Lie and Noether Symmetries of Some Spacetimes Using RIF Tree Approach

A Dissertation Submitted in Partial Fulfillment for the Requirement of the Degree of Doctor of Philosophy in Mathematics

By

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June, 2022

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Date: 06/06/2022

Dedicated To

My Parents, Brothers

and Sisters

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Abstract

This dissertation deals with the study of Lie and Noether symmetries of Kantowski-Sachs, static plane symmetric and locally rotationally symmetric Bianchi type I spacetimes using Rif tree approach. In each case, instead of directly integrating the symmetry equations, a computer algorithm is used to transform these equations to a simplified form. The interesting feature of this algorithm is that it provides all metrics admitting Lie and Noether symmetries other than the minimum ones. The set of Lie and Noether symmetry equations is solved for all these metrics to find the explicit form of symmetry vector fields. Moreover, we have calculated the conservation laws for all the obtained symmetries.

Comparing our obtained results with the existing results of direct integration technique, it is observed that this new approach of Rif algorithm recovers all the metrics obtained by direct integration technique and also this approach gives rise to some new physically realistic metrics.

To add some physical implications, the obtained metrics are used in Einstein's field equations to compute their energy-momentum tensor and it is shown how the parameters involved in the obtained spacetime metrics are associated with certain important energy conditions.

Acknowledgements

All praise to **Almighty Allah**, the most merciful, who gave me potential, courage and knowledge in completing this thesis successfully.

It is a great pleasure to acknowledge, my deepest thanks and gratitude to **Dr. Tahir Hussain** for his kind supervision, guidance and helping me in compiling this work. It is a great honor to work under his supervision.

I would like to express my deepest thanks and appreciation to **Prof. Ashfaque H. Bukhari** Department of Mathematics and Statistics King Fahad University of Petroleum and Minerals, Saudi Arbia, for his useful comments throughout my research studies.

I would like to thank the honorable Chairman **Dr. Imran Aziz** Department of Mathematics, University of Peshawar for providing us better atmosphere to complete the research work.

I humbly extend my thanks to all the faculty members of the Department who taught me and guide me during my course work and research project.

I would like to express my extreme sincere gratitude to **Sumaira Saleem** for her endless help, generous advice and support during research.

In the end I acknowledge my family, without whom I would not be able to finish this work.

Uzma Nasib

List of Publications

Out of the research work presented in this dissertation, the following papers have been published in different reputed journals.

- **U. Nasib**, T. Hussain and A. H. Bokhari, Homothetic Vector Fields of LRS Bianchi Type I Spacetimes via the Rif Tree Approach, *Theor. Math. Phys.* **209** (2021) 1673-1682.

- T. Hussain, **U. Nasib**, M. Farhan and A. H. Bokhari, A Study of Energy Conditions in Kantowski-Sachs Spacetimes via Homothetic Vector Fields, *Int. J. Geom. Meth. Mod. Phys.* **17** (2020) 2050035 (1-15).

- T. Hussain, **U. Nasib** and M. Farhan, A Study of Energy Conditions in Static Plane Symmetric Spacetimes via Homothetic Symmetries, *Int. J. Mod. Phys. A* **34** (2019) 1950238 (1-15).

- T. Hussain, **U. Nasib**, F. Khan and M. Farhan, An Efficient Rif Algorithm for the Classification of Kantowski-Sachs Spacetimes via Conformal Vector Fields, *J. Kor. Phys. Soc.* **76** (2020) 286-291.

- A. H. Bokhari, T. Hussain, J. Khan and **U. Nasib**, Proper Homothetic Vector Fields of Bianchi Type I Spacetimes via Rif Tree Approach, *Results Phys.* **25** (2021) 104299 (1-5).

Chapter 1

Introduction

Newton is believed to be the first one to study gravitation and its nature. According to him, gravity was nothing but an attractive force between massive objects. He proposed his law of gravitation in 1687 which was accepted by all. It was believed that mass is the real source of gravity. According to Newton's physics, space and time are absolute and do not depend on the aspects of objective reality. His law was very successful in explaining the mechanics of solar system and astronomy in general for more than two centuries [1].

In 1905, Einstein gave the idea of unification of space and time as a single 4-dimensional entity known as spacetime. This was the first step towards construction of his special theory of relativity. This theory puts a universal speed limit on every possible physical effects and on gravity as well. This was in conflict with the Newton's notion that gravity acts instantaneously over large distances. Thus these two theories couldn't be combined.

It took Einstein ten more years to present a generalized version of both Newton's classical theory and special relativity, as a result he proposed the general theory of relativity in 1915. According to this theory, gravity is the curvature of spacetime which defines a gravitational field. That gravitational field acts on neighboring matter, causing it to move. Moreover, spacetime's geometric properties are determined by the matter and exclusively by a mass, such properties make the curvature of spacetime. Therefore in general relativity, object's mass does not affect the position of the object in the spacetime (with the assumption that it is not that much large to change the spacetime curvature) and depends on the spacetime's geometry. This leads us to the idea that gravity is a geometrical phenomenon rather than simply a force.

In 1979, Johan Wheeler summarized the relationship between space and matter in general theory of relativity in a single sentence: "Space tells matter how to move and matter tells space how to curve" [2].

The mathematical expression of the above relationship is provided by the Einstein's field equations (EFEs) [3]:

$$G_{ij} = R_{ij} - \frac{R}{2} g_{ij} = kT_{ij}, \tag{1.0.1}$$

where G_{ij} and T_{ij} denote the Einstein and energy-momentum tensors respectively and k defines the gravitational coupling. Moreover, R_{ij}, g_{ij} and R symbolize the Ricci tensor, metric tensor and the Ricci scalar respectively. Though their appearance is simple, it is quite challenging to obtain the exact solutions of EFEs because they are highly non-linear partial differential equations. If we consider the most simple case of vacuum, that is $T_{ij} = 0$, even there the EFEs may be very difficult to solve. Due to this reason, only few exact solutions of EFEs have been found in the literature [3–5].

Spacetime symmetries are not only used to find new exact solutions of EFEs, they are also helpful in the classification of the known solutions. Besides this, these symmetries have a direct relation with conservation laws in a dynamical system. Symmetries usually demand some form of preserving property such as preserving geodesics of a spacetime, metric and the curvature tensors.

Depending upon these preserving properties, spacetime symmetries are defined in terms of specific type of vector fields such as Killing, homothetic, conformal Killing vector fields and Ricci, matter, affine and curvature collineations. All these symmetries are defined by some relations involving Lie derivatives, therefore these symmetries are usually known as Lie symmetries.

Noether symmetries, also known as variational symmetries, play a pivotal role in finding solutions and conservation laws admitted by differential equations (DEs) [6]. In case of ordinary differential equations (ODEs), they are helpful in reducing order of DEs, while for partial differential equations (PDEs) these symmetries are used to reduce the number of independent variables [7]. Apart from this, these symmetries are also used in the linearization of nonlinear DEs [8, 9]. In addition, Noether symmetries are useful because they provide double reduction in case of DEs and are directly related to conservation laws via Noether theorem [10].

To find the Lie and Noether symmetries of spacetimes, one needs to solve a system of PDEs. In literature, these equations are usually solved by direct integration technique. In this approach of studying Lie and Noether symmetries, there is always a chance of losing interesting metrics. Instead of this technique, here we use a new method, known as Rif tree approach, to study Lie and Noether symmetries of some spacetimes. In this method, first

a computer algorithm is developed to convert the determining equations to a simplified form and getting a tree, known as Rif tree, imposing the conditions on the metric functions. These conditions are then used to solve the determining equations, giving the explicit form of symmetry vector fields.
We came upto the conclusion of general theory of relativity and in the following chapter, the discussion will revolve around some basic concepts of general relativity supporting our thesis. Furthermore, the spacetimes and Noether symmetries literature and details of Rif algorithm will be discussed in details.

Chapter 2

Preliminaries

In this chapter, we present some basic concepts of general relativity, which will give a deep insight of the knowledge. Also the spacetimes and Noether symmetries along with their literature will be discussed as well as a brief introduction of a computer program, Rif algorithm will be explained.

2.1 Tensors

Tensor is a concept in mathematical physics which is the generalization of vectors and dual vectors. Tensors play a central role in solving physical problems by providing a complete mathematical background. In general relativity, tensors play a pivotal role because EFEs are tensor equations.

Tensors as Mulltilinear Map

Let V be a finite dimensional vector space. A tensor T of type (m, n) is a multilinear map:

$$T : \underbrace{V^* \times V^* \times V^* \times ...V^* \times}_{m-times} \underbrace{V \times V \times V \times ...V}_{n-times} \longrightarrow R, \tag{2.1.1}$$

where "$\times$" represents the Cartesian product, m and n are non-negative integers and $m + n$ represents rank or order of the tensor T [11]. Moreover, V^* is the dual space of the vector space V. By multilinear, we mean that T satisfies the relation:

$$T(v_1, ..., \alpha_1 p+\alpha_2 q, ..., v_{m+n}) = \alpha_1 T(v_1, ..., p, ..., v_{m+n})+\alpha_2 T(v_1, ..., q, ..., v_{m+n}), \tag{2.1.2}$$

where α_1 and α_2 are scalars and $v_1, ..., v_{m+n}$, p and q are elements of V or V^* as appropriate.

Types of Tensors

Tensors of type $(0, 1)$, $(1, 0)$, $(0, m)$, $(n, 0)$, (m, n) are called covariant vector, contravariant vector, covariant tensor, contravariant tensor and mixed tensor respectively and a tensor of type $(0, 0)$ is defined as a scalar.

Tensor Product

Tensor product of two covariant tensors T of rank m and T' of rank n is denoted by $T \otimes T'$ and is defined as [12]:

$$T \otimes T'(v_1, v_2, ..., v_m, w_1, w_2, ..., w_n) = T(v_1, v_2, ..., v_m)T'(w_1, w_2, ..., w_n). \tag{2.1.3}$$

Similar expressions are valid for all types of tensors. The tensor product is not commutative but it is distributive and bilinear.

Symmetric and Antisymmetric Tensors

If the indices of a tensor remain unchanged by exchanging any two of its contravariant or covariant indices then it is said to be symmetric, while it is antisymmetric when it changes sign with the exchange of any two of its contravariant or covariant indices [11].

Some Important Tensors

There is a vast variety of tensors but in general relativity the most important tensors are metric, Riemannian curvature, Ricci, Weyl, energy-momentum and Einstein tensors.

Metric Tensor: The metric tensor of type $(0, 2)$ is a symmetric bilinear scalar function of two vectors. It takes two vectors from tangent space and returns a scalar.
It is one of the important tensors in general relativity which shows the infinitesimal squared distance along a curve. Let ds be the infinitesimal distance between two points on a curve. Let $d\vec{r}$ be the vector joining the two points, then ds^2 may be written as:

$$ds^2 = d\vec{r} \cdot d\vec{r}. \tag{2.1.4}$$

If dx^i represent contravariant components of $d\vec{r}$ and $\vec{e_i}$ are the coordinate basis vectors, then we can write:

$$d\vec{r} = \vec{e_i}\ dx^i. \tag{2.1.5}$$

Similarly, for covariant components dx_i and the dual basis vectors $\vec{e^i}$, one can write:

$$d\vec{r} = \vec{e^i}\ dx_i. \tag{2.1.6}$$

Thus for the contravariant components dx^i, we get:

$$\begin{aligned} ds^2 &= d\vec{r} \cdot d\vec{r} \\ &= \vec{e_i}\ dx^i \cdot \vec{e_j}\ dx^j \\ &= (\vec{e_i} \cdot \vec{e_j})\ dx^i\ dx^j \\ &= g_{ij}\ dx^i\ dx^j, \end{aligned} \tag{2.1.7}$$

where g_{ij} denote the covariant components of the metric tensor. Similarly, one may use the covariant components dx_i to get:

$$ds^2 = g^{ij}\ dx_i\ dx_j. \tag{2.1.8}$$

The signs of $+$ and $-$ appearing in the metric represent signature of the metric. Particularly, a metric of signature $(-, +, ..., +)$ or $(+, -, ..., -)$ is called a Lorentzian metric and a metric with signature $(+, +, ..., +)$ is positive definite or Riemannian metric. Spacetime metric is always Lorentzian [13].

Riemannian curvature tensor: Another important tensor in general relativity is the Riemannian curvature tensor which is of type $(1, 3)$ and is expressed as [13]:

$$R^m_{ijk} = \Gamma^m_{ik,j} - \Gamma^m_{ij,k} + \Gamma^n_{ik}\Gamma^m_{nj} - \Gamma^n_{ij}\Gamma^m_{nk}, \tag{2.1.9}$$

where Γ^m_{ij} represents Christoffel symbol which is defined as follows:

$$\Gamma^m_{ij} = \frac{1}{2} g^{md} \left(g_{id,j} + g_{dj,i} - g_{ij,d} \right). \tag{2.1.10}$$

Riemannian curvature tensor vanishes if a spacetime is flat.

Ricci Tensor: The contraction of the Riemann curvature tensor R^m_{ijk} in the first and third indices gives rise to the Ricci tensor of type $(0, 2)$, which is defined as:

$$R_{ij} = R^k_{ikj} = \Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^n_{ij}\Gamma^k_{nk} - \Gamma^n_{ik}\Gamma^k_{nj}. \tag{2.1.11}$$

Ricci Scalar: The contraction of the Ricci tensor with metric is known as Ricci scalar:

$$R = g^{ij} R_{ij}, \tag{2.1.12}$$

where R represents the Ricci scalar.

Weyl Tensor: Weyl tensor is a type $(0, 4)$ tensor, also known as conformal tensor, which is defined as:

$$C_{ijkm} = R_{ijkm} + \frac{1}{2}[g_{im}R_{jk} + g_{jk}R_{im} - g_{ik}R_{jm} - g_{jm}R_{ik}] + \frac{1}{6}R[g_{ik}g_{jm} - g_{im}g_{jk}]. \tag{2.1.13}$$

If all the components of C_{ijkm} vanishes, then a spacetime is said to be conformally flat.

Energy-Momentum Tensor: The energy-momentum tensor, which is also termed as stress-energy tensor is symbolized by T_{ij} and it describes the pressure, energy density and flux of momentum in a spacetime. The spacetime is said to be vacumm if $T_{ij} = 0$.

Einstein Tensor: Einstein tensor, which is denoted by G_{ij}, is a symmetric tensor of rank 2 which explains the spacetime curvature and it is defined as:

$$G_{ij} = R_{ij} - \frac{1}{2} R g_{ij}. \tag{2.1.14}$$

2.2 Tangent vector and Tangent spaces

A tangent vector w at a point p of manifold M is a mapping $w : C^{\infty}(M) \longrightarrow R$, which satisfies the following properties [3]:

i. $w(k_1 + k_2) = w(k_1) + w(k_2),\ k_1,\ k_2 \in C^{\infty}(M)$

ii. $w(k_1 k_2) = k_2 w(k_1) + k_1 w(k_2)$,

iii. $w(c_1 k_1) = c_1 w(k_1)$, where c_1 represents a constant.

A tangent space is the collection of all tangent vectors at a point p on M, which is denoted as T_pM and a tangent bundle is the union of all T_pM at all points of M and is given by:

$$TM = \bigcup_{p \varepsilon M} T_pM. \tag{2.2.1}$$

2.3 Vector Field

A vector field ξ on a smooth manifold M is a map $\xi : M \longrightarrow TM$, which associates a tangent vector $\xi_p \in T_pM$ to each point p of M. Moreover, the vector field ξ is said to be global or local if it is defined on the whole M or on some subset of M respectively [14].

Lie Brackets: Lie bracket is an operator, used to combine two vector fields and get another vector field. For example, if ξ_i and ξ_j are two vector fields on a manifold M, then their Lie bracket (also known as Lie commutator) is defined as [15]:

$$[\xi_i, \xi_j](f) = \xi_i(\xi_j(f)) - \xi_j(\xi_i(f)),$$

for all smooth functions $f : M \to R$. The Lie bracket has the following properties:

i. Bilinearity: If ξ_i, ξ_j, ξ_k, ξ_l are vector fields on M and α, β, γ, δ are constants , then:

$$[\alpha\xi_i + \beta\xi_j, \gamma\xi_k + \delta\xi_l] = \alpha\gamma[\xi_i, \xi_k] + \alpha\delta[\xi_i, \xi_l] + \beta\gamma[\xi_j, \xi_k] + \beta\delta[\xi_j, \xi_l].$$

ii. Skew symmetry: If ξ_i and ξ_j are vector fields on a manifold M, then:

$$[\xi_i, \xi_j] = -[\xi_j, \xi_i].$$

iii. Jacobi identity: If ξ_i, ξ_j, ξ_k are vector fields on M, then:

$$[[\xi_i, \xi_j], \xi_k] + [[\xi_j, \xi_k], \xi_i] + [[\xi_k, \xi_i], \xi_j] = 0.$$

2.4 Lie Derivative

For a real-valued function $F(t)$, its derivative at some point $p \in R$ is defined as:

$$F'(p) = \lim_{t \to 0} \frac{1}{t}(F(t + p) - F(p)). \tag{2.4.1}$$

This definition cannot be generalized to define the derivative of a vector field on a manifold M because if ξ is a vector field on M and p_1 and p_2 are two nearby points on M, then the tangent vectors ξ_{p_1} and ξ_{p_2} belong to the tangent spaces $T_{p_1}M$ and $T_{p_2}M$, which are different vector spaces. Thus it is not possible to subtract ξ_{p_1} and ξ_{p_2}. To overcome this difficulty, the local flow ϕ_t of another vector field Y is used to transport the vector ξ_{p_2} to the tangent space $T_{p_1}M$ at p_1. This leads to the definition of Lie derivative of a vector field.

For a smooth manifold M, Let T be a global smooth tensor field on M and ξ be a global smooth vector field on M. For some appropriate t, Let ϕ_t be the local diffeomorphisms of ξ with ϕ_t^* as the corresponding pullback maps. Then the following limit defines the Lie derivative of T along ξ at a point $p \in M$ [14].

$$\mathcal{L}_\xi T_p = \lim_{t\to 0} \frac{\phi_t^* T_p - T_p}{t}. \tag{2.4.2}$$

It is to be mention here that $\mathcal{L}_\xi T$ is a global smooth tensor field on M and its type is same as that of T. If in some coordinate system, $T^{a\ldots b}_{c\ldots d}$ and ξ^a are the components of T and ξ respectively, then the components of $\mathcal{L}_\xi T$ are given by:

$$(\mathcal{L}_\xi T)^{a\ldots b}_{c\ldots d} = T^{a\ldots b}_{c\ldots d,e}\xi^e - T^{e\ldots b}_{c\ldots d}\xi^a_{,e\ldots} - T^{a\ldots e}_{c\ldots d}\xi^b_{,e} + T^{a\ldots b}_{e\ldots d}\xi^e_{,c\ldots} + T^{a\ldots b}_{c\ldots e}\xi^e_{,d}. \tag{2.4.3}$$

The left hand side of above equation can also be written as $\mathcal{L}_\xi T^{a\ldots b}_{c\ldots d}$. Moreover, Lie derivative has the following properties:

i. $\mathcal{L}_\xi g = \xi(g)$.

ii. $\mathcal{L}_{\xi_1}\xi_2 = [\xi_1, \xi_2]$.

iii. $\mathcal{L}_\xi(T_1 \bigotimes T_2) = \mathcal{L}_\xi T_1 \bigotimes T_2 + T_1 \bigotimes \mathcal{L}_\xi T_2$.

iv. $\mathcal{L}_{a\ \xi_1} +_{b\ \xi_2} T = a\mathcal{L}_{\xi_1} T + b\mathcal{L}_{\xi_2} T$.

v. $\mathcal{L}_{[\xi_1,\xi_2]} T = \mathcal{L}_{\xi_1}(\mathcal{L}_{\xi_2} T) - \mathcal{L}_{\xi_2}(\mathcal{L}_{\xi_1} T)$.

2.5 Spacetime Symmetries in General Relativity

The EFEs governe the theory of general relativity which are highly non-linear differential equations. These equations are the core of general relativity but due to their non-linear nature, finding the exact solutions for these equations is a cumbersome work. Though the solution of EFEs is a difficult task but is very important. To solve these equations, one needs to assume the possession of certain symmetries in these equations. Symmetries are important in finding not only the exact solutions of EFEs but in their classification as well.

In 2004, Hall [14] defined spacetime symmetries as vector fields preserving some features of a spacetime, like metric, geodesics, Ricci or Riemann curvature tensor. Generally, such vector fields satisfy the following relation:

$$\mathcal{L}_\xi \Psi = \Lambda, \tag{2.5.1}$$

where $\mathcal{L}$ denotes the Lie derivative operator, ξ is the symmetry vector field and Ψ is one of the quantities g_{mn}, T_{mn} or R_{mn} and the tensor Λ has the same index symmetries as Ψ.

For $\Psi = g_{mn}$ and $\Lambda = 2\alpha(x^a)g_{mn}$, Eq. (2.5.1) defines conformal vector fields (CVFs) which reduce to homothetic vector fields (HVFs) if α is constant and to Killing vector fields (KVFs) if $\alpha = 0$. Similarly, if $\Psi = T_{mn}$ and $\Lambda = 2\alpha(x^a)T_{mn}$ in Eq. (2.5.1), it gives rise to conformal symmetries of the matter tensor, which reduce to matter and homothetic matter collineations for $\alpha = 0$ and α constant respectively [16]. Moreover, if we put $\Psi = R_{mn}$ and $\Lambda = 2\alpha(x^a)R_{mn}$ in Eq. (2.5.1), the corresponding spacetime symmetries are called conformal Ricci collineations, which specialize to Ricci collineations when $\alpha = 0$ and homothetic Ricci collineations when α is a constant.
Since the focus of this thesis is an Lie symmetries of the metric tensor, so here we discuss these symmetries in detail.

2.5.1 Killing Vector Fields

A vector field ξ is said to be a KVF if the metric tensor g_{ij} is invariant under the Lie operator, that is [14]:

$$\mathcal{L}_\xi g_{ij} = g_{ij,}\xi^a + g_{ia}\xi^a_{,j} + g_{ja}\xi^a_{,i} = 0. \tag{2.5.2}$$

Wilhelm Killing in 1892 obtained the above equations and are known as Killing equations, where the solutions of Killing equations are termed as KVFs. For an m-dimensional manifold M, the maximum dimension of $K(M)$ is $\frac{m(m+1)}{2}$, where $K(M)$ denotes the collection of all KVFs on M which forms a Lie algebra. For a spacetime, dim $K(M) \leq 10$. Killing algebra attains the maximum dimension if the spacetime is flat [3].
In the literature, KVFs are used not only in constructing the new exact solutions of EFEs, but also in the classification of known solutions. Moreover,

the physical importance of KVFs cannot be ignored as they give rise to the conservation laws and their study is pivotal in understanding the physics of the gravitational fields. For a more comprehensive study of KVFs and their physical importance, we refer [5].

KVFs have been studied by different researchers. Petrov [5] was the first who solved the Killing equations and found Killing vectors in four-dimensional spaces. Bokhari and Qadir [17] studied Killing symmetries for static spherically symmetric spacetimes. Ali and his collaborators [18] explored KVFs of Bianchi type $VI_\circ$ and $VII_\circ$ spacetimes. Ali et. al. [19] investigated KVFs for non-static spherically symmetric spacetimes. Bokhari and his collaborators [20] studied Killing symmetries in three- dimensional circularly symmetric static metric. Feroze et. al. [21] got complete classification of plane symmetric Lorentzian manifolds after solving Killing equations for these symmetries. Khan et. al. [22] investigated Killing vector fields for LTB spacetime. The classification of static cylindrically symmetric and non-static spherically symmetric spacetimes via KVFs was given by Qadir and Ziad [23, 24].

2.5.2 Homothetic Vector Fields

A vector filed ξ on a manifold M which preserves the metric of spacetime up to a constant factor is known as a HVF. Such vector fields satisfy the relation [14]:

$$\mathcal{L}_\xi g_{ij} = g_{ij,}\xi^a + g_{ia}\xi^a_{,j} + g_{ja}\xi^a_{,i} = 2\alpha g_{ij}, \tag{2.5.3}$$

where α is a constant. For an m-dimensional manifold M, the maximum dimension of $H(M)$ is $\frac{m(m+1)}{2} + 1$, where $H(M)$ denotes the collection of all HVFs on M which forms a finite-dimensional Lie algebra, called homothetic algebra. Particularly, for a spacetime manifold M, we have dim $H(M) \leq 11$. The maximum dimension of homothetic algebra $H(M)$ is when the spacetime is of constant curvature or it is flat.

As for as the Lie symmetries are concerned, they have useful applications in both general relativity and mathematical physics. For example, HVFs are important in the study of singularities in general relativity. Also, HVFs are helpful in finding the new exact solutions of EFEs and in classifying them by their conservation laws. Though few solutions of EFEs have been found by assuming, in advance, that they possess a proper homothety, there are many solutions of these equations which admit proper homothety whose existence accounts for the comparatively simple forms of their line elements and consequently for their discovery [3]. Moreover, HVFs are regarded as a kind self-similar solutions of EFEs, known as similarity of the first kind. The study of self similar solutions of EFEs is important because of two main reasons. First, the self-similarity reduces the mathematical complexity of these equations, usually leading to the reduction of partial differential equations to ordinary differential equations, which are then comparatively easy to study. Second, self-similar solutions play important role in describing asymptotic behaviors of more general non-self-similar solutions. Apart from this, various astrophysical and cosmological applications of self similar solutions (homotheties) can be found in literature [25].

Like KVFs, different researchers explored HVFs for different spacetimes. Proper HVFs of Bianchi type I spacetime were explored by Shabbir and Amur [26]. Ali et al. [27,28] studied proper HVFs of Bianchi type IV and V spacetimes. Ahmad and Ziad [29] investigated the HVFs of spherically symmetric spacetimes. The contribution of Shabir and Ramzan [30] was their exploration of HVFs of static cylindrically symmetric spacetimes. Hall et al. [31] explored that the maximum dimension of homothetic algebra is 11 in four- dimensional spacetime. Shabbir and Fouzia [32] studied Kantowski-Sachs and Bianchi type III spacetimes and concluded that these spacetimes admit five independent HVFs. Qadir et al. [33] studied HVFs of cylindrically symmetric static manifolds and their global extension. Ali et al. [34] investigated proper HVFs for circularly symmetric static spacetimes on a three-dimensional Lorentzian manifold. Ziad [35] studied the plane symmetric spacetimes' classification via their HVFs.

2.5.3 Conformal Killing Vector Fields

A vector field ξ is said to be a CVF if it satisfies the following relation [14]:

$$\mathcal{L}_\xi g_{ij} = 2\alpha g_{ij}, \tag{2.5.4}$$

where $\alpha : M \to R$ is some smooth function, known as conformal function of ξ. Some particular forms of CVFs are HVFs and KVFs for which $\alpha = const.$ and $\alpha = 0$, respectively. The CVFs other than the homothetic and Killing vector fields are known as proper CVFs. Moreover, if $\alpha_{;ij} = 0$ then a CVF ξ is called a special CVF. One drawback of CVF is that they do not keep the Einstein's tensor invariant as homothetic and Killing vector fields do, but they preserves the casual character of the spacetime manifold [36]. If CVF(M) denotes the set consisting all CVFs on M, then CVF(M) gives the structure of a finite-dimensional Lie algebra of smooth vector fields, called the conformal algebra, such that dim $C(M) \leq 15$. The conformal algebra attains its maximum dimension if and only if the spacetime is conformally flat; that is, its Weyl tensor vanishes. For non-conformally flat spacetimes, dim $C(M) \leq 7$.

Earlier the conformal symmetry was considered just a mathematical tool for integrating EFEs but its physical usage was ignored in the study of cosmology and astrophysics. Recently, some work has been done that shows the usage of conformal symmetry in the study of cosmology and astrophysics [37]. Chrobok et al. [38], made an assumption for temperature vector to be CKV in the theory of irreversible and he got satisfying results thermodynamical processes and he got interesting results. Bohmer et al. [39] proved that the conformal factor for conformally symmetric spacetimes with moveable vector fields can be explained in terms of tangential velocity of the test particles that move in circular orbits. Its according to an assumption of spherically symmetry that admits one parameter group of conformal vector. Mak et al. [40] discovered a correct solution that describes and explains the inner part of charged strange quark star. Moreover, Usmani and other scientists suggested an astrophysical model named as gravastar, that admits CKV [41]. Arising of KVFs, usually ensures the conservation laws in spacetimes. However, in some circumstances, KVFs almost fail to find the conservation laws and in these situations, conformal transformations are used to find conservation laws in place of KVFs. For instance, in Friedman metric case, translation invariance is not present to provide the energy with conservation law, instead conformal analogue of energy conservation law is ensured by conformal time-translation invariance. Conformal motion symmetry is also a source of study of kinematic variables including but not limited to expansion, rotation and shear in spacetimes [42–45].

Shortly, conformal symmetry has many usages which are helpful in understanding physical and geometrical properties of spacetimes physics.

Like KVFs and HVFs, different researchers explored conformal symmetry for some spacetimes. CVFs are explored by Maartens et al. [46] for Friedmann-Robertson-Walker spacetimes. Hall and Steele [47] pointed out that if a spacetime has zero curvature (flat), then conformal symmetries will be 15 while for non-conformally flat spacetimes, the dimension of conformal algebra will be less than or equal to 7. Khan et al. [48, 49] investigated the CVFs on a four-dimensional Lorentzian manifolds of plane symmetric and locally rotationally symmetric (LRS) Bianchi type V spacetimes. Hall and Capocci [50] worked on the maximum dimension of conformal algebra for three-dimensional spacetimes and found that dim $C(M) \leq 14$ for non-conformally flat three-dimensional spacetime. Saifullah and Yazdan [51] studied conformal symmetries of static plane symmetric spacetimes and concluded that these spacetimes do not admit any proper CVFs. Later on, Hussain et al. [52] investigated that non-conformally flat static plane symmetric spacetime metrics have proper CVF. Coley and Tupper [53,54] studied proper inheriting CKVF of spherically symmetric spacetimes in both perfect and anisotropic fluids. Moreover, Coley et al. [55] studied perfect fluid, plane symmetric spacetimes admitting a proper inheriting CKVF. For the analysis of conformal motions of some other spacetimes, we refer [56–61].

2.6 Noether Symmetries

Apart from the conventional symmetries, there exist some other symmetries which are associated with differential equations (DEs). These symmetries are known as Noether symmetries, whose idea was given by Emmy Noether in 1918 [10]. Such symmetries admit conservation laws and describe the physical features of DEs in terms of these conservation laws. To study in detail the basic theory along with the advanced concepts of Noether symmetries and conservation laws of DEs, one can refer to the references [7, 10, 15, 62–70]. Noether proved a theorem, on the basis of Euler-Lagrangian (geodesic) system, which states that there is always a conservation law for every continuous symmetry admitted by the Lagrangian of a physical system. Consequently, this theorem gives conservation of energy and linear and angular momenta of a physical system if it is invariant under time translation and spacial translations and rotations. Noether's theorem helps to find out the conserved quantity for a continuous transformation of a symmetry that makes the action constant.

A vector field ξ of the form $\xi = \eta\frac{\partial}{\partial s} + \xi^a\frac{\partial}{\partial x^a}$, defines a Noether symmetry which leaves the Lagrangian L of a dynamical system invariant and satisfies the following condition [15]:

$$\xi^{[1]}L + LD(\eta) = DF. \tag{2.6.1}$$

In the above expression, $\xi^{[1]}$ is the first prolongation of ξ which is defined as $\xi^{[1]} = \xi + \xi^i_s\frac{\partial}{\partial \dot{x}^i}$, where $\xi^i_s = D\xi^i - \dot{x}^i D\eta$ and $D = \frac{\partial}{\partial s} + \dot{x}^i\frac{\partial}{\partial x^i}$. Moreover, the functions η, ξ^i and F (gauge function) all depend on five variables (s, x^0, x^1, x^2, x^3) and $x^i = (x^0, x^1, x^2, x^3)$ is dependent on the affine parameter

s such that $\dot{x^i} = \frac{\partial x^i}{\partial s}$.

Moreover, there exist some well known relations of Noether symmetries with KVFs and HVFs in such a way that every KVF is a Noether symmetry but the converse is not true. If ξ is a HVF if and only if $\xi + 2cs\partial_s$ is a Noether symmetry, where c represents the homothety constant. A Noether symmetry which is not a KVF and does not correspond to a HVF is known as a proper Noether symmetry. The most important feature of the Noether symmetries is that each Noether symmetry vector field corresponds to a conservation law and using the Noether's theorem, such conservation law is given by the expression [10]:

$$I = \eta L + \left(\xi^i - \dot{x^i}\eta\right)\frac{\partial L}{\partial \dot{x^i}} - F. \tag{2.6.2}$$

As the algebra of Noether symmetries contains the set of Killing and homothetic vector fields, we always expect to get some conservation laws in spacetimes with the help of Noether symmetries which are not given by Killing and homothetic vector fields.

The Noether symmetries attain the maximum dimension of Noether algebra that is 17, if the spacetime is flat.

The importance of Noether's work is not limited to its feature that for every symmetry there is a conservation law, but is applicable in different fields such as classical and quantum mechanics, electromagnetism, continuous group, particle physics and general relativity etc.

In literature, Noether symmetries are explored for some well known spacetimes. Hickman and Yazdan [71] explored Noether symmetries of Bianchi type II spacetimes. Ali et al. [72–74] investigated Noether symmetries of static spherical, cylindrical and plane symmetric spacetimes. A detailed study of Bianchi type V spacetimes via Noether symmetries was given in [75]. Bokhari and Kara [76] gave a complete classification of conformally flat Friedmann metric via Noether symmetries and their results were compared with KVFs. They pointed out that this type of metric admits additional conservation laws not given by KVFs. A similar comparison of Noether symmetries and KVFs was given by Bokhari et al. [77] and shown that the Noether symmetries obtained by considering the Larangians give additional symmetries which are not given by the Killing vectors. Camci [78] gave a classification of Gödel type spacetimes according to Noether symmetries of their geodesic Lagrangian. Camci et al. [79] explored Noether symmetries of the Lagrangian for some classes of pp-wave spacetimes. A relationship between Lie symmetries of Klein-Gordon equation and conformal Killing vectors of the underlying geometry was established by Paliathanasis et al. [80], where they also stated that the resulting Lie symmetries of the conformal algebra are also Noether symmetries. Usamah et al. [81] explored the Noether symmetries of non-static plane symmetric spacetimes. Jamil and his collaborators [82] gave the geometrical and physical interpretation of the conserved quantities corresponding to each Noether symmetry of the geodetic Lagrangian of plane symmetric spacetimes. Hussain et al. [83, 84] investigated Noether symmetries for the Lagrangians of Kantowski-Sachs and non-static spherically symmetric spacetimes. A complete classification of the LRS Bianchi type I and V spacetimes via Noether symmetries was given by Hussain and Akhtar [85, 86].

Moreover, the role of Noether symmetry approach is also noticeable in the classification of exact solutions of EFEs, for details we refer [87–96].

2.7 Energy Conditions

All the states of matter possess certain properties which are described by the energy conditions and are strong enough to rule out many un-physical solutions of the EFEs. The energy conditions have different forms i.e. null, weak, dominant and strong energy conditions which are actually restrictions on the eigenvalues and eigenvectors of the energy-momentum tensor. For a known spacetime metric, one can determine the source of matter through energy-momentum tensor, which in turn can be used to find bounds for different energy conditions. Consequently, the physical importance of the models can be determined through these energy conditions. The inequalities for different energy conditions for an anisotropic fluid matter are as follows [97]:

$$\begin{aligned}
\text{Null energy condition } (NEC) \quad &: \quad \rho + p_{||} \geq 0, \quad \rho + p_{\perp} \geq 0, \\
\text{Weak energy condition } (WEC) \quad &: \quad \rho \geq 0, \quad \rho + p_{||} \geq 0, \quad \rho + p_{\perp} \geq 0, \\
\text{Strong energy condition } (SEC) \quad &: \quad \rho + p_{||} \geq 0, \quad \rho + p_{\perp} \geq 0, \\
&\quad\;\; \rho + p_{||} + 2p_{\perp} \geq 0, \\
\text{Dominant energy condition } (DEC) \quad &: \quad \rho \geq 0, \quad \rho \geq |p_{||}|, \quad \rho \geq |p_{\perp}|,
\end{aligned} \tag{2.7.1}$$

where ρ denotes the energy density and $p_{\perp}$ and $p_{||}$ respectively represent the perpendicular and parallel pressures to spacelike unit vector n_a.

Moreover, if $p_{||} = p_{\perp}$, then the above expressions give the bounds for energy conditions for a perfect fluid.

2.8 Rif Algorithm

In different fields of science and technology, DEs play an important role. Mathematically, DEs are studied from different perspectives, but the main focus always remains on their solutions. Moreover, DEs may be linear or non-linear. It requires less effort to get the solution of DEs for linear problems as compared to non-linear problems. As far as non-linear problems are concerned, the construction of their solution requires some general approach. The most general approach is to study their Lie group of symmetries for constructing such solutions [98]. Sophus Lie was the first who started the symmetry analysis of DEs in 1870 and Lie symmetry methods become major tools for finding solutions of ordinary and partial differential equations [15, 99, 100].

The involvement of systematic algebric manipulation and tedious calculation in symmetry analysis makes method of symmetry analysis relevant to computer algebra. Moreover, for symmetry analysis, a number of computer algebra packages have been developed which give a basis to analyze and solve DEs [101–103].

Computer algebra usage for symmetry analysis started in 1980 which developed some new packages used for symmetry analysis and finding the symmetries of DEs. These symmetry analysis packages were used for three purposes: (i) to find the determining equations of DEs, (ii) to reduce these determining equations, and (iii) to solve these reduced determining equations to get symmetries. The purpose (ii) of symmetry analysis packages,was to to simplify the determining equations and getting much chances for their solutions. For this purpose, differential reduction and completion (DRC) methods [104–106] were included in symmetry analysis. The DRC methods help to reduce DEs into a simple form, called reduced form, having some information about the solutions of DEs. Several kinds of DRC methods including Rosenfeld-Gröbner algorithm [104], differential Gröbner basis [105] and Rif algorithm [106] are implemented using computer algebra systems such as Maple. These packages are widely used by the users. For example, to solve DEs, the Maple package "rifsimp" has achieved the status of front end procedure, which itself uses Rif algorithm [107].

Rif algorithm is one of the DRC algorithms which transforms the analytic systems of non-linear PDEs into a reduced involutive form, also referred to as Rif form [106]. The Rif algorithm has some interesting features. To begin with, the algorithm ends in a finite number of steps [108]. Second, the process consists only of differentiation and elimination with no integration. Furthermore, the Rif form which is the outcome of the Rif process, incorporates geometric properties of PDE systems. Despite the fact that Rif form is coordinate-dependent, it can easily be converted into a system with involutive geometric properties.

The Rif algorithm was first developed by Reid et al. [106] in 1996. To deal with linear systems, the standard form of Rif algorithm is used, where as for non-linear cases, further extensions of Rif algorithm are used. Reid et al. implemented the Rif algorithm as a Maple package, known as "rifsimp" and later on it modified by Wittkof [109]. The advantages of this algorithm include reducing the complexity of the system and extracting the information from the system, such as the number of its solutions without solving it. The output of the Rif algorithm can be viewed graphically by a command

'caseplot'. Such a plot is known as classification tree or Rif tree. The nodes of the Rif tree denote pivots which are actually the coefficients of the highest order derivatives involved in the system of equations.
While developing the Rif algorithm to get a workable Rif tree, some important operational issues need to be addressed. The most important issue is the variable ordering. Different ordering of the variables produces different Rif trees and no theory is available that assists in variable ordering and getting a simplified Rif tree. Trial and error is the only option to get a workable Rif tree.

