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Coupled-Bunch Longitudinal Instabilities

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Summary

The excitation of coupled-bunch longitudinal modes in PEP by the accelerating cavities has been examined. It is found that some modes may be driven unstable either by the impedance of the main resonance of the accelerating cavities, or by the higher-order parasitic resonances. Estimates of the driving strengths are given.

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1. INTRODUCTION

Coupled-bunch longitudinal instabilities have been observed in many synchrotrons and storage rings -- notably in Adone[1] which, like PEP, has three bunches per beam. The theory of the instability mechanism is well known -- see, for instance, Sacherer[2] or Pellegrini and Renieri[3] where references to earlier work will also be found. In this note we make some estimates of the damping coefficients of the possibly unstable modes in PEP. We find that anti-damping effects are sufficiently strong that some modes may become unstable at stored currents below the design goals.

Longitudinal instabilities have been controlled[4,5,6] by passive cavities or by active feedback systems. It is likely that some such feedback system will be required for PEP.

The mathematical formalism becomes quite cumbersome for the general case of two counter-rotating beams each containing many bunches.[4] For our estimates we consider only the special case of a single beam of three equal bunches in which the "rigid bunch" oscillations -- the so-called "dipole" motions -- are coupled and driven by their interactions with longitudinal impedances in the vacuum chamber. (We are not concerned with coherent motion within the bunches.) We may expect, however, that the driving terms for beams of somewhat unequal bunches, and for two-beam modes will not be very different from those for our special case. We give in Appendix A a simplified -- and, hopefully, relatively transparent -- treatment of the theory for our special case.

We present in the following section the result of the theory for the most general longitudinal impedance $Z(\omega)$, and then apply the results for the expected impedances of PEP. We shall be assuming that the efforts to make a "smooth" chamber for PEP will be successful and that the longitudinal impedance will be dominated by the impedance of the accelerating cavities. In our estimates we take into account the impedance contributed both by the fundamental (accelerating) mode, and by the higher, parasitic modes. For the parasitic modes -- see Sect. 5 -- a statistical approach is used.

2. NORMAL MODES AND DAMPING COEFFICIENTS

The coupled, longitudinal oscillations of a beam of three bunches are most conveniently described as the superposition of the motion of three normal modes in each of which the oscillations of each bunch have the same frequency Ω_μ and the same damping coefficient α_μ (the inverse of the damping time constant) where the index μ identifies each mode by its mode number. For the special case of three equal bunches, equally spaced around the ring the oscillation amplitudes of the three bunches are also equal for each mode, and the phase of the oscillation of each bunch relative to the preceeding bunch is the same for all bunches, and is, in particular, an integral multipule of $120^o = 2\pi/3$.

We choose to identify each bunch by a bunch number ℓ, which we give the values 0, +1 or -1. Bunch $\ell = 0$ is the "reference" bunch, bunch $\ell = +1$ preceeds it and bunch $\ell = -1$ follows it. Similarly we label the modes by the number $\mu = 0, \pm 1$, chosen so that the phase shift (of the longitudinal motion) from the reference bunch to bunch $\ell = 1$ is zero for mode $\mu = 0$, is $+2\pi/3 = 120^o$ for mode $\mu = +1$, and is $-2\pi/3$ for mode $\mu = -1$. In general, the phase shift from bunch "0" to bunch "± 1" can be written

$$(\Delta\Theta_{\pm 1})_\mu = \pm \frac{2\pi}{3} \mu . \quad (1)$$

Adopting the time displacement τ as our longitudinal coordinate[7], we can, then, express the motion of bunch ℓ in each mode μ by,

$$\tau_{\ell\mu} = \hat{\tau}_\mu e^{-\alpha_\mu t} \cos (\Omega_\mu t + \frac{2\pi}{3} \ell\mu), \quad (2)$$

where $\hat{\tau}_\mu$ is the amplitude of the oscillation of each bunch in mode μ. The total motion of bunch ℓ is the sum of the $\tau_{\ell\mu}$ for the three modes $\mu = 0, \pm 1$.

We are interested primarily in the damping coefficients α_μ. We may consider that each consists of two parts: a part α_{rad} due to radiation damping of the motion of each particle and a second part $\Delta\alpha_\mu$ that is driven by the chamber impedance:

$$\alpha_\mu = \alpha_{rad} + \Delta\alpha_\mu \quad . \tag{3}$$

A <u>negative</u> $\Delta\alpha_\mu$ is destabilizing, and an unstable oscillation will occur when

$$\Delta\alpha_\mu < -\alpha_{rad} \quad . \tag{4}$$

As we shall see, $\Delta\alpha_\mu$ is proportional to the <u>average</u> <u>beam</u> <u>current</u> I_{av} so the inequality (4) also defines a threshold current for the instability. If we write

$$\Delta\alpha_\mu = K_\mu I_{av}, \tag{5}$$

the mode can become unstable if K_μ is negative; and the onset of unstable oscillations will occur in mode μ at the threshold current $\bar{I}$ given by

$$\bar{I}_\mu = \frac{-\alpha_{rad}}{K_\mu} \tag{6}$$

We consider that any feedback system can be treated as a part of the chamber "impedance" and will, therefore, be included in $\Delta\alpha_\mu$.

The mode frequencies Ω_μ will be, in general, different from Ω, the frequency of the single particle longitudinal oscillations at low currents.

We shall, for our present purposes, however, ignore this difference and take $\Omega_\mu = \Omega$. So long as the "beam loading" of the rf system is small, we may expect that the difference $|\Omega_\mu - \Omega|$ will be of the same order as $|\Delta\alpha_\mu|$, which we shall assume is always much less than Ω. (See the end of Appendix A.) If beam loading is large (that is, if the power to the beam is comparable to or larger than the power dissipated in the cavities) we may still expect that the frequency shift will be comparable to $\Delta\alpha_\mu$ for mode "±1", it may, however, be much larger for mode "0". See, for instance, Baryshev and Kheifetz[8] or Sands[9].

With these preliminaries we can write a relatively simple expression for the damping coefficients:

$$\Delta\alpha_\mu = \frac{e I_{av} \alpha_c \omega_o}{4\pi E \nu_s} S_\mu \qquad (7)$$

where

I_{av} = average beam current (all bunches)

α_c = momentum compaction factor

ω_o = $2\pi/T_o$ = rotation angular frequency

E = beam particle energy

ν_s = Ω/ω_o = synchrotron number

The chamber <u>driving resistance</u> S_μ is

$$S_\mu = \sum_{m=1}^{\infty} \left\{ (3m - \mu) R_m^- - (3m + \mu) R_m^+ \right\} \qquad (8)$$

with

$$R_m^- = R\left[(3m - \mu - \nu_s)\,\omega_o\right]$$

$$R_m^+ = R\left[(3m + \mu + \nu_s)\,\omega_o\right],$$

where m is any integer, $\mu = 0, \pm 1$ is the mode number, and $R[\omega]$ is the real part of the chamber impedance $Z[\omega]$ at the frequency . Evidently, the resistances R_m^- and R_m^+ depend also on the mode μ, but we shall simplify our notation by leaving that dependance implicit.

