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Consistent Deformations for a Non-Standard $D = 6$ Topological BF Model from a BRST-Symmetry-Based Cohomological Approach

Solange-Odile Saliu , Eugen-Mihăiţă Cioroianu * and Constantin Bizdadea

Department of Physics, Faculty of Sciences, University of Craiova, 13 Al. I. Cuza Str., 200585 Craiova, Romania; osaliu@central.ucv.ro (S.-O.S.); bizdadea@central.ucv.ro (C.B.)
* Correspondence: cioroianu.eugen@ucv.ro

Abstract: This paper falls under the heading of constructing consistent self-couplings in topological BF models. Our endeavor is of interest in the context of pure gravity, General Relativity, and super-gravity in Ashtekar formalism, which allow for certain economic formulations in terms of self-coupled BF theories in the presence of certain extra-constraints. More precisely, herein we address the construction of a special class of $D = 6$ self-interactions for a collection of topological BF models with a non-standard field spectrum. Our methodology relies on a deformation method based on the relationship between antifield–BRST symmetry and the non-trivial gauge symmetries of a given field theory and implemented via the computation of certain precise spaces of the local BRST cohomology corresponding to the free limit. This cohomological BRST approach is applied to the starting free model under standard "selection rules" from Quantum Field Theory. Our findings are completely new and reveal a self-interacting topological BF model in $D = 6$ with a complex gauge structure that is entirely read from the expression of the fully deformed solution to the classical master equation (the canonical generator of the antifield–BRST symmetry), and includes a generalization of the famous $D = 2$ gravity in BF formulation.

Keywords: gauge symmetries; topological BF models; consistent interactions in QFT; BRST symmetry; local BRST cohomology

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1. Introduction

One of the striking features of topological field theories [1] is the relationship of certain interacting non-Abelian versions of Poisson algebra [2] present in various versions of Poisson sigma models [3–9], which are essential in the correct description of two-dimensional gravity [10–19]. Moreover, pure three-dimensional gravity is just a topological BF theory of the Schwarz type. (The designation "BF" is not an abbreviation; rather, it refers to a Lagrangian functional with the basic field form spectrum present in pairs oh which the form degrees differ by one unit (A_{k-1}, B_k) and displaying the free-limit integrand formally written as a sum among $B_k \wedge \star F_k$, with $F_k = dA_{k-1}$ and $\star F_k$ the dual Hodge of F_k.) Concerning the higher dimensional case, it is known that General Relativity and supergravity in Ashtekar formalism may be formulated as topological BF models with certain extra constraints [20–23]. More precisely, the idea of Ashtekar related to the first-order reformulation of Einstein–Hilbert gravity as a constrained topological BF theory is implemented in the $D = 4$ and $N = 1, 2$ SUGRA theories in [20]. The latter results have been generalized to an arbitrarily high dimension in [21]. The same approach has been used in the holographic formulations of quantum gravity and quantum supergravity in [22,23]. This suggests that the construction of self-interacting BF theories may be crucial in understanding higher-dimensional gravity and supergravity theories.

The main aim of this paper is to generate all consistent, non-trivial $D = 6$ self-interactions that can be added to a finite collection of free topological BF models with a non-standard field spectrum, consisting of four sets of form fields (with the form degree ranging between 0 and 3) in the presence of several selection rules typical of gauge field theories, namely, analyticity in the coupling constant, space–time locality, Lorentz covariance, Poincaré invariance, and preservation of the differential order of each field equation with respect to its free limit. (In an arbitrary space-time dimension D, a standard or maximal BF field form spectrum comprises the pairs (A_{k-1}, B_k), with $k = \overline{1, k(D)}$, where the maximum value of k is related to an integer depending on D, $k(D) = [(D+1)/2]$ such that the union between the original pair set and the set obtained by its Hodge dualization taken in reverse order, that is, $(A_{k-1}, B_k) \to (\star B_k, \star A_{k-1}) \equiv (A_{D-k}, B_{D-k+1})$, covers the entire set of form field pairs in D dimensions of which the form degrees differ by one unit, $\{A_{j-1}, B_j\}_{j=\overline{1,D}}$, and the intersection of these two sets contains at most one such pair. This definition can be equivalently reformulated via the condition that there is no pair, such that the partial reducibility order of the free-limit gauge transformations of the corresponding A_{k-1}-type form field, $\delta_\epsilon A_{k-1} = d\epsilon_{k-2}$, is strictly greater than that of the gauge transformations of the associated B_k partner, $\delta_\epsilon B_k = \star[d(\star\xi_{k+1})]$, which reduces to $k-2 \leq D-k-1$.) We opted for the most elegant approach, which in our opinion relies on antifield–BRST symmetry [24–27] and, more precisely, on the deformation of its canonical generator [28–30] by means of cohomological techniques adapted to the computation of specific sectors of the local BRST cohomology [31–33]. The results exposed here are completely new and add to the previous ones related to various self-couplings in single or several topological BF models in different dimensions emerging from distinct approaches, such as those based on the antifield–BRST deformation method [34–43], on superfield BV formalism [44,45], on a generalized complex structure [46], or on a Hamiltonian variant of the BRST deformation method [47–50], to cite only a few.

Our paper is divided into three main sections complemented by introductory and conclusive notes. Section 2 provides insight into the particularities of both the Lagrangian formulation and BRST symmetry for the considered finite collection of A free topological BF models evolving on a Minkowski $D = 6$ space-time of a mostly positively defined signature. The main characteristics of this analysis can be synthesized by the presence of a stationary surface defined by a set of linear first-order field equations and by generating a set of gauge symmetries that underlies an Abelian gauge algebra and is an off-shell reducible of order 4, therefore defining a linear gauge theory of Cauchy order 6. The associated antifield-BRST differential reduces to the sum between the Koszul–Tate differential and the exterior longitudinal operator (in this case, a differential), while the generator of the antifield-BRST symmetry decomposes into a sum of simple terms of antifield numbers varying between 0 and 5.

Section 3 represents the core of the paper, and exposes all the results regarding the construction of the fully deformed non-trivial solution to a master equation that complies with all the imposed selection rules via detailed computation of the necessary cohomological ingredients. This section is organized in turn into three subsections. Section 3.1 provides a synthetic overview on the antifield-BRST deformation method employed, with the main deformation equations provided in both functional and local forms. Next, in Section 3.2, we construct the first-order deformation of the solution to the master equation corresponding to the chosen $D = 6$ BF model by means of various BRST cohomological aspects. More accurately, this refers to the cohomology of the exterior longitudinal differential in antifield number 0 and the local homology of the Koszul–Tate differential in pure ghost number 0, both computed in a restriction of the BRST algebra that enforces all the required selection rules. The main properties of the first-order deformation may be synthesized as follows:

- It splits into non-trivial pieces of antifield numbers ranging between 0 and 6;
- It is parameterized by several function sets depending only on the undifferentiated scalar BF fields from the collection, namely:
  - $\{M^{abc}\}_{a,b,c=\overline{1,A}}$ is completely symmetric
  - $\{Z_{ab}\}_{a,b=\overline{1,A}}$ is antisymmetric

   – $V(\varphi)$ has a general smooth 'potential'

- All the terms from the first-order deformation except the 'potential' depend on the two sorts of function sets or their derivatives (with respect to the scalar fields) of orders varying between 1 and 6;
- Their detailed structure reveals one generic class of elements from the invariant local homology of the Koszul–Tate differential as well as three other particular classes belonging to the gauge non-invariant local homology of the Koszul–Tate differential in pure ghost number 0.

With all these ingredients at hand, Section 3.3 brings into focus the higher-order deformation equations and establishes the existence of certain obstructions to the existence of the second-order deformation due to the space-time locality selection rule. These obstructions are solved by enforcing the two parameterizing function sets mentioned above in order to satisfy a set of algebraic equations equivalent to the annihilation of their simple cross-contraction. The 'potential' $V(\varphi)$ is not obstructed in any way by the consistency of deformations in higher orders of perturbation theory. Under these circumstances, it is shown that the fully deformed solution to the master equation can be taken to stop at order one in the coupling constant and is indeed consistent at each and every order of perturbation theory.

Finally, Section 4 extracts all the properties of the Lagrangian formulation of the resulting $D = 6$ self-interacting BF theory from the expression of the deformed BRST generator. Thus, two key interaction vertices are revealed. One is quadratic in the BF 3-form fields and generalizes the famous BF self-couplings present in the $D = 2$ gravity formulation via topological BF theories. The other mimics an extended $\lambda\phi^3$-type coupling specific only to the space-time dimension $D = 6$, in the sense that it is cubic in the 2-form fields. Succinctly, we only mention that all the components of the deformed gauge theory are modified through the deformation procedure with respect to their free limit. In this manner, a generating set of gauge symmetries for the $D = 6$ self-coupled model with an open gauge algebra and certain on-shell reducibility relations is disclosed, revealing a self-interacting gauge theory that is strongly nonlinear and possesses highly non-trivial higher-order structure functions.

2. Lagrangian Formulation and BRST Symmetry for a Collection of $D = 6$ Free Topological BF Models

The starting point is provided by the Lagrangian action for a non-standard (finite) collection of Abelian topological BF models in $D = 6$

$$S^L\left[\overset{[0]}{\varphi}_a, \overset{[1]}{B}{}^a, \overset{[2]}{A}_a, \overset{[3]}{B}{}^a\right] = \int d^6x\left(B^a_\mu \partial^\mu \varphi_a + B^a_{\mu\nu\rho}\partial^{[\mu} A_a^{\nu\rho]}\right) \tag{1}$$

evolving on a 6-dimensional Minkowski space-time manifold endowed with a metric of 'mostly positive' signature, $\sigma = (- + \cdots +)$. The six-dimensional Levi-Civita symbol $\varepsilon_{\mu_1\cdots\mu_6}$ is defined by $\varepsilon_{01\ldots 5} = +1$ plus complete antisymmetry, such that its various identities in $D = 6$ can be synthesized by

$$\varepsilon_{\mu_1\cdots\mu_{6-k}\lambda_1\cdots\lambda_k}\varepsilon^{\rho_1\cdots\rho_{6-k}\lambda_1\cdots\lambda_k} = -k!\delta^{[\rho_1}_{\mu_1}\cdots\delta^{\rho_{6-k}]}_{\mu_{6-k}}, \quad k = \overline{0,6}. \tag{2}$$

We assume a finite collection of BF fields in $D = 6$, namely, the scalar–vector pairs $\{\overset{[0]}{\varphi}_a, \overset{[1]}{B}{}^a\}$ and the two-form–three-form pairs $\{\overset{[2]}{A}_a, \overset{[3]}{B}{}^a\}$, with $a = \overline{1, A}$ ($A \geq 2$), the coefficients of which are to be denoted without reference to their form degree simply by φ_a, B^a_μ, $A_a^{\mu\nu}$, and $B^a_{\mu\nu\rho}$, respectively. This BF field spectrum is non-standard in the sense that we discarded the allowed vector–two-form pairs $\{\overset{[1]}{A}_a, \overset{[2]}{B}{}^a\}$ due to our aim of exhibiting a special class of self-interactions that generalize those from $D = 2$ BF-based gravity and therefore depend only on the three-form coefficients $B^{a}_{\mu\nu\rho}$ in a background of the undifferentiated scalar fields $\{\varphi_a\}$. Everywhere in this paper, the notation $[\mu_1 \cdots \mu_k]$ signifies the operation of full antisymmetrization with respect to the (Lorentz) indices between brackets, defined via the next conventions: only the independent terms are taken once without further normalization factors, the expression $f_{\mu_1\cdots\mu_0}$ is identified with a scalar ($f_{\mu_1\cdots\mu_0} = f$), the quantity $f_{\mu_1\cdots\mu_1}$ with a 6-vector ($f_{\mu_1\cdots\mu_1} = f_{\mu_1}$), and any negative label of a Lorentz index defines a vanishing term, $f_{\mu_1\cdots\mu_{-1}} = 0$. For instance, the latter expression from (1) contains precisely three terms for each value of a, $\partial^{[\mu} A^{\nu\rho]}_a = \partial^{\mu} A^{\nu\rho}_a + \partial^{\nu} A^{\rho\mu}_a + \partial^{\rho} A^{\mu\nu}_a$ due to the pre-existing antisymmetry of the components of the two-forms $\overset{[2]}{A}_a$. For further computations, it is useful to denote the BF field spectrum in a collective manner by

$$\Phi^{\alpha_0} \equiv \left\{\varphi_a, A^{\mu\nu}_a, B^{a}_{\mu\nu\rho}, B^{a}_{\mu}\right\}. \tag{3}$$

The stationary surface of this free non-interacting BF theory is defined via linear field equations of a derivative order equal to one,

$$\Sigma : \frac{\delta S^L}{\delta\Phi^{\alpha_0}} \equiv \begin{cases} \frac{\delta S^L}{\delta\varphi_a} & = -\partial^{\lambda} B^{a}_{\lambda} \\ \frac{\delta S^L}{\delta A^{\mu\nu}_a} & = -3\partial^{\lambda} B^{a}_{\lambda\mu\nu} \\ \frac{\delta S^L}{\delta B^{a}_{\mu\nu\rho}} & = \partial^{[\mu} A^{\nu\rho]}_a \\ \frac{\delta S^L}{\delta B^{a}_{\mu}} & = \partial^{\mu}\varphi_a \end{cases} \approx 0, \tag{4}$$

where "$\approx$" is the symbol of weak equality.

We generate set of (non-trivial) gauge symmetries of action (1) such as

$$\delta_{\Omega^{\alpha_1}}\Phi^{\alpha_0} \equiv \begin{cases} \delta_{\Omega^{\alpha_1}}\varphi_a & = 0 \\ \delta_{\Omega^{\alpha_1}} A^{\mu\nu}_a & = \partial^{[\mu}\epsilon^{\nu]}_{(2,0)a} \\ \delta_{\Omega^{\alpha_1}} B^{a}_{\mu\nu\rho} & = -4\partial^{\lambda}\xi^{a}_{(3,0)\lambda\mu\nu\rho} \\ \delta_{\Omega^{\alpha_1}} B^{a}_{\mu} & = -2\partial^{\lambda}\xi^{a}_{(1,0)\lambda\mu} \end{cases}, \tag{5}$$

where the gauge parameters are collectively denoted by

$$\Omega^{\alpha_1} \equiv \left\{\epsilon^{\lambda}_{(2,0)a}, \xi^{a}_{(3,0)\lambda\mu\nu\rho}, \xi^{a}_{(1,0)\lambda\mu}\right\} \tag{6}$$

and represent a collection of (antisymmetric where appropriate) components of arbitrary form-fields of degrees 1, 4, and 2, respectively, defined on the chosen $D = 6$ Minkowski space-time manifold. The supplementary two-index pair $(m, 0)$ marks the form degree of the BF field of which the gauge transformations depend on the corresponding gauge parameters (for instance, $m = 2$ in $\epsilon^{\lambda}_{(2,0)a}$ signifies that these are precisely the coefficients of the one-forms $\overset{[1]}{\epsilon}_{(2,0)a}$ involved in the gauge transformations of the components of the two-forms $\overset{[2]}{A}_a$) and the fixed (second) label "0" refers to the reducibility level (the gauge parameters are known as the zeroth order reducibility parameters). We note that all the BF scalars, $\{\varphi_a\}_{a=\overline{1,A}}$, can be viewed as purely matter fields in the context of the starting free model, as they bear no non-trivial gauge symmetries of the Lagrangian action. The above generating set of gauge transformations is *Abelian* (the commutators of the gauge transformations of all fields vanish strongly, i.e., everywhere on the space of field histories)

$$\left[\delta_{\Omega^{(1)\alpha_1}}, \delta_{\Omega^{(2)\alpha_1}}\right]\Phi^{\alpha_0} = 0 \tag{7}$$

for any two arbitrary sets of gauge parameters of the type (6) denoted by $\Omega^{(1)\alpha_1}$ and $\Omega^{(2)\alpha_1}$.

It is important to observe that the considered generating set of gauge transformations is also *reducible* (the gauge generators are not all independent), with the overall *reducibility order* equal to 4. Indeed, due to fact that no gauge generator depends on the BF fields, we can assume that all the reducibility relations hold off-shell, that is, everywhere on the space of field histories, in contrast to on-shell ones, which only take place on the stationary surface (4). With this observation at hand, it can be shown that the non-trivial gauge variations from (5) vanish if we perform the following transformations (6) on the gauge parameters:

$$\delta_{\Omega^{\alpha_1}}\Phi^{\alpha_0}|_{\text{nontriv}} = 0 \Leftrightarrow \Omega^{\alpha_1} \to \Omega^{\alpha_1}(\Omega^{\alpha_2}) \equiv \begin{cases} \epsilon^{\lambda}_{(2,0)a}(\Omega^{\alpha_2}) & = \partial^{\lambda}\epsilon_{(2,1)a} \\ \xi^{a}_{(3,0)\mu\nu\rho\sigma}(\Omega^{\alpha_2}) & = -5\partial^{\lambda}\xi^{a}_{(3,1)\lambda\mu\nu\rho\sigma} \\ \xi^{a}_{(1,0)\mu\nu}(\Omega^{\alpha_2}) & = -3\partial^{\lambda}\xi^{a}_{(1,1)\lambda\mu\nu} \end{cases}, \tag{8}$$

using the compact notation

$$\Omega^{\alpha_2} \equiv \left\{\epsilon_{(2,1)a}, \xi^{a}_{(3,1)\mu\nu\rho\sigma\tau}, \xi^{a}_{(1,1)\mu\nu\rho}\right\}. \tag{9}$$