2.9 Dissertation's Outline

In this dissertation, the main focus is on finding the Lie and Noether symmetries of different spacetimes by using Rif tree approach and to compare our results with those of direct integrating technique. Finding the bounds for different energy conditions for the obtained metrics during classification is also the focus of this dissertation. The dissertation is designed as follows:

1. In chapter 1, a brief discussion of Einstein's theory of relativity is given.
2. In chapter 2, the basics of general relativity, spacetime and Noether symmetries and their related literature and an overview of Rif algorithm are given.
3. The complete classification of Kantowski-Sachs spacetimes, static plane symmetric and LRS Bianchi type I spacetimes with respect to their Lie and Noether symmetries are given in chapters 3, 4 and 5 respectively.

The bounds for different energy conditions for the obtained metrics and the comparison of our findings with the conventional method are also given in these chapters.

3. The last chapter concludes the whole dissertation.

Chapter 3

Lie and Noether Symmetries of Kantowski-Sachs Spacetimes

In this chapter, we have investigated homothetic, conformal and Noether symmetries of Kantowski-Sachs spacetimes. Instead of directly integrating the determining equations of each symmetry, first we have developed an algorithm in Maple, which reduces the system of these equations to the reduced involutive form (Rif) and produces a tree, known as Rif tree. The branches of this Rif tree yield the potential metrics that may possess the symmetries other than the minimum ones. The set of determining equations is integrated for each branch of the Rif tree to obtain the final form of the homothetic, conformal and Noether symmetries admitted by the corresponding metrics.

3.1 Homothetic Symmetries

In this section, we calculate the homothetic symmetries of Kantowski-Sachs spacetimes. This work has been published in an ISI journal [110]. The Kantowski-Sachs cosmological model is given by [111]:

$$ds^2 = -dt^2 + G^2\ dr^2 + H^2[\ d\theta^2 + \sin^2\theta\ d\phi^2], \tag{3.1.1}$$

where $G = G(t) \neq 0$ and $H = H(t) \neq 0$. The minimum KVFs for this metric are:

$$K_4 = \left\{ \frac{\partial}{\partial_r}, \frac{\partial}{\partial_\phi}, \sin\phi\frac{\partial}{\partial_\theta} + \cot\theta\cos\phi\frac{\partial}{\partial_\phi}, -\cos\phi\frac{\partial}{\partial_\theta} + \cot\theta\sin\phi\frac{\partial}{\partial_\phi} \right\}.$$

Without mentioning any particular source, the non-zero components of energymomentum tensor for the metric (3.1.1) are:

$$\begin{aligned} T_{00} &= 2\frac{G'H'}{GH} + \frac{H'^2}{H^2} + \frac{1}{H^2}, \\ T_{11} &= -G^2\left(2\frac{H''}{H} + \frac{H'^2}{H^2} + \frac{1}{H^2}\right), \\ T_{22} &= -H^2\left(\frac{G''}{G} + \frac{H''}{H} + \frac{G'H'}{GH}\right), \\ T_{33} &= \sin^2\theta T_{22}, \end{aligned} \tag{3.1.2}$$

which take specific form for different sources of matter. For example, if the source is an anisotropic fluid, then $T_{ab} = (\rho + p_\perp)u_a u_b + (p_{||} - p_\perp)n_a n_b + p_\perp g_{ab}$, where ρ denotes the energy density, u_a and n_a respectively are the four-velocity and spacelike unit vector and $p_\perp$ and $p_{||}$ respectively represent the perpendicular and parallel pressures to n_a. For this particular source, the components of energy-momentum tensor (3.1.2) become:

$$T_{00} = \rho,\ T_{11} = G^2 p_{||},\ T_{22} = H^2 p_\perp,\ T_{33} = \sin^2\theta T_{22}. \tag{3.1.3}$$

Moreover, if $p_{||} = p_{\perp}$, then these expressions give a perfect fluid matter. Comparing (3.1.2) and (3.1.3), we have:

$$\begin{aligned} \rho &= \frac{2G'H'}{GH} + \frac{H'^2}{H^2} + \frac{1}{H^2}, \\ p_{||} &= -\left(\frac{2H''}{H} + \frac{H'^2}{H^2} + \frac{1}{H^2}\right), \\ p_{\perp} &= -\left(\frac{G''}{G} + \frac{H''}{H} + \frac{G'H'}{GH}\right). \end{aligned} \tag{3.1.4}$$

Thus, for any Kantowski-Sachs metric with an anisotropic fluid source, one can find the bounds for null, strong, weak and dominant energy conditions by using the above values in the inequalities given in (2.7.1). Using the metric (3.1.1) in Eq. (2.5.3), we obtain:

$$G^0_{,t} = \alpha, \tag{3.1.5}$$

$$G^0_{,r} - G^2\ \xi^1_{,t} = 0, \tag{3.1.6}$$

$$G^0_{,\theta} - H^2\ \xi^2_{,t} = 0, \tag{3.1.7}$$

$$G^0_{,\phi} - H^2\ \sin^2\theta\xi^3_{,t} = 0, \tag{3.1.8}$$

$$G'\ \xi^0 + G\ \xi^1_{,r} = \alpha G, \tag{3.1.9}$$

$$G^2\ \xi^1_{,\theta} + H^2\ \xi^2_{,r} = 0, \tag{3.1.10}$$

$$G^2\ \xi^1_{,\phi} + H^2\ \sin^2\theta\xi^3_{,r} = 0, \tag{3.1.11}$$

$$H'\ \xi^0 + H\ \xi^2_{,\theta} = \alpha H, \tag{3.1.12}$$

$$\xi^2_{,\phi} + \sin^2\theta\xi^3_{,\theta} = 0, \tag{3.1.13}$$

$$H'\ \xi^0 + H\ \cot\theta\xi^2 + H\ \xi^3_{,\phi} = \alpha H. \tag{3.1.14}$$

where α is a constant. To find the HVFs of Kantowski-Sachs spacetimes, we need to solve the above system of equations. Certainly, if one tries to integrate the system of these equations directly, many cases will arise depending upon the nature of the metric functions G and H. In literature, such systems for finding the HVFs for different spacetimes are usually solved by direct integration technique [26–30]. However, here we develop a Maple algorithm which uses the Exterior package and transforms the set of these equations to the reduced involutive form. The details about the procedure of transforming a system of equations to the reduced involutive form is given in Ref. [106]. As a result of the developed Maple Rif algorithm, we have obtained the Rif tree given in Fig. 3.1 and the corresponding pivots.

$$
\begin{aligned}
p_1 &= H', \\
p_2 &= GH' + G'H, \\
p_3 &= GG''' - G'G'', \\
p_4 &= G''H - GH'', \\
p_5 &= GH' - G'H, \\
p_6 &= GH'' - G'H', \\
p_7 &= GG'' - G'^2, \\
p_8 &= G^2H'' - G'^2H, \\
p_9 &= G', \\
p_{10} &= HH'' - H'^2, \\
p_{11} &= H'H'' - HH''', \\
p_{12} &= HH'' - H'^2 - 1, \\
p_{13} &= G'H' - G''H,
\end{aligned}
$$

$$p_{14} = G'H'H - GH'^2 - G.$$

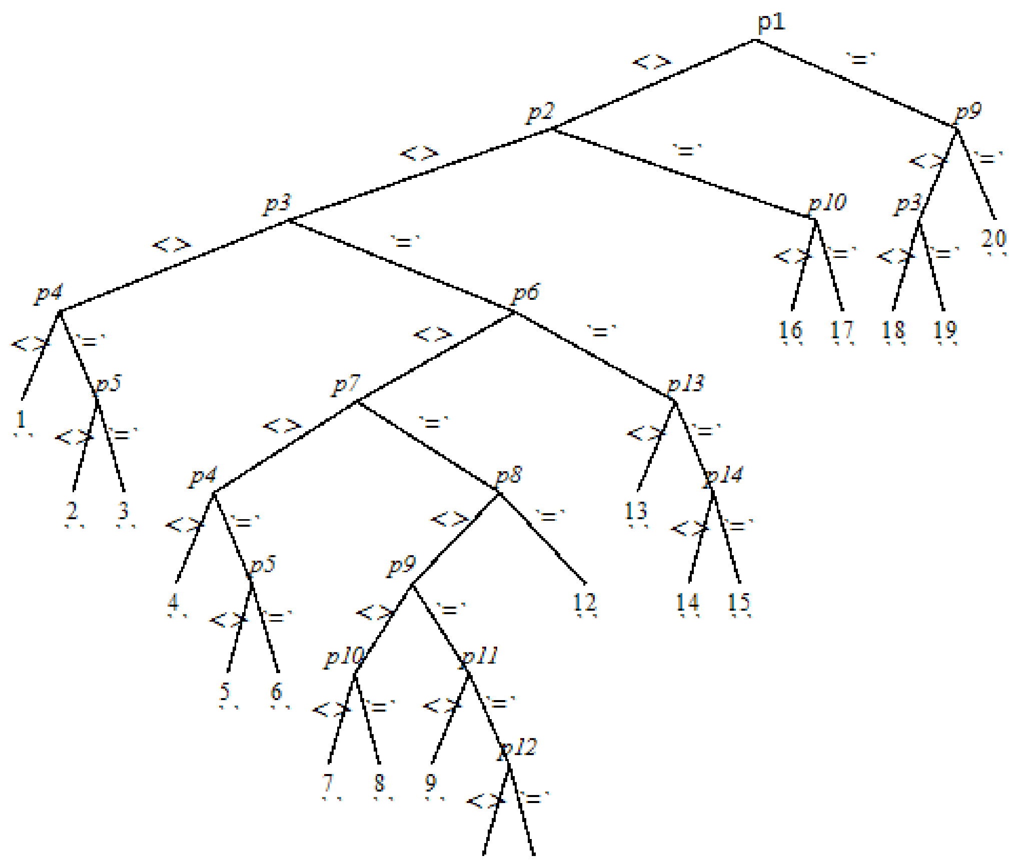

Figure 3.1: Rif Tree for HVFs

The branches of the Rif tree impose restrictions on the pivots, represented by the signs "=" and "<>", which, respectively signify whether the corresponding pivot is zero or non-zero.

As explained in chapter 2, variable ordering is most important in obtaining the simplified Rif tree and it can only be obtained by trial and error. For this particular problem we have deduced that the ordering $\xi^0 > \xi^1 > \xi^2 > \xi^3$

for dependent variables and $t > r > \theta > \phi$ for independent variables gives the simplified Rif tree. By these orderings, we mean that the rank derivative of ξ^0 is considered higher than equivalent derivative of ξ^1 and so on. Similarly, the derivative with respect to t is ranked higher than a derivative with respect to r and so on.

Another issue that we faced during the development of the Rif algorithm is that some branches in the Rif tree give expressions like $H^2 + 1 = 0$. Such expressions do not give any real value of the metric function H, hence the corresponding metric is not meaningful for our study. Such situations arise because the Rif algorithm works in the field of complex numbers. To exclude such cases, we have added the constraints like $H^2 + 1 \neq 0$ in the algorithm. For a complete classification, we have solved Eqs. (3.1.5)- (3.1.14) for the constraints of all branches of the Rif tree. The branches other than those labeled by 1, 6, 11, 14, 15, 16, 19 and 20 yield the minimum four KVFs, which are already given in the set K_4. Thus, we exclude these cases from our classification, while the results of the remaining branches are summarized in the coming sections.

3.1.1 Five HVFs

The branches labeled by 1, 6, 14 and 16 give 5-dimensional algebra of HVFs. In each case, we have obtained one proper HVF and four KVFs, same as K_4. In Table 3.1, we present metrics of these four cases along with their proper HVFs.

The metric 5a represents an anisotropic fluid with $\rho = \frac{3\alpha^2 - 2\alpha c_2 + 1}{(\alpha t + c_1)^2}$, $p_{||} =$

Table 3.1: Metrics admitting five HVFs

No.	Metric	Vector Field Components	Proper HVF
5a (Branch 1)	$G = (\alpha t + c_1)^{1-\frac{c_2}{\alpha}}$, $H = \alpha t + c_1$, where $c_2 \neq \alpha \neq 0$.	$\xi^0 = \alpha t + c_1$, $\xi^1 = c_2 r + c_3$, $\xi^2 = c_4 \cos\phi + c_5 \sin\phi$, $\xi^3 = \cot\theta(-c_4 \sin\phi + c_5 \cos\phi) + c_6$.	$\xi_{(5)} = t\frac{\partial}{\partial t}$.
5b (Branch 6)	$G = H = \alpha_1 t + \alpha_2$, where $\alpha_1 \neq 0$.	$\xi^0 = \frac{\alpha}{\alpha_1}(\alpha_1 t + \alpha_2)$, $\xi^1 = c_1$, ξ^2, ξ^3 are same as for the metric 5a.	$\xi_{(5)} = \frac{G}{G'}\frac{\partial}{\partial t}$.
5c (Branch 14)	$G = c_1 \neq 0$, $H = c_2 t + c_3$, where $c_2 \neq 0$.	$\xi^0 = \frac{\alpha}{c_2}(c_2 t + c_3)$, $\xi^1 = \alpha r + c_4$, ξ^2, ξ^3 are same as for the metric 5a.	$\xi_{(5)} = \frac{H}{H'}\frac{\partial}{\partial t}$ $+r\frac{\partial}{\partial r}$.
5d (Branch 16)	$H = c_2(\alpha t + c_3)$, $G = \frac{c_1}{H}$, where $c_1 \neq 0$, $c_2 \neq 0$.	$\xi^0 = \alpha t + c_3$, $\xi^1 = 2\alpha r + c_4$, ξ^2, ξ^3 are same as for the metric 5a, where we have chosen $c_1 = c_2 = 1$.	$\xi_{(5)} = t\frac{\partial}{\partial t}$ $+2r\frac{\partial}{\partial r}$.

$-\frac{1+\alpha^2}{(\alpha t+c_1)^2}$ and $p_\perp = -\frac{(\alpha-c_2)^2}{(\alpha t+c_1)^2}$. For this model to be physically realistic, we must have $\rho \geq 0$, that is $3\alpha^2 - 2\alpha c_2 + 1 \geq 0$. Moreover, the DEC is satisfied if $3\alpha^2 - 2\alpha c_2 + 1 \geq 0$, $\alpha > c_2$, $2\alpha^2 - \alpha c_2 + 1 \geq 0$, $2\alpha^2 - c_2^2 + 1 \geq 0$ and $(2\alpha - c_2)^2 + 1 \geq 0$. Similarly, the SEC and NEC are satisfied if $\alpha > c_2$ and $2\alpha^2 - c_2^2 + 1 \geq 0$, while for the WEC we must have $3\alpha^2 - 2\alpha c_2 + 1 \geq 0$, $\alpha > c_2$ and $2\alpha^2 - c_2^2 + 1 \geq 0$.

The metric 5b represents an anisotropic fluid such that $\rho = \frac{3\alpha_1^2+1}{(\alpha_1 t+\alpha_2)^2}$, $p_{||} = -\frac{\alpha_1^2+1}{(\alpha_1 t+\alpha_2)^2}$ and $p_\perp = -\frac{\alpha_1^2}{(\alpha_1 t+\alpha_2)^2}$. Here, the SEC and NEC are satisfied if $2\alpha_1^2 + 1 \geq 0$, while the WEC holds if $3\alpha_1^2 + 1 \geq 0$ and $2\alpha_1^2 + 1 \geq 0$. Finally, the DEC requires $3\alpha_1^2 + 1 \geq 0$, $2\alpha_1^2 + 1 \geq 0$ and $4\alpha_1^2 + 1 \geq 0$.

The metric 5c represents an anisotropic fluid with $\rho = \frac{1+c_2^2}{(c_2t+c_3)^2}$, $p_{||} = -\frac{1+c_2^2}{(c_2t+c_3)^2}$ and $p_\perp = 0$. All the energy conditions are identically satisfied here.

Finally, the metric 5d, being an anisotropic fluid, is given by $\rho = \frac{1-\alpha^2}{(\alpha t+c_3)^2}$, $p_{||} = -\frac{1+\alpha^2}{(\alpha t+c_3)^2}$ and $p_\perp = -\frac{\alpha^2}{(\alpha_1 t+c_3)^2}$. This model is physically realistic if the homothety constant belongs to the interval $[-1, 1]$. For non-zero α, the strong, weak, null and dominant energy conditions are failed for this model.

The metric 5a obtained here is same as the metric (17) of Ref. [32] obtained by using direct integration technique with $\alpha = C$. The metric 5b can be transformed to the metric (21) of Ref [32]. by taking $\alpha_1 = C$ and $\alpha_2 = 0$. One can see that the metric 5c is the generalized form of the metric (19) of Ref. [32] and it can be transformed to the mentioned metric of the Ref. [32] by taking $\alpha = C$ and $c_3 = 0$. However, the metric 5d is not listed in Ref. [32].

3.1.2 Six KVFs

There arise two cases, labeled by branches 19 and 20 in the Rif tree, where we have obtained 6-dimensional algebra of KVFs with no proper homothety. The set of these six KVFs contains the minimal set, given in the set K_4, while the extra two symmetries for both the metrics are given in Table 3.2. Each of these metrics is an anisotropic fluid. For the metric 6a, we obtain $\rho = \frac{1}{\gamma^2}$, $p_{||} = -\frac{1}{\gamma^2}$ and $p_\perp = -k$. The energy density is clearly positive, so the model is physically realistic. Here, the SEC is failed, while the remaining energy conditions are satisfied provided that $k\gamma^2 \leq 1$.

Similarly, the model 6b gives $\rho = -p_{||} = \frac{1}{c_3^2}$ and $p_\perp = 0$. The energy density is clearly positive and all the energy conditions are trivially satisfied

Table 3.2: Metrics admitting six KVFs

No.	Metric	Vector Field Components	Additional KVFs
6a (Branch 19)	$G=\beta_1 e^{\sqrt{k}t}+\beta_2 e^{-\sqrt{k}t}$, $H=\gamma\neq 0,\ k>0$.	$\xi^0=c_1\sin 2r+c_2\cos 2r$, $\xi^1=\frac{G'}{2G}(c_1\cos 2r-c_2\sin 2r)+c_3$, ξ^2,ξ^3 are as for the metric 5a.	$\xi_{(5)}=\sin 2r\frac{\partial}{\partial_t}+\frac{G'}{2G}\cos 2r\frac{\partial}{\partial_r}$, $\xi_{(6)}=\cos 2r\frac{\partial}{\partial_t}-\frac{G'}{2G}\sin 2r\frac{\partial}{\partial_r}$.
6b (Branch 19)	$G=c_1 t+c_2$, $H=c_3$, where $c_1\neq 0,\ c_3\neq 0$.	$\xi^0=c_4e^r+c_5e^{-r}$, $\xi^1=\frac{-1}{G}(c_4e^r-c_5e^{-r})+c_6$, ξ^2,ξ^3 are same as for the metric 5a, where we have chosen $c_1=1$.	$\xi_{(5)}=e^r\frac{\partial}{\partial_t}-\frac{e^r}{G}\frac{\partial}{\partial_r}$, $\xi_{(6)}=e^{-r}\frac{\partial}{\partial_t}+\frac{e^{-r}}{G}\frac{\partial}{\partial_r}$.
6c (Branch 20)	$G=\beta\neq 0$, $H=\gamma\neq 0$,	$\xi^0=\beta c_1 r+c_2$, $\xi^1=c_1 t+c_3$, ξ^2,ξ^3 are same as for the metric 5a.	$\xi_{(5)}=\frac{\partial}{\partial_t}$, $\xi_{(6)}=r\frac{\partial}{\partial_t}+t\frac{\partial}{\partial_r}$, where we have used $\beta=1$.

for this model.

Finally, for the metric 6c, we have $\rho = \frac{1}{\gamma^2}$, $p_{||} = -\frac{1}{\gamma^2}$ and $p_\perp = 0$. The model is physically realistic with positive energy density and it identically satisfies all the energy conditions.

3.1.3 Seven KVFs

The metric given by branch 11 admits seven KVFs with no proper homothety. Four KVFs are same as already given in the set K_4, while the extra three KVFs are presented in Table 3.3.

Table 3.3: Metric admitting seven KVFs

Metric	Vector Field Components	Additional KVFs
$G = c_1$, $H = c_2 e^{\sqrt{k}t} + c_3 e^{-\sqrt{k}t}$, where $c_1 \neq 0$, $k > 0$ and $4kc_2c_3 = 1$.	$\xi^0 = \sin\theta(c_4 \sin\phi - c_5\cos\phi)$ $-c_6\cos\theta$, $\xi^1 = c_7$, $\xi^2 = \frac{H'}{H}\cos\theta(c_4\sin\phi - c_5\cos\phi)$ $+c_6\frac{H'}{H}\sin\theta + c_8\cos\phi + c_9\sin\phi$, $\xi^3 = \frac{H}{H'}\csc\theta(c_4\cos\phi + c_5\sin\phi)$ $+\cot\theta(-c_8\sin\phi + c_9\cos\phi) + c_{10}$.	$\xi_{(5)} = \csc\theta\frac{H'}{H}\cos\phi\frac{\partial}{\partial_\phi} + \frac{H'}{H}\cos\theta\sin\phi\frac{\partial}{\partial_\theta}$ $+\sin\theta\sin\phi\frac{\partial}{\partial_t}$ $\xi_{(6)} = -\csc\theta\frac{H'}{H}\sin\phi\frac{\partial}{\partial_\phi} + \frac{H'}{H}\cos\theta\cos\phi\frac{\partial}{\partial_\theta}$ $+\sin\theta\cos\phi\frac{\partial}{\partial_t}$ $\xi_{(7)} = -\frac{H'}{H}\sin\theta\frac{\partial}{\partial_\theta} + \cos\theta\frac{\partial}{\partial_t}$.

The metric of this case represents an anisotropic fluid such that:

$$\begin{aligned}
\rho &= \frac{k(c_2^2 e^{2\sqrt{k}t} + c_3^2 e^{-2\sqrt{k}t} - 2c_2c_3) + 1}{(c_2 e^{2\sqrt{k}t} + c_3 e^{-2\sqrt{k}t})^2}, \\
p_{||} &= -\frac{3kc_2^2 e^{2\sqrt{k}t} + 3kc_3^2 e^{-2\sqrt{k}t} + 2kc_2c_3 + 1}{(c_2 e^{2\sqrt{k}t} + c_3 e^{-2\sqrt{k}t})^2}, \\
p_\perp &= -k.
\end{aligned}$$

One may use these quantities in the inequalities given in (2.7.1) to obtain the bounds for energy conditions.

3.1.4 Ten KVFs

Solving the set of homothetic equations for the constraints of branch 15, we obtain a Kantowski-Sachs metric admitting ten KVFs, out of which four are same as given in the set K_4, while the extra six KVFs are listed in Table 3.4. The metric of this case gives an anisotropic fluid model with $\rho = -p_{||} = \frac{1+G^2+2H^2}{H^2}$ and $p_\perp = -3$. Here, the energy density is clearly positive, the SEC is violated and all other energy conditions are satisfied.

3.2 Conformal Symmetries

In this section, we explore the conformal symmetries of Kantowski-Sachs spacetimes using Rif tree approach. A research paper has been published on the basis of this work [112].

For conformal symmetry, we obtain the same set of partial differential equations as given in Eqs. (3.1.5)-(3.1.14) by using the metric (3.1.1) in Eq. (2.5.3) with $\alpha = \alpha(t, r, \theta, \phi)$. The non-zero components of the Weyl tensor for the metric given in (3.1.1) are:

$$\begin{aligned} C^1_{010} &= \frac{\Gamma}{3GH^2}, \quad C^0_{202} = \frac{\Gamma}{6G}, \\ C^0_{303} &= \frac{\Gamma \sin^2\theta}{6G}, \quad C^1_{212} = C^0_{202}, \\ C^1_{313} &= C^0_{303}, \quad C^2_{323} = -2C^0_{303}, \end{aligned} \tag{3.2.1}$$

Table 3.4: Metric admitting ten KVFs

Metric	Vector Field Components	Additional KVFs
$G=\beta_1 e^t+\beta_2 e^{-t}$, $H=\beta_1 e^t-\beta_2 e^{-t}$, where β_1 and β_2 are constants such that $4\beta_1\beta_2+1=0$.	$\xi^0=\sin\theta\Big((c_1e^r+c_2e^{-r})\sin\phi-(c_3e^r+c_4e^{-r})\cos\phi\Big)$ $-(c_5e^r+c_6e^{-r})\cos\theta$, $\xi^1=-\frac{H}{G}\sin\theta\Big((c_1e^r-c_2e^{-r})\sin\phi-(c_3e^r-c_4e^{-r})\cos\phi\Big)$ $+\frac{H}{G}(c_5e^r-c_6e^{-r})\cos\theta+c_7$, $\xi^2=\frac{G}{H}\cos\theta\Big((c_1e^r+c_2e^{-r})\sin\phi-(c_3e^r+c_4e^{-r})\cos\phi\Big)$ $+\frac{G}{H}(c_5e^r+c_6e^{-r})\sin\theta+c_8\cos\phi+c_9\sin\phi$, $\xi^3=\frac{G}{H}\csc\theta\Big((c_1e^r+c_2e^{-r})\cos\phi+(c_3e^r+c_4e^{-r})\sin\phi)\Big)$ $+\cot\theta(-c_8\sin\phi+c_9\cos\phi)+c_{10}$.	$\xi_{(5)}=\frac{G}{H}e^r(\cos\theta\sin\phi\frac{\partial}{\partial_\theta}+\csc\theta\cos\phi\frac{\partial}{\partial_\phi})$ $+\sin\theta\sin\phi e^r(\frac{\partial}{\partial_t}-\frac{H}{G}\frac{\partial}{\partial_r})$, $\xi_{(6)}=\frac{G}{H}e^{-r}(\cos\theta\sin\phi\frac{\partial}{\partial_\theta}+\csc\theta\cos\phi\frac{\partial}{\partial_\phi})$ $+\sin\theta\sin\phi e^{-r}(\frac{\partial}{\partial_t}+\frac{H}{G}\frac{\partial}{\partial_r})$, $\xi_{(7)}=-\frac{G}{H}e^r(\cos\theta\cos\phi\frac{\partial}{\partial_\theta}-\csc\theta\sin\phi\frac{\partial}{\partial_\phi})$ $-\sin\theta\cos\phi e^r(\frac{\partial}{\partial_t}-\frac{H}{G}\frac{\partial}{\partial_r})$, $\xi_{(8)}=-\frac{G}{H}e^{-r}(\cos\theta\cos\phi\frac{\partial}{\partial_\theta}-\csc\theta\sin\phi\frac{\partial}{\partial_\phi})$ $-\sin\theta\cos\phi e^{-r}(\frac{\partial}{\partial_t}+\frac{H}{G}\frac{\partial}{\partial_r})$, $\xi_{(9)}=\frac{G}{H}\sin\theta e^r\frac{\partial}{\partial_\theta}-\cos\theta e^r(\frac{\partial}{\partial_t}-\frac{H}{G}\frac{\partial}{\partial_r})$, $\xi_{(10)}=\frac{G}{H}\sin\theta e^{-r}\frac{\partial}{\partial_\theta}-\cos\theta e^{-r}(\frac{\partial}{\partial_t}+\frac{H}{G}\frac{\partial}{\partial_r})$.

where $\Gamma = GHH'' - GH'^2 - G''H^2 + G'HH' - G$. Thus the Kantwoski-Sachs spacetime is conformally flat if $\Gamma = 0$.

To explore all possible cases where the Kantowski-Sachs spacetimes may posses CVFs, we follow the same approach as we have used in case of finding HVFs, where the corresponding Rif tree and pivots are obtained as follows:

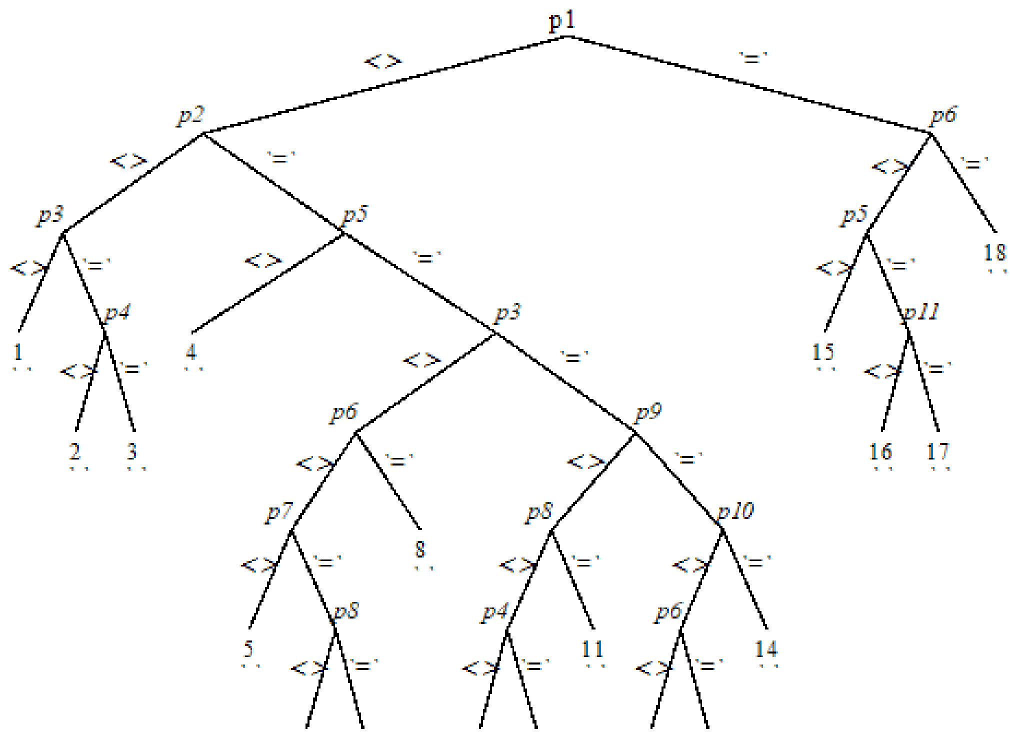

Figure 3.2: Rif Tree for CVFs

$$
\begin{aligned}
p_1 &= G', \\
p_2 &= G'''G - G'G'', \\
p_3 &= G''H - GH'', \\
p_4 &= GH' - G'H, \\
p_5 &= HH''' - H'H'', \\
p_6 &= H', \\
p_7 &= GG'' - G'^2, \\
p_8 &= HH'' - H'^2, \\
p_9 &= GH'' - G'H', \\
p_{10} &= G'H'H - GH'^2 - G, \\
p_{11} &= HH'' - H'^2 - 1.
\end{aligned}
$$

The Rif algorithm for CVFs is developed in the same way as that for HVFs with the difference that here we have used the ordering $\alpha > \xi^0 > \xi^1 > \xi^2 > \xi^3$ for dependent variables and $t > r > \theta > \phi$ for independent variables. Note that the Weyl tensor of the Kantowski-Sachs metric vanishes if and only if $\Gamma = 0$. In such a case, the conformal algebra is clearly 15-dimensional. Examples of such cases are labeled by the branches 14 and 17 of the Rif tree. Moreover, the constraints of some other branches of the Rif tree give rise to two sub-cases, depending upon whether the spacetime is conformally flat or not. Due to the fact that each conformally flat case gives 15 CVFs, we have only considered the cases where $\Gamma \neq 0$. Furthermore, the solution of conformal symmetry equations for some branches of the Rif tree give minimum four KVFs. Such cases are labeled in the Rif tree by 2, 5, 6, 7, 9, 11 and 15. The cases where the Kantowski-Sachs metric possesses proper CVFs or KVFs other than the minimum ones are discussed in the forthcoming sections. Here, the comparison of our findings with those of Ref. [113] is also given which to show that this new approach gives more general metrics as compared to those obtained by direct integration technique. Moreover, this Rif approach also provides some new metrics which remained undetermined by direct integration technique.

3.2.1 Six CVFs

Six cases arise where the dimension of conformal algebra is 6. These cases are labeled by branches 1, 3, 4, 10, 12 and 16 in the Rif tree.

In branch 1, we have $p_1 \neq 0, p_2 \neq 0$ and $p_3 \neq 0$. Thus, the metric functions are restricted to satisfy the conditions $G' \neq 0,\ GG''' - G'G'' \neq 0$ and $GH'' - HG'' \neq 0$. Under these restrictions, the solution of Eqs. (3.1.5)-(3.1.14) gives:

$$\xi^0 = H\left(c_1 e^{\sqrt{\lambda} r} + c_2 e^{-\sqrt{\lambda} r}\right),$$

$$\xi^1 = \sqrt{\lambda}\int \frac{H}{G^2}dt\left(c_1 e^{\sqrt{\lambda} r} - c_2 e^{-\sqrt{\lambda} r}\right) + c_3,$$

$\xi^2,\ \xi^3$ are same as for the metric 5a in the section of HVF,

$$\alpha = H'\left(c_1 e^{\sqrt{\lambda} r} + c_2 e^{-\sqrt{\lambda} r}\right), \tag{3.2.2}$$

where λ is a positive constant satisfying the relation $\lambda \int \frac{H}{G^2}dt = G\left(\frac{H}{G}\right)'$. Here, the constants c_1 and c_2 correspond to two proper CVFs, which can be expressed as $\xi_{(5)} = e^{\sqrt{\lambda} r}(H\frac{\partial}{\partial_t} + \sqrt{\lambda}\int \frac{H}{G^2}dt\ \frac{\partial}{\partial_r})$ and $\xi_{(6)} = e^{-\sqrt{\lambda} r}(H\frac{\partial}{\partial_t} - \sqrt{\lambda}\int \frac{H}{G^2}dt\ \frac{\partial}{\partial_r})$. The remaining four constants give the minimum four KVFs, as given in set K_4.

The branch 4 is a special case of branch 1 that gives the same six CVFs except that the metric function G satisfies the equation $GG''' - G'G'' = 0$. This mean that G is either a linear function or $G = \beta e^t + \gamma e^{-t}$, where β and γ are constants which are not zero simultaneously.

In branch 3, we have $p_1 \neq 0, p_2 \neq 0$ and $p_3 = p_4 = 0$. The conditions $p_3 = p_4 = 0$ relate the metric functions as $H = \beta G$, where β is a non-zero constant, and the solution of Eqs. (3.1.5)-(3.1.14) yield:

$$\begin{aligned}
\xi^0 &= \beta G(c_1 r + c_2), \\
\xi^1 &= \beta c_1 \int \frac{1}{G} dt + c_3,
\end{aligned}$$

ξ^2, ξ^3 are same as for the metric 5a in the section of HVF,

$$\alpha = \beta G'(c_1 r + c_2). \qquad (3.2.3)$$

We can express the two proper CVFs as $\xi_{(5)} = rG\frac{\partial}{\partial_t} + \int \frac{1}{G} dt \, \frac{\partial}{\partial_r}$ and $\xi_{(6)} = G\frac{\partial}{\partial_t}$. The remaining four CVFs are same as the minimum KVFs given in set K_4. The metric of this case can be transformed to the metric (33) of Ref. [113] by taking $\beta = 1$.

The branch 10 is a special case of branch 3, where the metric functions take the values $G = \gamma_1 t + \gamma_2$ or $G = \gamma_1 e^t + \gamma_2 e^{-t}$ and $H = \beta G$, where β, γ_1 and γ_2 are non-zero constants. The CVFs in this case are same as those of branch 3.

In one case of branch 10, when $G = \gamma_1 t + \gamma_2$, we have $\rho = \frac{1+3\gamma_1^2\beta^2}{\beta^2(\gamma_1 t+\gamma_2)^2}$, $p_{||} = -\frac{1+\gamma_1^2\beta^2}{\beta^2(\gamma_1 t+\gamma_2)^2}$ and $p_\perp = -\frac{\gamma_1^2}{\beta^2(\gamma_1 t+\gamma_2)^2}$. The metric is physically realistic with positive energy density ρ. Moreover, the quantities ρ, $p_{||}$ and $p_\perp$ identically satisfy all the energy conditions.

In another sub-case, that is when $G = \gamma_1 e^t + \gamma_2 e^{-t}$, the energy density is $\rho = \frac{1+3\beta^2(\gamma_1 e^t - \gamma_2 e^{-t})^2}{\beta^2(\gamma_1 e^t + \gamma_2 e^{-t})^2}$, which is clearly positive. Moreover, the parallel and perpendicular pressures are given by $p_{||} = -\frac{3\beta^2\gamma_1^2 e^{2t} + 3\beta^2\gamma_2^2 e^{-2t} + 2\beta^2\gamma_1\gamma_2 + 1}{\beta^2(\gamma_1 e^t + \gamma_2 e^{-t})^2}$ and $p_{\perp} = -\frac{3\gamma_1^2 e^{2t} + 3\gamma_2^2 e^{-2t} + 2\gamma_1\gamma_2}{\beta^2(\gamma_1 e^t + \gamma_2 e^{-t})^2}$. Here, the SEC fails while the DEC, NEC and WEC are satisfied if $\gamma_1\gamma_2 < 0$ and $8\beta^2\gamma_1\gamma_2 \leq 1$.

The constraints of branch 12 are $p_2 = p_3 = p_5 = p_9 = 0$, $p_1 \neq 0$, $p_6 \neq 0$, and $p_{10} \neq 0$. Simplifying these constraints, we get $G = \gamma_1 e^t + \gamma_2 e^{-t}$ and $H = \gamma_3 e^t + \gamma_4 e^{-t}$, where the constants γ_is satisfy the relation $\gamma_1\gamma_4 + \gamma_2\gamma_3 = 0$. Moreover, $GH' - HG' = 4\gamma_2\gamma_3 = \lambda$ (say) and because $p_9 = 0$, the function $\frac{H}{G'}$ is constant, say $\frac{H}{G'} = \beta$. Under these restrictions, we obtain the following solution of Eqs. (3.1.5)-(3.1.14):

$$\xi^0 = H\Big(c_1 \cos\sqrt{\frac{\lambda}{\beta}}r + c_2 \sin\sqrt{\frac{\lambda}{\beta}}r\Big),$$

$$\xi^1 = \frac{\sqrt{\lambda\beta}}{G}\Big(c_1 \sin\sqrt{\frac{\lambda}{\beta}}r - c_2 \cos\sqrt{\frac{\lambda}{\beta}}r\Big) + c_3,$$

ξ^2, ξ^3 are same as for the metric 5a in the section of HVF,

$$\alpha = H'\Big(c_1 \cos\sqrt{\frac{\lambda}{\beta}}r + c_2 \sin\sqrt{\frac{\lambda}{\beta}}r\Big). \qquad (3.2.4)$$

Here, we have two CVFs, $\xi_{(5)} = H\cos\sqrt{\frac{\lambda}{\beta}}r\ \frac{\partial}{\partial_t} + \frac{\sqrt{\lambda\beta}}{G}\sin\sqrt{\frac{\lambda}{\beta}}r\ \frac{\partial}{\partial_r}$ and $\xi_{(6)} = H\sin\sqrt{\frac{\lambda}{\beta}}r\ \frac{\partial}{\partial_t} - \frac{\sqrt{\lambda\beta}}{G}\cos\sqrt{\frac{\lambda}{\beta}}r\ \frac{\partial}{\partial_r}$ along with the four KVFs, same as given in the set K_4.

The metric of this case represents an anisotropic fluid with $p_{||} = -\rho$, $p_{\perp} = -3$ and $\rho = \frac{1+3\beta^2\gamma_1^2 e^{2t} + 3\beta^2\gamma_2^2 e^{-2t} - 2\beta^2\gamma_1\gamma_2}{\beta^2(\gamma_1 e^t + \gamma_2 e^{-t})^2}$. Here, the energy is positive if $\gamma_1\gamma_2 \leq 0$.