The driven contribution to the longitudinal damping, $\Delta\alpha_\mu$, of Eq. (7) is proportional to the stored current I_{av}, to a factor $\alpha_c\,\omega_o/E\nu_s$ that is a characteristic of the ring, and to the factor S_μ that we have called the "driving resistance". This factor is a sum of terms each proportional to the resistive parts of the chamber impedance evaluated at certain particular frequencies that are all near the harmonics of the rotation frequency ω_o. The specific frequencies are, in fact, all separated from the harmonics of ω_o by plus or minus the synchrotron frequency $\Omega = \nu_s\,\omega_o$. Let's look at how the terms of S_μ go for each mode.

Mode $\mu = 0$

In this mode the three bunches all oscillate in phase. (It is sometimes called the "baricentric" mode since it is the only mode in which there is an oscillation of the "center of gravity", that is, of the average longitudinal displacement, of the three bunches.) For this mode

$$S_o = \sum_m 3m\left\{R\left[(3m - \nu_s)\omega_o\right] - R\left[(3m + \nu_s)\,\omega_o\right]\right\} \qquad (9)$$

Each term in the sum is the difference between the resistances at the lower and upper sidebands of all harmonics of the bunch frequency $3\omega_o$ -- not of the rotation frequency ω_o. Since the difference is of resistances at two frequencies separated by the relatively small interval 2Ω, the "zero" mode is driven only by impedances that vary relatively rapidly. An instability of the "zero" mode was first discussed by Robinson[10], who considered, however, only the impedance of the fundamental of the accelerating cavity.

The contribution of the terms $R_m^{\pm}$ are shown schematically in Fig. 1, where an upward pointing arrow identifies the side-band where a resistance is damping and a downward arrow, one which is anti-damping. (The uppermost set of arrows refers to mode $\mu = 0$.)

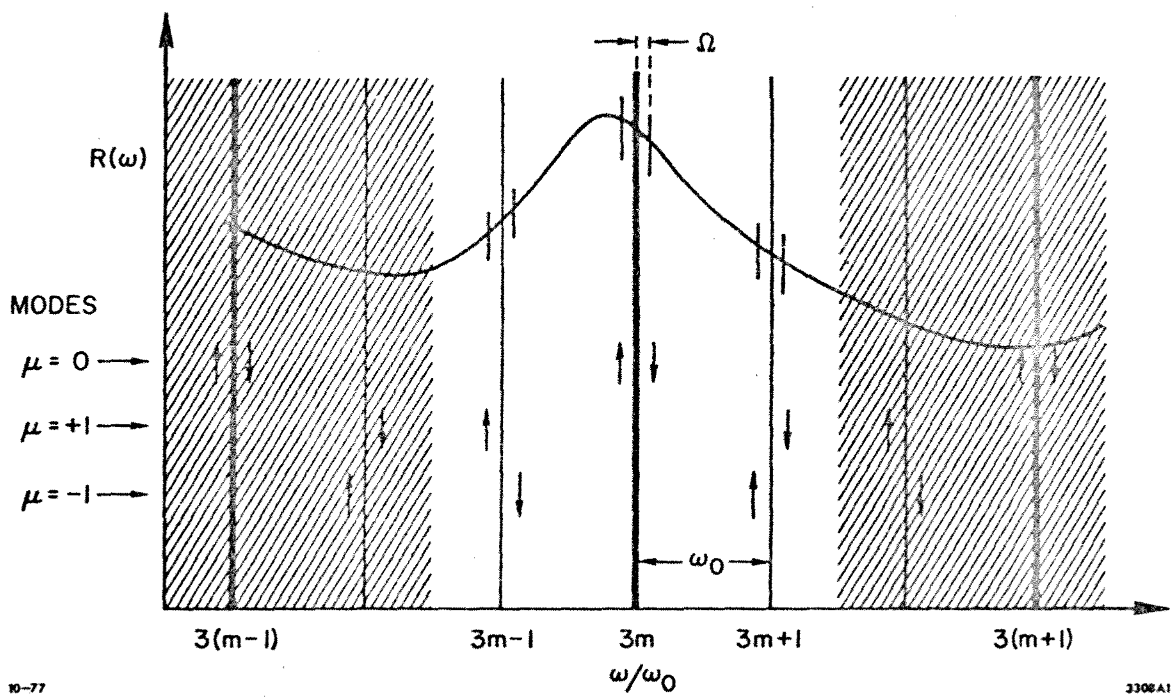

Fig. 1 A plot of $R(\omega)$ against ω/ω_0 showing the values that contribute to the sum S_μ. Upward arrows indicate a positive contribution to the damping of the particular mode labled at the left. Downward, arrows indicate a negative contribution to the damping ... and, therefore, anti-damping. Frequencies in the middle zone without cross-hatching belong to the order m of the sum. Those in the hatched zones belong to orders (m-1) and (m+1).

Mode $\mu = +1$

In this mode synchrotron oscillation of each bunch lags by 120° the oscillation of the bunch that preceeds it around the ring. The driving resistance for this mode is

$$S_{+1} = \sum_m \Big\{ (3m - 1)\ R\ \big[(3m - 1 - \nu_s)\ \omega_o\big] - (3m + 1)\ R\ \big[(3m + 1 + \nu_s)\ \omega_o\big] \Big\} \tag{10}$$

The damping of this mode is increased by a resistance at the lower sideband of the harmonic of ω_o just below $3m\ \omega_o$ and decreased by the resistance at the upper sideband of the harmonic just above $3m\ \omega_o$. The contributions of the two terms on either side of the harmonic $3m\ \omega_o$ are also indicated in Fig. 1, using the same convention as for the "zero" mode.

Mode μ = -1

In this mode the oscillation of each bunch leads by 120^o the oscillation of the bunch that preceeds it. The driving resistance is

$$S_{-1} = \sum_m \Big\{ (3m + 1)\ R\ \big[(3m + 1 - \nu_s)\ \omega_o\big] - (3m - 1)\ R\ \big[(3m - 1 + \nu_s)\ \omega_o\big] \Big\} \tag{11}$$

Notice that the damping term is now at the negative sideband of $(3m+1)\omega_o$ and the anti-damping term is at the positive sideband of $(3m-1)\omega_o$. See, again, Fig. 1.

