SOLUTIONS OF THE YANG-MILLS EQUATIONS AND A CLIFFORD ALGEBRAS

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Abstract. We consider the Yang-Mills and the Klein-Gordon equations in the external Yang-Mills fields in the spaces $\mathbb{R}^n$. Using the generators of the Clifford algebra, we constract the ansätze for the Yang-Mills potentials and for the scalar field. New classes of solutions of the Klein-Gordon and Yang-Mills equations in the spaces $\mathbb{R}^n$ with $n \geq 4$ are described.

1. Introduction

We will show that the Clifford algebras may be used in constructing the solutions of the Yang-Mills (YM) equations in $\mathbb{R}^n$. Our goal is to find some solutions of the equations for a pure classical YM theory in the Euclidean space $\mathbb{R}^n$ with the metric δ_{ab}, $a, b, ... = 1, \dots, n$. Let A_a be the YM potentials with values in the semisimple Lie algebra $\mathcal{G}$ of the Lie group G and $F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]$ be the curvature tensor for A_a.

The YM equations for the gauge potentials A_a have the form

$$\partial_a F_{ab} + [A_a, F_{ab}] = 0. \tag{1.1}$$

The Einstein summation convention is used throughout, if not stated otherwise.

Some solutions of Eqs.(1.1) in the spaces $\mathbb{R}^7, \mathbb{R}^8$ and $\mathbb{R}^{4k}$ were obtained in [1, 2, 3, 4, 5, 6] (see also [7]). In what follows we shall show that it is possible to obtain other classes of solutions of the YM equations in the spaces of dimension $n \geq 4$ using the properties of Clifford algebras.

2. Ansatz for Gauge Potentials

Let us suppose that in the space $\mathbb{R}^n$ with metric δ_{ab} there are q constant tensors $J^1_{ab}, \dots, J^q_{ab}$ that are antisymmetric in indices a and b and obey the relations

$$J^\alpha_{ac} J^\beta_{bc} = \delta^{\alpha\beta}\delta_{ab} + \Sigma^{\alpha\beta}_{ab}, \tag{2.1}$$

where $\Sigma^{\alpha\beta}_{ab}$ are some constant antisymmetric in a and b tensors, $\alpha,\beta,... = 1,...,q$. From (2.1) it follows that

$$J^{\alpha}_{ac}J^{\beta}_{cb} + J^{\beta}_{ac}J^{\alpha}_{cb} = -2\delta^{\alpha\beta}\delta_{ab},$$

i.e., $J^{\alpha} = (J^{\alpha}_{ab})$ give a real matrix representation for the generators J^{α} of the Clifford algebra for the space $I\!R^q$ with the metric $g_{\alpha\beta} = -\delta_{\alpha\beta}$.

We shall look for solutions of the YM equations (1.1) in the form

$$A_a = -J^{\alpha}_{ac}T_{\alpha}(\varphi)\partial_c\varphi, \tag{2.2}$$

where the real antisymmetric tensors J^{α}_{ab} satisfy (2.1); φ is an arbitrary function of coordinates $x_a \in I\!R^n$; $T_1,...,T_q$ depend only on φ, take values in the Lie algebra $\mathcal{G}$ and satisfy the Rouhani-Ward (RW) equations (see [4, 5, 6, 7, 8, 9]):

$$f_{\alpha\beta\gamma}\dot{T}_{\gamma} + [T_{\alpha}, T_{\beta}] = 0. \tag{2.3}$$

Here $f_{\alpha\beta\gamma}$ is some totally antisymmetric three-index tensor in $I\!R^q$ satisfying $f_{\alpha\gamma\delta}f_{\beta\gamma\delta} = 2\delta_{\alpha\beta}$ and $\dot{T}_{\gamma} \equiv dT_{\gamma}/d\varphi$. If q coincides with the dimension of the simple compact Lie algebra $\mathcal{H}$, then as $f_{\alpha\beta\gamma}$ one may take the structure constants of $\mathcal{H}$.