In branch 16, we have $p_1 = p_5 = 0$, $p_6 \neq 0$ and $p_{11} \neq 0$. Simplifying these conditions, we get $G = \beta \neq 0$ and the value of H depends upon the condition whether $H'' \neq 0$ or $H'' = 0$. When $H'' \neq 0$, we get $H = \gamma_1 e^t + \gamma_2 e^{-t}$, where

$4\gamma_1\gamma_2 + 1 \neq 0$. When $H'' = 0$, then $H = \gamma_1 t + \gamma_2$, where $\gamma_1 \neq 0$. In both cases we have obtained six CVFs in which four are the minimum KVFs, same as given in the set K_4, while the remaining two are proper CVFs which are listed in Table 3.5.

Table 3.5: Metrics admitting six CVFs

No.	Vector Field Components	Proper CVFs
6a (Branch 16)	$\xi^0 = H\left(c_1 e^{\beta r} + c_2 e^{-\beta r}\right)$, $\xi^1 = \frac{H'}{\beta}\left(c_1 e^{\beta r} - c_2 e^{-\beta r}\right) + c_3$, ξ^2, ξ^3 are same as for the metric 5a given in section of HVF, $\alpha = H'\left(c_1 e^{\beta r} + c_2 e^{-\beta r}\right)$.	$\xi_{(5)} = He^{\beta r}\frac{\partial}{\partial_t} + \frac{H'}{\beta}e^{\beta r}\frac{\partial}{\partial_r}$, $\xi_{(6)} = He^{-\beta r}\frac{\partial}{\partial_t} - \frac{H'}{\beta}e^{-\beta r}\frac{\partial}{\partial_r}$.
6b (Branch 16)	$\xi^0 = H\left(c_1 r + c_2\right)$, $\xi^1 = c_1\left(\frac{H^2}{2\beta^2\gamma_1} + \frac{\gamma_1 r^2}{2}\right) + c_2\gamma_1 r + c_3$, ξ^2, ξ^3 are same as for the metric 5a in the section of HVF, $\alpha = H'\left(c_1 r + c_2\right)$.	$\xi_{(5)} = Hr\frac{\partial}{\partial_t} + \left(\frac{H^2}{2\beta^2\gamma_1} + \frac{\gamma_1 r^2}{2}\right)\frac{\partial}{\partial_r}$, $\xi_{(6)} = H\frac{\partial}{\partial_t} + \gamma_1 r\frac{\partial}{\partial_r}$.

The metric 6a is same as the metric (36) of Ref. [113] with $\beta = 1$ and one can see that the metric 6b was missing there. For the metric 6a, we get $\rho = \frac{1+(\gamma_1 e^t - \gamma_2 e^{-t})^2}{(\gamma_1 e^t + \gamma_2 e^{-t})^2}$, $p_{||} = -\frac{3\gamma_1^2 e^{2t} + 3\gamma_2^2 e^{-2t} + 2\gamma_1\gamma_2 + 1}{(\gamma_1 e^t + \gamma_2 e^{-t})^2}$ and $p_\perp = -1$ and this Kantowski-Sachs metric is physically meaningful because the energy density is positive. One can easily simplify the energy conditions for this metric.

Similarly, the metric 6b represents an anisotropic fluid with $\rho = \frac{\gamma_1^2 + 1}{(\gamma_1 t + \gamma_2)^2}$,

$p_{||} = -\frac{\gamma_1^2+1}{(\gamma_1 t+\gamma_2)^2}$ and $p_\perp = 0$. Here all the energy conditions are identically satisfied.

3.2.2 Six KVFs

The branches 8, 13 and 18 give 6-dimensional Killing algebras with no proper CVF.

In branch 8, we have $p_1 \neq 0$, $p_2 = p_5 = 0$, $p_3 \neq 0$ and $p_6 = 0$. The simplification of these constraints gives $G = c_1 e^t + c_2 e^{-t}$ and $H = \beta$, where $\beta \neq 0$ is a constant. This metric admits six KVFs which are presented in Table 3.6.

Table 3.6: Metrics admitting six KVFs

Vector Field Components	Additional KVFs
$\xi^0 = c_3 \cos\lambda r + c_4 \sin\lambda r$, $\xi^1 = -\frac{G'}{\lambda G}\left(c_3 \sin\lambda r - c_4 \cos\lambda r\right) + c_5$, ξ^2, ξ^3 are same as for the metric 5a given in section of HVF, where $\lambda = 2\sqrt{c_1 c_2}$.	$\xi_{(5)} = \cos\lambda r \frac{\partial}{\partial_t} - \frac{G'}{\lambda G} \sin\lambda r \frac{\partial}{\partial_r}$, $\xi_{(6)} = \sin\lambda r \frac{\partial}{\partial_t} + \frac{G'}{\lambda G} \cos\lambda r \frac{\partial}{\partial_r}$.

This metric can be transformed to the metric (39) of Ref. [113] by setting $c_1 = c_2 = \frac{1}{2}$. For this metric, the energy density, and the parallel and perpendicular pressures are given by $\rho = -p_{||} = \frac{1}{\beta^2}$, and $p_\perp = -1$ respectively. The metric is physically realistic with positive energy. The SEC is violated, while the remaining energy conditions hold, provided that $\beta^2 \leq 1$. For $\beta = 1$, the metric represents a perfect fluid.

In branch 13, we have $p_1 \neq 0$, $p_2 = p_3 = p_5 = p_6 = p_9 = 0$ and $p_{10} \neq 0$, which give $G = c_1 t + c_2$ and $H = c_3$, where $c_1 \neq 0$ and $c_3 \neq 0$. For these values of G and H, the CVFs are reduced to KVFs. The obtained six KVFs are same as given for metric 6b in Table 3.2. It has been observed that this metric was missing in Ref. [113].

Finally, the constraints of branch 18 are $p_1 = 0$ and $p_6 = 0$, showing that both the metric functions are constants, say $G = \beta \neq 0$ and $H = \gamma \neq 0$. In this branch, the CVFs are reduced to KVFs which are same as given in Table 3.2 for the metric 6c of the section of HVFs.

This metric is same as the metric (34) of Ref. [113] with $\beta = 1$.

3.3 Noether Symmetries

In this section, we explore Noether symmetries of the following Lagrangian corresponding to the Kantowski- Sachs metric (3.1.1).

$$L = -\dot{t}^2 + G^2(t)\dot{r}^2 + H^2(t)[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2], \tag{3.3.1}$$

where a dot denotes derivative w.r.t the affine parameter s. Following are the minimum Noether symmetries admitted by the above Lagrangian:

$$N_5 = \left\{ \frac{\partial}{\partial_s}, \frac{\partial}{\partial_r}, \frac{\partial}{\partial_\phi}, \cos\phi\frac{\partial}{\partial_\theta} - \cot\theta\sin\phi\frac{\partial}{\partial_\phi}, \sin\phi\frac{\partial}{\partial_\theta} + \cot\theta\cos\phi\frac{\partial}{\partial_\phi} \right\}.$$

This set clearly contains the set of four minimum KVFs of Kantowski- Sachs metric, given in the set K_4, while $\frac{\partial}{\partial_s}$ is the Noether symmetry corresponding to the Lagrangian.

Using the Lagrangian (3.3.1) in the Noether symmetry equation (2.6.1), we obtain:

$$F_{,s} = \eta_{,t} = \eta_{,r} = \eta_{,\theta} = \eta_{,\phi} = 0, \tag{3.3.2}$$

$$2\xi^0_{,t} = \eta_{,s}, \tag{3.3.3}$$

$$2G'\xi^0 + 2G\xi^1_{,r} = G\eta_{,s}, \tag{3.3.4}$$

$$2H'\xi^0 + 2H\xi^2_{,\theta} = H\eta_{,s}, \tag{3.3.5}$$

$$2H'\xi^0 + 2H\cot\theta\xi^2 + 2H\xi^3_{,\phi} = H\eta_{,s}, \tag{3.3.6}$$

$$\xi^0_{,r} - G^2\xi^1_{,t} = 0, \tag{3.3.7}$$

$$\xi^0_{,\theta} - H^2\xi^2_{,t} = 0, \tag{3.3.8}$$

$$\xi^0_{,\phi} - H^2\sin^2\theta\xi^3_{,t} = 0, \tag{3.3.9}$$

$$G^2\xi^1_{,\theta} + H^2\xi^2_{,r} = 0, \tag{3.3.10}$$

$$G^2\xi^1_{,\phi} + H^2\sin^2\theta\xi^3_{,r} = 0, \tag{3.3.11}$$

$$\xi^2_{,\phi} + \sin^2\theta\xi^3_{,\theta} = 0, \tag{3.3.12}$$

$$2\xi^0_{,s} - F_{,t} = 0, \tag{3.3.13}$$

$$2G^2\xi^1_{,s} - F_{,r} = 0, \tag{3.3.14}$$

$$2H^2\xi^2_{,s} - F_{,\theta} = 0, \tag{3.3.15}$$

$$2H^2\sin^2\theta\xi^3_{,s} - F_{,\phi} = 0. \tag{3.3.16}$$

Like the cases of HVFs and CVFs, one needs to solve the above system of equations to find the explicit form of Noether symmetries. In literature, such systems are solved using direct integrating technique [83]. However, we follow the same Rif tree approach as we have used for finding homothetic and conformal symmetries. First we develop an algorithm in Maple which transforms the set of Eqs. (3.3.2)-(3.3.16) to the reduced involutive form and

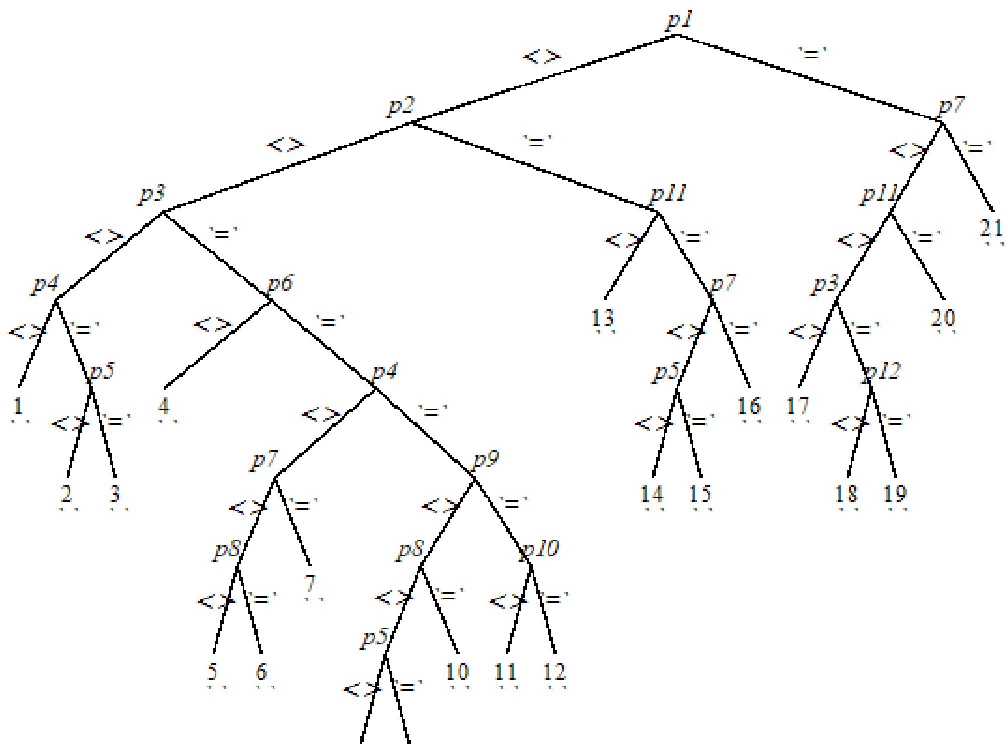

Figure 3.3: Rif Tree for Noether Symmetries

produces the Rif tree given in Fig. 3.3 and the list of pivots.

$$
\begin{aligned}
p_1 &= G', \\
p_2 &= G'', \\
p_3 &= HH''' - H''H', \\
p_4 &= HG''H - GH'', \\
p_5 &= HG' - GH', \\
p_6 &= G'''G - G''G',
\end{aligned}
$$

$$
\begin{aligned}
p_7 &= H', \\
p_8 &= H'^2 - HH'', \\
p_9 &= GH'' - G'H', \\
p_{10} &= G'H'H - GH'^2 - G, \\
p_{11} &= H'', \\
p_{12} &= HH'' - H'^2 - 1.
\end{aligned}
$$

For a complete classification, we have solved Eqs. (3.3.2)- (3.3.16) for each branch of the Rif tree given in Fig. 3.3. The branches other than those labeled by 4, 7, 12, 15, 16, 17, 18, 19, 20 and 21 give five Noether symmetries as given in the set N_5. We exclude all such cases from our classification, while the results of the remaining branches are summarized in the coming sections. A brief comparison of our results with those of direct integrating technique of Ref. [83] is also presented.

3.3.1 Six Noether Symmetries

The branches labeled by 4, 17 and 18 give 6-dimensional algebra of Noether symmetries.

In branch 4, we have $p_1 \neq 0, p_2 \neq 0, p_3 = 0$ and $p_6 \neq 0$. Under these conditions, the solution of Eqs. (3.3.2)-(3.3.16) gives the same metric as 5a given in Table 3.1. For this metric, we obtain six Noether symmetries. These six symmetries include the minimal set of five Noether symmetries along with one extra Noether symmetry $\xi_{(5)} = s\frac{\partial}{\partial_s} + \frac{t}{2}\frac{\partial}{\partial_t}$, whose corresponding conserved form is $I_{(5)} = -sL - t\dot{t}$. Clearly, $\xi_{(5)}$ corresponds to a homothetic vector, given by $t\frac{\partial}{\partial_t}$ which is same as given in Table 3.1 for the metric 5a. The physical interpretation of this metric is already presented in the section of HVFs. The metric of this branch is same as the metric (7vii) of Ref. [83]. However, the authors of Ref. [83] mentioned one additional symmetry $r\frac{\partial}{\partial_r}$ which seems to be wrong.

In branch 17, we have $p_1 = 0, p_3 \neq 0, p_7 \neq 0$ and $p_{11} \neq 0$, which yield $G = \beta$, where $\beta \neq 0$ and $H' \neq 0, H'' \neq 0, HH''' - H'H'' \neq 0$. The solution of Eqs. (3.3.2)- (3.3.16) gives six Noether symmetries in which five are same as given in the set N_5 and one extra Noether symmetry is given below along with its conserved form:

$$\xi_{(5)} = \frac{s}{2\beta^2}\frac{\partial}{\partial_r}; \qquad F = r.$$

$$I_{(5)} = s\dot{r} - r.$$

Clearly, $\xi_{(5)}$ is a proper Noether symmetry. This metric is exactly same as the metric (27) of Ref. [83]. The metric of branch 17 represents an anisotropic fluid model with energy density $\rho = \frac{H'^2+1}{H^2}$, which is clearly positive. Moreover, the parallel and perpendicular pressures are given by $p_{||} = -(\frac{2H''}{H} + \frac{H'^2}{H^2} + \frac{1}{H^2})$ and $p_\perp = -(\frac{H''}{H})$. For this model, SEC and NEC holds if $\frac{H''}{H} \leq 0$ and $H'^2+1-HH'' \geq 0$. Moreover, the DEC requires $\frac{H''}{H} \leq 0$, $H'^2 + 1 - HH'' \geq 0$ and $H'^2 + 1 + HH'' \geq 0$. Similarly, the WEC holds if $\frac{H'^2+1}{H^2} \geq 0$, $\frac{H''}{H} \leq 0$ and $H'^2 + 1 - HH'' \geq 0$.

In branch 18, the constraints are $p_1 = 0, p_3 = 0, p_7 \neq 0, p_{11} \neq 0$ and $p_{12} \neq 0$. The simplification of these constraints gives $G = \beta$ and $H = \alpha_1 e^{\sqrt{k}t} + \alpha_2 e^{-\sqrt{k}t}$, where $\beta \neq 0$, $k > 0$ and $k \neq \frac{1}{4\alpha_1\alpha_2}$. For this metric, we obtain the same Noether symmetries as obtained for the metric of branch 17. Comparing these results with Ref. [83], we can see that this case was not listed there.

The metric of branch 18 is an anisotropic fluid model with $\rho = \frac{k(\alpha_1 e^{\sqrt{k}t} - \alpha_2 e^{-\sqrt{k}t})^2+1}{(\alpha_1 e^{\sqrt{k}t} + \alpha_2 e^{-\sqrt{k}t})^2}$,

$p_{||} = -(\frac{3k(\alpha_1^2 e^{2\sqrt{k}t}+\alpha_2^2 e^{-2\sqrt{k}t})+2k\alpha_1\alpha_2+1}{(\alpha_1 e^{\sqrt{k}t}+\alpha_2 e^{-\sqrt{k}t})^2})$ and $p_\perp = -k$. This model is physically meaningful as $\rho \geq 0$. Moreover, the energy conditions for this model are conditionally satisfied.

3.3.2 Seven Noether Symmetries

Two branches of the Rif tree, labeled by 7 and 15 give seven Noether symmetries. In each case, we have two extra Noether symmetries other than the minimal Noether symmetries given in N_5.

In branch 7, we have $p_1 \neq 0$, $p_2 \neq 0$, $p_4 \neq 0$ and $p_3 = p_6 = p_7 = 0$. The simplification of these constraints gives $G = c_1 e^{\sqrt{k}t} + c_2 e^{-\sqrt{k}t}$ and $H = \beta$, where $\beta \neq 0$ and $k > 0$. Below are the two extra Noether symmetries (KVFs) and their conserved forms for this metric:

$$\xi_{(5)} = \cos(\lambda r)\frac{\partial}{\partial_t} - \frac{G'}{\lambda G}\sin(\lambda r)\frac{\partial}{\partial_r},$$
$$\xi_{(6)} = \sin(\lambda r)\frac{\partial}{\partial_t} + \frac{G'}{\lambda G}\cos(\lambda r)\frac{\partial}{\partial_r}.$$

$$I_{(5)} = -2\cos(\lambda r)\dot{t} - \frac{2}{\lambda}GG'\sin(\lambda r)\dot{r},$$
$$I_{(6)} = -2\sin(\lambda r)\dot{t} + \frac{2}{\lambda}GG'\cos(\lambda r)\dot{r}.$$

where $\lambda = 2\sqrt{c_1c_2}$. The metric of this case is same as the metric 6a presented in Table 3.2, admitting the same six KVFs. Moreover, this metric recovers all the metrics given by (7i-7v) of Ref. [83].

In branch 15, the constraints are $p_1 \neq 0$, $p_2 = p_5 = 0$, $p_7 \neq 0$ and $p_{11} = 0$. The simplification of these constraints gives $G = H = \alpha_1 t + \alpha_2$, where $\alpha_1 \neq 0$. The solution of the set of determining equations (3.3.2)- (3.3.16) yields seven Noether symmetries in which five are given in the set N_5 and two are extra Noether symmetries. The obtained two extra Noether symmetries and their conserved forms are given in Table 3.7, where $\xi_{(5)}$ is a proper Noether symmetry and $\xi_{(6)}$ is a Noether symmetry corresponding to a homothetic vector $\frac{\alpha_1 t+\alpha_2}{\alpha_1}\frac{\partial}{\partial_t}$. One can see that the metric of this case is same as the metric 5b, given in Table 3.1, admitting five HVFs. The physical interpretation of this metric is already given earlier.

Table 3.7: Metric admitting seven Noether Symmetries

Noether symmetry generators	Invariants
$\xi_{(5)} = \frac{s^2}{2}\frac{\partial}{\partial_s} + \frac{s(\alpha_1 t+\alpha_2)}{2\alpha_1}\frac{\partial}{\partial_t}$; $F = -(\frac{(\alpha_1 t+\alpha_2)\alpha_2}{\alpha_1^2} + \frac{t^2}{2})$, $\xi_{(6)} = s\frac{\partial}{\partial_s} + \frac{\alpha_1 t+\alpha_2}{2\alpha_1}\frac{\partial}{\partial_t}; F = r.$	$I_{(5)} = -\frac{s^2}{2}L - \frac{s(\alpha_1 t+\alpha_2)}{\alpha_1}\dot{t}$ $\frac{(\alpha_1 t+\alpha_2)\alpha_2}{\alpha_1^2} + \frac{t^2}{2}$, $I_{(6)} = -sL - \frac{s(\alpha_1 t+\alpha_2)}{\alpha_1}\dot{t} - r.$

Moreover, the same metric was also obtained in Ref. [83], see the metric (7vi) therein, by direct integration technique.

3.3.3 Eight Noether Symmetries

The branch 20 whose constraints are $p_1 = p_{11} = 0$ and $p_7 \neq 0$, gives the metric functions $G = \beta$ and $H = \alpha_1 t + \alpha_2$, where $\alpha_1 \neq 0$, $\beta \neq 0$. Solving the set of determining equations, we obtain eight Noether symmetries in which five are same as given in the set N_5, while the remaining three symmetries

Table 3.8: Metric admitting eight Noether Symmetries

Noether symmetry generators	Invariants
$\xi_{(5)}=\frac{s^2}{2}\frac{\partial}{\partial_s}+\frac{sr}{2}\frac{\partial}{\partial_r}+\frac{s(\alpha_1 t+\alpha_2)}{2\alpha_1}\frac{\partial}{\partial_t}$; $F=\frac{t^2}{2}-\frac{r^2\beta^2}{2}+\frac{\alpha_2 t}{\alpha_1}$,	$I_{(5)}=-\frac{s^2}{2}L+sr\beta^2\dot{r}$ $-\frac{s(\alpha_1 t+\alpha_2)}{\alpha_1}\dot{t}-\frac{t^2}{2}-\frac{\alpha_2 t}{\alpha_1}+\frac{r^2\beta^2}{2}$,
$\xi_{(6)}=\frac{s}{2\beta^2}\frac{\partial}{\partial_r}$; $F=r$,	$I_{(6)}=s\dot{r}-r$,
$\xi_{(7)}=s\frac{\partial}{\partial_s}+\frac{(\alpha_1 t+\alpha_2)}{2\alpha_1}\frac{\partial}{\partial_t}+\frac{r}{2}\frac{\partial}{\partial_r}$.	$I_{(7)}=-sL-\frac{\alpha_1 t+\alpha_2}{\alpha_1}\dot{t}+r\beta^2\dot{r}$.

along with their conserved forms are listed in Table 3.8.

Clearly, $\xi_{(5)}$ and $\xi_{(6)}$ are proper Noether symmetries, while $\xi_{(7)}$ corresponds to a homothetic vector $\frac{(\alpha_1 t+\alpha_2)}{\alpha_1}\frac{\partial}{\partial_t}+r\frac{\partial}{\partial_r}$ which is same as obtained for the metric 5c, given in Table 3.1.

This metric can be transformed to the metric (31) of Ref. [83] by replacing $\beta=\gamma$, $\alpha_1=\alpha$ and $\alpha_2=\beta$.

3.3.4 Nine Noether Symmetries

The branches 16, 19 and 21 give 9-dimensional algebra of Noether symmetries. In each case, we have four extra Noether symmetries other than the set N_5. In Table 3.9, we present metrics of these three cases along with their extra Noether symmetries and the corresponding conserved forms.

For metric 9a, $\xi_{(5)}$ and $\xi_{(6)}$ are proper Noether symmetries, while $\xi_{(7)}$ and $\xi_{(8)}$ are additional KVFs. One can see that the metric in this case is same as the metric 6b, given in Table 3.2, admitting the same KVFs.

In case of metric 9b, $\xi_{(5)}, \ldots, \xi_{(7)}$ are the additional KVFs and $\xi_{(8)}$ is a proper Noether symmetry. The metric in this case is same as the metric given in Table 3.3, admitting the same seven KVFs.

Similarly, in case of metric 9c, $\xi_{(5)}$ and $\xi_{(6)}$ are KVFs, while $\xi_{(7)}$ and $\xi_{(8)}$ are proper Noether symmetries. In this case, the metric is same as the metric 6c, given in Table 3.2.

The metric 9a, can be transformed to the metric (40) of Ref. [83] by taking $c_1 = \alpha$, $c_2 = \beta$ and $c_3 = \xi$. The metric 9b is the generalized form of the metric (34) of Ref. [83] and the metric 9c is same as the metric (37) of Ref. [83].

3.3.5 Eleven Noether Symmetries

In branch 12, the pivots are $p_1 \neq 0$, $p_2 \neq 0$ and $p_3 = p_4 = p_6 = p_9 = p_{10} = 0$. The simplification of these conditions gives the metric functions $G = \beta_1 e^t + \beta_2 e^{-t}$ and $H = \beta_1 e^t - \beta_2 e^{-t}$, where β_1 and β_2 are constants such that $4k\beta_1\beta_2 + 1 = 0$. This metric is same as the metric presented in Table 3.4, admitting ten KVFs.

The number of Noether symmetries for this metric turned out to be 11, in which five are same as given in set N_5, while the remaining six symmetries and their conserved forms are given in the Table 3.10. One can see that these six additional KVFs are same as listed in Table 3.4.

Table 3.9: Metrics admitting nine Noether Symmetries

No.	Metric	Noether symmetry generators	Invariants
9a (Branch 16)	$G = c_1 t + c_2$, $H = c_3$, where $c_1 \neq 0$, and $c_3 \neq 0$.	$\xi_{(5)} = -\frac{s}{2} e^r \frac{\partial}{\partial_t} + \frac{s}{2G} e^r \frac{\partial}{\partial_r}$; $F = Ge^r$, $\xi_{(6)} = \frac{s}{2} e^{-r} \frac{\partial}{\partial_t} - \frac{s}{2G} e^{-r} \frac{\partial}{\partial_r}$; $F = -Ge^{-r}$, $\xi_{(7)} = e^r \frac{\partial}{\partial_t} - \frac{1}{G} e^r \frac{\partial}{\partial_r}$, $\xi_{(8)} = e^{-r} \frac{\partial}{\partial_t} + \frac{1}{G} e^{-r} \frac{\partial}{\partial_r}$, where we have chosen $c_1 = 1$.	$I_{(5)} = e^r (s\ \dot{t} + G\ \dot{r} - G)$, $I_{(6)} = -e^{-r} (s\ \dot{t} + G\ \dot{r} - G)$, $I_{(7)} = -2e^r (\dot{t} + G\ \dot{r})$, $I_{(8)} = 2e^{-r} (-\dot{t} - \frac{1}{G}\ \dot{r})$.
9b (Branch 19)	$G = c_1$, $H = c_2 e^{\sqrt{k}t} + c_3 e^{-\sqrt{k}t}$, where $c \not= 0$, $k > 0$ and $4kc_2c_3 = 1$.	$\xi_{(5)} = \sin\theta \sin\phi \frac{\partial}{\partial_t} + \frac{H'}{H} \cos\theta \sin\phi \frac{\partial}{\partial_\theta}$ $+ \frac{H'}{H} \csc\theta \cos\phi \frac{\partial}{\partial_\phi}$, $\xi_{(6)} = \sin\theta \cos\phi \frac{\partial}{\partial_t} + \frac{H'}{H} \cos\theta \cos\phi \frac{\partial}{\partial_\theta}$ $- \frac{H'}{H} \csc\theta \sin\phi \frac{\partial}{\partial_\phi}$, $\xi_{(7)} = \cos\theta \frac{\partial}{\partial_t} + \frac{H'}{H} \sin\theta \frac{\partial}{\partial_\theta}$, $\xi_{(8)} = \frac{s}{2c_1^2} \frac{\partial}{\partial_r}$. $F = r$.	$I_{(5)} = -2 \sin\theta \sin\phi \dot{t}$ $+2HH' \left(\cos\theta \cos\phi \dot{\theta} + \sin\theta \sin\phi \dot{\phi} \right)$, $I_{(6)} = -2 \sin\theta \sin\phi \dot{t}$ $2HH' \left(\cos\theta \cos\phi \dot{\theta} - \sin\theta \sin\phi \dot{\phi} \right)$, $I_{(7)} = -2 \cos\theta \dot{t} + 2HH' \sin\theta \dot{\theta}$, $I_{(8)} = s\dot{r} - r$.
9c (Branch 21)	$G = \beta$, $H = \gamma$, where $\beta \neq 0$ and $\gamma \neq 0$.	$\xi_{(5)} = \frac{\partial}{\partial_t}$, $\xi_{(6)} = r \frac{\partial}{\partial_t} + t \frac{\partial}{\partial_r}$, $\xi_{(7)} = -\frac{s}{2} \frac{\partial}{\partial_t}$; $F = t$, $\xi_{(8)} = \frac{s}{2} \frac{\partial}{\partial_r}$; $F = r$. where we have chosen $\beta = 1$.	$I_{(5)} = -2\ \dot{t}$, $I_{(6)} = 2t\dot{r} - 2r\dot{t}$, $I_{(7)} = s\dot{t} - t$, $I_{(8)} = s\ \dot{r} - r$.

Table 3.10: Metric admitting eleven Noether Symmetries

Additional KVFs	Invariants
$\xi_{(5)}=\sin\theta\sin\phi e^{r}(\frac{\partial}{\partial_t}-\frac{H}{G}\frac{\partial}{\partial_r})$ $+\frac{G}{H}e^{r}(\cos\theta\sin\phi\frac{\partial}{\partial_\theta}+\csc\theta\cos\phi\frac{\partial}{\partial_\phi})$,	$I_{(5)}=-2\sin\theta\sin\phi e^{r}(\dot{t}+GH\dot{r}$ $+2GHe^{r}(\cos\theta\sin\phi\dot{\theta}+\csc\theta\cos\phi\dot{\phi}$,
$\xi_{(6)}=\sin\theta\sin\phi e^{-r}(\frac{\partial}{\partial_t}+\frac{H}{G}\frac{\partial}{\partial_r})$ $+\frac{G}{H}e^{-r}(\cos\theta\sin\phi\frac{\partial}{\partial_\theta}+\csc\theta\cos\phi\frac{\partial}{\partial_\phi})$,	$I_{(6)}=-2\sin\theta\sin\phi e^{-r}(\dot{t}-GH\dot{r})$ $+2GHe^{-r}(\cos\theta\sin\phi\dot{\theta}+\csc\theta\cos\phi\dot{\phi})$,
$\xi_{(7)}=-\sin\theta\cos\phi e^{r}(\frac{\partial}{\partial_t}-\frac{H}{G}\frac{\partial}{\partial_r})$ $-\frac{G}{H}e^{r}(\cos\theta\cos\phi\frac{\partial}{\partial_\theta}-\csc\theta\sin\phi\frac{\partial}{\partial_\phi})$,	$I_{(7)}=2\sin\theta\cos\phi e^{r}(\dot{t}+GH\dot{r})$ $-2GHe^{r}(\cos\theta\cos\phi\dot{\theta}-\csc\theta\sin\phi\dot{\phi})$,
$\xi_{(8)}=-\sin\theta\cos\phi e^{-r}(\frac{\partial}{\partial_t}+\frac{H}{G}\frac{\partial}{\partial_r})$ $-\frac{G}{H}e^{-r}(\cos\theta\cos\phi\frac{\partial}{\partial_\theta}-\csc\theta\sin\phi\frac{\partial}{\partial_\phi})$,	$I_{(8)}=2\sin\theta\cos\phi e^{-r}(\dot{t}-GH\dot{r})$ $-2GHe^{-r}(\cos\theta\cos\phi\dot{\theta}-\csc\theta\sin\phi\dot{\phi}$,
$\xi_{(9)}=-\cos\theta e^{r}(\frac{\partial}{\partial_t}-\frac{H}{G}\frac{\partial}{\partial_r})$ $+\frac{G}{H}\sin\theta e^{r}\frac{\partial}{\partial_\theta}$,	$I_{(9)}=2\cos\theta e^{r}(\dot{t}+GH\dot{r})$ $+GH\sin\theta e^{r}\dot{\theta}$,
$\xi_{(10)}=-\cos\theta e^{-r}(\frac{\partial}{\partial_t}+\frac{H}{G}\frac{\partial}{\partial_r})$ $+\frac{G}{H}\sin\theta e^{-r}\frac{\partial}{\partial_\theta}$.	$I_{(10)}=2\cos\theta e^{-r}(\dot{t}-GH\dot{r})$ $+2GH\sin\theta e^{-r}\dot{\theta}$.

As none of these six symmetries involve the affine parameter s, it is clear that all these are KVFs. Thus the total number of KVFs in this case is ten. The metric of this branch is the generalized form of the metric (43) of Ref. [83].

3.4 Stability of the Obtained Models

In this section, we discuss the stability of the cosmological models which are obtained in the process of our classification in the section of HVFs. It is well known that the motion of a test particle in a background gravitational field is described by the geodesic equations

$$\ddot{x}+\Gamma^{a}_{bc}\dot{x^{b}}\dot{x^{c}}=0, \tag{3.4.1}$$

varying along the geodesics and Γ^a_{bc} are the Christoffel symbols. For the metric (3.1.1), the geodesic equations (3.4.1) reduce to the following four equations:

$$\begin{aligned}
\ddot{t} + GG'\dot{r}^2 + HH'(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = 0,\\
G\ddot{r} + 2G'\dot{t}\dot{r} = 0,\\
H\ddot{\theta} + 2H'\dot{t}\dot{\theta} - H\sin\theta\cos\theta\dot{\phi}^2 = 0,\\
H\ddot{\phi} + 2H'\dot{t}\dot{\phi} + 2H\cot\theta\dot{\theta}\dot{\phi} = 0.
\end{aligned} \tag{3.4.2}$$

The perturbation of the geodesic equations leads to the geodesic deviation equations, given by [114, 115]:

$$\ddot{\zeta}^a + 2\Gamma^a_{bc}\dot{x}^b\dot{\zeta}^c + \Gamma^a_{bc,e}\dot{x}^b\dot{x}^c\xi^e = 0, \tag{3.4.3}$$

where ζ^a is the deviation four-vector from the geodesic motion. For the metric (3.1.1), these equations become:

$$\begin{aligned}
\zeta^0_{,ss} + 2GG'\dot{r}\zeta^1_{,s} + 2HH'\dot{\theta}\zeta^2_{,s} + 2HH'\sin^2\theta\dot{\phi}\zeta^3_{,s} + (GG')'\dot{r}^2\zeta^0 & \\
+(HH')'\dot{\theta}^2\zeta^2 + (HH')'\sin^2\theta\dot{\phi}^2\zeta^0 + 2HH'\sin\theta\cos\theta\dot{\phi}^2\zeta^2 &= 0,\\
\zeta^1_{,ss} + \frac{2G'}{G}(\dot{t}\zeta^1_{,s} + \dot{r}\zeta^0_{,s}) + 2\left(\frac{G'}{G}\right)'\dot{t}\dot{r}\zeta^0 &= 0,\\
\zeta^2_{,ss} + \frac{2H'}{H}(\dot{t}\zeta^2_{,s} + \dot{\theta}\zeta^0_{,s}) - 2\sin\theta\cos\theta\dot{\phi}\zeta^3_{,s} + 2\left(\frac{H'}{H}\right)'\dot{t}\dot{\theta}\zeta^0 - (\cos^2\theta - \sin^2\theta)\dot{\phi}^2\zeta^2 &= 0,\\
\zeta^3_{,ss} + \frac{2H'}{H}(\dot{t}\zeta^3_{,s} + \dot{\phi}\zeta^0_{,s}) + 2\cot\theta(\dot{\theta}\zeta^3_{,s} + \dot{\phi}\zeta^2_{,s}) + 2\left(\frac{H'}{H}\right)'\dot{t}\dot{\phi}\zeta^0 - 2\csc^2\theta\ \dot{\theta}\dot{\phi}\zeta^2 &= 0.
\end{aligned} \tag{3.4.4}$$

The above geodesic deviation equations can be used to study the stability of cosmological models. Solving the geodesic equations (3.4.2), one may find the velocity 4-vector $\dot{x}_a$, and using it in Eq. (3.4.4), one gets ζ^a. Now, if $[a, b]$

is an interval in which the functions ζ^a behave monotonically, then the limit, $\lim_{t\to b}\zeta^a$, is considered as an indicator for the stability of the model under consideration. If $\lim_{t\to b}\zeta^a$ is finite, then the model is stable, otherwise it is unstable. This approach was recently used to discuss the stability of some spherically symmetric cosmological models [116, 117].
If we use the comoving coordinates, then one can choose $\dot{r} = \dot{\theta} = \dot{\phi} = 0$ and $\dot{t} = 1$. These quantities can be considered as a solution of the geodesic equations (3.4.2), while the geodesic deviation equations (3.4.4) become

$$\begin{aligned} \zeta^0_{,ss} &= 0, \\ \zeta^1_{,ss} + \frac{2G'}{G}\zeta^1_{,s} &= 0, \\ \zeta^2_{,ss} + \frac{2H'}{H}\zeta^2_{,s} &= 0, \\ \zeta^3_{,ss} + \frac{2H'}{H}\zeta^3_{,s} &= 0. \end{aligned} \tag{3.4.5}$$

It is well known that the first integral of the geodesic deviation equation is given by:

$$g_{ab}\dot{x}^a\zeta^b = C, \tag{3.4.6}$$

where C is some constant. Using $\dot{r} = \dot{\theta} = \dot{\phi} = 0$ and $\dot{t} = 1$ in this equation, we have $\zeta^0 = C$. Moreover, solving Eqs. (3.4.5), we get

$$\zeta^1 = \lambda_1 \int \frac{1}{G^2}dt,\ \zeta^2 = \lambda_2 \int \frac{1}{H^2}dt,\ \zeta^3 = \lambda_3 \int \frac{1}{H^2}dt, \tag{3.4.7}$$

where λ_1, λ_2 and λ_3 are constants. Thus, a KantowskiSachs metric is stable if the limits $\lim_{t\to\infty}\int \frac{1}{G^2}dt$ and $\lim_{t\to\infty}\int \frac{1}{H^2}dt$ are finite. This criteria can be used to check the stability of the models obtained in our classification. As a result, it can be seen that the models 5b and 10a are stable, the model 5a is stable if $1-\frac{2c_2}{\alpha}>0$ and the models 5c, 5d, 6a, 6b, 6c and 7a are unstable. Similarly, if we use $\dot{\theta}=\dot{\phi}=0$ and $\dot{t}=1$, then the geodesic equations (3.4.2) reduce to

$$
\begin{aligned}
G'\dot{r^2} &= 0, \\
G\ddot{r}+2G'\dot{r} &= 0,
\end{aligned}
\tag{3.4.8}
$$

and the geodesic deviation equations given in (3.4.4) become

$$
\begin{aligned}
\zeta^0_{,ss}+2GG'\dot{r}\zeta^1_{,s}+(GG')'\dot{r^2}\zeta^0 &= 0, \\
\zeta^1_{,ss}+\frac{2G'}{G}(\zeta^1_{,s}+\dot{r}\zeta^0_{,s})+2(\frac{G'}{G})'\dot{r}\zeta^0 &= 0, \\
\zeta^2_{,ss}+\frac{2H'}{H}\zeta^2_{,s} &= 0, \\
\zeta^3_{,ss}+\frac{2H'}{H}\zeta^3_{,s} &= 0.
\end{aligned}
\tag{3.4.9}
$$

As $\dot{t}=1$, we can take $t=s$. Moreover, from Eq. (3.4.8), we can see that either $\dot{r}=0$ or $G'=0$. In the former case, the stability analysis is same as presented above. In the latter case, the metric function P becomes constant and the solution of geodesic deviation equations is obtained as:

$$
\zeta^0=\lambda_1 t+\lambda_2,\ \zeta^1=\lambda_3 t+\lambda_4,\ \zeta^2=\lambda_5\int\frac{1}{H^2}dt,\ \zeta^3=\lambda_6\int\frac{1}{H^2}dt,
\tag{3.4.10}
$$

where $\lambda_1...,\lambda_6$ are constants. We can see that as t increases, ζ^0 and ζ^1 become infinite. Thus, the models where the metric function G is constant are unstable, for example, the models given by 5c, 6c and 7a. However, there is no indication of the stability/unstability of the remaining models using this approach.