We call attention to two important aspects of the damping of modes "±1". First, notice that there is a separation of very nearly ω_o between the frequencies that make opposite contributions to the damping -- which is larger by the factor $\omega_o/2\Omega$ than the separation for mode "zero". This means that more slowly varying impedances will be much more important for the mode "±1" than for the mode "0".

Second, notice that resistances at each harmonic $(3m \pm 1)\omega_0$ are damping for one of the "±1" modes and anti-damping by the same amount for the other. There is only a difference in the effects if the resistance is varying rapidly between the two side-bands displaced from $(3m \pm 1)\, \omega_0$ by $\pm\Omega$. Generally, then, we may expect that incidental impedances in the ring will damp one of the "±1" modes and antidamp the other -- as was, for example, observed at Adone.

We consider in the next two sections the influence of the fundamental resonance of the rf cavities on the various modes.

3. DAMPING OF MODE $\mu = 0$ BY THE CAVITY FUNDAMENTAL

The impedance in the ring will be highest at the fundamental resonant frequency of the accelerating cavities. We consider first the case in which all cavities are tuned to the same frequency ω_r near $h\omega_0$ -- with h the operating harmonic. (For PEP, h = 3 x 864 = 2592.) The real part of the impedance of a high-Q resonant cavity is described very nearly by

$$R_r(\omega) = \frac{Z_0}{2} \frac{\Gamma\omega_r}{(\omega-\omega_r)^2 + \Gamma^2} \tag{12}$$

where ω_r is the resonant frequency, Γ is the half-width of the resonance, $\Gamma = \omega_r/2Q_\ell$, with Q_ℓ the loaded quality factor, and Z_0 is the characteristic impedance of the cavity. (The parallel resistance at resonance is $Z_0\, \omega_r/2\Gamma = Q_\ell\, Z_0$.)

In PEP there are N_r = 24 separate resonators. When they are all tuned to the same frequency, the total resistive impedance is just the resistance of Eq. (12) multiplied by the number of resonators:

$$R(\omega) = N_r\, R_r\, (\omega). \tag{13}$$

For the PEP resonators we may take that

$$\begin{aligned} Z_0 &= 750 \text{ ohm} \\ \omega_r &= 2\pi f_r = 2.23 \times 10^9 \text{ s}^{-1} \\ Q_\ell &= 8{,}000 \\ \Gamma &= 0.139 \times 10^6 \text{ s}^{-1} \\ N_r &= 24 \\ \omega_0 &= 0.86 \times 10^6 \text{ s}^{-1} \end{aligned} \tag{14}$$

Notice that Γ is about three times as large as the synchrotron (angular) frequency $\Omega \simeq 0.04 \times 10^6$ s^{-1} at 15 GeV, and about 1/6 of the rotation frequency ω_0. The relative magnitudes are illustrated roughly in the graph of Fig. 2, where the "detuning" of the cavities, $\Delta\omega = h\omega_0 - \omega_r$ has been taken about equal to Ω. This is approximately the situation that would obtain when the cavity is detuned enough to "compensate" for the reactive component of the beam current -- for two 55 mA beams.

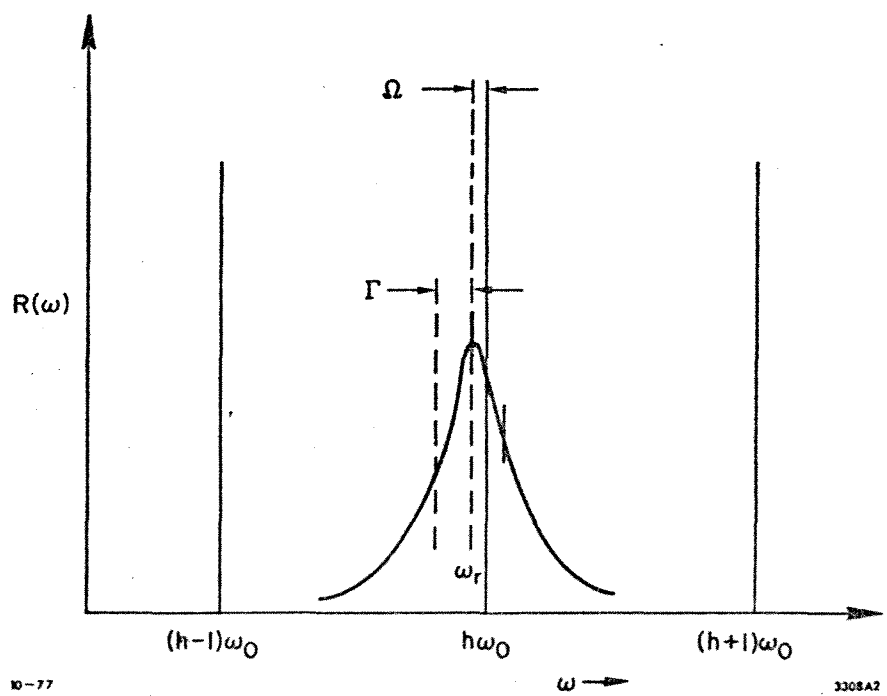

Fig. 2 The resistive impedance of the fundamental resonance of the rf system, with a typical detuning, $\Delta\omega = (h\omega_0 - \omega_r)$, about equal to Ω.

In these more-or-less representative circumstances the difference in $R(\omega)$ between the two side-band frequencies is about equal to one-half the peak value of R. The driving resistance of Eq. (9) is then dominated by the m = h/3 term, and we may take that

$$S_o \simeq h \frac{R_{max}}{2} = \frac{N_r Z_o \omega_r^2}{4\Gamma \omega_o} . \tag{15}$$

Then the damping coefficient of Eq. (7) becomes

$$\Delta\alpha_o \simeq \frac{e\alpha_c I_{av} N_r}{4\pi E} \frac{\omega_r^2 Z_o}{4\nu_s \Gamma} . \tag{16}$$

Let's take a look at 15 GeV operation. We take

$$\begin{aligned} E/e &= 15 \times 10^9 \text{ volt}, \\ I_{av} &= 0.055 \text{ amp}, \\ \alpha_c &= 0.0030, \\ \nu_s &= 0.0051. \end{aligned} \tag{17}$$

Using, as well, the values in (14) we get

$$\Delta\alpha_o \simeq 2800 \text{ s}^{-1}. \tag{18}$$

This term is much larger than the radiation damping $\alpha_{rad} = 250 \text{ s}^{-1}$, so the driven damping dominates, and coherent synchrotron oscillations will be damped with a time-constant of about 360 μsec.

The driven coherent damping is quite strong -- with several consequences. First, if the cavities were to be detuned even a rather small amount to the "wrong" side ($\omega_r > h\,\omega_0$) the negative driven term $\Delta\alpha_\mu$ would swamp the radiation damping α_{rad} and the beam would become unstable. Second, the strong damping will provide insurance against any anti-damping effects from other impedances in the ring and, in particular, from the many parasitic resonances of the accelerating cavities. We may expect that so long as the parasitic resonances have somewhat randomized resonant frequencies, their combined effect would not be as strong as the damping provided by the fundamental resonance, so the total effect will be damping.

