ALGEBRAS OF THE SU(n) VECTOR INVARIANTS AND SOME OF THEIR APPLICATIONS

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ABSTRACT: We examine new algebraic structures which are generated by the SU(n) vector invariants consisted of oscillator creation and annihilation operators. For analyzing bosonic oscillator systems with internal SU(n) symmetry we introduce both infinite-dimensional Lie algebras and nonstandard polynomial deformations and mutations of finite-dimensional oscillator Lie algebras. A spectral analysis of the Fock spaces of initial oscillators is given with respect to the SU(n) invariant algebras under consideration. Some physical applications in composite models of many-body systems are pointed out.

1. Introduction.

The symmetry approach based on the use of mathematical formalism of represantation theory of Lie groups and Lie algebras is widely and successfully used in quantum theory of many-body systems (see, e.g., [1-6] and references therein). Specifically, analysis of many-body problems within the second quantization method introduces in a natural way a symmetry formalism associated with oscillators of bosonic and fermionic types: represantation theory of oscillator Lie algebras and superalgebras in the Fock spaces [1,4-8].

Such an approach is especially fruitful in examining composite models with an internal symmetry since it allows to display some hidden symmetries and other peculiarities of systems under consideration [8-11]. Besides, within this analysis we obtain some new algebraic structures which differ from usual Lie algebras and and groups and repreesent their specific deformations and mutations (cf.[12-15]).

Indeed, let us consider many-body quantum oscillator systems (of bosonic or fermionic types) which are associated with the creation and annihilation operators x_i^{α} and $x_i^{-\alpha}=(x_i^{\alpha})^{+}$, respectively ($\alpha=1,2,\ldots,n$; $i=1,2,\ldots,m<\infty$, the superscript "+" denotes the Hermitian conjugation). Here the superscript "α" labels "internal" components of one-particle states that transform in a accordance with the vector (fundamental) irreducible representation (irrep) $D^1(G)$ of a classical group G:

$$x_i^\alpha \overset{G}{\to} u^{\alpha\beta}x_i^\beta,\ \bar{x}_i^\alpha \overset{G}{\to} (u^{\alpha\beta}x_i^\beta)^+,\quad \|u^{\alpha\beta}\| \in D^1(G), \tag{1.1}$$

where from here on the summation is implied over repeated Greek superscripts. The operator $x_i^\alpha, \bar{x}_j^\beta$ satisfy the standard commutation relations (CR)

$$[x_i^\alpha, x_j^\beta]_{\sigma(\lambda)} = x_i^\alpha x_j^\beta + \lambda x_j^\beta x_i^\alpha = 0 = [\bar{x}_i^\alpha, \bar{x}_j^\beta]_{\sigma(\lambda)}, \tag{1.2}$$

$$[\bar{x}_i^\alpha, x_j^\beta]_{\sigma(\lambda)} = \delta^{\alpha\beta}\delta_{ij},\quad \sigma(\lambda)=\mathrm{sgn}\lambda,$$

where $\lambda=-1$ and 1 for bosonic and fermionic systems, respectively. The Hilbert space for these systems are the Fock spaces L_F spanned by the basic vectors

$$|\{n_i^\alpha\}\rangle = N(\{n_i^\alpha\}) \underset{\{\alpha\}}{G} (x_1^{\alpha_1})^{n_1^{\alpha_1}} (x_2^{\alpha_2})^{n_2^{\alpha_2}} \ldots (x_m^{\alpha_m})^{n_m^{\alpha_m}} |0\rangle, \tag{1.3}$$

where $|0\rangle$ is the vacuum vector: $\bar{x}_i^\alpha|0\rangle=0\ \ \alpha, i$, N is a normalization constant; the range of the exponents $\{n_i^\alpha\}$ depends on the type of the oscillator statistics. All physical operators including Hamiltonian H are polynomials in variables x_i^α, $\bar{x}_j^\beta$, e.g.

$$H = \sum_{i,j} \omega_{ij}^{\alpha\beta} x_i^\alpha \bar{x}_j^\beta + \sum_i (c_i^\alpha x_i^\alpha + c_i^{\alpha *} \bar{x}_i^\alpha) + \text{higher powers}, \tag{1.4}$$

where the asterisk * denotes the complex conjugation.