3.5 Summary

In this chapter, we have presented a complete classification of Kantowski-Sachs spacetimes via HVFs, CVFs and Noether symmetries by adopting a new approach. Instead of directly integrating the set of determining equations, an algorithm is developed in Maple which reduces the set of determining equations to the simplified form and yields a tree known as Rif tree and a list of pivots. The branches of the Rif tree give all possible cases where the Kantowski-Sachs spacetimes may possess proper homothetic and conformal vector fields, KVFs and the Noether symmetries other than the minimum ones.

Out of the 20 branches of the Rif tree for HVFs, 12 branches give the minimum four KVFs which we have excluded from our discussion, while the remaining branches produce some Kantowski-Sachs metrics possessing proper HVFs or additional KVFs.

In 18 branches of the Rif tree for CVFs, we only considered those cases where the spacetime is non-conformally flat. Solving the conformal symmetry equations for each branch, we have concluded that non-conformally flat Kantowski-Sachs spacetimes possess at most two proper CVFs along with four minimum KVFs. In some cases, we also have six KVFs with zero conformal factor, giving no proper CVF.

For Noether symmetries, out of 21 branches of the Rif tree, 11 branches give the minimal set of Noether symmetry in which four are the basic KVFs and one is the symmetry corresponding to the Lagrangian. The remaining branches produce 6, 7, 8, 9 and 11-dimensionl Lie algebra of Noether symmetries.

We have also given a comparison of our results of Rif tree approach with those of direct integration technique. It is observed that, the metrics which we have obtained by Rif tree approach are more generalized than those obtained by direct integration technique. Moreover, we have also noticed that some metrics were missing there.

For most of the obtained metrics, we have checked that the energy density is positive, which means that these metrics are physically realistic. Moreover, we have found the bounds for energy conditions and the stability of the obtained metrics is checked with the help of geodesic equations and the perturbed geodesic deviation equations.

Chapter 4

Lie and Noether Symmetries of Static Plane Symmetric Spacetimes

In this chapter, we find the homothetic, conformal and Noether symmetries of static plane symmetric spacetimes. Like the previous chapter, here we again use the Rif tree approach instead of direct integrating technique for solving the determining equations of these symmetries. First we develop a Rif algorithm in Maple which reduces the set of determining equations to the simplified form and yields a Rif tree. Like the previous chapter, the integration of determining equations is carried out for the metric of every branch of the Rif tree for obtaining the final form of homothetic, conformal and Noether symmetries.

4.1 Homothetic Symmetries

In this section, we explore homothetic symmetries of static plane symmetric spacetimes. This work has been published in an ISI journal [118]. The metric of static plane symmetric spacetimes has the form [3]:

$$ds^2 = -e^{2G(x)}dt^2 + \ dx^2 + e^{2H(x)}\ [dy^2 + dz^2]. \tag{4.1.1}$$

The four linearly independent KVFs for the above metric are:

$$K_4 = \left\{\frac{\partial}{\partial_t}, \frac{\partial}{\partial_y}, \frac{\partial}{\partial_z}, z\frac{\partial}{\partial_y} - y\frac{\partial}{\partial_z}\right\},$$

and the non-zero components of its energy-momentum tensor are:

$$\begin{aligned} T_{00} &= -e^{2G}\left(2H'' + 3H'^2\right), \\ T_{11} &= H'^2 + 2G'H', \\ T_{22} &= T_{33} = e^{2H}\left(H'' + H'^2 + G'' + G'^2 + H'G'\right). \end{aligned} \tag{4.1.2}$$

For an anisotropic fluid, the above components become:

$$T_{00} = \rho e^{2G},\ T_{11} = p_{||},\ T_{22} = \ T_{33} = e^{2H}p_{\perp}, \tag{4.1.3}$$

where ρ denotes the energy-density, while $p_{||}$ and $p_{\perp}$ represent parallel and perpendicular pressures of the fluid respectively. The comparison of the Eqs. (4.1.2) and (4.1.3) yields:

$$\begin{aligned} \rho &= -2H'' - 3H'^2, \\ p_{||} &= H'^2 + 2G'H', \\ p_{\perp} &= H'' + H'^2 + G'' + G'^2 + H'G'. \end{aligned} \tag{4.1.4}$$

Using the metric (4.1.1) in the equation satisfied by homothetic vector fields, given in (2.5.3), we obtain:

$$G'\xi^1 + \xi^0_{,t} = \alpha, \tag{4.1.5}$$

$$\xi^1_{,t} - e^{2G}\xi^0_{,x} = 0, \tag{4.1.6}$$

$$e^{2H}\xi^2_{,t} - e^{2G}\xi^0_{,y} = 0, \tag{4.1.7}$$

$$e^{2H}\xi^3_{,t} - e^{2G}\xi^0_{,z} = 0, \tag{4.1.8}$$

$$\xi^1_{,x} = \alpha, \tag{4.1.9}$$

$$e^{2H}\xi^2_{,x} + \xi^1_{,y} = 0, \tag{4.1.10}$$

$$e^{2H}\xi^3_{,x} + \xi^1_{,z} = 0, \tag{4.1.11}$$

$$H'\xi^1 + \xi^2_{,y} = \alpha, \tag{4.1.12}$$

$$\xi^2_{,z} + \xi^3_{,y} = 0, \tag{4.1.13}$$

$$H'\xi^1 + \xi^3_{,z} = \alpha, \tag{4.1.14}$$

where α represent a constant. Like the previous chapter, to investigate the HVFs of static plane symmetric spacetimes, we have followed the same approach based on a computer algorithm and obtained the Rif tree given in Fig. 4.1 and the following list of pivots.

$$\begin{aligned}
p_1 &= H'', \\
p_2 &= G' + H', \\
p_3 &= H'^2 + H'' - G'^2 - G'', \\
p_4 &= H' - G', \\
p_5 &= H', \\
p_6 &= G'',
\end{aligned}$$

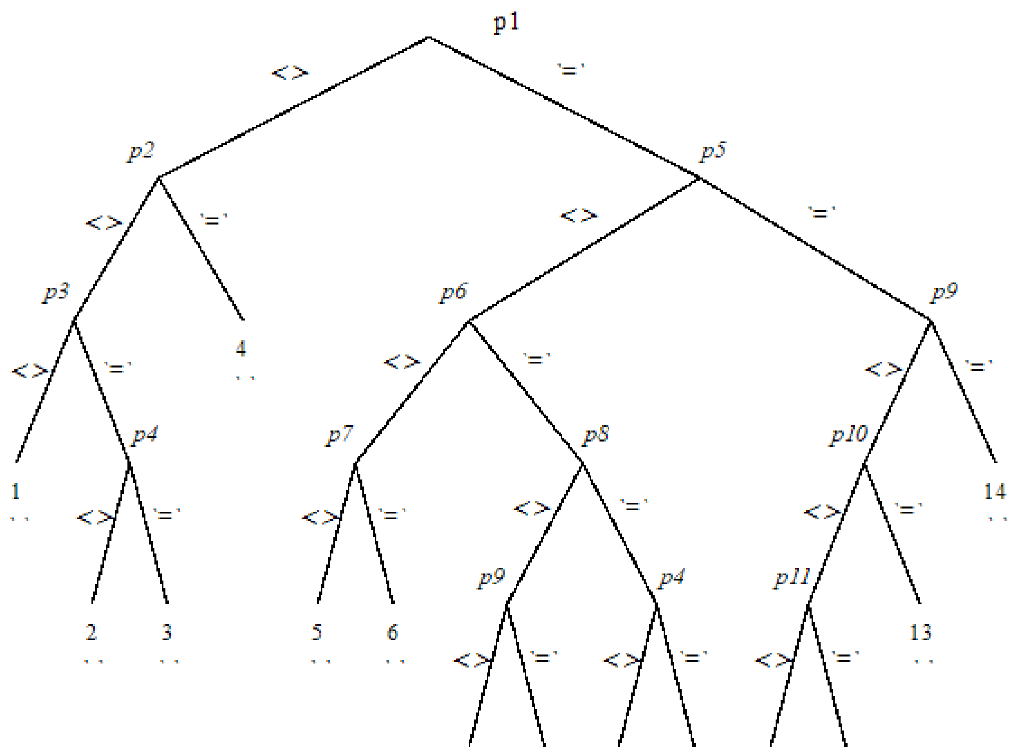

Figure 4.1: Rif Tree for HVFs

$$
\begin{aligned}
p_7 &= H'^2 - G'^2 - G'', \\
p_8 &= H'^2 - G'^2, \\
p_9 &= G', \\
p_{10} &= G'^2 + G'', \\
p_{11} &= G''' + 2G'G''.
\end{aligned}
$$

For a complete classification, we use the constraints of the branches of the Rif tree and solve the system of equations (4.1.5)- (4.1.14). Consequently, some branches yield homothetic algebras of dimension 5, 7 and 11, while the remaining branches give Killing algebras of dimension 4, 5, 6, 7 and 10. We skip to write the basic calculations and summarize the obtained results in the forthcoming sections. The branches labeled by 5 and 6 give minimum four KVFs and are therefore excluded from the classification. Here, we also compare our results with the existing results of Ref. [119], which were obtained by using direct integration technique.

4.1.1 Five HVFs

The branches 1, 2, 4 and 11 give 5-dimensional algebra of HVFs. In each case, we have four KVFs which are same as given in the set K_4, and one proper HVF. Table 4.1 presents the metrics of these cases along with their proper HVFs.

The metrics 5a(i), 5b(i), 5c(i) and 5d admitting five HVFs are more generalized than the only one metric, given by Eq. (3.10) in Ref. [119] possessing the same number of homothetic symmetries. In fact, these metrics can be reduced to the metric (3.10) of Ref. [119]. For example, in case of the metric 5c(i), we may use $\alpha = 1$, $c_1 = 0$ and $c_2 = 1 + \frac{A}{2} = 1 - \frac{C}{2}$ to get the metric (3.10), where A and C are constants used in Ref. [119]. Moreover, the metrics 5a(ii), 5b(ii) and 5c(ii) are not listed in Ref. [119].

The model 5a(i) is an anisotropic fluid with $p_{||} = \frac{3\alpha^2+c_4^2-4\alpha c_4-2\alpha c_1+2c_1c_4}{(\alpha x+c_3)^2}$, $p_\perp = \frac{\alpha^2+c_1^2+c_4^2-2\alpha c_1-2\alpha c_4+c_1c_4}{(\alpha x+c_3)^2}$ and $\rho = \frac{4\alpha c_4-\alpha^2-3c_4^2}{(\alpha x+c_3)^2}$. Here, the energy density is non-negative provided that $4\alpha c_4 - \alpha^2 - 3c_4^2 \geq 0$. Using these values of ρ, $p_{||}$ and $p_\perp$, one can easily simplify the inequalities given in (2.7.1) to find the

Table 4.1: Metrics admitting five HVFs

No.	Metric	Vector Field Components	Proper HVF
5a(i) (Branch 1)	$G = \ln(\alpha x + c_3)^{1-\frac{c_1}{\alpha}}$, $H = \ln(\alpha x + c_3)^{1-\frac{c_4}{\alpha}}$, where $\alpha \neq c_1 \neq c_4$, and $\alpha - c_1 \neq c_4$.	$\xi^0 = c_1 t + c_2$, $\xi^1 = \alpha x + c_3$, $\xi^2 = c_4 y + c_5 z + c_6$, $\xi^3 = c_4 z - c_5 y + c_7$.	$\xi_{(5)} = x\frac{\partial}{\partial_x}$.
5a(ii) (Branch 1)	$G = \beta$, $H = \ln(\alpha x + c_1)^{1-\frac{c_3}{\alpha}}$, where $\alpha \neq c_3 \neq 0$ and $\beta \neq 0$.	$\xi^0 = \alpha t + c_2$, $\xi^1 = \alpha x + c_1$, $\xi^2 = c_3 y - c_4 z + c_5$, $\xi^3 = c_3 z + c_4 y + c_6$.	$\xi_{(5)} = t\frac{\partial}{\partial_t} + x\frac{\partial}{\partial_x}$.
5b(i) (Branch 2)	$G = \ln(\alpha x + c_1)^{\frac{c_4}{\alpha}}$, $H = \ln(\alpha x + c_1)^{1-\frac{c_4}{\alpha}}$, where $\alpha \neq c_4$.	$\xi^0 = \alpha t - c_4 t + c_2$, $\xi^1 = \alpha x + c_3$, $\xi^2 = c_4 y + c_5 z + c_6$, $\xi^3 = c_4 z - c_5 y + c_7$.	$\xi_{(5)} = t\frac{\partial}{\partial_t} + x\frac{\partial}{\partial_x}$.
5b(ii) (Branch 2)	$G = \beta$, $H = \ln(\gamma_1 x + \gamma_2)$, where $\beta \neq 0$ and $\gamma_1 \neq 0$.	$\xi^0 = \alpha t + c_1$, $\xi^1 = \frac{\alpha_1 x + \alpha_2}{\gamma_1}\alpha$, $\xi^2 = -c_2 z + c_3$, $\xi^3 = c_2 y + c_4$.	$\xi_{(5)} = \frac{\alpha_1 x + \alpha_2}{\gamma_1}\frac{\partial}{\partial_x} + t\frac{\partial}{\partial_t}$.
5c(i) (Branch 4)	$H = \ln(\alpha x + c_1)^{1-\frac{c_2}{\alpha}}$, where $\alpha \neq c_2$, and $G = -H$.	$\xi^0 = 2\alpha t - c_2 t + c_3$, $\xi^1 = \alpha x + c_1$, $\xi^2 = c_2 y + c_4 z + c_5$, $\xi^3 = c_2 z - c_4 y + c_6$.	$\xi_{(5)} = 2t\frac{\partial}{\partial_t} + x\frac{\partial}{\partial_x}$.
5c(ii) (Branch 4)	$G = \ln(\alpha_1 x + \alpha_2)^{1-\frac{c_1}{\alpha}}$, $H = \ln(\alpha_1 x + \alpha_2)$, where $\alpha \neq 0$ and $\alpha = \frac{c_1}{2}$.	$\xi^0 = c_1 x + c_2$, $\xi^1 = \frac{\alpha_1 x + \alpha_2}{\alpha_1}\alpha$, $\xi^2 = -c_3 z + c_4$, $\xi^3 = c_3 y + c_5$.	$\xi_{(5)} = \frac{\alpha_1 x + \alpha_2}{\alpha_1}\frac{\partial}{\partial_x}$.
5d (Branch 11)	$G = \ln(\alpha x + c_2)^{\frac{c_3}{\alpha}}$, $H = c_1$, where $c_3 \neq \alpha \neq 0$.	$\xi^0 = \alpha t - c_3 t + c_4$, $\xi^1 = \alpha x + c_2$, $\xi^2 = \alpha y + c_5 z + c_6$, $\xi^3 = \alpha z - c_5 y + c_7$.	$\xi_{(5)} = t\frac{\partial}{\partial_t} + x\frac{\partial}{\partial_x} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}$

bounds for all energy conditions. For model 5a(ii), we have $\rho = \frac{4\alpha c_3 - \alpha^2 - 3c_3^2}{(\alpha x + c_1)^2}$, $p_{||} = \frac{(\alpha - c_3)^2}{(\alpha x + c_1)^2}$ and $p_{\perp} = \frac{c_3(c_3 - \alpha)}{(\alpha x + c_1)^2}$. The SEC and NEC require $c_3(c_3 - \alpha) > 0$ and $3\alpha c_3 - \alpha^2 - 2c_3^2 \geq 0$, while the WEC holds if $4\alpha c_3 - \alpha^2 - 3c_3^2 \geq 0$, $c_3(c_3 - \alpha) > 0$ and $3\alpha c_3 - \alpha^2 - 2c_3^2 \geq 0$. Finally, the DEC is satisfied if $4\alpha c_3 - \alpha^2 - 3c_3^2 \geq 0$, $c_3(c_3 - \alpha) > 0$, $3\alpha c_3 - \alpha^2 - 2c_3^2 \geq 0$ and $5\alpha c_3 - \alpha^2 - 4c_3^2 \geq 0$.

Similarly, for the model 5b(i) being an anisotropic fluid, the energy density is same as in case 5a(i) and the pressures in parallel and perpendicular directions are given by $p_{||} = \frac{\alpha^2 - c_4^2}{(\alpha x + c_1)^2}$ and $p_{\perp} = \frac{c_4(c_4 - \alpha)}{(\alpha x + c_1)^2}$. These quantities satisfy the SEC and NEC if $c_4(\alpha - c_4) > 0$ and $3\alpha c_4 - \alpha^2 - 2c_4^2 \geq 0$, while the WEC holds if along with these inequalities being true, we further have $4\alpha c_4 - \alpha^2 - 3c_4^2 \geq 0$. The DEC is clearly violated.

For the model 5b(ii), we obtain $p_{||} = \frac{\gamma_1^2}{(\gamma_1 x + \gamma_2)^2} = -\rho$ and $p_{\perp} = 0$, which give an unphysical model having negative energy density and violating all the energy conditions.

Moreover, for the metric 5c(i) we have $\rho = \frac{4\alpha c_2 - \alpha^2 - 3c_2^2}{(\alpha x + c_1)^2}$ and $p_{||} = p_{\perp} = -\frac{(\alpha - c_2)^2}{(\alpha x + c_1)^2}$, giving a perfect fluid model for which one can easily simplify the inequalities given in (2.7.1) to find the bounds for energy conditions.

The model 5c(ii) represents an anisotropic fluid with $\rho = -\frac{\alpha_1^2}{(\alpha_1 x + \alpha_2)^2}$, $p_{||} = \frac{\alpha_1^2(3\alpha - 2c_1)}{\alpha(\alpha_1 x + \alpha_2)^2}$ and $p_{\perp} = \frac{\alpha_1^2(3\alpha^2 + c_1^2 - 4\alpha c_1)}{\alpha^2(\alpha_1 x + \alpha_2)^2}$. The model is physically unrealistic with the negative energy density and like the previous case, the bounds for different energy conditions can be found by using these quantities in (2.7.1).

Finally, for the metric 5d, we obtain $\rho = p_{||} = 0$ and $p_{\perp} = \frac{c_3(c_3 - \alpha)}{(\alpha x + c_2)^2}$. The DEC is violated here, while all other energy conditions are satisfied subject to the inequality $c_3(c_3 - \alpha) > 0$.

4.1.2 Seven HVFs

There is only one metric, given by branch 3, which admits 7-dimensional homothetic algebra. In Table 4.2, we present this metric along with the vector field components and its three additional symmetries other than those given in the set K_4. Here $\xi_{(5)}$ and $\xi_{(6)}$ are two additional KVFs, while $\xi_{(7)}$ is a proper HVF.

Table 4.2: Metric admitting seven HVFs

Metric	Vector Field Components	Additional Symmetries
$G = H = \ln(\alpha x + c_1)^{1-\frac{c_2}{\alpha}}$, where $\alpha \neq c_2$.	$\xi^0 = c_2 t + c_3 z + c_4 y + c_5$, $\xi^1 = \alpha x + c_1$, $\xi^2 = c_2 y + c_4 t + c_6 z + c_7$, $\xi^3 = c_2 z + c_3 t - c_6 y + c_8$.	$\xi_{(5)} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}$, $\xi_{(6)} = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}$, $\xi_{(7)} = x\frac{\partial}{\partial x}$.

This metric can be reduced to the metric (3.11) of Ref. [119] by taking $\alpha = 1$, $c_1 = 0$ and $c_2 = 1 - \frac{A}{2}$, where A is the constant used in Ref. [119].
The energy density and, parallel and perpendicular pressures for this metric are found to be $\rho = \frac{4\alpha c_2 - \alpha^2 - 3c_2^2}{(\alpha x + c_1)^2}$, $p_{||} = \frac{3(\alpha - c_2)^2}{(\alpha x + c_1)^2}$ and $p_{\perp} = \frac{\alpha^2 - 4\alpha c_2 + 3c_2^2}{(\alpha x + c_1)^2}$. The simplified form of energy bounds can be easily obtained by substituting these values in (2.7.1).

4.1.3 Eleven HVFs

In branches labeled by 13 and 14, we obtain eleven HVFs, out of which four are same as given in the set K_4 and the additional seven symmetries for each metric are given in Table 4.3. For each metric, $\xi_{(5)}$ represents a proper HVF, while $\xi_{(6)}...\xi_{(11)}$ give six additional KVFs.

As for the metric 11a is concerned, it is same as the metric (2.29) of Ref. [119] with $c_1 = 1$ and $c_2 = A$. The metric 11b is the flat Minkowski metric and therefore attains the maximum dimension of homothetic algebra.

Moreover, both the metrics 11a and 11b represent vacuum solutions of EFEs with zero energy density and vanishing pressure.

4.1.4 Five KVFs

Two branches, given by branch 7 and 9, yield five KVFs with no proper homothety. Four KVFs are same as given in the set K_4 and the fifth one is given in Table 4.4.

The metric 5a can be transformed to the metric given by Eq. (2.32) in Ref. [119] by setting $c_1 = \frac{A}{2}$, $c_3 = \frac{C}{2}$ and $c_2 = c_4 = 0$. Under the same conditions, the metric 5b can be regarded as a special case of the metric (2.32) of Ref. [119].

As for the physical implications of these metrics, the metric 5a is an anisotropic fluid with $\rho = -3c_3^2$, $p_{||} = c_3^2 + 2c_1c_3$ and $p_\perp = c_1^2 + c_3^2 + c_1c_3$. For these values, the DEC and WEC are failed, while NEC holds if $c_3(c_1 - c_3) > 0$ and $c_1^2 - 2c_3^2 + c_1c_3 \geq 0$. Along with these conditions, if the inequality $c_1(c_1 + 2c_3) \geq 0$ also holds, then the SEC is satisfied.

Similarly, for the model 5b, we have $\rho = -3c_1^2$, $p_{||} = -c_1^2$ and $p_\perp = c_1^2$. However, none of the energy conditions is satisfied here. Moreover, both of these models are un-physical because their energy densities are negative.

Table 4.3: Metrics admitting eleven HVFs

No.	Metric	Vector Field Components	Proper HVF	Additional KVFs
11a (Branch 13)	$G=\ln(c_1x+c_2)$, $H=c_3$, where $c_1\neq 0$.	$\xi^0=\frac{1}{c_1x+c_2}[z(c_4e^{c_1t}-c_5e^{-c_1t})$, $+y(c_6e^{c_1t}-c_7e^{-c_1t})-\frac{1}{c_1}(c_8e^{c_1t}+c_9e^{-c_1t})]+c_{10}$, $\xi^1=-z(c_4e^{c_1t}+c_5e^{-c_1t})-y(c_6e^{c_1t}+c_7e^{-c_1t})$, $+\frac{1}{c_1}(c_8e^{c_1t}-c_9e^{-c_1t})+\frac{\alpha}{c_1}(c_1x+c_2)$, $\xi^2=\alpha y+\frac{c_1x+c_2}{c_1}(c_6e^{c_1t}+c_7e^{-c_1t})+c_{11}z+c_{12}$, $\xi^3=\alpha z+\frac{c_1x+c_2}{c_1}(c_4e^{c_1t}+c_5e^{-c_1t})-c_{11}y+c_{13}$.	$\xi_{(5)}=\frac{1}{G'}\frac{\partial}{\partial_x}+y\frac{\partial}{\partial_y}+z\frac{\partial}{\partial_z}$.	$\xi_{(6)}=e^{c_1t}\left(ze^{-G}\frac{\partial}{\partial_t}-z\partial_x+\frac{1}{G'}\partial_z\right)$, $\xi_{(7)}=e^{-c_1t}\left(-ze^{-G}\frac{\partial}{\partial_t}-z\frac{\partial}{\partial_x}+\frac{1}{G'}\frac{\partial}{\partial_z}\right)$, $\xi_{(8)}=e^{c_1t}\left(ye^{-G}\frac{\partial}{\partial_t}-y\frac{\partial}{\partial_x}+\frac{1}{G'}\frac{\partial}{\partial_y}\right)$, $\xi_{(9)}=e^{-c_1t}\left(-ye^{-G}\frac{\partial}{\partial_t}-y\frac{\partial}{\partial_x}+\frac{1}{G'}\frac{\partial}{\partial_y}\right)$, $\xi_{(10)}=\frac{1}{c_1}e^{c_1t}(\frac{\partial}{\partial_x}-e^{-G}\frac{\partial}{\partial_t})$, $\xi_{(11)}=-\frac{1}{c_1}e^{-c_1t}(\frac{\partial}{\partial_x}+e^{-G}\frac{\partial}{\partial_t})$.
11b (Branch 14)	$G=Const.$, $H=Const.$,	$\xi^0=\alpha t+c_1x+c_2y+c_3z+c_4$, $\xi^1=\alpha x+c_1t-c_5y-c_6z+c_7$, $\xi^2=\alpha y+c_2t+c_5x+c_8z+c_9$, $\xi^3=\alpha z+c_3t+c_6x-c_8y+c_{10}$.	$\xi_{(5)}=t\frac{\partial}{\partial_t}+x\frac{\partial}{\partial_x}+y\frac{\partial}{\partial_y}+z\frac{\partial}{\partial_z}$.	$\xi_{(6)}=x\frac{\partial}{\partial_t}+t\frac{\partial}{\partial_x}$, $\xi_{(7)}=y\frac{\partial}{\partial_t}+t\frac{\partial}{\partial_y}$, $\xi_{(8)}=z\frac{\partial}{\partial_t}+t\frac{\partial}{\partial_z}$, $\xi_{(9)}=x\frac{\partial}{\partial_y}-y\frac{\partial}{\partial_x}$, $\xi_{(10)}=x\frac{\partial}{\partial_z}-z\frac{\partial}{\partial_x}$, $\xi_{(11)}=\frac{\partial}{\partial_x}$.

Table 4.4: Metrics admitting five KVFs

No.	Metric	Vector Field Components	Additional KVF
5a (Branch 7)	$G=c_1x+c_2$, $H=c_3x+c_4$, where $c_1\neq 0$, $c_3\neq 0$ and $c_1^2\neq c_3^2$.	$\xi^0=\frac{c_1c_5}{c_3}t+c_6$, $\xi^1=-\frac{c_5}{c_3}$, $\xi^2=c_5y+c_7z+c_8$, $\xi^3=c_5z-c_7y+c_9$.	$\xi_{(5)}=2t\frac{\partial}{\partial_t}-\frac{\partial}{\partial_x}$ $+y\frac{\partial}{\partial_y}+z\frac{\partial}{\partial_z}$, where we have used $c_1=2, c_3=1$.
5b (Branch 9)	$G=c_1x+c_2$, $H=c_3x+c_4$, where $c_1\neq 0$, $c_3\neq 0$ and $c_3=-c_1$.	Same as in case 5a with $c_3=-c_1$.	$\xi_{(5)}=-t\frac{\partial}{\partial_t}-\frac{\partial}{\partial_x}$ $+y\frac{\partial}{\partial_y}+z\frac{\partial}{\partial_z}$, where we have used $c_3=1$.

4.1.5 Six KVFs

The procedure of solving Eqs. (4.1.5)-(4.1.14) for the constraints of branch 12 splits the problem in to further sub cases depending upon whether $G''(x)\neq 0$ or $G''(x)=0$. In both the cases, we have obtained 6-dimensional Killing algebra with no proper homothety. The results of these cases are summarized in Table 4.5, where the additional two KVFs are represented by $\xi_{(5)}$ and $\xi_{(6)}$. Like the previous cases, here we have obtained most generalized metrics than those presented in Ref. [119]. One can see that the metrics (2.26) and (2.27) of Ref. [119] are the special cases of the metric 6a. Similarly, the metric 6b can be reduced to the metric (2.28) of Ref. [119] by choosing $c_1=\frac{A}{2}$ and

Table 4.5: Metrics admitting six KVFs

No.	Metric	Vector Field Components	Additional KVFs
6a (Branch 12)	$G = \ln(c_2 e^{\sqrt{m}x} + c_3 e^{-\sqrt{m}x})$, $H = c_1$, where $m > 0$.	$\xi^0 = -\frac{G'}{\lambda}(c_4 \sin\lambda t - c_5 \cos\lambda t)$ $+c_6$, $\xi^1 = c_4 \cos\lambda t + c_5 \sin\lambda t$, $\xi^2 = c_7 z + c_8$, $\xi^3 = -c_7 y + c_9$. where $\lambda = \sqrt{4mc_2c_3}$.	$\xi_{(5)} = -\frac{G'}{\lambda} \sin\lambda t \frac{\partial}{\partial_t}$ $+\cos\lambda t \frac{\partial}{\partial_x}$, $\xi_{(6)} = \frac{P'}{\lambda} \cos\lambda t \frac{\partial}{\partial_t}$ $+\sin\lambda t \frac{\partial}{\partial_x}$.
6b (Branch 12)	$G = c_1 x + c_2$, $H = c_3$, where $c_1 \neq 0$.	$\xi^0 = -(\frac{c_4}{2} t^2 + c_5 t)$ $-\frac{c_4}{2} e^{2(-x+c_2)} + c_7$, $\xi^1 = c_4 t + c_5$, $\xi^2 = c_8 z + c_9$, $\xi^3 = -c_8 y + c_{10}$, where we have chosen $c_1 = 1$.	$\xi_{(5)} = \left(-\frac{t^2}{2} - \frac{1}{2} e^{-2(x+c_2)}\right) \frac{\partial}{\partial_t}$ $+t \frac{\partial}{\partial_x}$, $\xi_{(6)} = -t \frac{\partial}{\partial_t} + \frac{\partial}{\partial_x}$.

$c_2 = 0$.

The model 6a represents an anisotropic fluid such that $\rho = p_{||} = 0$ and $p_\perp = m$. These values clearly satisfy all the energy conditions except the dominant energy condition.

Similarly, for the metric 6b, we have $\rho = p_{||} = 0$ and $p_\perp = c_1^2$, which violate the DEC, while all other energy conditions are identically satisfied.

4.1.6 Seven KVFs

The metric given by branch 8 admits seven KVFs with no proper homothety. Four KVFs are same as given in the set K_4 and the extra three are presented in Table 4.6.

The metric given in Eq. (2.34) of Ref. [119] can be recovered from the above metric by setting $c_2 = \frac{C}{2}$ and $c_3 = 0$.

The physical quantities for this metric are found to be $\rho = -3c_2^2$ and $p_{||} = p_\perp = c_2^2$, which give an unphysical model having negative energy density and violating all the energy conditions.

4.1.7 Ten KVFs

There is only one metric, given by branch 10, admitting ten KVFs with no proper homothety. Out of these ten, four are already given in the set K_4, while the extra six KVFs are listed in Table 4.7. One can easily reduce this metric to the metric (2.33) of Ref. [119] by taking $c_1 = \frac{C}{2}$ and $c_2 = 0$.

Table 4.6: Metric admitting seven KVFs

Metric	Vector Field Components	Additional KVFs
$G = c_1$,	$\xi^0 = c_4$,	$\xi_{(5)} = y\frac{\partial}{\partial_x} - yz\frac{\partial}{\partial_z} + \frac{1}{2}\left(z^2 - y^2 + e^{-2(x+c_3)}\right)\frac{\partial}{\partial_y}$,
$H = c_2 x + c_3$,	$\xi^1 = c_5 y - c_6 z - c_7$,	$\xi_{(6)} = -z\frac{\partial}{\partial_x} + yz\frac{\partial}{\partial_y} + \frac{1}{2}\left(z^2 - y^2 - e^{-2(x+c_3)}\right)\frac{\partial}{\partial_z}$,
where $c_2 \neq 0$.	$\xi^2 = c_7 y + c_6 yz + c_8 z + c_9$	$\xi_{(7)} = -\frac{\partial}{\partial_x} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}$.
	$\xi^3 = c_7 z - c_5 yz - c_8 y + c_{10}$	
	where we have chosen $c_2 = 1$.	

Table 4.7: Metric admitting ten KVFs

Metric	Vector Field Components	Additional KVFs
$G = H = c_1 x + c_2$, where $c_1 \neq 0$.	$\xi^0 = \frac{c_3}{2}\left(t^2 + y^2 + z^2 + e^{-2(x+c_2)}\right)$ $+c_4 t + c_5 y + c_6 z - c_7 ty + c_8 tz + c_9$, $\xi^1 = c_7 y - c_8 z - c_3 t - c_4$, $\xi^2 = \frac{c_7}{2}\left(z^2 - y^2 - t^2 + e^{-2(x+c_2)}\right)$ $+c_3 yt + c_4 y + c_5 t + c_{10} z + c_8 yz + c_{11}$, $\xi^3 = \frac{c_8}{2}\left(z^2 - y^2 + t^2 - e^{-2(x+c_2)}\right)$ $+c_3 zt + c_4 z + c_6 t - c_{10} y - c_7 yz + c_{12}$, where we have chosen $c_1 = 1$.	$\xi_{(5)} = \frac{1}{2}\left(t^2 + y^2 + z^2 + e^{-2(x+c_2)}\right)\frac{\partial}{\partial_t}$ $-t\frac{\partial}{\partial_x} + yt\frac{\partial}{\partial_y} + zt\frac{\partial}{\partial_z}$, $\xi_{(6)} = t\frac{\partial}{\partial_t} - \frac{\partial}{\partial_x} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}$, $\xi_{(7)} = y\frac{\partial}{\partial_t} + t\frac{\partial}{\partial_y}$, $\xi_{(8)} = z\frac{\partial}{\partial_t} + t\frac{\partial}{\partial_z}$, $\xi_{(9)} = -ty\frac{\partial}{\partial_t} + y\frac{\partial}{\partial_x} - yz\frac{\partial}{\partial_z}$ $+\frac{1}{2}\left(z^2 - y^2 - t^2 + e^{-2(x+c_2)}\right)\frac{\partial}{\partial_y}$, $\xi_{(10)} = tz\frac{\partial}{\partial_t} - z\frac{\partial}{\partial_x} + yz\frac{\partial}{\partial_y}$, $+\frac{1}{2}\left(z^2 - y^2 + t^2 - e^{-2(x+c_2)}\right)\frac{\partial}{\partial_z}$.

For this metric, we have $\rho = -3c_1^2$ and $p_{||} = p_{\perp} = 3c_1^2$, which identically satisfy the SEC and NEC, while the DEC and WEC are violated here. Moreover, since $\rho = -3c_1^2$, the model is not physically realistic.

4.2 Conformal Symmetries

For conformal symmetry, we use the metric (4.1.1) in Eq. (2.5.3) and obtain the same set of determining equations as given in Eqs. (4.1.5)-(4.1.14) with $\alpha = \alpha(t, x, y, z)$. The non-zero components of the Weyl tensor for the metric (4.1.1) are:

$$
\begin{aligned}
C^1_{010} &= \frac{e^{2G}\Gamma}{3}, \quad C^2_{202} = -\frac{e^{2G}\Gamma}{6} = C^3_{303}, \\
C^1_{212} &= \frac{e^{2H}\Gamma}{6} = C^1_{313}, \quad C^2_{323} = -\frac{e^{2H}\Gamma}{3},
\end{aligned} \tag{4.2.1}
$$

where $\Gamma = G'(H' - G') - G'' + H''$. If $\Gamma = 0$, then the static plane symmetric spacetime is conformally flat, otherwise non-conformally flat.
To explore all possible cases where the spacetimes under consideration may possess proper CVFs, we follow the same Rif tree approach as we have used in case of obtaining HVFs in the previous section, where the obtained Rif tree given in Fig. 4.2 and pivots are given below:

$$\begin{aligned}
p1 &= H', \\
p2 &= H'', \\
p3 &= G'' + G'^2 - H'' - H'^2, \\
p4 &= G' - H', \\
p5 &= G'', \\
p6 &= G'' + H'^2 - H'^2, \\
p7 &= G', \\
p8 &= G'' + G'^2, \\
p9 &= G''' + 2G'G''.
\end{aligned}$$

To get a complete classification of static plane symmetric spacetimes, we solve the set of determining equations (4.1.5)-(4.1.14) for the conditions imposed by each branch of the Rif tree given in Fig. 4.2. If the spacetime is conformally flat, i.e. $\Gamma = 0$, in such a case the conformal algebra is clearly 15-dimensional. Such cases are labeled by the branches 3, 7, 8, 11 and 12. Moreover, there are some other branches of the Rif tree which give two sub-cases, depending upon the spacetime is conformally flat or not. Like the previous chapter, we focus only on the cases where the spacetime is non-conformally flat. Moreover,

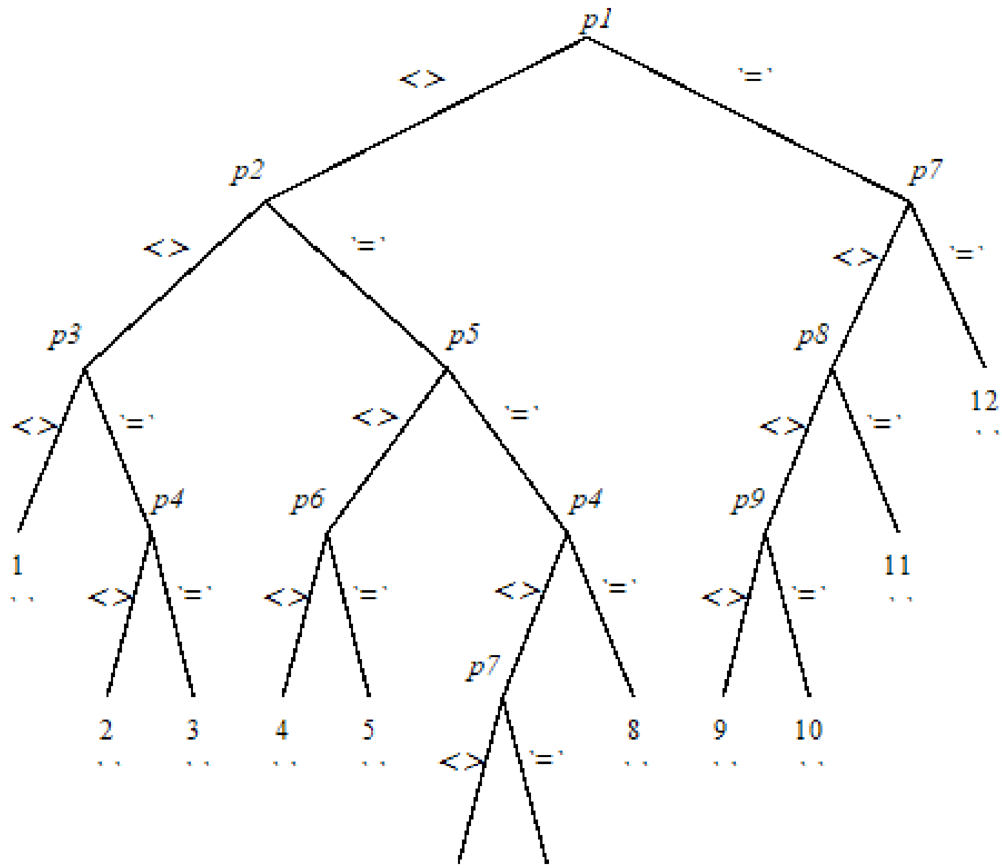

Figure 4.2: Rif Tree for CVFs

some branches of the Rif tree yield sub-cases, some of them give minimum four KVFs. Also branch 5 gives the minimum KVFs. Such cases are also skipped from the discussion, while the results of the remaining sub-cases yield five, six or seven CVFs as summarized in the forthcoming sections.

