary invariants at degrees two, four, and six in terms of which all higher degree invariants can be written.

7.2 31 entanglement families

While we retain some hopes for our covariant based classification, it is with the normal forms of the family perspective that we have met with unambiguous success. We now describe this approach.

Black hole side The string theoretic framework we need to consider is that of timelike dimensional reduction of 4D supergravity theories to 3D. In four dimensions the moduli parameterise a symmetric space of the form $M_4 = G_4/H_4$, with global U-duality group G_4 and maximal compact subgroup H_4. Post-reduction the moduli space becomes a pseudo-Riemannian symmetric space $M_3^* = G_3/H_3^*$, with 3D U-duality group G_3 and where H_3^* is a non-compact form of the maximal compact subgroup H_3. One finds that geodesic motion on M_3^* corresponds to stationary solutions of the $D = 4$ theory [20, 171–175]. These geodesics are parameterised by the Lie algebra valued matrix of Noether charges Q and the problem of classifying the spherically symmetric extremal (resp. non-extremal) black hole solutions consists of classifying the nilpotent (resp. semisimple) orbits of Q (Nilpotent means $Q^n = 0$ for some sufficiently large n.)

As one might expect, the case we want to consider is the STU model and the nilpotent classification of STU black holes. Through a procedure called scalar-dressing, the eight charges of the model can be grouped into the $\mathcal{N} = 2$ central charge z and three "matter charges" z_a ($a = 1, 2, 3$), which exhibit a triality (corresponding to permutation of three of the qubits). The black holes are divided as shown in Figure 7.3. The small extremal black holes are termed lightlike, critical, or doubly critical according to whether the minimal number of representative electric or magnetic charges is 3, 2 or 1. The lightlike case is split into one 1/2-BPS solution, where the charges satisfy $z_1 = 0, |z|^2 = 4|z_2|^2 = 4|z_3|^2$ and three non-BPS solutions, where the central charges satisfy $z = 0, |z_1|^2 = 4|z_2|^2 = 4|z_3|^2$ or $z_2 = 0, |z_3|^2 = 4|z_1|^2 = 4|z|^2$ or $z_3 = 0, |z_2|^2 = 4|z_1|^2 = 4|z|^2$. The critical case splits into three 1/2-BPS solutions with $z = z_a \neq 0, z_b = z_c = 0$ and three non-BPS cases with $z = z_a = 0, z_b = z_c \neq 0$, where $a \neq b \neq c$. The doubly critical

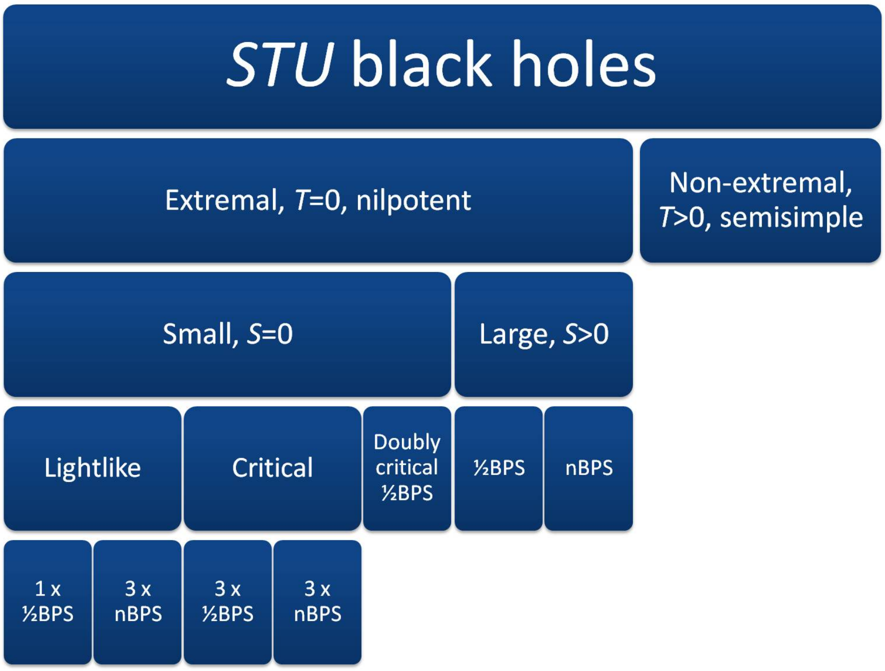


Figure 7.3: Nilpotent classification of *STU* Black holes. They are divided into extremal or non-extremal according to whether the temperature is zero or not, with the orbits being nilpotent or semisimple, respectively. Depending on the values of the charges, the extremal black holes are further divided into small or large according to whether the entropy is zero or not, with still further subdivisions described in the text.

case is always 1/2-BPS with $|z|^2 = |z_1|^2 = |z_2|^2 = |z_3|^2$ and vanishing sum of the z_a phases. The large black holes may also be 1/2-BPS or non-BPS. One subtlety is that some extremal cases, termed "extremal", cannot be obtained as limits of non-extremal black holes.

Correspondence The post reduction moduli space G_3/H_3^* for the STU model is $\mathrm{SO}_{4,4}\,/\,\mathrm{SL}_2(\mathbb{R})^4$, which yields the Lie algebra decomposition

$$\mathfrak{so}_{4,4} \cong \mathfrak{sl}_2(\mathbb{R})^4 \oplus (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}). \tag{7.19}$$

We had already pointed to the significance of (7.19) for four qubits in [23] but this was recently elucidated by Levay in [20] where he related four qubits to $D = 4$ STU black holes.

Critically, the Kostant-Sekiguchi correspondence [176] then implies that the nilpotent orbits of $\mathrm{SO}_{4,4}$ acting on the adjoint representation $\mathbf{28}$ are in one-to-one correspondence with the nilpotent orbits of $\mathrm{SL}_2(\mathbb{C})^4$ acting on the fundamental representation $(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2})$ and hence with the classification of four-qubit entanglement. In particular it is the complex qubits that appear, relaxing the restriction to rebits that featured in our earlier considerations.

More specifically, the nilpotent orbits required by the Kostant-Sekiguchi theorem are those of $\mathrm{SO}^0_{4,4}$, where the 0 superscript denotes the identity component. These orbits may be labelled by "signed" Young tableaux, often referred to as *ab*-diagrams in the mathematics literature. See [177] and the references therein. Each signed Young tableau actually corresponds to a single nilpotent $\mathrm{O}_{4,4}$ orbit of which the $\mathrm{SO}^0_{4,4}$ nilpotent orbits are the connected components. Since $\mathrm{O}_{4,4}$ has four components, for each nilpotent $\mathrm{O}_{4,4}$ orbit there may be either 1, 2 or 4 nilpotent $\mathrm{SO}^0_{4,4}$ orbits. This number can also be determined by the corresponding signed Young tableau. If there are 2 orbits one labels the diagram to its left (right) with a I or a II. If there are 4 orbits one labels the diagram to both its left and right with a I or a II. If it is none of these it is said to be stable and there is only one orbit. We summarise the partition structure of the signed Young tableaux, but retain their labellings. There are a total of 31 nilpotent $\mathrm{SO}^0_{4,4}$ orbits, which are summarised in Figure 7.4. We also supply the complete list of the associated cosets in Table 7.4, some of which may be found in [174].

Qubit side Our main result, as summarised in Table 7.4, is that via the Kostant-Sekiguchi theorem we find 31 nilpotent orbits for the SLOCC-equivalence group acting

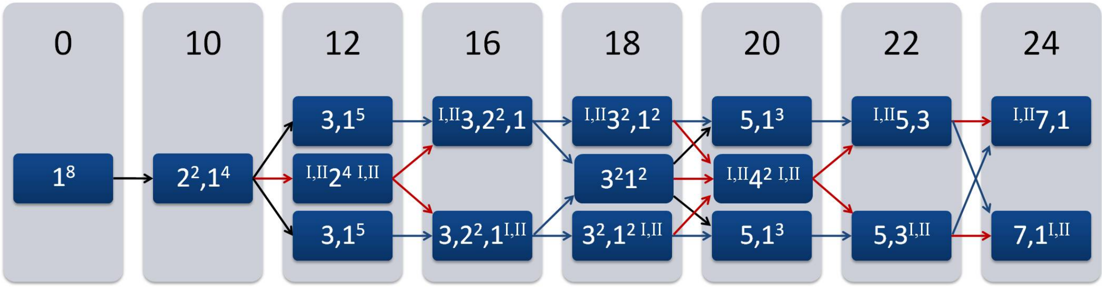


Figure 7.4: $SO_{4,4}$ Hasse diagram. Numbers along the top are dimensions of the real orbit, integers inside the boxes indicate the structure of the appropriate Young tableau, arrows indicate their closure ordering defining a partial order [177], blue arrows count as double and red arrows count as quadruple.

on the representation space of four qubits. For each nilpotent orbit there is precisely one family of SLOCC orbits since each family contains one nilpotent orbit on setting all invariants to zero. The nilpotent orbits and their associated families are summarised in Table 7.4, which is split into upper and lower sections according to whether the nilpotent orbits belong to parameter-dependent or parameter-independent families.

If one allows for the permutation of the four qubits the connected components of each $O_{4,4}$ orbit are re-identified reducing the count to 17. Moreover, these 17 are further grouped under this permutation symmetry into just nine nilpotent orbits. In other words there are 31 entanglement families which reduce to nine up to permutations of the four qubits. From Table 7.1 we see that the nine agrees with [167, 168] while the 31 is new. It is not difficult to show that the nine cosets really match the nine families, as listed in the final column of Table 7.4 (provided we adopt the version of L_{ab_3} presented in [168] rather than in [167]). For example, the state representative $L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}} = |0111\rangle + |0000\rangle$ is left invariant by the $SO_2(\mathbb{C})^2 \times \mathbb{C}$ subgroup, where $SO_2(\mathbb{C})^2$ is the stabiliser of the three-qubit GHZ state [24]. In contrast, the four-way entangled family $L_{0_{7\oplus\bar{1}}}$, which is the "principal" nilpotent orbit [176], is not left invariant by any subgroup. Note that the total of 31 does not follow trivially by permuting the qubits in these nine. Naive permutation produces far more than 31 candidates which then have to be reduced to SLOCC inequivalent families. To the best of our knowledge, this constitutes the only calculation the nine four-qubit $SL_2(\mathbb{C})^4$ cosets. The matching of the *STU* extremal classes to the nilpotent orbits is also given in Table 7.4.

These results are actually consistent with the covariant approach. For example, the permutation sensitive covariant classification has four biseparable classes A-GHZ, B-GHZ, C-GHZ and D-GHZ which are then identified as a single class under the permutation symmetry. These four classes are in fact the four nilpotent orbits corresponding to the families $L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$ in Table 7.4, which are also identified as a single nilpotent orbit under permutations. Similarly, each of the four A-W classes is a nilpotent orbit belonging to one of the four families labelled $L_{a_2 0_{3\oplus\bar{1}}}$ which are again identified under permutations. A less trivial example is given by the six A-B-EPR classes of the covariant classification. These all lie in the single family $L_{a_2b_2}$ of [167], which is defined up to permutation. Consulting Table 7.4 we see that, when not allowing permutations, this family splits into six pieces, each containing one of the six A-B-EPR classes. Finally, the single totally separable class A-B-C-D is the single nilpotent orbit inside the single family L_{abc_2} which maps into itself under permutations.

Table 7.2: The transposed adjacency matrix of the four-qubit covariant lattice up to and including all degree six covariants.

	A_{1111}	B_{0000}	$6B_{2200}$	$3C_{1111}$	$4C_{3111}$	$3D_{0000}$	$12D_{2200}$	$12D_{2220}$	D_{2222}	$4D_{4000}$	$12D_{4220}$	$10E_{1111}$	$68E_{3111}$	$30E_{3311}$	$16E_{3331}$	$4E_{5111}$	$12E_{5311}$
A_{1111}																	
B_{0000}	1																
$6B_{2200}$	6																
$3C_{1111}$			3														
$4C_{3111}$			4														
$3D_{0000}$				3													
$12D_{2200}$				6	6												
$12D_{2220}$				3	9												
D_{2222}					1												
$4D_{4000}$					4												
$12D_{4220}$					12												
$10E_{1111}$							6	3	1								
$68E_{3111}$							24	24	4	4	12						
$30E_{3311}$								12	6		12						
$16E_{3331}$									4		12						
$4E_{5111}$											4						
$12E_{5311}$											12						
$4F_{0000}$												4					
$114F_{2200}$												18	72	24			
$160F_{2220}$												12	89	52	7		
$93F_{2222}$													56	21	16		
$4F_{4000}$													3			1	
$12F_{4200}$														9		1	2
$300F_{4220}$													138	90	36	12	24
$76F_{4222}$														24	36	4	12
$6F_{4400}$																	6
$84F_{4420}$														36	36		12
$30F_{4422}$															24		6
$4F_{4440}$															4		
$12F_{6200}$																4	8
$4F_{6222}$																	4
$24F_{6420}$																	24

Table 7.3: Four qubit covariant classification up to degree six.

Class	A_{1111}	$6B_{2200}$	B_{0000}	$4C_{3111}$	$3C_{1111}$	$4D_{4000}$	D_{2222}	$12D_{2200}$	$3D_{0000}$	$4F_{4000}$	$4F_{0000}$
Null											
A-B-C-D	1										
A-B-EPR	1	1									
A-W	1	3		1							
A-GHZ	1	3		1		1					
L_{ab_3}	1	6		4			1				
?	1	6		4	3	2	1	2			
?	1	6		4	3	3	1	6			
$L_{0_{7\oplus\bar{1}}}$	1	6		4	3	4	1	12			
?	1	6		4	3	4	1	12		1	
?	1	6		4	3	4	1	12		4	4
?	1	6		4	3	4	1	12	3	4	4
GHZ	1	6	1	4	3	4	1	12	3	4	4

Table 7.4: Each black hole nilpotent $SO^0_{4,4}$ orbit corresponds to a 4-qubit nilpotent $SL_2(\mathbb{C})^4$ orbit. z_H is the horizon value of the $\mathcal{N}=2$, $D=4$ central charge.

STU black holes			$\dim_{\mathbb{R}}$	Four qubits		
description	orbit	$SO^0_{4,4}$ coset		$[SL_2(\mathbb{C})]^4$ coset	family	nilpotent rep
trivial	$[1^8]$	$\frac{SO^0_{4,4}}{SO^0_{4,4}}$	1	$\frac{[SL_2(\mathbb{C})]^4}{[SL_2(\mathbb{C})]^4}$	G_{abcd}	0
doubly-critical $\frac{1}{2}$BPS	$[2^2,1^4]$	$\frac{SO^0_{4,4}}{[SL_2(\mathbb{R})\times SO_{2,2}(\mathbb{R})]\ltimes[(\mathbf{2},\mathbf{4})^{(1)}\oplus\mathbf{1}^{(2)}]}$	10	$\frac{[SL_2(\mathbb{C})]^4}{[SO_2(\mathbb{C})]^3\ltimes\mathbb{C}^4}$	L_{abc_2}	$\|0110\rangle$
critical, $\frac{1}{2}$BPS and non-BPS	$[3,1^5]$ $[3,1^5]$ ${}^{I,II}[2^4]^{I,II}$	$\frac{SO^0_{4,4}}{SO_{3,2}(\mathbb{R})\ltimes[(\mathbf{5}\oplus\mathbf{1})^{(2)}]}$ $\frac{SO^0_{4,4}}{SO_{2,3}(\mathbb{R})\ltimes[(\mathbf{5}\oplus\mathbf{1})^{(2)}]}$ $\frac{SO^0_{4,4}}{Sp_4(\mathbb{R})\ltimes[(\mathbf{5}\oplus\mathbf{1})^{(2)}]}$	12	$\frac{[SL_2(\mathbb{C})]^4}{[SO_3(\mathbb{C})\times\mathbb{C}]\times[SO_2(\mathbb{C})\ltimes\mathbb{C}]}$	$L_{a_2b_2}$	$\|0110\rangle+\|0011\rangle$
lightlike $\frac{1}{2}$BPS and non-BPS	${}^{I,II}[3,2^2,1]$ $[3,2^2,1]^{I,II}$	$\frac{SO^0_{4,4}}{SL_2(\mathbb{R})\ltimes[(2\times\mathbf{2})^{(1)}\oplus(3\times\mathbf{1})^{(2)}\oplus\mathbf{2}^{(3)}]}$	16	$\frac{[SL_2(\mathbb{C})]^4}{[SO_2(\mathbb{C})\ltimes\mathbb{C}]\times\mathbb{C}^2}$	$L_{a_20_{3\oplus\bar{1}}}$	$\|0110\rangle+\|0101\rangle+\|0011\rangle$
large non-BPS $Z_H\neq0$	$[3^2,1^2]$	$\frac{SO^0_{4,4}}{SO_{1,1}(\mathbb{R})\times SO_{1,1}(\mathbb{R})\ltimes[((\mathbf{2},\mathbf{2})\oplus(\mathbf{3},\mathbf{1}))^{(2)}\oplus\mathbf{1}^{(4)}]}$	18	$\frac{[SL_2(\mathbb{C})]^4}{\mathbb{C}^3}$	L_{ab_3}	$\frac{i}{\sqrt{2}}(\|0001\rangle+\|0010\rangle-\|0111\rangle-\|1011\rangle)$
"extremal"	$[5,1^3]$ $[5,1^3]$ ${}^{I,II}[4^2]^{I,II}$	$\frac{SO^0_{4,4}}{SO_{2,1}(\mathbb{R})\ltimes[\mathbf{1}^{(2)}\oplus\mathbf{3}^{(4)}\oplus\mathbf{1}^{(6)}]}$ $\frac{SO^0_{4,4}}{SO_{1,2}(\mathbb{R})\ltimes[\mathbf{1}^{(2)}\oplus\mathbf{3}^{(4)}\oplus\mathbf{1}^{(6)}]}$ $\frac{SO^0_{4,4}}{Sp(2,\mathbb{R})\ltimes[\mathbf{1}^{(2)}\oplus\mathbf{3}^{(4)}\oplus\mathbf{1}^{(6)}]}$	20	$\frac{[SL_2(\mathbb{C})]^4}{SO_2(\mathbb{C})\times\mathbb{C}}$	L_{a_4}	$i\|0001\rangle+\|0110\rangle-i\|1011\rangle$
large $\frac{1}{2}$BPS and non-BPS $Z_H=0$	${}^{I,II}[3^2,1^2]$ $[3^2,1^2]^{I,II}$	$\frac{SO^0_{4,4}}{SO_2(\mathbb{R})\times SO_2(\mathbb{R})\ltimes[((\mathbf{2},\mathbf{2})\oplus(\mathbf{3},\mathbf{1}))^{(2)}\oplus\mathbf{1}^{(4)}]}$	18	$\frac{[SL_2(\mathbb{C})]^4}{SO_2(\mathbb{C})^2\times\mathbb{C}}$	$L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$	$\|0000\rangle+\|0111\rangle$
"extremal"	${}^{I,II}[5,3]$ $[5,3]^{I,II}$	$\frac{SO^0_{4,4}}{\mathbb{R}^{3(2)}\oplus\mathbb{R}^{1(4)}\oplus\mathbb{R}^{2(6)}}$	22	$\frac{[SL_2(\mathbb{C})]^4}{\mathbb{C}}$	$L_{0_{5\oplus\bar{3}}}$	$\|0000\rangle+\|0101\rangle+\|1000\rangle+\|1110\rangle$
"extremal"	${}^{I,II}[7,1]$ $[7,1]^{I,II}$	$\frac{SO^0_{4,4}}{\mathbb{R}^{(2)}\oplus\mathbb{R}^{2(6)}\oplus\mathbb{R}^{(10)}}$	24	$\frac{[SL_2(\mathbb{C})]^4}{\mathrm{id}}$	$L_{0_{7\oplus\bar{1}}}$	$\|0000\rangle+\|1011\rangle+\|1101\rangle+\|1110\rangle$