(We have made a rough estimate of the possible effect of the parasitic resonances on the $\mu = 0$ mode -- using arguments similar to those described below in Sect. 5. Our result is that we expect to have a contribution -- either damping or anti-damping -- to $\Delta\alpha_0$ from the parasitic resonances of about 4 percent of the contribution from the principle resonance.)

If the detuning is somewhat larger than Ω, both sidebands will be on the same shoulder of the resonance curve and we can approximate ΔR by

$$\Delta R = R^- - R^+ = -\frac{dR}{d\omega} \cdot 2\Omega_s \qquad (19)$$

where the derivative is to be evaluated at $h\omega_0$. When R is given by N_r times the resonance function of Eq. (12), we get that

$$S_0 = -2h\Omega \left(\frac{dR}{d\omega}\right)_{h\omega_0} = 2h\Omega \frac{N_r Z_0 \omega_r}{\Gamma^2} \frac{x}{(1+x^2)^2} \qquad (20)$$

where $x = \Delta\omega/\Gamma$, and $\Delta\omega = h\omega_0 - \omega_r$.

Then

$$\Delta\alpha_o = \frac{e\alpha_c \, I_{av} \, N_r \, Z_o}{4\pi E} \cdot \frac{h^2 \, \omega_o{}^3}{\Gamma^2} \cdot \frac{2x}{(1 + x^2)^2} \tag{21}$$

Notice that now $\Delta\alpha_o$ is independent of ν_s, and would also be independent of energy in PEP -- where I_{av} is proportional to E. We can get a representative value by taking $\Delta\omega = \Gamma$, then

$$\Delta\alpha_o = \frac{e \, I_{av} N_r \, Z_o}{4 \pi E} \quad \frac{h^2 \, \omega_o{}^3}{2 \Gamma^2} \tag{22}$$

Taking values from (14) and (17)

$$\Delta\alpha_o \simeq 1700 \text{ s}^{-1}. \tag{23}$$

The coherent damping time would be about 600 microseconds at all energies.

One further remark: we expect that when PEP is operated at low energies -- say 4 GeV -- it will be expedient to deactivate many of the rf stations. The cavities of these stations would then be tuned well off resonance, $\Delta\omega > \Gamma$, to avoid their being driven by the beam. The contributions of these cavities to $\Delta\alpha_o$ will be decreased -- and may, in fact, be negative if they are detuned on the other side of $h\omega_o$. Since the ratio of I/E is expected to be constant, the only change in Eq. (22, at low energies, will be, in effect, a reduction of N_r. The driven damping will then be less at lower energy. And if N_r is decreased too far, an antidamping contribution from the parasitic cavity resonances -- which still remain in a detuned cavity -- might dominate.

4. DRIVING OF MODES $\mu = \pm 1$ BY THE CAVITY FUNDAMENTAL

Imagine the resonant curve for $R_r(\omega)$ of Fig. 2 superimposed on Fig. 1. The tails of the fundamental resonance of the rf cavities will extend to the frequencies $(h \pm 1)\omega_0$, and so long as the cavities are tuned away from $h\omega_0$, the resistive impedance, $R(\omega)$, will be different at the two relevant driving frequencies for modes $\mu = \pm 1$. From Figs. (1) and (2) we see that for "normal" detuning, mode $\mu = 1$ will be damped and mode $\mu = -1$ will be anti-damped. We now estimate the magnitude of this effect.

In the sums of Eqs. (10) and (11), we keep only the terms for $m = h/3$, we neglect 1 in comparison with h, and we evaluate $R(\omega)$ at $(h \pm 1)\omega_0$. (The displacement of the sidebands by Ω will not be significant here.) We may write, then, that

$$S_{\pm 1} = \mp h(R^+ - R^-) \tag{24}$$

where $R^\pm$ we mean $R\ (h \pm 1)\omega_0$ and for $R(\omega)$ we take N_r times the resonance function $R_r(\omega)$ of Eq. (12), with $\omega_r = (h\omega_0 - \Delta\omega)$. Then $R^\pm$ is given by

$$R^\pm = N_r R_r (\omega_r + \Delta\omega \pm \omega_0) \tag{25}$$

Using a Taylor expansion of R_r for frequencies near $(\omega_r \pm \omega_0)$, and remembering that $R_r(\omega)$ is symmetric about ω_r, we get that

$$S_{\pm 1} = \mp 2h\ \Delta\omega\ N_r R_r{}'\ (\omega_r + \omega_0) \tag{26}$$

where $R_r{}'$ means $dR_r/d\omega$. Since $\omega_0 \gg \Gamma$, we may take

$$R_r{}'\ (\omega_r + \omega_0) = -\frac{N_r Z_0 \omega_r \Gamma}{\omega_0{}^3} . \tag{27}$$

Then

$$S_{\pm 1} = \pm 2 N_r Z_o \frac{h^2 \Gamma}{\omega_o^2} \Delta\omega \tag{28}$$

So for the driven damping due to the fundamental resonance we have

$$\Delta\alpha_{\pm 1} = \pm \frac{e\alpha_c I_{av} N_r Z_o}{4\pi E} \cdot \frac{2 h^2 \Gamma \Delta\omega}{\nu_s \omega_o} \tag{29}$$

Suppose we choose $\Delta\omega = \Gamma$ as we did for $\Delta\alpha_o$ at the end of the preceeding section. This circumstance corresponds to high-energy operation with all rf stations on-line. Then the ratio of the damping coefficients of (29) and (22) is

$$\frac{\Delta\alpha_{\pm 1}}{\Delta\alpha_o} = \pm \frac{4}{\nu_s} \left(\frac{\Gamma}{\omega_o}\right)^4. \tag{30}$$

From (14) the ratio Γ/ω_o is 0.162, so

$$\frac{\Delta\alpha_{\pm 1}}{\Delta\alpha_o} \simeq \pm \frac{2.7 \times 10^{-3}}{\nu_s}, \tag{31}$$

In the range of energies from 4 GeV to 15 GeV, ν_s will vary from, perhaps, 0.010 to 0.051, and the ratio of (31) will vary from 0.27 to 0.053. Typical values might then be as follows:

E (GeV)	I_{av} (mA)	ν_s	$\Delta\alpha_0$ (s^{-1})	$\Delta\alpha_{\pm1}$ (s^{-1})	α_{rad} (s^{-1})
4	15	0.010	1700	480	11
15	55	0.051	1700	95	245

Notice that at high energies the radiation damping dominates; so that, for normal tuning, all modes will be stable. On the contrary, at the lowest operating energy the mode $\mu = -1$ will (for normal tuning) be highly antidamped and be unstable. Using Eqs. (5) and (6) we would estimate that the threshold current for unstable oscillation at 4 GeV would be about 0.34 mA!