Now we suppose that Hamiltonian H is invariant with respect to the action (1.1) of the "internal" symmetry group G. Then, according to the vector invariant theory [16], H depends polynomially only on some elementary G-invariants $I_r(\{x_i^\alpha, \bar{x}_j^\beta\})$ constructed in terms of G-vectors $x_i=(x_i^\alpha)$ and $\bar{x}_i=(\bar{x}_i^\alpha)$. Further, this G-invariance of H provokes a possibility of picking out the G-invariant subspaces in L_F that one may interpret as a existence of kinematically coupled ("confined") in internal variable subsystems with the G-invariant dynamics. In order to examine such composite subsystems within the general symmetry approach [3,4] we need in constructing C^*-algebras [17] of the G-invariant observables $k_m(G)$ and the G-invariant dynamic symmetry algebras $k_m^{(\lambda)}(G)$ in terms of

$\{I_r(\{x_i^{\alpha},\bar{x}_j^{\beta}\})\}$ as well as studying representations of these algebras in the spaces L_F.

Efficient tools for solving these problems are the vector invariant theory [16] and the conception of complementary groups and algebras [10,18]. Specifically, the complementarity theory allows us to decompose the space L_F into direct sum (with a simple spectrum)

$$L_F=\oplus_{\alpha} L_F^{\alpha}, \tag{1.5}$$

where the subspaces L_F^{α} are irreducible with respect to an action of the algebra $g\oplus k_m^{(\lambda)}(G)$ ("g" being the Lie algebra of G) and furthermore the label "α" determines simultaneously both an irrep $D^{\alpha}(g)$ of g and an dual irrep $D^{\alpha}(k_m^{(\lambda)}(G))$ of $k_m^{(\lambda)}(G)$. From the physical point of view the decomposition (1.5) gives rise to some superselection rules [19] since the single spaces L_F^{α} with different "α" do not "mix" under the time-evolution governed by a Hamiltonian $H \in k_m^{(\lambda)}(G)$. Thus the "internal" symmetry algebra g "induces" the "hidden" dynamic symmetry algebra $k_m^{(\lambda)}(G)$.

This program is simply and fruitfully realized in many-body physics for the groups G=O(n) and Sp(n) since in these cases the basic invariants $I_r(\{\ldots\})$ are bilinear combinations of the operators x_i^{α} and $\bar{x}_j^{\beta}$, and therefore algebras $k_m^{(\lambda)}(G)$ are well-known finite-dimensional Lie-algebras (see, e.g., [18,20-22] and references therein). However for the groups G=SU(n) and SO(n) the situation is more complicated. Specifically, for $n\geq3$ the algebras $k_m^{(-1)}(SU(n))$ and $k_m^{(-1)}(SO(n))$ belong to new classes of infinite-dimensional Lie algebras [8,9,23] associated with some deformations of the universal enveloping algebras of generalized oscillator algebras [11]. A theory of these structures has as yet been developed not quite enough.

The main aim of the present paper is to examine the situation in more detail for the case G=SU(n), $D^1(G)=D(1\dot{0}_{n-2})$, $\lambda=-1$, where $[p_1,\ldots,p_n]$ is the highest weight of the SU(n) irrep $D(p_1,\ldots,p_{n-1})$and dot as a superscript over "a" in "$\dot{a}_r$" means the repetition of "a" r times. The paper is organized as follows. In Sec.2 we investigate some properties of algebras $k_m^{(-1)}(SU(n))=k_m^{(-)}(n)$ and associated structures. In Sec.3 we study their representations in the spaces L_F. Sec.4 is devoted to certain physical applications of the algebras under consideration. In Sec.5 some problems and generalizations are discussed.

2. Bosonic algebras of the SU(n) vector invariants.

So, specialize our further analysis for bosonic systems ($\lambda=-1$). As is well known [8,16] the set of the basic vector invariants $I_r(\{x_i,\bar{x}_j\})$ for the group SU(n) consists of the following constructions:

$$E_{ij}\equiv(x_i\bar{x}_j)=x_i^{\alpha}\bar{x}_j^{\alpha}=(E_{ji})^+,\quad i,j=1,\dots,m, \tag{2.1a}$$

$$X_{i_1\dots i_n}\equiv[x_{i_1}\dots x_{i_n}]=\varepsilon^{\alpha_1\dots\alpha_n}x_{i_1}^{\alpha_1}\dots x_{i_n}^{\alpha_n},\quad i_1<i_2<\dots<i_n, \tag{2.1b}$$