Chapter 8

Superqubits and superentanglement

Upon encountering the construction of a supersymmetric version of Cayley's hyperdeterminant in [178], we immediately regarded it as a candidate "superentanglement" measure for a hypothetical super analogue of a three qubit system. We were inspired to generalise the qubit to the *superqubit* by adding fermionic degrees of freedom, taking the first steps towards generalising quantum information theory to super quantum information theory. So, although the topic of superqubits grew organically out of the black hole/qubit correspondence, strictly speaking it is logically independent of it. We discuss it as an interesting parallel development.

In order to accommodate a super analogue of entanglement from the outset it was necessary to appropriately generalise the (S)LOCC group. We thus promoted the SLOCC equivalence group $\mathrm{SL}_2(\mathbb{C})^n$ to its minimal supersymmetric extension, the orthosymplectic supergroup $\mathrm{OSp}_{2|1}^n$ [179, 180], concurrently promoting the LOCC equivalence group SU_2^n to the supergroup $\mathrm{uOSp}_{2|1}^n$.

In ordinary quantum information theory, n-qubit states lie in the fundamental representation of the SLOCC equivalence group, so a single superqubit was constructed as a 3-dimensional representation of $\mathrm{OSp}_{2|1}$ consisting of two commuting "bosonic" components and one anticommuting "fermionic" component:

$$|\Psi\rangle = a_0|0\rangle + a_1|1\rangle + a_\bullet|\bullet\rangle, \tag{8.1}$$

where the bullet $\bullet$ here denotes the new fermionic component, in contrast to it's use in other chapters. This forced us to consider a super Hilbert space defined over the

Grassmann numbers. We label the Grassmann algebra as Λ and the even (resp. odd) sub-algebra of commutative (resp. anti-commutative) elements as Λ_0 (resp. Λ_1) where $a_0, a_1 \in \Lambda_0$ and $a_\bullet \in \Lambda_1$. It was then possible to introduce the appropriate supersymmetric generalisations of the conventional entanglement measures for the cases of $n = 2$ and $n = 3$: the *superdeterminant* (distinct from the Berezinian) and *superhyperdeterminant* respectively. In particular, super-Bell and super-GHZ states were characterised by the nonvanishing of these invariants.

We review the superqubit concept in two parts: 1) the promotion of the entanglement equivalence groups to their minimal supersymmetric extensions (accompanied by the necessary supporting material) followed by an examination of the transformation properties of one, two, and three superqubits under these groups. 2) The derivation of the superinvariants that are the obvious candidates for supersymmetric entanglement measures for two and three superqubits.

8.1 Super analogues of qubits and entanglement

8.1.1 Superlinear algebra

Grassmann numbers are the 2^n-dimensional vectors populating the Grassmann algebra Λ_n, which is generated by n mutually anticommuting elements $\{\theta^i\}_{i=1}^n$.

Any Grassmann number z may be decomposed into "body" $z_\mathcal{B} \in \mathbb{C}$ and "soul" $z_\mathcal{S}$ viz.

$$\begin{aligned} z &= z_\mathcal{B} + z_\mathcal{S} \\ z_\mathcal{S} &= \textstyle\sum_{k=1}^{\infty} \frac{1}{k!} c_{a_1\cdots a_k} \theta^{a_1} \cdots \theta^{a_k}, \end{aligned} \tag{8.2}$$

where $c_{a_1\cdots a_k} \in \mathbb{C}$ are totally antisymmetric. For finite dimension n the sum terminates at $k = 2^n$ and the soul is nilpotent $z_\mathcal{S}^{n+1} = 0$.

One may also decompose z into even and odd parts u and v

$$\begin{aligned} u &= z_\mathcal{B} + \textstyle\sum_{k=1}^{\infty} \frac{1}{(2k)!} c_{a_1\cdots a_{2k}} \theta^{a_1} \cdots \theta^{a_{2k}} \\ v &= \textstyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} c_{a_1\cdots a_{2k+1}} \theta^{a_1} \cdots \theta^{a_{2k+1}}, \end{aligned} \tag{8.3}$$

which may also be expressed as the direct sum decomposition $\Lambda_n = \Lambda_n^0 \oplus \Lambda_n^1$. Furthermore, analytic functions f of Grassmann numbers are defined via

$$f(z) := \sum_{k=0}^{\infty} \tfrac{1}{k!} f^{(k)}(z_{\mathcal{B}}) z_{\mathcal{S}}^k, \tag{8.4}$$

where $f^{(k)}(z_{\mathcal{B}})$ is the k^{th} derivative of f evaluated at $z_{\mathcal{B}}$ and is well defined if f is nonsingular at $z_{\mathcal{B}}$ [181].

One defines the *grade of a Grassmann number* as

$$\deg x := \begin{cases} 0 & x \in \Lambda_n^0 \\ 1 & x \in \Lambda_n^1, \end{cases} \tag{8.5}$$

where the grades 0 and 1 are referred to as even and odd, respectively.

Define the star * and superstar $^\#$ operators [180, 182, 183] satisfying the following properties:

$$\begin{gathered} (\Lambda_n^0)^\star = \Lambda_n^0, \quad (\Lambda_n^1)^\star = \Lambda_n^1, \\ (\Lambda_n^0)^\# = \Lambda_n^0, \quad (\Lambda_n^1)^\# = \Lambda_n^1, \\ (x\theta_i)^\star = x^*\theta_i^\star, \qquad \theta_i^{\star\star} = \theta_i, \qquad (\theta_i\theta_j)^\star = \theta_j^\star\theta_i^\star, \\ (x\theta_i)^\# = x^*\theta_i^\#, \quad \theta_i^{\#\#} = -\theta_i, \quad (\theta_i\theta_j)^\# = \theta_i^\#\theta_j^\#, \end{gathered} \tag{8.6}$$

where $x \in \mathbb{C}$ and * is ordinary complex conjugation, which means

$$\alpha^{\star\star} = \alpha, \qquad\qquad \alpha^{\#\#} = (-)^{\deg\alpha}\alpha \tag{8.7}$$

for pure even/odd Grassmann α. The impure case follows by linearity.

Following [181] one may, if so desired, take the formal limit $n \to \infty$ defining the infinite dimensional vector space Λ_∞. Elements of Λ_∞ are called *supernumbers*. Our results are independent of the dimension of the underlying Grassmann algebra and one can use supernumbers throughout, but for the sake of simplicity we restrict to finite dimensional algebra by assigning just one Grassmann generator θ and its superconjugate $\theta^\#$ to every superqubit.

The grade definition applies to the components $T_{X_1\cdots X_k}$ of any k-index array of Grassmann numbers T, but one may also define $\deg X_i$, the *grade of an index*, for such an array by specifying a characteristic function from the range of the index X_i to the set $\{0, 1\}$. In general the indices can have different ranges and the characteristic functions can be arbitrary for each index. It is then possible to define $\deg T$, the *grade of an array*, as long as the compatibility condition

$$\deg T \equiv \deg(T_{X_1 \cdots X_k}) + \sum_{i=1}^{k} \deg X_i \mod 2 \quad \forall\ X_i \tag{8.8}$$

is satisfied. In precisely such cases the entries of T satisfy

$$\begin{aligned} \deg(T_{X_1 \cdots X_k}) &= \deg T + \sum_{i=1}^{k} \deg X_i \mod 2, \\ \implies \deg T &= \deg(T_{\underbrace{1 \ldots 1}_{k}}), \\ \deg(T_1 T_2) &= \deg T_1 + \deg T_2 \mod 2, \end{aligned} \tag{8.9}$$

so that in other words T is partitioned into blocks with definite grade such that the nearest neighbours of any block are of the opposite grade to that block. The array grade simply distinguishes the two distinct ways of accomplishing such a partition (i.e. the two possible grades of the first element $T_{1 \cdots 1}$). Grassmann numbers and the Grassmann number grade may be viewed as special cases of arrays and the array grade.

Special care must be taken not to confuse this notion of array grade with whether the array entries at even/odd index positions vanish. An array T may be decomposed as

$$T = T_E + T_O, \tag{8.10}$$

where the pure even part T_E is obtained from T by setting to zero all entries satisfying $\deg(T_{X_1 \cdots X_k}) = 1$, and similarly *mutatis mutandis* for T_O. The property of being pure even or pure odd is therefore independent of the array grade as defined above.

The various grades commonly appear in formulae as powers of -1 and the shorthand

$$(-)^X := (-1)^{\deg X} \tag{8.11}$$

is often used. The indices of superarrays may be supersymmetrised as follows:

$$\begin{aligned} &T_{X_1 \cdots [[X_i | \cdots | X_j]] \cdots X_k} := \\ &\tfrac{1}{2}[T_{X_1 \cdots X_i \cdots X_j \cdots X_k} + (-)^{X_i X_j} T_{X_1 \cdots X_j \cdots X_i \cdots X_k}]. \end{aligned} \tag{8.12}$$

While we require these definitions for some of our considerations, one typically only uses arrays with 0, 1, or 2 indices where the characteristic functions are monotonic: supernumbers, supervectors, and supermatrices, respectively. Functions of grades extend to mixed superarrays (with nonzero even *and* odd parts) by linearity.

A $(p|q) \times (r|s)$ supermatrix is just an $(p+q) \times (r+s)$-dimensional block partitioned matrix

$$M = \begin{array}{c|c} & r \quad | \quad s \\ \begin{array}{c} p \\ q \end{array} & \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \end{array} \tag{8.13}$$

where entries in the A and D blocks are grade $\deg M$, and those in the B and C blocks are grade $\deg M + 1 \mod 2$. The special cases $s = 0$ or $q = 0$ can be permitted to make the definition encapsulate row and column supervectors. Supermatrix multiplication is defined as for ordinary matrices; however, the trace, transpose, adjoint, and determinant have distinct super versions [180, 184].

The supertrace $\operatorname{str} M$ of a supermatrix is M defined as

$$\operatorname{str} M := \sum_X (-)^{(X+M)X} M_{XX} \tag{8.14}$$

and is linear, cyclic modulo sign, and insensitive to the supertranspose

$$\begin{aligned} \operatorname{str}(M+N) &= \operatorname{str}(M) + \operatorname{str}(N) \\ \operatorname{str}(MN) &= (-)^{MN} \operatorname{str}(NM) \\ \operatorname{str} M^{st} &= \operatorname{str} M. \end{aligned} \tag{8.15}$$

The supertranspose M^{st} of a supermatrix M is defined componentwise as

$$M^{st}{}_{X_1X_2} := (-)^{(X_2+M)(X_1+X_2)} M_{X_2X_1}. \tag{8.16}$$

Unlike the transpose the supertranspose is not idempotent; instead,

$$\begin{aligned} M^{st\,st}{}_{X_1X_2} &= (-)^{(X_1+X_2)} M_{X_1X_2}, \\ M^{st\,st\,st}{}_{X_1X_2} &= (-)^{(X_1+M)(X_1+X_2)} M_{X_2X_1}, \\ M^{st\,st\,st\,st}{}_{X_1X_2} &= M_{X_1X_2}, \end{aligned} \tag{8.17}$$

so that it is of order 4. The supertranspose also satisfies

$$(MN)^{st} = (-)^{MN} N^{st} M^{st}. \tag{8.18}$$

The adjoint † and superadjoint ‡ of a supermatrix are defined as

$$\begin{aligned} M^\dagger &:= M^{\star t} \\ M^\ddagger &:= M^{\# st}, \end{aligned} \tag{8.19}$$

and satisfy

$$\begin{aligned} M^{\dagger\dagger} &= M, & M^{\ddagger\ddagger} &= (-)^M M, \\ (MN)^\dagger &= N^\dagger M^\dagger, & (MN)^\ddagger &= (-)^{MN} N^\ddagger M^\ddagger. \end{aligned} \tag{8.20}$$

The preservation of anti-super-Hermiticity, $M^\ddagger = -M$, under scalar multiplication by Grassmann numbers, as required for the proper definition of $\mathfrak{uosp}(1|2)$ [185], necessitates the left/right multiplication rules:

$$\begin{aligned} (\alpha M)_{X_1X_2} &= (-)^{X_1\alpha}\alpha M_{X_1X_2}, \\ (M\alpha)_{X_1X_2} &= (-)^{X_2\alpha} M_{X_1X_2}\alpha. \end{aligned} \tag{8.21}$$

The Berezinian is defined as

$$\begin{aligned} \operatorname{Ber} M &:= \det(A - BD^{-1}C)/\det(D) \\ &= \det(A)/\det(D - CA^{-1}B) \end{aligned} \tag{8.22}$$

and is multiplicative, insensitive to the supertranspose, and generalises the relationship between trace and determinant

$$\begin{aligned} \operatorname{Ber}(MN) &= \operatorname{Ber}(M)\operatorname{Ber}(N) \\ \operatorname{Ber} M^{st} &= \operatorname{Ber} M \\ \operatorname{Ber} e^M &= e^{\operatorname{str} M}. \end{aligned} \tag{8.23}$$

The direct sum and super tensor product are unchanged from their ordinary versions. As such, the dimension of the tensor product of two superqubits is given by

$$(2|1) \otimes (2|1) = (2|1|2|3|1), \tag{8.24}$$

while the threefold product is

$$(2|1)^3 = (2|1|2|3|3|1|2|3|1|2|1|2|3|1), \tag{8.25}$$

with similar results holding for the associated density matrices. In analogy with the ordinary case we have

$$\begin{aligned}(M \otimes N)^t &= M^t \otimes N^t \\ (M \otimes N)^{st} &= M^{st} \otimes N^{st} \\ \operatorname{str}(M \otimes N) &= \operatorname{str} M \operatorname{str} N.\end{aligned} \tag{8.26}$$

These definitions are manifestly compatible with Hermiticity and super-Hermiticity.

Denoting the total number of bosonic elements in the product of n superqubits by B_n, and similarly the total number of fermionic elements by F_n, we know that B_n (F_n) is given by the total number of basis kets with an even (odd) number of $\bullet$'s:

$$\begin{aligned}B_n &= \binom{n}{0}2^n + \binom{n}{2}2^{n-2} + \cdots = \frac{3^n+1}{2} \\ F_n &= \binom{n}{1}2^{n-1} + \binom{n}{3}2^{n-3} + \cdots = \frac{3^n-1}{2}\end{aligned} \tag{8.27}$$

so that, in particular, $B_n - F_n = 1$: the number of bosonic elements is always one more than the number of fermionic ones.

In supermatrix representations of superalgebras, one may represent the superbracket of generators M and N as

$$[[M, N]] := MN - N_E M - N_O(M_E - M_O). \tag{8.28}$$

One may also consider supermatrices M and N whose components are themselves supermatrices. Provided the component supermatrices are pure even (odd) at even (odd) index positions (e.g. M_{11} is a pure even supermatrix for even M), one may write the superbracket of such supermatrices as

$$\begin{gathered}[[M_{X_1X_2}, N_{X_3X_4}]] = \\ M_{X_1X_2}N_{X_3X_4} - (-)^{(X_1+X_2)(X_3+X_4)}N_{X_3X_4}M_{X_1X_2},\end{gathered} \tag{8.29}$$

where the final two indices are suppressed. This grouping of supermatrices into supermatrices is useful for summarising the superbrackets of superalgebras.