The discussion above assumed that all accelerating cavities were "on-line" at low energies. As mentioned in Section 3, there are reasons for wanting to de-activate most of the 24 stations when running at low energies. We might expect to take cavities out of operation by detuning them so that $\Delta\omega >> \Gamma$. If this is done, however, $\Delta\alpha_{-1}$ will have its magnitude increased and the mode will be <u>even more</u> unstable than just calculated. An alternative would be to detune one half of the de-activated cavities on one side of $h\omega_0$ and the other half in the opposite direction. As you can see by reference to Fig. 1, the contribution of each pair of detuned cavities to S_μ would be zero. In effect, the number N_r of cavities contributing to $\Delta\alpha_\mu$ would be reduced to the number of <u>active</u> cavities. Unfortunately, even reducing N_r from 24 to, say 4, would only increase the threshold of instability by a factor of six -- to 1 mA.

One can, of course, gain a little by decreasing somewhat the detuning $\Delta\omega$. If one goes too far in that direction however -- especially if one had only 4 active cavities, the $\mu = 0$ mode might then become unstable.

It seems clear that some more specific corrective measures will be required to stabilize the $\mu = -1$ mode. The results of the next section reinforce this conclusion.

5. DRIVING OF MODES $\mu = \pm 1$ BY PARASITIC RESONANCES

We have until now considered only the influence of the fundamental resonances of the accelerating cavities. The remaining large contribution to the longitudinal impedance of the vacuum chamber is expected to come from the higher-order, parasitic modes of the accelerating cavities. We will now make an estimate of the contribution of these resonances to the anti-damping of the coupled longitudinal oscillations. As mentioned in Section 3, we have estimated that the effects on the baricentric mode $\mu = 0$ will be much less than the driven damping from the fundamental cavity resonance; so we consider here only the damping of the mode $\mu = \pm 1$.

For our estimate we shall make several simplifying, but reasonable, assumptions. First, we consider that the parasitic resonances have Q's at least similar to the main resonance, so that the resonance widths are at least somewhat less than ω_0. A typical resonant frequency might be 1 GHz, and a Q of 10,000 would give a Γ of 0.3×10^6 which is less than $\omega_0 = 0.86 \times 10^6$. Next, we assume that the resonances of each cell of the accelerating structures act independently. (Actually, there may be strong coupling of corresponding modes across the 5 cells of an rf station. But this coupling would probably not change the results by a large factor.) Finally, we assume that the resonant frequencies of any given parasitic mode are different from one cavity (cell) to the next, and are, in particular, spread more or less randomly across a range that is at least as large as $3\omega_0$. This means that the relative frequency variations are assumed to be about 4×10^{-3}, or somewhat more.

This last assumption means that the resonant frequency of each cavity (cell) is equally likely to fall near each of the 3 harmonics of ω_0 where one of the 3 modes of the longitudinal oscillation is driven. (See again Fig. 1).

Consider a given parasitic resonance of frequency ω_r and width Γ. We assume its impedance varies with ω according to Eq. (12), which has a peak value of

$$R_s = \frac{Z_o \, \omega_r}{2 \, \Gamma} = QZ_o \, , \tag{32}$$

and, roughly speaking, a width of 2Γ. Now ask: What is the likely effect of this resonance on the $\mu = \pm 1$ mode of the longitudinal oscillations? The probability is $2\Gamma/3\omega_o$ that this resonance will land on some $(3m - 1)$ harmonic of ω_o and give damping to mode $\mu = 1$. There is equal probability that it will land on some $(3m + 1)$ harmonic of ω_o and give anti-damping.

If we now consider the combined effect of all of the N_c corresponding resonances in the N_c cavity cells*, we can say that the probable number n of resonances landing on the harmonic $(3m - 1)$ is N_c times the probability just mentioned.

$$n = N_c \cdot \frac{2\,\Gamma}{3\omega_o} \tag{33}$$

In the sum of Eq. (10) these resonances will make an expected contribution of $3m \, R^+$, where the expectation value of R^+ is

$$\langle R^+ \rangle = n \, R_s \tag{34}$$

We are here assuming that all the resonances will be near the same 3m harmonic, with $3m >> 1$ so that a factor $(3m - 1)$ can be replaced by 3m.

Similarly, n resonances will, on the average, land on the $(3m + 1)$ harmonics and give a contribution to the sum of Eq. (10) whose expected value is $3m \langle R^- \rangle$, with

$$\langle R^- \rangle = n \, R_s \tag{35}$$

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* For PEP $N_c = 5 \times 24 = 120$.

The expectation value for $S_{\pm 1}$ is then

$$\langle S_{\pm 1} \rangle = 3m \left\{ \langle R^- \rangle - \langle R^+ \rangle \right\} \tag{36}$$

which is clearly zero. Not surprisingly the expected damping is zero, since there is equal likelihood that the damping will be positive or negative.

On the other hand, we must expect that any given set of resonances will have a damping or anti-damping effect somewhere in the range of the standard deviation of S, which we shall call here σ_S. We may write that

$$\sigma_S{}^2 = (3m)^2 \left[(\delta R^+)^2 + (\delta R^-)^2 \right] \tag{37}$$

where $(\delta R^+)^2$ and $(\delta R^-)^2$ are the variances of R^+ and R^-.

Under our assumptions, the two variances are uncorrelated, and we can take that the number of resonances contributing to R^+ (or to R^-) follows a Poisson distribution with a variance given by $n\, R_S{}^2$. Then

$$\sigma_S{}^2 = (3m)^2\, (2n\, R_S{}^2) \; . \tag{38}$$

We may take for 3m the ratio ω_r/ω_o. Then taking R_S and n from Eqs. (32) and (33), we get that

$$\sigma_S{}^2 = \frac{2}{3}\, N_c Q\, Z_o{}^2 \left(\frac{\omega_r}{\omega_o}\right)^3 \tag{39}$$

This, remember, is the contribution from one particular cavity resonance. The contributions to $\sigma_S{}^2$ from each family of resonances must be added to get the overall $\sigma_s{}^2$. We can approximate that sum by assuming that Eq.(39) refers to the parameters of a "typical" resonance, and that there are N_p such resonances in each cavity. Then for the total effect, we can take that

$$\sigma_s{}^2 \simeq \frac{2}{3} N_p N_c Q Z_o{}^2 \left(\frac{\omega_r}{\omega_o}\right)^3 \tag{40}$$

A <u>typical</u> value of the driven damping of mode $\mu = +1$ is obtained by using σ_s in place of S in Eq. (7).