$$\bar{X}_{i_1\dots i_n}\equiv[\bar{x}_{i_1}\dots\bar{x}_{i_n}]=(X_{i_1\dots i_n})^+ \text{ for } \lambda=-1, \tag{2.1c}$$

where $\varepsilon^{\dots}$ is the invariant antisymmetric tensor. The entities (2.1) are generators of the C^*-algebra $k_m(SU(n))\equiv k_m(n)$ of the SU(n)-invariant conceptual observables [17] whose elements are formal power series in the variables (2.1) with their certain ordering. The ordering is dictated by existence some relations between the quantities (2.1).

Specifically, from the second Hilbert theorem of the vector invariant theory [16] for $\lambda=-1$ we have the identities ("syzygies")

$$X_{i_1\dots i_n}X_{j_1\dots j_n}-X_{j_1i_2\dots i_n}X_{i_1j_2\dots j_n}+\dots+ \\ +(-1)^nX_{j_1i_1\dots i_{n-1}}X_{i_nj_2\dots j_n}=0, \tag{2.2a}$$

$$X_{i_1\dots i_n}E_{rj}-X_{ri_2\dots i_n}E_{i_1j}+\dots+(-1)^nX_{ri_1\dots i_{n-1}}E_{i_nj}=0, \tag{2.2b}$$

$$X_{i_1\dots i_n}\bar{X}_{j_1\dots j_n}=P_n(\{E_{ij}\}) \tag{2.2c}$$

and those obtained by the Hermitian conjugation of eqs (2.2). Here $P_n(\{E_{ij}\})$ are polynomials of the n-th order in variables (2.1a) whose explicit form can be found from the well-known algebraic identity [16]

$$\varepsilon^{\alpha_1\dots\alpha_n}\varepsilon_{\beta_1\dots\beta_n}=\det\|\delta_{\beta_j}^{\alpha_i}\|. \tag{2.3}$$

For example, for the case n=2 we have

$$P_2(\{E_{ij}\})=E_{i_1j_1}E_{i_2j_2}-E_{i_1j_2}E_{i_2j_1}-\delta_{i_2j_1}E_{i_1j_2}+\delta_{i_2j_2}E_{i_1j_1}. \tag{2.4}$$

The identies (2.2) allow us to identify the algebras $k_m(n)$ as PI-algebras [24] on the Grassmann manifolds with the Plücker coordinates $X_{i_1 \dots i_n}$ [25,26].

Further, from the CRs (1.2) with $\lambda=-1$ we easily find CRs for the quantities (2.1):

$$[E_{ij}, E_{rs}] \equiv [E_{ij}, E_{rs}]_{-} = \delta_{jr} E_{is} - \delta_{is} E_{rj}, \tag{2.5a}$$

$$[X_{i_1 \dots i_n}, X_{j_1 \dots j_n}] = 0 = [X_{i_1 \dots i_n}, X_{j_1 \dots j_n}], \tag{2.5b}$$

$$[E_{rj}, X_{i_1 \dots i_n}] = \delta_{ji_1} X_{ri_2 \dots i_n} + \delta_{ji_2} X_{i_1 r i_3 \dots i_n} + \dots, \tag{2.5c}$$

$$[E_{rj}, \bar{X}_{i_1 \dots i_n}] = -(\delta_{ri_1} \bar{X}_{ji_2 \dots i_n} + \delta_{ri_2} \bar{X}_{i_1 j i_3 \dots i_n} + \dots), \tag{2.5d}$$

$$[\bar{X}_{i_1 \dots i_n}, X_{j_1 \dots j_n}] = P_n'(\{E_{ij}\}), \tag{2.5e}$$

where $P_n'(\{E_{ij}\})$ are polynomials of the (n-1)-th order in variables E_{ij} which are obtained by using the explicit form of the polynomials $P_n(\{E_{ij}\})$ in eq. (2.2c). Specifically, for the case n=2 we have

$$[\bar{X}_{ij}, X_{kl}] = P_2'(\{E_{ij}\}) = -2(\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl}) - \delta_{il} E_{kj} + \delta_{ik} E_{lj} +$$

$$+ \delta_{lj} E_{ki} - \delta_{jk} E_{li}, \tag{2.5e'}$$

that allow us to close the CRs (2.5) and to introduce the $SO^*(2m)$ Lie algebra structure on the set $I_m(2) \equiv \{X_{ij}, \bar{X}_{kl}, E_{ij}\}$ [8,11].