8.1.2 Orthosymplectic superalgebras and super (S)LOCC

In promoting the conventional SLOCC equivalence group $\mathrm{SL}_2(\mathbb{C})$ with Lie algebra

$$[P_{A_1A_2}, P_{A_3A_4}] = 2\varepsilon_{(A_1(A_3}P_{A_4)A_2)}, \tag{8.30}$$

to its minimal supersymmetric extension $\mathrm{OSp}_{2|1}$, we begin with a discussion of orthosymplectic superalgebras in general. Supermatrix representations of the orthosymplectic supergroup $\mathrm{OSp}_{p|2q}$ consist of supermatrices $M \in \mathrm{GL}_{p|2q}$ satisfying

$$M^{st} E M = E, \tag{8.31}$$

but for convenience we choose instead to use supermatrices $M \in \mathrm{GL}_{2q|p}$ satisfying (8.31). In this convention, the invariant supermatrix E is defined by

$$E := \left(\begin{array}{c|c} \mathbb{J}_{2q} & 0 \\ \hline 0 & \mathbb{1}_p \end{array}\right), \qquad \mathbb{J}_{2q} := \begin{pmatrix} 0 & \mathbb{1}_q \\ -\mathbb{1}_q & 0 \end{pmatrix}. \tag{8.32}$$

Definitions of supermatrices, the supertranspose, and further details of superlinear algebra were supplied in section 8.1.1.

Writing a generic supermatrix $\mathfrak{M}$ of the super Lie algebra $\mathfrak{osp}(p|2q)$ as

$$\mathfrak{M} = \left(\begin{array}{c|c} A & B \\ C & D \end{array}\right) \tag{8.33}$$

permits (8.31) to be rewritten as the following conditions on the blocks of the algebra supermatrices:

$$A^t \mathbb{J} = -\mathbb{J} A, \qquad C = B^t \mathbb{J}, \qquad D^t = -D. \tag{8.34}$$

Depending on the value of p, the superalgebra falls into one of three basic, "classical" families

$$\mathfrak{osp}_{p|2q} = \begin{cases} B(r,q) & p = 2r+1,\ r \geq 0 \\ C(q+1) & p = 2 \\ D(r,q) & p = 2r,\ r \geq 2. \end{cases} \tag{8.35}$$

Clearly it is the first case that will concern us, in particular, with $r = 0, q = 1$. $B(r,q)$ has rank $q + r$, dimension $2(q+r)^2 + 3q + r$, and even part $\mathfrak{so}_p \oplus \mathfrak{sp}_{2q}$, which for $\mathfrak{osp}_{1|2}$ are 1, 5, and $\mathfrak{sl}_2$, respectively.

One generates $\mathfrak{osp}_{p|2q}$ as a matrix superalgebra by defining the supermatrices U and

G

$$\begin{aligned}(U_{X_1X_2})_{X_3X_4} &:= \delta_{X_1X_4}\delta_{X_2X_3},\\ G &:= \left(\begin{array}{c|c}\mathbb{J}_{2q} & 0\\ \hline 0 & H_p\end{array}\right),\end{aligned} \tag{8.36}$$

where

$$H_p := \begin{cases}\sigma_1 \otimes \mathbb{1}_r & p = 2r\\ [\sigma_1 \otimes \mathbb{1}_r] \oplus (1) & p = 2r+1\end{cases} \tag{8.37}$$

with σ_1 being the first Pauli matrix. Here the indices X_i range from 1 to $2q + p$ and are partitioned as $X_i = (\bar{X}_i, \dot{X}_i)$ with $\bar{X}_i$ ranging from 1 to $2q$, and $\dot{X}_i$ taking on the remaining p values. Note that under (8.36), G has the following symmetry properties

$$\begin{aligned}G_{\bar{X}_1\bar{X}_2} = -G_{\bar{X}_2\bar{X}_1}, \quad G_{\dot{X}_1\dot{X}_2} = +G_{\dot{X}_2\dot{X}_1},\\ G_{\bar{X}_1\dot{X}_2} = 0 = G_{\dot{X}_2\bar{X}_1},\end{aligned} \tag{8.38}$$

which are shared with the invariant supermatrix E. In the special case $p = 1$, G reduces to E.

The generators T are obtained as

$$T_{X_1X_2} = 2G_{[[X_1|X_3}U_{X_3|X_2]]}, \tag{8.39}$$

where T has array grade zero and the index grades are monotonically increasing:

$$\deg X := \begin{cases}0 & X \in \{1, \ldots, 2q\}\\ 1 & X \in \{2q+1, \ldots, 2q+p\}.\end{cases} \tag{8.40}$$

Clearly T has symmetry properties $T_{X_1X_2} = T_{[[X_1X_2]]}$. The $2q(2q+1)/2$ generators $T_{\bar{X}_1\bar{X}_2}$ generate $\mathfrak{sp}_{2q}$, the $p(p-1)/2$ generators $T_{\dot{X}_1\dot{X}_2}$ generate $\mathfrak{so}_p$, and both are even (bosonic), while the $2pq$ generators $T_{\bar{X}_1\dot{X}_2}$ are odd (fermionic). These supermatrices yield the $\mathfrak{osp}_{p|2q}$ superbrackets

$$[[T_{X_1X_2}, T_{X_3X_4}]] := 4G_{[[X_1[[X_3}T_{X_2]]X_4]]}, \tag{8.41}$$

where the supersymmetrisation on the right-hand side is over pairs X_1X_2 and X_3X_4 as on the left-hand side. The action of the generators on $(2q|p)$-dimensional supervectors

a_X is given by

$$(T_{X_1X_2})_{X_3X_4}a_{X_4} \equiv (T_{X_1X_2}a)_{X_3} = 2G_{[[X_1|X_3}a_{X_2]]} \tag{8.42}$$

This action may be generalised to an N-fold super tensor product of $(2q|p)$ supervectors by labelling the indices with integers $k = 1, 2, \ldots, N$

$$\begin{aligned}(T_{X_kY_k}a)_{Z_1\cdots Z_k\cdots Z_N} &= \\ (-)^{(X_k+Y_k)\sum_{i=1}^{k-1}|Z_i|}2G_{[[X_k|Z_k}a_{Z_1\cdots|Y_k]]\cdots Z_N}&.\end{aligned} \tag{8.43}$$

In our special case $p = 1$ we denote the lone dotted index $\dot{X}_i$ by a bullet $\bullet$ and start counting the barred indices at zero so that $X_i = (0, 1, \bullet)$. Obviously the $T_{\bullet\bullet}$ generator vanishes identically, leaving only the following superbrackets:

$$\begin{aligned}[T_{A_1A_2}, T_{A_3A_4}] &= 4E_{(A_1(A_3}T_{A_2)A_4)} \\ [T_{A_1A_2}, T_{A_3\bullet}] &= 2E_{(A_1|A_3}T_{A_2)\bullet} \\ \{T_{A_1\bullet}, T_{A_2\bullet}\} &= T_{A_1A_2},\end{aligned} \tag{8.44}$$

which are written out in Table 8.1 with $T_A \equiv T_{A\bullet} \equiv T_{\bullet A}$. Explicitly the generators are

Table 8.1: $\mathfrak{osp}_{1|2}$ superbrackets.

	T_{01}	T_{00}	T_{11}	T_0	T_1
T_{01}	0	$-2T_{00}$	$2T_{11}$	$-T_0$	T_1
T_{00}	$2T_{00}$	0	$4T_{01}$	0	$2T_0$
T_{11}	$-2T_{11}$	$-4T_{01}$	0	$-2T_1$	0
T_0	T_0	0	$2T_1$	T_{00}	T_{01}
T_1	$-T_1$	$-2T_0$	0	T_{01}	T_{11}

$$\begin{gathered}T_{01} = \left(\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \\ T_{00} = \left(\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad T_{11} = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ -2 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \\ T_0 = \left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 1 & 0 \end{array}\right), \quad T_1 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline -1 & 0 & 0 \end{array}\right).\end{gathered} \tag{8.45}$$

In order to make contact with [178], we rescale the generators into a new supermatrix P

$$P_{X_1X_2} := \tfrac{1}{2}T_{X_1X_2} \equiv E_{[[X_1|X_3}U_{X_3|X_2]]} \tag{8.46}$$

to yield the superbrackets

$$\begin{aligned} [P_{A_1A_2}, P_{A_3A_4}] &= 2\varepsilon_{(A_1(A_3}P_{A_2)A_4)} \\ [P_{A_1A_2}, Q_{A_3}] &= \varepsilon_{(A_1|A_3}Q_{A_2)} \\ \{Q_{A_1}, Q_{A_2}\} &= \tfrac{1}{2}P_{A_1A_2}, \end{aligned} \tag{8.47}$$

where $Q_A \equiv P_A$, which are summarised as

$$[[P_{X_1X_2}, P_{X_3X_4}]] = 2E_{[[X_1[[X_3}P_{X_2]]X_4]]}. \tag{8.48}$$

The rescaled generators have the action

$$\begin{aligned} (P_{X_1X_2}a)_{X_3} &= E_{[[X_1|X_3}a_{X_2]]} \\ (P_{X_kY_k}a)_{Z_1\cdots Z_k\cdots Z_N} &= \\ (-)^{(X_k+Y_k)\sum_{i=1}^{k-1}Z_i}&E_{[[X_k|Z_k}a_{Z_1\cdots|Y_k]]\cdots Z_N}, \end{aligned} \tag{8.49}$$

which, it can be checked, summarises Tables 8.2, 8.3 and 8.4.

The three even elements $P_{A_1A_2}$ form an $\mathfrak{sl}_2$ subalgebra generating the bosonic SLOCC equivalence group, under which Q_A transforms as a spinor.

The supersymmetric generalisation of the conventional group of local unitaries is given by $\mathrm{uOSp}_{2|1}$, a compact subgroup of $\mathrm{OSp}_{2|1}$ [180, 183]. It has a supermatrix representation as the subset of $\mathrm{OSp}_{2|1}$ supermatrices satisfying the additional superunitary condition

$$M^{\ddagger}M = \mathbb{1}, \tag{8.50}$$

where ‡ is the superadjoint given by

$$M^{\ddagger} = (M^{st})^{\#}. \tag{8.51}$$

The $\mathrm{uOSp}_{2|1}$ algebra is given by

$$\mathfrak{uosp}_{2|1} := \{X \in \mathfrak{osp}_{2|1} | X^{\ddagger} = -X\}. \tag{8.52}$$

An arbitrary element $X \in \mathfrak{uosp}(2|1)$ may be written as

$$X = \xi_i A_i + \eta^{\#} Q_0 + \eta Q_1, \tag{8.53}$$

where ξ_i and η are pure even/odd Grassmann numbers respectively and

$$\begin{gathered} A_1 = \tfrac{i}{2}(P_{00} - P_{11}), \qquad A_2 = \tfrac{1}{2}(P_{00} + P_{11}), \\ A_3 = iP_{01}, \\ Q_A^{\ddagger} = \varepsilon_{AA'} Q_{A'}, \qquad A_i^{\ddagger} = -A_i. \end{gathered} \tag{8.54}$$

8.1.3 Super Hilbert space and $\mathrm{uOSp}_{2|1}$

The dual space

We now generalise the ordinary concept of a Hilbert space, described in section 3.1. With one important difference, explained below, our definition of a super Hilbert space follows that of DeWitt [181]. We define a super Hilbert space to be a supervector space $\mathcal{H}$ equipped with an injection to its dual space $\mathcal{H}^{\ddagger}$,

$$\begin{aligned} {}^{\ddagger} &: \mathcal{H} \to \mathcal{H}^{\ddagger}, \\ |\psi\rangle &\mapsto (|\psi\rangle)^{\ddagger} := \langle\psi|. \end{aligned} \tag{8.55}$$

Details of even and odd Grassmann numbers and supervectors may be found in section 8.1.1. A basis in which all basis vectors are pure even or odd is said to be pure. Such a basis may always be found [181].

The map $^{\ddagger} : \mathcal{H} \to \mathcal{H}^{\ddagger}$ defines an inner product $\langle\psi|\phi\rangle$ and satisfies the following axioms:

1. ‡ sends pure bosonic (fermionic) supervectors in $\mathcal{H}$ into bosonic (fermionic) supervectors in $\mathcal{H}^{\ddagger}$.

2. ‡ is linear
$$(|\psi\rangle + |\phi\rangle)^{\ddagger} = \langle\psi| + \langle\phi|. \tag{8.56}$$

3. For pure even/odd α and $|\psi\rangle$
$$(|\psi\rangle\alpha)^{\ddagger} = (-)^{\alpha\psi}\alpha^{\#}\langle\psi| \tag{8.57}$$
and
$$(\alpha\langle\psi|)^{\ddagger} = (-)^{\psi+\alpha\psi}|\psi\rangle\alpha^{\#}, \tag{8.58}$$

where $^{\#}$ is the superstar introduced in section 8.1.1. In particular

$$|\psi\rangle^{\ddagger\ddagger} = (-)^{\psi}|\psi\rangle. \tag{8.59}$$

Note, an α (or ψ and the like) appearing in the exponent of $(-)$ is shorthand for its grade, $\deg(\alpha)$, which takes the value 0 or 1 according to whether α is even or odd. The impure case follows from the linearity of ‡.

In a pure even/odd orthonormal basis $\{|i\rangle\}$ we adopt the following convention:

$$|\psi\rangle = |i\rangle\psi_i \tag{8.60}$$

so that for pure even/odd ψ (8.57) and (8.58) imply

$$\begin{aligned} (|i\rangle\psi_i)^{\ddagger} &= (-)^{\psi_i i}\psi_i^{\#}\langle i| = (-)^{i+i\psi}\psi_i^{\#}\langle i| \\ ((-)^{i+i\psi}\psi_i^{\#}\langle i|)^{\ddagger} &= (-)^{\psi}|i\rangle\psi_i \end{aligned} \tag{8.61}$$

where we have used $\deg(\psi_i) = \deg(i) + \deg(\psi)$. This is consistent with (8.21).

Inner product

For all pure even/odd $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ the inner product $\langle\psi|\phi\rangle$ satisfies

$$\langle\psi|\phi\rangle^{\#} = (-)^{\psi+\psi\phi}\langle\phi|\psi\rangle. \tag{8.62}$$

Consequently,

$$\langle\psi|\phi\rangle^{\#\#} = (-)^{\psi+\phi}\langle\phi|\psi\rangle, \tag{8.63}$$

as would be expected of a pure even/odd Grassmann number since $\deg(\langle\phi|\psi\rangle) = \deg(\psi) + \deg(\phi)$. In a pure even/odd orthonormal basis we find

$$\langle\phi|\psi\rangle = (-)^{i+i\phi}\phi_i^{\#}\psi_i. \tag{8.64}$$

In using the superstar we depart from the formalism presented in [181] which uses the ordinary star. A comparison of the star and superstar may be found in section 8.1.1. The use of the superstar is tailored to the implementation of $\mathrm{uOSp}_{2|1}$ as the compact subgroup of $\mathrm{OSp}_{2|1}$ as explained in section 8.1.2.

Physical states

For all $|\psi\rangle \in \mathcal{H}$

$$\langle\psi|\psi\rangle_{\mathcal{B}} \geq 0. \tag{8.65}$$

Here $z_{\mathcal{B}} \in \mathbb{C}$ denotes the purely complex number component of the Grassmann number z and is referred to as the *body*, a terminology introduced in [181]. The *soul* of z, denoted $z_{\mathcal{S}}$, is the purely Grassmannian component. Any Grassmann number may be decomposed into body and soul, $z = z_{\mathcal{B}} + z_{\mathcal{S}}$.

A Grassmann number has an inverse iff it has a non-vanishing body. Consequently, a state $|\psi\rangle$ is normalisable iff $\langle\psi|\psi\rangle_{\mathcal{B}} > 0$. The state may then be normalised,

$$|\hat{\psi}\rangle = N_\psi|\psi\rangle, \quad N_\psi = \langle\psi|\psi\rangle^{-1/2}, \tag{8.66}$$

where N_ψ is given by the general definition of an analytic function f on the space of Grassmann numbers (8.4). Explicitly,

$$\langle\psi|\psi\rangle^{-1/2} = \sum_{k=0}^{\infty} \frac{1}{k!2^k} \prod_{j=0}^{k} (1-2j) \langle\psi|\psi\rangle_{\mathcal{B}}^{-\frac{2k+1}{2}} \langle\psi|\psi\rangle_{\mathcal{S}}^{k}. \tag{8.67}$$

Motivated by the above considerations a state $|\psi\rangle$ is said to be *physical* iff $\langle\psi|\psi\rangle_{\mathcal{B}} > 0$. We restrict our attention to physical states throughout.

We must acknowledge that this choice presents us with the problem of interpreting "physical" states with non-vanishing soul for which probabilities are no longer real numbers but elements of a Grassmann algebra. (The probabilities still add up to one, however.) The examples of section 8.2 avoid this problem, being pure body. DeWitt advocates retaining only such pure body states in the Hilbert space [181], but we view this as too draconian, preferring the approaches of [186].

8.1.4 One superqubit

The one superqubit system is described by the state

$$|\Psi\rangle = |A\rangle a_A + |\bullet\rangle a_\bullet, \tag{8.68}$$

where a_A is commuting with $A = 0, 1$ and $a_\bullet$ is anticommuting. That is to say, the state vector is promoted to a supervector. The super Hilbert space has dimension 3, two "bosons" and one "fermion". In more compact notation we may write,

$$|\Psi\rangle = |X\rangle a_X, \tag{8.69}$$

where $X = (A, \bullet)$.

The super SLOCC equivalence group for a single qubit is $\mathrm{OSp}_{2|1,A}$. Under the $\mathrm{SL}_{2,A}$ subgroup a_A transforms as a **2** while $a_\bullet$ is a singlet as shown in Table 8.2. The super LOCC entanglement equivalence group, i.e. the group of local unitaries, is given by $\mathrm{uOSp}_{2|1,A}$, the unitary subgroup of $\mathrm{OSp}_{2|1,A}$.