4.2.1 Branch 1

The constraints of branch 1 are $p_1 \neq 0, p_2 \neq 0$, and $p_3 \neq 0$. Further simplification of Eqs. (4.1.5)-(4.1.14) using these constraints produces two cases, depending upon whether $\Gamma = 0$ or $\Gamma \neq 0$. The non-conformally flat case ($\Gamma \neq 0$) is further divided into three sub-cases. Out of these three sub-cases, one gives the minimum four KVFs, while the remaining two cases give five and six CVFs. In case of five CVFs, one is a proper CVF while the remaining four CVFs are same as given in set K_4. Similarly, in case of six CVFs, there are two proper CVFs while the remaining four CVFs are same as given in the set K_4. In Table 4.8, we have shown the results of these cases. It is to be noted

Table 4.8: Metrics admitting five and six CVFs

No.	Vector Field Components	Proper CVFs
5a (Branch 1)	$\xi^0 = c_1 t(1-\frac{1}{\beta^2}) + c_2$, $\xi^1 = c_1 e^Q \int e^{-H} dx$, $\xi^2 = c_1 y + c_4 z + c_3$, $\xi^3 = c_1 z - c_4 y + c_5$, $\alpha = c_1\Big(H' e^H \int e^{-H} dx + 1\Big)$.	$\xi_{(5)} = t(1-\frac{1}{\beta^2})\frac{\partial}{\partial_t} + e^H \int e^{-Q} dx \ \frac{\partial}{\partial_x} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}$.
6a (Branch 1)	$\xi^0 = (c_1 \cos kt + c_2 \sin kt)\Big(\int e^{H-2G} dx + \frac{1}{k^2}\Big) + c_3$, $\xi^1 = \frac{e^H}{k}(c_1 \sin kt - c_2 \cos kt)$, $\xi^2 = c_4 z + c_5$, $\xi^3 = -c_4 y + c_5$, $\alpha = \frac{H' e^H}{k}\Big(c_1 \sin kt - c_2 \cos kt\Big)$.	$\xi_{(5)} = \cos kt\Big(\int e^{H-2G} dx + \frac{1}{k^2}\Big) \ \frac{\partial}{\partial_t} + \frac{e^H}{k} \sin kt \frac{\partial}{\partial_x}$, $\xi_{(6)} = \sin kt\Big(\int e^{H-2G} dx + \frac{1}{k^2}\Big)\frac{\partial}{\partial_t} - \frac{e^H}{k} \cos kt \frac{\partial}{\partial_x}$.

that the obtained five and six CVFs satisfy the Eqs. (4.1.5)-(4.1.14) subject to the conditions $(G'-H')e^{H}\int e^{-H}dx=\frac{1}{\beta^2}$, and $(G'-H')e^{H}=k^2\int e^{H-2G}dx$ respectively, where k and β are some constants.

As the metric functions are not explicitly known, we cannot find the simplified values of ρ, $p_{||}$ and $p_{\perp}$ in this case.

4.2.2 Branch 2

This branch gives 6-dimensional conformal algebra. The constraints for this branch are $p_1\neq 0, p_2\neq 0, p_3=0$ and $p_4\neq 0$. Solving the set of Eqs. (4.1.5)-(4.1.14) under these conditions , we obtain the following components of CVF, subject to the condition $(G'-H')e^{H}=k^2\int e^{H-2G}dx$, where k is a constant.

$$\begin{aligned}
\xi^0 &= \int e^{H-2G}dx\big(c_1\cos kt+c_2\sin kt\big)+c_3,\\
\xi^1 &= \frac{e^H}{k}\big(c_1\sin kt-c_2\cos kt\big),\\
\xi^2 &= c_4 z+c_5,\\
\xi^3 &= -c_4 y+c_6,\\
\alpha &= H'\frac{e^H}{k}\big(c_1\sin kt-c_2\cos kt\big).
\end{aligned}$$

Here, we can express the two proper CVFs as $\xi_{(5)}=\cos kt\int e^{H-2G}dx\ \frac{\partial}{\partial_t}+\frac{e^H}{k}\sin kt\ \frac{\partial}{\partial_x}$ and $\xi_{(6)}=\sin kt\int e^{H-2G}dx\ \frac{\partial}{\partial_t}-\frac{e^H}{k}\cos kt\ \frac{\partial}{\partial_x}$.

4.2.3 Branch 4

In branch 4, we have $p_1\neq 0, p_2=0, p_5\neq 0$ and $p_6\neq 0$, which give $Q=\alpha_1 x+\alpha_2$, where $\alpha_1\neq 0$. This branch produces two cases. The first case splits into two sub-cases, one giving four KVFs, same as given in the set K_4,

while the second sub-case gives five CVFs. The second case of branch 4 again splits into two sub-cases, one produces six CVFs, while the other gives seven CVFs. The details of these results are given in Table 4.9.
The energy density and parallel and perpendicular pressures for the metric 5a are found to be:

$$\begin{aligned}
\rho &= -3\alpha_1^2, \\
p_{||} &= \frac{(3\alpha_1^2 e^H - \alpha_1^2\beta^2 + 2\alpha_1\eta^2)}{e^H - \beta^2}, \\
p_\perp &= \frac{\alpha_1^2(e^H - \beta^2)^2 - \alpha_1 e^H(\alpha_1\beta^2 + \eta^2) + (\alpha_1 e^H + \eta^2)^2 - \alpha_1(\alpha_1 e^H + \eta^2)(e^H - \beta^2)}{(e^H - \beta^2)^2}.
\end{aligned}$$

Similarly, for the model 6a we have:

$$\begin{aligned}
\rho &= -3\alpha_1^2, \\
p_{||} &= \alpha_1^2 + 2\alpha_1 G', \\
p_\perp &= \alpha_1^2 + G'' + G'^2 + \alpha_1 G'.
\end{aligned}$$

For both models, energy density is negative, so the DEC and WEC are clearly violated for these model, while the bounds for other energy conditions can be easily obtained by substituting these values in (2.7.1).

The metric 7a is an anisotropic fluid with the following quantities:

$$\begin{aligned}
\rho &= -3\alpha_1^2, \\
p_{||} &= \frac{3\alpha_1^2 e^H - \alpha_1^2 k^2\beta^2}{e^H - k^2\beta^2}, \\
p_\perp &= \frac{\alpha_1^2(3e^{2H} + k^4\beta^4 - 4k^2\beta^2 e^H)}{(e^H - k^2\beta^2)^2}.
\end{aligned}$$

These quantities satisfy the SEC and NEC if $e^H > k^2\beta^2$, while the DEC and WEC are failed for these values.

Table 4.9: CVFs of Branch 4

No.	Metric	Vector Field Components	Proper CVFs
5a (Branch 4)	$H=\alpha_1 x+\alpha_2$, $G=\int\frac{\alpha_1 e^H+\eta^2}{e^H-\beta^2}dx$, where $\alpha_1\neq 0$, and β,η are some constants.	$\xi^0=c_1 t+c_2$, $\xi^1=\frac{\beta^2-e^H}{\eta^2}c_1$, $\xi^2=-c_1\frac{\alpha_1\beta^2}{\eta^2}y+c_4 z+c_3$, $\xi^3=-c_1\frac{\alpha_1\beta^2}{\eta^2}z-c_4 y+c_5$, $\alpha=-c_1\frac{H'e^H}{\eta^2}$.	$\xi_{(5)}=t\frac{\partial}{\partial_t}+\frac{\beta^2-e^H}{\eta^2}\ \frac{\partial}{\partial_x}$, $-\frac{\beta^2}{\eta^2}y\ \frac{\partial}{\partial_y}-\frac{\beta^2}{\eta^2}z\ \frac{\partial}{\partial_z}$.
6a (Branch 4)	$H=\alpha_1 x+\alpha_2$, where $\alpha_1\neq 0$.	$\xi^0=\frac{1}{k^2}(G'-H')e^H\Big(c_1\cos kt+c_2\sin kt\Big)dx+c_3$, $\xi^1=\frac{e^H}{k}(c_1\sin kt-c_2\cos kt)$, $\xi^2=c_5+c_4 z$, $\xi^3=c_6-c_4 y$, $\alpha=\frac{H'e^H}{k}\Big(c_1\sin kt-c_2\cos kt\Big)$.	$\xi_{(5)}=\frac{e^H}{k^2}(G'-H')\cos kt\ \frac{\partial}{\partial_t}$ $+\frac{e^H}{k}\sin kt\ \frac{\partial}{\partial_x}$, $\xi_{(6)}=\frac{e^H}{k^2}(G'-H')\sin kt\ \frac{\partial}{\partial_t}$ $-\frac{e^H}{k}\cos kt\ \frac{\partial}{\partial_x}$.
7a (Branch 4)	$H=\alpha_1 x+\alpha_2$, $G=ln(e^H-k^2\beta^2)$, where k,β are some constants and $\alpha_1\neq 0$.	$\xi^0=\frac{1}{k^2}(G'-\xi')e^H\Big(c_1\cos kt+c_2\sin kt\Big)+c_3$, $\xi^1=c_5(e^H-k^2\beta^2)+\frac{e^H}{k}\Big(c_1\sin kt-c_2\cos kt\Big)$, $\xi^2=c_5k^2\alpha_1^2\beta^2 y+c_4 z+c_6$, $\xi^3=c_5k^2\alpha_1^2\beta^2 z-c_4 y+c_7$, $\alpha=c_5H'e^H+\frac{H'e^H}{k}\Big(c_1\sin kt-c_2\cos kt\Big)$.	$\xi_{(5)}=\frac{e^H}{k^2}(G'-H')\cos kt\ \frac{\partial}{\partial_t}$ $+\frac{e^H}{k}\sin kt\ \frac{\partial}{\partial_x}$, $\xi_{(6)}=\frac{e^H}{k^2}(G'-H')\sin kt\ \frac{\partial}{\partial_t}$ $-\frac{e^H}{k}\cos kt\ \frac{\partial}{\partial_x}$, $\xi_{(7)}=(e^H-k^2\beta^2)\ \frac{\partial}{\partial_x}$ $+k^2\beta^2\Big(y\frac{\partial}{\partial_y}+z\frac{\partial}{\partial_z}\Big)$.

4.2.4 Branch 6

The constraints of branch 6 are $p_1 \neq 0$, $p_2 = p_5 = 0$, $p_4 \neq 0$ and $p_7 \neq 0$. The simplification of these conditions gives $G = \alpha_1 x + \alpha_2$, $H = \alpha_3 x + \alpha_4$, where $\alpha_1 \neq 0$, $\alpha_3 \neq 0$ and $\alpha_1 - \alpha_3 \neq 0$. This metric is same as the metric 5a of the section of HVFs given in Table 4.4. Using these values of G and H, the solution of conformal symmetry equations yields $\alpha = 0$ and thus we have obtained no proper CVF in this case. The obtained five KVFs for this metric are same as given in Table 4.4 of the previous section.

4.2.5 Branch 9

In branch 9, we have $p_1 = 0$, $p_7 \neq 0, p_8 \neq 0$ and $p_9 \neq 0$. These constraints give $H = Const. = c_1$ and $G'' + G'^2 \neq 0$, which yield a non-conformally flat metric. Solving the set of conformal symmetry equations under these conditions, we get $G = \ln(\alpha_1 x + c_2)^{1-\frac{c_3}{\alpha_1}}$, where $c_3 \neq \alpha_1 \neq 0$. The CVFs in this case are reduced to HVFs which are given below:

$$
\begin{aligned}
\xi^0 &= c_3 t + c_4, \\
\xi^1 &= \alpha_1 x + c_2, \\
\xi^2 &= \alpha_1 y + c_5 z + c_6, \\
\xi^3 &= \alpha_1 z - c_5 y + c_7, \\
\alpha &= c_3.
\end{aligned}
$$

The above five HVFs include four KVFs, already given in the set K_4 and one proper homothety $\xi_{(5)} = x\frac{\partial}{\partial_x} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}$, which is a special case of metric 5d, given in Table 4.1.

4.2.6 Branch 10

The constraints of branch 10 are $p_1 = 0$, $p_7 \neq 0, p_8 \neq 0$, and $p_9 = 0$. Simplification of these constraints yields two cases depending upon whether $G'' = 0$ or $G'' \neq 0$. In the former case, the metric functions get the values $G = c_1 x + c_2$ and $H = c_3$, where $c_1 \neq 0$ and $c_3 \neq 0$. This metric is same as the metric 6b given in Table 4.5. In the later case, that is when $G'' \neq 0$, the metric becomes $G = \ln(c_1 e^{\sqrt{m}x} + c_2 e^{-\sqrt{m}x})$ and $H = c_3 \neq 0$, where $m > 0$, which is same as the metric 6a given in Table 4.5.
For both of these metrics, the solution of conformal symmetry equations gives $\alpha = 0$ and the CVFs become KVFs, already listed in Table 4.5.

4.3 Noether Symmetries

For Noether symmetries, we consider the following Lagrangian corresponding to the line element of static plane symmetric metric :

$$L = -e^{G(x)}\dot{t}^2 + \dot{x}^2 + e^{H(x)}(\dot{y}^2 + \dot{z}^2). \tag{4.3.1}$$

The minimal set of Noether symmetries for this Lagrangian is:

$$N_5 = \left\{ \frac{\partial}{\partial_s}, \frac{\partial}{\partial_t}, \frac{\partial}{\partial_y}, \frac{\partial}{\partial_z}, z\frac{\partial}{\partial_y} - y\frac{\partial}{\partial_z} \right\}.$$

Like the previous chapter, here N_5 contains the same minimum four KVFs of static plane symmetric metric, given in set K_4 and $\frac{\partial}{\partial_s}$ represents the Noether symmetry corresponding to the Lagrangian.

By using the Lagrangian (4.3.1) in Eq. (2.6.1), we obtain:

$$F_{,s} = \eta_{,t} = \eta_{,x} = \eta_{,y} = \eta_{,z} = 0, \tag{4.3.2}$$

$$2e^G \xi^0_{,s} = F_{,t}, \tag{4.3.3}$$

$$-2\xi^1_{,s} = F_{,x}, \tag{4.3.4}$$

$$-2e^H \xi^2_{,s} = F_{,y}, \tag{4.3.5}$$

$$-2e^H \xi^3_{,s} = F_{,z}, \tag{4.3.6}$$

$$2\xi^1_{,x} - \eta_{,s} = 0, \tag{4.3.7}$$

$$\xi^2_{,z} + \eta^3_{,y} = 0, \tag{4.3.8}$$

$$\xi^1_{,y} + e^H \xi^2_{,x} = 0, \tag{4.3.9}$$

$$\xi^1_{,z} + e^H \xi^3_{,x} = 0, \tag{4.3.10}$$

$$e^G \xi^0_{,x} - \xi^1_{,t} = 0, \tag{4.3.11}$$

$$e^G \xi^0_{,y} - e^H \xi^2_{,t} = 0, \tag{4.3.12}$$

$$e^G \xi^0_{,z} - e^H \xi^3_{,t} = 0, \tag{4.3.13}$$

$$G' \xi^1 + 2\xi^0_{,t} = \eta_{,s}, \tag{4.3.14}$$

$$H' \xi^1 + 2\xi^2_{,y} = \eta_{,s}, \tag{4.3.15}$$

$$H' \xi^1 + 2\xi^3_{,z} = \eta_{,s}. \tag{4.3.16}$$

The next step is to develop an algorithm in Maple which reduces the above set of equations to the simplified form and produces the Rif tree given in Fig. 4.3 along with the list of pivots.

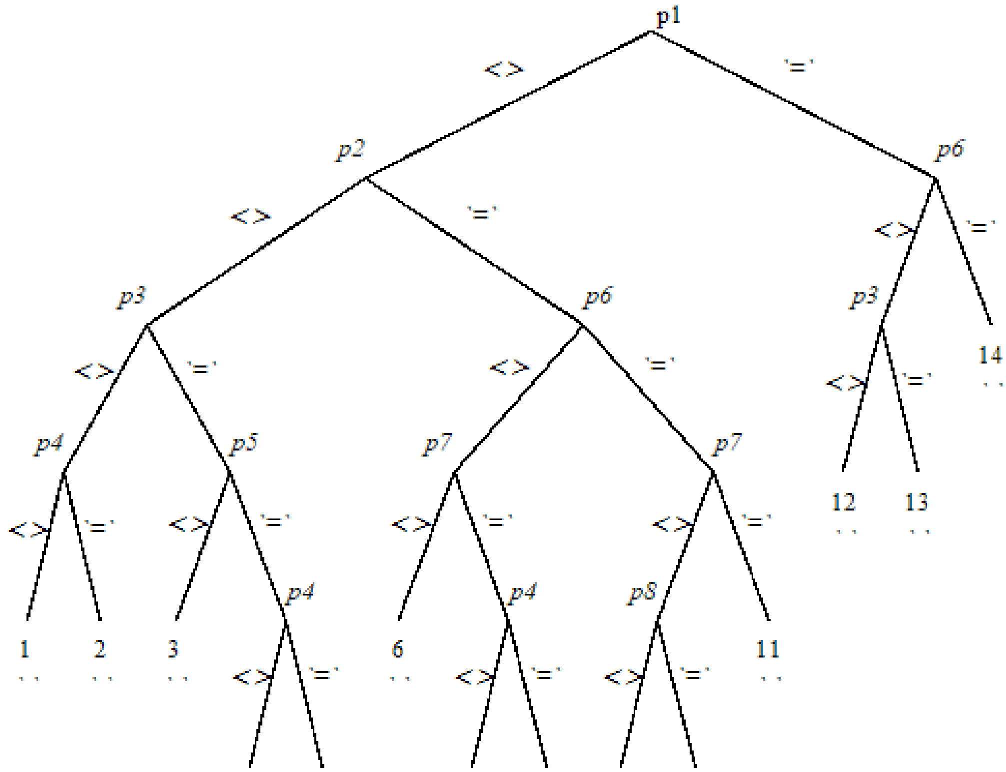

Figure 4.3: Rif Tree for Noether Symmetries

$$
\begin{aligned}
p_1 &= G', \\
p_2 &= H'^2 + 2G'', \\
p_3 &= H'', \\
p_4 &= H' - G', \\
p_5 &= G'', \\
p_6 &= H', \\
p_7 &= G'^2 + 2G'', \\
p_8 &= G'G'' + G'''.
\end{aligned}
$$

To get a complete classification via Noether symmetries, we have solved the set of determining equations for the conditions imposed by every branch of the Rif tree and it is concluded that the possible dimension of Noether algebra for these spacetimes is 5, 6, 7, 8, 9, 10, 11 and 17. In our classification, we omit those cases which give the minimal set of Noether symmetries, given in the set N_5. Such cases are labeled by branches 3 and 7 in the Rif tree. Moreover, there are some other branches of the Rif tree which give two sub-cases. In one of each sub-case, we get the set N_5, so we have skipped such cases from our discussion. We only focus on those cases which give symmetries other than the minimal set of Noether symmetries. The Noether symmetries and the corresponding conservation laws for these cases are discussed in the forthcoming sections. We also compare our results with those of Ref. [121] to show that this new approach gives more general metrics as compared to those obtained by direct integration technique.

4.3.1 Branch 1

In branch 1, the constraints are $p_1 \neq 0, p_2 \neq 0, p_3 \neq 0$ and $p_4 \neq 0$. This case splits into two sub-cases, one giving the minimal set of Noether symmetries, while the other case gives 6-dimensional Noether algebra. Out of these six Nother symmetries, five are same as given in the set N_5 and one is an extra symmetry. The extra symmetry along its conserved form are listed in Table 4.10. Clearly, the extra symmetry corresponds to a homothetic vector $x\frac{\partial}{\partial_x}$. The above metric is same as the metric 5a(i), given in Table 4.1, admitting the same HVF. Moreover, the metric of branch 1 can be transformed to the

Table 4.10: Metric admitting six Noether Symmetries.

Metric	Noether Symmetry generator	Invariant
$G=\ln(\alpha_1 x+2c_3)^{2-4\frac{c_1}{\alpha_1}}$, $H=\ln(\alpha_1 x+2c_3)^{2-4\frac{c_4}{\alpha_1}}$, where $c_4\neq c_1\neq\frac{\alpha_1}{2}$, and $c_1\neq 0$.	$\xi_{(5)}=s\frac{\partial}{\partial_s}+\frac{x}{2}\frac{\partial}{\partial_x}$.	$I_{(5)}=-sL+x\,\dot{x}$.

metric (31) of Ref. [121] by setting $b=2-\frac{4c_1}{\alpha_1}$ and $a=2-\frac{4c_4}{\alpha_1}$. The physical interpretation of this metric is already presented in the section of HVFs.

4.3.2 Branch 2

The constraints of branch 2 are $p_1\neq 0, p_2\neq 0, p_3\neq 0$ and $p_4=0$. From $p_4=0$, we get $G=H$. While solving the set of determining equations under these constraints, we get two sub-cases. In the first sub-case, we have obtained seven Noether symmetries, while the second sub-case gives eight Noether symmetries. Among the obtained symmetries in both cases, five are same as given in the set N_5, while the remaining two and three extra symmetries along with their conserved forms are given in Table 4.11.

Clearly, $\xi_{(5)}$ and $\xi_{(6)}$ are the additional KVFs for both metrics, while $\xi_{(7)}$ is a Noether symmetry corresponding to a homothetic vector $x\frac{\partial}{\partial_x}$ for the second metric. The metric 8a is same as the metric presented in Table 4.2, admitting seven HVFs.

The metric 7a can be transformed to the metrics (34-36) of Ref. [121] by

Table 4.11: Metrics admitting seven and eight Noether Symmetries.

No.	Metric	Noether Symmetry generators	Invariants
7a (Branch 2)	$G = H \neq \ln(\alpha_1 x + 2c_1)^{2-\frac{4c_2}{\alpha_1}}$.	$\xi_{(5)} = y\frac{\partial}{\partial_t} + t\frac{\partial}{\partial_y}$, $\xi_{(6)} = z\frac{\partial}{\partial_t} + t\frac{\partial}{\partial_z}$.	$I_{(5)} = -2e^G(y\dot{t} - t\dot{y})$, $I_{(6)} = -2e^G(z\dot{t} - t\dot{z})$.
8a (Branch 2)	$G = H = \ln(\alpha_1 x + 2c_1)^{2-\frac{4c_2}{\alpha_1}}$, where $\alpha_1 \neq 0$ and $c_2 \neq \frac{\alpha_1}{2}$.	$\xi_{(5)}, \xi_{(6)}$ are same as in case of 7a, $\xi_{(7)} = s\frac{\partial}{\partial_s} + \frac{x}{2}\frac{\partial}{\partial_x}$.	$I_{(5)}, I_{(6)}$ are same as in case of 7a, $I_{(7)} = -sL + x\dot{t}$.

setting some suitable substitutions, while the metric 8a gives the metric (42) of Ref. [121] by putting $a = 2 - \frac{4c_2}{\alpha_1}$ and $c_1 = 0$.

4.3.3 Branch 4

In branch 4 we have, $p_1 \neq 0, p_2 \neq 0, p_3 = 0, p_5 = 0$ and $p_4 \neq 0$. These constraints give the metric functions $G = c_1 x + c_2$, $H = c_3 x + c_4$, where $c_1 \neq 0, c_3 \neq 0$, and $c_1 \neq c_3$. For these values of G and H, we have obtained six Noether symmetries. These six Noether symmetries include five basic Noether symmetries, mentioned in set N_5 and one is extra Noether symmetery (which is actually a KVF), given by $\xi_{(5)} = 2t\,\frac{\partial}{\partial_t} - 2\,\frac{\partial}{\partial_x} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}$. The corresponding conserved form is $I_{(5)} = -2e^G t\;\dot{t} - 4\;\dot{x} + 2e^H(y\;\dot{y} + z\;\dot{z})$. One can see that this additional KVF is same as given in Table 4.4 for the metric 5a, admitting five KVFs.

The metric of this branch give the exact form of the metric (19) of Ref. [121]

if $c_1 = \frac{1}{a}$, $c_3 = \frac{1}{b}$ and $c_2 = c_4 = 0$. For this metric, the physical interpretation is already given in the section of HVFs.

4.3.4 Branch 5

The constraints for branch 5 are $p_1 \neq 0, p_2 \neq 0$ and $p_3 = p_4 = p_5 = 0$. The simplification of these constraints gives $G = H = c_1x + c_2$, where $c_1 \neq 0$. This metric is same as the metric given in Table 4.7, admitting ten KVFs. Using these values of G and H to solve the set of determining equations, we get eleven Noether symmetries in which five are given in the minimal set and six are extra Noether symmetries. These six Noether symmetries (which are actually KVFs) along with their conserved forms are given in Table 4.12.

Table 4.12: Metric admitting eleven Noether Symmetries

Additional Symmetries	Invariants
$\xi_{(5)} = -2t\frac{\partial}{\partial_x} + yt\frac{\partial}{\partial_y} + zt\frac{\partial}{\partial_z} + \left(\frac{t^2+y^2+z^2+4e^{-(x+c_2)}}{2}\right)\frac{\partial}{\partial_t}$,	$I_{(5)} = -4t\dot{x} + 2te^{-(x+c_2)}(y\dot{y} + z\,\dot{z}) - \left(t^2 + y^2 + z^2 - 4e^{-(x+c_2)}\right)e^{x+c_2}\,\dot{t}$,
$\xi_{(6)} = t\frac{\partial}{\partial_t} - 2\frac{\partial}{\partial_x} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}$,	$I_{(6)} = -2e^{x+c_2}t\dot{t} - 4\,\dot{x} + 2e^{x+c_2}(y\,\dot{y} + z\,\dot{z})$,
$\xi_{(7)} = y\frac{\partial}{\partial_t} + t\frac{\partial}{\partial_y}$,	$I_{(7)} = 2e^{x+c_2}(t\,\dot{y} - y\,\dot{t})$,
$\xi_{(8)} = z\frac{\partial}{\partial_t} + t\frac{\partial}{\partial_z}$,	$I_{(8} = 2e^{x+c_2}(t\,\dot{z} - z\,\dot{t})$,
$\xi_{(9)} = -yt\frac{\partial}{\partial_t} + 2y\frac{\partial}{\partial_x} - yz\frac{\partial}{\partial_z} + \left(\frac{z^2-y^2-t^2+4e^{-(x+c_2)}}{2}\right)\frac{\partial}{\partial_y}$,	$I_{(9)} = 2tye^{x+c_2}\dot{t} + 4y\dot{x} - 2e^{-(x+c_2)}yz\dot{z} + \left(z^2 - y^2 - t^2 + 4e^{-(x+c_2)}\right)e^{x+c_2}\,\dot{y}$,
$\xi_{(10)} = zt\frac{\partial}{\partial_t} - 2z\frac{\partial}{\partial_x} + yz\frac{\partial}{\partial_y} + \left(\frac{z^2-y^2+t^2-4e^{-(x+c_2)}}{2}\right)\frac{\partial}{\partial_z}$.	$I_{(10)} = -2e^{x+c_2}ty\dot{t} - 4z\dot{x} + 2e^{x+c_2}yz\dot{y} + \left(z^2 - y^2 + t^2 - 4e^{-(x+c_2)}\right)e^{x+c_2}\,\dot{z}$.

Moreover, this metric is the generalized form of metric (63) of Ref. [121] and can be transformed to the metric (63) by taking $c_1 = \frac{1}{a}$ and $c_2 = 0$.

4.3.5 Branch 6

The constraints for branch 6 are $p_1 \neq 0$, $p_2 = 0$, $p_6 \neq 0$ and $p_7 \neq 0$. The constraint $p_2 = 0$ gives $H = 2\ln(\alpha_1 x + c_1)$, where $\alpha_1 \neq 0$. The calculation of this branch is divided into two cases, one giving the minimal set of Noether symmetries, while the second case gives six Noether symmetries. The metric function G gets the value $G = \ln x^{(2-\frac{4c_2}{c_3})}$, where $c_2 \neq \frac{c_3}{4}$ and $c_3 \neq 0$. The obtained six Noether symmetries include the minimal set of five Noether symmetries and one extra Noether symmetry is given by $\xi_{(5)} = s\frac{\partial}{\partial_s} + \frac{t}{4}\frac{\partial}{\partial_t} + \frac{x}{2}\frac{\partial}{\partial_x}$. The corresponding conserved form is $I_{(5)} = -sL - \frac{t}{2}\dot{t} + x\dot{x}$.
Clearly, $\xi_{(5)}$ represent a Noether symmetry corresponding to a homothetic vector $\frac{t}{2}\frac{\partial}{\partial_t} + x\frac{\partial}{\partial_x}$. The metric of this branch is a special case of the metric 5b(i), given in Table 4.1. Moreover, one can easily see that this metric is the generalized form of the metric (28) of Ref. [121] and can be transformed to the metric (28) by choosing $a = 2 - \frac{4c_2}{c_3}$.

4.3.6 Branch 8

In this branch, we have $p_1 \neq 0, p_6 \neq 0$ and $p_2 = p_4 = p_7 = 0$. These constraints give the metric functions $G = H = \ln(\alpha_1 x + \alpha_2)^2$, where $\alpha_1 \neq 0$. The solution of the set of determining equations (5.3.2)-(5.3.16) yields nine Noether symmetries in which five are same as given in the set N_5 and four are the extra Noether symmetries. The obtained extra Noether symmetries and their conserved forms are listed in Table 4.13.

Table 4.13: Metric admitting nine Noether Symmetries

Noether Symmetry generators	Invariants
$\xi_{(5)} = \frac{s^2}{2}\frac{\partial}{\partial_s} + \frac{s(\alpha_1 x+\alpha_2)}{2\alpha_1}\frac{\partial}{\partial_x}$; $F = \frac{-x^2}{2} + \frac{\alpha_2}{\alpha_1^2}(\alpha_1 x + \alpha_2)$,	$I_{(5)} = -\frac{s^2}{2}L + \frac{s}{\alpha_1}(\alpha_1 x + \alpha_2)\,\dot{x}$ $+\frac{x^2}{2} - \frac{\alpha_2}{\alpha_1^2}(\alpha_1 x + \alpha_2)$,
$\xi_{(6)} = s\frac{\partial}{\partial_s} + \frac{(\alpha_1 x+\alpha_2)}{2\alpha_1}\frac{\partial}{\partial_x}$,	$I_{(6)} = -sL + \frac{(\alpha_1 x+\alpha_2)}{\alpha_1}\,\dot{x}$,
$\xi_{(7)} = y\frac{\partial}{\partial_t} + t\frac{\partial}{\partial_y}$,	$I_{(7)} = 2e^G(t\dot{y} - y\dot{t})$,
$\xi_{(8)} = z\frac{\partial}{\partial_t} + t\frac{\partial}{\partial_z}$.	$I_{(8)} = 2e^G(t\dot{z} - z\dot{t})$.

Here, $\xi_{(5)}$ is a proper Noether symmetry, $\xi_{(6)}$ is a Noether symmetry corresponding to a homothetic vector $\frac{(\alpha_1 x+\alpha_2)}{\alpha_1}\frac{\partial}{\partial_x}$ and $\xi_{(7)}$ and $\xi_{(8)}$ are additional KVFs. This metric is a special case of the metric 7a, given in Table 4.2. The metric of this branch can be transformed to the metric (60) of Ref. [121] by taking $\alpha_1 = \frac{1}{a}$ and $\alpha_2 = 0$.

4.3.7 Branch 9

In branch 9, the pivots are $p_1 \neq 0$, $p_2 = p_6 = 0$, $p_7 \neq 0$ and $p_8 \neq 0$, which give $H = Const. = \beta$. This branch produces two metrics, admitting 7 and 8-dimensional Noether algebras. In both cases, five Noether symmetries are same as given in the set N_5, while the extra two and three symmetries along with their conserved forms are given in Table 4.14. For the first metric, $\xi_{(5)}$ and $\xi_{(6)}$ are proper Noether symmetries. The second metric admits one Noether symmetry corresponding to a homothetic vector $x\frac{\partial}{\partial_x} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}$, while $\xi_{(6)}$ and $\xi_{(7)}$ are proper Noether symmetries. The metric 8a is a special

Table 4.14: Metrics admitting seven and eight Noether Symmetries

No.	Metric	Noether Symmetry generators	Invariants
7a (Branch 9)	$G \neq \ln(c_1x + 2c_2)^{2-4\frac{c_3}{c_1}}$, $H = \beta$, where $\beta \neq 0$.	$\xi_{(5)} = s\frac{\partial}{\partial y}$; $F = -2y$, $\xi_{(6)} = s\frac{\partial}{\partial z}$; $F = -2z$.	$I_{(5)} = 2(s\dot{y} + y)$, $I_{(6)} = 2(s\dot{z} + z)$.
8a (Branch 9)	$G = \ln(c_1x + 2c_2)^{2-4\frac{c_3}{c_1}}$, $H = \beta$, where $c_3 \neq \frac{c_1}{2}$, $c_1 \neq 0$ and $\beta \neq 0$.	$\xi_{(5)} = s\partial_s + \frac{1}{2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$, $\xi_{(6)} = s\frac{\partial}{\partial y}$, $F = -2y$, $\xi_{(7)} = s\frac{\partial}{\partial z}$, $F = -2z$.	$I_{(5)} = -sL + (x\ \dot{x} + y\ \dot{x} + z\ \dot{z})$, $I_{(5)} = 2(s\ \dot{y} + y)$, $I_{(6)} = 2(s\ \dot{z} + z)$.

case of the metric 5d, presented in Table 4.1, admitting the same HVF.

One can see that the metric 7a is same as the metric (40) of Ref. [121], while the metric (47) in Ref. [121] can be recovered from the metric 8a by setting $a = 2 - \frac{4c_3}{c_1}$ and $c_2 = 0$.

4.3.8 Branch 10

The constraints of this branch are $p_1 \neq 0, p_7 \neq 0$ and $p_2 = p_6 = p_8 = 0$, which give $H = c_1 \neq 0$. This case is splitted into two sub-cases depending upon whether $G'' \neq 0$ or $G'' = 0$. In both cases we have obtain nine Noether symmetries. The obtained nine Noether symmetries, in first case are subject to the condition $e^G G'' = \lambda$, where λ is some constant. However, in second case we also get the explicit form of the metric function G as $G = c_2 x + c_3$. Moreover, the metric of this second case is same as the metric 6b given in Table 4.5, admitting six KVFs. The extra symmetries and their conservation laws for both cases are given in Table 4.15. For both metrics, $\xi_{(5)}$ and $\xi_{(6)}$ are KVFs, while $\xi_{(7)}$ and $\xi_{(8)}$ are proper Noether symmetries. The metrics (54- 55) of Ref. [121] are the special case of the metric 9(i), while the metric 9(ii) can be transformed to the metric (52) of Ref. [121] by taking $c_1 = \frac{1}{a}$ and $c_2 = 0$.

In case of metric 9(i), we have $\rho = p_{||} = 0$ and $p_\perp = G'' + G'^2$. The metric is physically realistic with the non-negative energy density and the bounds for different energy conditions can be found by using these values in (2.7.1). For model 9(ii), the physical interpretation is already given in the previous section.

Table 4.15: Metrics admitting nine Noether Symmetries

No.	Metrics	Noether Symmetry generators	Invariants
9(i) Branch(10)	$e^G G'' = \lambda$, $H = c_3$, where λ and β are some constants.	$\xi_{(5)} = -\frac{G'}{\sqrt{2\lambda}} \sin\sqrt{\frac{\lambda}{2}}t \frac{\partial}{\partial t}$ $+\cos\sqrt{\frac{\lambda}{2}}t\ \frac{\partial}{\partial x}$, $\xi_{(6)} = \frac{G'}{\sqrt{2\lambda}} \cos\sqrt{\frac{\lambda}{2}}t\partial_t$ $+\sin\sqrt{\frac{\lambda}{2}}t\ \frac{\partial}{\partial x}$. $\xi_{(7)} = s\frac{\partial}{\partial y},\ F = -2y$, $\xi_{(8)} = s\frac{\partial}{\partial z},\ F = -2z$,	$I_{(5)} = \sqrt{\frac{2}{\lambda}} \sin\sqrt{\frac{\lambda}{2}}tP'e^G\ \dot{t}$ $+2\cos\sqrt{\frac{\lambda}{2}}t\ \dot{x}$, $I_{(6)} = -\sqrt{\frac{2}{\lambda}} \cos\sqrt{\frac{\lambda}{2}}tG'e^G\ \dot{t}$ $+2\sin\sqrt{\frac{\lambda}{2}}t\ \dot{x}$, $I_{(7)} = 2(s\ \dot{y} + y)$, $I_{(8)} = 2(s\ \dot{z} + z)$,
9(ii) (Branch 10)	$H = c_1$, $G = c_2 x + c_3$, where $c_1 \neq 0$ and $c_2 \neq 0$. where we have chosen $c_2 = 1$.	$\xi_{(5)} = (-e^{-(x+c_2)} - \frac{t^2}{4})\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}$, $\xi_{(6)} = \frac{\partial}{\partial x} - \frac{1}{2}t\ \frac{\partial}{\partial t}$, $\xi_{(7)}, \xi_{(8)}$ are same as in case of 9(i).	$I_{(5)} = 2(t\dot{x} + (1 + \frac{t^2 e^G}{4}))\dot{t}$, $I_{(6)} = t\dot{t} + 2\dot{x}$ $I_{(7)}, I_{(8)}$ are same as in case of 9(i).

4.3.9 Branch 11

In branch 11, we have $p_1 \neq 0$ and $p_2 = p_6 = p_7 = 0$. These constraints give $G = \ln(c_1x + c_2)^2$ and $H = c_3$, where $c_1 \neq 0$, which is a flat metric having zero curvature. Using these values to solve the system of determining equations (5.3.2)-(5.3.16), we obtain 17-dimensional Noether algebra containing the five minimum Noether symmetries, already given in the set N_5 and the remaining twelve symmetries with their conserved forms are given in Table 4.16.
Here, $\xi_{(5)}$ corresponds to a homothetic vector $\frac{1}{\alpha_1}(\alpha_1 x + \alpha_2)\frac{\partial}{\partial_x} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}$, $\xi_{(6)}..., \xi_{(11)}$ are the additional KVFs and $\xi_{(12)}..., \xi_{(16)}$ represent proper Noether symmetries. One can see that these six additional KVFs and one proper homothety are same as given in Table 4.3 of metric 11a. The metric of this branch is missing in Ref. [121].