The variables (9) define the components of the arbitrary forms of degree 0, 5, and 3, respectively, and are called the *reducibility parameters of order one*, symbolized by the lower 2-index pair $(m,1)$, where m has the same meaning as in the case of notation (3) and the index "1" marks the reducibility order. Consequently, the identities

$$\delta_{\Omega^{\alpha_1}(\Omega^{\alpha_2})}\Phi^{\alpha_0}\Big|_{\text{nontriv}} = 0 \Leftrightarrow \begin{cases} \delta_{\Omega^{\alpha_1}(\Omega^{\alpha_2})}A^{\mu\nu}_{a} \\ \delta_{\Omega^{\alpha_1}(\Omega^{\alpha_2})}B^{a}_{\mu\nu\rho} \\ \delta_{\Omega^{\alpha_1}(\Omega^{\alpha_2})}B^{a}_{\mu} \end{cases} = 0 \tag{10}$$

are known in the literature as the *first-order reducibility relations*. Next, we note that the transformed gauge parameters from (8) vanish strongly if we realize the next transformations on the first-order reducibility parameters (9)

$$\Omega^{\alpha_1}(\Omega^{\alpha_2}) = 0 \Leftrightarrow \Omega^{\alpha_2} \to \begin{cases} \epsilon_{(2,1)a} & = 0, \\ \Omega^{\alpha_2}(\Omega^{\alpha_3})|_{\text{nontriv}} & \equiv \begin{cases} \xi^{a}_{(3,1)\mu\nu\rho\sigma\tau}(\Omega^{\alpha_3}) & = -6\partial^{\lambda}\xi^{a}_{(3,2)\lambda\mu\nu\rho\sigma\tau} \\ \xi^{a}_{(1,1)\mu\nu\rho}(\Omega^{\alpha_3}) & = -4\partial^{\lambda}\xi^{a}_{(1,2)\lambda\mu\nu\rho} \end{cases}, \end{cases} \tag{11}$$

where

$$\Omega^{\alpha_3} \equiv \left\{\xi^{a}_{(3,2)\lambda\mu\nu\rho\sigma\tau}, \xi^{a}_{(1,2)\mu\nu\rho\sigma}\right\} \tag{12}$$

are named the *reducibility parameters of order two*, symbolized by the lower index pair $(m,2)$, and represent the components of the arbitrary forms of degree 6 and 4, respectively, defined on the space-time manifold. Accordingly, we find that the relations $\Omega^{\alpha_1}(\Omega^{\alpha_2}) = 0$ split into two categories

$$\Omega^{\alpha_1}(\Omega^{\alpha_2}) = 0 \Leftrightarrow \begin{cases} \epsilon^{\lambda}_{(2,0)a}(\Omega^{\alpha_2})\Big|_{\epsilon_{(2,1)a}=0} & = 0, \\ \Omega^{\alpha_1}(\Omega^{\alpha_2}(\Omega^{\alpha_3}))|_{\text{nontriv}} & \equiv \begin{cases} \xi^{a}_{(3,0)\lambda\mu\nu\rho}(\Omega^{\alpha_2}(\Omega^{\alpha_3})|_{\text{nontriv}}) \\ \xi^{a}_{(1,0)\lambda\mu}(\Omega^{\alpha_2}(\Omega^{\alpha_3})|_{\text{nontriv}}) \end{cases} = 0. \end{cases} \tag{13}$$

The former type indicates that the reducibility of the gauge transformations of the two-forms $\overset{[2]}{A}_a$ ends at order one, while the latter category, $\Omega^{\alpha_1}(\Omega^{\alpha_2}(\Omega^{\alpha_3}))|_{\text{nontriv}} = 0$, comprises the genuine *second-order reducibility relations*. Next, we observe that the non-trivially transformed first-order reducibility parameters from (11) are strongly annihilated if we enforce the next transformations on the second-order reducibility parameters (12)

$$\Omega^{\alpha_2}(\Omega^{\alpha_3})|_{\text{nontriv}} = 0 \Leftrightarrow \Omega^{\alpha_3} \to \begin{cases} \xi^{a}_{(3,2)\lambda\mu\nu\rho\sigma\tau} & = 0 \\ \Omega^{\alpha_3}(\Omega^{\alpha_4})|_{\text{nontriv}} & \equiv \xi^{a}_{(1,2)\mu\nu\rho\sigma}(\Omega^{\alpha_4}) = -5\partial^{\lambda}\xi^{a}_{(1,3)\lambda\mu\nu\rho\sigma} \end{cases}, \tag{14}$$

where

$$\Omega^{\alpha_4} \equiv \left\{ \xi^a_{(1,3)\mu\nu\rho\sigma\tau} \right\} \tag{15}$$

represent the *reducibility parameters of order three*, symbolized by the unique lower index pair $(1,3)$, which are nothing other than the components of the arbitrary forms of degree 5 defined on the space-time manifold. Consequently, we see that relations $\Omega^{\alpha_2}(\Omega^{\alpha_3})|_{\text{nontriv}} = 0$ split again into two categories

$$\Omega^{\alpha_2}(\Omega^{\alpha_3})|_{\text{nontriv}} = 0 \Leftrightarrow \begin{cases} \xi^a_{(3,1)\mu\nu\rho\sigma\tau}(\Omega^{\alpha_3})\Big|_{\xi^a_{(3,2)\lambda\mu\nu\rho\sigma\tau}=0} & = 0, \\ \Omega^{\alpha_2}(\Omega^{\alpha_3}(\Omega^{\alpha_4}))|_{\text{nontriv}} & \equiv \xi^a_{(1,1)\mu\nu\rho}(\Omega^{\alpha_3}(\Omega^{\alpha_4})|_{\text{nontriv}}) = 0. \end{cases} \tag{16}$$

The former identities stipulate that the reducibility of the gauge transformations of the three-forms $\overset{[3]}{B}{}^a$ stop individually at order two, while the latter, $\Omega^{\alpha_2}(\Omega^{\alpha_3}(\Omega^{\alpha_4}))|_{\text{nontriv}} \equiv \xi^a_{(1,1)\mu\nu\rho}(\Omega^{\alpha_3}(\Omega^{\alpha_4})|_{\text{nontriv}}) = 0$, signify the true *third-order reducibility relations*. Finally, we notice that the non-trivially transformed second-order reducibility parameters from (14), $\Omega^{\alpha_3}(\Omega^{\alpha_4})|_{\text{nontriv}} \equiv \xi^a_{(1,2)\mu\nu\rho\sigma}(\Omega^{\alpha_4})$ vanish strongly if we transform the third-order reducibility parameters (15) into

$$\Omega^{\alpha_3}(\Omega^{\alpha_4})|_{\text{nontriv}} \equiv \xi^a_{(1,2)\mu\nu\rho\sigma}(\Omega^{\alpha_4}) = 0 \Leftrightarrow \Omega^{\alpha_4} \to \Omega^{\alpha_4}(\Omega^{\alpha_5}) \equiv \xi^a_{(1,3)\mu\nu\rho\sigma\tau}(\Omega^{\alpha_5}) = -6\partial^\lambda \xi^a_{(1,4)\lambda\mu\nu\rho\sigma\tau} \tag{17}$$

with

$$\Omega^{\alpha_5} \equiv \left\{ \xi^a_{(1,4)\lambda\mu\nu\rho\sigma\tau} \right\}, \tag{18}$$

the *reducibility parameters of order four*, symbolized by the single lower index pair $(1,4)$, and represented by a collection of components of the arbitrary forms of degree 6 defined on the space-time manifold. The relations expressing the annihilation of the non-trivially transformed second-order reducibility parameters in (14)

$$\Omega^{\alpha_3}(\Omega^{\alpha_4}(\Omega^{\alpha_5}))|_{\text{nontriv}} \equiv \xi^a_{(1,2)\mu\nu\rho\sigma}(\Omega^{\alpha_4}(\Omega^{\alpha_5})) = 0 \tag{19}$$

define the *fourth-order reducibility relations*. It is easy to see that the reducibility of the gauge transformations of the components of the 1-forms $\overset{[1]}{B}{}^a$ stop at order four, as the transformed third-order reducibility parameters in (17) vanish strongly if all the reducibility parameters of order four, (18), vanish strongly:

$$\Omega^{\alpha_4}(\Omega^{\alpha_5}) \equiv -6\partial^\lambda \xi^a_{(1,4)\lambda\mu\nu\rho\sigma\tau} = 0 \Leftrightarrow \Omega^{\alpha_5} \equiv \xi^a_{(1,4)\lambda\mu\nu\rho\sigma\tau} = 0. \tag{20}$$

In conclusion, the collection of considered free topological BF models in $D = 6$ is described at the Lagrangian level by a set of linear field equations and an Abelian generating set of gauge transformations that is reducible of order four or, in other words, by a *linear gauge theory of Cauchy order equal to* 6.

In the final part of this section we construct the antifield-BRST symmetry for this model, which can be shown to decompose into

$$s = \delta + \gamma, \tag{21}$$

where s signifies the BRST differential, δ the Koszul–Tate (co)differential, and γ the exterior longitudinal differential (which may simply be a differential modulo delta in more general cases), with the essential properties synthesized as follows:

$$\varepsilon(s) = \varepsilon(\delta) = \varepsilon(\gamma) = 1, \tag{22}$$

$$\text{ant}(\delta) = -1, \qquad \text{ant}(\gamma) = 0, \qquad \text{pgh}(\delta) = 0, \qquad \text{pgh}(\gamma) = +1, \tag{23}$$

$$\text{gh} \equiv \text{pgh} - \text{ant}, \qquad \text{gh}(\delta) = \text{gh}(\gamma) = \text{gh}(s) = +1, \tag{24}$$

$$s^2 = 0 \quad \Leftrightarrow \quad \left\{\delta^2 = 0,\ \delta\gamma + \gamma\delta = 0,\ \gamma^2 = 0\right\}, \tag{25}$$

$$H_*(\delta) = H_0(\delta) \simeq C^\infty(\Sigma) \otimes \mathbb{C}[[\text{ghosts}]], \qquad H^0(\gamma|H_0(\delta)) = \{\text{physical obs.}\}, \tag{26}$$

$$H^0(s) \simeq H^0(\gamma|H_0(\delta)) = \{\text{physical obs.}\}. \tag{27}$$

In the above, ε represents the Grassmann parity while ant and pgh stand for two different $\mathbb{N}$-graduations of the BRST algebra on which the operators δ, γ, and s act (ant, known as the antifield number, is specific to the Koszul–Tate differential and pgh—the pure ghost number—to the exterior longitudinal differential, while their difference, pgh $-$ ant $\equiv$ gh, is named the ghost number and provides a $\mathbb{Z}$-graduation of the BRST algebra, especially designed with reference to the BRST differential). The nilpotency of order two of the BRST differential s is completely equivalent with the co-differential property of δ, the anticommutativity between δ and γ, and, in this case, with the purely differential feature of γ (Equation (25)). The Koszul–Tate differential is required in order to provide a homological resolution of the algebra of smooth functions defined on the stationary surface (4), $C^\infty(\Sigma)$ tensor product with the space of polynomials in the ghost fields (to be introduced below) and their space-time derivatives up to a finite order (in order to ensure the space-time locality of the BRST formalism for field theories) according to the first relations in (26), where $H_*(\delta)$ denotes the homology of δ and $H_0(\delta)$ the homology space of δ at antifield number zero. In the present paper, by $f([u])$ we understand that f depends on u and its space-time derivatives up to a finite order, while $\mathbb{C}[v]$ signifies the space of polynomials in v with complex coefficients. Moreover, the cohomology of γ restricted to $H_*(\delta)$ at pure ghost number zero, $H^0(\gamma|H_0(\delta))$, is precisely the algebra of physical observables of the free theory (1), denoted by $\{\text{physical obs.}\}$, that is, the last relation from (26). This is due to the main isomorphism emphasized in (27), in terms of which one establishes the key required property of the BRST differential associated with a given gauge theory, namely, to be isomorphic to the algebra of physical observables of that theory via its cohomology in ghost number zero, $H^0(s)$.

In order to implement the previously discussed properties and construct the differential BRST algebra $(\mathcal{A}, s)$, we initially introduce the BRST generators, which are of two kinds: fields/ghosts and their antifields. Related to the first kind, we associate ghost fields with all the gauge and reducibility parameters of various orders, (6), (9), (12), (15), and (18)

$$\Omega^{\alpha_1} \equiv \left\{\epsilon^{\lambda}_{(2,0)a}, \xi^a_{(3,0)\lambda\mu\nu\rho}, \xi^a_{(1,0)\lambda\mu}\right\} \quad \rightarrow \quad \eta^{\alpha_1} \equiv \left\{\eta^{\lambda}_{(2,0)a}, C^a_{(3,0)\lambda\mu\nu\rho}, C^a_{(1,0)\lambda\mu}\right\}, \tag{28}$$

$$\Omega^{\alpha_2} \equiv \left\{\epsilon_{(2,1)a}, \xi^a_{(3,1)\mu\nu\rho\sigma\tau}, \xi^a_{(1,1)\mu\nu\rho}\right\} \quad \rightarrow \quad \eta^{\alpha_2} \equiv \left\{\eta_{(2,1)a}, C^a_{(3,1)\mu\nu\rho\sigma\tau}, C^a_{(1,1)\mu\nu\rho}\right\}, \tag{29}$$

$$\Omega^{\alpha_3} \equiv \left\{\xi^a_{(3,2)\lambda\mu\nu\rho\sigma\tau}, \xi^a_{(1,2)\mu\nu\rho\sigma}\right\} \quad \rightarrow \quad \eta^{\alpha_3} \equiv \left\{C^a_{(3,2)\lambda\mu\nu\rho\sigma\tau}, C^a_{(1,2)\mu\nu\rho\sigma}\right\}, \tag{30}$$

$$\Omega^{\alpha_4} \equiv \left\{\xi^a_{(1,3)\mu\nu\rho\sigma\tau}\right\} \quad \rightarrow \quad \eta^{\alpha_4} \equiv \left\{C^a_{(1,3)\mu\nu\rho\sigma\tau}\right\}, \tag{31}$$

$$\Omega^{\alpha_5} \equiv \left\{\xi^a_{(1,4)\lambda\mu\nu\rho\sigma\tau}\right\} \quad \rightarrow \quad \eta^{\alpha_5} \equiv \left\{C^a_{(1,4)\lambda\mu\nu\rho\sigma\tau}\right\}, \tag{32}$$

such that the generators of the BRST algebra are precisely

$$\Phi^A \equiv \{\Phi^{\alpha_0}, \eta^{\alpha_1}, \eta^{\alpha_2}, \eta^{\alpha_3}, \eta^{\alpha_4}, \eta^{\alpha_5}\}, \tag{33}$$

where Φ^{α_0} are the original BF fields (3). The second type of BRST generators are the antifields that respectively correspond to the field and ghost spectra

$$\Phi^*_A \equiv \{\Phi^*_{\alpha_0}, \eta^*_{\alpha_1}, \eta^*_{\alpha_2}, \eta^*_{\alpha_3}, \eta^*_{\alpha_4}, \eta^*_{\alpha_5}\}, \tag{34}$$

with

$$\Phi^*_{\alpha_0} \equiv \left\{\varphi^{*a}, A^{*a}_{\mu\nu}, B^{*\mu\nu\rho}_a, B^{*\mu}_a\right\}, \qquad \eta^*_{\alpha_1} \equiv \left\{\eta^{*a}_{(2,0)\lambda}, C^{*\lambda\mu\nu\rho}_{(3,0)a}, C^{*\lambda\mu}_{(1,0)a}\right\}, \tag{35}$$

$$\eta^*_{\alpha_2} \equiv \left\{\eta^{*a}_{(2,1)}, C^{*\mu\nu\rho\sigma\tau}_{(3,1)a}, C^{*\mu\nu\rho}_{(1,1)a}\right\}, \qquad \eta^*_{\alpha_3} \equiv \left\{C^{*\lambda\mu\nu\rho\sigma\tau}_{(3,2)a}, C^{*\mu\nu\rho\sigma}_{(1,2)a}\right\}, \tag{36}$$

$$\eta^*_{\alpha_4} \equiv \left\{C^{*\mu\nu\rho\sigma\tau}_{(1,3)a}\right\}, \qquad \eta^*_{\alpha_5} \equiv \left\{C^{*\lambda\mu\nu\rho\sigma\tau}_{(1,4)a}\right\}. \tag{37}$$

Meanwhile, according to the antifield-BRST method, we endow the field/ghost spectrum with the following properties:

$$\varepsilon(\eta^{\alpha_k}) = k \bmod 2, \qquad \mathrm{pgh}(\Phi^{\alpha_0}) = 0, \qquad \mathrm{pgh}(\eta^{\alpha_k}) = k, \tag{38}$$

$$\varepsilon(\Phi^*_A) = \left(\varepsilon\left(\Phi^A\right) + 1\right) \bmod 2, \qquad \mathrm{ant}(\Phi^*_{\alpha_0}) = 1, \qquad \mathrm{ant}\left(\eta^*_{\alpha_k}\right) = k+1, \tag{39}$$

$$\mathrm{ant}\left(\Phi^A\right) = 0, \qquad \mathrm{pgh}(\Phi^*_A) = 0, \tag{40}$$

with $k = \overline{1,5}$. An arbitrary element of the BRST algebra $\mathcal{A}$ will be a polynomial in the ghosts, antifields, and their space-time derivatives, with coefficients that are smooth functions of the BF fields (3) and their space-time derivatives. We introduce a natural involution on $\mathcal{A}$, to be denoted by "$\star$", as follows:

$$(\Phi^{\alpha_0})^\star = \Phi^{\alpha_0}, \quad (\eta^{\alpha_k})^\star = -(-)^k \eta^{\alpha_k}, \quad \left(\Phi^*_{\alpha_0}\right)^\star = -\Phi^*_{\alpha_0}, \quad \left(\eta^*_{\alpha_k}\right)^\star = (-)^k \eta^*_{\alpha_k}. \tag{41}$$

We can write properties (38)–(40) in a compact manner with reference to the lower index pair (m,l). In view of this, we extend this notation to the original field spectrum (3) and their antifields $\Phi^*_{\alpha_0}$ from (35):

$$A^{\mu\nu}_a \equiv \eta^{\mu\nu}_{(2,-1)a}, \qquad B^a_{\mu\nu\rho} \equiv C^a_{(3,-1)\mu\nu\rho}, \qquad B^a_\mu \equiv C^a_{(1,-1)\mu}, \tag{42}$$

$$A^{*a}_{\mu\nu} \equiv \eta^{*a}_{(2,-1)\mu\nu}, \qquad B^{*\mu\nu\rho}_a \equiv C^{*\mu\nu\rho}_{(3,-1)a}, \qquad B^{*\mu}_a \equiv C^{*\mu}_{(1,-1)a}, \tag{43}$$

such that we can alternatively reorganize (33) and (34) as follows:

$$\Phi^A \equiv \left\{\varphi_a, \left\{\eta^{\mu_1\cdots\mu_{1-l(2)}}_{(2,l(2))a}\right\}_{l(2)=\overline{-1,1}}, \left\{\left\{C^a_{(m,l(m))\mu_1\cdots\mu_{m+l(m)+1}}\right\}_{l(m)=\overline{-1,5-m}}\right\}_{m=1,3}\right\}, \tag{44}$$

$$\Phi^*_A \equiv \left\{\varphi^{*a}, \left\{\eta^{*a}_{(2,l(2))\mu_1\cdots\mu_{1-l(2)}}\right\}_{l(2)=\overline{-1,1}}, \left\{\left\{C^{*\mu_1\cdots\mu_{m+l(m)+1}}_{(m,l(m))a}\right\}_{l(m)=\overline{-1,5-m}}\right\}_{m=1,3}\right\}. \tag{45}$$