Although we have been discussing mode $\mu = +1$, it is clear that precisely the same arguments would apply to mode $\mu = -1$. In a particular situation either mode may be damped or anti-damped, but, since S_{+1} and S_{-1} will be almost exactly equal in magnitude but with opposite signs, <u>one of the two</u> modes will almost certainly be anti-damped with a typical driving strength equal to the σ_s of Eq. (40). We write then that

$$\Delta\alpha_1 \simeq \frac{e\alpha_c I_{av} \omega_o}{4 E \nu_s} \cdot \sigma_s. \tag{41}$$

We want now to evaluate σ_s. We could make some reasonable guesses for the parameters, but we will, rather, make use of some information provided us by Perry Wilson from computer calculations he has made to determine the properties of higher-order modes in a model PEP cavity. Wilson has found that there are some 17 resonances with longitudinal impedance on the axis and with frequencies between the fundamental (358 MHz) and the cut-off frequency of the beam pipe (near 2,000 MHz), and has listed their resonant frequencies and characteristic impedances. There are 8 resonances with characteristic impedances Z_0 between 2 ohms and 30 ohms, with a typical value like 5 ohms, and a typical frequency like 1,000 MHz. The quality factor Q is harder to come by. Since the unloaded Q of the fundamental is 26,000, we may guess that the Q's of the higher modes* should also be some 20,000 -- or, probably more. For our estimate of $\Delta\alpha_1$, we then take the following numbers:

$$N_p = 8 \qquad Q = 2 \times 10^4$$
$$N_c = 120 \qquad \omega_r = 6 \times 10^9 \ s^{-1} \tag{42}$$
$$Z_o = 5 \text{ ohms} \qquad \omega_o = 0.86 \times 10^6 \ s^{-1}$$

From these we obtain

$$\sigma_s \simeq 1.1 \times 10^{10} \text{ ohms.} \tag{43}$$

We have also made a more refined calculation of σ_s that yields a similar result. A more careful statistical calculation shows that the factor of 2 in Eq. (39) should be replaced by 1.56. Also, using Wilson's information, and assuming that the Q of each mode is proportional to the square root of its frequency, we have calculated the sum of all the individual contributions to σ_s. The result we get is a little larger than (43), namely, that*

$$\sigma_s \simeq 2.5 \times 10^{10} \text{ ohms.} \tag{44}$$

(The larger value comes primarily from larger assumed Q's.)

Using this last value in Eq. (41) we get the following results.

E(GeV)	I_{av}(mA)	ν_s	$\Delta\alpha_1(s^{-1})$	$\alpha_{rad}(s^{-1})$
4	15	0.010	2,000	11
15	55	0.051	380	245

* See "note added in proof" at end of the text.

We see that we must expect an instability of one of the modes $\mu = \pm 1$ at all energies. And our best estimate for the threshold currents would be 35 mA at 15 GeV, and 0.08 mA at 4 GeV. We emphasize, however, that because of the statistical nature of our estimate -- as well as uncertainties in the relevant parameters -- the actual driving strengths encountered could well be different from our estimates by a factor of three or more.

The driving strength of $\Delta\alpha \simeq 2{,}000$ obtained above is quite large. It is, in fact, already becoming comparable to the synchrotron frequency $\Omega = \nu_s\omega_0 \simeq 9{,}000\ s^{-1}$ at 4 GeV. Under such circumstances the theoretical formalism leading to the formula of Section 2 is no longer quite applicable. (See discussion of Section 2 and of Appendix A.) Our results should, however, be roughly correct, nevertheless.

One final remark. The contributions of the parasitic resonance is, statistically speaking, independent of the tuning of the main resonance. It will be the same whether cavities are tuned on-line for acceleration or not. (In fact, of course, tuning the main resonance will shift the parasitic resonances around, and the actual $\Delta\alpha_{\pm 1}$ will vary in some unpredictable way, but the rms fluctuation σ_s will remain unchanged.)

6. CONCLUSION

We have made estimates of the driven damping or anti-damping of the three normal modes $\mu = 0, \pm 1$ of the dipole oscillations of the three bunches in one stored beam. We found that (a) the baricentric mode $\mu = 0$ is adequately damped by the normal detuning of the main resonance of the accelerating cavities; (b) the mode $\mu = -1$ will be driven unstable at low energies by the normal detuning of the main cavity resonance; (c) the stastical effects of parasitic cavity resonances is likely to drive either mode $\mu = +1$ or mode $\mu = -1$ unstable at any energy. The last effect is the strongest of all considered, and might give anti-damping coefficients $\Delta\alpha$ of perhaps 400 sec^{-1} at 15 GeV & 2,000 sec^{-1} at 4 GeV.

We have not considered complications that may result from the interactions between two stored beams, nor have we looked at the various possible feedback systems that might be used to control the potential instabilities. It seems certain that some such system will be required in PEP for stable three-bunch operation, particularly at low energies.

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Note Added in Proof: It now appears that the information on parasitic resonances given us by Wilson may contain an arithmetic error, which, if corrected, would increase σ_s of Eq.(44) by a factor of 1.2. On the other hand, we now think that it may be incorrect to use the unloaded Q's for determining σ_s (on p. 20), and that, perhaps a somewhat lower Q should be taken -- say, one-half or so of the unloaded Q. Since σ_s depends on $\sqrt{Q}$, using a lower effective Q would lower the σ_s of Eqs.(43) or (44) by 1.2, or so.

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Appendix A. Derivation of the Damping Coefficients

We consider the coupled longitudinal oscillations of a beam of three idealized, point-like bunches which have equal charges and are equally spaced around the ring. The bunches are assumed to interact via the longitudinal impedance of the vacuum chamber, and the coupling forces are assumed to be weak so that the normal (rf) forces dominate and each bunch executes nearly sinusoidal oscillations about its synchronous position with the synchrotron frequency Ω.

For the ideal case being considered here, symmetry allows us to identify immediately the normal modes. Since all bunches are equivalent, the phase advance of the longitudinal oscillations must be the same from one bunch to the next. There are, clearly, only three possibilities, and they are the three modes identified in Section 2. Ignoring for the moment the damping (which is assumed to be small) the longitudinal motion of bunch ℓ (ℓ = 0, ±1) in mode μ is -- see Eq.(2) --

$$\tau_{\ell\mu} = \hat{\tau}_\mu \cos\left(\Omega\ t + \frac{2\pi}{3}\,\ell_\mu\right). \tag{A1}$$

We are also setting $\Omega_\mu = \Omega$, the low current synchrotron frequency.