It may be shown that after substituting (2.2) into (1.1) and using the identities (2.1), the YM equations are reduced to the following system of linear equations:

$$f^{\alpha}_{\beta\gamma}\Sigma^{\beta\gamma}_{ac}\partial_c\partial_b\varphi - 2J^{\alpha}_{bc}\partial_c\partial_a\varphi + 2f^{\alpha}_{\beta\gamma}J^{\beta}_{ac}J^{\gamma}_{be}\partial_c\partial_e\varphi + J^{\alpha}_{ab}\Box\varphi = 0, \tag{2.4}$$

where $\Box \equiv \partial_c\partial_c$.

PROPOSITION: If tensors J^{α}_{ab} satisfy the relations (2.1) and $q = \dim\mathcal{H}$, then to each solution of system $\{(2.3),(2.4)\}$ one may correspond the solution (2.2) of the YM equations (1.1) for gauge fields A_a of an arbitrary semisimple Lie group G in the Euclidean space $I\!R^n$.

3. Explicit Form of Tensors J^{α}_{ab}

To find solutions of Eqs. (2.4), one should give the concrete expressions to the tensors J^{α}_{ab} and $\Sigma^{\alpha\beta}_{ab}$. The theory of Clifford algebras gives the examples of such tensors.

Let us denote by $Cl(0,q)$ the Clifford algebra for the space $I\!R^q$ with the metric $g_{\alpha\beta} = -\delta_{\alpha\beta}$, $\alpha,\beta,\ldots = 1,\ldots,q$. It has been known for a long time that the algebra $Cl(0,q)$ can be realized in terms of matrices. In particular, $Cl(0,6) \cong M(8,I\!R)$ and $Cl(0,8) \cong M(16,I\!R)$(see, e.g., [10]), where through

$M(s,\mathbb{R})$ the full $s \times s$ matrix algebra over $\mathbb{R}$ is denoted. Let us give some examples of tensors J^{α}_{ab} .

Example 1: Consider the algebra $Cl(0,2)$ with generators γ^1 and γ^2. It is well-known [10] that $Cl(0,2)$ is isomorphic to the algebra of quaternions **H**, and elements $\gamma^1, \gamma^2, \gamma^3 \equiv \gamma^1\gamma^2$ can be realized in terms of real antisymmetric 4×4 matrices η^1, η^2, η^3 with components: $\eta^{\alpha}_{\beta\gamma} = \epsilon^{\alpha}_{\beta\gamma}$, $\eta^{\alpha}_{\mu 4} = -\eta^{\alpha}_{4\mu} = \delta^{\alpha}_{\mu}$, where $\epsilon_{\alpha\beta\gamma}$ are structure constants of $SU(2)$, $\alpha, \beta, \gamma, \delta = 1,2,3$; $\mu, \nu, \ldots = 1,\ldots,4$. Tensors $\eta^1_{\mu\nu}, \eta^2_{\mu\nu}$ and $\eta^3_{\mu\nu}$ coincide with the well-known 't Hooft tensors that obey the relations (2.1) with $\Sigma^{\alpha\beta}_{\mu\nu} = \epsilon^{\alpha\beta\gamma}\eta^{\gamma}_{\mu\nu}$.

Now, let us introduce the tensors

$$J^{\alpha}_{(\mu i)(\nu j)} = \delta_{ij}\eta^{\alpha}_{\mu\nu} \tag{3.1}$$

with the double indices $(\mu i), (\nu j), \ldots$, where $i, j, \ldots = 1, \ldots, p$. If we denote the double indices by $a, b, \ldots = 1, \ldots, 4p$, then it is not difficult to verify that the tensors J^{α}_{ab} will satisfy the relations (2.1) with $\Sigma^{\alpha\beta}_{ab} = \epsilon^{\alpha\beta\gamma}J^{\gamma}_{ab}$. Thus, in the spaces $\mathbb{R}^{4p}$ one may always introduce three tensors J^{α}_{ab} satisfying (2.1).