In this alternative notational framework, properties (38) and (39) are written as

$$\varepsilon(\varphi_a) = 0, \qquad \varepsilon\left(\eta^{\mu_1\cdots\mu_{1-l(2)}}_{(2,l(2))a}\right) = (l(2)+1) \bmod 2, \quad l(2) = \overline{-1,1}, \tag{46}$$

$$\varepsilon\left(C^a_{(m,l(m))\mu_1\cdots\mu_{m+l(m)+1}}\right) = (l(m)+1) \bmod 2, \quad l(m) = \overline{-1,5-m}, \quad m = 1,3, \tag{47}$$

$$\mathrm{pgh}(\varphi_a) = 0, \qquad \mathrm{pgh}\left(\eta^{\mu_1\cdots\mu_{1-l(2)}}_{(2,l(2))a}\right) = l(2)+1, \quad l(2) = \overline{-1,1}, \tag{48}$$

$$\mathrm{pgh}\left(C^a_{(m,l(m))\mu_1\cdots\mu_{m+l(m)+1}}\right) = l(m)+1, \quad l(m) = \overline{-1,5-m}, \quad m = 1,3, \tag{49}$$

$$\varepsilon(\varphi^{*a}) = 1, \qquad \varepsilon\left(\eta^{*a}_{(2,l(2))\mu_1\cdots\mu_{1-l(2)}}\right) = l(2) \bmod 2, \quad l(2) = \overline{-1,1}, \tag{50}$$

$$\varepsilon\left(C^{*\mu_1\cdots\mu_{m+l(m)+1}}_{(m,l(m))a}\right) = l(m) \bmod 2, \quad l(m) = \overline{-1,5-m}, \quad m = 1,3, \tag{51}$$

$$\mathrm{ant}(\varphi^{*a}) = 1, \qquad \mathrm{ant}\left(\eta^{*a}_{(2,l(2))\mu_1\cdots\mu_{1-l(2)}}\right) = l(2)+2, \quad l(2) = \overline{-1,1}, \tag{52}$$

$$\mathrm{ant}\left(C^{*\mu_1\cdots\mu_{m+l(m)+1}}_{(m,l(m))a}\right) = l(m)+2, \quad l(m) = \overline{-1,5-m}, \quad m = 1,3. \tag{53}$$

The actions of the operators δ and γ on the BRST generators (44) and (45) that implement the desired properties (22)–(27) are defined by

$$\delta\Phi^A = 0, \quad \gamma\Phi^*_A = 0 \tag{54}$$

together with

$$\delta\varphi^{*a} \equiv -\frac{\delta S^L}{\delta\varphi_a} = \partial^\lambda B^a_\lambda, \qquad \delta\eta^{*a}_{(2,-1)\mu_1\mu_2} \equiv -\frac{\delta S^L}{\delta A^{\mu_1\mu_2}_a} = 3\partial^\lambda B^a_{\lambda\mu_1\mu_2}, \tag{55}$$

$$\delta\eta^{*a}_{(2,l(2))\mu_1\cdots\mu_{1-l(2)}} = (-)^{l(2)+1}(2-l(2))\partial^\lambda\eta^{*a}_{(2,l(2)-1)\lambda\mu_1\cdots\mu_{1-l(2)}}, \quad l(2) = 0,1, \tag{56}$$

$$\delta C^{*\mu_1\mu_2\mu_3}_{(3,-1)a} \equiv -\frac{\delta S^L}{\delta B^a_{\mu_1\mu_2\mu_3}} = -\partial^{[\mu_1}A^{\mu_2\mu_3]}_a, \qquad \delta C^{*\mu_1}_{(1,-1)a} \equiv -\frac{\delta S^L}{\delta B^a_{\mu_1}} = -\partial^{\mu_1}\varphi_a, \tag{57}$$

$$\delta C^{*\mu_1\cdots\mu_{m+l(m)+1}}_{(m,l(m))a} = (-)^{l(m)}\partial^{[\mu_1}C^{*\mu_2\cdots\mu_{m+l(m)+1}]}_{(m,l(m)-1)a}, \quad l(m) = \overline{0,5-m}, \quad m = 1,3 \tag{58}$$

and, respectively,

$$\gamma\varphi_a = 0, \qquad \gamma\eta^{\mu_1\cdots\mu_{1-l(2)}}_{(2,l(2))a} = \partial^{[\mu_1}\eta^{\mu_2\cdots\mu_{1-l(2)}]}_{(2,l(2)+1)a}, \quad l(2) = -1,0, \qquad \gamma\eta_{(2,1)a} = 0, \tag{59}$$

$$\gamma C^a_{(m,l(m))\mu_1\cdots\mu_{m+l(m)+1}} = -(m+l(m)+2)\partial^\lambda C^a_{(m,l(m)+1)\lambda\mu_1\cdots\mu_{m+l(m)+1}}, \quad l(m) = \overline{-1,4-m}, \quad m = 1,3, \tag{60}$$

$$\gamma C^a_{(m,5-m)\mu_1\cdots\mu_6} = 0, \quad m = 1,3. \tag{61}$$

Obviously, the actions of the BRST differential on the BRST generators follow from (54)–(61) via expansion (21). We mention that all the operators (the (co)-differentials) from (21) are assumed to act as right derivations.

A major feature of the antifield-BRST formalism is provided by its *canonical action* in a structure named *antibracket*, which is denoted by $(,)$ and is defined by decreeing the fields/ghosts respectively conjugated with the corresponding antifields

$$\left(\Phi^A, \Phi^*_B\right) = \delta^A_B, \tag{62}$$

where the DeWitt condensed notation is understood and δ^A_B contains a projector on the tensor subspace for each and every form degree. The properties of the antibracket are complementary to those of the Poisson bracket from the Hamiltonian formalism, excepting the bilinearity in its arguments, which is preserved. The *canonical generator of the antifield-BRST symmetry*, S, is a bosonic functional depending on the fields/ghosts and antifields of ghost number 0, purely real with respect to the natural involution (41), in terms of which the (right derivation) action of the BRST operator s is recovered precisely via the antibracket

$$\forall F \in \mathcal{A},\ sF = (F,S), \quad \varepsilon(S) = 0, \quad (S)^\star = S, \quad \mathrm{gh}(S) = 0 \tag{63}$$

and the second-order nilpotency of s is equivalent to the famous classical master equation satisfied by S:

$$s^2 = 0 \Leftrightarrow (S,S) = 0. \tag{64}$$

In view of this, S is usually referred to as the solution to the classical master equation. We cannot stress enough that the solution to the classical master equation exists for all theories that satisfy standard regularity conditions and contains *all* the information on the Lagrangian formulation of a given gauge theory, including the Lagrangian action, the chosen set of its generating set of gauge symmetries, the underlying gauge algebra, and the reducibility of its generating set of gauge transformations (if any), as well as the expressions of the structure functions of all types and orders. In the case of the model under study, the solution to the classical master equation can be taken as

$$\begin{aligned} S \;=\; \int d^6x \Big(& B^a_\mu\partial^\mu\varphi_a + B^a_{\mu\nu\rho}\partial^{[\mu}A^{\nu\rho]}_a + A^{*a}_{\mu\nu}\partial^{[\mu}\eta^{\nu]}_{(2,0)a} - 4B^{*\mu\nu\rho}_a\partial^\lambda C^a_{(3,0)\lambda\mu\nu\rho} \\ & -2B^{*\mu}_a\partial^\lambda C^a_{(1,0)\lambda\mu} + \eta^{*a}_{(2,0)\mu}\partial^\mu\eta_{(2,1)a} - 5C^{*\mu\nu\rho\sigma}_{(3,0)a}\partial^\lambda C^a_{(3,1)\lambda\mu\nu\rho\sigma} \\ & -3C^{*\mu\nu}_{(1,0)a}\partial^\lambda C^a_{(1,1)\lambda\mu\nu} - 6C^{*\mu\nu\rho\sigma\tau}_{(3,1)a}\partial^\lambda C^a_{(3,2)\lambda\mu\nu\rho\sigma\tau} - 4C^{*\mu\nu\rho}_{(1,1)a}\partial^\lambda C^a_{(1,2)\lambda\mu\nu\rho} \end{aligned}$$

$$-5C^{*\mu\nu\rho\sigma}_{(1,2)a}\partial^{\lambda}C^{a}_{(1,3)\lambda\mu\nu\rho\sigma}-6C^{*\mu\nu\rho\sigma\tau}_{(1,3)a}\partial^{\lambda}C^{a}_{(1,4)\lambda\mu\nu\rho\sigma\tau}\Big). \tag{65}$$

We organized S according to the increasing values of the antifield number of its components, and thus it contains pieces of ant ranging from 0 to 5. The component of antifield number zero always reduces to the Lagrangian action of the considered gauge theory, that is, the first two terms from the right-hand side of (65) provide precisely (1)). The elements of antifield number one are always written as the antifields of the original fields 'times' the gauge transformations of the corresponding fields, where the gauge parameters are replaced with the associated ghosts of pure ghost number 1; the next three terms from the right-hand side of (65) are related to the gauge transformations (5), with the gauge parameters Ω^{α_1} from (6) substituted by the corresponding ghosts η^{α_1} introduced in (28). The components of antifield numbers strictly greater than 1 from the solution to the classical master equation (if any) are present only if the chosen generating set of gauge transformations for the theory under study is reducible and/or generates a non-Abelian gauge algebra. The terms related to the reducibility functions and relations of various orders specific to the generating set are always linear in the ghosts of pure ghost numbers strictly greater than 1. In the case of our $D=6$ BF model, this type of component covers the remaining seven elements from the right-hand side of (65) of antifield number ranging between 2 and 5. They are linear in the antifields of the ghost fields, as all the reducibility relations connected to the generating set (5) hold off-shell, and it is easy to see that they account for all the non-trivial transformations of the gauge/reducibility parameters present in (8), (11), (14), and (17), with the various parameters replaced by the associated ghosts. Finally, the remaining possible pieces from the solution to the classical master equation, which are at least quadratic in the ghosts, are due to the non-Abelianity of the gauge algebra induced by the generating set and involve all the non-vanishing structure functions of various orders. In the case of the model under study, there are no such terms in (65) because the gauge algebra generated by (5) is Abelian. It is useful to rewrite solution (65) in terms of the alternative notations and structuring (42)–(45), as follows:

$$\begin{aligned} S \;=\; & S^{L}+\int d^{6}x\Bigg(\sum_{l(2)=-1}^{0}\eta^{*a}_{(2,l(2))\mu_1\cdots\mu_{1-l(2)}}\partial^{[\mu_1}\eta^{\mu_2\cdots\mu_{1-l(2)}]}_{(2,l(2)+1)a} \\ & -\sum_{m=1,3}\sum_{l(m)=-1}^{4-m}(m+l(m)+2)C^{*\mu_1\cdots\mu_{m+l(m)+1}}_{(m,l(m))a}\partial^{\lambda}C^{a}_{(m,l(m)+1)\lambda\mu_1\cdots\mu_{m+l(m)+1}}\Bigg), \end{aligned} \tag{66}$$

with S^L the free Lagrangian action (1).

3. Antifield-BRST Deformation Method and Its Application to a Collection of $D=6$ BF Models

3.1. Brief Review of the Antifield–BRST Deformation Method

It is possible to reformulate the long standing problem of generating consistent interactions in gauge field theories via the antifield–BRST deformation method [28–30] based on the observation that, if consistent couplings can be added, then the solution to the classical master equation of the original gauge theory, S, can be deformed into a solution to the classical master equation for the coupled gauge theory, $\bar{S}$:

$$\bar{S}=S+\lambda S_1+\lambda^2 S_2+\cdots,\qquad \tfrac{1}{2}(\bar{S},\bar{S})=0, \tag{67}$$

with λ the coupling constant or deformation parameter. Regarding the interacting theory, we preserve the field, ghost, and antifield spectra of the initial theory in order to maintain the number of physical degrees of freedom with respect to the free limit. In addition, we do not alter either the antibracket or the standard features of $\bar{S}$ compared to those of the initial theory, only the canonical generator itself; thus, $\bar{S}$ remains a real bosonic functional of fields, ghosts, and antifields, with the ghost number equal to 0. The projection of the key equation $\frac{1}{2}(\bar{S},\bar{S})=0$ on the various increasing powers in the coupling constant λ is equivalent to the chain of equations

$$\begin{aligned}
\lambda^0 &: \tfrac{1}{2}(S,S)=0, && (68)\\
\lambda^1 &: (S_1,S)=0, && (69)\\
\lambda^2 &: (S_2,S)+\tfrac{1}{2}(S_1,S_1)=0, && (70)\\
\lambda^3 &: (S_3,S)+(S_1,S_2)=0, && (71)\\
&\vdots
\end{aligned}$$

known as the *equations of the antifield–BRST deformation method*. The functionals S_i, $i\geq 1$ are known as the *deformations of order* i of the solution to the classical master equation. The first equation is fulfilled by assumption, while the remaining ones may be expressed via the canonical action $s\cdot=(\cdot,S)$ as follows:

$$\begin{aligned}
\lambda^1 &: sS_1=0, && (72)\\
\lambda^2 &: sS_2+\tfrac{1}{2}(S_1,S_1)=0, && (73)\\
\lambda^3 &: sS_3+(S_1,S_2)=0, && (74)\\
&\vdots
\end{aligned}$$

The solutions to (72) always exist as long as they pertain to the cohomology of the BRST differential s in ghost number 0 computed in the space of *all functionals* (local and non-local) of fields, ghosts, and antifields, $H^0(s)$ (this cohomology space is generically non-empty due to its isomorphism to the algebra of physical observables of the original gauge theory). All *trivial first-order deformations*, defined via s-exact elements of $H^0(s)$, must be *discarded* as they produce trivial interactions (these can be removed by several possibly non-linear field redefinitions). The existence of solutions to the remaining deformation equations, (73), (74), etc., has been proved to exist [28] on behalf of the triviality of the antibracket map in the BRST cohomology $H(s)$ as *computed in the space of all functionals*. Thus, if we enforce *no restrictions on the interactions* (such as the space-time locality or other properties), then the antifield–BRST deformation procedure proceeds unobstructed.

On the other hand, if we impose restrictions on the deformations, such as, for instance that $\bar{S}$ (and implicitly all its components) should be a local functional, then the construction of consistent interactions via the antifield–BRST method must be approached differently, as in this context there is no guarantee that there exist non-trivial solutions to the main equations, (72)–(74), etc. that comply with these constraints. Assuming the space-time locality of deformations, if we make the notations

$$S_1=\int d^6x\, a,\qquad S_2=\int d^6x\, b,\qquad S_3=\int d^6x\, c, \tag{75}$$

$$\tfrac{1}{2}(S_1,S_1)=\int d^6x\,\Delta,\qquad (S_1,S_2)=\int d^6x\,\Gamma, \tag{76}$$

then Equations (72)–(74), etc. take the local form

$$\begin{aligned}
sa &= \partial_\mu j^\mu, && (77)\\
sb+\Delta &= \partial_\mu k^\mu, && (78)\\
sc+\Gamma &= \partial_\mu l^\mu, && (79)\\
&\vdots
\end{aligned}$$

All the general properties of $\bar{S}$ are transferred to its non-integrated local density, thus,

$$\varepsilon(a)=\varepsilon(b)=\varepsilon(c)=\cdots=0, \tag{80}$$

$$a^\star = a, \qquad b^\star = b, \qquad c^\star = c, \qquad \cdots, \tag{81}$$
$$\mathrm{gh}(a) = \mathrm{gh}(b) = \mathrm{gh}(c) = \cdots = 0, \tag{82}$$
$$\varepsilon(j^\mu) = \varepsilon(k^\mu) = \varepsilon(l^\mu) = \cdots = 1, \tag{83}$$
$$(j^\mu)^\star = j^\mu, \qquad (k^\mu)^\star = k^\mu, \qquad (l^\mu)^\star = l^\mu, \qquad \cdots, \tag{84}$$
$$\mathrm{gh}(j^\mu) = \mathrm{gh}(k^\mu) = \mathrm{gh}(l^\mu) = \cdots = 1. \tag{85}$$

All the non-integrated densities of the deformations of various orders of perturbation theory, namely, a, b, c, etc., are restricted now to be bosonic real elements of ghost number 0 of the BRST algebra of local "functions", i.e., simultaneously polynomials in the ghosts, antifields, and their space-time derivatives up to some finite orders, with coefficients that are local functions of the original fields and their space-time derivatives, $\mathcal{A}_{\mathrm{local}}$. In addition, all the currents j^μ, k^μ, l^μ, etc., are essentially required to be fermionic real elements of ghost number 1 from $\mathcal{A}_{\mathrm{local}}$.

Thus, Equation (77), which is now responsible for the non-integrated density of the first-order deformation, is equivalent to the fact that a should be a (non-trivial) element of the local cohomology of the BRST differential at ghost number 0, $a \in H^0(s|d)$. By non-trivial we mean that a does not reduce to an s-exact modulo of d-exact local quantities. In the next subsection, we construct the general non-trivial solution to the first-order deformation Equation (77) in an even more restricted BRST algebra than $\mathcal{A}_{\mathrm{local}}$ in order to comply with all the standard "selection rules" imposed on field theories.

3.2. Construction of the First-Order Deformation

The goal of our paper is to generate all the non-trivial consistent self-interactions that can be added to the free model exposed in Section 2 with the help of the antifield–BRST deformation method briefly reviewed in the previous subsection. We adopt the standard selection rules from field theory on the deformed solution to the classical master Equation (67), namely, *analyticity in the coupling constant, space-time locality, Lorentz covariance, Poincaré invariance, and conservation of the differential order of the interacting field equations with respect to their free limit* ($\lambda \to 0$). In detail, the last requirement means that all of the interacting vertices are asked to possess the maximum derivative order of the free Lagrangian density at all orders in the coupling constant, i.e., one. Due to the space-time locality hypothesis and based on the first notation from (75), it follows that the non-integrated density of the first-order deformation, a, should be a non-trivial solution to Equation (77), and hence a non-trivial element of the local BRST cohomology $H^0(s|d)$. The last cohomology space will be computed in the BRST algebra of local "functions", which, in addition, must be Lorentz scalars independent of the space-time coordinates (in order to ensure the Poincaré invariance of deformations), and should contain at most one space-time derivative acting on the BF field spectrum (3) at the level of its antifield-independent component (this is ghost-independent due to the fact that $\mathrm{gh}(a) = 0$, and thus can only involve the BF fields and their space-time derivatives). We denote this restricted space of the BRST algebra as $\mathcal{A}_{\mathrm{local}}^{\mathrm{restricted}}$.