4.3.10 Branch 12

The constraints of branch 12 are $p_1 = 0$, $p_3 \neq 0$ and $p_6 \neq 0$. Since $p_1 = 0$, so $G = Const.$, where $\beta \neq 0$. The calculation of this branch is divided into three sub-cases depending upon the value of H. In this way we have obtained three different metrics, admitting six, seven and eight Noether symmetries. The extra one, two and three symmetries for these metrics along with their conserved forms are given in Table 4.17. The exact form of these three metrics are also presented in the same table. For the metric 6a, $\xi_{(5)}$ is a proper Noether symmetry. For the metric 7a, $\xi_{(5)}$ corresponds to a homothetic vector, given by $t\frac{\partial}{\partial_t} + x\frac{\partial}{\partial_x}$ and $\xi_{(6)}$ is a proper Noether symmetry, while for the metric 8a, $\xi_{(5)}$ and $\xi_{(6)}$ are proper Noether symmetries, while $\xi_{(7)}$ is a

Table 4.16:

Noether Symmetry generators	Invariants
$\xi_{(5)} = s\frac{\partial}{\partial_s} + \frac{1}{G'}\partial_x + \frac{1}{2}(y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z});\;\; F = -\frac{y^2+z^2}{2}$,	$I_{(5)} = -sL + \frac{2}{G'}\dot{x} + y\dot{y} + z\dot{z} + \frac{y^2+z^2}{2}$,
$\xi_{(6)} = e^{c_1 t}\left(ze^{-G}\frac{\partial}{\partial_t} - z\frac{\partial}{\partial_x} + \frac{1}{G'}\frac{\partial}{\partial_z}\right)$,	$I_{(6)} = -e^{c_1 t}\left(2z\dot{t} + 2z\dot{x} - \frac{2}{G'}e^{G}\ \dot{z}\right)$,
$\xi_{(7)} = e^{-c_1 t}\left(-ze^{-G}\frac{\partial}{\partial_t} - z\frac{\partial}{\partial_x} + \frac{1}{G'}\frac{\partial}{\partial_z}\right)$,	$I_{(7)} = e^{-c_1 t}\left(2z\dot{t} - 2z\dot{x} + \frac{2}{G'}e^{G}\ \dot{z}\right)$,
$\xi_{(8)} = e^{c_1 t}\left(ye^{-G}\frac{\partial}{\partial_t} - y\frac{\partial}{\partial_x} + \frac{1}{G'}\frac{\partial}{\partial_y}\right)$,	$I_{(8)} = -2e^{c_1 t}\left(y\dot{t} + \dot{x} - \frac{e^G}{G'}\dot{y}\right)$,
$\xi_{(9)} = e^{-c_1 t}\left(-ye^{-G}\frac{\partial}{\partial_t} - y\frac{\partial}{\partial_x} + \frac{1}{G'}\frac{\partial}{\partial_y}\right)$,	$I_{(9)} = 2e^{-c_1 t}\left(y\ \dot{t} - y\dot{x} - \frac{e^G}{G'}\dot{y}\right)$,
$\xi_{(10)} = \frac{1}{c_1}e^{c_1 t}(\frac{\partial}{\partial_x} - e^{-G}\frac{\partial}{\partial_t})$,	$I_{(10)} = 2\frac{1}{c_1}e^{c_1 t}(\dot{x} + \dot{t})$,
$\xi_{(11)} = -\frac{1}{c_1}e^{-c_1 t}(\frac{\partial}{\partial_x} + e^{-G}\frac{\partial}{\partial_t})$,	$I_{(11)} = \frac{1}{c_1}e^{-c_1 t}(\dot{t} - \dot{x})$.
$\xi_{(12)} = \frac{s^2}{2}\frac{\partial}{\partial_s} + \frac{s}{2G'}\frac{\partial}{\partial_x} + \frac{s}{2}y\frac{\partial}{\partial_y}$	$I_{(12)} = -\frac{s^2}{2}L + \frac{s}{G'}\dot{x} + sy\dot{y}$
$+\frac{s}{2}z\frac{\partial}{\partial_z};\;\; F = -(\frac{x^2}{2} + \frac{\alpha_2}{\alpha_1 G'})$,	$+sz\dot{z} + \frac{x^2}{2} + \frac{\alpha_2}{\alpha_1 G'}$,
$\xi_{(13)} = \frac{s}{\alpha_1}(-e^{-G}\frac{\partial}{\partial_t} + \frac{\partial}{\partial_x})$,	$I_{(13)} = \frac{2s}{\alpha_1}(\dot{t} + \dot{x})$,
$\xi_{(14)} = -\frac{s}{\alpha_1}(e^{-G}\frac{\partial}{\partial_t} + \frac{\partial}{\partial_x})$,	$I_{(14)} = \frac{2s}{\alpha_1}(\dot{t} - \dot{x})$,
$\xi_{(15)} = s\frac{\partial}{\partial_y};\;\; F = -2y$,	$I_{(15)} = 2(s\dot{y} + y)$,
$\xi_{(16)} = s\frac{\partial}{\partial_z};\;\; F = -2z$.	$I_{(16)} = 2(s\dot{z} + z)$.

Noether symmetry corresponding to a homothetic vector $t\frac{\partial}{\partial_t} + x\frac{\partial}{\partial_x}$.
The metric 7a is same as the metric 5a(ii), presented in Table 4.1, admitting the same five HVF, where as the metric 8a is same as the metric 5b(ii), given in Table 4.1.

One can easily see that the metric 6a is same as the metrics (22-23) of Ref. [121]. The metric 7a can be transformed to the metric (38) of Ref. [121] by choosing $\alpha_1 = \frac{1}{d}$ and $a = 2 - \frac{4c_3}{\alpha_1}$, while the metric 8a gives the exact form of the metric (45) of Ref. [121] by choosing $\gamma_1 = \frac{1}{a}$.

Table 4.17: Metric admitting six, seven and eight Noether Symmetries

No.	Metrics	Noether Symmetry generators	Invariants
6a (Branch 12)	$G = Const.$, $H = \ln\cosh^2 x$ or $H = \ln\cos x^2$.	$\xi_{(5)} = s\frac{\partial}{\partial_t}$; $F = -2t$.	$I_{(5)} = -2(s\dot{t} - t)$.
7a (Branch 12)	$G = Const.$, $H = \ln(\alpha_1 x + 2c_1)^{2-4\frac{c_3}{\alpha_1}}$, where $\alpha_1 \neq 0$ and $c_3 \neq \frac{\alpha_1}{2}$.	$\xi_{(5)} = s\frac{\partial}{\partial_s} + \frac{1}{2}(t\frac{\partial}{\partial_t} + x\frac{\partial}{\partial_x}); F = -2t$, $\xi_{(6)} = -s\frac{\partial}{\partial_t}$; $F = -2t$,	$I_{(5)} = -sL - t\dot{t} + x\dot{x}$, $I_{(6)} = 2(s\dot{t} + t)$,
8a (Branch 12)	$G = Const.$, $H = 2\ln(\gamma_1 x)$, where $\gamma_1 \neq 0$.	$\xi_{(5)} = \frac{s}{2}\left(s\frac{\partial}{\partial_s} + t\frac{\partial}{\partial_t} + x\frac{\partial}{\partial_x}\right)$; $F = \frac{t^2-x^2}{2}$, $\xi_{(6)} = -s\frac{\partial}{\partial_t}$; $F = -2t$. $\xi_{(7)} = s\frac{\partial}{\partial_s} + \frac{1}{2}t\frac{\partial}{\partial_t} + x\frac{\partial}{\partial_x}$.	$I_{(5)} = -\frac{s^2}{2}L - st\dot{t} + sx\dot{x}$ $+(\frac{x^2-t^2}{2})$, $I_{(6)} = 2(s\,\dot{t} + t)$. $I_{(7)} = -sL - t\,\dot{t} + x\,\dot{x}$.

4.3.11 Branch 13

The constraints of this branch are $p_1 = 0$, $p_6 \neq 0$ and $p_3 = 0$. These conditions give the metric functions $G = c_1$ and $H = c_2 x + c_3$, where $c_2 \neq 0$. This metric admits nine Noether symmetries in which five are same as given in the set N_5, while the remaining four symmetries along with their conservation laws are given in Table 4.18. The metric of this case is same as the metric 7a given in Table 4.6, admitting the same additional KVFs, denoted by $\xi_{(5)}$, $\xi_{(6)}$ and $\xi_{(7)}$, while $\xi_{(8)}$ represent a proper Noether symmetry. The metric of this branch can be transformed to the metric (49) of Ref. [121] by taking $c_2 = \frac{1}{a}$ and $c_3 = 0$. The physical interpretation of this metric is already given in the section of HVFs.

4.3.12 Branch 14

For this branch, we have $p_1 = p_6 = 0$, which give $G = H = Const$. Thus the metric (4.1.1) reduces to the well known Minkowski metric, admitting 17-dimensional Noether algebra. Five Noether symmetries of this metric are same as given in the set N_5, while the remaining twelve symmetries and their conserved forms are given in Table 4.19. Here, $\xi_{(5)}$ corresponds to homothety, given by $t\frac{\partial}{\partial_t} + x\frac{\partial}{\partial_x} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}$, while $\xi_{(6)}, \ . \ . \ . \ ,\xi_{(11)}$ are the additional KVFs. Moreover, $\xi_{(12)}, ..., \xi_{(16)}$ are proper Noether symmetries. The six additional KVFs and one proper homothety obtained here are same as given in Table 4.3.

Table 4.18: Metric admitting nine Noether Symmetries

Noether Symmetry generators	Invariants
$\xi_{(5)} = 2y\frac{\partial}{\partial_x} - yz\frac{\partial}{\partial_z},$	$I_{(5)} = 4y\dot{x} - 2e^{-(x+c_3)}yz\ \dot{z},$
$+\left(\frac{z^2-y^2+e^{-(x+c_3)}}{2}\right)\partial_y,$	$+e^{x+c_3}\left(z^2 - y^2 + 4e^{-(x+c_3)}\right)\ \dot{y},$
$\xi_{(6)} = -2z\frac{\partial}{\partial_x} + yz\frac{\partial}{\partial_y}$	$I_{(6)} = -4z\dot{x} + 2yze^{x+c_3}\ \dot{y},$
$+\left(\frac{z^2-y^2-e^{-(x+c_3)}}{2}\right)\partial_z,$	$+e^{x+c_3}\left(z^2 - y^2 - \frac{4}{\beta_1^2}e^{-(x+c_3)}\right)\ \dot{z},$
$\xi_{(7)} = -2\frac{\partial}{\partial_x} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}.$	$I_{(7)} = -4\ \dot{x} + 2e^{x+c_3}(y\ \dot{z} + z\ \dot{z}).$
$\xi_{(8)} = -s\frac{\partial}{\partial_t},\ \ F = -2t,$	$I_{(8)} = 2(s\ \dot{t} + t).$
where we have chosen $c_2 = 1$.	

Table 4.19: Metric admitting seventeen Noether Symmetries

Noether Symmetry generators	Invariants
$\xi_{(5)} = 2s\frac{\partial}{\partial_s} + t\frac{\partial}{\partial_t} + x\partial_x + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}.$	$I_{(5)} = -2sL - 2(t\dot{t} - x\dot{x} - y\dot{y} - z\dot{z}),$
$\xi_{(6)} = x\frac{\partial}{\partial_t} + t\frac{\partial}{\partial_x},$	$I_{(6)} = -2(x\dot{t} - t\dot{x}),$
$\xi_{(7)} = y\frac{\partial}{\partial_t} + t\frac{\partial}{\partial_y},$	$I_{(7)} = 2(t\dot{y} - y\dot{t}),$
$\xi_{(8)} = z\frac{\partial}{\partial_t} + t\frac{\partial}{\partial_z},$	$I_{(8)} = 2(t\dot{z} - z\dot{t}),$
$\xi_{(9)} = x\frac{\partial}{\partial_y} - y\frac{\partial}{\partial_x},$	$I_{(9)} = 2(x\dot{y} - y\dot{x}),$
$\xi_{(10)} = x\frac{\partial}{\partial_z} - z\frac{\partial}{\partial_x},$	$I_{(10)} = 2(x\dot{z} - z\ \dot{x}),$
$\xi_{(11)} = \frac{\partial}{\partial_x},$	$I_{(11)} = 2\dot{x},$
$\xi_{(12)} = s^2\frac{\partial}{\partial_s} + st\frac{\partial}{\partial_t} + sx\frac{\partial}{\partial_x}$	$I_{(12)} = -s^2L - 2s(t\ \dot{t} - x\ \dot{x} - y\ \dot{y} - z\ \dot{z})$
$+sy\frac{\partial}{\partial_y} + sz\frac{\partial}{\partial_z};\ F = t^2 - (z^2 + y^2 + x^2),$	$-t^2 + z^2 + y^2 + x^2.$
$\xi_{(13)} = -s\frac{\partial}{\partial_t};\ F = -2t,$	$I_{(13)} = 2(s\dot{t} + t),$
$\xi_{(14)} = s\frac{\partial}{\partial_x};\ F = -2x,$	$I_{(14)} = 2(s\dot{x} + x),$
$\xi_{(15)} = s\frac{\partial}{\partial_y};\ F = -2y,$	$I_{(15)} = 2(s\dot{y} + y),$
$\xi_{(16)} = s\frac{\partial}{\partial_z};\ F = -2z,$	$I_{(16)} = 2(s\dot{z} + z).$

4.4 Summary

In this chapter, we have studied HVFs, CVFs and Noether symmetries of static plane symmetric spacetimes using Rif tree approach. Instead of directly integrating the set of determining equations, an algorithm is developed in Maple which reduces these equations to the simplified form and yields a Rif tree and a list of pivots. The set of determining equations is then solved for the conditions of all branches of the Rif tree.

Out of the 14 branches of the Rif tree for HVFs, two branches give the minimum four KVFs, while the remaining produce some static plane symmetric metrics possessing proper homothetic algebras of dimension 5, 7 and 11. Our classification also shows that the Killing algebra for these spacetimes is 4, 5, 6, 7 and 10-dimensional.

For a complete classification of static plane symmetric spacetimes via CVFs, we have categorized our results branch-wise, because some branches of the Rif tree yield sub-cases. Out of 12 branches of the Rif tree for CVFs, we have only considered those cases where the spacetime is non-conformally flat. Solving the conformal symmetry equations for each branch, we have concluded that non-conformally flat static plane symmetric spacetimes possess at most three proper CVFs along with four minimum KVFs. In some cases, the CVFs also reduce to HVFs and KVFs, as the conformal factor in such cases becomes constant or zero.

Like CVFs, we have also categorized our results for Noether symmetries branch-wise. Out of 14 branches of the Rif tree, 3 give the minimal set of Noether symmetries. The remaining branches yield 6, 7, 8, 9, 10, 11 and 17 Noether symmetries.

Comparing our results (HVFs and Noether symmetries) with those obtained by conventional method, we have observed that the Rif approach gives the same algebras of KVFs, HVFs and Noether symmetries with more generalized metrics than those produced by direct integration technique. Also we observed that some metrics were missing in the earlier study by direct integration technique.

For CVFs, we compared our results with those of Ref. [51] and found the same dimensional algebras of KVFs and HVF. Moreover, through Rif approach we also obtained proper CVFs which were not listed in Ref. [51].

To add some physical implications, we have found the energy-momentum tensor for all the obtained metrics and it is seen that all the metrics arising during our classification are anisotropic or perfect fluid models satisfying different energy conditions.

Chapter 5

Lie and Noether Symmetries of LRS Bianchi type I Spacetimes

In this chapter, we discuss Lie and Noether symmetries of LRS Bianchi type I spacetimes. Rather than using direct integration technique, first a Rif algorithm is developed in Maple which reduces the symmetry equations to the simplified form. Consequently, we get a Rif tree and list of pivots. Like the previous chapters, the integration of the system of symmetry equations is carried out for every branch of the Rif tree for obtaining the final form of homothetic, conformal and Noether symmetries.

5.1 Homothetic Symmetries

In this section, we explore HVFs of LRS Bianchi type I spacetimes. This work has been published in the journal "Theoretical and Mathematical Physics"

[120]. The metric of LRS Bianchi type I spacetimes is given by:

$$ds^2 = -dt^2 + G^2\ dx^2 + H^2\ [dy^2 + dz^2], \tag{5.1.1}$$

where $G = G(t) \neq 0$ and $H = H(t) \neq 0$. Following are the set of minimum KVFs for the above metric :

$$K_4 = \left\{ \frac{\partial}{\partial_x}, \frac{\partial}{\partial_y}, \frac{\partial}{\partial_z}, z\frac{\partial}{\partial_y} - y\frac{\partial}{\partial_z} \right\}.$$

The non-zero components of energy-momentum tensor for the metric (5.1.1) are found to be:

$$\begin{aligned} T_{00} &= \frac{2G'H'}{GH} + \frac{H'^2}{H^2}, \\ T_{11} &= -\frac{G^2}{H^2}\left(2HH'' + H'^2\right), \\ T_{22} &= T_{33} = -\frac{H}{G}\left(GH'' + HG'' + G'H'\right). \end{aligned} \tag{5.1.2}$$

For different sources, the above components take specific form. For example, for an anisotropic fluid, these components take the form:

$$T_{00} = \rho,\ T_{11} = G^2 p_{||},\ T_{22} = T_{33} = H^2 p_{\perp}, \tag{5.1.3}$$

where $\rho, p_{||}$ and $p_{\perp}$ respectively represent the energy density and parallel and perpendicular pressures of the fluid. Moreover, if $p_{||} = p_{\perp}$, then the above components represent a perfect fluid. Comparing Eqs. (5.1.2) and (5.1.3), we get:

$$\begin{aligned} \rho &= \frac{2G'H'}{GH} + \frac{H'^2}{H^2}, \\ p_{||} &= -\left(\frac{2H''}{H} + \frac{H'^2}{H^2}\right), \\ p_{\perp} &= -\left(\frac{H''}{H} + \frac{G''}{G} + \frac{G'H'}{GH}\right). \end{aligned} \tag{5.1.4}$$

Using the metric (5.1.1) in Eq. (2.5.3), we obtain the following set of determining equations:

$$\xi^0_{,t} = \alpha, \tag{5.1.5}$$

$$\xi^0_{,x} - G^2\ \xi^1_{,t} = 0, \tag{5.1.6}$$

$$\xi^0_{,y} - H^2\ \xi^2_{,t} = 0, \tag{5.1.7}$$

$$\xi^0_{,z} - H^2\ \xi^3_{,t} = 0, \tag{5.1.8}$$

$$G'\ \xi^0 + G\ \xi^1_{,x} = \alpha G, \tag{5.1.9}$$

$$G^2\ \xi^1_{,y} + H^2\ \xi^2_{,x} = 0, \tag{5.1.10}$$

$$G^2\ \xi^1_{,z} + H^2\ \xi^3_{,x} = 0, \tag{5.1.11}$$

$$G'\ \xi^0 + H\ \xi^2_{,y} = \alpha H, \tag{5.1.12}$$

$$\xi^2_{,z} + \xi^3_{,y} = 0, \tag{5.1.13}$$

$$H'\ \xi^0 + H\ \xi^3_{,z} = \alpha H. \tag{5.1.14}$$

The same approach which we have used in the the previous chapters for finding the HVFs of Kantowski-Sachs and static plane symmetric spacetimes is used here for investigating the HVFs of LRS Bianchi type I spacetimes. As a result of the Maple algorithm for the above set of equations, we obtain the Rif tree given in Fig. 5.1 and the following pivots:

$$p_1 = H',$$

$$p_2 = HH'' - H'^2,$$

$$p_3 = GH' + G'H,$$

$$p_4 = GH'' - G''H,$$

$$p_5 = GH' - G'H,$$

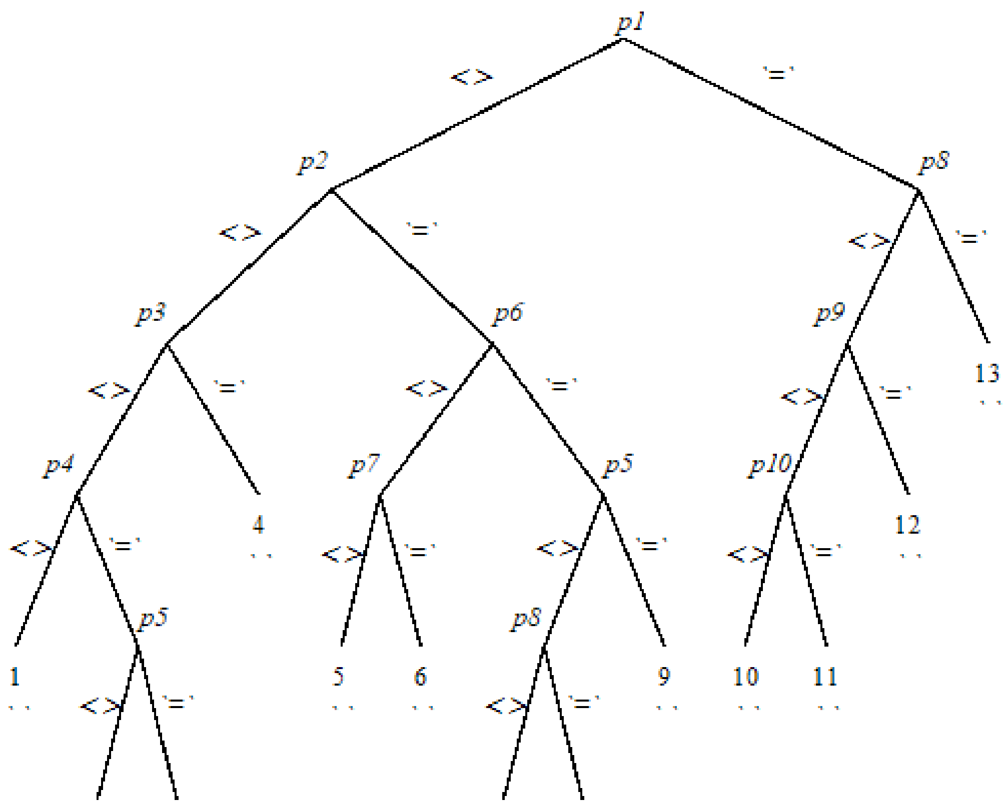

Figure 5.1: Rif Tree for HVFs

$$
\begin{aligned}
p_6 &= GG'' - G'^2, \\
p_7 &= GH'^2 - H^2G'', \\
p_8 &= G', \\
p_9 &= G'', \\
p_{10} &= GG''' - G'G''.
\end{aligned}
$$

We present a complete classification of LRS Bianchi type I spacetimes via HVFs by solving the system of Eqs. (5.1.5)-(5.1.14) for all branches of the Rif tree given in Fig. 5.1. This procedure also gives the classification of these spacetimes via KVFs, since in some cases the homothety constant α vanishes and the solution of Eqs. (5.1.5)-(5.1.14) produces no proper homothety. Such cases are labeled by the branches 7, 8, 9 and 11 in the Rif tree. Moreover, the branches of the Rif tree labeled by 5 and 6 yield minimum four KVFs. These cases are excluded from the current classification. The results of the remaining branches are summarized in the forthcoming sections. We also compare our results with the existing results of Ref. [122] where HVFs were obtained by directly integrating the homothetic symmetry equations for these spacetimes.

5.1.1 Five HVFs

Four branches of the Rif tree, namely 1, 2, 4 and 10 give 5-dimensional algebra of HVFs with one proper homothety and four minimum KVFs. In Table 5.1, we present the exact form of the metrics of all these cases along with their proper HVFs.

Comparing these results with the results of direction integration technique presented in Ref. [122], one can easily see that the metrics 5a(ii), 5b(i-ii) and 5c(i-ii) were not listed in Ref. [122]. However, the metrics 5a(i) and 5d obtained here are same as presented in Ref. [122].

Both the models 5a(i) and 5a(ii) represent anisotropic fluids. For model 5a(i),

Table 5.1: Metrics admitting five HVFs

No.	Metric	Vector Field Components	Proper HVF
5a(i) (Branch 1)	$G=(\alpha t+c_1)^{1-\frac{c_2}{\alpha}}$, $H=(\alpha t+c_1)^{1-\frac{c_3}{\alpha}}$, where $\alpha\neq c_2\neq c_3\neq 0$, $\alpha\neq c_2+c_3$ and $2\alpha\neq c_2+c_3$.	$\xi^0=\alpha t+c_1$, $\xi^1=c_2x+c_4$ $\xi^2=c_3y-c_5z+c_6$, $\xi^3=c_3z+c_5y+c_7$.	$\xi_{(5)}=t\frac{\partial}{\partial t}$.
5a(ii) (Branch 1)	$G=\beta$, $H=(\alpha t+c_1)^{1-\frac{c_3}{\alpha}}$, where $\alpha\neq c_3\neq 0$ and $\beta\neq 0$.	$\xi^0=\alpha t+c_1$, $\xi^1=\alpha x+c_2$, $\xi^2=c_3y-c_4z+c_5$, $\xi^3=c_3z+c_4y+c_6$.	$\xi_{(5)}=t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$.
5b(i) (Branch 2)	$G=(\alpha t+c_1)^{\frac{c_2}{\alpha}}$, $H=(\alpha t+c_1)^{1-\frac{c_2}{\alpha}}$, where $\alpha\neq 0$, $\alpha\neq c_2$, and $\alpha\neq 2c_2$.	$\xi^0=\alpha t+c_1$, $\xi^1=\alpha x-c_2x+c_3$, $\xi^2=c_2y-c_4z+c_5$, $\xi^3=c_2z+c_4y+c_6$.	$\xi_{(5)}=t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$.
5b(ii) (Branch 2)	$G=\beta$, $H=\gamma_1t+\gamma_2$, where $\beta\neq 0$ and $\gamma_1\neq 0$.	$\xi^0=\frac{Q}{\gamma_1}\alpha$, $\xi^1=\alpha x+c_1$, $\xi^2=-c_2z+c_3$, $\xi^3=c_2y+c_34$.	$\xi_{(5)}=\frac{H}{\gamma_1}\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}$.
5c(i) (Branch 4)	$G=(\alpha t+c_1)^{\frac{c_2}{\alpha}-1}$, $H=(\alpha t+c_1)^{1-\frac{c_2}{\alpha}}$, where $\alpha\neq 0$ and $\alpha\neq c_2$.	$\xi^0=\alpha t+c_1$, $\xi^1=2\alpha x-c_2x+c_3$, $\xi^2=c_2y-c_4z+c_5$, $\xi^3=c_2z+c_4y+c_6$.	$\xi_{(5)}=t\frac{\partial}{\partial t}+2x\frac{\partial}{\partial x}$.
5c(ii) (Branch 4)	$G=(\alpha_1t+\alpha_2)^{1-\frac{c_1}{\alpha}}$, $H=\alpha_1t+\alpha_2$, where $\alpha\neq 0$ and $\alpha=\frac{c_1}{2}$.	$\xi^0=\frac{H}{\alpha_1}\alpha$, $\xi^1=c_1x+c_2$, $\xi^2=-c_3z+c_4$, $\xi^3=c_3y+c_5$.	$\xi_{(5)}=\frac{H}{\alpha_1}\frac{\partial}{\partial t}$.
5d (Branch 10)	$G=(\alpha t+c_1)^{1-\frac{c_2}{\alpha}}$, $H=Const.$, where $c_2\neq\alpha\neq 0$.	$\xi^0=\alpha t+c_1$, $\xi^1=c_2x+c_3$, $\xi^2=\alpha y-c_4z+c_5$, $\xi^3=\alpha z+c_4y+c_6$.	$\xi_{(5)}=t\frac{\partial}{\partial t}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}$.

we have:

$$\begin{aligned}\rho &= \frac{3\alpha^2 + c_3^2 - 4c_3\alpha - 2c_2\alpha + 2c_2c_3}{(\alpha t + c_1)^2}, \\ p_{||} &= -\frac{\alpha^2 + 3c_3^2 - 4c_3\alpha}{(\alpha t + c_1)^2}, \\ p_{\perp} &= -\frac{\alpha^2 + c_3^2 + c_2^2 - 2c_3\alpha - 2c_2\alpha + c_2c_3}{(\alpha t + c_1)^2}.\end{aligned}$$

One may use these values in the inequalities given in (2.7.1) to obtain the bounds for energy conditions.

For model 5a(ii), we obtain $\rho = \frac{(\alpha - c_3)^2}{(\alpha t + c_1)^2}$, $p_{||} = -\frac{\alpha^2 + 3c_3^2 - 4\alpha c_3}{(\alpha t + c_1)^2}$ and $p_{\perp} = \frac{c_3(\alpha - c_3)}{(\alpha t + c_1)^2}$. The energy density is clearly non-negative, so the model is physically meaningful. The SEC, WEC and NEC are satisfied if $c_3(\alpha - c_3) > 0$ and $\alpha(\alpha - c_3) > 0$, while the DEC requires $c_3(\alpha - c_3) > 0$, $\alpha(\alpha - c_3) > 0$ and $(\alpha - c_3)(\alpha - 2c_3) \geq 0$.

Similarly, for the metric 5b(i), we have $\rho = \frac{\alpha^2 - c_2^2}{(\alpha t + c_1)^2}$, $p_{||} = -\frac{\alpha^2 + 3c_2^2 - 4\alpha c_2}{(\alpha t + c_1)^2}$ and $p_{\perp} = \frac{c_2(\alpha - c_2)}{(\alpha t + c_1)^2}$. These quantities satisfy the SEC and NEC if $c_2(\alpha - c_2) > 0$ and $\alpha^2 - 2c_2^2 + \alpha c_2 \geq 0$. Moreover, the WEC is satisfied if $\alpha > c_2$, $c_2(\alpha - c_2) > 0$ and $\alpha^2 - 2c_2^2 + \alpha c_2 \geq 0$, while the DEC requires $\alpha > c_2$, $c_2(\alpha - c_2) > 0$, $(\alpha - c_2)^2 \geq 0$, $\alpha^2 - 2c_2^2 + \alpha c_2 \geq 0$ and $\alpha(\alpha - c_2) > 0$.

The model 5b(ii) represents an anisotropic fluid with $\rho = -p_{||} = \frac{\gamma_1^2}{(\gamma_1 t + \gamma_2)^2}$ and $p_{\perp} = 0$. Here, all the energy conditions are identically satisfied. Moreover, the energy density is non-negative ,so the model is physically realistic.

For the metric 5c(i), we found $\rho = -\frac{(\alpha - c_2)^2}{(\alpha t + c_1)^2}$, $p_{||} = -\frac{\alpha^2 + 3c_2^2 - 4\alpha c_2}{(\alpha t + c_1)^2}$ and $p_{\perp} = -\frac{(\alpha - c_2)^2}{(\alpha t + c_1)^2}$. Here, the energy density is negative, so it clearly violates the DEC and WEC. Also the WEC and NEC are violated because $\rho + p_{\perp} < 0$.

For the model 5c(ii), we have $\rho = \alpha_1^2 \frac{(3\alpha - 2c_1)}{\alpha(\alpha_1 t + \alpha_2)^2}$, $p_{||} = -\frac{\alpha_1^2}{(\alpha_1 t + \alpha_2)^2}$ and $p_{\perp} = -\alpha_1^2 \frac{(\alpha^2 + c_1^2 - 4\alpha c_1)}{\alpha^2(\alpha_1 t + \alpha_2)^2}$. This model is to be physically realistic, provided that,

$\frac{3\alpha-2c_1}{\alpha} \geq 0$. Moreover, one may use these values in the inequalities given in (2.7.1) to obtain the bounds for energy conditions like previous cases.

Finally, for the metric 5d, we have $\rho = p_{||} = 0$ and $p_{\perp} = \frac{c_2(\alpha-c_2)}{(\alpha t+c_1)^2}$. The dominant energy condition is failed for these values, while all other energy conditions are satisfied provided that $c_2(\alpha - c_2) > 0$.

5.1.2 Seven HVFs

For branch 3, we have $p_1 \neq 0$, $p_2 \neq 0$, $p_3 \neq 0$ and $p_4 = p_5 = 0$. The constraint $p_5 = 0$ gives $G = H$. The calculation of this branch is divided into two cases, the first giving six KVFs, which can be seen in the section of KVFs (Section 4.1.5). The second case leads to two sub-cases, both giving 7-dimensional homothetic algebra. The metrics of both cases along with their components of vector field and the additional symmetries other than the minimum ones are presented in Table 5.2, where $\xi_{(5)}$ represents a proper HVF, while $\xi_{(6)}$ and $\xi_{(7)}$ are two additional KVFs.

The metric 7a is same as listed in Ref. [122], while the metric 7b is missing there.

The model 7a represents a perfect fluid with $\rho = \frac{3(\alpha-c_2)^2}{(\alpha t+c_1)^2}$ and $p_{||} = p_{\perp} = \frac{-\alpha^2-3c_2^2+4\alpha c_2}{(\alpha t+c_1)^2}$. The model is physically realistic as the energy density is clearly positive. Here the WEc and NEC are satisfied if $\alpha(\alpha - c_2) > 0$. Along with this condition, if the inequality $c_2(\alpha - c_2) > 0$ also holds, then the SEC is satisfied. Finally, the DEC holds provided that $\alpha(\alpha - c_2) > 0$ and $(\alpha - c_2)(2\alpha - 3c_2) \geq 0$.

The model 7b represents a perfect fluid with $\rho = \frac{3\beta_1^2}{(\beta_1 t+\beta_2)^2}$ and $p_{||} = p_{\perp} =$

Table 5.2: Metrics admitting seven HVFs

No.	Metric	Vector Field Components	Additional Symmetries
7a	$G=(\alpha t+c_1)^{1-\frac{c_2}{\alpha}}$, $H=G$, where $\alpha\neq c_2$ and $\alpha\neq 0$.	$\xi^0=\alpha t+c_1$, $\xi^1=c_2x-c_3y-c_4z+c_5$, $\xi^2=c_2y+c_3x-c_6z+c_7$, $\xi^3=c_2z+c_4x+c_6y+c_8$.	$\xi_{(5)}=t\frac{\partial}{\partial_t}$, $\xi_{(6)}=y\frac{\partial}{\partial_x}-x\frac{\partial}{\partial_y}$, $\xi_{(7)}=z\frac{\partial}{\partial_x}-x\frac{\partial}{\partial_z}$.
7b	$G=\beta_1t+\beta_2$, $H=G$, where $\beta_1\neq 0$.	$\xi^0=\frac{G}{\beta_1}\alpha$, $\xi^1=-c_1y-c_2z+c_3$, $\xi^2=c_1x-c_4z+c_5$, $\xi^3=c_2x+c_4y+c_6$.	$\xi_{(5)}=\frac{G}{\beta_1}\frac{\partial}{\partial_t}$, $\xi_{(6)}$, $\xi_{(7)}$ are same for the metric 7a.

$-\frac{\beta_1^2}{(\beta_1t+\beta_2)^2}$. The energy density is clearly positive so the model is physically meaningful and hence all the energy conditions are identically satisfied here.

5.1.3 Eleven HVFs

The branches 12 and 13 of the Rif tree yield 11-dimensional homothetic algebras. In branch 13, both the metric functions G and H become constant and hence the LRS Bianchi type I metric reduces to the well known flat Minkowski metric. This metric admits eleven HVFs which are already discussed in chapter 4. This metric was also presented in Ref. [122].

For branch 12, we have $G=\beta_1t+\beta_2$ and $H=\beta_3$, where $\beta_1\neq 0$ and $\beta_3\neq 0$. The components of HVFs for this metric along with the additional symmetries other than the minimum ones are presented in Table 5.3, where $\xi_{(5)}$

denotes a proper HVF ,while $\xi_{(6)}, ..., \xi_{(11)}$ are six additional KVFs. One can easily see that this metric is not given in Ref. [122]. Moreover, the energy-

Table 5.3: Metric admitting eleven HVFs

Vector Field Components	Additional Symmetries
$\xi^0 = y(c_1e^{\beta_1 x} - c_2e^{-\beta_1 x}) + z(c_3e^{\beta_1 x} - c_4e^{-\beta_1 x})$ $+\frac{1}{\beta_1}(c_5e^{\beta_1 x} - c_6e^{-\beta_1 x}) + \frac{\alpha}{\beta_1}(\beta_1 t + \beta_2),$ $\xi^1 = -\frac{1}{\beta_1 t+\beta_2}\Big[y(c_1e^{\beta_1 x} + c_2e^{-\beta_1 x})$ $+z(c_3e^{\beta_1 x} + c_4e^{-\beta_1 x}) + \frac{1}{\beta_1}(c_5e^{\beta_1 x} + c_6e^{-\beta_1 x})\Big] + c_7,$ $\xi^2 = \alpha y + \frac{\beta_1 t+\beta_2}{\beta_1}(c_1e^{\beta_1 x} - c_2e^{-\beta_1 x}) - c_8 z + c_9,$ $\xi^3 = \alpha z + \frac{\beta_1 t+\beta_2}{\beta_1}(c_3e^{\beta_1 x} - c_4e^{-\beta_1 x}) + c_8 y + c_{10}.$	$\xi_{(5)} = \frac{G}{\beta_1}\frac{\partial}{\partial_t} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}.$ $\xi_{(6)} = e^{\beta_1 x}(y\frac{\partial}{\partial_t} - \frac{y}{G}\frac{\partial}{\partial_x} + \frac{G}{\beta_1}\frac{\partial}{\partial_y}),$ $\xi_{(7)} = -e^{-\beta_1 x}(y\frac{\partial}{\partial_t} + \frac{y}{G}\frac{\partial}{\partial_x} + \frac{G}{\beta_1}\frac{\partial}{\partial_y}),$ $\xi_{(8)} = e^{\beta_1 x}(z\frac{\partial}{\partial_t} - \frac{z}{G}\frac{\partial}{\partial_x} + \frac{G}{\beta_1}\frac{\partial}{\partial_z}),$ $\xi_{(9)} = -e^{-\beta_1 x}(z\frac{\partial}{\partial_t} + \frac{z}{G}\frac{\partial}{\partial_x} + \frac{G}{\beta_1}\frac{\partial}{\partial_z}),$ $\xi_{(10)} = \frac{1}{\beta_1}e^{\beta_1 x}(\frac{\partial}{\partial_t} - \frac{1}{G}\frac{\partial}{\partial_x}),$ $\xi_{(11)} = -\frac{1}{\beta_1}e^{-\beta_1 x}(\frac{\partial}{\partial_t} + \frac{1}{G}\frac{\partial}{\partial_x}).$

momentum tensor for both of these metrics vanishes and hence they represent vacuum solutions satisfying all the energy conditions.

5.1.4 Five KVFs

For the metric of branch 7, the homothety constant vanishes and hence there is no proper HVF in this case. The number of KVFs turned out to be five in which four are the minimum KVFs, while the fifth KVF is given in Table 5.4 along with the exact form of the metric and the components of the vector field.

Table 5.4: Metric admitting five KVFs

Metric	Vector Field Components	Additional KVF
$G = e^{c_1 t}$, $H = e^{c_2 t}$, where $c_1 > 0$, $c_2 > 0$ and $c_1 \neq c_2$.	$\xi^0 = -\frac{c_3}{c_2}$, $\xi^1 = \frac{c_1}{c_2} c_3 x + c_4$, $\xi^2 = c_3 y - c_4 z + c_5$, $\xi^3 = c_3 z + c_4 y + c_6$.	$\xi_{(5)} = -\frac{1}{c_2}\frac{\partial}{\partial t} + \frac{c_1}{c_2} x \frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$.

For the above model, being an anisotropic fluid, we have found $\rho = 2c_1c_2 + c_2^2$, $p_{||} = -3c_2^2$, and $p_{\perp} = -(c_1^2 + c_2^2 + c_1c_2)$. As c_1 and c_2 are positive, so clearly $\rho > 0$. Thus the model is physically realistic. The NEC and WEC are satisfied if $c_2(c_1 - c_2) > 0$ and $c_1(c_2 - c_1) > 0$. The DEC requires $c_2(c_1 - c_2) > 0$, $c_2(c_1 + 2c_2) \geq 0$, $c_1(c_2 - c_1) > 0$ and $c_1^2 + 2c_2^2 + 3c_1c_2 \geq 0$, while the SEC violates.

5.1.5 Six KVFs

Like branch 7, the metric given by branch 11 also does not possess any proper HVF, while it admits a 6-dimensional algebra of KVFs. In Table 5.5, we present the exact form of this metric and its additional two KVFs.

Here, we have also listed the exact form of the metric given by a sub-case of branch 3 and its additional two KVFs.