Consider the passage of the reference bunch (ℓ = 0) through an element of impedance Z located at some point in the ring. When there are no longitudinal oscillations the bunch passes at regular intervals with the period $T_0 = 2\pi/\omega_0$. If we choose our origin of t at one of these passages the current I(t) through Z can be written as the Fourier series

$$I(t) = \frac{I_{av}}{3} + \frac{2I_{av}}{3} \sum_{n=1}^{\infty} e^{-i\,n\,\omega_0\,t}, \tag{A2}$$

where it is understood that we are to take the real part of the right-hand side, and where I_{av} means the average current carried by all three bunches.

We are, here, not interested in this "normal" current, but, rather, want to consider only the effect of the perturbation current i(t) which is the difference between the current when the beam is advanced by the time displacement τ and the normal current of (A2). The spectrum of the perturbation can, when τ is small, be written as

$$i(t) = I(t+\tau) - I(t) = -\frac{2I_{av}}{3}\tau\sum_{n=1}^{\infty} i\, n\, \omega_0\, e^{-i\, n\, \omega_0\, t}. \qquad (A3)$$

So long as the longitudinal displacement varies slowly -- that is for $\Omega << \omega_0$ -- we may still use Eq.(A3) for the current. We have then that the effective perturbing current of bunch $\ell = 0$ can be taken as

$$i(t) = -\frac{2I_{av}}{3}\hat{\tau}\cos\Omega t \sum i\, n\, \omega_0\, e^{-i\, n\, \omega_0\, t} \qquad (A4)$$

The bunch $\ell = -1$ which follows the reference bunch arrives at Z with a delay of one-third of the rotation period. The n-th Fourier component of its current is shifted in phase by $-2\pi n/3$. Similarly, the bunch $\ell = 1$ which preceeds the reference bunch will have its phases advanced by $2\pi n/3$. When the bunches are oscillating <u>in one of the normal modes</u> μ described by EQ.(A1), the perturbing current of each bunch ℓ at the impedance Z is

$$i_\ell(t) = -\frac{2I_{av}\hat{\tau}_\mu}{3}\cos\left(\Omega t + \frac{2\pi}{3}\ell\mu\right)\sum_n i\, n\, \omega_0\, e^{-i\, n(\omega_0 t - \frac{2\pi}{3}\ell)}. \qquad (A5)$$

The total current through Z is the sum of the three bunch currents, which we write as

$$i_{total} = -\frac{2I_{av}}{3}\tau_\mu \, i\omega_o \, J_\mu(t) \tag{A6}$$

with

$$J_\mu(t) = \sum_n \sum_\ell \cos(\Omega t + \frac{2\pi}{3}\mu\ell)\, n e^{-i\, n(\omega_o t - \frac{2\pi}{3}\ell)}. \tag{A7}$$

In evaluating the sum over ℓ it is convenient to express the cosine factor in exponential form. Then

$$J_\mu(t) = \sum_n \frac{n}{2} \sum_\ell \left\{ e^{i(\Omega t + \frac{2\pi}{3}\mu\ell)} + e^{-i(\Omega t + \frac{2\pi}{3}\mu\ell)} \right\} \cdot e^{-in(\omega_o t - \frac{2\pi}{3}\ell)} \tag{A8}$$

or

$$J_\mu(t) = \sum_n \frac{n}{2} \sum_\ell \left\{ e^{-i\left[(n\omega_o - \Omega)t - \frac{2\pi}{3}\ell(n+\mu)\right]} + e^{-i\left[(n\omega_o + \Omega)t - \frac{2\pi}{3}\ell(n-\mu)\right]} \right\}. \tag{A9}$$

This final form shows that the phase modulation at the synchrotron frequency Ω splits each Fourier component at $n\omega_o$ into a pair of components displaced to the "side-band" frequencies $n\omega_o \pm \Omega$.

Equation (A9) shows another important aspect of the currents from the three bunches. The lower side-band component (at the frequency $3n\omega_o - \Omega$) contributed by each bunch is shifted in phase by $2\pi(n + \mu)$ from one bunch to another. That means that when we sum over ℓ for a given mode μ we are adding three terms whose relative phases are integer multiples of $2\pi/3$. Such a sum will give zero unless the integer $(n + \mu)$ is a whole multiple of 3 -- let's say, unless

$$n + \mu = 3m \tag{A10}$$

with m any positive integer.

In other words, when we sum over ℓ we get zero for two of every three values of n, and triple the single bunch amplitude for the third value of n, with the surviving value of n depending on the mode μ being considered. The lower side-band energy of each mode is concentrated at a different harmonic of ω_0 according to the following scheme:

Lower Sideband

Mode Number μ :	- 1	0	+ 1
Surviving Harmonic n :	3m + 1	3m	3m - 1

If you look now at the upper sideband terms in Eq.(A9), you will see that the same arguments hold again with the exception that after summing over ℓ the surviving terms will be those with

$$n - \mu = 3m. \tag{A11}$$

And the energy of the upper side-band appears at the harmonics of ω_0 given by this scheme:

Upper Sidebands

Mode Number μ :	- 1	0	+ 1
Surviving Harmonic n :	3m - 1	3m	3m + 1

The location of the non-zero components is also shown in Fig. 1 of Section 2.

Since for any particular mode there is only one contribution to each sideband for one of every three harmonics, we can replace the sum over n by a sum over m. When we have summed over ℓ, Eq.(A9) becomes

$$J_\mu(t) = \sum_m \frac{3}{2} \left\{ (3m - \mu)\, e^{-i\left[(3m-\mu)\omega_o - \Omega\right]t} + (3m+\mu)\, e^{-i\left[(3m+\mu)\omega_o + \Omega\right]t} \right\} \tag{A12}$$

Apart from the common factor of (A6) these are the components of the beam current through the impedance Z.

The beam current $i(\omega)$ at a particular frequency ω will give a voltage drop across Z of

$$v(\omega) = i(\omega)\, Z(\omega) = i(\omega)\left[R(\omega) + i\, X(\omega)\right] \tag{A13}$$

where $Z(\omega)$ is the complex impedance at the frequency ω and $R(\omega)$ is its real part. The first term in the curly bracket of A(12) will give a voltage proportional to the impedance at the frequency $\left[(3m-\mu)\omega_o - \Omega\right]$, and the second term, at the frequency $\left[(3m+\mu)\omega_o - \Omega\right]$. Let's define

$$\begin{aligned} Z_m^- &= Z\left[(3m-\mu)\,\omega_o - \Omega\right] \\ Z_m^+ &= Z\left[(3m+\mu)\,\omega_o + \Omega\right]. \end{aligned} \tag{A14}$$

Putting (A6), (A12) and (A13) together we write the voltage drop as

$$v = -I_{av}\, \tau_\mu\, i\, \omega_o\, H_\mu(t) \tag{A15}$$

with

$$H_\mu(t) = \sum_m \left\{ (3m-\mu)\, Z_m^-\, e^{-i\left[(3m-\mu)\omega_o - \Omega\right]t} + (3m+\mu)\, Z_m^+\, e^{-i\left[(3m+\mu)\omega_o + \Omega\right]t} \right\}. \tag{A16}$$

This is the voltage that couples the longitudinal motion, since it produces energy changes proportional to τ_μ.