In order to compute the general non-trivial solution to Equation (77) in compliance with all imposed selection rules, we simultaneously decompose the first-order deformation a and the associated current j^μ according to the antifield number k

$$a = \sum_{k=0}^{k_{\max}} a_k, \qquad j^\mu = \sum_{k=0}^{k_{\max}} j_k^\mu, \tag{86}$$

and assume, without loss of generality, that the above decomposition stops at some finite value of the non-negative integer $k_{\max}$. The components of a and j^μ are subject to the following properties, which are induced by (80)–(85):

$$\varepsilon(a_k) = 0, \qquad \mathrm{gh}(a_k) = 0, \qquad \mathrm{ant}(a_k) = k, \qquad \mathrm{pgh}(a_k) = k, \tag{87}$$

$$\varepsilon\left(j_k^\mu\right) = 1, \qquad \text{gh}\left(j_k^\mu\right) = 1, \qquad \text{ant}\left(j_k^\mu\right) = k, \qquad \text{pgh}\left(j_k^\mu\right) = k+1. \tag{88}$$

Inserting (86) into (77) and taking into account expansion (21) of the BRST differential, it is easy to see that the first-order deformation equation becomes equivalent to the next tower of equations:

$$\gamma a_{k_{\max}} = \partial_\mu j^\mu_{k_{\max}}, \tag{89}$$
$$\delta a_k + \gamma a_{k-1} = \partial_\mu j^\mu_{k-1}, \quad k = \overline{1, k_{\max}}. \tag{90}$$

Due to the fact that the starting $D = 6$ collection of Abelian BF models is a linear gauge theory of Cauchy order equal to 6, several standard results from the literature [31,32] adapted to this case stipulate that one can, without loss of non-trivial terms, take expansion (86) to stop at antifield number $k_{\max} = 6$. Moreover, it can be shown (see for instance [32,37,41]) that the last component, a_6, can be taken as a non-trivial element of the cohomology of the longitudinal exterior differential $H(\gamma)$ computed in $\mathcal{A}^{\text{restricted}}_{\text{local}}$ instead of its local version, $H(\gamma|d)$, computed in the same space, such that (86), (89), and (90) can be safely replaced with

$$a = \sum_{k=0}^{6} a_k, \qquad j^\mu = \sum_{k=0}^{5} j^\mu_k, \tag{91}$$
$$\gamma a_6 = 0, \tag{92}$$
$$\delta a_k + \gamma a_{k-1} = \partial_\mu j^\mu_{k-1}, \quad k = \overline{1,6}. \tag{93}$$

If we manipulate Equations (92) and (93) in a cohomological fashion that takes into account the specificities of the initial $D = 6$ collection of Abelian BF models (such as, for example in [41]), we reach the conclusion that the non-trivial solution to the Equation (92) satisfied by the component of maximum antifield number from (91) can be generated, without loss of non-trivial terms, by 'gluing' the ghost basis of the pure ghost number equal to 6 from the cohomology of the exterior longitudinal differential computed in $\mathcal{A}^{\text{restricted}}_{\text{local}}$, further projected on antifield number 0, denoted by $H^6_0(\gamma, \mathcal{A}^{\text{restricted}}_{\text{local}})$ (by convention, the elements of the ghost basis from $H^*_0(\gamma, \mathcal{A}^{\text{restricted}}_{\text{local}})$ display strictly positive values of the pure ghost number, do not depend on either the original BF fields (3) or the antifields or their space-time derivatives, and are assumed to be non-trivial; thus, they may depend only on those γ-closed combinations of the ghosts and possibly certain of their space-time derivatives that are not γ-exact), to the non-trivial elements of the local homology of the Koszul–Tate differential at antifield number 6 and pure ghost number 0 computed in the space of the gauge-invariant component of $\mathcal{A}^{\text{restricted}}_{\text{local}}$, denoted by $H^{\text{inv}}_6(\delta|d, \mathcal{A}^{\text{restricted}}_{\text{local}})$:

$$a_6 : e^6 \in H^6_0(\gamma, \mathcal{A}^{\text{restricted}}_{\text{local}}) \longleftrightarrow \alpha^{\text{inv}}_6 \in H^{\text{inv}}_6(\delta|d, \mathcal{A}^{\text{restricted}}_{\text{local}}). \tag{94}$$

On the one hand, from actions (59)–(61) of the exterior longitudinal differential on the BRST generators it follows that the ghost basis from $H^*_0(\gamma, \mathcal{A}^{\text{restricted}}_{\text{local}})$ at pure ghost number $n > 0$ is spanned by the monomials of partial orders (n_1, n_2, n_3) in the *undifferentiated* γ-closed non-trivial ghosts $\eta_{(2,1)a}$, $\overset{[6]}{C}{}^b{}_{(3,2)}$, and $\overset{[6]}{C}{}^c{}_{(1,4)}$, respectively, obtained as all distinct solutions (in $\mathbb{N}$) to the Diophantine equation $2n_1 + 3n_2 + 5n_3 = n > 0$

$$e^n \equiv \left\{ \eta_{(2,1)a_1} \cdots \eta_{(2,1)a_{n_1}} \overset{[6]}{C}{}^{b_1}{}_{(3,2)} \cdots \overset{[6]}{C}{}^{b_{n_2}}{}_{(3,2)} \overset{[6]}{C}{}^{c_1}{}_{(1,4)} \cdots \overset{[6]}{C}{}^{c_{n_3}}{}_{(1,4)},\ 2n_1 + 3n_2 + 5n_3 = n > 0,\ n_1, n_2, n_3 \in \mathbb{N} \right\}. \tag{95}$$

The space-time derivatives of all orders of these ghosts, although γ-closed according to the last formula in (59) and (61), respectively, are nevertheless γ-exact due to the middle relation from (59) for $l(2) = 0$, $\partial^{\mu_1}\eta_{(2,1)a} = \gamma\eta^{\mu_1}_{(2,0)a}$ and, respectively, (60) for $m = 1,3$

and $l(m) = 4 - m$, which produce $\partial^{\lambda} C^{a}_{(m,5-m)\lambda\mu_1\cdots\mu_5} = \gamma[(-1/6)C^{a}_{(m,4-m)\mu_1\cdots\mu_5}]$ and are therefore trivial in the cohomology of γ. For $n = 6$, there are only two distinct solutions to the equation $2n_1 + 3n_2 + 5n_3 = 6$, namely, the triplets $(3,0,0)$ and $(0,2,0)$, finally yielding

$$e^{6} \equiv \left\{ \eta_{(2,1)a}\eta_{(2,1)b}\eta_{(2,1)c}, C^{a}_{(3,2)\mu_1\cdots\mu_6}C^{b}_{(3,2)\nu_1\cdots\nu_6} \right\}. \tag{96}$$

On the other hand, it can be shown (see for instance [40] for an antifield-BRST approach or [49] for a Hamiltonian BRST analysis) that both the spaces $H^{\rm inv}_{6}(\delta|d, \mathcal{A}^{\rm restricted}_{\rm local})$ and $H^{\rm inv}_{5}(\delta|d, \mathcal{A}^{\rm restricted}_{\rm local})$ are entirely spanned by non-trivial elements that depend *only on the antifields* $\left\{ C^{*\mu_1\cdots\mu_{l(1)+2}}_{(1,l(1))a} \right\}_{l(1)=\overline{-1,4}}$ corresponding to the one-forms $\overset{[1]}{B}{}^{a}$ and to their ghosts (the latter originating in the gauge transformations of $\overset{[1]}{B}{}^{a}$ and their reducibility relations). Actually, in each remaining space of the local homology of the Koszul–Tate differential at strictly positive values of the antifield number, j, and pure ghost number 0 computed in the space of the gauge-invariant component of $\mathcal{A}^{\rm restricted}_{\rm local}$, $\left\{ H^{\rm inv}_{j}(\delta|d, \mathcal{A}^{\rm restricted}_{\rm local}) \right\}_{j=\overline{1,4}}$, there exist non-trivial elements depending only on the antifields $\left\{ C^{*\mu_1\cdots\mu_{l(1)+2}}_{(1,l(1))a} \right\}_{l(1)=\overline{-1,4}}$, although they no longer span the entirety of $H^{\rm inv}_{j}(\delta|d, \mathcal{A}^{\rm restricted}_{\rm local})$. The general expressions of these elements from $\left\{ H^{\rm inv}_{j}(\delta|d, \mathcal{A}^{\rm restricted}_{\rm local}) \right\}_{j=\overline{1,6}}$ are provided below. For $j = 6$, we have

$$\begin{aligned} \alpha^{\rm inv}_{6} \to P^{\Delta|\mu_1\cdots\mu_6}_{6}(f) \equiv\; & \frac{\partial f^{\Delta}}{\partial \varphi_{d_1}} C^{*\mu_1\cdots\mu_6}_{(1,4)d_1} + \frac{\partial^2 f^{\Delta}}{\partial \varphi_{d_1}\partial \varphi_{d_2}} \left(C^{*[\mu_1\cdots\mu_5}_{(1,3)d_1} C^{*\mu_6]}_{(1,-1)d_2} + C^{*[\mu_1\cdots\mu_4}_{(1,2)d_1} C^{*\mu_5\mu_6]}_{(1,0)d_2} + C^{*[\mu_1\mu_2\mu_3}_{(1,1)d_1} C^{*\mu_4\mu_5\mu_6]}_{(1,1)d_2} \right) \\ & + \frac{\partial^3 f^{\Delta}}{\partial \varphi_{d_1}\partial \varphi_{d_2}\partial \varphi_{d_3}} \left(C^{*[\mu_1\cdots\mu_4}_{(1,2)d_1} C^{*\mu_5}_{(1,-1)d_2} C^{*\mu_6]}_{(1,-1)d_3} + C^{*[\mu_1\mu_2\mu_3}_{(1,1)d_1} C^{*\mu_4\mu_5}_{(1,0)d_2} C^{*\mu_6]}_{(1,-1)d_3} + C^{*[\mu_1\mu_2}_{(1,0)d_1} C^{*\mu_3\mu_4}_{(1,0)d_2} C^{*\mu_5\mu_6]}_{(1,0)d_3} \right) \\ & + \frac{\partial^4 f^{\Delta}}{\partial \varphi_{d_1}\cdots\partial \varphi_{d_4}} \left(C^{*[\mu_1\mu_2\mu_3}_{(1,1)d_1} C^{*\mu_4}_{(1,-1)d_2} C^{*\mu_5}_{(1,-1)d_3} C^{*\mu_6]}_{(1,-1)d_4} + C^{*[\mu_1\mu_2}_{(1,0)d_1} C^{*\mu_3\mu_4}_{(1,0)d_2} C^{*\mu_5}_{(1,-1)d_3} C^{*\mu_6]}_{(1,-1)d_4} \right) \\ & + \frac{\partial^5 f^{\Delta}}{\partial \varphi_{d_1}\cdots\partial \varphi_{d_5}} C^{*[\mu_1\mu_2}_{(1,0)d_1} C^{*\mu_3}_{(1,-1)d_2} \cdots C^{*\mu_6]}_{(1,-1)d_5} + \frac{\partial^6 f^{\Delta}}{\partial \varphi_{d_1}\cdots\partial \varphi_{d_6}} C^{*\mu_1}_{(1,-1)d_1} \cdots C^{*\mu_6}_{(1,-1)d_6}, \end{aligned} \tag{97}$$

where $f^{\Delta} = f^{\Delta}(\varphi)$ stand for some arbitrary, smooth functions allowed to depend only on the undifferentiated scalar fields $\{\varphi_a\}$. Although gauge-invariant, the space-time derivatives of all orders of the scalar fields φ_a are δ-exact from some local quantities, in agreement with the latter definition from (57), and hence can be safely removed from the non-trivial elements of the local homology of the Koszul–Tate differential in strictly positive values of the antifield number. Exactly the same argument is in fact valid with respect to all the remaining gauge-invariant quantities constructed out of the BF fields and their space-time derivatives, namely, $\partial^{[\mu_1} A^{\mu_2\mu_3]}_{a}$ in agreement with the former relation in (57) and $\partial^{\lambda} B^{a}_{\lambda}$ together with $\partial^{\lambda} B^{a}_{\lambda\mu_1\mu_2}$ according to formula (55). The only exceptions are represented by the undifferentiated scalar fields, φ_a, which, while gauge-invariant, cannot be expressed as δ-exact terms from *objects that are local in space-time* because $\varphi_a = -\delta(\partial_{\mu_1} B^{*\mu_1}_{a}/\Box)$, as can be seen from the last relation in (57). In antifield number 5, the class of elements that span $H^{\rm inv}_{5}(\delta|d, \mathcal{A}^{\rm restricted}_{\rm local})$ can be represented as

$$\begin{aligned} \alpha^{\rm inv}_{5} \to P^{\Delta|\mu_1\cdots\mu_5}_{5}(f) \equiv\; & \frac{\partial f^{\Delta}}{\partial \varphi_{d_1}} C^{*\mu_1\cdots\mu_5}_{(1,3)d_1} + \frac{\partial^2 f^{\Delta}}{\partial \varphi_{d_1}\partial \varphi_{d_2}} \left(C^{*[\mu_1\cdots\mu_4}_{(1,2)d_1} C^{*\mu_5]}_{(1,-1)d_2} + C^{*[\mu_1\mu_2\mu_3}_{(1,1)d_1} C^{*\mu_4\mu_5]}_{(1,0)d_2} \right) \\ & + \frac{\partial^3 f^{\Delta}}{\partial \varphi_{d_1}\partial \varphi_{d_2}\partial \varphi_{d_3}} \left(C^{*[\mu_1\mu_2\mu_3}_{(1,1)d_1} C^{*\mu_4}_{(1,-1)d_2} C^{*\mu_5]}_{(1,-1)d_3} + C^{*[\mu_1\mu_2}_{(1,0)d_1} C^{*\mu_3\mu_4}_{(1,0)d_2} C^{*\mu_5]}_{(1,-1)d_3} \right) \\ & + \frac{\partial^4 f^{\Delta}}{\partial \varphi_{d_1}\cdots\partial \varphi_{d_4}} C^{*[\mu_1\mu_2}_{(1,0)d_1} C^{*\mu_3}_{(1,-1)d_2} C^{*\mu_4}_{(1,-1)d_3} C^{*\mu_5]}_{(1,-1)d_4} + \frac{\partial^5 f^{\Delta}}{\partial \varphi_{d_1}\cdots\partial \varphi_{d_5}} C^{*\mu_1}_{(1,-1)d_1} \cdots C^{*\mu_5}_{(1,-1)d_5}. \end{aligned} \tag{98}$$

For the remaining strictly positive values of ant $= \overline{1,4}$, the elements under discussion read

$$\alpha_4^{\mathrm{inv}}\left(C^{*\mu_1\cdots\mu_{l(1)+2}}_{(1,l(1))a}\right)\to P_4^{\Delta|\mu_1\cdots\mu_4}(f) \equiv \frac{\partial f^\Delta}{\partial\varphi_{d_1}}C^{*\mu_1\cdots\mu_4}_{(1,2)d_1}+\frac{\partial^2 f^\Delta}{\partial\varphi_{d_1}\partial\varphi_{d_2}}\left(C^{*[\mu_1\mu_2\mu_3}_{(1,1)d_1}C^{*\mu_4]}_{(1,-1)d_2}+C^{*[\mu_1\mu_2}_{(1,0)d_1}C^{*\mu_3\mu_4]}_{(1,0)d_2}\right)$$
$$+\frac{\partial^3 f^\Delta}{\partial\varphi_{d_1}\partial\varphi_{d_2}\partial\varphi_{d_3}}C^{*[\mu_1\mu_2}_{(1,0)d_1}C^{*\mu_3}_{(1,-1)d_2}C^{*\mu_4]}_{(1,-1)d_3}$$
$$+\frac{\partial^4 f^\Delta}{\partial\varphi_{d_1}\cdots\partial\varphi_{d_4}}C^{*\mu_1}_{(1,-1)d_1}\cdots C^{*\mu_4}_{(1,-1)d_4}, \tag{99}$$

$$\alpha_3^{\mathrm{inv}}\left(C^{*\mu_1\cdots\mu_{l(1)+2}}_{(1,l(1))a}\right)\to P_3^{\Delta|\mu_1\mu_2\mu_3}(f) \equiv \frac{\partial f^\Delta}{\partial\varphi_{d_1}}C^{*\mu_1\mu_2\mu_3}_{(1,1)d_1}+\frac{\partial^2 f^\Delta}{\partial\varphi_{d_1}\partial\varphi_{d_2}}C^{*[\mu_1\mu_2}_{(1,0)d_1}C^{*\mu_3]}_{(1,-1)d_2}$$
$$+\frac{\partial^3 f^\Delta}{\partial\varphi_{d_1}\partial\varphi_{d_2}\partial\varphi_{d_3}}C^{*\mu_1}_{(1,-1)d_1}C^{*\mu_2}_{(1,-1)d_2}C^{*\mu_3}_{(1,-1)d_3}, \tag{100}$$

$$\alpha_2^{\mathrm{inv}}\left(C^{*\mu_1\cdots\mu_{l(1)+2}}_{(1,l(1))a}\right)\to P_2^{\Delta|\mu_1\mu_2}(f) \equiv \frac{\partial f^\Delta}{\partial\varphi_{d_1}}C^{*\mu_1\mu_2}_{(1,0)d_1}+\frac{\partial^2 f^\Delta}{\partial\varphi_{d_1}\partial\varphi_{d_2}}C^{*\mu_1}_{(1,-1)d_1}C^{*\mu_2}_{(1,-1)d_2}, \tag{101}$$

$$\alpha_1^{\mathrm{inv}}\left(C^{*\mu_1\cdots\mu_{l(1)+2}}_{(1,l(1))a}\right)\to P_1^{\Delta|\mu_1}(f) \equiv \frac{\partial f^\Delta}{\partial\varphi_{d_1}}C^{*\mu_1}_{(1,-1)d_1}. \tag{102}$$