It is not the case, however, for $n \geq 3$ because repeated CRs of $P_n'(\{E_{ij}\})$ with elements of the set $I_m(n) \equiv \{E_{ij}, X_{i_1 \dots i_n}, \bar{X}_{i_1 \dots i_n} | i=1, \dots m\}$ contain elements of the $k_m(n)$ algebras with higher powers of $E_{ij}, X_{i_1 \dots i_m}, \bar{X}_{j_1 \dots j_m}$, and thus result in infinite-dimensional Lie algebras $k_m^{(-)}(n)$ [11]. But if we restrict ourselves by considering only the initial CRs (2.5) we obtain a new class of Lie-algebraic structures $I_m^{(-)}(n)$ which are some deformations of usual Lie algebras (cf. [12-14]). Indeed, the CRs (2.5a)-(2.5d) are similar to those for elements of usual bosonic oscillator Lie algebras $u(m) \oplus h(m)$ [2-4] while the CR (2.5e) represents a polynomial deformation of the canonical CR for usual bosonic oscillator operators. For example, in the case n=m=3 we have

$$[\bar{X}_{123},X_{123}]=$$
$$=3!+3\sum_{i=1}^{3}E_{ii}+E_{11}E_{22}-E_{21}E_{12}+E_{22}E_{33}-E_{32}E_{23}+E_{33}E_{11}-E_{13}E_{31}. \quad (2.6)$$

Therefore, taking also into account the eqs.(2.2), we may call algebras $I_m^{(-)}(n)$ as Grassmann deformations of bosonic oscillator algebras.

We also note that with each algebra $I_m^{(-)}(n)$ one may associate a Lie algebra $k_m^{(*)}(n)$ if instead of the usual Lie bracket $[\cdot,\cdot]$ we use a new Lie bracket $[\cdot,\cdot]_*\equiv Pr_{Im}[\cdot,\cdot]$ where the symbol Pr_{Im} stands for the projection onto $\mathrm{Span}I_m(n)\cup\{cI\}$ (with "I" being the identity operator). In a sense the algebras $k_m^{(*)}(n)$ may be considered as peculiar ("linearized") mutations [15] of the algebras $k_m^{(-)}(n)$which can be used for a descriptions of a generalized dynamics of the SU(n)-clusters (cf.[11,15]).

Thus, the set (2.1) generates mutually related Lie algebraic structures $k_m^{(-)}(n)$, $I_m^{(-)}(n)$ and $k_m^{(*)}(n)$ of three different types which are connected with Grassmann oscillators. Any of these algebras has two mutually conjugate finite-dimensional Lie subalgebras $b_+^{(m,n)}=\mathrm{Span}\{E_{ij},X_{i_1\ldots i_n}\}$ and $b_-^{(m,n)}=\mathrm{Span}\{E_{ij},\bar{X}_{i_1\ldots i_n}\}$ of the Grassmann oscillator algebras. In addition, the algebras $k_m^{(-)}(n)$ have a characteristic property of nilpotency

$$ad_A^{n+1}B=0,\quad A\in X_-=\mathrm{Span}\{X^-_{i_1\ldots i_n}\},\quad B\in X_+=\mathrm{Span}\{X_{i_1\ldots i_n}\}, \qquad (2.7)$$

$$ad_A B=[A,B],\quad ad_A^n B=ad_A^{n-1}(ad_A B),$$

which is useful for summing up the Baker-Campbell-Hausdorf series [11].

3. Representations of the algebras $k_m^{(-)}(n)$, $I_m^{(-)}(n)$ and $k_m^{(*)}(n)$ in the Fock spaces L_F.

For the physical applications we need in constructing representations of above defined algebras of the SU(n)-vector invariants, particularly, in the spaces L_F. Below we outline a general scheme of the spectral analysis of the spaces L_F with respect to actions of the algebras $su(n)\oplus k_m^{(-)}(n)$ that determine also appropriate irreps of the algebras $I_m^{(-)}(n)$ and $k_m^{(*)}(n)$. For this aim we use the concept of complementary algebras and groups [18,22].