Table 8.2: The action of the $\mathfrak{osp}_{2|1}$ generators on the superqubit fields.

Generator	Field acted upon	
	a_{A_3}	$a_\bullet$
$P_{A_1A_2}$	$\varepsilon_{(A_1\|A_3}a_{\|A_2)}$	0
$2Q_{A_1}$	$\varepsilon_{A_1A_3}a_\bullet$	a_{A_1}

The norm squared $\langle\Psi|\Psi\rangle$ is given by

$$\langle\Psi|\Psi\rangle = \delta^{A_1A_2}a^\#_{A_1}a_{A_2} - a^\#_\bullet a_\bullet, \tag{8.70}$$

where $\langle\Psi| = (|\Psi\rangle)^\ddagger$ and $\langle\Psi|\Psi\rangle$ is the conventional inner product which is manifestly $\mathrm{uOSp}_{2|1}$ invariant.

The one-superqubit state may then be normalised. When presenting examples of state vector normalisation, we take the underlying Grassmann algebra to have one generator per superqubit for simplicity, however our other results are independent of the dimension. Hence the n-superqubit Hilbert space is defined over a 2^n-dimensional Grassmann algebra. Consequently, $z_\mathcal{S}^{n+1} = 0$ for all z and (8.4) terminates after a finite number of terms. Using (8.67) for n superqubits one obtains

$$\langle\Psi|\Psi\rangle^{-1/2} = \sum_{k=0}^{2n} \frac{1}{k!2^k} \prod_{j=0}^{k}(1-2j)\langle\Psi|\Psi\rangle_\mathcal{B}^{-\frac{2k+1}{2}}\langle\Psi|\Psi\rangle_\mathcal{S}^k, \tag{8.71}$$

where the sum runs to $2n$ since an arbitrary Grassmann α and its superstar conjugate $\alpha^\#$ are independent.

The one-superqubit density matrix in component form is given by

$$\begin{aligned} \rho_{X_1X_2} &= \langle X_1|\rho|X_2\rangle \\ &= (-)^{X_2} a_{X_1} a^{\#}_{X_2}. \end{aligned} \tag{8.72}$$

The density matrix is self-superadjoint,

$$\begin{aligned} \rho^{\ddagger}_{X_1X_2} &= (\rho^{st}_{X_1X_2})^{\#} \\ &= (-)^{X_2+X_1X_2}\rho^{\#}_{X_2X_1} \\ &= (-)^{X_2+X_1X_2}(-)^{X_1} a^{\#}_{X_2} a^{\#\#}_{X_1} \\ &= (-)^{X_2} a_{X_1} a^{\#}_{X_2} \\ &= \rho_{X_1X_2}. \end{aligned} \tag{8.73}$$

The norm squared is then given by the supertrace

$$\begin{aligned} \mathrm{str}(\rho) &= (-)^{X_1}\delta^{X_1X_2}\langle X_1|\rho|X_2\rangle \\ &= \sum_X a_X a^{\#}_X \\ &= \sum_X (-)^X a^{\#}_X a_X \\ &= \langle\Psi|\Psi\rangle \end{aligned} \tag{8.74}$$

as one would expect.

Unnormalised pure state super density matrices satisfy $\rho^2 = \mathrm{str}(\rho)\rho$,

$$\begin{aligned} \rho^2 &= (-)^{X_2} a_{X_1} a^{\#}_{X_2} \delta^{X_2X_3}(-)^{X_4} a_{X_3} a^{\#}_{X_4} \\ &= \delta^{X_2X_3} a_{X_2} a^{\#}_{X_3}(-)^{X_4} a_{X_1} a^{\#}_{X_4} \\ &= \mathrm{str}(\rho)\rho, \end{aligned} \tag{8.75}$$

the appropriate supersymmetric version of the conventional pure state density matrix condition.

8.1.5 Two superqubits

The two superqubit system is described by the state

$$|\Psi\rangle = |AB\rangle a_{AB} + |A\bullet\rangle a_{A\bullet} + |\bullet B\rangle a_{\bullet B} + |\bullet\bullet\rangle a_{\bullet\bullet} \tag{8.76}$$

where a_{AB} is commuting, $a_{A\bullet}$ and $a_{\bullet B}$ are anticommuting and $a_{\bullet\bullet}$ is commuting. The super Hilbert space has dimension 9: 5 "bosons" and 4 "fermions". The super SLOCC group for two superqubits is $\mathrm{OSp}_{2|1,A} \times \mathrm{OSp}_{2|1,B}$. Under the $\mathrm{SL}_{2,A} \times \mathrm{SL}_{2,B}$ subgroup a_{AB} transforms as a $(\mathbf{2},\mathbf{2})$, $a_{A\bullet}$ as a $(\mathbf{2},\mathbf{1})$, $a_{\bullet B}$ as a $(\mathbf{1},\mathbf{2})$ and $a_{\bullet\bullet}$ as a $(\mathbf{1},\mathbf{1})$ as summarised in Table 8.3. The coefficients may also be assembled into a $(2|1) \times (2|1)$ supermatrix

Table 8.3: The action of the $\mathfrak{osp}_{2|1} \oplus \mathfrak{osp}_{2|1}$ generators on the two-superqubit fields.

Generator	Field acted upon			
	Bosons		Fermions	
	$a_{A_3B_3}$	$a_{\bullet\bullet}$	$a_{A_3\bullet}$	$a_{\bullet B_3}$
$P_{A_1A_2}$	$\varepsilon_{(A_1\vert A_3}a_{\vert A_2)B_3}$	0	$\varepsilon_{(A_1\vert A_3}a_{\vert A_2)\bullet}$	0
$P_{B_1B_2}$	$\varepsilon_{(B_1\vert B_3}a_{A_3\vert B_2)}$	0	0	$\varepsilon_{(B_1\vert B_3}a_{\bullet\vert B_2)}$
$2Q_{A_1}$	$\varepsilon_{A_1A_3}a_{\bullet B_3}$	$a_{A_1\bullet}$	$\varepsilon_{A_1A_3}a_{\bullet\bullet}$	$a_{A_1B_3}$
$2Q_{B_1}$	$\varepsilon_{B_1B_3}a_{A_3\bullet}$	$-a_{\bullet B_1}$	$a_{A_3B_1}$	$-\varepsilon_{B_1B_3}a_{\bullet\bullet}$

$$\langle XY|\Psi\rangle = a_{XY} = \left(\begin{array}{cc|c} a_{AB} & & a_{A\bullet} \\ \hline a_{\bullet B} & & a_{\bullet\bullet} \end{array}\right). \tag{8.77}$$

See Figure 8.1.

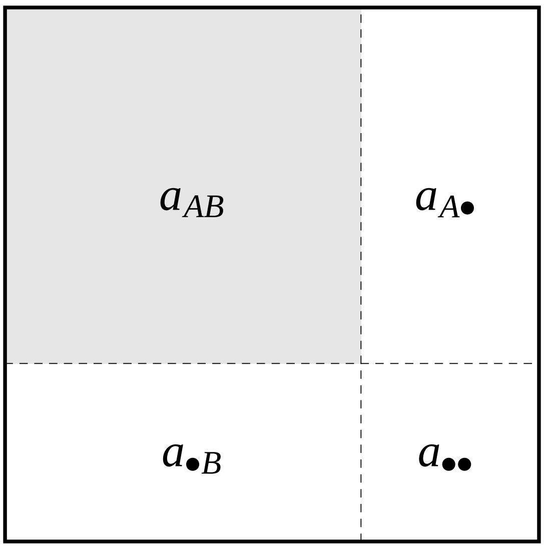


Figure 8.1: The 3×3 square supermatrix, which generalises the ordinary qubit square of Figure 3.1.

The norm squared $\langle\Psi|\Psi\rangle$ is given by

$$\begin{aligned}\langle\Psi|\Psi\rangle &= (-)^{X_1+Y_1}\delta^{X_1X_2}\delta^{Y_1Y_2}a^{\#}_{X_1Y_1}a_{X_2Y_2}\\ &= \delta^{A_1A_2}\delta^{B_1B_2}a^{\#}_{A_1B_1}a_{A_2B_2}\\ &\quad - \delta^{A_1A_2}a^{\#}_{A_1\bullet}a_{A_1\bullet} - \delta^{B_1B_2}a^{\#}_{\bullet B_1}a_{\bullet B_1}\\ &\quad + a^{\#}_{\bullet\bullet}a_{\bullet\bullet},\end{aligned} \tag{8.78}$$

where $\langle\Psi| = (|\Psi\rangle)^{\ddagger}$ and $\langle\Psi|\Psi\rangle$ is the conventional inner product which is manifestly $\mathrm{uOSp}_{2|1,A} \times \mathrm{uOSp}_{2|1,B}$ invariant.

The two-superqubit density matrix is given by

$$\begin{aligned}\rho &= |\Psi\rangle\langle\Psi|\\ &= (-)^{X_2+Y_2}|X_1Y_1\rangle a_{X_1Y_1}a^{\#}_{X_2Y_2}\langle X_2Y_2|.\end{aligned} \tag{8.79}$$

In component form the reduced density matrices for A and B are given by the partial supertraces:

$$\begin{aligned}(\rho_A)_{X_1X_2} &= \sum_Y (-)^{X_2}a_{X_1Y}a^{\#}_{X_2Y},\\ (\rho_B)_{Y_1Y_2} &= \sum_X (-)^{Y_2}a_{XY_1}a^{\#}_{XY_2},\end{aligned} \tag{8.80}$$

and

$$\operatorname{str}\rho_A = \operatorname{str}\rho_B = \langle\Psi|\Psi\rangle. \tag{8.81}$$

8.1.6 Three superqubits

The three superqubit system is described by the state

$$\begin{aligned}|\Psi\rangle &= |ABC\rangle a_{ABC}\\ &+|AB\bullet\rangle a_{AB\bullet} + |A\bullet C\rangle a_{A\bullet C} + |\bullet BC\rangle a_{\bullet BC}\\ &+|A\bullet\bullet\rangle a_{A\bullet\bullet} + |\bullet B\bullet\rangle a_{\bullet B\bullet} + |\bullet\bullet C\rangle a_{\bullet\bullet C}\\ &+|\bullet\bullet\bullet\rangle a_{\bullet\bullet\bullet}\end{aligned} \tag{8.82}$$

where a_{AB} is commuting, $a_{AB\bullet}$ $a_{A\bullet C}$ $a_{\bullet BC}$ are anticommuting, $a_{A\bullet\bullet}$ $a_{\bullet B\bullet}$ $a_{\bullet\bullet C}$ are commuting and $a_{\bullet\bullet\bullet}$ is anticommuting. The super Hilbert space has dimension 27: 14 "bosons" and 13 "fermions". The super SLOCC group for three superqubits is $\mathrm{OSp}_{2|1,A} \times \mathrm{OSp}_{2|1,B} \times \mathrm{OSp}_{2|1,C}$. Under the $\mathrm{SL}_{2,A} \times \mathrm{SL}_{2,B} \times \mathrm{SL}_{2,C}$ subgroup a_{ABC} transforms as a $(\mathbf{2},\mathbf{2},\mathbf{2})$, $a_{AB\bullet}$ as a $(\mathbf{2},\mathbf{1},\mathbf{1})$, $a_{A\bullet C}$ as a $(\mathbf{2},\mathbf{1},\mathbf{2})$, $a_{\bullet BC}$ as a $(\mathbf{1},\mathbf{2},\mathbf{2})$, $a_{A\bullet\bullet}$ as a $(\mathbf{2},\mathbf{1},\mathbf{1})$, $a_{\bullet B\bullet}$ as a $(\mathbf{1},\mathbf{2},\mathbf{1})$, $a_{\bullet\bullet C}$ as a $(\mathbf{1},\mathbf{1},\mathbf{2})$ and $a_{\bullet\bullet\bullet}$ as a $(\mathbf{1},\mathbf{1},\mathbf{1})$ as summarised in Table 8.4.The coefficients may also be assembled into a $(2|1)\times(2|1)\times(2|1)$ superhypermatrix

$$\langle XYZ|\Psi\rangle = a_{XYZ}. \tag{8.83}$$

See Figure 8.2.

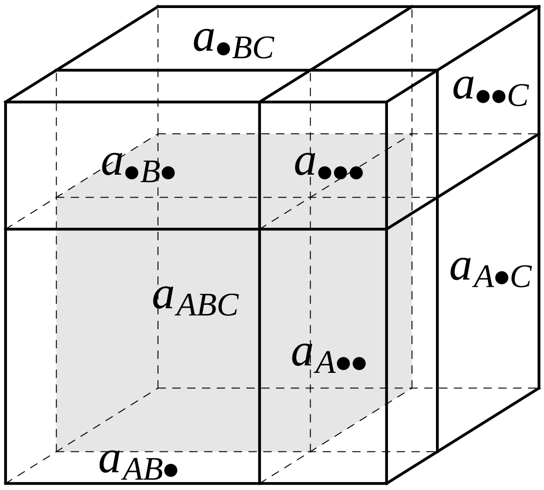


Figure 8.2: The $3\times3\times3$ cubic superhypermatrix, which generalises the ordinary qubit cube of Figure 3.1.

The norm squared $\langle\Psi|\Psi\rangle$ is given by

$$\begin{aligned}
\langle\Psi|\Psi\rangle &= (-)^{X_1+Y_1+Z_1}\delta^{X_1X_2}\delta^{Y_1Y_2}\delta^{Z_1Z_2}a^{\#}_{X_1Y_1Z_1}a_{X_2Y_2Z_2}\\
&= \delta^{A_1A_2}\delta^{B_1B_2}\delta^{C_1C_2}a^{\#}_{A_1B_1C_1}a_{A_2B_2C_2}\\
&\quad - \delta^{A_1A_2}\delta^{B_1B_2}a^{\#}_{A_1B_1\bullet}a_{A_2B_2\bullet}\\
&\quad - \delta^{A_1A_2}\delta^{C_1C_2}a^{\#}_{A_1\bullet C_1}a_{A_2\bullet C_2}\\
&\quad - \delta^{B_1B_2}\delta^{C_1C_2}a^{\#}_{\bullet B_1C_1}a_{\bullet B_2C_2}\\
&\quad + \delta^{A_1A_2}a^{\#}_{A_1\bullet\bullet}a_{A_2\bullet\bullet}\\
&\quad + \delta^{B_1B_2}a^{\#}_{\bullet B_1\bullet}a_{\bullet B_2\bullet}\\
&\quad + \delta^{C_1C_2}a^{\#}_{\bullet\bullet C_1}a_{\bullet\bullet C_2}\\
&\quad - a^{\#}_{\bullet\bullet\bullet}a_{\bullet\bullet\bullet},
\end{aligned} \tag{8.84}$$

where $\langle\Psi| = (|\Psi\rangle)^{\ddagger}$ and $\langle\Psi|\Psi\rangle$ is the conventional inner product which is manifestly

Table 8.4: The action of the $\mathfrak{osp}_{2|1} \oplus \mathfrak{osp}_{2|1} \oplus \mathfrak{osp}_{2|1}$ generators on the three-superqubit fields.

Generator	Bosons acted upon			
	$a_{A_3B_3C_3}$	$a_{A_3\bullet\bullet}$	$a_{\bullet B_3\bullet}$	$a_{\bullet\bullet C_3}$
$P_{A_1A_2}$	$\varepsilon_{(A_1\|A_3}a_{\|A_2)B_3C_3}$	$\varepsilon_{(A_1\|A_3}a_{\|A_2)\bullet\bullet}$	0	0
$P_{B_1B_2}$	$\varepsilon_{(B_1\|B_3}a_{A_3\|B_3)C_2}$	0	$\varepsilon_{(B_1\|B_3}a_{\bullet\|A_2)\bullet}$	0
$P_{C_1C_2}$	$\varepsilon_{(C_1\|C_3}a_{A_3B_3\|C_2)}$	0	0	$\varepsilon_{(C_1\|C_3}a_{\bullet\bullet\|C_2)}$
$2Q_{A_1}$	$\varepsilon_{A_1A_3}a_{\bullet B_3C_3}$	$\varepsilon_{A_1A_3}a_{\bullet\bullet\bullet}$	$a_{A_1B_3\bullet}$	$a_{A_1\bullet C_3}$
$2Q_{B_1}$	$\varepsilon_{B_1B_3}a_{A_3\bullet C_3}$	$a_{A_3B_1\bullet}$	$-\varepsilon_{B_1B_3}a_{\bullet\bullet\bullet}$	$-a_{\bullet B_1C_3}$
$2Q_{C_1}$	$\varepsilon_{C_1C_3}a_{A_3B_3\bullet}$	$-a_{A_3\bullet C_1}$	$-a_{\bullet B_3C_1}$	$\varepsilon_{C_1C_3}a_{\bullet\bullet\bullet}$
	Fermions acted upon			
	$a_{A_3B_3\bullet}$	$a_{A_3\bullet C_3}$	$a_{\bullet B_3C_3}$	$a_{\bullet\bullet\bullet}$
$P_{A_1A_2}$	$\varepsilon_{(A_1\|A_3}a_{\|A_2)B_3\bullet}$	$\varepsilon_{(A_1\|A_3}a_{\|A_2)\bullet C_3}$	0	0
$P_{B_1B_2}$	$\varepsilon_{(B_1\|B_3}a_{A_3\|B_3)\bullet}$	0	$\varepsilon_{(B_1\|B_3}a_{\bullet\|B_3)C_2}$	0
$P_{C_1C_2}$	0	$\varepsilon_{(C_1\|C_3}a_{A_3\bullet\|C_2)}$	$\varepsilon_{(C_1\|C_3}a_{\bullet B_3\|C_2)}$	0
$2Q_{A_1}$	$\varepsilon_{A_1A_3}a_{\bullet B_3\bullet}$	$\varepsilon_{A_1A_3}a_{\bullet\bullet C_3}$	$a_{A_1B_3C_3}$	$a_{A_1\bullet\bullet}$
$2Q_{B_1}$	$\varepsilon_{B_1B_3}a_{A_3\bullet\bullet}$	$a_{A_3B_1C_3}$	$-\varepsilon_{B_1B_3}a_{\bullet\bullet C_3}$	$-a_{\bullet B_1\bullet}$
$2Q_{C_1}$	$a_{A_3B_3C_1}$	$-\varepsilon_{C_1C_3}a_{A_3\bullet\bullet}$	$-\varepsilon_{C_1C_3}a_{\bullet B_3\bullet}$	$a_{\bullet\bullet C_1}$

$\mathrm{uOSp}_{2|1,A} \times \mathrm{uOSp}_{2|1,B} \times \mathrm{uOSp}_{2|1,C}$ invariant.