The quantities from (97)–(102) can be written in a more compact form, as follows:

$$\left\{\alpha_j^{\mathrm{inv}}\left(C^{*\mu_1\cdots\mu_{l(1)+2}}_{(1,l(1))a}\right)\right\}_{j=\overline{1,6}}\to$$
$$P_j^{\Delta|\mu_1\cdots\mu_j}(f)\equiv\sum_{l=1}^{j}\frac{\partial^l f^\Delta}{\partial\varphi_{d_1}\cdots\partial\varphi_{d_l}}\left(\sum_{\substack{j-l+1\geq k_1\geq k_2\geq\cdots\geq k_l\geq 1\\ k_1+k_2+\cdots+k_l=j}}C^{*[\mu_1\cdots\mu_{k_1}}_{(1,k_1-2)d_1}C^{*\mu_{k_1+1}\cdots\mu_{k_1+k_2}}_{(1,k_2-2)d_2}\cdots C^{*\mu_{k_1+\cdots+k_{l-1}+1}\cdots\mu_{k_1+\cdots+k_l}]}_{(1,k_l-2)d_l}\right), \tag{103}$$

where the antisymmetrization operation $[\mu_1\cdots\mu_j]$ must be effectively realized only for the terms with $6\geq j\geq 3$ and $l=\overline{2,j-1}$. This due to the fact that: 1. for $l=1$ and $j=\overline{2,6}$ the antifields $C^{*\mu_1\cdots\mu_j}_{(1,j-2)d_1}$ from $\frac{\partial f^\Delta}{\partial\varphi_{d_1}}C^{*\mu_1\cdots\mu_j}_{(1,j-2)d_1}$ are fully antisymmetric by definition; 2. for $l=j$ and $j=\overline{2,6}$ the elements $\frac{\partial^j f^\Delta}{\partial\varphi_{d_1}\cdots\partial\varphi_{d_j}}C^{*\mu_1}_{(1,-1)d_1}\cdots C^{*\mu_j}_{(1,-1)d_j}$ are again fully antisymmetric with respect to their set of Lorentz indices due to the simultaneous anticommutation among all $C^{*\mu}_{(1,-1)d}$ and the full symmetry of the front coefficients with respect to the internal (collection) BF indices $\{d_1,\cdots,d_j\}$; 3. for $j=1$ there appears a single Lorentz index, and thus no antisymmetrization is necessary; and 4. for $j=2$ there appear only two kinds of terms, which belong to the situations already discussed with respect to items 1 and 2. The status of (97)–(102) with respect to belonging to $H_*^{\mathrm{inv}}(\delta|d,\mathcal{A}^{\mathrm{restricted}}_{\mathrm{local}})$ in strictly positive values of the antifield number is translated, via the action of the Koszul–Tate differential, into the relations

$$\delta P_j^{\Delta|\mu_1\cdots\mu_j}(f) = (-)^j\partial^{[\mu_1}P_{j-1}^{\Delta|\mu_2\cdots\mu_j]}(f),\quad j=\overline{2,6}, \tag{104}$$

$$\delta P_1^{\Delta|\mu_1}(f) = -\partial^{\mu_1}P_0^\Delta(f),\qquad P_0^\Delta(f)\equiv f^\Delta(\varphi). \tag{105}$$

Returning to the construction of the general non-trivial component of maximum antifield number from the first-order deformation, we notice that by inserting results (96) and (97) into (94) it follows that a_6 reduces to the sum between two kinds of terms

$$a_6=P_6^{\Delta_1|\mu_1\cdots\mu_6}(f_1)N^{abc}_{\Delta_1|\mu_1\cdots\mu_6}\eta_{(2,1)a}\eta_{(2,1)b}\eta_{(2,1)c}+P^{\rho_1\cdots\rho_6}_{6,\Delta_2}(f_2)N^{\Delta_2|\mu_1\cdots\mu_6;\nu_1\cdots\nu_6}_{ab|\rho_1\cdots\rho_6}C^a_{(3,2)\mu_1\cdots\mu_6}C^b_{(3,2)\nu_1\cdots\nu_6}, \tag{106}$$

where $P_6^{\Delta_1|\mu_1\cdots\mu_6}(f_1)$ and $P_6^{\Delta_2|\rho_1\cdots\rho_6}(f_2)$ are read as in (97), with $f^\Delta(\varphi)$ replaced by $f_1^{\Delta_1}(\varphi)$ and $f_{2,\Delta_2}(\varphi)$, respectively, and the quantities denoted by N are real field-independent coefficients. In addition, the commutativity of the ghosts $\eta_{(2,1)a}$ and the respective anticommutativity of the ghosts $\overset{[6]}{C}{}^{a}{}_{(3,2)}$ restricts these coefficients in order to satisfy the symmetry/antisymmetry properties:

$$N^{abc}_{\Delta_1|\mu_1\cdots\mu_6} = N^{bac}_{\Delta_1|\mu_1\cdots\mu_6} = N^{acb}_{\Delta_1|\mu_1\cdots\mu_6}, \tag{107}$$

$$N^{\Delta_2|\mu_1\cdots\mu_6;\nu_1\cdots\nu_6}_{ab|\rho_1\cdots\rho_6} = -N^{\Delta_2|\nu_1\cdots\nu_6;\mu_1\cdots\mu_6}_{ba|\rho_1\cdots\rho_6}. \tag{108}$$

Finally, the assumptions of Lorentz covariance and Poincaré invariance together with properties (107) and (108) allow us to represent the general form of both the functions and (constant) coefficients entering (106) as follows:

$$f_1^{\Delta_1}(\varphi) \to f_1^{a'b'c'}(\varphi), \qquad f_{2,\Delta_2}(\varphi) \to f_{2,a'b'}(\varphi), \tag{109}$$

$$N^{abc}_{\Delta_1|\mu_1\cdots\mu_6} \to c_1\delta^{(a}_{a'}\delta^{b}_{b'}\delta^{c)}_{c'}\varepsilon_{\mu_1\cdots\mu_6}, \tag{110}$$

$$N^{\Delta_2|\mu_1\cdots\mu_6;\nu_1\cdots\nu_6}_{ab|\rho_1\cdots\rho_6} \to c_2\delta^{a'}_{[a}\delta^{b'}_{b]}\varepsilon_{\rho_1\cdots\rho_6}\varepsilon^{\mu_1\cdots\mu_6}\varepsilon^{\nu_1\cdots\nu_6}, \tag{111}$$

where c_1 and c_2 are two real constants while the notation (abc) signifies the operation of complete symmetrization with respect to the collection indices between brackets, and $[ab]$ that of complete antisymmetrization. Replacing (109)–(111) in (106), making a convenient choice on c_1 and c_2, and using (2) for $k = 0$, we determine the concrete expression of the component of the highest antifield number from the first-order deformation as the general non-trivial solution to Equation (92), which further complies with all the imposed selection rules, as follows:

$$a_6 = -\frac{1}{2}\varepsilon^{\nu_1\cdots\nu_6}P^{\mu_1\cdots\mu_6}_{6,ab|}(Z)C^{a}_{(3,2)\mu_1\cdots\mu_6}C^{b}_{(3,2)\nu_1\cdots\nu_6} - \frac{1}{90\cdot 3!}\varepsilon_{\mu_1\cdots\mu_6}P^{abc|\mu_1\cdots\mu_6}_{6}(M)\eta_{(2,1)a}\eta_{(2,1)b}\eta_{(2,1)c} \equiv a_6(Z) + a_6(M). \tag{112}$$

The elements $P^{\mu_1\cdots\mu_6}_{6,ab|}(Z)$ and $P^{abc|\mu_1\cdots\mu_6}_{6}(M)$ read as in (97), with

$$f^{\Delta}(\varphi) \to Z_{ab}(\varphi) \equiv \frac{1}{2!}\delta^{a'}_{[a}\delta^{b'}_{b]}f_{2,a'b'}(\varphi), \qquad Z_{ab} = -Z_{ba}, \tag{113}$$

$$f^{\Delta}(\varphi) \to M^{abc}(\varphi) \equiv \frac{1}{3!}\delta^{(a}_{a'}\delta^{b}_{b'}\delta^{c)}_{c'}f_1^{a'b'c'}(\varphi), \qquad M^{abc} = M^{bac} = M^{acb}. \tag{114}$$

We observe that (112) is parameterized in terms of two sets of smooth functions depending on the undifferentiated scalar fields, with complementary symmetry/antisymmetry properties related to their internal collection indices: $\{Z_{ab}(\varphi)\}_{a,b=\overline{1,A}}$ antisymmetric and $\{M^{abc}(\varphi)\}_{a,b,c=\overline{1,A}}$ completely symmetric.

Inserting (112) into Equation (93) for $k = 6$ and then recursively solving the remaining equations for k in reverse order (from higher to lower values), we obtain all the components of the first-order deformation from decomposition (91) via the actions of the Koszul–Tate and exterior longitudinal differentials on the BRST generators, (55)–(61). The decomposition of a_6 into a sum between the two kinds of terms depending either on the parameterizing functions Z_{ab} (and their derivatives of various orders with respect to the scalar fields) or involving M^{abc} and their derivatives is preserved at each antifield number from (91); thus, we can finally state that a can be conveniently expressed as

$$a = a(Z) + a(M), \qquad a(Z) = \sum_{j=0}^{6} a_j(Z), \qquad a(M) = \sum_{j=0}^{6} a_j(M), \tag{115}$$

with $a_6(Z)$ and $a_6(M)$ provided in (112). The inner structure of $a(Z)$ and $a(M)$ reveals the presence, apart from the elements emphasized previously, namely, (97)–(102) from $H^{\mathrm{inv}}_{*}(\delta|d,\mathcal{A}^{\mathrm{restricted}}_{\mathrm{local}})$ in strictly positive values of the antifield number with f^{Δ} replaced by Z_{ab}

and M^{abc}, respectively, of several classes of non-trivial elements from $H_*(\delta|d, \mathcal{A}^{\text{restricted}}_{\text{local}})$ in strictly positive values of the antifield number and at pure ghost number zero that are no longer gauge-invariant, and therefore actually belong to $H_*(\delta|d, \mathcal{A}^{\text{restricted}}_{\text{local}}) \backslash H^{\text{inv}}_*(\delta|d, \mathcal{A}^{\text{restricted}}_{\text{local}})$. More precisely, $a(Z)$ contains a single class of such components and $a(M)$ two distinct types, each kind present only to selected values of the antifield number (j)

$$a(Z) = a_0(Z) + \sum_{j=1}^{3} a_j(Q(Z, \overset{[3]}{B})) + \sum_{j=2}^{6} a_j(P(Z)) \equiv a_0(Z) + a(Q(Z, \overset{[3]}{B})) + a(P(Z)), \tag{116}$$

$$\begin{aligned} a(M) &= a_0(M) + \sum_{j=1}^{2} a_j(R(M, \overset{[2]}{A}, \overset{[2]}{A})) + \sum_{j=2}^{4} a_j(R(M, \overset{[2]}{A})) + \sum_{j=3}^{6} a_j(P(M)) \\ &\equiv a_0(M) + a(R(M, \overset{[2]}{A}, \overset{[2]}{A})) + a(R(M, \overset{[2]}{A})) + a(P(M)). \end{aligned} \tag{117}$$

The concrete form of the various quantities from (116) are provided below

$$a_0(Z) = \frac{1}{2}\varepsilon^{\mu_1\cdots\mu_6} B^a_{\mu_1\mu_2\mu_3} B^b_{\mu_4\mu_5\mu_6} Z_{ab}(\varphi), \tag{118}$$

$$a(Q(Z, \overset{[3]}{B})) = \sum_{j=1}^{3} (-)^{\frac{j(j-1)}{2}} \varepsilon^{\mu_1\cdots\mu_6} Q_{j,b|\mu_1\cdots\mu_{3-j}}(Z, \overset{[3]}{B}) C^b_{(3,j-1)\mu_{4-j}\cdots\mu_6}, \tag{119}$$

$$\begin{aligned} a(P(Z)) = -\varepsilon^{\mu_1\cdots\mu_6} \Big[& 3P^{\lambda_1\lambda_2}_{2,ab|}(Z) C^a_{(3,0)\lambda_1\lambda_2\mu_1\mu_2} C^b_{(3,0)\mu_3\cdots\mu_6} + 4P^{\lambda_1\lambda_2\lambda_3}_{3,ab|}(Z) C^a_{(3,0)\lambda_1\lambda_2\lambda_3\mu_1} C^b_{(3,1)\mu_2\cdots\mu_6} \\ & + P^{\lambda_1\cdots\lambda_4}_{4,ab|}(Z) \left(C^a_{(3,0)\lambda_1\cdots\lambda_4} C^b_{(3,2)\mu_1\cdots\mu_6} - \frac{5}{2} C^a_{(3,1)\lambda_1\cdots\lambda_4\mu_1} C^b_{(3,1)\mu_2\cdots\mu_6} \right) \\ & + P^{\lambda_1\cdots\lambda_5}_{5,ab|}(Z) C^a_{(3,1)\lambda_1\cdots\lambda_5} C^b_{(3,2)\mu_1\cdots\mu_6} + \frac{1}{2} P^{\lambda_1\cdots\lambda_6}_{6,ab|}(Z) C^a_{(3,2)\lambda_1\cdots\lambda_6} C^b_{(3,2)\mu_1\cdots\mu_6} \Big]. \end{aligned} \tag{120}$$

As a side note, we mention that (120) can be rewritten in a more compact manner as

$$a(P(Z)) = \varepsilon^{\mu_1\cdots\mu_6} \sum_{k=0}^{2} \left[\sum_{j=2k+2}^{k+4} \left((-)^{k(j+1)+1} C^j_{k+4} c(j) P^{\lambda_1\cdots\lambda_j}_{j,ab|}(Z) C^a_{(3,k)\lambda_1\cdots\lambda_j\mu_1\cdots\mu_{k+4-j}} C^b_{(3,j-k-2)\mu_{k+5-j}\cdots\mu_6} \right) \right], \tag{121}$$

$$c(j) = \begin{cases} \frac{1}{2}, & j = 2k+2 \\ 1, & j > 2k+2 \end{cases}. \tag{122}$$

In the above, $\left\{ P^{\lambda_1\cdots\lambda_j}_{j,ab|}(Z) \right\}_{j=\overline{2,6}}$ belong to $H^{\text{inv}}_j(\delta|d, \mathcal{A}^{\text{restricted}}_{\text{local}})$ and should be read as in the appropriate formulas (97)–(101), with $f^{\Delta}(\varphi) \to Z_{ab}(\varphi)$. In addition, C^k_n denotes the number of combinations of n items taken k together. The objects denoted by $\left\{ Q_{j,b|\mu_1\cdots\mu_{3-j}}(Z, \overset{[3]}{B}) \right\}_{j=\overline{1,3}}$ display the antifield number j, the pure ghost number 0, and are non-trivial representatives from $H_j(\delta|d, \mathcal{A}^{\text{restricted}}_{\text{local}}) \backslash H^{\text{inv}}_j(\delta|d, \mathcal{A}^{\text{restricted}}_{\text{local}})$ depending on the BF form fields, $\overset{[3]}{B}{}^a$, of the form

$$\begin{aligned} Q_{j,b|\mu_1\cdots\mu_{3-j}}(Z, \overset{[3]}{B}) \equiv & \sum_{l=0}^{j-1} (-)^{\frac{l(l+1)}{2}} C^l_{3-j+l} \eta^{*a}_{(2,j-l-2)\mu_1\cdots\mu_{3-j}\lambda_1\cdots\lambda_l} P^{\lambda_1\cdots\lambda_l}_{l,ab|}(Z) \\ & + (-)^{\frac{j(j+1)}{2}} C^j_3 B^a_{\mu_1\cdots\mu_{3-j}\lambda_1\cdots\lambda_j} P^{\lambda_1\cdots\lambda_j}_{j,ab|}(Z), \quad j = \overline{1,3}, \end{aligned} \tag{123}$$

where the quantities $\left\{P^{\lambda_1\cdots\lambda_j}_{j,ab|}(Z)\right\}_{j=\overline{1,3}}$ and $\left\{\left\{P^{\lambda_1\cdots\lambda_l}_{l,ab|}(Z)\right\}_{l=\overline{0,j-1}}\right\}_{j=\overline{1,3}}$ are again provided by the relevant formula among (100)–(102), with $f^\Delta \to Z_{ab}$, while

$$P_{0,ab}(Z) \equiv Z_{ab}(\varphi). \tag{124}$$

They satisfy the recursive relations showing their affiliation to $H_j(\delta|d,\mathcal{A}^{\text{restricted}}_{\text{local}})$

$$\delta Q_{j,b|\mu_1\cdots\mu_{3-j}}(Z,\overset{[3]}{B}) = (-)^{j-1}(4-j)\partial^\lambda Q_{j-1,b|\lambda\mu_1\cdots\mu_{3-j}}(Z,\overset{[3]}{B}), \quad j=\overline{1,3}, \tag{125}$$

with

$$Q_{0,b|\mu_1\mu_2\mu_3}(Z,\overset{[3]}{B}) \equiv B^a_{\mu_1\mu_2\mu_3}P_{0,ab}(Z). \tag{126}$$

It is easy to see that $Q_{j,b|\mu_1\cdots\mu_{3-j}}(Z,\overset{[3]}{B})$, provided in (123), *are not gauge-invariant* due to the last terms from their structure depending on the BF form fields, $\overset{[3]}{B}{}^a$, endowed with the non-trivial gauge transformations placed on the third line from (5).