We start from the simplest case n=2 when we have $k_m^{(-)}(2)=k_m^{(*)}(2)$. As is known the algebra $k_m^{(-)}(2)=so^*(2m)$ acts complementarily to the algebra $su(2)_{\sim}sp(2)$ on the space L_F [11,22, 27] and the decomposition (1.5) takes the form

$$L_F = \oplus_{J\geq 0} L(J), \tag{3.1}$$

where the label J specifies both the SU(2) irrep D(2J) and the appropriate $so^*(2m)$ irrep $D^J(so^*(2m))$ [11].

The subspaces L(J) are spanned by the basic vectors $|J;M;\nu\rangle$ where the lables "M" and "ν" distinguish basic vectors within irreps D(2J0 and $D^J(so^*(2m))$ respectively. Thus, the decomposition (3.1) represents a vector bundle [28]. The vectors $|J;M;\nu\rangle$ are linear combinations of the Fock states (1.3). In the papers [10,29] a simple algorithm has been developed for explicit constructing these vectors by using the techniques of the generating invariants and generalized coherent states. We consider such constructions for the case m=2 which, however, elucidiates the situation in the general case..

For m=n=2 the algebra $k_2^{(-)}(2)=so^*(4)$ decomposes into the direct sum $so^*(4)=su_{inv}(2)\oplus su(1,1)$ (with generators x_{12}, $\bar{x}_{12}$, $(1/2)(E_{11}+E_{22})+1$ and E_{12}, E_{21}, $(1/2)(E_{11}-E_{22})$ for the subalgebras su(1,1) and su(2), respectively) and the basis vectors $|J;M;\nu\rangle$ have the form

$$|J;M;\nu\rangle \equiv |J;M;\{T,t\}\rangle = N(J,M,T,t)(e_1\bar{u})^{J-M}(e_2\bar{u})^{J+M}[x_1u]^{J+t}[x_2u]^{J-t}\times \times(x_{12})^{T-J}|0\rangle, \qquad (e_i\bar{u})\equiv e_i^{\alpha}\bar{u}^{\alpha} \tag{3.2}$$

where N is a normalization factor, "u" and "$\bar{u}$" are some intermediate boson operators, $e_i=(\delta_i^{\alpha})$ are the reference vectors. Specifically, for the G=SU(2)-scalar subspace L(0) the vectors (3.2) take the form [11]

$$|T\rangle \equiv |0;0;\{T,0\}\rangle = [T!(T+1)!]^{-1/2}(x_{12})^T|0\rangle, \tag{3.3}$$

which is generated by measns of action of powers of the SU(2)-invariant cluster (Grassmann oscillator) creation operators x_{12} on the vacuum vector $|0\rangle$ by analogy with usual one-mode Fock states (1.3). Comparing the appearence of the vectors (3.2) and (3.3) we observe that they have a similiar structure. The only discrepancy is in that the vectors (3.2) are generated, unlike (3.3) by action of the operators X_{12}^s on the $(2J+1)^2$-dimensional "vacuum" subspace $L_\nu(J)=\mathrm{Span}\{|J;M;\{Jt\}\rangle : J=\mathrm{const}\}$ with characteristic property

$$\bar{X}_{12}|\nu\rangle=0, \quad |\nu\rangle \in L_\nu(J). \tag{3.4}$$

In turn the space $L_\nu(J)$ is generated by means of action the lowering operators $\sum_{i=1}^{2} x_i^2 \bar{x}_i^1 = E^{21}$ and E_{21} of two subalgebras su(2) ⊂ su(2)⊕so*(4) on the highest vector |J;J;{JJ}>.

Now we consider an action of the above algebra so*(4) on the vectors (3.2) using the CRs (2.5). We note that because of the definition of $k_2^{(-)}(2)$=so*(4) its action does not change the values of numbers J,M "controlled" by the "internal" algebra $su_{int}(2)$. Hence each space L(J), J≠0, decomposes into the direct sum

$$L(J)=\oplus_M L(J,M)=\oplus_M \mathrm{Span}\{|JM;\{Tt\}>: J,M=\mathrm{const}\} \qquad (3.5)$$