Once the modes have been defined, as we have done, we can obtain the growth rate of any given mode by looking at the motion of any one bunch -- say the reference bunch, $\ell = 0$. This bunch crosses the impedance element Z at the times $t_j = j\ T_o = j\ 2\pi/\omega_o$, so on each revolution it will gain the energy $-e\ v(t_j)$. The sign is negative because v is a voltage drop. Also, we are ignoring the small displacement τ in the time of the bunch passage, since this would give us a term that is second order in τ -- one order from v itself and the other from the time-derivative of v with the displacement τ. For the reference bunch, then, the terms $(3m \pm \mu)\omega_o\ t$ in the exponents are always multiples of 2π and can be dropped, and we are left with only the slowly varying terms $\pm\,\Omega t$. We get that the energy gain $\delta\varepsilon$ per revolution of the reference bunch (for a given mode μ) is

$$\delta\varepsilon = e\, I_{av}\, \tau_\mu\, i\, \omega_o \sum_m \left\{ (3m-\mu)\, Z_m^-\, e^{i\Omega t} + (3m+\mu)\, Z_m^+\, e^{-i\Omega t} \right\} \tag{A17}$$

We remind you now that for the actual energy change we are to take the real part of the right-hand side. Writing $Z_m^\pm$ as $R_m^\pm + i\ X_m^\pm$. we get

$$\delta\varepsilon = e\, I_{av}\, \tau_\mu\, \omega_o \sum_m \left\{ (3m-\mu)(-X_m^- \cos \Omega t - R_m^- \sin \Omega t) + (3m+\mu)(-X_m^+ \cos \Omega t + R_m^+ \sin \Omega t) \right\}. \tag{A18}$$

Or, rearranging terms,

$$\delta\varepsilon = e\, I_{av}\, \tau_\mu\, \omega_o \sum_m \left\{ - \left[(3m-\mu)X^- + (3m+\mu)X^+ \right] \cos \Omega t - \left[(3m-\mu)R^- - (3m+\mu)R^+ \right] \sin \Omega t \right\} \tag{A19}$$

We are almost there! We have one set of terms in the sum that is proportional to τ itself (the terms in $\cos \Omega t$) and another set proportional to $d\tau/dt$ (the terms in $\sin \Omega t$). The first set will produce a change in the coherent synchrotron frequency -- which is not our main interest here. The second set gives us the contribution to the damping term $\Delta\alpha_\mu$ defined in Section 2. To see how that works we make a brief diversion to look at the equations of motion of a bunch.

Using the notation of Sands[7], Section 3, the energy deviation of the bunch changes according to

$$\frac{d\varepsilon}{dt} = \frac{e\dot{V}}{T_o}\tau + \frac{\delta\varepsilon}{T_o} \tag{A20}$$

and the longitudinal displacement, according to

$$\frac{d\tau}{dt} = -\frac{\alpha_c}{E}\varepsilon. \tag{A21}$$

We are interested in energy perturbations $\delta\varepsilon$ that vary like τ and $d\tau/dt$, so let's write

$$\delta\varepsilon = A\,\tau + B\, d\tau/dt \tag{A22}$$

Pulling the pieces together, we get

$$\frac{d^2\tau}{dt^2} + \frac{\alpha_c B}{ET_o}\frac{d\tau}{dt} + \frac{\alpha_c}{ET_o}(e\dot{V} + A)\tau = 0. \tag{A23}$$

As we said, A gives a contribution to the frequency and B gives damping. The coefficient of $d\tau/dt$ is twice the damping coefficient, so

$$\Delta\alpha_\mu = \frac{\alpha_c B}{2E\,T_o} = \frac{\alpha_c\,\omega_o}{4\pi\;E}\cdot B. \tag{A24}$$

Referring back to the perturbation energy gain of (A19), B is $1/\Omega$ times the factor that multiplies $\tau_\mu \sin \Omega t$. Our result is that

$$\Delta\alpha_\mu = \frac{e\,\alpha_c\,I_{av}\,\omega_o^2}{4\pi\,E\,\Omega}\sum_m\left\{(3m-\mu)R_m^- - (3m+\mu)R_m^+\right\}. \tag{A25}$$

Our derivation assumed that Z was an impedance element localized at some azimuth of the ring. Since we have used a perturbation treatment, keeping only linear terms in τ, the damping coefficients $\Delta\alpha_\mu$ and also the $\Delta\Omega$ due to successive impedance elements will just add. We can then reinterpret $\Delta\alpha_\mu$ in Eq.(A24) as representing the <u>total</u> damping due to all longitudinal impedances, and the resistances $R_m^\pm$ as representing the <u>total</u> longitudinal impedance.

For completeness, we point out that the term with A in Eq.(A22) gives us the coherent frequency shift of mode μ. Specifically, the frequency of mode Ω_μ is obtained from the coefficient of τ in (A22):

$$\Omega_\mu^2 = \frac{\alpha_c}{E\,T_o}(e\dot{V} + A). \tag{A26}$$

With our weak coupling approximation we can write

$$\Omega_\mu{}^2 = (\Omega + \Delta\Omega_\mu)^2 \simeq \Omega^2 + 2\Omega\,\Delta\Omega_\mu, \tag{A27}$$

where Ω is the undisturbed synchrotron frequency. Clearly,

$$\Delta\Omega_\mu = \frac{\alpha_c}{2E\,T_o\,\Omega}\,A = \frac{\alpha_c}{4\pi E\,\nu_s}\,A. \tag{A28}$$

Taking for A the factor multiplying τ_μ cos Ωt in (A19), we find that

$$\Delta\Omega_\mu = -\frac{e\,\alpha_c\,I_{av}\,\omega_o{}^2}{4\pi E\,\Omega}\sum_m \left\{(3m-\mu)X_m^- + (3m+\mu)X_m^+\right\}. \tag{A29}$$

Comparing this equation with (A25), we see that so long as X and R are comparable in magnitude (as we would normally expect), $\Delta\Omega_\mu$ and $\Delta\alpha_\nu$ will have similar magnitudes -- as we affirmed in Section 2. The statement is not valid for $\mu = 0$ because (A25) contains the difference of resistances of two nearby frequencies, while (A29) contains the sum of the corresponding reactances. For modes $\mu = \pm 1$, the two frequencies are far apart and $R^- - R^+$ is likely to be similar in magnitude to $X^- + X^+$.