The various quantities from (117) read as

$$a_0(M) = \frac{1}{180}\varepsilon_{\mu_1\cdots\mu_6}A_a^{[\mu_1\mu_2}A_b^{\mu_3\mu_4}A_c^{\mu_5\mu_6]}M^{abc}(\varphi), \tag{127}$$

$$a(R(M,\overset{[2]}{A},\overset{[2]}{A})) = \frac{1}{90}\varepsilon_{\mu_1\cdots\mu_6}\left(R_1^{c|[\mu_1\cdots\mu_5}(M,\overset{[2]}{A},\overset{[2]}{A})\eta^{\mu_6]}_{(2,0)c} - R_2^{c|\mu_1\cdots\mu_6}(M,\overset{[2]}{A},\overset{[2]}{A})\eta_{(2,1)c}\right), \tag{128}$$

$$\begin{aligned} a(R(M,\overset{[2]}{A})) = {}& \frac{1}{90}\varepsilon_{\mu_1\cdots\mu_6}\left(R_2^{bc|[\mu_1\cdots\mu_4}(M,\overset{[2]}{A})\eta^{\mu_5}_{(2,0)b}\eta^{\mu_6]}_{(2,0)c} - R_3^{bc|[\mu_1\cdots\mu_5}(M,\overset{[2]}{A})\eta^{\mu_6]}_{(2,0)b}\eta_{(2,1)c}\right. \\ & \left.+\frac{1}{2}R_4^{bc|\mu_1\cdots\mu_6}\eta_{(2,1)b}\eta_{(2,1)c}\right), \end{aligned} \tag{129}$$

$$\begin{aligned} a(P(M)) = {}& \frac{1}{90}\varepsilon_{\mu_1\cdots\mu_6}\left(P_3^{abc|[\mu_1\mu_2\mu_3}(M)\eta^{\mu_4}_{(2,0)a}\eta^{\mu_5}_{(2,0)b}\eta^{\mu_6]}_{(2,0)c} - P_4^{abc|[\mu_1\cdots\mu_4}(M)\eta^{\mu_5}_{(2,0)a}\eta^{\mu_6]}_{(2,0)b}\eta_{(2,1)c}\right. \\ & \left.+\frac{1}{2}P_5^{abc|[\mu_1\cdots\mu_5}(M)\eta^{\mu_6]}_{(2,0)a}\eta_{(2,1)b}\eta_{(2,1)c} - \frac{1}{3!}P_6^{abc|\mu_1\cdots\mu_6}(M)\eta_{(2,1)a}\eta_{(2,1)b}\eta_{(2,1)c}\right), \end{aligned} \tag{130}$$

where $\left\{P_j^{abc|\mu_1\cdots\mu_j}(M)\right\}_{j=\overline{3,6}}$ belong to $H^{\text{inv}}_j(\delta|d,\mathcal{A}^{\text{restricted}}_{\text{local}})$ and read as in formulas (97)–(100), with $f^\Delta(\varphi) \to M^{abc}(\varphi)$. We recall that $[\mu_1\cdots\mu_6]$ signifies full antisymmetry with respect to the indices between brackets, defined according to the conventions explained in the paragraph following Formula (2). The quantities denoted by $\left\{R_j^{c|\mu_1\cdots\mu_{j+4}}(M,\overset{[2]}{A},\overset{[2]}{A})\right\}_{j=\overline{1,2}}$ and $\left\{R_j^{bc|\mu_1\cdots\mu_{j+2}}(M,\overset{[2]}{A})\right\}_{j=\overline{2,4}}$, respectively, possess an antifield number equal to j, a pure ghost number equal to 0, and define non-trivial elements from $H_j(\delta|d,\mathcal{A}^{\text{restricted}}_{\text{local}})\setminus H^{\text{inv}}_j(\delta|d,\mathcal{A}^{\text{restricted}}_{\text{local}})$ that involve the BF form fields $\overset{[2]}{A}$, expressed as follows:

$$\begin{aligned} R_2^{c|\mu_1\cdots\mu_6}(M,\overset{[2]}{A},\overset{[2]}{A}) \equiv {}& A_a^{[\mu_1\mu_2}A_b^{\mu_3\mu_4}P_2^{abc|\mu_5\mu_6]}(M) + C^{*[\mu_1\mu_3\mu_3}_{(3,-1)a}A_b^{\mu_4\mu_5}P_1^{abc|\mu_6]}(M) \\ & +\left(C^{*[\mu_1\cdots\mu_4}_{(3,0)a}A_b^{\mu_5\mu_6]} + C^{*[\mu_1\mu_3\mu_3}_{(3,-1)a}C^{*\mu_4\mu_5\mu_6]}_{(3,-1)b}\right)P_0^{abc}(M), \end{aligned} \tag{131}$$

$$R_1^{c|\mu_1\cdots\mu_5}(M,\overset{[2]}{A},\overset{[2]}{A}) \equiv A_a^{[\mu_1\mu_2}A_b^{\mu_3\mu_4}P_1^{abc|\mu_5]}(M) + C^{*[\mu_1\mu_3\mu_3}_{(3,-1)a}A_b^{\mu_4\mu_5]}P_0^{abc}(M), \tag{132}$$

$$R_j^{bc|\mu_1\cdots\mu_{j+2}}(M,\overset{[2]}{A}) \equiv A_a^{[\mu_1\mu_2}P_j^{abc|\mu_3\cdots\mu_{j+2}]}(M)+\sum_{l=-1}^{j-2}C_{(3,l)a}^{*[\mu_1\cdots\mu_{l+4}}P_{j-l-2}^{abc|\mu_{l+5}\cdots\mu_{j+2}]}(M), \quad j=\overline{1,4}. \tag{133}$$

The notation $P_0^{abc}(M)$ implied in (131)–(133) is defined by

$$P_0^{abc}(M)\equiv M^{abc}(\varphi). \tag{134}$$

We stress that only the elements from (133) with the antifield number $j=\overline{2,4}$ are present in (117) and (129); there is a more general chain in $H_j(\delta|d,\mathcal{A}_{\text{local}}^{\text{restricted}})\backslash H_j^{\text{inv}}(\delta|d,\mathcal{A}_{\text{local}}^{\text{restricted}})$ that includes the value $j=1$. The above quantities satisfy the next recursive relations, marking their inclusion in $H_j(\delta|d,\mathcal{A}_{\text{local}}^{\text{restricted}})$

$$\delta R_j^{c|\mu_1\cdots\mu_{j+4}}(M,\overset{[2]}{A},\overset{[2]}{A}) = (-)^j\partial^{[\mu_1}R_{j-1}^{c|\mu_2\cdots\mu_{j+4}]}(M,\overset{[2]}{A},\overset{[2]}{A}), \quad j=1,2, \tag{135}$$

$$R_0^{c|\mu_1\cdots\mu_4}(M,\overset{[2]}{A},\overset{[2]}{A}) \equiv A_a^{[\mu_1\mu_2}A_b^{\mu_3\mu_4]}P_0^{abc}(M), \tag{136}$$

$$\delta R_j^{bc|\mu_1\cdots\mu_{j+2}}(M,\overset{[2]}{A}) = (-)^j\partial^{[\mu_1}R_{j-1}^{bc|\mu_2\cdots\mu_{j+2}]}(M,\overset{[2]}{A}), \quad j=\overline{1,4}, \tag{137}$$

$$R_0^{bc|\mu_1\mu_2}(M,\overset{[2]}{A}) \equiv A_a^{\mu_1\mu_2}P_0^{abc}(M). \tag{138}$$

Just as in relation to (123), *neither* of the elements (131)–(133) *is gauge-invariant*, due to its dependence on the BF form fields $\overset{[2]}{A}_a$ displaying non-trivial gauge transformations, as can be seen from the second line in (5).

At each value of the antifield number strictly less than its maximum value from decomposition (91), $j=\overline{0,5}$, we must investigate the supplementary non-trivial solutions to the first-order deformation Equation (77) that stop at antifield number j and are independently consistent of $a(Z)$, $a(M)$, and among themselves (of course, only at order one of perturbation theory, as at higher orders their parameterizing coefficients may be restricted to satisfy certain identities that ensure the existence of the remaining deformation equations, (78) and (79), etc.):

$$a \to a+\sum_{j=0}^{5}a^{(j)}, \quad \left\{a^{(0)}=a_0^{(0)},\ l^{(0)\mu}=l_0^{(0)\mu}\right\}_{j=0}, \quad \left\{a^{(j)}=\sum_{k=0}^{j}a_k^{(j)},\ l^{(j)\mu}=\sum_{k=0}^{j-1}l_k^{(j)\mu}\right\}_{j=\overline{1,5}}, \tag{139}$$

$$j = \overline{0,5}: sa^{(j)}=\partial_\mu l^{(j)\mu} \Leftrightarrow \begin{cases} j=\overline{1,5}: \gamma a_j^{(j)}=0,\ \delta a_k^{(j)}+\gamma a_{k-1}^{(j)}=\partial_\mu l_{k-1}^{(j)\mu},\ k=\overline{1,j} \\ j=0: \gamma a_0^{(0)}=\partial_\mu l_0^{(0)\mu} \end{cases}, \tag{140}$$

where the main degrees of the quantities involved in (139) and (140) read as in (87) and (88). The solutions to the last (homogeneous) equation from each descent for $j=\overline{1,5}$, $\gamma a_j^{(j)}=0$, fall into two classes. A cohomological BRST analysis for $j=\overline{2,5}$ reveals that such homogeneous solutions may be safely constructed as at the maximum value of the antifield number, 6, namely, by "gluing" the non-trivial representatives of $H_j^{\text{inv}}(\delta|d,\mathcal{A}_{\text{local}}^{\text{restricted}})$ (in antifield number j and pure ghost number 0) to the ghost basis from $H_0^j(\gamma,\mathcal{A}_{\text{local}}^{\text{restricted}})$ (in pure ghost number j and antifield number 0), where the latter is taken to be field-independent. For $j=1$, by coupling the gauge-invariant non-trivial representatives of $H_1(\delta|d,\mathcal{A}_{\text{local}}^{\text{restricted}})$ (again in antifield number 1 and pure ghost number 0) to the field-independent, ghost basis from $H_0^1(\gamma,\mathcal{A}_{\text{local}}^{\text{restricted}})$,

$$a_j^{(j)} : e^j\in H_0^j(\gamma,\mathcal{A}_{\text{local}}^{\text{restricted}}) \longleftrightarrow \alpha_j^{\text{inv}}\in H_j^{\text{inv}}(\delta|d,\mathcal{A}_{\text{local}}^{\text{restricted}}), \quad j=\overline{2,5}, \tag{141}$$

$$a_1^{(1)} : e^1\in H_0^1(\gamma,\mathcal{A}_{\text{local}}^{\text{restricted}}) \longleftrightarrow \alpha_1\in H_1^0(\delta|d,\mathcal{A}_{\text{local}}^{\text{restricted}})\cap H^*(\gamma,\mathcal{A}_{\text{local}}^{\text{restricted}}). \tag{142}$$

The main difference between the two situations resides in the fact that, while for $j = \overline{2,5}$ the elements $\alpha_j^{\rm inv}$ are subject to the equation $\delta\alpha_j^{\rm inv} = \partial_\mu \bar{k}_{j-1}^{{\rm inv}\,\mu}$ (with $\bar{k}_{j-1}^{{\rm inv}\,\mu}$ necessarily a gauge-invariant current of antifield number $(j-1)$ and pure ghost number 0, $\delta_{\Omega^{\alpha_1}} \bar{k}_{j-1}^{{\rm inv}\,\mu} = 0$), in antifield number 1, although still gauge-invariant because it is ghost-independent and belongs to $H^*(\gamma, \mathcal{A}_{\rm local}^{\rm restricted})$, α_1 is only subject to the more general equation $\delta\alpha_1 = \partial_\mu k_0^\mu$, where k_0^μ may depend only on the BF fields and their space-time derivatives (as both its antifield and pure ghost numbers are equal to 0); however, it is no longer required to be gauge-invariant, and thus $\delta_{\Omega^{\alpha_1}} k_0^\mu$ may be non-vanishing. Finally, in antifield number 0 we can search for the non-trivial non-homogeneous solutions to the first-order deformation equation $s a_0^{(0)} \equiv \gamma a_0^{(0)} = \partial_\mu l_0^{(0)\mu}$, where $l_0^{(0)\mu}$ may be non-trivial ($\partial_\mu l_0^{(0)\mu} \neq 0$); thus, we can write that

$$a_0^{(0)} \in H_0^0\left(\gamma | d, \mathcal{A}_{\rm local}^{\rm restricted}\right), \tag{143}$$

where $H_0^0\left(\gamma | d, \mathcal{A}_{\rm local}^{\rm restricted}\right)$ denotes the local cohomology of γ in pure ghost number 0 (upper index) and antifield number 0 (lower index) computed in $\mathcal{A}_{\rm local}^{\rm restricted}$. Taking into account the previous remarks and the imposed selection rules (among which the Lorentz covariance is particularly important, as it helps with ruling out the components from (139) with $j = 5$, $j = 2$, and $j = 1$), it can be shown that $\mathcal{A}_{\rm local}^{\rm restricted}$ hosts only three kinds of such non-trivial solutions that can extend the first-order deformation, which stop individually in antifield numbers 4, 3, and 0, respectively:

$$a \rightarrow a + a^{(4)} + a^{(3)} + a^{(0)}, \tag{144}$$

$$a^{(4)} = \sum_{k=0}^{4} a_k^{(4)}, \quad \begin{cases} a_4^{(4)} = \frac{1}{180} \varepsilon_{\mu_1 \cdots \mu_6} g^{abc} C_{(3,2)a}^{*\mu_1 \cdots \mu_6} \eta_{(2,1)b} \eta_{(2,1)c} \\ a_3^{(4)} = -\frac{1}{15} \varepsilon_{\mu_1 \cdots \mu_6} g^{abc} C_{(3,1)a}^{*\mu_1 \cdots \mu_5} \eta_{(2,0)b}^{\mu_6} \eta_{(2,1)c} \\ a_2^{(4)} = -\frac{1}{3} \varepsilon_{\mu_1 \cdots \mu_6} g^{abc} \left(\frac{1}{2} C_{(3,0)a}^{*\mu_1 \cdots \mu_4} A_b^{\mu_5 \mu_6} + \frac{1}{3} B_a^{*\mu_1 \mu_2 \mu_3} B_b^{*\mu_4 \mu_5 \mu_6} \right) \eta_{(2,1)c} \\ \qquad + \frac{1}{6} \varepsilon_{\mu_1 \cdots \mu_6} g^{abc} C_{(3,0)a}^{*\mu_1 \cdots \mu_4} \eta_{(2,0)b}^{\mu_5} \eta_{(2,0)c}^{\mu_6} \\ a_1^{(4)} = \frac{2}{3} \varepsilon_{\mu_1 \cdots \mu_6} g^{abc} B_a^{*\mu_1 \mu_2 \mu_3} A_b^{\mu_4 \mu_5} \eta_{(2,0)c}^{\mu_6} \\ a_0^{(4)} = \frac{1}{6} \varepsilon_{\mu_1 \cdots \mu_6} g^{abc} A_a^{\mu_1 \mu_2} A_b^{\mu_3 \mu_4} A_c^{\mu_5 \mu_6} \end{cases}, \tag{145}$$

$$a^{(3)} = \sum_{k=0}^{3} a_k^{(3)}, \quad \begin{cases} a_3^{(3)} = -\varepsilon^{\mu_1 \cdots \mu_6} f_{ab} \eta_{(2,1)}^{*a} C_{(3,2)\mu_1 \cdots \mu_6}^b \\ a_2^{(3)} = -\varepsilon^{\mu_1 \cdots \mu_6} f_{ab} \eta_{(2,0)\mu_1}^{*a} C_{(3,1)\mu_2 \cdots \mu_6}^c \\ a_1^{(3)} = \varepsilon^{\mu_1 \cdots \mu_6} f_{ab} A_{\mu_1 \mu_2}^{*a} C_{(3,0)\mu_3 \cdots \mu_6}^b \\ a_0^{(3)} = \frac{1}{2} \varepsilon^{\mu_1 \cdots \mu_6} f_{ab} B_{\mu_1 \mu_2 \mu_3}^a B_{\mu_4 \mu_5 \mu_6}^b \end{cases}, \tag{146}$$

$$a^{(0)} \equiv a_0^{(0)} = V(\varphi), \tag{147}$$

where g^{abc} and f_{ab} are some real constants with all g fully symmetric and all f antisymmetric, while $V(\varphi)$ stands for an arbitrary smooth function depending only on the undifferentiated scalar fields $\{\varphi_a\}$. Solutions $a^{(4)}$ and $a^{(3)}$, although non-trivial, may be absorbed into (115) via simple shift-like redefinitions of the parameterizing functions M and Z, as follows:

$$M^{abc}(\varphi) \rightarrow M^{abc}(\varphi) + g^{abc}, \tag{148}$$

$$Z_{ab}(\varphi) \rightarrow Z_{ab}(\varphi) + f_{ab}, \tag{149}$$

while $V(\varphi)$ represents the only new functionally independent component that can be added to the first-order deformation (115). It is interesting to observe that even in antifield number 0 it is possible in principle to work with non-trivial solutions to the non-homogeneous equation $\gamma a_0^{(0)} = \partial_\mu l_0^{(0)\mu} \neq 0$, although in the case of the model under study we are confined only to a single class of solutions to the homogeneous version, $\gamma a_0^{(0)} = 0$, depending only on the undifferentiated scalar BF fields, $a_0^{(0)} = V(\varphi)$. This result is not obvious, and can be proved following a line similar to that employed, for instance, in [41] (Appendix A, beginning with Formula (243)) or [51] (Appendix A, Section 2, starting from Equation (A42)). The main consequence of the results obtained in the above is that the most general form of the non-integrated density of the first-order deformation actually decomposes as in (115), to which we must add the arbitrary 'potential', $V(\varphi)$

$$a \to a(Z) + a(M) + V(\varphi). \tag{150}$$

The final conclusion of the approach developed thus far is that the most general non-integrated density of the first-order deformation of the solution to the master equation related to all consistent self-interactions that can be added to a non-standard collection of topological BF models described in the free limit by action (1), which, in addition, satisfies all the imposed selection rules discussed in the beginning of Section 3.2, decomposes as in (150), where the first two components are provided in (115) and are structured as in (116) and (117), their concrete expressions being contained in (118)–(120) and (127)–(130), while the various elements from the local homology of the Koszul–Tate differential $H_j(\delta|d, \mathcal{A}^{\text{restricted}}_{\text{local}})$ involved therein are detailed in (123) and (131)–(133). It is important to underline that the entire first-order deformation is parameterized in terms of two sets of functions, $\{Z_{ab}\}$ and $\{M^{abc}\}$, and a 'potential', $V(\varphi)$, all depending smoothly only on the undifferentiated scalar fields from the BF spectrum, among which the former set is antisymmetric and the latter set completely symmetric. Apart from their special symmetry properties, the two function sets (as well as the 'potential') are otherwise arbitrary. We will see in the sequel that while the consistency of the first-order deformation at order two of perturbation theory imposes certain relations among the two function sets, it does not constrain the 'potential', $V(\varphi)$.

3.3. Consistency of the First-Order Deformation. Obstructions. Complete Deformed Solution to the Master Equation

Having completed the construction of the first-order deformation, we pass to the next step, namely, investigating the existence of the second-order deformation, S_2, as a solution to Equation (73). On behalf of notations (75) and (76), the local form of (73) is expressed by Equation (78). Using relations (150), (115)–(124), and (127)–(134), we infer by direct computation that the non-integrated density of the antibracket $(1/2)(S_1, S_1)$ reads as

$$\Delta = \left(M^{abd}(\varphi)Z_{dc}(\varphi)\right)Y^c_{ab}\left(\Phi^A, \Phi^*_A\right) + \sum_{k=1}^{6} \frac{\partial^k\left(M^{abd}(\varphi)Z_{dc}(\varphi)\right)}{\partial\varphi_{e_1}\cdots\partial\varphi_{e_k}} Y^c_{ab|e_1\cdots e_k}\left(\Phi^A, \Phi^*_A\right), \tag{151}$$

where Y^c_{ab} and $\left\{Y^c_{ab|e_1\cdots e_k}\right\}_{k=\overline{1,6}}$ denote some purely fermionic quantities of ghost number equal to 1 that depend *only on the undifferentiated BRST generators* (33) and (34), excepting the BF scalar fields, the dependence of which is entirely contained in the front coefficients (which multiply the Ys). Inspecting (78), we notice that the existence of the non-integrated density of the second-order deformation, b, is equivalent to Δ belonging to a trivial (s-exact modulo d-exact terms) class from the local cohomology of the BRST differential in ghost number 1, which is precisely computed in the algebra $\mathcal{A}^{\text{restricted}}_{\text{local}}$ in order to comply with all the imposed selection rules. Nevertheless, this is not possible due to the obvious fact that neither the Ys nor the coefficients that multiply them contain space-time derivatives of the BRST generators, while the actions of both d and s (see definitions (55)–(61)) acting on $\mathcal{A}^{\text{restricted}}_{\text{local}}$ produce at least one space-time derivative acting on the fields, ghosts, and/or antifields. In conclusion, *the deformed solution to the master equation is obstructed at order two in the coupling constant due to the space-time locality hypothesis*. Moreover, because the Ys are functionally independent quantities, the only solution to solving this obstruction is to annihilate (151), which is equivalent to setting as zero all the coefficients in front of Y^c_{ab} (then, all the other coefficients are contracted to the remaining Ys, and will vanish strongly)

$$\Delta = 0 \Leftrightarrow \left\{M^{abd}(\varphi)Z_{dc}(\varphi) = 0, \quad a,b,c = \overline{1,A}\right\}. \tag{152}$$

Assuming there exist any non-vanishing solutions to the algebraic equations

$$M^{abd}(\varphi)Z_{dc}(\varphi) = 0, \quad a,b,c = \overline{1,A} \tag{153}$$

with respect to the functions of type M and Z, (henceforth called the consistency equations), we can safely use the second-order deformation to vanish

$$S_2 = 0. \tag{154}$$

Substituting (154) into (71) and analyzing the remaining higher-order deformation equations, we can further deduce that we can set

$$S_3 = S_4 = \cdots = 0. \tag{155}$$

It is important to observe that the 'potential', $V(\varphi)$, is *not* restricted by the consistency of the deformations of the solution to the master equation, and remains completely arbitrary up to the smoothness condition.