of the disjoint spaces L(J,M) which are equivalent with respect to the action of the algebra $k_2^{(-)}(2)$=so*(4). Further, the action of the subalgebra $su(2)_{inv}$ ⊂ so*(4) does not change the quantum number T while the operators X_{12} and $\bar{X}_{12}$ of the subalgebra su(1,1) ⊂ so*(4) raise and lower its value by one respectively. Thus each space L(J,M) is a conjuction of the disjoint $su_{inv}(2)$-equivalent subspaces L(J;M;T)=Span{|J;M;{Tt}>: J,M,T≥J =const} which are "intertwined" by the operators $X_{12}, \bar{X}_{12}$. Such the action of the algebra $k_2^{(-)}(2)$=so*(4) on the space L(J,M) resembles that of usual oscillator algebra on the Fock space (cf. [2,4]) and allows us to obtain the space-carrier of the so*(4) irrep D^J(so*(4)) starting from any vector of the "vacuum space" $L_\nu(J)$. Similiarly, one can show that all spaces L(J,M) are the carrier-spaces of equivalent irreps of the algebra $I_2^{(-)}(2)$.

The above analysis provides a sample for realizing spectral analysis of the spaces L_F in the case of arbitrary "m" and "n" [11]. Therefore we outline its logical scheme and point out some peculiarities in the general case.

For arbitrary "m" and "n" the algorithm consist of determining "vacuum spaces" $L_\nu(p_1 \ldots p_{n-1})$ and next constructing their su(m)-equivalent replicas (su(m) ⊂ $k_m^{(-)}(n)$, $I_m^{(-)}(n)$, $k_m^{(-)}(n)$) by means action of operators $X_{i_1 \ldots i_n}$ on the vectors |v> ∈ $L_\nu(p_1,\ldots,p_{n-1})$. In turn the spaces $L_\nu(\ldots)$ are generated by means of action of lowering operators of the algebras $su_{int}(n)=\mathrm{Span}\{E^{ij} \equiv \sum_{r=1}^{m} x_r^i \bar{x}_r^j,\ i \neq j,\ \tilde{E}^{ii}=E^{ii}-E^{i+1,i+1}\}$ and $su_{inv}(m)=\mathrm{Span}\{E_{ij},\ i\neq j,\ \tilde{E}_{ii}=E_{ii}-E_{i+1,i+1}\} \subset k_m^{(-)}(n)$ on the common highest vectors $|p_1 \ldots p_{n-1};max\rangle \equiv |\langle p_i\rangle\rangle$satisfying the following equations

a) $\bar{X}_{i_1\ldots i_n}|<p_i>>=0,$ (3.6a)

b) $\tilde{E}^{ii}|<p_i>>=p_i|<p_i>>=\tilde{E}_{ii}|<p_i>>,\ i=1,\ldots,n-1,$ (3.6b)

c) $E^{ij}|<p_i>>=0=E_{ij}|<p_i>>,\ i<j.$ (3.6c)

As a result we obtain at final step of the algorithm the following specialization of eq.(1.5) [11]:

$$L_F=\oplus_{<p_i>}L(<p_i>)=\oplus_{<p_i>,\mu',\mu'',\gamma}L(<p_i>;\mu';\{\mu';\gamma\}), \tag{3.7}$$

where $L(<p_i>;\mu';\{\mu'';\gamma\})$=Span$\{|<p_i>;\mu';\{\mu'';\gamma\}>\}$ are carrier-spaces of the $su_{int}(n)$ irreps $D(<p_i>)$ and of associated (dual to $D(<p_i>)$) irreps of the algebras $k_m^{(-)}(n), I_m^{(-)}(n)$ and $k_m^{(*)}(n)$; μ' and μ'' are the Gel'fand -Tsetlin patterns for the algebras $su_{int}(n)$ and $su_{inv}(m)$, respectively; γ is an extra label for distinguishing vectors within irreps of $k_m^{(-)}(n)$ etc [10]. Basic vectors $|<p_i>;\mu';\{\mu'';\gamma\}>$ resemble in their appearance the structure of the vectors (3.2) but instead of monomials X_{12}^s we obtain some polynomials in variables $X_{i_1\ldots i_n}$. An algorithm for obtaining an explicit (quasimonomial in vector invariants consisted of x_i and some intermediate boson vectors u_i, $\bar{u}_j$) form has been developed in our papers [10,29].