The three-superqubit density matrix is given by

$$\begin{aligned}\rho &= |\Psi\rangle\langle\Psi| \\ &= (-)^{X_2+Y_2+Z_2}|X_1Y_1Z_1\rangle a_{X_1Y_1Z_1}a^{\#}_{X_2Y_2Z_2}\langle X_2Y_2Z_2|.\end{aligned} \tag{8.85}$$

The singly reduced density matrices are defined using the partial supertraces

$$\begin{aligned}\rho_{AB} &= \sum_Z (-)^{X_2+Y_2}|X_1Y_1\rangle a_{X_1Y_1Z}a^{\#}_{X_2Y_2Z}\langle X_2Y_2|, \\ \rho_{BC} &= \sum_X (-)^{Y_2+Z_2}|Y_1Z_1\rangle a_{XY_1Z_1}a^{\#}_{XY_2Z_2}\langle Y_2Z_2|, \\ \rho_{CA} &= \sum_Y (-)^{X_2+Z_2}|X_1Z_1\rangle a_{X_1YZ_1}a^{\#}_{X_2YZ_2}\langle X_2Z_2|.\end{aligned} \tag{8.86}$$

The doubly reduced density matrices for A, B and C are given by the partial supertraces

$$\begin{aligned}\rho_A &= \sum_{Y,Z}(-)^{X_2}|X_1\rangle a_{X_1YZ}a^{\#}_{X_2YZ}\langle X_2|,\\ \rho_B &= \sum_{X,Z}(-)^{Y_2}|Y_1\rangle a_{XY_1Z}a^{\#}_{XY_2Z}\langle Y_2|,\\ \rho_C &= \sum_{X,Y}(-)^{Z_2}|Z_1\rangle a_{XYZ_1}a^{\#}_{XYZ_2}\langle Z_2|.\end{aligned} \tag{8.87}$$

8.2 Superinvariant superentanglement measures

8.2.1 Two superqubits

In seeking a supersymmetric generalisation of the 2-tangle (3.14) one might be tempted to replace the determinant of a_{AB} by the Berezinian of a_{XY}

$$\operatorname{Ber} a_{XY} = \det(a_{AB} - a_{A\bullet}a_{\bullet\bullet}^{-1}a_{\bullet B})a_{\bullet\bullet}^{-1}. \tag{8.88}$$

See section 8.1.1. However, although the Berezinian is the natural supersymmetric extension of the determinant, it is not defined for vanishing $a_{\bullet\bullet}$, making it unsuitable as an entanglement measure.

A better candidate follows from writing

$$\begin{aligned}\det a_{AB} = \tfrac{1}{2}a^{AB}a_{AB} &= \tfrac{1}{2}\operatorname{tr}(a^t\varepsilon a\varepsilon^t)\\ &= \tfrac{1}{2}\operatorname{tr}[(a\varepsilon)^t\varepsilon a],\end{aligned} \tag{8.89}$$

This expression may be generalised by a straightforward promotion of the trace and transpose to the supertrace and supertranspose and replacing the SL_2 invariant tensor ε with the $\mathrm{OSp}_{2|1}$ invariant tensor E. See section 8.1.1. This yields a quadratic polynomial, which we refer to as the superdeterminant, denoted sdet:

$$\begin{aligned}\operatorname{sdet} a_{XY} &= \tfrac{1}{2}\operatorname{str}[(aE)^{st}Ea]\\ &= \tfrac{1}{2}(a^{AB}a_{AB} - a^{A\bullet}a_{A\bullet} - a^{\bullet B}a_{\bullet B} - a^{\bullet\bullet}a_{\bullet\bullet})\\ &= (a_{00}a_{11} - a_{01}a_{10} + a_{0\bullet}a_{1\bullet} + a_{\bullet 0}a_{\bullet 1}) - \tfrac{1}{2}{a_{\bullet\bullet}}^2,\end{aligned} \tag{8.90}$$

which is clearly not equal to the Berezinian, but is nevertheless supersymmetric since Q_A annihilates $a^{AB}a_{AB} - a^{\bullet B}a_{\bullet B}$ and $a^{A\bullet}a_{A\bullet} + a^{\bullet\bullet}a_{\bullet\bullet}$, while Q_B annihilates $a^{AB}a_{AB} - a^{A\bullet}a_{A\bullet}$ and $a^{\bullet B}a_{\bullet B} + a^{\bullet\bullet}a_{\bullet\bullet}$. Satisfyingly, (8.90) reduces to $\det a_{AB}$ when $a_{A\bullet}$, $a_{\bullet B}$ and $a_{\bullet\bullet}$ are set to zero. We then define the super 2-tangle as:

$$\tau_{XY} = 4 \operatorname{sdet} a_{XY} (\operatorname{sdet} a_{XY})^{\#}. \tag{8.91}$$

In summary, 2-superqubit entanglement seems to have the same two entanglement classes as 2-qubits with the invariant $\det a_{AB}$ replaced by its supersymmetric counterpart $\operatorname{sdet} a_{XY}$.

Non-superentangled states are given by product states for which $a_{AB} = a_A b_B$, $a_{A\bullet} = a_A b_\bullet$, $a_{\bullet B} = a_\bullet b_B$, $a_{\bullet\bullet} = a_\bullet b_\bullet$ and $\operatorname{sdet} a_{XY}$ vanishes. This provides a non-trivial consistency check.

An example of a normalised physical superentangled state is given by

$$|\Psi\rangle = \tfrac{1}{\sqrt{3}}(|00\rangle + |11\rangle + i|\bullet\bullet\rangle) \tag{8.92}$$

for which

$$\operatorname{sdet} a_{XY} = \tfrac{1}{3} + \tfrac{1}{2} \cdot \tfrac{1}{3} = \tfrac{1}{2} \tag{8.93}$$

and

$$\tau_{XY} = 4 \operatorname{sdet} a_{XY} (\operatorname{sdet} a_{XY})^{\#} = 1. \tag{8.94}$$

So this state is not only entangled but maximally entangled, just like the Bell state

$$|\Psi\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{8.95}$$

for which $\operatorname{sdet} a_{XY} = 1/2$ and $\tau_{XY} = 1$. Another more curious example is

$$|\Psi\rangle = i|\bullet\bullet\rangle \tag{8.96}$$

which is not a product state since $a_{\bullet\bullet}$ is pure body and hence could never be formed by the product of two odd supernumbers. In fact $\operatorname{sdet} a_{XY} = 1/2$ and $\tau_{XY} = 1$, so this state is also maximally entangled.

We may interpolate between these two examples with the normalised state

$$(|\alpha|^2 + |\beta|^2)^{-1/2}[\alpha|\Psi\rangle_{\text{Bell}} + \beta|\bullet\bullet\rangle], \tag{8.97}$$

where $\alpha, \beta \in \mathbb{C}$, for which we have

$$\begin{aligned} \operatorname{sdet} a_{XY} &= \frac{1}{2} \frac{\alpha^2 - \beta^2}{|\alpha|^2 + |\beta|^2}, \\ \tau_{XY} &= \frac{|\alpha^2 - \beta^2|^2}{(|\alpha|^2 + |\beta|^2)^2}. \end{aligned} \tag{8.98}$$

The entanglement for this state is displayed as a function of the complex parameter β in Figure 8.3 for the case $\alpha = 1$. Note in particular that while the state is maximised for

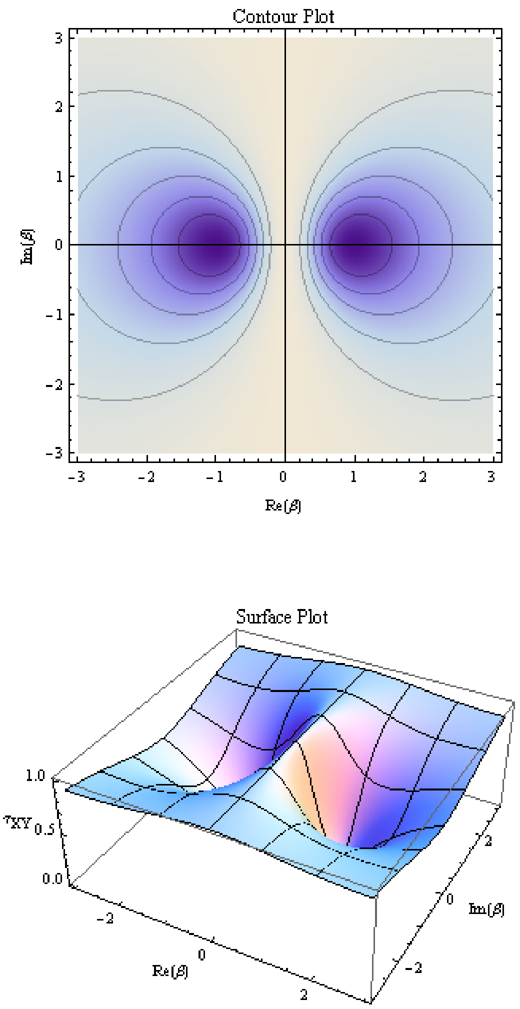


Figure 8.3: The 2-tangle τ_{XY} for the state (8.97) for a complex parameter β.

arbitrary pure imaginary β the state has its minimum value on the real axis at $\beta = \pm 1$

as shown in Figure 8.4.

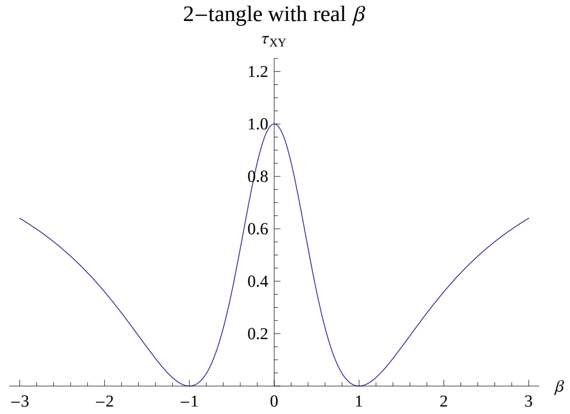


Figure 8.4: The 2-tangle τ_{XY} for the state (8.97) for a real parameter β.

8.2.2 Three superqubits

In seeking to generalise the 3-tangle (3.18), invariant under SL_2^3, to a supersymmetric object, invariant under $[OSp(2|1)]^3$, we need to find a quartic polynomial which reduces to Cayley's hyperdeterminant when $a_{AB\bullet}$, $a_{A\bullet C}$, $a_{\bullet BC}$, $a_{A\bullet\bullet}$, $a_{\bullet B\bullet}$, $a_{\bullet\bullet C}$ and $a_{\bullet\bullet\bullet}$ are set to zero. We do this by generalising the γ matrices:

$$\begin{aligned}\gamma_{A_1A_2} := {}& a_{A_1}{}^{BC}a_{A_2BC} - a_{A_1}{}^{B\bullet}a_{A_2B\bullet} \\ & - a_{A_1}{}^{\bullet C}a_{A_2\bullet C} - a_{A_1}{}^{\bullet\bullet}a_{A_2\bullet\bullet},\end{aligned} \tag{8.99}$$

$$\begin{aligned}\gamma_{A_1\bullet} := {}& a_{A_1}{}^{BC}a_{\bullet BC} + a_{A_1}{}^{B\bullet}a_{\bullet B\bullet} \\ & + a_{A_1}{}^{\bullet C}a_{\bullet\bullet C} - a_{A_1}{}^{\bullet\bullet}a_{\bullet\bullet\bullet},\end{aligned} \tag{8.100}$$

$$\begin{aligned}\gamma_{\bullet A_2} := {}& a_{\bullet}{}^{BC}a_{A_2BC} - a_{\bullet}{}^{B\bullet}a_{A_2B\bullet} \\ & - a_{\bullet}{}^{\bullet C}a_{A_2\bullet C} - a_{\bullet}{}^{\bullet\bullet}a_{A_2\bullet\bullet},\end{aligned} \tag{8.101}$$

together with their B and C counterparts; notice that the building blocks with two indices are bosonic and those with one index are fermionic. The final bosonic possibility,

$\gamma_{(\bullet\bullet)}$, vanishes identically. The simple supersymmetry relations are given by:

$$\begin{aligned} Q_{A_1}\gamma_{A_2A_3} &= \varepsilon_{A_1(A_2}\gamma_{A_3)\bullet} \\ Q_{A_1}\gamma_{A_2\bullet} &= \tfrac{1}{2}\gamma_{A_1A_2} \\ Q_B\gamma_{A_1A_2} &= 0 = Q_C\gamma_{A_1A_2} \\ Q_B\gamma_{A\bullet} &= 0 = Q_C\gamma_{A\bullet}. \end{aligned} \tag{8.102}$$

Using these expressions we define the superhyperdeterminant, denoted sDet a:

$$\operatorname{sDet} a_{XYZ} = \tfrac{1}{2}(\gamma^{A_1A_2}\gamma_{A_1A_2} - \gamma^{A\bullet}\gamma_{A\bullet} - \gamma^{\bullet A}\gamma_{\bullet A}) \tag{8.103}$$

which is invariant under the action of the superalgebra. The corresponding expressions singling out superqubits B and C are also invariant and equal to (8.103). sDet a_{XYZ} can be seen as the definition of the super-Cayley determinant of the cubic superhypermatrix given in Figure 8.2.

Writing

$$\Gamma^A := \left(\begin{array}{c|c} \gamma_{A_1A_2} & \gamma_{A_1\bullet} \\ \hline \gamma_{\bullet A_2} & \gamma_{\bullet\bullet} \end{array}\right) = \left(\begin{array}{c|c} \gamma_{A_1A_2} & \gamma_{A_1\bullet} \\ \hline \gamma_{A_2\bullet} & 0 \end{array}\right), \tag{8.104}$$

we obtain an invariant analogous to (8.90)

$$\operatorname{sDet} a_{XYZ} = \tfrac{1}{2}\operatorname{str}[(\Gamma^A E)^{st} E\Gamma^A] \tag{8.105}$$

so that

$$\operatorname{sDet} a_{XYZ} = -\operatorname{sdet}\Gamma^A \tag{8.106}$$

in analogy to the conventional three-qubit identity (3.21).

Finally, using Γ^A we are able to define the supersymmetric generalisation T_{XYZ} of the 3-qubit tensor T_{ABC} as encountered in (A.2.2) (in other words, super-C_{111}),

$$T_{XYZ} = \Gamma^A_{XX'}a^{X'}{}_{YZ}. \tag{8.107}$$

It is not difficult to verify that T_{XYZ} transforms in precisely the same way as a_{XYZ} (as given in Table 8.4) under $\mathfrak{osp}_{2|1} \oplus \mathfrak{osp}_{2|1} \oplus \mathfrak{osp}_{2|1}$. The superhyperdeterminant may then also be written as,

$$
\begin{aligned}
\mathrm{sDet}\, a_{XYZ} = {} & T_{ABC}a^{ABC} + T_{\bullet BC}a^{\bullet BC} \\
& - T_{A\bullet C}a^{A\bullet C} - T_{AB\bullet}a^{AB\bullet} \\
& - T_{A\bullet\bullet}a^{A\bullet\bullet} + T_{\bullet B\bullet}a^{\bullet B\bullet} \\
& + T_{\bullet\bullet C}a^{\bullet\bullet C} - T_{\bullet\bullet\bullet}a^{\bullet\bullet\bullet}.
\end{aligned} \tag{8.108}
$$

In this sense $\mathrm{sDet}\, a_{XYZ}$, $(\Gamma^A)_{X_1X_2}$ and T_{XYZ} are the natural supersymmetric generalisations of the hyperdeterminant, $\mathrm{Det}\, a_{ABC}$, and the covariant tensors, $(\gamma^A)_{A_1A_2}$ and T_{ABC}, of the conventional 3-qubit treatment summarised in Table A.5. Finally we are in a position to define the super 3-tangle:

$$
\tau_{XYZ} = 4\sqrt{\mathrm{sDet}\, a_{XYZ}(\mathrm{sDet}\, a_{XYZ})^{\#}}. \tag{8.109}
$$

In summary 3-superqubit entanglement seems to have the same five (seven) entanglement classes as that of 3-qubits shown in Table A.5, with the covariants $a_{ABC}, \gamma^A, \gamma^B, \gamma^C, T_{ABC}$ and $\mathrm{Det}\, a_{ABC}$ replaced by their supersymmetric counterparts $a_{XYZ}, \Gamma^A, \Gamma^B, \Gamma^C, T_{XYZ}$ and $\mathrm{sDet}\, a_{ABC}$.