Combining the results deduced thus far via (67), we conclude that *the most general non-trivial deformation of the solution to the master equation that is consistent to all orders of perturbation theory, complies with all the working hypotheses, and provides* $D = 6$ *self-interactions among a non-standard collection of topological BF models ends at order one in the deformation parameter*

$$\bar{S} = S + \lambda S_1, \tag{156}$$

where S is the solution to the master equation for the starting free model, that is, (66), the functional S_1 is expressed via (150), (115)–(124), and (127)–(134), and the parameterizing function sets $\{Z_{ab}\}$ and $\left\{M^{abc}\right\}$ as well as the 'potential', $V(\varphi)$, depend smoothly only on the undifferentiated scalar fields from the BF spectrum, with the function sets being non-vanishing solutions to the consistency conditions (153). Assembling (156) according to its components organized along the increasing values of the antifield number, we can write that

$$\bar{S} = \sum_{j=0}^{6}\left(\int d^6x\, \mathcal{L}_j\right), \quad \varepsilon(\mathcal{L}_j) = 0, \quad \mathcal{L}_j^\star = \mathcal{L}_j, \quad \mathrm{gh}(\mathcal{L}_j) = 0, \quad \mathrm{ant}(\mathcal{L}_j) = j. \tag{157}$$

For further interpretation and analysis of the Lagrangian formulation of the self-interacting BF model corresponding to (156), we present below the concrete expressions of each component of $\bar{S}$ following from the previous subsection without full antisymmetrization with respect to each set of contracted Lorentz indices. This means, for instance, that a former expression written previously as $\varepsilon_{\mu_1\cdots\mu_6}t^{[\mu_1\cdots\mu_5}u^{\mu_6]}$, with $t^{\mu_1\cdots\mu_5}$ fully antisymmetric, may be replaced by $6\varepsilon_{\mu_1\cdots\mu_6}t^{\mu_1\cdots\mu_5}u^{\mu_6}$ in agreement with our antisymmetrization conventions. The pieces of antifield number 0 and 1 read, respectively,

$$\mathcal{L}_0 = B^a_{\mu_1}\partial^{\mu_1}\varphi_a + B^a_{\mu_1\mu_2\mu_3}\left(\partial^{[\mu_1}A^{\mu_2\mu_3]}_a + \frac{\lambda}{2}\varepsilon^{\mu_1\cdots\mu_6}Z_{ab}B^b_{\mu_4\mu_5\mu_6}\right) + \frac{\lambda}{6}\varepsilon_{\mu_1\cdots\mu_6}M^{abc}A^{\mu_1\mu_2}_a A^{\mu_3\mu_4}_b A^{\mu_5\mu_6}_c + \lambda V(\varphi), \tag{158}$$

$$\begin{aligned}\mathcal{L}_1 = {}& A^{*a}_{\mu_1\mu_2}\left(\partial^{[\mu_1}\eta^{\mu_2]}_{(2,0)a} + \lambda\varepsilon^{\mu_1\cdots\mu_6}Z_{ab}C^b_{(3,0)\mu_3\cdots\mu_6}\right) + B^{*\mu_1\mu_2\mu_3}_a\left(-4\partial^\rho C^a_{(3,0)\rho\mu_1\mu_2\mu_3} + \frac{2\lambda}{3}\varepsilon_{\mu_1\cdots\mu_6}M^{abc}A^{\mu_4\mu_5}_b\eta^{\mu_6}_{(2,0)c}\right) \\ &+B^{*\mu_1}_a\left(-2\partial^\rho C^a_{(1,0)\rho\mu_1} - 4\lambda\varepsilon^{\rho_1\cdots\rho_6}\frac{\partial Z_{bc}}{\partial\varphi_a}B^b_{\rho_1\rho_2\rho_3}C^c_{(3,0)\mu_1\rho_4\rho_5\rho_6} + \lambda\varepsilon_{\mu_1\cdots\mu_6}\frac{\partial M^{bcd}}{\partial\varphi_a}A^{\mu_2\mu_3}_b A^{\mu_4\mu_5}_c\eta^{\mu_6}_{(2,0)d}\right).\end{aligned} \tag{159}$$

The terms of antifield number 2 are structured as follows:

$$\begin{aligned}\mathcal{L}_2 &= \eta^{*a}_{(2,0)\mu_1}\left(\partial^{\mu_1}\eta_{(2,1)a} - \lambda\varepsilon^{\mu_1\cdots\mu_6}Z_{ab}C^c_{(3,1)\mu_2\cdots\mu_6}\right) + C^{*\mu_1\cdots\mu_4}_{(3,0)a}\left(-5\partial^\rho C^a_{(3,1)\rho\mu_1\cdots\mu_4} - \frac{\lambda}{6}\varepsilon_{\mu_1\cdots\mu_6}M^{abc}A^{\mu_5\mu_6}_b\eta_{(2,1)c}\right)\\
&+C^{*\mu_1\mu_2}_{(1,0)a}\left(-3\partial^\rho C^a_{(1,1)\rho\mu_1\mu_2} + 10\lambda\varepsilon^{\rho_1\cdots\rho_6}\frac{\partial Z_{bc}}{\partial\varphi_a}B^b_{\rho_1\rho_2\rho_3}C^c_{(3,1)\mu_1\mu_2\rho_4\rho_5\rho_6} - \frac{\lambda}{2}\varepsilon_{\mu_1\cdots\mu_6}\frac{\partial M^{bcd}}{\partial\varphi_a}A^{\mu_3\mu_4}_b A^{\mu_5\mu_6}_c\eta_{(2,1)d}\right)\\
&+\lambda\varepsilon^{\mu_1\cdots\mu_6}\left(2\frac{\partial Z_{ab}}{\partial\varphi_d}B^{*\rho}_d A^{*a}_{\rho\mu_1} + 3\frac{\partial^2 Z_{ab}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}B^{*\rho_1}_{d_1}B^{*\rho_2}_{d_2}B^a_{\rho_1\rho_2\mu_1}\right)C^b_{(3,1)\mu_2\cdots\mu_6}\\
&-\lambda\varepsilon_{\mu_1\cdots\mu_6}\left[\frac{1}{3}B^{*\mu_1\mu_2\mu_3}_a\left(\frac{1}{3}M^{abc}B^{*\mu_4\mu_5\mu_6}_b + 2\frac{\partial M^{abc}}{\partial\varphi_d}B^{*\mu_4}_d A^{\mu_5\mu_6}_b\right) + \frac{1}{2}\frac{\partial^2 M^{abc}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}B^{*\mu_1}_{d_1}B^{*\mu_2}_{d_2}A^{\mu_3\mu_4}_a A^{\mu_5\mu_6}_b\right]\eta_{(2,1)c}\\
&-3\lambda\varepsilon^{\mu_1\cdots\mu_6}\left(\frac{\partial Z_{ab}}{\partial\varphi_d}C^{*\rho_1\rho_2}_{(1,0)d} + \frac{\partial^2 Z_{ab}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}B^{*\rho_1}_{d_1}B^{*\rho_2}_{d_2}\right)C^a_{(3,0)\rho_1\rho_2\mu_1\mu_2}C^b_{(3,0)\mu_3\cdots\mu_6} + \lambda\varepsilon_{\mu_1\cdots\mu_6}\left(\frac{1}{6}M^{abc}C^{*\mu_1\cdots\mu_4}_{(3,0)a}\right.\\
&\left.+\frac{\partial M^{abc}}{\partial\varphi_d}C^{*\mu_1\mu_2}_{(1,0)d}A^{\mu_3\mu_4}_a + \frac{2}{3}\frac{\partial M^{abc}}{\partial\varphi_d}B^{*\mu_1\mu_2\mu_3}_a B^{*\mu_4}_d + \frac{\partial^2 M^{abc}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}B^{*\mu_1}_{d_1}B^{*\mu_2}_{d_2}A^{\mu_3\mu_4}_a\right)\eta^{\mu_5}_{(2,0)b}\eta^{\mu_6}_{(2,0)c}. \end{aligned} \tag{160}$$

The non-integrated density with the antifield number equal to 3 is provided by

$$\begin{aligned}\mathcal{L}_3 &= -\lambda\varepsilon^{\mu_1\cdots\mu_6}\eta^{*a}_{(2,1)}Z_{ab}C^b_{(3,2)\mu_1\cdots\mu_6} - 6C^{*\mu_1\cdots\mu_5}_{(3,1)a}\partial^\rho C^a_{(3,2)\rho\mu_1\cdots\mu_5}\\
&-C^{*\mu_1\mu_2\mu_3}_{(1,1)a}\left(4\partial^\rho C^a_{(1,2)\rho\mu_1\mu_2\mu_3} + 20\lambda\varepsilon^{\rho_1\cdots\rho_6}\frac{\partial Z_{bc}}{\partial\varphi_a}B^b_{\rho_1\rho_2\rho_3}C^c_{(3,2)\mu_1\mu_2\mu_3\rho_4\rho_5\rho_6}\right)\\
&-\lambda\varepsilon^{\mu_1\cdots\mu_6}\left[\left(3\frac{\partial^2 Z_{ab}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}C^{*\rho_1\rho_2}_{(1,0)d_1}B^{*\rho_3}_{d_2} + \frac{\partial^3 Z_{ab}}{\partial\varphi_{d_1}\partial\varphi_{d_2}\partial\varphi_{d_3}}B^{*\rho_1}_{d_1}B^{*\rho_2}_{d_2}B^{*\rho_3}_{d_3}\right)B^a_{\rho_1\rho_2\rho_3}\right.\\
&\left.-\left(\frac{\partial Z_{ab}}{\partial\varphi_d}C^{*\rho_1\rho_2}_{(1,0)d_1} + \frac{\partial^2 Z_{ab}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}B^{*\rho_1}_{d_1}B^{*\rho_2}_{d_2}\right)A^{*a}_{\rho_1\rho_2} - \frac{\partial Z_{ab}}{\partial\varphi_d}B^{*\rho}_d\eta^{*a}_{(2,0)\rho}\right]C^b_{(3,2)\mu_1\cdots\mu_6}\\
&-4\lambda\varepsilon^{\mu_1\cdots\mu_6}\left(\frac{\partial Z_{ab}}{\partial\varphi_d}C^{*\rho_1\rho_2\rho_3}_{(1,1)d} + 3\frac{\partial^2 Z_{ab}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}C^{*\rho_1\rho_2}_{(1,0)d_1}B^{*\rho_3}_{d_2} + \frac{\partial^3 Z_{ab}}{\partial\varphi_{d_1}\partial\varphi_{d_2}\partial\varphi_{d_3}}B^{*\rho_1}_{d_1}B^{*\rho_2}_{d_2}B^{*\rho_3}_{d_3}\right)C^a_{(3,0)\rho_1\rho_2\rho_3\mu_1}C^b_{(3,1)\mu_2\cdots\mu_6}\\
&-\frac{\lambda}{3}\varepsilon_{\mu_1\cdots\mu_6}\left[\frac{1}{5}M^{abc}C^{*\mu_1\cdots\mu_5}_{(3,1)a} + \frac{\partial M^{abc}}{\partial\varphi_d}C^{*\mu_1\cdots\mu_4}_{(3,0)a}B^{*\mu_5}_d + 2B^{*\mu_1\mu_2\mu_3}_a\left(\frac{\partial M^{abc}}{\partial\varphi_d}C^{*\mu_4\mu_5}_{(1,0)d} + \frac{\partial^2 M^{abc}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}B^{*\mu_4}_{d_1}B^{*\mu_5}_{d_2}\right)\right.\\
&\left.+2A^{\mu_1\mu_2}_a\left(\frac{\partial M^{abc}}{\partial\varphi_d}C^{*\mu_3\mu_4\mu_5}_{(1,1)d} + 3\frac{\partial^2 M^{abc}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}C^{*\mu_3\mu_4}_{(1,0)d_1}B^{*\mu_5}_{d_2} + \frac{\partial^3 M^{abc}}{\partial\varphi_{d_1}\partial\varphi_{d_2}\partial\varphi_{d_3}}B^{*\mu_3}_{d_1}B^{*\mu_4}_{d_2}B^{*\mu_5}_{d_3}\right)\right]\eta^{\mu_6}_{(2,0)b}\eta_{(2,1)c}\\
&+\frac{2\lambda}{9}\varepsilon_{\mu_1\cdots\mu_6}\left(\frac{\partial M^{abc}}{\partial\varphi_d}C^{*\mu_1\mu_2\mu_3}_{(1,1)d} + 3\frac{\partial^2 M^{abc}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}C^{*\mu_1\mu_2}_{(1,0)d_1}B^{*\mu_3}_{d_2} + \frac{\partial^3 M^{abc}}{\partial\varphi_{d_1}\partial\varphi_{d_2}\partial\varphi_{d_3}}B^{*\mu_1}_{d_1}B^{*\mu_2}_{d_2}B^{*\mu_3}_{d_3}\right)\eta^{\mu_4}_{(2,0)a}\eta^{\mu_5}_{(2,0)b}\eta^{\mu_6}_{(2,0)c}. \end{aligned} \tag{161}$$

Related to the piece of antifield number 4, we have

$$\begin{aligned}\mathcal{L}_4 &= -5C^{*\mu_1\cdots\mu_4}_{(1,2)a}\partial^\rho C^a_{(1,3)\rho\mu_1\cdots\mu_4} - \lambda\varepsilon^{\mu_1\cdots\mu_6}\left[\frac{\partial Z_{ab}}{\partial\varphi_d}C^{*\rho_1\cdots\rho_4}_{(1,2)d} + \frac{\partial^2 Z_{ab}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}\left(4C^{*\rho_1\rho_2\rho_3}_{(1,1)d_1}B^{*\rho_4}_{d_2} + 3C^{*\rho_1\rho_2}_{(1,0)d_1}C^{*\rho_3\rho_4}_{(1,0)d_2}\right)\right.\\
&\left.+6\frac{\partial^3 Z_{ab}}{\partial\varphi_{d_1}\partial\varphi_{d_2}\partial\varphi_{d_3}}C^{*\rho_1\rho_2}_{(1,0)d_1}B^{*\rho_3}_{d_2}B^{*\rho_4}_{d_3} + \frac{\partial^4 Z_{ab}}{\partial\varphi_{d_1}\cdots\partial\varphi_{d_4}}B^{*\rho_1}_{d_1}\cdots B^{*\rho_4}_{d_4}\right]\left(C^a_{(3,0)\rho_1\cdots\rho_4}C^b_{(3,2)\mu_1\cdots\mu_6} - \frac{5}{2}C^a_{(3,1)\rho_1\cdots\rho_4\mu_1}C^b_{(3,1)\mu_2\cdots\mu_6}\right)\\
&+\frac{\lambda}{180}\varepsilon_{\mu_1\cdots\mu_6}\left\{15A^{\mu_1\mu_2}_a\left[\frac{\partial M^{abc}}{\partial\varphi_d}C^{*\mu_3\cdots\mu_6}_{(1,2)d} + \frac{\partial^2 M^{abc}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}\left(4C^{*\mu_3\mu_4\mu_5}_{(1,1)d_1}B^{*\mu_6}_{d_2} + 3C^{*\mu_3\mu_4}_{(1,0)d_1}C^{*\mu_5\mu_6}_{(1,0)d_2}\right)\right.\right.\\
&\left.+6\frac{\partial^3 M^{abc}}{\partial\varphi_{d_1}\partial\varphi_{d_2}\partial\varphi_{d_3}}C^{*\mu_3\mu_4}_{(1,0)d_1}B^{*\mu_5}_{d_2}B^{*\mu_6}_{d_3} + \frac{\partial^4 M^{abc}}{\partial\varphi_{d_1}\cdots\partial\varphi_{d_4}}B^{*\mu_3}_{d_1}\cdots B^{*\mu_6}_{d_4}\right]\\
&+20B^{*\mu_1\mu_2\mu_3}_a\left(\frac{\partial M^{abc}}{\partial\varphi_d}C^{*\mu_4\mu_5\mu_6}_{(1,1)d} + 3\frac{\partial^2 M^{abc}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}C^{*\mu_4\mu_5}_{(1,0)d_1}B^{*\mu_6}_{d_2} + \frac{\partial^3 M^{abc}}{\partial\varphi_{d_1}\partial\varphi_{d_2}\partial\varphi_{d_3}}B^{*\mu_4}_{d_1}B^{*\mu_5}_{d_2}B^{*\mu_6}_{d_3}\right)\\
&\left.+15C^{*\mu_1\cdots\mu_4}_{(3,0)a}\left(\frac{\partial M^{abc}}{\partial\varphi_d}C^{*\mu_5\mu_6}_{(1,0)d} + \frac{\partial^2 M^{abc}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}B^{*\mu_5}_{d_1}B^{*\mu_6}_{d_2}\right) + 6\frac{\partial M^{abc}}{\partial\varphi_d}C^{*\mu_1\cdots\mu_5}_{(3,1)a}B^{*\mu_6}_d + M^{abc}C^{*\mu_1\cdots\mu_6}_{(3,2)a}\right\}\eta_{(2,1)b}\eta_{(2,1)c}\\
&-\frac{\lambda}{6}\varepsilon_{\mu_1\cdots\mu_6}\left[\frac{\partial M^{abc}}{\partial\varphi_d}C^{*\mu_1\cdots\mu_4}_{(1,2)d} + \frac{\partial^2 M^{abc}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}\left(4C^{*\mu_1\mu_2\mu_3}_{(1,1)d_1}B^{*\mu_4}_{d_2} + 3C^{*\mu_1\mu_2}_{(1,0)d_1}C^{*\mu_3\mu_4}_{(1,0)d_2}\right)\right. \end{aligned}$$

$$+6\frac{\partial^3 M^{abc}}{\partial\varphi_{d_1}\partial\varphi_{d_2}\partial\varphi_{d_3}}C^{*\mu_1\mu_2}_{(1,0)d_1}B^{*\mu_3}_{d_2}B^{*\mu_4}_{d_3}+\frac{\partial^4 M^{abc}}{\partial\varphi_{d_1}\cdots\partial\varphi_{d_4}}B^{*\mu_1}_{d_1}\cdots B^{*\mu_4}_{d_4}\bigg]\eta^{\mu_5}_{(2,0)a}\eta^{\mu_6}_{(2,0)b}\eta_{(2,1)c}. \quad (162)$$