4. Some physical applications.

A natural area of applications of the above results is in developing composite models with internal SU(n)-symmetries within both quantum mechanics and quantum field theory [2,11,30-32]. Such models are governed by SU(n)-invariant Hamiltonian H_{inv} formulated in terms of elements of the algebras $k_m(n)$:

$$H_{inv}=cI+\sum_i \omega_i E_{ii}+\sum_{i,j} c_{ij}E_{ij}+\ \sum\ d_{i_1\ldots i_n}\ X_{i_1\ldots i_n}\ +$$

$$+\ \sum\ d^*_{i_1\ldots i_n}\ \bar{X}_{i_1\ldots i_n}\ \ \text{+higher powers.} \tag{4.1}$$

Specifically, some effective Hamiltonians in quantum polarization optics have this form [11,32].

The quantities $X_{i_1\ldots i_n}$ and $\bar{X}_{i_1\ldots i_n}$ may be interpreted as operators of creation and annihilation, respectively, of SU(n)-invariant clusters. But, unlike usual quantum particles (bosons and fermions) these clusters have unusual statistics as it follows fromthe CRs (2.5). In particular, in the case n=2 we obtain from (2.5) trilinear CRs

$$[\bar{X}_{rs},[\bar{X}_{ij},X_{kl}]]=(\delta_{jl}\delta_{rk}-\delta_{jk}\delta_{rl})\bar{X}_{is}+(\delta_{jl}\delta_{sk}-\delta_{jk}\delta_{sl})\bar{X}_{ri}+\ldots, \qquad (4.2)$$

which generalize the Green's trilinear CR for parafields and paraparticles [2]. The CRs (2.5) imply also the general form of the number operator N_{cl} of such clusters [11]

$$N_{cl}=(1/n)\sum_i E_{ii}-C(\{E^{\alpha\beta}\})=(1/n)\sum_i E_{ii}-\tilde{C}(\{E_{ij}\}), \qquad (4.3)$$

where C(...) are some SU(n)-invariant nonlinear functions of the SU(n) generators $E^{\alpha\beta}$ which are multiple to the identity operator I on each subspace $L(<p_i>)$ from (3.7). Specifically, for m=n=2 we have

$$C(\{E^{\alpha\beta}\})=-1/2+(1/2)(1+2(E^{12}E^{21}+E^{21}E^{12})+(E^{11}-E^{22})^2)^{1/2}. \qquad (4.4)$$

Thus, taking also into account (2.2), we see that internal SU(n)-symmetry yields us a scheme of a generalized paraquantization with constraints (cf.[30,33]) on the spaces $L_F=\oplus L(<p_i>)$. Because of nontrivial dimensions of the "vacuum subspaces" $L_\nu(<p_i>)$ we can develope models with spontaneously broken and hidden symmetries (cf.[31]) within above formalism.

Another interesting line of investigations here is in examining possibilities of constructing canonical bases of observables Y_a, $\bar{Y}_b$ ($[\bar{Y}_b,Y_a]=\delta_{ab}$) in terms of elements of algebras $k_m(n)$. This way seems to be perspective since, following the general scheme [34], we obtained in [11] explicit expressions for Y, $\bar{Y}$ in the case m=n:

$$Y=\sum_{J\geq 0} C_J(X_{12\ldots n})^{J+1}(\bar{X}_{12\ldots n})^J,\ \bar{Y}=(Y)^+, \qquad (4.5)$$

where the coefficients C_r are determined from a set of reccurence relations depending on signatures $<p_i>$ of subspacies $L(<p_i>)$.

Such developments can be useful in analyzing composite models of many-body quantum systems of arbitrary physical nature (photons, phonons etc.). Some examples of solving certain problems in polarisation quantum optics have been considered within this approach in [11].

5. Conclusion.

In conclusion we point out some problems and generalizations of the above developments.

The results obtained provide a mathematical tool for analyzing composite models with internal SU(n)-symmetry only at algebraic level. However, for examining time evolution governed by hamiltonians (4.1) we need in developing group-theoretical aspects of the theory, in particular, generalized coherent states of algebras $k_m^{(-)}(n)$ etc.

It is also of interest to extend our analysis by common considering both internal and the space-time Poincare symmetries. The "Grassmann nature" of the SU(n)-clusters $X_{i_1\ldots i_n}$ gives hope that we can obtain along this line certain results which are useful for some developments in string theory (cf. [25,26]) and for analyzing nonlinear phenomena and coherent structures in strongly interacting many-body systems [35].

Finally we note that formal aspects of the above analysis may be extended completely for the case G=SO(n). Another generalisationis obtained by involving in consideration other than $D^1(G)$ irreps of "internal" groups G.

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