Completely separable non-superentangled states are given by product states for which $a_{ABC} = a_A b_B c_C, a_{AB\bullet} = a_A b_B c_\bullet, a_{A\bullet C} = a_A b_\bullet c_C, a_{\bullet BC} = a_\bullet b_B c_C, a_{A\bullet\bullet} = a_A b_\bullet c_\bullet, a_{\bullet B\bullet} = a_\bullet b_B c_\bullet, a_{\bullet\bullet C} = a_\bullet b_\bullet c_C, a_{\bullet\bullet\bullet} = a_\bullet b_\bullet c_\bullet$ and $\mathrm{sDet}\, a_{XYZ}$ vanishes. This provides a non-trivial consistency check.

An example of a normalised physical biseparable state is provided by

$$
|\Psi\rangle = \tfrac{1}{\sqrt{3}}(|000\rangle + |011\rangle + |0\bullet\bullet\rangle) \tag{8.110}
$$

for which

$$
(\Gamma^A)_{00} = \tfrac{1}{3} \tag{8.111}
$$

and Γ^B, Γ^C, T_{XYZ} and $\mathrm{sDet}\, a_{XYZ}$ vanish. More generally, one can consider the combination

$$
|\Psi\rangle = (|\alpha|^2 + |\beta|^2)^{-1/2}[\tfrac{1}{\sqrt{2}}\alpha(|000\rangle + |011\rangle) + \beta|0\bullet\bullet\rangle] \tag{8.112}
$$

for which

$$
(\Gamma^A)_{00} = \frac{\alpha^2 - \beta^2}{|\alpha|^2 + |\beta|^2} \tag{8.113}
$$

and the other covariants vanish.

An example of a normalised physical W state is provided by

$$|\Psi\rangle = \tfrac{1}{\sqrt{6}}(|110\rangle + |101\rangle + |011\rangle + |\bullet\bullet 1\rangle + |\bullet 1\bullet\rangle + |1\bullet\bullet\rangle) \tag{8.114}$$

for which

$$(\Gamma^A)_{11} = (\Gamma^B)_{11} = (\Gamma^C)_{11} = -\tfrac{1}{2} \tag{8.115}$$

and

$$T_{111} = \tfrac{1}{2\sqrt{6}} \tag{8.116}$$

while sDet a_{XYZ} vanishes. One could also consider

$$|\Psi\rangle = \tfrac{1}{\sqrt{3}}(|\alpha|^2 + |\beta|^2)^{-1/2}[\alpha(|110\rangle + |101\rangle + |011\rangle) + \beta(|\bullet\bullet 1\rangle + |\bullet 1\bullet\rangle + |1\bullet\bullet\rangle)] \tag{8.117}$$

for which

$$(\Gamma^A)_{11} = (\Gamma^B)_{11} = (\Gamma^C)_{11} = -\frac{2\alpha^2 + \beta^2}{3(|\alpha|^2 + |\beta|^2)} \tag{8.118}$$

and

$$T_{111} = \frac{\alpha(2\alpha^2 + \beta^2)}{3\sqrt{3}(|\alpha|^2 + |\beta|^2)^{3/2}} \tag{8.119}$$

while the other T components and sDet a_{XYZ} vanish.

An example of a normalised physical tripartite superentangled state is provided by

$$|\Psi\rangle = \tfrac{1}{\sqrt{8}}(|000\rangle + |\bullet\bullet 0\rangle + |\bullet 0\bullet\rangle + |0\bullet\bullet\rangle + |111\rangle + |\bullet\bullet 1\rangle + |\bullet 1\bullet\rangle + |1\bullet\bullet\rangle) \tag{8.120}$$

for which

$$\mathrm{sDet}\, a_{XYZ} = \tfrac{1}{64} \tag{8.121}$$

and

$$\tau_{XYZ} = 4\sqrt{\mathrm{sDet}\, a_{XYZ}(\mathrm{sDet}\, a_{XYZ})^{\#}} = \tfrac{1}{16}. \tag{8.122}$$

Chapter 9

Conclusion

9.1 Summary

We have reviewed our recent discoveries and developments in the black hole/qubit correspondence, concentrating on those aspects more closely related to supersymmetry than FTS technology, namely supersymmetric black hole/entanglement classification and the superqubit development. Here follows a recap of our discussion:

Chapter 2 We began with a brief introduction to the black hole side of the correspondence. In particular describing the origin of black hole entropy, and the fact that it is expressed in terms of dyonic charges transforming in a representation of the U-duality group of the attendant theory. This was followed up by describing the theories of interest: the STU model and generalisations to $\mathcal{N} = 8$ and $D = 5$, along with the corresponding supersymmetric classifications, and a mention of magic supergravities.

Chapter 3 We reviewed the basics of entanglement and entanglement classification including the SLOCC paradigm, and the classification of three qubits into seven classes by appropriate entanglement measures. supported by some elementary concepts in classical invariant theory.

Chapter 4 The correspondences for the theories discussed in chapter 2 were presented, with the $\mathcal{N} = 2$ STU model and three qubits, $\mathcal{N} = 8$ and tripartite entangled "seven qubits" described by the Fano plane, and $D = 5$ and bipartite entangled "three qutrits". We observed that in the $\mathcal{N} = 2$ theory the three-qubit entanglement classification is matched by the black hole classification into small ($S = 0$), with 1/2 of supersymmetry preserved, and large ($S \neq 0$), with either 1/2 or 0.

Chapter 5 Further developments in the correspondences of chapter 4 were presented, starting with the dictionaries used in transforming between three important bases of Cartan's quartic $E_{7(7)}$ invariant. More examples of the SUSY/entanglement classifications were then given. For $\mathcal{N} = 8$, as for $\mathcal{N} = 2$, the large black holes correspond to the two classes of GHZ-type (entangled) states and small black holes to the separable or W class, but this time the large black holes can be 1/8- or non-BPS, while the small black holes can be 1/8-, 1/4-, or 1/2-BPS. Lastly, we described the octonions (and their split-signature cousins), which play a role in one of the aforementioned dictionaries, amongst other roles throughout the correspondence.

Chapter 6 We examined the brane wrapping paradigm in four and five dimensions. In four dimensions matching the eight states of the three-qubit system to the eight ways of wrapping four D3-branes around three 2-tori of Type II string theory compactified to four dimensions. Similarly, the nine states of the 2-qutrit system correspond to nine ways of wrapping two M2-branes around two 3-tori of M-theory compactified to five dimensions.

Chapter 7 Attempts at four qubit entanglement classification were discussed, including our own in-progress covariant classification approach. By invoking the Kostant-Sekiguchi correspondence, the U-duality orbits resulting from timelike reduction of string theory from $D = 4$ to $D = 3$ are mapped to 31 entanglement families (which reduce to nine modulo permutations) for four qubits. In the process it is shown that the nilpotent classification of $\mathcal{N} = 2$ black holes is identical to the nilpotent classification of complex qubits, bypassing the need for a restriction to rebits in earlier treatments.

Chapter 8 The super generalisations of Hilbert space and entanglement were explored, specifically by promoting the SLOCC group to it's minimal supersymmetric extension. The significant body of auxiliary definitions and structures were introduced to provide the means to define superentanglement measures in the cases of two and three superqubits, these being generalisations of the determinant and hyperdeterminant respectively. The non-intuitive nature of superentanglement is demonstrated by the maximal entanglement of a nominally separable state.

Appendix A The definitions of (integral) cubic Jordan algebras and Freudenthal triple systems are provided, followed by telegraphic descriptions of their uses in classifying qubit entanglement, defining new black hole dualities, and examining integral U-duality orbits.

9.2 Further work

Taking our existing research [21–28] as a starting point, one can divide the future research prospects within the black hole/qubit correspondence into four categories:

- The continued development of the superqubit idea.
- Continued matching of entanglement and supersymmetric black hole classification, potentially extending the brane wrapping perspective.
- Technological developments in the Freudenthal/Jordan machinery enabling the classification of charge orbits in more theories.
- Broader outstanding issues regarding the physical foundations of the correspondence.

9.2.1 Superqubits

Our treatment of superqubits left one or two loose ends with regard to the formalism and numerous tasks to investigate:

- Formalising super-Hilbert spaces and the pernicious problem of Grassmann valued norms to be acceptable to realistic systems.
- Identifying appropriate physical systems that admit a superqubit description.
- Generalising superqubits to other semisimple supergroups like e.g. $\mathrm{SL}_{2|1}$.
- Generalising entanglement invariants and classifications of the tensor product of multiple superqubits.
- Generalising, if possible, some of the physically important results of ordinary quantum theory: are super-Bell inequalities and a super-Kochen-Specker theorem possible?
- Classifying fully the two- and three-superqubit entanglement classes and their corresponding orbits as was done for the two- and three-qubit entanglement classes in [4, 24, 99]

While one can easily go to town on super generalisations of familiar quantum concepts, the principle problem remains the need for physical scenarios demanding such a superqubit description. Are superqubits a solution looking for a problem? We can go

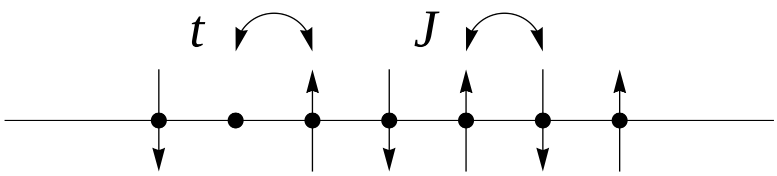


Figure 9.1: The t-J model which shows how the t interaction corresponds to electron hopping and the J to anti-ferromagnetic exchange.

some way to allaying any apprehensions on this point by identifying a couple of candidates. One's first instinct may be to look to photinos or selectrons, but since the supersymmetrisation of the (S)LOCC equivalence groups is quite distinct from the supersymmetrisation of the spacetime Poincaré group these are not obvious candidates. Supergroups of the form $\mathrm{OSp}_{2|1}$ do in fact show up in some models of strongly correlated electrons in condensed matter physics. In particular in the t-J model [187,188], which is a specialisation of the Hubbard model. It consists of a one dimensional anti-ferromagnetic lattice containing holes and where double occupancy is energetically hindered. The t corresponds to electron hopping and the J corresponds to anti-ferromagnetic exchange, see Figure 9.1. The field theory describes a condensate of holons and spinons which becomes supersymmetric when $J = 2t$, with the holons and spinons transforming in the fundamental representation of the supergroup [189]. Thus, in the same way that two polarisations of a photon can be a realisation of a qubit, we can anticipate that the holons and spinons form a valid realisation of a superqubit. If true, the lattice would contain long chains of superqubits which could be superentangled. Our techniques could then be applied to quantify the superentanglement and, speculatively, determine if the supersymmetry is able to control the decoherence times of these systems. While entirely hypothetical this could have a significant impact on quantum error correction and quantum computing, whose super analogues are already being investigated by Castellani et al [190].

Another example where our superqubits can be used is the *supersymmetric quantum Hall effect* [191], observed in two-dimensional electron systems subjected to low temperatures and strong magnetic fields, whereby conductivity is quantised.

9.2.2 Classification: entanglement vs. supersymmetric black holes

In section 6.1 four D3-branes of Type IIB string theory wrapped the (469), (479), (569), (578) cycles of a six-torus and intersected over a string. However, in the string literature one finds *D-brane intersection rules* which specify how N branes can intersect over one another and the fraction ν of supersymmetry that they preserve. Up to $N = 4$ the results are given by [192]

$$\begin{array}{ccccccccc}
 & & & & N=4,\nu=1/8 & & & & \\
 & & N=3,\nu=1/8 & & & & N=3,\nu=1/8 & & \\
N=2,\nu=1/4 & & & & N=2,\nu=1/4 & & & & N=2,\nu=1/4 \\
 & & N=1,\nu=1/2 & & & & N=1,\nu=1/2 & & \\
 & & & & N=0,\nu=1 & & & &
\end{array} \tag{9.1}$$

Our case is clearly $N = 4, \nu = 1/8$ and corresponds to a three qubit GHZ state, however one is naturally led to consider the $N < 4$ cases. Using our dictionary, we see that string theory predicts the three qubit entanglement classification (9.2), in complete agreement with the standard results of quantum information theory.

$$\begin{array}{ccccc}
 & & \text{GHZ} & & \\
 & \text{W} & & \text{W} & \\
A\text{-}BC & & A\text{-}BC & & A\text{-}BC \\
 & A\text{-}B\text{-}C & & A\text{-}B\text{-}C & \\
 & & \text{Null} & &
\end{array} \tag{9.2}$$

Allowing for different p-branes wrapping tori of different dimensions, we are led to speculate that the classifications of intersecting branes in string theory in D-dimensions are related to the entanglement classifications of qubits, qutrits and more generally qudits, with our three qubit case of section 6.1 being just one particular example. While still under development, this perspective yields positive results for the well-documented cases of 2×2, 2×3, 3×3, $2 \times 2 \times 3$ and $2 \times 2 \times 4$, in which the brane intersection rules match the classification of the respective quantum information systems. As we have seen, the $2 \times 2 \times 2 \times 2$ four qubit system has a less well understood or, at any rate, less well accepted entanglement structure [124, 163, 167] leading one to consider the brane wrapping perspective in this scenario. For example, the (un-normalised) W state in four dimensions is given in Table 9.1 by

$$|W\rangle = |0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle. \tag{9.3}$$

Using the same dictionary as in Table 6.1 we find that the intersection of four D4 branes

Table 9.1: Four D4 branes intersecting in $D = 2$ with 1/16 conserved supersymmetry looks like a 4 qubit W state $|0000\rangle + |1100\rangle + |1010\rangle + |1001\rangle$ using the same dictionary as in Table 6.1.

0	1	2	3	4	5	6	7	8	9	brane	State
x	o	o	x	x	o	x	o	x	o	D4	$\|1000\rangle$
x	o	x	o	o	x	x	o	x	o	D4	$\|0100\rangle$
x	o	x	o	x	o	o	x	x	o	D4	$\|0010\rangle$
x	o	x	o	x	o	x	o	o	x	D4	$\|0001\rangle$

is supersymmetric. Similarly for the 4 qubit GHZ state $|0000\rangle + |1111\rangle$ in Table 9.2. Our recent successes at classifying four qubit entanglement reduce the urgency of this

Table 9.2: Two D4 branes intersecting in $D = 2$ with 1/2 conserved supersymmetry looks like a 4 qubit GHZ $|0000\rangle + |1111\rangle$ state using the same dictionary as in Table 6.1.

0	1	2	3	4	5	6	7	8	9	brane	State
x	o	x	o	x	o	x	o	x	o	D4	$\|0000\rangle$
x	o	o	x	o	x	o	x	o	x	D4	$\|1111\rangle$

particular approach, and indeed supersedes it in that we need not trouble ourselves with rebits, nevertheless our goal remains to use the allowed wrapping configurations and D-brane intersection rules to predict new qubit entanglement classifications. We also have in mind to produce a more detailed four qubit analysis that incorporates the covariant classification approach and provides the details of the computation of the cosets.

9.2.3 Freudenthal/Jordan ranks and duals

The microscopic stringy interpretation of F- and J-duality is not clear, in part due to the dualities being defined on black hole charge vectors rather that the component fields of the lowest order action. Furthermore we are as yet unaware of the broader significance of the spaces of black holes admitting F- and J-duals.

We have seen that the 4D/5D lift of [193] relates 4D and 5D black holes through the exceptional Jordan algebras and the Freudenthal triple systems, but these black holes are themselves related to three qubits and two qutrits respectively. What then are the implications for the relationship between qubits and qutrits? In addition, the 4D/5D lift involves nonvanishing angular momentum and the quantum information analogue of rotating black holes remains unclear. In $D \geq 6$ dimensions there are no black holes with nonvanishing entropy, but there are black strings and other intersecting brane configurations with entropies given by U-duality invariants, all awaiting QI interpretations.

We are also presently engaged in the extension of the formalism of Jordan/Freudenthal ranks and canonical forms to the case of $\mathcal{N} = 2$ black hole charge orbits in 4 and 5D.

9.2.4 Outstanding issues

There are a number of research avenues broached by other authors contributing to the correspondence which we have not yet approached: distillation protocols, error correction codes, twistors and the geometry of entanglement to name a few. Within our own body of work there also remain a couple of loose ends.

We confirmed the form of the $\mathrm{E}_{7(7)} \supset \mathrm{SL}_2^7$ invariant written in terms of state vector amplitudes, but a longstanding issue has been its monotonicity or lack thereof. Our early investigations suggested the latter, meaning that it may not in fact be a valid entanglement measure, but this needs to be thoroughly checked and the outcome published.

Cayley's hyperdeterminant has already proven itself to be a bit of a Swiss-army-knife invariant, but we have neglected one of it's other important physical applications. Namely that it provides the Lagrangian of the Nambu-Goto string in spacetime signature $(2, 2)$ [194]. It remains to be seen whether this is related to it's role in the black hole/qubit correspondence.

Despite all the progress already made, the fundamental basis of the correspondence - why black holes should be related to qubits - remains murky. Keeping an open mind, one must accept the possibility that they may not in fact be dual descriptions of the same physical phenomenon. Nevertheless the research conducted has been genuinely cross-disciplinary, with insights in one field finding ready application in the other. For this reason the work has enduring value even in the absence of a more physically substantive duality. We need not be so pessimistic however, as the continually arising correlations between entanglement classification and the supersymmetric classification of black holes hint that the answer may lie with a more developed version of the brane wrapping perspective. The story for black holes and qubits, it seems, is far from over.

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Appendix A

Cubic Jordan algebras and Freudenthal triple systems

The quantised charges A of five dimensional black strings and quantised charges B of five dimensional black holes are assigned to elements of an *integral Jordan algebra* $\mathfrak{J}$ whose cubic norm N determines the lowest order entropy

$$S_{5(\text{black string})} = 2\pi\sqrt{N(A)}, \qquad S_{5(\text{black holes})} = 2\pi\sqrt{N(B)}, \tag{A.1}$$

and whose *reduced structure group* $\mathrm{Str}_0(\mathfrak{J})$ is the U-duality group. Integral cubic Jordan algebras are defined in section A.1.1 and the reduced structure group is described in section A.1.2. We proceed to define the Jordan duality operation in section A.1.3.