Example 2: Let us consider the algebra $Cl(0,6)$ with generators $\gamma^1, \ldots, \gamma^6$ and also introduce $\gamma^7 \equiv \gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6$. It is known [10] that γ^{α} ($\alpha = 1, \ldots, 7$) can be realized in terms of real antisymmetric 8×8 matrices. The components $\gamma^{\alpha}_{\mu\nu}$ ($\mu, \nu, \ldots = 1, \ldots, 8$) of these matrices satisfy the relations (2.1) with $\Sigma^{\alpha\beta}_{\mu\nu} = \frac{1}{2}\gamma^{[\alpha}_{\mu\lambda}\gamma^{\beta]}_{\nu\lambda} \equiv \frac{1}{2}(\gamma^{\alpha}_{\mu\lambda}\gamma^{\beta}_{\nu\lambda} - \gamma^{\beta}_{\mu\lambda}\gamma^{\alpha}_{\nu\lambda})$.

Now we introduce the tensors

$$J^{\alpha}_{(\mu i)(\nu j)} = \delta_{ij}\gamma^{\alpha}_{\mu\nu}, \tag{3.2}$$

where $\mu, \nu, \ldots = 1, \ldots, 8$; $i, j, \ldots = 1, \ldots, p$. Numbering the components of these tensors by the indices $a, b, \ldots = 1, \ldots, 8p$, in the space $\mathbb{R}^{8p}$ we obtain seven tensors J^{α}_{ab} satisfying (2.1) with $\Sigma^{\alpha\beta}_{ab} = \frac{1}{2}J^{[\alpha}_{ac}J^{\beta]}_{bc}$. It is clear that for ansatz (2.2) one can choose not all seven tensors but only q of them with $4 \le q \le 7$.

Example 3: Let us consider the algebra $Cl(0,8)$ with generators γ^{α}, $\alpha, \beta, \ldots = 1, \ldots, 8$. It is known [10] that γ^{α} can be realized in terms of real antisymmetric 16×16 matrices. The components $\gamma^{\alpha}_{\mu\nu}$ ($\mu, \nu, \ldots = 1, \ldots, 16$) of these matrices satisfy (3.1) with $\Sigma^{\alpha\beta}_{\mu\nu} = \frac{1}{2}\gamma^{[\alpha}_{\mu\lambda}\gamma^{\beta]}_{\nu\lambda}$. Let us also introduce the tensors $J^{\alpha}_{(\mu i)(\nu j)}$ defined by (3.2) but with $\mu, \nu, \ldots = 1, \ldots, 16$; $i, j, \ldots = 1, \ldots, p$. Numbering the components of these tensors by the indices $a, b, \ldots = 1, \ldots, 16p$, we obtain eight tensors J^{α}_{ab}. In the space $\mathbb{R}^{16p}$ all these tensors satisfy the relations (2.1) with $\Sigma^{\alpha\beta}_{ab} = \frac{1}{2}J^{[\alpha}_{ac}J^{\beta]}_{bc}$ and can be used in constructing of the ansatz (2.2).

And finally, we point out that in the spaces $\mathbb{R}^n$ one may introduce q tensors J^{α}_{ab} satisfying (2.1) in the following cases:

$$n = p2^{2+4m} \Rightarrow 1 + 8m \leq q \leq 3 + 8m, \tag{3.3a}$$

$$n = p2^{3+4m} \Rightarrow 4 + 8m \leq q \leq 7 + 8m, \tag{3.3b}$$

$$n = p2^{4+4m} \Rightarrow \qquad q = 8 + 8m, \tag{3.3c}$$

where $m = 0, 1, 2, \ldots$; $p = 1, 2, \ldots$. Proof may be obtained with the help of formula [10]:

$$Cl(0, s + 8m) = Cl(0, s) \otimes Cl(0, 8) \underbrace{\otimes \ldots \otimes}_{m\ times} Cl(0, 8), \tag{3.4}$$

where $1 \leq s \leq 8$. Using the recurrence relations given in [10], one can easily obtain the explicit form of tensors $J^1_{ab}, \ldots, J^q_{ab}$ in the spaces of dimension n indicated in (3.3).

4. Constructing of Solutions for the Scalar Field Equations

Substituting the explicit form of J^α_{ab} into Eqs.(2.4), one may try to solve (2.4). Solutions exist. Rather then make an exhaustive study of all the possibilities we shall restrict ourselves to the case of $n = 4p$ and $q = 3$.