The metric 6a represents an anisotropic fluid with $\rho = p_{||} = 0$ and $p_{\perp} = -k^2$. The energy density is clearly non-negative, so the model is physically

Table 5.5: Metrics admitting six KVFs

No.	Metric	Vector Field Components	Additional KVFs
6a (Branch 11)	$G = \beta_1 e^{kt} + \beta_2 e^{-kt}$, $H = Const.$, where $k > 0$.	$\xi^0 = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$, $\xi^1 = -\frac{G'}{\lambda G}[c_1 \sin(\lambda x) - c_2 \cos(\lambda x)] + c_3$, $\xi^2 = -c_4 z + c_5$, $\xi^3 = c_4 y + c_6$, where $\lambda = \sqrt{4\beta_1\beta_2}k$.	$\xi_{(5)} = \cos(\lambda x)\frac{\partial}{\partial_t} - \frac{G'}{\lambda G}\sin(\lambda x)\frac{\partial}{\partial_x}$, $\xi_{(6)} = \sin(\lambda x)\frac{\partial}{\partial_t} + \frac{G'}{\lambda G}\cos(\lambda x)\frac{\partial}{\partial_x}$.
6b (Branch 3)	$G = H$. $G' \neq 0$ and $GG'' - G'^2 \neq 0$.	$\xi^0 = 0$, $\xi^1 = c_1 y + c_2 z - c_3$, $\xi^2 = c_1 x - c_4 z + c_5$, $\xi^3 = c_2 x + c_4 y + c_6$.	$\xi_{(5)} = x\frac{\partial}{\partial_y} + y\frac{\partial}{\partial_x}$, $\xi_{(6)} = x\frac{\partial}{\partial_z} + z\frac{\partial}{\partial_x}$.

meaningful. All the energy conditions are failed for this model.

Similarly, the metric 6b represents a perfect fluid with:

$$\rho = \frac{3H'^2}{H^2},$$
$$p_{||} = p_{\perp} = -(\frac{2H''}{H} + \frac{H'^2}{H^2}).$$

The energy conditions for this metric are satisfied conditionally and can be easily simplified by using the above expression in (2.7.1).

5.1.6 Seven KVFs

The branch 8 of the Rif tree produces a metric admitting seven KVFs, out of which four are the minimum and the extra three KVFs are given in Table 5.6.

Table 5.6: Metric admitting seven KVFs

Metric	Vector Field Components	Additional KVFs
$G = Const.$, $H = e^{kt}$, where $k > 0$	$\xi^0 = \frac{1}{k}(-c_1 y + c_2 z - c_3)$, $\xi^1 = c_4$, $\xi^2 = c_3 y + \frac{c_1}{2k^2}e^{-2kt} + \frac{c_1}{2}(y^2 - z^2)$ $-c_2 yz - c_5 z + c_6$, $\xi^3 = c_3 z - \frac{c_2}{2k^2}e^{-2kt} + \frac{c_2}{2}(y^2 - z^2)$ $+c_1 yz + c_5 y + c_7$.	$\xi_{(5)} = -\frac{y}{k}\frac{\partial}{\partial_t} + \left(\frac{e^{-2kt}}{2k^2} + \frac{y^2 - z^2}{2}\right)\frac{\partial}{\partial_y} + yz\frac{\partial}{\partial_z}$, $\xi_{(6)} = \frac{z}{k}\frac{\partial}{\partial_t} + \left(\frac{-e^{-2kt}}{2k^2} + \frac{y^2 - z^2}{2}\right)\frac{\partial}{\partial_z} - yz\frac{\partial}{\partial_y}$, $\xi_{(7)} = -\frac{1}{k}\frac{\partial}{\partial_t} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}$.

For this metric, being an anisotropic fluid, the values of $\rho, p_{||}$ and $p_{\perp}$ are found to be $\rho = k^2, p_{||} = -3k^2$ and $p_{\perp} = -k^2$. Clearly $\rho > 0$, so the model is physically realistic. All other energy conditions are violated here.

5.1.7 Ten KVFs

The metric of branch 9 possesses 10-dimensional Killing algebra, containing the 4-dimensional minimal algebra as a sub algebra and the six additional KVFs are presented in Table 5.7.

The metric of this case is a flat metric representing a perfect fluid with $\rho = 3k^2$ and $p_{||} = p_{\perp} = -3k^2$. The energy density is positive, showing that the model is physically realistic. For this model, the SEC is violated, while the WEC, NEC and DEC are satisfied.

5.2 Conformal Symmetries

For conformal symmetry, we use the metric (5.1.1) in Eq. (2.5.3) and get the same set of equations as given in Eqs. (5.1.5)-(5.1.14) with $\alpha = \alpha(t, x, y, z)$. We have found the following non-zero components of Weyl tensor for the metric (5.1.1):

$$\begin{aligned} C^1_{212} &= C^1_{313} = \frac{\Gamma}{6G}, \quad C^1_{010} = C^0_{220} = \frac{\Gamma}{3GH^2}, \\ C^2_{020} &= C^3_{030} = -\frac{\Gamma}{6GH}, \quad C^2_{323} = -\frac{\Gamma}{3G}, \end{aligned} \tag{5.2.1}$$

where $\Gamma = GHH'' - GH'^2 - G''H^2 + G'HH'$. Thus the LRS Bianchi type I spacetime is conformally flat if and only if $\Gamma = 0$.

To explore a complete list of LRS Bianchi type I metrics admitting conformal symmetries, we need to solve the system of equations (5.1.5)-(5.1.14) with

Table 5.7: Metric admitting ten KVFs

Metric	Vector Field Components	Additional KVFs
$G = H = e^{kt}$, where $k > 0$.	$\xi^0 = -\frac{1}{k}(c_1 y - c_2 z + c_3 x + c_4)$,	$\xi_{(5)} = -\frac{y}{k}\frac{\partial}{\partial_t} + xy\frac{\partial}{\partial_x} + \frac{1}{2}\big(\frac{e^{-2kt}}{k^2} - x^2 + y^2 - z^2\big)\frac{\partial}{\partial_y} + yz\frac{\partial}{\partial_z}$,
	$\xi^1 = \frac{c_3}{2}\big(\frac{e^{-2kt}}{k^2} + x^2 - y^2 - z^2\big)$	$\xi_{(6)} = \frac{z}{k}\frac{\partial}{\partial_t} - xz\frac{\partial}{\partial_x} - yz\frac{\partial}{\partial_y} + \frac{1}{2}\big(\frac{-e^{-2kt}}{k^2} + x^2 + y^2 - z^2\big)\frac{\partial}{\partial_z}$,
	$+c_1 xy - c_2 xz + c_4 x - c_5 y - c_6 z + c_7$,	$\xi_{(7)} = -\frac{x}{k}\frac{\partial}{\partial_t} + \frac{1}{2}\big(\frac{e^{-2kt}}{k^2} + x^2 - y^2 - z^2\big)\frac{\partial}{\partial_x} + xy\frac{\partial}{\partial_y} + xz\frac{\partial}{\partial_z}$,
	$\xi^2 = \frac{c_1}{2}\big(\frac{e^{-2kt}}{k^2} - x^2 + y^2 - z^2\big)$	$\xi_{(8)} = -\frac{1}{k}\frac{\partial}{\partial_t} + x\frac{\partial}{\partial_x} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}$,
	$-c_2 yz + c_3 xy + c_5 x + c_4 y - c_8 z + c_9$,	$\xi_{(9)} = -y\frac{\partial}{\partial_x} + x\frac{\partial}{\partial_y}$,
	$\xi^3 = \frac{c_2}{2}\big(-\frac{e^{-2kt}}{k^2} + x^2 + y^2 - z^2\big)$	$\xi_{(10)} = -z\frac{\partial}{\partial_x} + x\frac{\partial}{\partial_z}$.
	$+c_3 xz + c_1 yz + c_6 x + c_8 y + c_4 z + c_{10}$.	

$\alpha = \alpha(t, x, y, z)$. For the solution of these equations, we use the same Rif tree approach as we have used in the previous section for finding HVFs. As a result of the Rif algorithm, we obtain the Rif tree given in Fig. 5.2 and the following list of pivots:

$$
\begin{aligned}
p_1 &= G', \\
p_2 &= HH'' - H'^2, \\
p_3 &= GH' - G'H, \\
p_4 &= GG''' - G'G'', \\
p_5 &= H^2G'' - GH'^2, \\
p_6 &= H', \\
p_7 &= GG'' - G'^2, \\
p_8 &= G'(GH' - G'H)(GH' + G'H), \\
p_9 &= G'(GH' - G'H).
\end{aligned}
$$

To get a complete classification, we solve the set of conformal symmetry equations for each branch of the Rif tree. Like the previous chapters, we only focus on those cases which give non-conformally flat metrics. If $\Gamma = 0$, then the spacetime admits 15-dimensional conformal algebra. Such cases are labeled by the branches 2, 9, 10, 12 and 13. Further, there are some other branches of the Rif tree which give two sub-cases, depending upon whether $\Gamma = 0$ or $\Gamma \neq 0$. Again, here we have omitted the cases where $\Gamma = 0$. Moreover, the branches 1, 4 and 7 give the minimum four KVFs. Such cases are not part of our present discussion, while the results of the remaining cases yield one and two proper CVFs as summarized in the forthcoming sections.

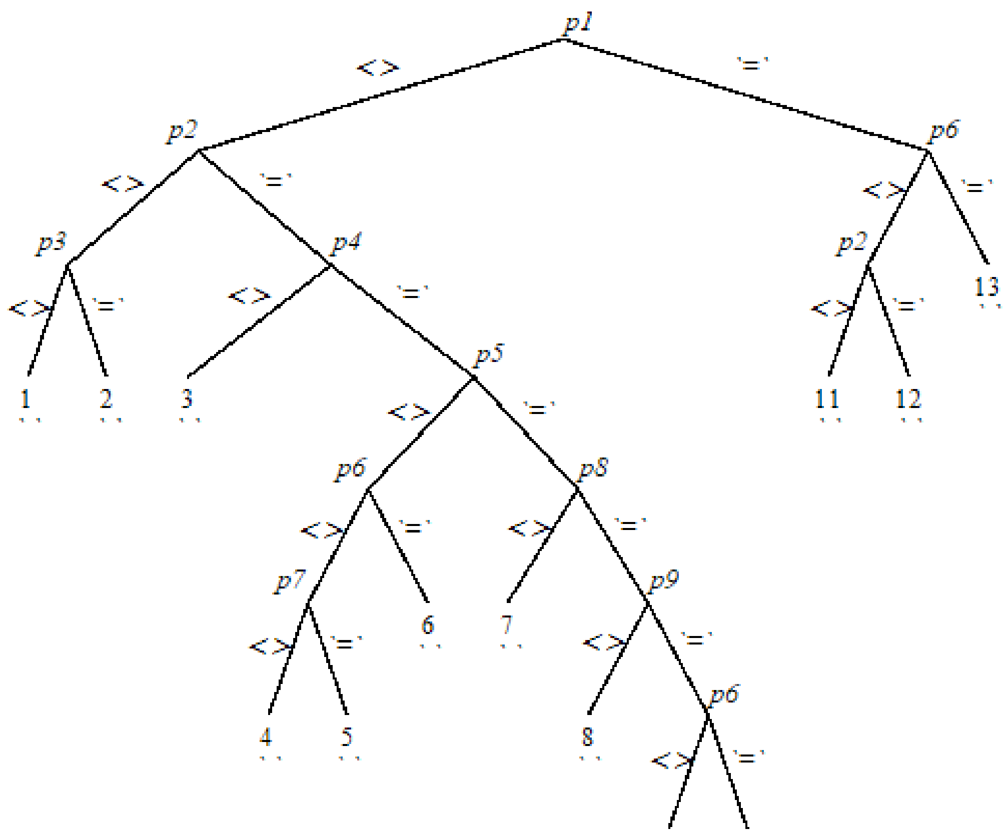

Figure 5.2: Rif Tree for CVFs

5.2.1 Branch 3

The constraints of this branch are $p_1 \neq 0, p_2 = 0$ and $p_4 \neq 0$. The condition $p_2 = 0$ gives $HH'' - H'^2 = 0$. This yields two sub-cases depending upon whether $H' \neq 0$ or $H' = 0$. When $H' \neq 0$, the solution of equation $HH'' - H'^2 = 0$ gives $H = e^{kt}$, where $k > 0$. Further simplification of the conformal symmetry equations by using this value of H gives rise to two sub-cases, one giving the minimum four KVFs, while the second sub-case produces a conformally flat metric admitting 15 CVFs.

When $H' = 0$, we get $H = \beta$, where $\beta \neq 0$. Solving the system of conformal symmetry equations, we obtain $G = (c_1 t + c_2)^{1-\frac{c_3}{c_1}}$, where $c_1 \neq 0$, $c_3 \neq 0$ and $c_1 \neq c_3$ and the CVFs in this case are reduced to HVFs which are same as given for the metric 5d in Table 5.1, with $c_1 = \alpha$.

5.2.2 Branch 5

In branch 5, we have, $p_1 \neq 0, p_2 = p_4 = 0, p_5 \neq 0, p_6 \neq 0$ and $p_7 = 0$. Simplification of these constraints gives the metric functions $G = e^{c_1 t}$ and $H = e^{c_2 t}$, where $c_1 > 0$, $c_2 > 0$, $c_1 \neq c_2$ and $c_1 \neq -c_2$. Using these values of the metric functions, the solution of conformal symmetry equations gives $\alpha = 0$ and hence the CVFs are reduced to KVFs. The obtained five KVFs are same as given in Table 5.4 of the previous section.

5.2.3 Branch 6

The constraints for this branch are $p_1 \neq 0, p_2 = p_4 = 0, p_5 \neq 0$ and $p_6 = 0$, which give $G = \beta_1 e^{kt} + \beta_2 e^{-kt}$ and $H = \gamma$, where $k > 0$ and $\gamma \neq 0$. The solution of conformal symmetry equations for these values of G and H gives $\alpha = 0$, showing that there exists no proper CVF, and we have obtained six KVFs which are same as give in Table 5.5 in the previous section.

5.2.4 Branch 8

In branch 8, we have $p_1 \neq 0$, $p_2 = p_4 = p_5 = p_8 = 0$ and $p_9 \neq 0$. Simplification of these constraints gives $G = e^{-kt}$ and $H = e^{kt}$, where $k \neq 0$. In this branch, the CVFs are reduced to KVFs, i.e. $\alpha = 0$. The obtained five KVFs are same as given in Table 5.4 with $c_1 = -c_2$.

5.2.5 Branch 11

The constraints of this branch are $p_1 = 0, p_2 \neq 0$ and $p_6 \neq 0$, which give $G = \beta$, where $\beta \neq 0$ and $H' \neq 0$, $HH'' - H'^2 \neq 0$. The calculation of this branch is divided into two cases. One of the cases again splits into two sub-cases, the first giving the minimum KVFs, while the second sub-case gives a metric same as the metric 5a(ii), given in Table 5.1. For this metric, there exist no proper CVF and the CVFs are reduced to HVFs, which are already presented in Table 5.1. Moreover, the second case gives six CVFs, which include four minimum KVFs and two proper CVFs, given in Table 5.8.

Table 5.8: Metrics admitting six CVFs

Metric	Vector Field Components	Proper CVFs
$G = \beta$, $H = \alpha_1 e^{kt} + \alpha_2 e^{-kt}$, where $k > 0$ $\beta \neq 0$, $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$.	$\xi^0 = \frac{H}{k}\Big(c_1 \sinh(kx) + c_2 \cosh(kx)\Big)$, $\xi^1 = \frac{H'}{k^2}\Big(c_1 \cosh(kx) + c_2 \sinh(kx)\Big) + c_3$, $\xi^2 = -c_4 z + c_5$, $\xi^3 = c_4 y + c_6$, $\alpha = \frac{H'}{k}\Big(c_1 \sinh(kx) + c_2 \cosh(kx)\Big)$.	$\xi_{(5)} = \frac{H}{k} \sinh(kx)\ \frac{\partial}{\partial t}$ $+\frac{H'}{k^2} \cosh(kx)\ \frac{\partial}{\partial x}$, $\xi_{(6)} = \frac{H}{k} \cosh(kx)\ \frac{\partial}{\partial t}$ $+\frac{H'}{k^2} \sinh(kx)\ \frac{\partial}{\partial x}$.

For the metric of this branch, we have $\rho = \frac{k^2(\alpha_1 e^{kt} - \alpha_2 e^{-kt})^2}{(\alpha_1 e^{kt} + \alpha_2 e^{-kt})^2}$, $p_\perp = -k^2$ and $p_{||} = \frac{-k^2(3\alpha_1^2 e^{2kt} + 3\alpha_2^2 e^{-2kt} + 2\alpha_1\alpha_2)}{(\alpha_1 e^{kt} + \alpha_2 e^{-kt})^2}$. These quantities clearly show that none of the energy conditions is satisfied except that energy density is non-negative, giving a physically meaningful model.

Since there is no published work on the conformal symmetries of LRS Bianchi type I via direct integration technique, so we could not compare the results obtained by Rif tree approach with those of direct integration technique.

5.3 Noether Symmetries

We consider the following Lagrangian corresponding to the metric (5.1.1) of LRS Bianchi type I spacetimes for the investigation of Noether symmetries:

$$L = -\dot{t}^2 + G^2(t)\dot{x}^2 + H^2(t)(\dot{y}^2 + \dot{z}^2). \tag{5.3.1}$$

For this Lagrangian, the minimal set of Noether symmetries is:

$$N_5 = \left\{ \frac{\partial}{\partial_s}, \frac{\partial}{\partial_x}, \frac{\partial}{\partial_y}, \frac{\partial}{\partial_z}, z\frac{\partial}{\partial_y} - y\frac{\partial}{\partial_z} \right\}.$$

We have obtained the following set of determining equations by using the Lagrangian (5.3.1) in Eq. (2.6.1):

$$F_{,s} = \eta_{,t} = \eta_{,x} = \eta_{,y} = \eta_{,z} = 0, \tag{5.3.2}$$

$$2\xi^0_{,t} = \eta_{,s}, \tag{5.3.3}$$

$$2G'\xi^0 + 2G\xi^1_{,x} = G\eta_{,s}, \tag{5.3.4}$$

$$2H'\xi^0 + 2H\xi^2_{,y} = H\eta_{,s}, \tag{5.3.5}$$

$$2H'\xi^0 + 2H\xi^3_{,z} = H\eta_{,s}, \tag{5.3.6}$$

$$\xi^0_{,x} - G^2\xi^1_{,t} = 0, \tag{5.3.7}$$

$$\xi^0_{,y} - H^2\xi^2_{,t} = 0, \tag{5.3.8}$$

$$\xi^0_{,z} - H^2\xi^3_{,t} = 0, \tag{5.3.9}$$

$$G^2\xi^1_{,y} + H^2\xi^2_{,x} = 0, \tag{5.3.10}$$

$$G^2\xi^1_{,z} + H^2\xi^3_{,x} = 0, \tag{5.3.11}$$

$$\xi^2_{,z} + \xi^3_{,y} = 0, \tag{5.3.12}$$

$$2\xi^0_{,s} = -F_{,t}, \tag{5.3.13}$$

$$2G^2\xi^1_{,s} = F_{,x}, \tag{5.3.14}$$

$$2H^2\xi^2_{,s} = F_{,y}, \tag{5.3.15}$$

$$2H^2\xi^3_{,s} = F_{,z}. \tag{5.3.16}$$

To find the Noether symmetries, these equations are analyzed through a Maple algorithm, which produces the Rif tree, given in Fig. 5.3 and the list of pivots.

$$\begin{aligned}
p_1 &= G', \\
p_2 &= G'', \\
p_3 &= H'', \\
p_4 &= H'^2 - HH'', \\
p_5 &= G'H - GH', \\
p_6 &= G'^2 - GG'', \\
p_7 &= H', \\
p_8 &= G'G'' - GG'''.
\end{aligned}$$

To achieve a complete classification, the Noether determining equations are solved for each branch of the Rif tree. As a result, we have obtained 5,

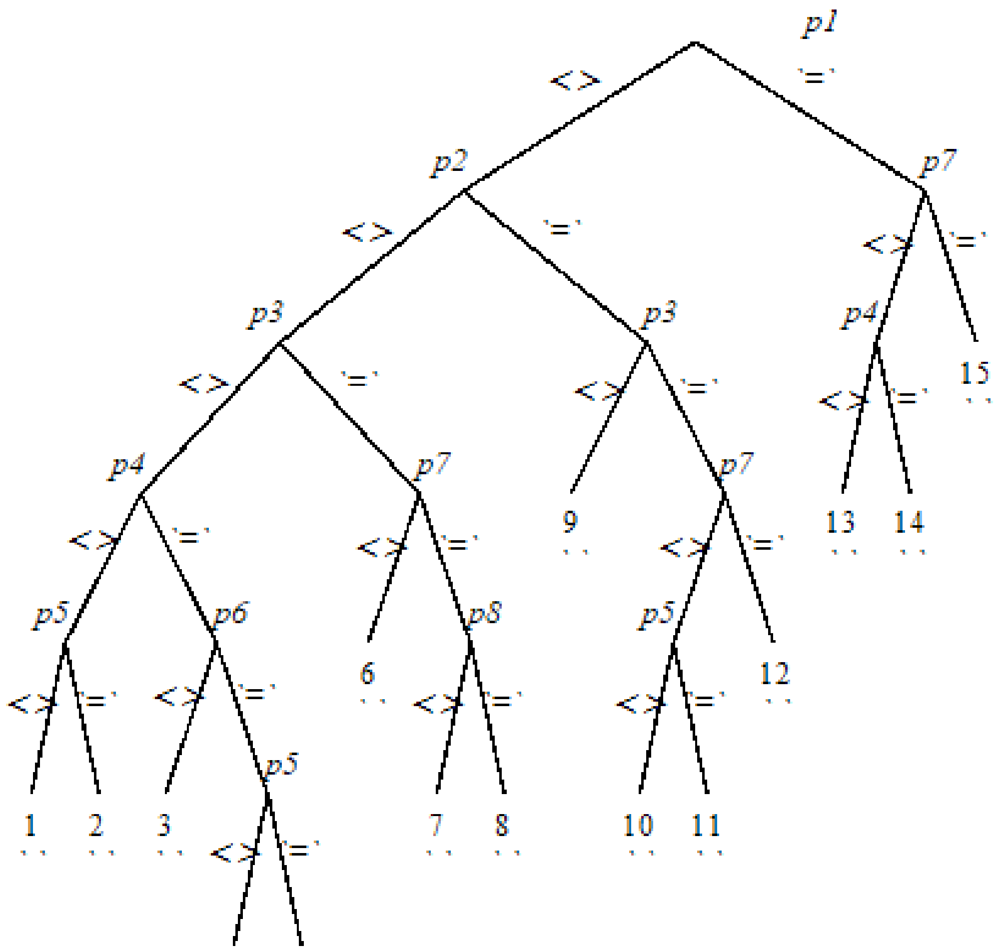

Figure 5.3: Rif Tree for Noether Symmetries

6, 7, 8, 9, 11 and 17-dimensional Noether algebras. We omit those cases that give the minimal set of Noether symmetries, given in the set N_5. Such cases are labeled by branches 3, 9 and 10 in the Rif tree. The cases which give symmetries other than the minimal set are discussed in the forthcoming sections. Here, we also compare our results with those of Ref. [85], where the same spacetimes were classified according to their Noether symmetries by direct integration technique.

5.3.1 Branch 1

The constraints of this branch are $p_1 \neq 0$, $p_2 \neq 0$, $p_3 \neq 0$, $p_4 \neq 0$ and $p_5 \neq 0$. The calculation of solving Eqs. (5.3.2)- (5.3.16) for this branch is divided into two cases. One of the cases gives the minimal set of Noether symmetries, given in the set N_5, while the second case gives six Noether symmetries. Out of these six, five are given in the minimal set, while one is the extra Noether symmetry presented in Table 5.9 along with the values of the metric functions. Clearly, this extra symmetry is a Noether symmetry corresponding to a homothetic vector $t\frac{\partial}{\partial_t}$. The metric of this branch can be transformed to the metric (8c) of Ref. [85] by setting $c_1 = a$, $c_2 = b$, $c_3 = d$ and $c_4 = c$. The authors of Ref. [85] obtained two extra symmetries for this metric, given by $\xi_{(6)} = x\frac{\partial}{\partial_x} - \frac{A}{\acute{A}}\ \frac{\partial}{\partial_t}$ and $\xi_{(7)} = y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}$, which seems to be not correct.

Table 5.9: Metric admitting six Noether Symmetries

Metric	Additional Noether Symmetry	Invariant
$G = (c_1 t + 2c_2)^{1-\frac{2c_3}{c_1}}$, $H = (c_1 t + 2c_2)^{1-\frac{2c_4}{c_1}}$, $c_1 \neq 0$, $c_3 \neq c_4 \neq 0$, $c_1 \neq 2c_3$ and $c_1 \neq 2c_4$.	$\xi_{(5)} = s\ \frac{\partial}{\partial_s} + \frac{t}{2}\ \frac{\partial}{\partial_t}$.	$I_{(5)} = -sL - t\ \dot{t}$.

One can see that the metric of this case is same as the metric 5a, given in

Table 5.1, admitting the same HVF.

5.3.2 Branch 2

In branch 2, we have $p_1 \neq 0$, $p_2 \neq 0$, $p_3 \neq 0$, $p_4 \neq 0$ and $p_5 = 0$. The condition $p_5 = 0$ gives $G = H$. Using these constraints to solve the system of determining equations (5.3.2)-(5.3.16) yields two cases during calculation. One of the cases gives 7-dimensional algebra of Noether symmetries, out of which five are given in the minimal set and two are the extra Noether symmetries. The second case gives eight Noether symmetries in which five are the same as given in the set N_5, while three are the extra Noether symmetries given in Table 5.10.

Here, $\xi_{(5)}$ and $\xi_{(6)}$ are the additional KVFs for both metrics and $\xi_{(7)}$ is a Noether symmetry corresponding to a homothetic vector $t\frac{\partial}{\partial_t}$ for the second metric. The metric 7a is same as the metric (7d) presented in Ref. [85], while for metric 8a, the obtained symmetries are same as obtained for the metric (9i) of Ref. [85] except that an additional symmetry $\xi_{(8)} = x\frac{\partial}{\partial_x} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z} - \frac{A}{\dot{A}}\frac{\partial}{\partial_t}$ was also listed in Ref. [85] which seems to be wrong.

The metric 8a is same as the metric 7a given in Table 5.2, admitting seven HVFs.

For model 7a, the metric functions are not explicitly known, we could not find the simplified values of ρ, $p_{||}$ and $p_{\perp}$.

For model 8a, the physical interpretation is already given in the section of HVFs.

Table 5.10: Metric admitting seven and eight Noether Symmetries

No.	Metric	Noether Symmetry generators	Invariants
7a (Branch 2)	$G = H \neq (c_1 t + 2c_2)^{1-\frac{2c_3}{c_1}}$.	$\xi_{(5)} = x\frac{\partial}{\partial_y} - y\frac{\partial}{\partial_x}$, $\xi_{(6)} = x\frac{\partial}{\partial_z} - z\frac{\partial}{\partial_x}$.	$I_{(5)} = -2(G^2 y\dot{x} - H^2 x\dot{y})$, $I_{(6)} = -2(G^2 z\ \dot{x} - H^2 x\ \dot{z})$.
8a (Branch 2)	$G = H$, $H = (c_1 t + 2c_2)^{1-\frac{2c_3}{c_1}}$, where $c_1 \neq 0$ and $c_1 \neq 2c_3$.	$\xi_{(5)}, \xi_{(6)}$ are same as in case of 7a $\xi_{(7)} = s\frac{\partial}{\partial_s} + \frac{t}{2}\frac{\partial}{\partial_t}$.	$I_{(5)}, I_{(6)}$ are same as in case of 7a $I_{(7)} = -sL - G^2 t\dot{t}$.

5.3.3 Branch 4

In this branch, we have $p_1 \neq 0$, $p_2 \neq 0$, $p_3 \neq 0$, $p_4 = p_6 = 0$ and $p_5 \neq 0$. The simplification of these constraints gives $G = e^{c_1 t}$ and $H = e^{c_2 t}$, where $c_1 > 0$ and $c_2 > 0$. Using these values to solve the system of determining equations (5.3.2)-(5.3.16), we get one extra symmetry (KVF) along with the five minimum Noether symmetries, given in the set N_5. The obtained extra symmetry is given by $\xi_{(5)} = -\frac{1}{c_2}\frac{\partial}{\partial_t} + \frac{c_1}{c_2}x\frac{\partial}{\partial_x} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}$ with corresponding invariant $I_{(5)} = \frac{2}{c_2}\dot{t} + \frac{2c_1}{c_2}G^2 x\dot{x} + 2H^2(y\dot{y} + z\dot{z})$. This metric can be transformed to the metric (7c) of Ref. [85] by setting $c_1 = -\frac{a}{b}$ and $c_2 = \beta$. Again, the authors of Ref. [85] have given an additional symmetry $\xi_{(6)} = x\frac{\partial}{\partial_x}$ for this metric, which is not correct.

Moreover, the metric of this branch is same as the metric presented in Table 5.4, admitting five KVFs.

5.3.4 Branch 5

For this branch, we have $p_1 \neq 0$, $p_2 \neq 0$, $p_3 \neq 0$ and $p_4 = p_5 = p_6 = 0$. The simplification of these constraints gives the metric functions $G = H = e^{kt}$, where $k > 0$. This metric is same as the metric presented in Table 5.7 admitting ten KVFs. Also, the same metric is listed as metric (37) in Ref. [85]. After solving Eqs. (5.3.2)-(5.3.16) for these values of G and H, we have obtained eleven Noether symmetries. Out of these eleven, five are same as given in the set N_5 set and six are extra symmetries (KVFs), which along with their conserved forms are given in Table 5.11. One can see these additional KVFs are same as listed in Table 5.7.

5.3.5 Branch 6

For branch 6, we have $p_1 \neq 0$, $p_2 \neq 0$, $p_3 = 0$ and $p_7 \neq 0$. The constraint $p_3 = 0$ gives $H'' = 0 \Rightarrow H = \alpha_1 t + \alpha_2$, where $\alpha_1 \neq 0$. Solving the set of Noether symmetry equations under these conditions, we get $G = (\alpha_1 t + \alpha_2)^{1-\frac{2c_1}{c_2}}$, where $c_2 \neq 0$ and $c_2 \neq 2c_1$ and six Noether symmetries are obtained. These six symmetries include the minimal set of five Noether symmetries and one extra Noether symmetry, given by $\xi_{(5)} = s\frac{\partial}{\partial_s} + \frac{\alpha_1 t+\alpha_2}{2\alpha_1}\ \frac{\partial}{\partial_t}$. The corresponding conserved form is $I_{(5)} = -sL - \frac{(\alpha_1 t+\alpha_2)}{\alpha_1} G^2\ \dot{t}$. The metric of this branch can be transformed to the metric (7a) of Ref. [**?**] by using $c_1 = c$ and $c_2 = b$, while in Ref. [85],the extra symmetry $\xi_{(5)} = x\frac{\partial}{\partial_x}$, seems not to be correct. Clearly, $\xi_{(5)}$ is a Noether symmetry corresponding to a homothetic vector $\frac{\alpha_1 t+\alpha_2}{\alpha_1}\ \frac{\partial}{\partial_t}$. One can see that the metric of this case is same as the metric 5c(ii) given in Table 5.1, admitting same HVF. The physical interpretation of this metric was already done in the section of HVFs.

5.3.6 Branch 7

The constraints for branch 7 are $p_1 \neq 0$, $p_2 \neq 0$, $p_3 = p_7 = 0$ and $p_8 \neq 0$. The condition $p_7 = 0$ gives $H = \beta$, where $\beta \neq 0$. The calculation of this branch produces eight Noether symmetries. In Table 5.12, we present the metric of this case along with its extra Noether symmetries, other than the minimum ones. For both metrics, $\xi_{(5)}$ and $\xi_{(6)}$ are proper Noether symmetries, while $\xi_{(7)}$ is a Noether symmetry corresponding to a homothetic vector $t\frac{\partial}{\partial_t} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}$ for the second metric, which is same as given in Table 5.1 for the same metric, denoted by 5d therein. The first metric of branch 7 is same as the metric

Table 5.11: Metric admitting eleven Noether Symmetries

Additional Symmetries	Invariants
$\xi_{(5)} = \frac{y}{k}\frac{\partial}{\partial_t} - xy\frac{\partial}{\partial_x} - yz\frac{\partial}{\partial_z} + \left(\frac{x^2-y^2+z^2}{2} - \frac{e^{-2kt}}{2k^2}\right)\frac{\partial}{\partial_y},$	$I_{(5)} = -\frac{2y}{k}\dot{t} - 2e^{2kt}y(x\dot{x} + z\dot{z}) - \frac{\dot{y}}{k^2} - e^{2kt}\dot{y}(x^2 - y^2 + z^2),$
$\xi_{(6)} = -\frac{z}{k}\frac{\partial}{\partial_t} + xz\frac{\partial}{\partial_x} + yz\frac{\partial}{\partial_y}\left(\frac{z^2-y^2-x^2}{2} + \frac{e^{-2kt}}{2k^2}\right)\frac{\partial}{\partial_z},$	$I_{(6)} = \frac{2z}{k}\dot{t} + 2e^{2kt}z(x\dot{x} + y\dot{y}) + \frac{\dot{z}}{k^2} + e^{2kt}\dot{z}(z^2 - y^2 - x^2),$
$\xi_{(7)} = \frac{x}{k}\frac{\partial}{\partial_t} - xy\frac{\partial}{\partial_y} - xz\frac{\partial}{\partial_z} + \left(\frac{y^2-x^2+z^2}{2} - \frac{e^{-2kt}}{2k^2}\right)\frac{\partial}{\partial_x},$	$I_{(7)} = -\frac{2x}{k}\dot{t} - 2e^{2kt}x(y\dot{y} + z\dot{z}) - \frac{\dot{x}}{k^2} + e^{2kt}\dot{x}(y^2 - x^2 + z^2),$
$\xi_{(8)} = -\frac{1}{k}\frac{\partial}{\partial_t} + x\frac{\partial}{\partial_x} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z},$	$I_{(8)} = \frac{2}{k}\dot{t} + 2e^{2kt}(x\ \dot{x} + y\ \dot{y} + z\ \dot{z}),$
$\xi_{(9)} = -y\frac{\partial}{\partial_x} + x\frac{\partial}{\partial_y},$	$I_{(9)} = 2e^{2kt}(x\dot{y} - y\dot{x}),$
$\xi_{(10)} = -z\frac{\partial}{\partial_x} + x\frac{\partial}{\partial_z}.$	$I_{(10)} = 2e^{2kt}(x\dot{z} - z\dot{x}).$

(7b) of Ref. [85]. The second metric can be transformed to the metric (9h) of Ref. [85] by setting $c_1 = \alpha$, $c_2 = b$ and $c_3 = c$. The obtained symmetries for metric 7a are same as those listed in Ref. [85]. However, an additional KVF $x\partial_x - \frac{A}{\dot{A}}\frac{\partial}{\partial_t}$ is listed for metric 8a in Ref. [85], which is not correct.

Table 5.12: Metrics admitting seven and eight Noether Symmetries

No.	Metric	Noether Symmetry generators	Invariants
7a (Branch 7)	$G \neq (c_1 t + c_2)^{1-\frac{2c_3}{c_1}}$, $H = \beta$, where $\beta \neq 0$.	$\xi_{(5)} = \frac{s}{\beta}\frac{\partial}{\partial_y}$; $F = 2\beta y$, $\xi_{(6)} = \frac{s}{\beta}\frac{\partial}{\partial_z}$; $F = 2\beta z$.	$I_{(5)} = 2\beta(s\ \dot{y} - y)$, $I_{(6)} = 2\beta(s\ \dot{z} - z)$.
8a (Branch 7)	$G = (c_1 t + c_2)^{1-\frac{2c_3}{c_1}}$, $H = \beta$, where $\beta \neq 0$, $c_1 \neq 0$, $c_3 \neq 0$ and $c_1 \neq 2c_3$.	$\xi_{(5)}, \xi_{(6)}$ are same as in case of 7a, $\xi_{(7)} = s\frac{\partial}{\partial_s} + \frac{1}{2}\left(t\ \frac{\partial}{\partial_t} + y\ \frac{\partial}{\partial_y} + z\ \frac{\partial}{\partial_z}\right)$.	$I_{(5)}, I_{(6)}$ are same as in case of 7a, $I_{(7)} = -sL - t\ \dot{t}$ $\beta^2(y\ \dot{y} + z\ \dot{z})$.

The first model represents an anisotropic fluid with $\rho = p_{||} = 0$ and $p_{\perp} = -\frac{G''}{G}$. The energy conditions including strong, null and weak energy conditions are satisfied if $\frac{G''}{G} < 0$, while the DEC violates.

For the second model, the physical interpretation is already presented in the section of HVFs.

5.3.7 Branch 8

For this branch, the pivots are restricted to satisfy the conditions $p_1 \neq 0$, $p_2 \neq 0$ and $p_3 = p_7 = p_8 = 0$. From the simplification of these constraints, we get $G = \alpha_1 e^{\sqrt{k}t} + \alpha_2 e^{-\sqrt{k}t}$ and $H = \beta$, where $k > 0$ and $\beta \neq 0$. This metric is same as the metric 6a, given in Table 5.5, admitting six KVFs. Solving the set of determining equations for this metric, we obtain nine Noether symmetries, out of which five are same as given in the set N_5, while the remaining four are extra Noether symmetries. The details of these four Noether symmetries along with their conserved forms are given in Table 5.13.

Table 5.13: Metric admitting nine Noether Symmetries

Noether Symmetry generators	Invariants
$\xi_{(5)} = \frac{s}{\beta}\frac{\partial}{\partial_y}; F = 2\beta y,$	$I_{(5)} = 2H(s\dot{y} - y),$
$\xi_{(6)} = \frac{s}{\beta}\frac{\partial}{\partial_z}; F = 2\beta z,$	$I_{(6)} = 2H(s\dot{z} - z),$
$\xi_{(7)} = \frac{1}{\lambda}\sin(\lambda x)\frac{\partial}{\partial_t} + \frac{G'}{\lambda^2 G}\cos(\lambda x)\frac{\partial}{\partial_x},$	$I_{(7)} = -\frac{2}{\lambda}\sin(\lambda x)\dot{t} + \frac{2GG'}{\lambda^2}\cos(\lambda x)\dot{x},$
$\xi_{(8)} = -\frac{1}{\lambda}\cos(\lambda x)\frac{\partial}{\partial_t} + \frac{G'}{\lambda^2 G}\sin(\lambda x)\frac{\partial}{\partial_x}.$	$I_{(8)} = \frac{2}{\lambda}(\cos\lambda x)\dot{t} + \frac{2GG'}{\lambda^2}\sin(\lambda x)\ \dot{x}.$

Here, $\xi_{(5)}$ and $\xi_{(6)}$ are proper Noether symmetries, while $\xi_{(7)}$ and $\xi_{(8)}$ are the additional KVFs. This metric is the generalized form of the metrics (9b-9f) of Ref. [**?**].

5.3.8 Branch 11

Here, the constraints are $p_1 \neq 0$, $p_2 = p_3 = p_5 = 0$ and $p_7 \neq 0$, which give $G = H = \beta_1 t + \beta_2$, where $\beta_1 \neq 0$. This metric is same as the metric 7b presented in Table 5.2, admitting seven HVFs. Solving the set of determining equations for this metric, we obtain nine Noether symmetries, in which five are listed in the minimal set and four are the extra Noether symmetries, which along with their conserved forms are given in Table 5.14.

Table 5.14: Metric admitting nine Noether Symmetries

Noether Symmetry generators	Invariants
$\xi_{(5)} = -y\frac{\partial}{\partial_x} + x\frac{\partial}{\partial_y}$,	$I_{(5)} = 2G^2(x\dot{y} - y\dot{x})$,
$\xi_{(6)} = -z\frac{\partial}{\partial_x} + x\frac{\partial}{\partial_z}$,	$I_{(6)} = 2H^2(x\dot{z} - z\dot{x})$,
$\xi_{(7)} = \frac{s^2}{2}\frac{\partial}{\partial_s} + \frac{s}{2\beta_1}G\frac{\partial}{\partial_t}$; $F = -t(\frac{t}{2} + \frac{\beta_2}{\beta_1})$,	$I_{(7)} = -\frac{s^2}{2}L - \frac{s}{\beta_1}G\dot{t} + t(\frac{t}{2} + \frac{\beta_2}{\beta_1})$,
$\xi_{(8)} = s\frac{\partial}{\partial_s} + \frac{G}{2\beta_1}\frac{\partial}{\partial_t}$.	$I_{(8)} = -sL - \frac{G}{\beta_1}\dot{t}$.