Similarly the quantised charges x of black holes of the four dimensional supergravities arising from string and M-theory are assigned to elements of an *integral Freudenthal triple system* (FTS) $\mathfrak{M}(\mathfrak{J})$ (where $\mathfrak{J}$ is the integral cubic Jordan algebra underlying the corresponding 5D supergravity [80–82, 169, 170, 195, 196]) whose quartic form $\Delta(x)$ determines the lowest order entropy

$$S_4 = \pi\sqrt{|\Delta(x)|}, \tag{A.2}$$

and whose *automorphism group* $\mathrm{Aut}(\mathfrak{M}(\mathfrak{J}))$ is the U-duality group. FTSs are defined in section A.2.1 and the automorphism group is described in section A.2.2. We proceed to define the Freudenthal duality operation in section A.2.3.

Examples of integral Jordan algebras and FTSs with the corresponding U-duality groups $\mathrm{Str}_0(\mathfrak{J})$ and $\mathrm{Aut}\,\mathfrak{M}(\mathfrak{J})$ are given in Table A.1. In particular this includes the cases $\mathcal{N}=2$ STU, $\mathcal{N}=2$ coupled to n vector multiplets, magic $\mathcal{N}=2$, and $\mathcal{N}=8$

[23,34,38,70,73,77,82,85,197,198]. The $\mathcal{N}=4$ heterotic string with $\mathrm{SL}_2(\mathbb{Z})\times\mathrm{SO}_{6,22}(\mathbb{Z})$ U-duality may also be included using $\mathfrak{J}=\mathbb{Z}\oplus Q_{5,21}$ [34,199]. The notation $J_3^{\mathbb{A}}$ denotes sets of 3×3 Hermitian matrices defined over the four division algebras $\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H}$ or $\mathbb{O}$ (or their split signature cousins). The notation $\mathbb{Z}\oplus Q_n$ denotes the infinite sequence of *spin factors* $\mathbb{Z}\oplus Q_n$, where Q_n is an n-dimensional vector space over $\mathbb{Z}$ [138,170,200–202].

Table A.1: The automorphism group $\mathrm{Aut}(\mathfrak{M}(\mathfrak{J}))$ and the dimension of its representation $\dim\mathfrak{M}(\mathfrak{J})$ given by the Freudenthal construction defined over the integral cubic Jordan algebra $\mathfrak{J}$ with dimension $\dim\mathfrak{J}$ and reduced structure group $\mathrm{Str}_0(\mathfrak{J})$. The quantised $\mathcal{N}=8$ theories in 5 and 4 dimensions have U-duality groups $\mathrm{E}_{6(6)}(\mathbb{Z})$ and $\mathrm{E}_{7(7)}(\mathbb{Z})$ respectively.

Jordan algebra $\mathfrak{J}$	$\mathrm{Str}_0(\mathfrak{J})$	$\dim\mathfrak{J}$	$\mathrm{Aut}(\mathfrak{M}(\mathfrak{J}))$	$\dim\mathfrak{M}(\mathfrak{J})$
$\mathbb{Z}$	$-$	1	$\mathrm{SL}_2(\mathbb{Z})$	4
$\mathbb{Z}\oplus\mathbb{Z}$	$\mathrm{SO}_{1,1}(\mathbb{Z})$	2	$\mathrm{SL}_2(\mathbb{Z})^2$	6
$\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$	$\mathrm{SO}_{1,1}(\mathbb{Z})\times\mathrm{SO}_{1,1}(\mathbb{Z})$	3	$\mathrm{SL}_2(\mathbb{Z})^3$	8
$\mathbb{Z}\oplus Q_n$	$\mathrm{SO}_{1,1}(\mathbb{Z})\times\mathrm{SO}_{n-1,1}(\mathbb{Z})$	$n+1$	$\mathrm{SL}_2(\mathbb{Z})\times\mathrm{SO}_{2,n}(\mathbb{Z})$	$2n+4$
$J_3^{\mathbb{Z}}$	$\mathrm{SL}_3(\mathbb{Z})$	6	$\mathrm{Sp}_6(\mathbb{Z})$	14
$J_3^{\mathbb{C}}$	$\mathrm{SL}_3(\mathbb{Z})$	9	$\mathrm{SU}_{3,3}(\mathbb{Z})$	20
$J_3^{\mathbb{H}}$	$\mathrm{SU}_6^\star(\mathbb{Z})$	15	$\mathrm{SO}_{12}^\star(\mathbb{Z})$	32
$J_3^{\mathbb{O}}$	$\mathrm{E}_{6(-26)}(\mathbb{Z})$	27	$\mathrm{E}_{7(-25)}(\mathbb{Z})$	56
$J_3^{\mathbb{O}^s}$	$\mathrm{E}_{6(6)}(\mathbb{Z})$	27	$\mathrm{E}_{7(7)}(\mathbb{Z})$	56

The Bekenstein-Hawking entropy and other macroscopic physical quantities are invariant under the continuous U-duality group of the underlying low energy supergravity action, but in the full quantum theory the symmetry is broken to a discrete subgroup due to the Dirac-Schwinger quantisation conditions. As a result, physical quantities in the quantised theory can depend on previously absent discrete invariants. Typically the discrete invariants are given by the gcds of particular dyon charge combinations [40,203–206]. These are described for 5D in section A.1.4 and for 4D in section A.2.4 along with their behaviour under J/F-duality.

A.1 Cubic Jordan algebras and 5D black holes/strings

A.1.1 Cubic Jordan algebras

The bilinear product $\circ$, of a Jordan algebra $\mathfrak{J}$ [200,201,207–209], satisfies

$$X\circ Y=Y\circ X,\qquad X^2\circ(X\circ Y)=X\circ(X^2\circ Y),\qquad \forall\, X,Y\in\mathfrak{J}. \tag{A.3}$$

Of particular interest to us is the class of integral cubic Jordan algebras, which possess a cubic map $N:\mathfrak{J}\to\mathbb{Z}$ homogeneous of degree three

$$N(\lambda X)=\lambda^3 N(X),\quad \forall\,\lambda\in\mathbb{Z},\ X\in\mathfrak{J}. \tag{A.4}$$

With the aid of a base point $\mathbb{1}\in\mathfrak{J}$ satisfying $N(\mathbb{1})=1$ one may determine the properties of $\mathfrak{J}$ from the cubic form through a construction due to Freudenthal, Springer and Tits [202, 210, 211]. First the cubic form is polarised to a trilinear map

$$N(X,Y,Z):=\tfrac{1}{6}\big[N(X+Y+Z)-N(X+Y)-N(X+Z)-N(Y+Z)+N(X)+N(Y)+N(Z)\big]. \tag{A.5}$$

Second the trilinear map defines the following four maps

$$\begin{aligned}
&\text{trace} && \mathrm{Tr}(X):=3N(X,\mathbb{1},\mathbb{1}), && \text{(A.6a)}\\
&\text{spur quadratic map} && S(X):=3N(X,X,\mathbb{1}), && \text{(A.6b)}\\
&\text{spur bilinear map} && S(X,Y):=6N(X,Y,\mathbb{1}), && \text{(A.6c)}\\
&\text{trace bilinear form} && \mathrm{Tr}(X,Y):=\mathrm{Tr}(X)\,\mathrm{Tr}(Y)-S(X,Y). && \text{(A.6d)}
\end{aligned}$$

Third the trace bilinear form uniquely defines the quadratic adjoint map $^{\#}:\mathfrak{J}\to\mathfrak{J}$ and its polarisation

$$\begin{aligned}
\mathrm{Tr}(X^{\#},Y)&:=3N(X,X,Y), && \text{(A.7a)}\\
X\#Y&:=(X+Y)^{\#}-X^{\#}-Y^{\#}. && \text{(A.7b)}
\end{aligned}$$

Finally the Jordan product is defined as

$$X\circ Y:=\tfrac{1}{2}\big(X\#Y+\mathrm{Tr}(X)Y+\mathrm{Tr}(Y)X-S(X,Y)\mathbb{1}\big). \tag{A.8}$$

The result is a cubic Jordan algebra provided the cubic form is *Jordan cubic*, to wit

1. The trace bilinear form (A.6d) is non-degenerate.

2. The quadratic adjoint map (A.7a) satisfies

$$(X^{\#})^{\#}=N(X)X,\qquad \forall X\in\mathfrak{J}. \tag{A.9}$$

For example, in the $J_3^{\mathbb{A}}$ case the Jordan product is $X\circ Y=\frac{1}{2}(XY+YX)$, where XY is just the conventional matrix product. See [201] for a comprehensive account. In all cases, one defines the Jordan triple product as

$$\{X, Y, Z\} := (X \circ Y) \circ Z + X \circ (Y \circ Z) - (X \circ Z) \circ Y. \tag{A.10}$$

In general an integral Jordan algebra is not closed under the Jordan product, but the cubic norm and trace bilinear form are integer valued, which are the crucial properties for our purposes. Furthermore $\mathfrak{J}$ is closed under the (un)polarised quadratic adjoint map as required.

A.1.2 Jordan ranks

The structure group, $\mathrm{Str}(\mathfrak{J})$, is composed of all linear bijections on $\mathfrak{J}$ that leave the cubic norm N invariant up to a fixed scalar factor,

$$N(g(X)) = \lambda N(X), \quad \forall g \in \mathrm{Str}(\mathfrak{J}). \tag{A.11}$$

The reduced structure group $\mathrm{Str}_0(\mathfrak{J})$ leaves the cubic norm invariant and therefore consists of those elements in $\mathrm{Str}(\mathfrak{J})$ for which $\lambda = 1$ [140, 198, 201]. The usual concept of matrix rank may be generalised to cubic Jordan algebras and is invariant under both $\mathrm{Str}(\mathfrak{J})$ and $\mathrm{Str}_0(\mathfrak{J})$ [170, 200]. The ranks are specified by the vanishing or not of three rank polynomials linear, quadratic, and cubic in A (resp. B) as shown in Table A.2. Large BPS black holes and strings correspond to rank 3 with $N(A), N(B) \neq 0$ and small BPS correspond to ranks 1 and 2 with $N(A), N(B) = 0$. In Table A.2 we have listed the fraction of unbroken supersymmetry for the $\mathcal{N} = 8$ case.

Table A.2: Partition of the space $\mathfrak{J}$ into four orbits of $\mathrm{Str}_0(\mathfrak{J})$ or ranks.

Rank	Condition			$\mathcal{N} = 8$ BPS
	A	$A^\#$	$N(A)$	
0	$= 0$	$= 0$	$= 0$	-
1	$\neq 0$	$= 0$	$= 0$	1/2
2	$\neq 0$	$\neq 0$	$= 0$	1/4
3	$\neq 0$	$\neq 0$	$\neq 0$	1/8

Table A.3: Orbit representatives of $(D = 5, \mathcal{N} = 8)$, see Table 2.5 for details. Each orbit is specified by a Jordan algebra element of a given rank.

Rank	Rep	Orbit
0	$\mathrm{diag}(0,0,0)$	$\{0\}$
1	$\mathrm{diag}(1,0,0)$	$\mathrm{E}_{6(6)}/(O(5,5) \ltimes \mathbb{R}^{16})$
2	$\mathrm{diag}(1,1,0)$	$\mathrm{E}_{6(6)}/(O(5,4) \ltimes \mathbb{R}^{16})$
3	$\mathrm{diag}(1,1,k)$	$\mathrm{E}_{6(6)}/F_{4(4)}$

A.1.3 Jordan dual

Given a black string with charges A or black hole with charges B, we define its Jordan dual by

$$A^\star := A^\# N(A)^{-1/3}, \qquad B^\star := B^\# N(B)^{-1/3}. \tag{A.12}$$

J-duality is well defined for large rank 3 strings for which both $A^\#$ and $N(A)$ are nonzero and large rank 3 holes for which both $B^\#$ and $N(B)$ are nonzero. It can be shown [25] that the Jordan dual leaves the cubic form invariant $N(A) = N(A^\star)$ and satisfies $A^{\star\star} = A$. For a valid dual $A^\star$, we require that $N(A)$ is a perfect cube. Despite the non-polynomial nature of the transformation, the J-dual scales linearly in the sense

$$A^\star(nA) = nA^\star(A), \qquad B^\star(nB) = nB^\star(B), \qquad n \in \mathbb{Z}. \tag{A.13}$$

The U-duality integral invariants $\mathrm{Tr}(X, Y)$ and $N(X, Y, Z)$ are not generally invariant under Jordan duality while $\mathrm{Tr}(A^\star, A)$ and $N(A)$, and hence the lowest-order black hole entropy are. However, higher order corrections to the black hole entropy depend on some of the discrete U-duality invariants, to which we now turn.

A.1.4 Discrete U-invariants

J-duality commutes with U-duality in the sense that $A^\star$ transforms contragredient to A. This follows from the property that a linear transformation s belongs to the norm preserving group if and only if

$$s(A)\#s(B) = s'(A\#B) \tag{A.14}$$

where s' is given by

$$\mathrm{Tr}(s(A), s'(B)) = \mathrm{Tr}(A, B) \tag{A.15}$$

and always belongs to the norm preserving group if s itself does [212]. This implies

$$(s(A))^\star = s'(A^\star). \tag{A.16}$$

The gcd of a collection of not all zero integral Jordan algebra elements is defined to be the greatest integer that divides them. By definition gcd is positive. The gcd may be used to define the following set of discrete U-duality invariants [169]:

$$d_1(A) = \gcd(A), \qquad d_2(A) = \gcd(A^\#), \qquad d_3(A) = |N(A)|, \tag{A.17}$$

which are the gcds of the rank polynomials. Clearly $d_3(A)$ is conserved as expected, but this is not necessarily the case for $d_1(A)$ and $d_2(A)$. Nevertheless the product $d_1(A)d_2(A)$ is preserved.

While we require that $N(A)$ is a perfect cube for a valid J-dual this is not a sufficient condition because we further require that

$$d_3(A) = \left[\frac{d_2(A)}{d_1(A^\star)}\right]^3 = \left[\frac{d_2(A^\star)}{d_1(A^)}\right]^3 = d_3(A^\star). \tag{A.18}$$

For $\mathfrak{J} = J_3^{\mathbb{A}}$ with $\mathbb{A} \in \{\mathbb{C}^s, \mathbb{H}^s, \mathbb{O}^s\}$ the orbit representatives of all black strings (holes) have been fully classified [169] and it can be shown that the three d_i uniquely determine the representative.

A.2 Freudenthal triple systems and 4D black holes

A.2.1 Freudenthal triple systems

The charges of various 4D supersymmetric black holes, e.g.

- $\mathcal{N} = 2$ STU model
- $\mathcal{N} = 2$ magic supergravities
- $\mathcal{N} = 4$ heterotic
- $\mathcal{N} = 8$

are assigned to elements of a Freudenthal triple system (FTS). Given an integral cubic Jordan algebra $\mathfrak{J}$, one is able to construct an integral FTS by defining the vector space $\mathfrak{M}(\mathfrak{J})$,

$$\mathfrak{M}(\mathfrak{J}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathfrak{J} \oplus \mathfrak{J}. \tag{A.19}$$

with elements $x \in \mathfrak{M}(\mathfrak{J})$ written

$$x = (\alpha, \beta, A, B), \quad \alpha, \beta \in \mathbb{Z}, \quad A, B \in \mathfrak{J}. \tag{A.20}$$

In the STU case the charges are assigned

$$\mathbf{8} \to (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{3}) \tag{A.21}$$

where the Jordan elements are 3-tuples. For the $\mathcal{N} = 8$ generalisation we have instead

$$\mathbf{56} \to (\mathbf{1}, \mathbf{1}, \mathbf{27}, \mathbf{27'}), \tag{A.22}$$

specifically

$$(-q_0, p^0, J_3(P), J_3(Q)) \tag{A.23}$$

where

$$\mathbf{27} \to \mathbf{1} + \mathbf{1} + \mathbf{1} + \mathbf{8_s} + \mathbf{8_c} + \mathbf{8_v} \tag{A.24}$$

so that **27** are assigned to a 3×3 Hermitian matrix over the split octonions.

$$J_3(P) = \begin{pmatrix} p^1 & P^v & \overline{P^s} \\ \overline{P^v} & p^2 & P^c \\ P^s & \overline{P^c} & p^3 \end{pmatrix}. \tag{A.25}$$

For convenience we identify the quadratic quantity

$$\kappa(x) := \tfrac{1}{2}(\alpha\beta - \operatorname{Tr}(A, B)). \tag{A.26}$$

The FTS comes equipped with a non-degenerate bilinear antisymmetric quadratic form, a quartic form and a trilinear triple product [170, 197, 198, 213, 214]:

1. Quadratic form $\{x, y\}$: $\mathfrak{M}(\mathfrak{J}) \times \mathfrak{M}(\mathfrak{J}) \to \mathbb{Z}$

$$\begin{aligned} \{x, y\} &= \alpha\delta - \beta\gamma + \operatorname{Tr}(A, D) - \operatorname{Tr}(B, C), \\ \text{where} \quad x &= (\alpha, \beta, A, B), \qquad y = (\gamma, \delta, C, D). \end{aligned} \tag{A.27a}$$

2. Quartic form $\Delta : \mathfrak{M}(\mathfrak{J}) \to \mathbb{Z}$

$$\Delta(x) = -4[\kappa(x)^2 + (\alpha N(A) + \beta N(B) - \operatorname{Tr}(A^\#, B^\#))]. \tag{A.27b}$$

The quartic norm $\Delta(x)$ is either $4k$ or $4k+1$ for some $k \in \mathbb{Z}$.