So, let us substitute (3.1) into Eqs.(2.4) where $\epsilon_{\alpha\beta\gamma}$ are taken instead of $f_{\alpha\beta\gamma}$ and $\Sigma^{\alpha\beta}_{ab} = \epsilon^{\alpha\beta\gamma} J^\gamma_{ab}$. We use the following identities for $\eta^\alpha_{\mu\nu}$ [11]:

$$\eta^\alpha_{\mu\lambda}\eta^\beta_{\nu\lambda} = \delta^{\alpha\beta}\delta_{\mu\nu} + \epsilon^{\alpha\beta\gamma}\eta^\gamma_{\mu\nu}, \tag{4.1a}$$

$$\epsilon^\alpha_{\beta\gamma}\eta^\beta_{\mu\lambda}\eta^\gamma_{\nu\sigma} = \delta_{\mu\nu}\eta^\alpha_{\lambda\sigma} - \delta_{\mu\sigma}\eta^\alpha_{\lambda\nu} - \delta_{\lambda\nu}\eta^\alpha_{\mu\sigma} + \delta_{\lambda\sigma}\eta^\alpha_{\mu\nu}, \tag{4.1b}$$

and obtain the equations:

$$\begin{aligned} 2\eta^\alpha_{\mu\lambda}(\partial_{\lambda i}\partial_{\nu j}\varphi - \partial_{\lambda j}\partial_{\nu i}\varphi) - 2\eta^\alpha_{\nu\lambda}(\partial_{\lambda j}\partial_{\mu i}\varphi - \partial_{\lambda i}\partial_{\mu j}\varphi) \\ +\delta_{\mu\nu}\eta^\alpha_{\lambda\sigma}(\partial_{\lambda i}\partial_{\sigma j}\varphi - \partial_{\lambda j}\partial_{\sigma i}\varphi) + \\ +\eta^\alpha_{\mu\nu}(2\partial_{\lambda i}\partial_{\lambda j}\varphi + \delta_{ij}\Box\varphi) = 0, \end{aligned} \tag{4.2}$$

where $\partial_{\lambda i} \equiv \partial/\partial x^{\lambda i}$. It is clear that Eqs.(4.2) are satisfied if φ obeys the following equations

$$\partial_{\mu i}\partial_{\nu j}\varphi = \partial_{\mu j}\partial_{\nu i}\varphi, \quad \partial_{\lambda i}\partial_{\lambda j}\varphi = 0, \tag{4.3}$$

where $\mu, \nu, \ldots = 1, \ldots, 4$; $i, j, \ldots = 1, \ldots, p$. Equations (4.3) are simpler then Eqs.(2.4) and appear in study of the hyper-Kähler manifolds of dimension $4p$ (see [12]). In principle, for Eqs.(4.3) one may write a general solution (see [12]), but we shall not do this here. As an example, we write out one of the particular solutions of Eqs.(4.3) (and Eqs.(4.2)):

$$\varphi = 1 + \sum_{I=1}^{N} \frac{B_I^2}{(X_\mu - C_\mu^I)(X_\mu - C_\mu^I)}, \tag{4.4}$$

where $X_\mu = x_{\mu i} p_i$, p_i=const, N is any integer number, B_I and C_μ^I are arbitrary constants. For a special case of the space $\mathbb{R}^8$ and group $G = SU(2)$ the solution of this type was obtained by Ward [1].

Equations (2.3) with $q = 3$ and $\mathcal{H} = su(2)$ coincide with the well-known Nahm equations (see [8, 9] and [13, 14]). These equations appeared in constructing the solutions of the YM equations in $\mathbb{R}^4$ [14, 15, 16] and of the model of chiral fields in $\mathbb{R}^2$ [17]. Nahm's equations have a Lax-type representation with a spectral parameter, and in terms of theta functions one can write a general solution of Nahm's equations for any semisimple Lie algebra $\mathcal{G}$ (see [13] and [9]). The explicit form of particular solutions of Nahm's equations may be found in [15] and [16]. We shall not write it here.

5. Solutions of the Massless Klein-Gordon Equation

In $\mathbb{R}^n$ let us consider the massless scalar field χ with values in the adjoint representation of the Lie algebra $\mathcal{G}$. The Klein-Gordon equation for χ in the external field A_a has the form

$$\Big(\partial_a + [A_a, \;]\Big)\Big(\partial_a + [A_a, \;]\Big)\chi = 0, \tag{5.1}$$

where $a, \ldots = 1, \ldots, n$.