$\xi_{(5)}$ and $\xi_{(6)}$ are the additional KVFs, $\xi_{(7)}$ is a proper Noether symmetry, while $\xi_{(8)}$ represent a Noether symmetry corresponding to a homothetic vector $\frac{G}{\beta_1}\frac{\partial}{\partial_t}$. One can easily see that these two additional KVFs and one proper homothety are same as presented in Table 5.2. To get the metric (9a) of Ref. [85] from our metric of this case, we may use $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2$.
The physical interpretation of this metric is already given in the section of HVFs.

5.3.9 Branch 12

For branch 12, we have $p_1 \neq 0$ and $p_2 = p_3 = p_7 = 0$. The simplification of these constraints gives $G = \beta_1 t + \beta_2$ and $H = \gamma$, where $\beta_1 \neq 0$ and $\gamma \neq 0$. This metric is same as the metric presented in Table 5.3, obtained during the classification via HVFs. It admits 17-dimensional Noether algebra and it was also presented in Ref. [85], see metric (40) therein. These seventeen Noether symmetries contain five minimum Noether symmetries, one proper homothety, six additional KVFs and five extra proper Noether symmetries. The proper HVF and six KVFs are same as given in Table 5.3, while the extra five proper Noether symmetries are listed below:

$$\xi_{(12)} = \frac{s^2}{2\beta^2}\frac{\partial}{\partial_s} + \frac{sG}{2\alpha_1}\frac{\partial}{\partial_t} + \frac{sy}{2}\frac{\partial}{\partial_y} + \frac{sz}{2}\frac{\partial}{\partial_z}; \; F = -\frac{t^2}{2} + \frac{\gamma^2(y^2+z^2)}{2} - \frac{\alpha_2^2 G}{\alpha_1^2},$$

$$\xi_{(13)} = \frac{s}{\gamma}\frac{\partial}{\partial_y}; \qquad F = 2\gamma y,$$

$$\xi_{(14)} = \frac{s}{\gamma}\frac{\partial}{\partial_z}; \qquad F = 2\gamma z,$$

$$\xi_{(15)} = -\frac{s}{2}\cosh(\alpha_1 x)\frac{\partial}{\partial_t} + \frac{s}{2G}\sinh(\alpha_1 x)\frac{\partial}{\partial_x}, \; F = \frac{G}{\alpha_1}\cosh(\alpha_1 x),$$

$$\xi_{(16)} = -\frac{s}{2}\sinh(\alpha_1 x)\frac{\partial}{\partial_t} + \frac{s}{2G}\cosh(\alpha_1 x)\frac{\partial}{\partial_x}, \; F = \frac{G}{\alpha_1}\sinh(\alpha_1 x).$$

The corresponding invariants are:

$$I_{(12)} = -\frac{s^2}{2}L - \frac{sG}{\alpha_1}\dot{t} + s\gamma^2(y\dot{y} + z\dot{z}) + \frac{t^2}{2} - \frac{\gamma^2(y^2+z^2)}{2} + \frac{\alpha_2^2 G}{\alpha_1^2},$$

$$I_{(13)} = 2\gamma(s\dot{y} - y),$$

$$I_{(14)} = 2\gamma(s\dot{z} - z),$$

$$I_{(15)} = s\cosh(\alpha_1 x)\dot{t} + sG\sinh(\alpha_1 x)\dot{x} - \frac{G}{\alpha_1}\cosh(\alpha_1 x),$$

$$I_{(16)} = s\sinh(\alpha_1 x)\dot{t} + sG\cosh(\alpha_1 x)\dot{x} - \frac{G}{\alpha_1}\sinh(\alpha_1 x).$$

5.3.10 Branch 13

For branch 13, the constraints are $p_1 = 0$, $p_4 \neq 0$ and $p_7 \neq 0$, that is $G' = 0$, $H' \neq 0$ and $H'^2 - HH'' \neq 0$, which give $G = \beta$, where $\beta \neq 0$. Solving the determining equations in this case leads us to two sub-cases depending upon whether $H'' \neq 0$ or $H'' = 0$. When $H'' \neq 0$, then the metric in this case admits seven Noether symmetries and H gets the value $H = (c_1 t + 2c_2)^{1-\frac{2c_3}{c_1}}$, where $c_1 \neq 0$, $c_3 \neq 0$ and $c_1 \neq 2c_3$. Out of these seven Noether symmetries, five are same as given in the set N_5, while two are the extra Noether symmetries. When $H'' = 0 \Rightarrow H = \gamma_1 t + \gamma_2$, where $\gamma_1 \neq 0$, then we get eight Noether symmetries, five are given in the minimal set, while three are the additional Noether symmetries. The extra symmetries of both cases are given in Table 5.15 along with their conserved forms.

Table 5.15: Metrics admitting seven and eight Noether Symmetries

Noether Symmetry generators	Invariants
$\xi_{(5)} = \frac{s}{2}\frac{\partial}{\partial_x}$, $\xi_{(6)} = s\frac{\partial}{\partial_s} + \frac{1}{2}(t\frac{\partial}{\partial_t} + x\frac{\partial}{\partial_x})$.	$I_{(5)} = s\beta^2\dot{x}$, $I_{(6)} = -sL - t\dot{t} + \beta^2 x\,\dot{x}$.
$\xi_{(5)} = \frac{s^2}{2}\frac{\partial}{\partial_s} + s(\frac{\gamma_1 t+\gamma_2}{2\gamma_1})\frac{\partial}{\partial_t}$ $+\frac{s}{2}x\frac{\partial}{\partial_x}$; $F = -(\frac{t^2-\beta^2x^2}{2} + t\frac{\gamma_2}{\gamma_1})$, $\xi_{(6)} = -\frac{s}{2\beta^2}\frac{\partial}{\partial_x}$; $F = -x$, $\xi_{(7)} = s\frac{\partial}{\partial_s} + (\frac{\gamma_1 t+\gamma_2}{2\gamma_1})\frac{\partial}{\partial_t}$ $+\frac{x}{2}\frac{\partial}{\partial_x}$.	$I_{(5)} = -\frac{s^2}{2}L - s(\frac{\gamma_1 t+\gamma_2}{\gamma_1})\dot{t}$ $+s\beta^2 x\,\dot{x} + \frac{t^2}{2} - \frac{\beta^2x^2}{2} + \frac{\gamma_2}{\gamma_1}$, $I_{(6)} = s\dot{x} + x$, $I_{(7)} = -sL - (\frac{\gamma_1 t+\gamma_2}{\gamma_1})\dot{t}$ $+\beta^2 x\dot{x}$.

For the first metric, $\xi_{(5)}$ is a proper Noether symmetry and $\xi_{(6)}$ is a Noether symmetry corresponding to a homothetic vector $t\frac{\partial}{\partial_t} + x\frac{\partial}{\partial_x}$. The metric in this case gives the same symmetries as the metric (8b) of Ref. [85]. For the second metric, $\xi_{(5)}$ and $\xi_{(6)}$ are proper Noether symmetries, while $\xi_{(7)}$ represents a

Noether symmetry corresponding to a homothetic vector $(\frac{\gamma_1 t+\gamma_2}{\gamma_1})\frac{\partial}{\partial_t} + x\frac{\partial}{\partial_x}$. This metric can be transformed to the metric (8a) of Ref. [85] by setting $\beta = \alpha$, $\gamma_1 = \alpha_1$ and $\gamma_2 = \alpha_2$.

The metrics of this branch are exactly same as the metrics 5a(ii) and 5b(ii), presented in Table 5.1 in the section of HVFs.

5.3.11 Branch 14

The conditions for branch 14 are $p_1 = p_4 = 0$ and $p_7 \neq 0$, which give $G = \beta$ and $H = e^{kt}$, where $\beta \neq 0$ and $k > 0$. This metric admits nine Noether symmetries in which five are given in the minimal set and four are additional Noether symmetries. In Table 5.16, we have listed these four symmetries along with their conserved forms.

Table 5.16: Metric admitting nine Noether Symmetries

Noether Symmetry generators	Invariants
$\xi_{(5)} = \frac{y}{k}\frac{\partial}{\partial_t} - yz\frac{\partial}{\partial_z} + \left(\frac{z^2-y^2}{2} - \frac{1}{2k^2e^{2kt}}\right)\frac{\partial}{\partial_y}$,	$I_{(5)} = -\frac{2y}{k}\ \dot{t} - 2yze^{2kt}\ \dot{z} + \left(e^{2kt}(z^2-y^2) - \frac{1}{k^2}\right)\ \dot{y}$,
$\xi_{(6)} = -\frac{z}{k}\frac{\partial}{\partial_t} + yz\frac{\partial}{\partial_y} + \left(\frac{z^2-y^2}{2} + \frac{1}{2k^2e^{2kt}}\right)\frac{\partial}{\partial_z}$,	$I_{(5)} = \frac{2z}{k}\ \dot{t} + 2yzH^2\ \dot{y} + \left(e^{2kt}(z^2-y^2) + \frac{1}{k^2}\right)\ \dot{z}$,
$\xi_{(7)} = -\frac{1}{k}\frac{\partial}{\partial_t} + y\frac{\partial}{\partial_y} + z\frac{\partial}{\partial_z}$,	$I_{(7)} = 2\dot{t} + 2ke^{2kt}(y\dot{y} + z\dot{z})$,
$\xi_{(8)} = \frac{s}{2\beta^2}\frac{\partial}{\partial_x}$, $F = -x$.	$I_{(8)} = s\dot{x} + x$.

Clearly, $\xi_{(5)}$, $\xi_{(6)}$ and $\xi_{(7)}$ represent additional KVFs, while $\xi_{(8)}$ is a proper Noether symmetry. This metric is same as the metric (9g) of Ref. [85] with

$\beta = \alpha$ and $k = \beta$.

Moreover, this metric is exactly same as the metric presented in Table 5.6. In both cases, we have obtained the same seven KVFs.

5.3.12 Branch 15

For branch 15, the constraints are $p_1 = 0$ and $p_7 = 0$, which means that both G and H are constant and the metric (5.1.1) reduces to the Minkowski metric, whose Noether symmetries were already calculated in the previous chapter.

5.4 Summary

In this chapter, we have given the classification of LRS Bianchi type I spacetimes via HVFs, CVFs and Noether symmetries by adopting the Rif tree approach. Instead of directly integrating the set of symmetry equations, an algorithm is developed in Maple which reduces these equations to the simplified form and yields a Rif tree and a list of pivots. Then the set of symmetry equations is solved under the conditions of each branch of the Rif tree.

Out of the 13 branches of the Rif tree for HVFs, two branches yield minimum four KVFs, while the remaining branches produce some LRS Bianchi type I metrics possessing proper HVF. In some cases, the homothety constant vanishes during calculation and HVFs become KVFs. Moreover, the Killing algebra for these spacetimes is found to be 4, 5, 6, 7 and 10-dimensional.

To investigate CVFs in these spacetimes, we have categorized our results branch-wise, due to the existence of sub-cases. Out of the 13 branches of the Rif tree for CVFs, we only considered the cases giving non-conformally spacetimes. Solving the conformal symmetry equations for each branch, we have concluded that non-conformally flat LRS Bianchi type I metrics possess at most two proper CVFs along with four minimum KVFs. In some cases, the CVFs also reduce to HVFs and KVFs, as the conformal factor in these cases becomes constant or zero.

In the section of Noether symmetries, two branches of the Rif tree give the minimal set of Noether symmetries. The remaining branches produce 6, 7, 8, 9, 10, 11 and 17 Noether symmetries. Also, we have calculated the conservation laws for all the Noether symmetry generators by using the Noether's theorem.

Comparing the obtained results with those of direct integration technique, it is observed that this new approach provides more metrics than those obtained by direct integration technique. Interestingly, most of these new metrics are physically realistic having non-negative energy density.

Chapter 6

Conclusion

In this thesis, we have investigated homothetic, conformal and Noether symmetries of Kantowski-Sachs, static plane symmetric and LRS Bianchi type I spacetimes by adopting Rif tree approach, instead of directly integrating the determining equations. For such investigations, first we have developed a Rif algorithm for reducing the system of determining equations for all the mentioned spacetimes to a simplified form. The algorithm gives all possible metrics possessing these symmetries in terms of a tree, known as Rif tree. The determining equations are solved for all these metrics and as a result, we have obtained the required symmetries of the spacetimes under consideration. We have deduced the following results.

For Kantowski-Sachs spacetime, out of 20 branches of the Rif tree for HVFs, 12 branches give the minimum four KVFs , while the remaining eight branches produce some Kantowski-Sachs metrics admitting proper HVFs. The number of KVFs turned out to be 4, 6, 7 and 10 for different values of the metric functions. Moreover, we have compared our results with those of direct integrating technique [32] and it is observed that Rif tree approach gives a new metric which was missing in Ref. [32].

In case of CVFs, Rif tree gives 18 branches but we have only considered those cases where the spacetime is non-conformally flat. Solving the conformal symmetry equations for each branch, we have concluded that non-conformally flat Kantowski-Sachs spacetimes possess at most two proper CVFs along with four minimum KVFs. In some cases, we also have six KVFs with zero conformal factor, giving no proper CVF. Comparing our results with those of Ref. [113], we have noticed that this approach gives more general metrics as compared to those obtained by direct integration technique, while some important metrics were missing in Ref. [113].

For Noether symmetries, out of 21 branches of the Rif tree, eleven give the minimal set of Noether symmetries, while the remaining branches produce 6, 7, 8, 9 and 11-dimensional Lie algebra of Noether symmetries. We have also compared obtained by Rif tree approach with the existing results of direct integrating technique [83]. One can easily see that here the obtained metrics are same as given in Ref. [83] but with more generalized form.

Similarly, for static plane symmetric spcetimes, two branches of the Rif tree for HVFs give the minimum four KVFs, while the remaining branches produce some static plan symmetric metrics possessing proper homothetic algebras of dimension 5, 7 and 11. Moreover, it is observed that these spacetimes admit Killing algebra of dimension 4, 5, 6, 7 and 10. Comparing our results with those obtained by conventional method in Ref. [119], we have observed that this new approach of Rif algorithm gives the same algebras of KVFs and HVFs with more generalized metrics than those produced by direct integration technique.

For CVFs, we have considered only those cases where the spacetime is non-conformally flat. Solving the determining equations for each branch of the Rif tree, we have concluded that non-conformally flat static plan symmetric spacetimes possess at most three proper CVFs along with four minimum KVFs. Moreover, we have compared our results with Ref. [51] where the authors claimed that non-conformal flat static plane symmetric spcetimes do not admit any proper CVFs. But in our analysis, we have found same non-conformally flat metrics admitting proper CVFs.

In section of Noether symmetries, we have found 5, 6, 7, 8, 9, 10, 11 and 17 Noether symmetries for static plan symmetric spacetimes. Comparing our results with Ref. [121], it is noticed that the present approach gives the same dimensional Noether algebras as obtained in Ref. [121] by diect integration method. However, the metric obtained in branch 11 was missing in Ref. [121].

For LRS Bianchi type I spacetime, two branches of Rif tree yield minimum four KVFs, while the remaining branches produce some LRS Bianchi type I metrics possessing proper HVF. In some cases, the homothety constant vanishes during calculation and HVFs become KVFs. Moreover, the Killing algebra for these spacetimes is found to be 4, 5, 6, 7 and 10-dimensional. Comparing our results with the existing results of Ref. [122], it is observed that the metrics which we have obtained by Rif tree approach are more generalized than those produced by conventional method. Also we get some new metrics during calculations which were not given in Ref. [122].

To investigate CVFs, we have only considered the cases giving non-conformally flat metrics. Solving the conformal symmetry equations for these metrics, we have concluded that non-conformally flat LRS Bianchi type I metrics possess at most two proper CVFs along with four minimum KVFs. We could not compare our results with the results of direct integration method because no published work about it could be found in the literature.

Finally, in the section of Noether symmetries, our analysis yield 5, 6, 7, 8, 9, 10, 11 and 17 Noether symmetries. We have also compared our results with those of Ref. [85] and it is observed that this new approach provides more metrics than those obtained by direct integration technique. Interestingly, most of these new metrics are physically realistic having non-negative energy density.

Summarizing, we can see that, Rif tree approach is a better option to find the Lie and Noether symmetries as compared to the direct integrating technique. The reason behind this is that, Rif tree approach recovers all the metrics obtained by direct integrating technique and produces those useful metrics which are not provided by direct integrating technique.

References

[1] P. Fleury, Gravitation: From Newton to Einstein, First Edition, Springer International, 2019.

[2] R. A. Serway, J. W. Jewett, Physics For Scientists and Engineers with Modern Physics, Sixth Edition, Thomson Brooks Cole, 2009.

[3] H. Stephani, D. Kramer, M. Maccallum, C. Hoenselaers and E. Herlt, Exact Solutions of Einstein's Field Equations (Second Edition), Cambridge University Press, Cambridge, 2003.

[4] C. W. Misnor, K. S. Thorne and J. A. Wheeler, Gravitation, W.H. Freeman and Company, San Francisco, 1973.

[5] A. Z. Petrov, Einstein Spaces, Pergamon Press, Oxford, 1969.

[6] N. H. Ibragimov, Elementary Lie Group Analysis and Ordinary Differential Equations, Wiely, Chichester, 1999.

[7] G. W. Bluman and S. Kumei, Symmetries and Differential Equations, Springer-Verlag, New York, 1989.

[8] S. C. Wafo, F. M. Mahomed, Linearization Criteria for a System of Second-Order Ordinary Differential Equations, *Int. J. Non-Linear Mech.* **36** (2001) 671-677.

[9] N. H. Ibragimov, S. V. Maleshko, Linearization of Third-Order Ordinary Differential Equations by Point Transformations, *J. Math. Anal. Appl.* **308** (2005) 266-289.

[10] E. Noether, Invariant Variations Problems, Nachr. Konig. Gissell. Wissen., Gottingen, *Math. -Phys. Kl.* **2** (1918) 235-257. (English Translation in Transport Theory and Statistical Physics **1** (1971)) 186-207.

[11] P. Renteln, Manifolds, Tensors and Forms An Introduction for Mathematicians and Physicists, Cambridge University Press, Cambridge, 2013.

[12] O. Gron and S. Hervi, Einsteins General Theory of Relativity, First Edition, Springer, 2010.

[13] W. R. Wald, General Relativity, The University of Chicago Press, Chicago and London, 1984.

[14] G. S. Hall, Symmetries and Curvature Structure in General Relativity, World Scientific, London, 2004.

[15] P. J. Olver, Application of Lie Groups to Differential Equations, Graduate Text in Mathematics, Second Edition, Springer-Verlag, New York, 1993.

[16] K. L. Duggal, Curvature Inheritance Symmetry in Riemannian Spaces with Applications to Fluid Spacetimes, *J. Math. Phys.* **33** (1992) 2989-2997.

[17] A. H. Bokhari and A. Qadir, Symmetries of Static Spherically Symmetric Spacetimes, *A. J. Math. Phys.* **28** (1987) 1019-1022.

[18] A. Ali, T. Hussain and J. Khan, Isometries in Bianchi Types VI and VII Spacetimes, *Adv. Stud. Theor. Phys.* **6** (2012) 607-613.

[19] A. Ali, Kamran and Z. Minullah, Classification of Spherically Symmetric Non-Static Spacetimes According to their Killing Vector Fields, *Appl. Math. sci.* **6** (2012) 2681-2686.

[20] A. H. Bokhari, M. Karim, D. N. Al-Sheikh and F. D. Zaman, Circularly Symmetric Static Metric in Three Dimensions and Its Killing Symmetry, *Int. J. Theor. Phys.* **47** (2008) 2672-2678.

[21] T. Feroze, A. Qadir and M. Ziad, The Classification of Plane Symmetric Spacetimes by Isometries *J. Math. Phys.* **42** (2001) 4947-4955.

[22] S. Khan, F. Muhammad, A. Ali and T. Hussain, A Note on Killing Symmetries of Lemaitre-Tolman-Bondi Metric, *J. Appl. Environ. Biol. Sci.* **4** (2014) 403-405.

[23] A. Qadir and M. Ziad, Classification of Static Cylindrically Symmetric Spacetimes, *Nuovo Cimento B* **110** (1995) 277-290.

[24] A. Qadir and M. Ziad, The Classification of Spherically Symmetric Spacetimes, *Nuovo Cimento B* **110** (1995) 317-334.

[25] B. J. Carr and A. A. Coley, Self-Similarity in General Relativity, *Class. Quant. Grav.* **16** (1999) 31-71.

[26] G. Shabbir and K. B. Amur, Proper Homothetic Vector Fields in Bianchi Type-I Spacetime, *App. Sci.* **8** (2006) 153-157.

[27] A. Ali, J. Khan and T. Hussain, A Note on Proper Homothetic Symmetry in Bianchi Type IV Spacetimes, *Adv. Studies Theor. Phys.* **6** (2012) 187-192.

[28] A. Ali, Z. Minullah and Kamran, Proper Homothetic Vector Field in Bianchi Type-V Spacetimes, *Adv. Studies Theor. Phys.* **6** (2012) 193-200.

[29] D. Ahmad and M. Ziad, Homothetic Motions of Spherically Symmetric Spacetimes, *J. Math. Phys.* **38** (1997) 2547-2552.

[30] G. Shabbir and M. Ramzan, Classification of Cylindrically Symmetric Static Spacetimes According to their Proper Homothetic Vector Fields, *App. Sci.* **9** (2007) 148-154.

[31] G. S. Hall and J. D. Steel, M. Ziad, Homothety Groups in Spacetimes, *Gen. Relativ. Grav.* **22** (1990) 457-468.

[32] G. Shabir and F. Iqbal, A Note on Classification of Proper Homothetic Vector Fields in Kantowski-Sachs and Bianchi Type III Lorentzian Manifolds https://arxiv.org/abs/1110.3171.

[33] A. Qadir, M. Sharif and M. Ziad, Homotheties of Cylindrically Symmetric Static Manifolds and their Global Extension, *Class. Quant. Grav.* **17** (2000) 345-349.

[34] A. Ali, M. Ramzan and S. Khan, Proper Homothetic Vector Fields on Three Dimensional Lorentzian Manifold of Circularly Symmetric Static Spacetimes, *Sindh Univ. Res. Jour. (Sci. Ser.)* **47** (2015) 739-742.

[35] M. Ziad, The Classification of Static Plane Symmetric Spacetimes, *Nuovo Cimento B* **114** (1999) 683-692.

[36] K. L. Duggal and R. Sharma, Symmetries of Spacetimes and Riemannian Manifolds, First Edition, Springer US, 1999.

[37] D. Kramer, J. Carot, Conformal Symmetry of Perfect Fluids in General Relativity, *J. Math. Phys.* **32** (1991) 1857-1860 .

[38] T. Chrobok, H.H. Borzeszkowski, Thermodinamical Equilibrium and Spacetime Geometry, *Gen. Relativ. Grav.* **38** (2006) 397-415.

[39] C.G. Bohmer, T. Harko, F.S.N. Lobo, Wormhole Geometries with Conformal Motions, *Class. Quant. Grav.* **25** (2008) 075016 (1-11).

[40] M.K. Mak, T. Harko, Quark Stars Admitting a One-Parameter Group of Conformal Motions, *Int. J. Mod. Phys. D* **13** (2004) 149-156.

[41] A. A. Usmani, F. Rahaman, S. Ray, K. K. Nandi, P. K. F. Kuhfittig, S. A. Rakib and Z. Hasan, Charged Gravastars Admitting Conformal Motion, *Phys. Lett. B* **701** (2011) 388-392.

[42] A. A. Coley and B. O. J. Tupper, Affine Conformal Vectors in Space-time, *J. Math. Phys.* **33** (1992) 1754-1764.

[43] R. Maartens, D. P. Mason, and M. Tsamparlis, Kinematic and Dynamic Properties of Conformal Killing Vectors in Anisotropic Fluids, *J. Math. Phys.* **27** (1986) 2987-2994.

[44] R. Marteens, S. D. Maharaj, and B. O. J. Tupper, Conformal Motions in Static Spherical Spacetimes, *Class. Quant. Grav.* **13** (1996) 317-318.

[45] M. Tsamparlis, A. Paliathanasis, and L. Karpathopoulos, Exact solutions of Bianchi I Spacetimes which Admit Conformal Killing Vectors, *Gen. Relativ. Grav.* **47** (2015) 15 (1-21).

[46] R. Maartens and S. D. Maharaj, Conformal Killing Vectors in Robertson-Walker Spacetimes, *Class. Quant. Grav.* **3** (1986) 1005-1011.

[47] G. S. Hall, J. D. Steele, Conformal Vector Fields in General Relativity, *J. Math. Phys.* **32** (1991) 1847-1853.

[48] S. Khan, T. Hussain, A. H. Bokhari and G. A. Khan, Conformal Killing Vectors of Plane Symmetric Four Dimensional Lorentzian Manifolds, *Eur. Phys. J. C* **75** (2015) 523 (1-9).

[49] S. Khan, T. Hussain, A. H. Bokhari and G. A. Khan, Conformal Killing Vectors in LRS Bianchi Type V Spacetimes, *Commun. Theor. Phys.* **65** (2016) 315-320.

[50] G. S. Hall and M. S. Capocci, Classification and Conformal Symmetry in Three Dimensional Spacetimes*J. Math. Phys.* **40** (1999) 1466-1478.

[51] K. Saifullah and Shair-E-Yazdan. Conformal Motions in Plane Symmetric Static Spacetimes, *Int. J. Mod. Phys. D* **18** (2009) 71-81.

[52] T. Hussain, S. Khan, A. H. Bokhari and G. A. Khan, Proper Conformal Killing Vectors in Static Plane Symmetric Spacetimes, *Theor. Math. Phys.* **191** (2017) 620-629.

[53] A. A. Coley and B. O. Tupper, Spherically Symmetric Spacetimes Admitting Inheriting Conformal Killing Vector Fields, *J. Class. Quant. Grav.* **7** (1990) 2195-2214.

[54] A. A. Coley and B. O. Tupper, Spherically Symmetric Anisotropic Fluid ICKV Spacetimes, *J. Class. Quant. Grav.* **11** (1994) 2553-2574.

[55] A. A. Coley and S. R. Czapor, Plane Symmetric Spacetimes Admitting Inheriting Conformal Killing Vector Fields, *Class. Quant. Grav.* **9** (1992) 1787-1811.

[56] G. Shabbir and A. Ali, Proper Conformal Motions in Bianchi Type II Spacetimes, *U.P.B. Sci. Bull. Series D* **73** (2011) 185-189.

[57] G. Shabbir and A. Ali, A Note on Proper Conformal Symmetry in Bianchi Types VIII and IX Spacetimes, *Adv. Stud. Theor. Phys.* **3** (2009) 93-97.

[58] G. Shabbir and A. Ali, Classiffication of Spatially Homogeneous Rotating Spacetimes According to their Conformal Vector Fields, *Appl. Math. Sci.* **3** (2009) 869-876.

[59] J. C. Amenedo and A. A. Coley, Exact Solutions with Conformal Killing Vector Fields, *Class. Quant. Grav.* **9** (1992) 2203-2215.

[60] A. Chudecki and M. Dobrski, Proper Conformal Symmetries in Self-Dual Einstein Spaces, *J. Math. Phys.* **55** (2014) 082502 (1-12).

[61] R. Marteens, S. D. Maharaj and B. O. J. Tupper, General Solution and Classification of Conformal Motions in Static Spherical Spacetimes, *Class. Quant. Grav.* **12** (1995) 2577-2586.

[62] B. J. Cantwell, An introduction to Symmetry Analysis, Cambridge University Press, 2002.

[63] H. Stephani, Differential Equations, their Solutions Using Symmetries, Cambridge University Press, 1989.

[64] N. H. Ibragimov, Elementary Lie Group Analysis and Ordinary Differential Equations, Wiley, New York, 1999.

[65] N. H. Ibragimov (Ed.), CRC Handbook of Lie Group Analysis of Differential Equations, Symmetries, Exact Solutions and Conservation Laws, First Edition, CRC Press, Boca Raton, 1993.

[66] N. H. Ibragimov (Ed.), CRC Handbook of Lie Group Analysis of Differential Equations, New Trends in Theoretical Developments and Computational Methods, Third Edition, CRC Press, 1995.

[67] N. H. Ibragimov, Slected Works, Volume 3, ALGA Publications, 2008.

[68] G. W. Bluman, A. F. Cheviakov and S. C. Anco, Applications of Symmetry Methods to Partial Differential Equations, Springer Science, New York, 2010.

[69] Y. Kosmann-Schwarzbach, The Noether Theorems: Invariance and Conservation Laws in the Twentieth Century, Translated by Bertram E. Schwarzbach, (Sources and Studies in the History of Mathematics and Physical Sciences), Springer, 2010.

[70] N. H. Ibragimov,A. H. Kara, F. M. Mahomed, Lie Backlund and Noether Symmetries with Applications, Non-Linear Dynamics **15** (1998) 115-136.

[71] M. Hickman and S. Yazdan, Noether Symmetries of Bianchi type II Spacetimes, *Gen. Relativ. Grav.* **49** (2017) 1-15.

[72] F. Ali, T. Feroze and S. Ali, Complete Classification of Spherically Symmetric Static Spacetimes via Noether Symmetries, *Theor. Math. Phys.* **184** (2015) 973-985.

[73] F. Ali and T. Feroze, Complete Classification of Cylindrically Symmetric Static Spacetimes and the Corresponding Conservation Laws, *Sigma Mathematics* **4** (2016) 50 (1-15).

[74] F. Ali and T. Feroze, Classification of Plane Symmetric Static Spacetimes According to their Noether Symmetries, *Int. J. Theor. Phys.* **52** (2013) 3329-3342.

[75] S. Ali and I. Hussain, A Study of Positive Condition in Bianchi V Spacetimes via Noether Symmetries, *IEur. Phys. J. C* **76** (2016) 63 (1-15).

[76] A. H. Bokhari and A. H. Kara, Noether Versus Killing Symmetry of Conformally Flat Friedmann Metric, *Gen. Relativ. Grav.* **39** (2007) 2053-2059.

[77] A. H. Bokhari, A. H. Kara, A. R. Kashif and F. D. Zaman, Noether Symmetries Versus Killing Vectors and Isometries of Spacetimes, *Int. J. Theor. Phys.* **45** (2006) 1029-1039.

[78] U. Camci, Symmetries of Geodesic Motion in Gödel-Type Spacetimes, *J. Cosmol. Astropart. Phys.* **7** (2014) 1-20.

[79] U. Camci and A. Yildirim, Lie and Noether Symmetries in Some Classes of PP-Wave Spacetimes, *Phys. Scr.* **89** (2014) 084003 (1-30).

[80] A. Paliathanasis, M. Tsamparlis and M. T. Mustafa, Symmetry Analysis of the Klein-Gordon Equation in Bianchi I Spacetimes, *Int. J. Geom. Meth. Mod. Phys.* **12** (2015) 1550033 (1-20).

[81] S. A. Usamah, A. H. Bokhari, A. H. Kara and G. Shabbir, Classification of Variational Conservation Laws of General Plane Symmetric Spacetimes, *Commun. Theor. Phys.* **68** (2017) 335 (1-11).

[82] B. Jamil, T. Feroze and A. Vargas, Geometrical/Physical Interpretation of the Conserved Quantities Corresponding to Noether Symmetries of Plane Symmetric Spacetimes, *Adv. Math. Phys.* **2017** (2017) 4384093 (1-40).

[83] S. S. Akhtar, T. Hussain and A. H. Bokhari, Positive Energy Condition and Conservation Laws in Kantowski-Sachs Spacetime via Noether Symmetries, *Symmetry* **10** (2018) 712 (1-12).

[84] T. Hussain, S. S. Akhtar and A. H. Bokhari, A Study of Energy Conditions in Non-Static Spherically Symmetric Spacetimes via Noether Symmetries, *J. Math. Anal. App.* **483** (2020) 123574 (1-16).

[85] S. S. Akhtar and T. Hussain, Energy Conditions and Conservation Laws in LRS Bianchi Type I Spacetimes via Noether Symmetries, *Commun. Theor. Phys.* **71** (2019) 298-306.

[86] S. S. Akhtar and T. Hussain, A Study of Noether Symmetries in LRS Bianchi Type V Spacetimes, *Mod. Phys. Lett. A* **35** (2019) 2050026 (1-15).

[87] U. Camci, Approximate Noether Gauge Symmetries of the Bardeen Model, *Eur. Phys. J. C* **74** (2014) 3201 (1-8).

[88] F. Ali and T. Feroze, Approximate Noether Symmetries from Geodetic Lagrangian for Plane Smmetric Spacetimes, *Int. J. Geo. Meth. Mod. Phys.* **12** (2015) 1550124 (1-9).

[89] I. Hussain, F. M. Mahomed and A. Qadir, Approximate Noether Symmetries of the Geodesic Equations for the Charged-Kerr Spacetime and Rescaling of Energy, *Gen. Relativ. Grav.* **41** (2009) 2399-2414.

[90] I. Hussain, F. M. Mahomed and A. Qadir, Approximate Partial Noether Operators of the Schwarzschild Spacetime, *J. Non-Linear Math. Phy.* **17** (2010) 13-25.

[91] B. Jamil and T. Feroze, Conservation Laws Corresponding to the Noether Symmetries of the Geodetic Lagrangian in Spherically Symmetric Spacetimes, *Int. J. Mod. Phys. D* **26** (2017) 1741006 (1-17).

[92] L. Karpathopoulos, A. Paliathanasis and M. Tsamparlis, Lie and Noether Point Symmetries for a Class of Non-Autonomous Dynamical Systems, *J. Math. Phys.* **58** (2017) 082901 (1-14).

[93] N. Dimakis, A. Giacomini, S. Jamal, G. Leon and A. Paliathanasis, Noether Symmetries and Stability of Ideal Gas Solution in Galileon Cosmology, *Phys. Rev. D* **95** (2017) 064031 (1-14).

[94] A. K. Halder, A. Paliathanasis and P. G. L. Leach, Noether's Theorem and Symmetry, *Symmetry* **10** (2018) 744 (1-21).

[95] K. F. Dialektopoulos and S. Capozziello, Noether Symmetries as a Geometric Criterion to Select Theories of Gravity, *Int. J. Geom. Meth. Mod. Phys.* **15** (2018) 1840007 (1-16).

[96] T. Hussain and S. S. Akhtar, Energy Conditions and Conservation Laws in LTB Metric via Noether Symmetries, *Eur. Phys. J. C* **78** (2018) 677 (1-9).

[97] A. A. Coley and B. 0. J. Tupper, Spherically Symmetric Anisotropic Fluid ICKV Spacetimes, *Class. Quant. Grav.* **11** (1994) 2553-2574.

[98] I. M. Yalglom, Felix Klein and Sophus Lie, Evolution of the Idea of Symmetry in the Nineteenth Century, Birkhäuser, Boston, 1988.

[99] G. Bluman and S. Kumei, Symmetries and Differential Equations, Springer-Verlag, New York, 1989.

[100] L. V. Ovsiannikov, Group Analysis of Differential Equations, Academic Press, New York, 1982.

[101] A. F. Cheviakov, GeM Software Package for Computation of Symmetries and Conservation Laws of Differential Equations, *Comp. Phys. Comm.* **176** (2007) 48-61.

[102] W. Hereman, Review of Symbolic Software for the Computation of Lie Symmetries of Differential Equations, *Euromath Bulletin* **1** (1994) 45-79.

[103] J. Sherring, A. K. Head and G. E. Prince, Dimsym and LIE:Symmetry Determination Packages, *Math. Comput. Model.* **25** (1997) 153-164.

[104] F. Boulier, D. Lazard, F. Olllivier and M. Petitot, Representation for the Radical of a Finitely Generated Differential Ideal, in Proc. ISSAC 95, New York, ACM Press, 1995.

[105] E. Mansfield, Differential Gröbner Bases, PhD Thesis, University of Sydney, 1991.

[106] G. Reid, A. Wittkopf and A. Boulton, Reduction of Systems of Nonlinear Partial Differential Equations to Simplified Involutive Forms, *Euro. J. Appl. Maths.* **7** (1996) 604-635.

[107] Waterloo Maple Software, Maple 13, Software package, 2009.

[108] C. Rust, Rankings of Derivatives for Elimination Algorithms and Formal Solvability of Analytic Partial Differential Equations, PhD thesis, University of Chicago, 1998.

[109] A. Wittkopf, Algorithms and Implementations for Differential Elimination, PhD thesis, Department of Mathematics, Simon Fraser University, Canada, 2004.

[110] T. Hussain, U. Nasib and M. Farhan, A Study of Energy Conditions in Kantowski-Sachs Spacetimes via Homothetic Vector Fields, *Int. J. Geom. Meth. Mod. Phys.* **17** (2020) 2050035 (1-15).

[111] R. Kantowski and R. K. Sachs, Some Spatially Homogeneous Anisotropic Relativistic Cosmological Models *J. Math. Phys.* **7** (1996) 443-446.

[112] T. Hussain, U. Nasib, F. Khan and M. Farhan, An Efficient Rif Algorithm for the Classification of Kantowski-Sachs Spacetimes via Conformal Vector Fields, *J. Kor. Phys. Soc.* **76** (2020) 286-291.

[113] T. Hussain and M. Farhan, Proper Conformal Killing Vectors in Kantowski-Sachs Metric, *Commun. Theor. Phys.* **69** (2018) 393-398.

[114] G. G. L. Nashed, Stability of Reissner Nordstrom Black Hole, *Acta Pol. Phys. A* **112** (2007) 13-21.

[115] R. dInverno, Introducing Einsteins Relativity, Oxford University Press, New York, 1992.

[116] G. G. L. Nashed, Stability of the Vacuum Non-Singular Black Hol, *Chaos, Solit. Frac.*, **15** (2003) 841-847.

[117] M. I. Wanas and M. A. Bakry, Stability of Cosmological Models, *Astro. Space Sci.* **228** (1995) 239-253.

[118] T. Hussain, U. Nasib and M. Farhan, A Study of Energy Conditions in Static Plane Symmetric Spacetimes via Homothetic Symmetries, *Int. J. Mod. Phys. A* **33** (2020) 1950238 (1-15).

[119] M. Ziad, The classification of Static Plane Symmetric Spacetimes, *Il Nuovo cimento* **114** (1999) 683-692.

[120] U. Nasib, T. Hussain and A. H. Bokhari, Homothetic Vector Fields of LRS Bianchi Type I Spacetimes via the Rif Tree Approach, *Theor. Math. Phys.* **209** (2021) 1673-1682.

[121] F. Ali and T. Feroze, Classification of Plane Symmetric Static Spacetimes According to their Noether Symmetries, *Int. J. Theor. Phys.* **52** (2013) 3329-3342.

[122] G. Shabir and KB. Amur Proper Homothetic Vector Fields in Bianchi type I Spacetime, *Appl Sci* **8** (2006) 153-157.

[123] D. McMahon, Relativity Demystifiled, McGraw Hill, New York, 2006.

[124] D. Fleisch, A Students Guide to Vectors and Tensors, Cambridge University Press, New york, 2012.

[125] G. Baumann, Symmetry Analysis of Differential Equations with Mathematica, Springer-Verlag, 2000.

[126] N. H. Ibragimov, Elementary Lie Group Analysis and Ordinary Differential Equations, Wiely, Chichester, 1999.

[127] I.M. Yaglom, Felix Klein and Sophus Lie Evolution of the Idea of Symmetry in the Nineteenth Century, Academy of Pedagogical Sciences NII SHOTSO Boston, Basel, Birkhiiuser, 1988.