3. Triple product $T : \mathfrak{M}(\mathfrak{J}) \times \mathfrak{M}(\mathfrak{J}) \times \mathfrak{M}(\mathfrak{J}) \to \mathfrak{M}(\mathfrak{J})$ which is uniquely defined by

$$\{T(w,x,y),z\} = 2\Delta(w,x,y,z), \tag{A.27c}$$

where $\Delta(w,x,y,z)$ is the fully polarised quartic form such that $\Delta(x,x,x,x) = \Delta(x)$.

Note that all the necessary definitions, such as the cubic and trace bilinear forms, are inherited from the underlying Jordan algebra $\mathfrak{J}$. When $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and $\mathfrak{J} = J_3^{\mathbb{A}}$, the group $\mathrm{Aut}(\mathfrak{M}(\mathfrak{J}))$ is generated by the following three maps [198]:

$$\phi(C) : \begin{pmatrix} \beta \\ A \\ B \\ \alpha \end{pmatrix} \mapsto \begin{pmatrix} \beta \\ A + \beta C \\ B + A \times C + \beta C^\sharp \\ \alpha + (B,C) + (A, C^\sharp) + \beta N(C) \end{pmatrix}, \tag{A.28a}$$

$$\psi(D) : \begin{pmatrix} \beta \\ A \\ B \\ \alpha \end{pmatrix} \mapsto \begin{pmatrix} \beta + (A,D) + (B,C^\sharp) + \alpha N(C) \\ A + B \times D + \alpha D^\sharp \\ B + \alpha D \\ \alpha \end{pmatrix}, \tag{A.28b}$$

$$T(s) : \begin{pmatrix} \beta \\ A \\ B \\ \alpha \end{pmatrix} \mapsto \begin{pmatrix} \lambda\beta \\ s(A) \\ s'^{-1}(B) \\ \lambda^{-1}\alpha \end{pmatrix} \tag{A.28c}$$

where $s \in \mathrm{Str}(\mathfrak{J})$ and s^* is its adjoint defined with respect to the trace bilinear form, $\mathrm{Tr}(X, s(Y)) = \mathrm{Tr}(s^*(X), Y)$.

A.2.2 FTS ranks

The automorphism group $\mathrm{Aut}(\mathfrak{M}(\mathfrak{J}))$ is composed of all invertible $\mathbb{Z}$-linear transformations that leave both the antisymmetric bilinear form $\{x,y\}$ and the quartic form $\Delta(x)$ invariant [198]. The conventional concept of matrix rank may be generalised to Freudenthal triple systems in a natural and $\mathrm{Aut}(\mathfrak{M}(\mathfrak{J}))$ invariant manner. The rank of an arbitrary element $x \in \mathfrak{M}(\mathfrak{J})$ is uniquely defined by the vanishing or not of four rank polynomials linear, quadratic (in essence), cubic, and quartic in x as shown in Table A.4 [170, 214]. Large BPS and large non-BPS black holes correspond to rank

4 with $\Delta(x) > 0$ and $\Delta(x) < 0$, respectively. Small BPS black holes correspond to ranks 1, 2 and 3 with $\Delta(x) = 0$. In Table A.4 we have listed the fraction of unbroken supersymmetry for the $\mathcal{N} = 8$ case.

Table A.4: Partition of the space $\mathfrak{M}(\mathfrak{J})$ into five orbits of $\mathrm{Aut}(\mathfrak{M}(\mathfrak{J}))$ or ranks.

Rank	Condition				$\mathcal{N} = 8$ BPS
	x	$3T(x,x,y) + \{x,y\}x$	$T(x,x,x)$	$\Delta(x)$	
0	$= 0$	$= 0\ \forall y$	$= 0$	$= 0$	-
1	$\neq 0$	$= 0\ \forall y$	$= 0$	$= 0$	1/2
2	$\neq 0$	$\neq 0$	$= 0$	$= 0$	1/4
3	$\neq 0$	$\neq 0$	$\neq 0$	$= 0$	1/8
4	$\neq 0$	$\neq 0$	$\neq 0$	> 0	1/8
4	$\neq 0$	$\neq 0$	$\neq 0$	< 0	0

Noting that the STU model may be consistently embedded in the $\mathcal{N} = 8$ theory one might anticipate an FTS characterisation of three qubits. Indeed, setting $\mathbb{F} = \mathbb{C}$ and $\mathfrak{J} = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ we may model the 3-qubit system using:

$$|\Psi\rangle = a_{ABC}|ABC\rangle \quad \leftrightarrow \quad \Psi = \begin{pmatrix} a_{111} & (a_{001}, a_{010}, a_{100}) \\ (a_{110}, a_{101}, a_{011}) & a_{000} \end{pmatrix}. \tag{A.29}$$

In this case the automorphism group is given by $\mathrm{SL}_{2,A}(\mathbb{C}) \times \mathrm{SL}_{2,B}(\mathbb{C}) \times \mathrm{SL}_{2,C}(\mathbb{C})$ together with a discrete triality that interchanges A, B and C. The quartic norm (A.27b) coincides with Cayley's hyperdeterminant (3.18),

$$\Delta(\Psi) = -\operatorname{Det} a_{ABC}, \tag{A.30}$$

and the entanglement of a state is determined precisely by its FTS rank, rank 1 separable through to rank 4 GHZ, as described in Table 5.3. A slightly modified (qubit-adapted) set of rank conditions is presented in Table A.5. here, the γ^i refer to the B^i covariants of section 3.3.1 and the trilinear form T can be identified as the covariant C_{111}. Embryonic attempts to generalise this classification scheme are described in section 7.1.

A.2.3 Freudenthal dual

Given a black hole with charges x, we define its Freudenthal dual by

$$\tilde{x} := T(x)|\Delta(x)|^{-1/2}, \tag{A.31}$$

Table A.5: FTS rank conditions.

Class	Rank	FTS rank condition	
		vanishing	non-vanishing
Null	0	Ψ	–
A-B-C	1	$3T(\Psi,\Psi,\Phi)+\{\Psi,\Phi\}\Psi$	Ψ
A-BC	2a	$T(\Psi,\Psi,\Psi)$	γ^A
B-CA	2b	$T(\Psi,\Psi,\Psi)$	γ^B
C-AB	2c	$T(\Psi,\Psi,\Psi)$	γ^C
W	3	$\Delta(\Psi)$	$T(\Psi,\Psi,\Psi)$
GHZ	4	–	$\Delta(\Psi)$

where $T(x) \equiv T(x,x,x) \in \mathfrak{M}(\mathfrak{J})$. F-duality is well defined for large rank 4 black holes for which both $T(x)$ and $\Delta(x)$ are nonzero. It can be shown that the Freudenthal dual leaves the quartic form invariant $\Delta(x) = \Delta(\tilde{x})$ and satisfies $\tilde{\tilde{x}} = -x$. For a valid $\tilde{x}$ we require that $\Delta(x)$is a perfect square. Despite the non-polynomial nature of the transformation, the F-dual scales linearly in the sense

$$\tilde{x}(nx) = n\tilde{x}(x), \qquad n \in \mathbb{Z}. \tag{A.32}$$

The U-duality integral invariants $\{x,y\}$ and $\Delta(x,y,z,w)$ are not generally invariant under Freudenthal duality while $\{\tilde{x},x\}$, $\Delta(x)$, and hence the lowest-order black hole entropy, are invariant. However, higher order corrections to the black hole entropy depend on some of the discrete U-duality invariants.

For $\mathfrak{J} = J_3^{\mathbb{A}}$ with $\mathbb{A} \in \{\mathbb{C}^s, \mathbb{H}^s, \mathbb{O}^s\}$, that is, when $\mathbb{A}$ is one of the three integral split composition algebras (i.e. including the all-important $\mathcal{N}=8$ case) we can rewrite the charge vector of any black hole in the diagonally reduced canonical form

$$x = \alpha(1, j, k(1,l,lm), \vec{0}), \tag{A.33}$$

where $\alpha > 0, k, l \geq 0$, and $\alpha, j, k, l, m \in \mathbb{Z}$. The quartic norm of this element is

$$\Delta(x) = -(j^2 + 4k^3l^2m)\alpha^4. \tag{A.34}$$

The F-dual is then explicitly given by

$$\tilde{x} = ((-n_1, n_1 j + n_2 klm), n_1 k(1,l,lm), n_2(lm, m, 1)), \tag{A.35}$$

where we have ensured an admissible dual by imposing the constraints

$$|j^2 + 4k^3l^2m|^{1/2} = n_0 \in \mathbb{N}, \tag{A.36a}$$

$$\alpha j/n_0 = n_1 \in \mathbb{Z}, \tag{A.36b}$$

$$2k^2 l\alpha/n_0 = n_2 \in \mathbb{N}_0, \tag{A.36c}$$

where $\operatorname{sgn} n_1 = \operatorname{sgn} j$. It is then possible to classify example F-duals as shown in Table A.6.

Table A.6: Conditions on parameters for several example FTSs, where $p \in \mathbb{Z}$. Parameters n_0, n_1 and n_2 are fixed by (A.36). Note that we still require $\alpha > 0$, $k, l \geq 0$ and $n_0 \neq 0$ in all cases.

Case	j	k	l	m	$\operatorname{sgn}\Delta$	n_0	n_1	n_2
1	0	$p^2\|m\|$	>0	$\neq 0$	$-\operatorname{sgn} m$	$2\|p^3\|m^2 l$	0	$\|p\|\alpha$
2.1	$\neq 0$	0	≥ 0	$\in \mathbb{Z}$	$-$	$\|j\|$	$\alpha \operatorname{sgn} j$	0
2.2	$\neq 0$	>0	0	$\in \mathbb{Z}$	$-$	$\|j\|$	$\alpha \operatorname{sgn} j$	0
2.3	$\neq 0$	>0	>0	0	$-$	$\|j\|$	$\alpha \operatorname{sgn} j$	$2k^2 l\alpha/\|j\|$
3.1.1	$2p$	1	1	$-(p^2 \pm 1)$	$\pm$	2	$\|p\|\alpha$	α
3.1.2	$2lr$	1	>0	$-(r^2 \pm q)$	$\pm$	$2l\|q\|$	nr	n
3.1.3	$2lr$	>0	>0	$4q(q \pm r)/k^3$	$-$	$2l\|2q \pm r\|$	nr	nk^2
3.2	$2p+1$	1	1	$-p(p+1)$	$-$	1	$(2p+1)\alpha$	2α

A.2.4 Discrete U-invariants

We make the important observation that since

$$T(\sigma(x), \sigma(y), \sigma(z)) = \sigma(T(x, y, z)), \qquad \forall \sigma \in \operatorname{Aut}\mathfrak{M}(\mathfrak{J}), \tag{A.37}$$

F-duality commutes with U-duality

$$\widetilde{\sigma(x)} = \sigma(\tilde{x}). \tag{A.38}$$

The gcd of a collection of not all zero integral FTS elements is defined to be the greatest integer that divides them. By definition gcd is positive. The gcd may be used to define the following[1] set of discrete U-duality invariants [40, 170]:

$$\begin{aligned} &d_1(x) = \gcd(x), && d_3(x) = \gcd(T(x,x,x)),\\ &d_2(x) = \gcd(3\,T(x,x,y) + \{x,y\}\,x) \quad \forall y, && d_4(x) = |\Delta(x)|,\\ &d_2'(x) = \gcd(B^\# - \alpha A,\ A^\# - \beta B,\ 2\kappa(x)C + 2\{A,B,C\}) \quad \forall C, && d_4'(x) = \gcd(x \wedge T(x)), \end{aligned} \tag{A.39}$$

where $\wedge$ denotes the antisymmetric tensor product. As in the 5D case, the unprimed invariants are gcds of the rank polynomials. Clearly $d_4(x)$ is conserved as expected and it can be shown [25] that $d_4'(x), d_2(x)$, and $d_2'(x)$ are also invariant, but this is not necessarily the case for $d_1(x)$ and $d_3(x)$. Nevertheless the product $d_1(x)d_3(x)$ is preserved.

Typically, the literature on exact 4D black hole degeneracies [32, 34, 40, 135, 203–206, 215–222] deals only with *primitive* black holes $d_1(x) = 1$. We are not required to impose this condition and generically do not do so. More generally a quantity is termed primitive if it has unit gcd. A related simplifying concept is *projectivity*, wherein a charge vector for the cases $\mathfrak{J} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, J_3^{\mathbb{C}^s}, J_3^{\mathbb{H}^s}, J_3^{\mathbb{O}^s}$ is projective if the components of it's quadratic rank tensor are primitive.

While we require that $|\Delta(x)|$ is a perfect square for a valid F-dual this is not a sufficient condition because we further require that

$$d_4(x) = \left[\frac{d_3(x)}{d_1(\tilde{x})}\right]^2 = \left[\frac{d_3(\tilde{x})}{d_1(x)}\right]^2 = d_4(\tilde{x}). \tag{A.40}$$

Unlike in 5D the invariants (A.39) are insufficient to uniquely determine the orbit representatives for the $\mathfrak{J} = J_3^{\mathbb{A}}$ with $\mathbb{A} \in \{\mathbb{C}^s, \mathbb{H}^s, \mathbb{O}^s\}$ cases.

A.3 4D/5D lift

A subset of 5D black holes admit a Jordan dual $A^\star$ preserving some, but not all, discrete U-invariants. Similarly, a subset of 4D black holes admit a Freudenthal dual $\tilde{x}$ preserving some, but not all, discrete U-invariants. In both the 4D and 5D cases, if the discrete invariant d_1 is preserved by the J/F-duality map, then all the listed discrete invariants are preserved. When d_1 isn't conserved the J/F-dual is not U-related to the original charge vector. In the simpler 5D case the preservation of d_1 ensures that $A^\star$ is U-related to A, but in 4D the analogous conclusion only holds in the projective case. For non-projective 4D black holes the situation is complicated by the absence of a complete orbit

[1] In the $\mathcal{N} = 8$ case Sen [40] denotes $d_2'(x)$ by ψ, and $d_4'(x)$ by χ.

classification and uncertainty regarding what invariants are relevant to the higher-order corrections. This situation is summarised in Table A.7. It would be interesting if in the

Table A.7: Are F or J duals related by U-duality?

Duality	d_1 conserved?		U-dual?
F	Yes	Projective	Yes
		Non-projective	?
	No		No
J	Yes		Yes
	No		No

non-projective case there were configurations with the same precision entropy that are F-related but not U-related.

From a physical standpoint F-duality and J-duality are defined on charge vectors rather than component fields of the lowest order action so their microscopic stringy interpretation remains unclear, but we remark that two black holes related by F-duality in 4D are related by J-duality when lifted to 5D. The 4D/5D lift [193] relates the entropy of non-rotating 4D black holes to the entropy of rotating 5D black holes and it can be shown [25] that the lift of the F-dual is related to the J-dual of the lift thus:

$$\begin{array}{ccc} \text{4D black hole } x & \xrightarrow{\text{4D/5D lift}} & \text{5D black string } A \sim \tilde{B}^\star \\ {\scriptstyle \text{Freudenthal dual}}\big\downarrow & & \big\downarrow{\scriptstyle \text{Jordan dual}} \\ \text{dual 4D black hole } \tilde{x} & \xrightarrow[\text{4D/5D lift}]{} & \text{dual 5D black hole } \tilde{B} \sim A^\star \end{array} . \tag{A.41}$$

A.4 Integral U-duality orbits

We recently leveraged the integral Jordan/Freudenthal technology to address the question of whether two a priori distinct extremal black p-brane solutions are actually U-duality related in the context of $\mathcal{N} = 8$ supergravity in four, five and six dimensions. The answer has been known for some time in the classical supergravity limit, but Dirac-Zwanziger-Schwinger charge quantisation breaks the U-duality group to a discrete subgroup, making the matter of U-duality orbits in the full quantum theory rather more subtle.

Not only is this an interesting mathematical question [170, 223], it is also of physical significance, with implications for a number of topics including the stringy origins of microscopic black hole entropy [206, 216, 224]. Moreover, following a conjecture for the finiteness of $D = 4, \mathcal{N} = 8$ supergravity [225], it has recently been observed that some of the orbits of $E_{7(7)}(\mathbb{Z})$ should play an important role in counting microstates of this theory [226], even if it may differ from its superstring or M-theory completion [227].

We were able to clarify the state of knowledge at the time, presenting the cascade of relationships between orbits (and the associated moduli spaces) as one descends from six to four dimensions in the continuous case. We were able to fill in a few gaps in the literature by adapting the work of Krutelevich [169, 170, 195, 196] in the discrete case.

The charge vector of the dyonic black string in $D = 6$ is $SO_{5,5}(\mathbb{Z})$ related to a two-charge reduced canonical form uniquely specified by a set of two arithmetic U-duality invariants. Similarly, the black hole (string) charge vectors in $D = 5$ are $E_{6(6)}(\mathbb{Z})$ equivalent to a three-charge canonical form, again uniquely fixed by a set of three arithmetic U-duality invariants. However, the situation in four dimensions is, perhaps predictably, less clear. While black holes preserving more than 1/8 of the supersymmetries may be fully classified by known arithmetic $E_{7(7)}(\mathbb{Z})$ invariants, 1/8-BPS and non-BPS black holes yield increasingly subtle orbit structures, which remain to be properly understood. However, for the very special subclass of *projective* black holes a complete classification is known. All projective black holes are $E_{7(7)}(\mathbb{Z})$ related to a four or five charge canonical form determined uniquely by Cartan's quartic $E_{7(7)}(\mathbb{R})$ invariant. Moreover, $E_{7(7)}(\mathbb{Z})$ acts transitively on the charge vectors of black holes with a given leading-order entropy.

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