Now, substitute the ansatz (2.2) for A_a into (5.1). Suppose that $T_\alpha(\varphi)$ and φ obey the equations (2.3) and (2.4). For χ let us consider the following ansatz:

$$\chi = \chi_\alpha T_\alpha(\varphi), \qquad \chi_\alpha = const. \tag{5.2}$$

In this case, the Klein-Gordon equation (5.1) is reduced to the following equation:

$$\chi_\alpha \dot{T}_\alpha \Box\varphi + \chi_\alpha \partial_c\varphi\partial_c\varphi\Big\{\ddot{T}_\alpha - [T_\beta, [T_\alpha, T_\beta]]\Big\} = 0. \tag{5.3}$$

Here we have used the identities (2.1); $\ddot{T}_\alpha \equiv d^2T_\alpha/d\varphi^2$.

Thus, if $T_\alpha(\varphi)$ satisfy the equations

$$\ddot{T}_\alpha - \Big[T_\beta, [T_\alpha, T_\beta]\Big] = 0. \tag{5.4}$$

and φ satisfies the Laplace equation

$$\Box\varphi = 0, \tag{5.5}$$

then the ansatz (5.2) gives the solution of the massless Klein-Gordon equation (5.1).

It is easy to see that each solution of the RW equations (2.3) satisfies Eqs. (5.4). Indeed, if one multiplies Eqs. (5.4) by $f_{\alpha\beta\delta}$ and differentiates these equations once more, then obtains

$$\ddot{T}_\alpha = -f_{\alpha\beta\gamma}[T_\beta, \dot{T}_\gamma].$$

At the same time, from Eqs.(2.3) it follows that

$$\Big[T_\beta, [T_\alpha, T_\beta]\Big] = -f_{\alpha\beta\gamma}[T_\beta, \dot{T}_\gamma].$$

Therefore, if T_α satisfy Eqs.(2.3), then T_α satisfy Eqs.(5.4). Remind that the function φ must satisfy Eqs.(2.4). Comparing Eqs.(2.4) with Eq.(5.5), we obtain the following system of equations:

$$f^\alpha_{\beta\gamma}\Sigma^{\beta\gamma}_{ac}\partial_c\partial_b\varphi - 2J^\alpha_{bc}\partial_c\partial_a\varphi + 2f^\alpha_{\beta\gamma}J^\beta_{ac}J^\gamma_{be}\partial_c\partial_e\varphi = 0, \tag{5.6a}$$

$$\Box\varphi = 0. \tag{5.6b}$$

Equations (5.6) have solutions. Some of them have been written out in Section IV (see also [6, 7]).

6. Conclusion

An example for $n = 4p$ and $q = 3$ shows that Eqs.(2.4) may have not only solution linear on coordinates x^a , but also more complicated solutions. It is interesting to study Eqs. (2.4) in the spaces $\mathbb{R}^n$ with q tensors J^α_{ab} and $n > 4p$ from (3.3) in the case when q coincides with the dimension of some simple Lie algebra $\mathcal{H}$. In this case, as $f_{\alpha\beta\gamma}$ in Eqs.(2.3) one may take structure constants of $\mathcal{H}$.

We have considered the case of Example 1 when $n = 4p$, $q = 3$ and $\mathcal{H} = su(2)$. If one takes eight tensors J^α_{ab} in $\mathbb{R}^{16p}$ from Example 3, then as $f_{\alpha\beta\gamma}$ one may choose the structure constants of the Lie algebra $su(3)$. In particular, from (3.3c) it follows that in spaces of dimension $n = 4096p$ one may introduce 24 tensors J^α_{ab} satisfying the relations (2.1), and as $f_{\alpha\beta\gamma}$ one may take the structure constants of the Lie algebra $su(5)$. All these cases need a special investigation.

Thus, we have shown that in constructing the solutions of the Yang-Mills equations in the spaces of dimension greater than four the technique of Clifford algebras plays an importent role. It permits one to reduce these equations to more simple system $\{(2.3), (2.4)\}$. Our results show strong evidence for detailed study of the integrability of the Rouhani-Ward equations (2.3) and Eqs.(2.4) for scalar field φ.

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