In the same manner, we can organize the terms of antifields number 5 and 6 as, respectively,

$$\begin{aligned}\mathcal{L}_5 = & -6C^{*\mu_1\cdots\mu_5}_{(1,3)a}\partial^\rho C^a_{(1,4)\rho\mu_1\cdots\mu_5}-\lambda\varepsilon^{\mu_1\cdots\mu_6}\bigg[\frac{\partial Z_{ab}}{\partial\varphi_d}C^{*\rho_1\cdots\rho_5}_{(1,3)d}+5\frac{\partial^2 Z_{ab}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}\Big(C^{*\rho_1\cdots\rho_4}_{(1,2)d_1}B^{*\rho_5}_{d_2}+2C^{*\rho_1\rho_2\rho_3}_{(1,1)d_1}C^{*\rho_4\rho_5}_{(1,0)d_2}\Big)\\
&+5\frac{\partial^3 Z_{ab}}{\partial\varphi_{d_1}\partial\varphi_{d_2}\partial\varphi_{d_3}}\Big(2C^{*\rho_1\rho_2\rho_3}_{(1,1)d_1}B^{*\rho_4}_{d_2}B^{*\rho_5}_{d_3}+3C^{*\rho_1\rho_2}_{(1,0)d_1}C^{*\rho_3\rho_4}_{(1,0)d_2}B^{*\rho_5}_{d_3}\Big)+10\frac{\partial^4 Z_{ab}}{\partial\varphi_{d_1}\cdots\partial\varphi_{d_4}}C^{*\rho_1\rho_2}_{(1,0)d_1}B^{*\rho_3}_{d_2}B^{*\rho_4}_{d_3}B^{*\rho_5}_{d_4}\\
&+\frac{\partial^5 Z_{ab}}{\partial\varphi_{d_1}\cdots\partial\varphi_{d_5}}B^{*\rho_1}_{d_1}\cdots B^{*\rho_5}_{d_5}\bigg]C^a_{(3,1)\rho_1\cdots\rho_5}C^b_{(3,2)\mu_1\cdots\mu_6}+\frac{\lambda}{30}\varepsilon_{\mu_1\cdots\mu_6}\bigg[\frac{\partial M^{abc}}{\partial\varphi_d}C^{*\mu_1\cdots\mu_5}_{(1,3)d}\\
&+5\frac{\partial^2 M^{abc}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}\Big(C^{*\mu_1\cdots\mu_4}_{(1,2)d_1}B^{*\mu_5}_{d_2}+2C^{*\mu_1\mu_2\mu_3}_{(1,1)d_1}C^{*\mu_4\mu_5}_{(1,0)d_2}\Big)+5\frac{\partial^3 M^{abc}}{\partial\varphi_{d_1}\partial\varphi_{d_2}\partial\varphi_{d_3}}\Big(2C^{*\mu_1\mu_2\mu_3}_{(1,1)d_1}B^{*\mu_4}_{d_2}+3C^{*\mu_1\mu_2}_{(1,0)d_1}C^{*\mu_3\mu_4}_{(1,0)d_2}\Big)B^{*\mu_5}_{d_3}\\
&+10\frac{\partial^4 M^{abc}}{\partial\varphi_{d_1}\cdots\partial\varphi_{d_4}}C^{*\mu_1\mu_2}_{(1,0)d_1}B^{*\mu_3}_{d_2}B^{*\mu_4}_{d_3}B^{*\mu_5}_{d_4}+\frac{\partial^5 M^{abc}}{\partial\varphi_{d_1}\cdots\partial\varphi_{d_5}}B^{*\mu_1}_{d_1}\cdots B^{*\mu_5}_{d_5}\bigg]\eta^{\mu_6}_{(2,0)a}\eta_{(2,1)b}\eta_{(2,1)c}, \quad (163)\\
\mathcal{L}_6 = & -\frac{\lambda}{2}\varepsilon^{\mu_1\cdots\mu_6}\bigg[\frac{\partial Z_{ab}}{\partial\varphi_d}C^{*\rho_1\cdots\rho_6}_{(1,4)d}+\frac{\partial^2 Z_{ab}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}\Big(6C^{*\rho_1\cdots\rho_5}_{(1,3)d_1}B^{*\rho_6}_{d_2}+15C^{*\rho_1\cdots\rho_4}_{(1,2)d_1}C^{*\rho_5\rho_6}_{(1,0)d_2}+10C^{*\rho_1\rho_2\rho_3}_{(1,1)d_1}C^{*\rho_4\rho_5\rho_6}_{(1,1)d_2}\Big)\\
&+15\frac{\partial^3 Z_{ab}}{\partial\varphi_{d_1}\partial\varphi_{d_2}\partial\varphi_{d_3}}\Big(C^{*\rho_1\cdots\rho_4}_{(1,2)d_1}B^{*\rho_5}_{d_2}B^{*\rho_6}_{d_3}+4C^{*\rho_1\rho_2\rho_3}_{(1,1)d_1}C^{*\rho_4\rho_5}_{(1,0)d_2}B^{*\rho_6}_{d_3}+C^{*\rho_1\rho_2}_{(1,0)d_1}C^{*\rho_3\rho_4}_{(1,0)d_2}C^{*\rho_5\rho_6}_{(1,0)d_3}\Big)\\
&+5\frac{\partial^4 Z_{ab}}{\partial\varphi_{d_1}\cdots\partial\varphi_{d_4}}\Big(4C^{*\rho_1\rho_2\rho_3}_{(1,1)d_1}B^{*\rho_4}_{d_2}B^{*\rho_5}_{d_3}B^{*\rho_6}_{d_4}+9C^{*\rho_1\rho_2}_{(1,0)d_1}C^{*\rho_3\rho_4}_{(1,0)d_2}B^{*\rho_5}_{d_3}B^{*\rho_6}_{d_4}\Big)+15\frac{\partial^5 Z_{ab}}{\partial\varphi_{d_1}\cdots\partial\varphi_{d_5}}C^{*\rho_1\rho_2}_{(1,0)d_1}B^{*\rho_3}_{d_2}\cdots B^{*\rho_6}_{d_5}\\
&+\frac{\partial^6 Z_{ab}}{\partial\varphi_{d_1}\cdots\partial\varphi_{d_6}}B^{*\rho_1}_{d_1}\cdots B^{*\rho_6}_{d_6}\bigg]C^a_{(3,2)\rho_1\cdots\rho_6}C^b_{(3,2)\mu_1\cdots\mu_6}-\frac{\lambda}{540}\varepsilon_{\mu_1\cdots\mu_6}\bigg[\frac{\partial M^{abc}}{\partial\varphi_d}C^{*\mu_1\cdots\mu_6}_{(1,4)d}\\
&+\frac{\partial^2 M^{abc}}{\partial\varphi_{d_1}\partial\varphi_{d_2}}\Big(6C^{*\mu_1\cdots\mu_5}_{(1,3)d_1}B^{*\mu_6}_{d_2}+15C^{*\mu_1\cdots\mu_4}_{(1,2)d_1}C^{*\mu_5\mu_6}_{(1,0)d_2}+10C^{*\mu_1\mu_2\mu_3}_{(1,1)d_1}C^{*\mu_4\mu_5\mu_6}_{(1,1)d_2}\Big)\\
&+15\frac{\partial^3 M^{abc}}{\partial\varphi_{d_1}\partial\varphi_{d_2}\partial\varphi_{d_3}}\Big(C^{*\mu_1\cdots\mu_4}_{(1,2)d_1}B^{*\mu_5}_{d_2}B^{*\mu_6}_{d_3}+4C^{*\mu_1\mu_2\mu_3}_{(1,1)d_1}C^{*\mu_4\mu_5}_{(1,0)d_2}B^{*\mu_6}_{d_3}+C^{*\mu_1\mu_2}_{(1,0)d_1}C^{*\mu_3\mu_4}_{(1,0)d_2}C^{*\mu_5\mu_6}_{(1,0)d_3}\Big)\\
&+5\frac{\partial^4 M^{abc}}{\partial\varphi_{d_1}\cdots\partial\varphi_{d_4}}\Big(4C^{*\mu_1\mu_2\mu_3}_{(1,1)d_1}B^{*\mu_4}_{d_2}B^{*\mu_5}_{d_3}B^{*\mu_6}_{d_4}+9C^{*\mu_1\mu_2}_{(1,0)d_1}C^{*\mu_3\mu_4}_{(1,0)d_2}B^{*\mu_5}_{d_3}B^{*\mu_6}_{d_4}\Big)\\
&+15\frac{\partial^5 M^{abc}}{\partial\varphi_{d_1}\cdots\partial\varphi_{d_5}}C^{*\mu_1\mu_2}_{(1,0)d_1}B^{*\mu_3}_{d_2}\cdots B^{*\mu_6}_{d_5}+\frac{\partial^6 M^{abc}}{\partial\varphi_{d_1}\cdots\partial\varphi_{d_6}}B^{*\mu_1}_{d_1}\cdots B^{*\mu_6}_{d_6}\bigg]\eta_{(2,1)a}\eta_{(2,1)b}\eta_{(2,1)c}. \quad (164)\end{aligned}$$

Due to the fact that the entire consistency of the deformed solution to the classical master equation (156) relies on the existence of non-vanishing solutions to the algebraic consistency equation (153), it is critical to provide viable situations where such classes of solutions exist. Therefore, we will assume that the parameterizing functions Z and M fall under one of the next two categories:

A. The Zs are degenerate, and thus there exist null vectors $\{u_\Delta\}_{\Delta=\overline{1,J}}$

$$Z_{ab}(\varphi)u^b_\Delta(\varphi)=0, \quad a=\overline{1,A}, \quad \Delta=\overline{1,J}. \quad (165)$$

Accordingly, we construct the M-type functions as solutions to (153) as follows:

$$M^{abc}(\varphi)=\frac{1}{3!}u^a_\Delta(\varphi)u^b_{\Delta'}(\varphi)u^c_{\Delta''}(\varphi)f^{\Delta\Delta'\Delta''}(\varphi), \quad a,b,c=\overline{1,A}, \quad (166)$$

where $f^{\Delta\Delta'\Delta''}$ denote arbitrary completely symmetric smooth functions depending on the undifferentiated BF scalar fields, $f^{\Delta\Delta'\Delta''}=\frac{1}{3!}f^{(\Delta\Delta'\Delta'')}$.

B. The Ms are degenerate, and thus there exist null vectors $\left\{v^{\bar{\Delta}}\right\}_{\bar{\Delta}=\overline{1,\bar{J}}}$

$$M^{abc}(\varphi)v^{\bar{\Delta}}_c(\varphi)=0, \quad a,b=\overline{1,A}, \quad \bar{\Delta}=\overline{1,\bar{J}}. \quad (167)$$

Consequently, we generate the Z-type functions as solutions to (153) as follows:

$$Z_{ab}(\varphi) = \frac{1}{2} v_a^{\bar{\Delta}}(\varphi) v_b^{\bar{\Delta}'}(\varphi) g_{\bar{\Delta}\bar{\Delta}'}(\varphi), \quad a, b = \overline{1, A}, \tag{168}$$

where $g_{\bar{\Delta}\bar{\Delta}'}$ represent arbitrary skew-symmetric smooth functions of the undifferentiated BF scalar fields, $g_{\bar{\Delta}\bar{\Delta}'} = -g_{\bar{\Delta}'\bar{\Delta}}$.

With all the above results at hand, in the sequel we address the defining properties of the Lagrangian formulation of the self-interacting $D = 6$ BF model behind the fully deformed solution to the master equation expressed by (156), the various components of which were introduced in expansion (157) and listed in Formulas (158)–(164).

4. Lagrangian Formulation of the Self-Interacting Model

Once the fully deformed solution to the master equation has been completed, (156), from its sectors of fixed antifield number $j = \overline{0, 6}$, provided by (158)–(164) we can read all of the information regarding the gauge structure of the associated self-interacting $D = 6$ BF theory. In view of this, we return to the interpretation of the various terms present in a generic solution to the classical master equation detailed in Section 2 (see the paragraph following Formula (65)). We recall that at each order of perturbation theory (0 and 1 in this case), all the ingredients related to the Lagrangian formulation of the self-coupled BF model are expressed via quantities that are local in space-time, Lorentz covariant, and Poincaré invariant, while the differential order of each interacting field equation with respect to its free limit is rigorously conserved. We cannot stress enough that everywhere in what follows the functions denoted by M and Z are assumed to satisfy the consistency conditions (153) as well as all further identities implied by them via taking their derivatives of various orders with respect to the BF scalar fields

$$M^{abe} Z_{ec} = 0 \Rightarrow \frac{\partial M^{abe}}{\partial \varphi_d} Z_{ec} = -M^{abe} \frac{\partial Z_{ec}}{\partial \varphi_d}, \quad \cdots . \tag{169}$$

The piece of antifield number 0 from (157), namely, (158), is both antifield- and ghost-independent, and thus involves only the BF field spectrum and its space-time derivatives. Moreover, it defines a crucial ingredient of the $D = 6$ BF self-coupled model, namely, its *Lagrangian action*:

$$\begin{aligned} \bar{S}^L\left[\overset{[0]}{\varphi}_a, \overset{[1]}{B}{}^a, \overset{[2]}{A}_a, \overset{[3]}{B}{}^a\right] = & \int d^6x \left[B^a_{\mu_1} \partial^{\mu_1} \varphi_a + B^a_{\mu_1\mu_2\mu_3} \left(\partial^{[\mu_1} A_a^{\mu_2\mu_3]} + \frac{\lambda}{2} \varepsilon^{\mu_1 \cdots \mu_6} Z_{ab}(\varphi) B^b_{\mu_4\mu_5\mu_6} \right) \right. \\ & \left. + \frac{\lambda}{6} \varepsilon_{\mu_1 \cdots \mu_6} M^{abc}(\varphi) A_a^{\mu_1\mu_2} A_b^{\mu_3\mu_4} A_c^{\mu_5\mu_6} + \lambda V(\varphi) \right]. \end{aligned} \tag{170}$$

We notice that there appears one *arbitrary potential describing all possible self-interactions among the BF scalar fields*, which may be further restricted (for instance, from renormalizability arguments), and *two kinds of consistent vertices* depending on the BF form fields of strictly positive form degrees, which can be added to the free Lagrangian, all at order one in the coupling constant. The last two vertex classes involve smooth functions of the undifferentiated BF fields with precise symmetry/antisymmetry properties, and break the PT-invariance (which was not imposed as an explicit selection rule), among which one is quadratic in the components of the BF 3-forms $\overset{[3]}{B}{}^a$ and the other is cubic in those of the 2-forms $\overset{[2]}{A}_a$ plus.

The former kind couples the two components of the BF 3-forms from each term via the elements of some antisymmetric functions $Z_{ab}(\varphi)$

$$\frac{\lambda}{2} \varepsilon^{\mu_1 \cdots \mu_6} Z_{ab}(\varphi) B^a_{\mu_1\mu_2\mu_3} B^b_{\mu_4\mu_5\mu_6}, \quad Z_{ab} = -Z_{ba} \tag{171}$$

and provides the $D=6$ generalization of the famous BF self-couplings present in the $D=2$ gravity formulation via topological BF theories [17]

$$D=2: \quad \frac{\lambda}{2}\varepsilon^{\mu_1\mu_2}Z_{ab}(\varphi)B^a_{\mu_1}B^b_{\mu_2}, \quad Z_{ab}=-Z_{ba}, \quad Z_{ad}\frac{\partial Z_{bc}}{\partial\varphi_d}+Z_{bd}\frac{\partial Z_{ca}}{\partial\varphi_d}+Z_{cd}\frac{\partial Z_{ab}}{\partial\varphi_d}=0, \quad a,b,c=\overline{1,A}. \tag{172}$$

We mention that the starting free Lagrangian action in $D=2$ displays the simplest BF field spectrum, consisting only in two kinds of BF forms, namely, $\overset{[0]}{\varphi}_a$ and $\overset{[1]}{B}{}^a$, while its Lagrangian density is similar to the first term from (170)

$$S^L\left[\overset{[0]}{\varphi}_a, \overset{[1]}{B}{}^a\right]=\int d^2x\, B^a_\mu\partial^\mu\varphi_a. \tag{173}$$

The last equations from (172), obtained via the BRST–antifield deformation procedure as the consistency conditions specific to the two-dimensional case, allow us, together with the antisymmetry of the Zs, to interpret the functions $Z_{ab}(\varphi)$ as the components of the (Poisson) two-tensor corresponding to a Poisson manifold with the scalar BF fields $\{\varphi_a\}$ viewed as the local coordinates on the target space, $[\varphi_a,\varphi_b]=Z_{ab}(\varphi)$, where these equations play the role of the associated Jacobi identities. Here, although the vertices in (171) generalize those from the $D=2$ case in (172) and the Zs remain antisymmetric, $Z_{ab}(\varphi)$ no longer has a precise geometric interpretation due to the lack of the Jacobi identities from (172).

The latter type of vertices present in (170) couples the three components of the BF 2-forms from each term via the elements of some fully symmetric functions $M^{abc}(\varphi)$

$$\frac{\lambda}{6}\varepsilon_{\mu_1\cdots\mu_6}M^{abc}(\varphi)A_a^{\mu_1\mu_2}A_b^{\mu_3\mu_4}A_c^{\mu_5\mu_6}. \tag{174}$$

These 'cubic' vertices are unique in the sense that they are present *only* in a $D=6$ space-time, and meanwhile describe true self-interactions among the BF 2-forms even if *all* the Ms are constant, in contrast to the previous vertex class.

We are still able to provide a good interpretation of (171) and (174) by means of solutions A and B, as emphasized in the final part of the previous section. Thus, if we adopt solution B, described by Formulas (167)–(168), further particularized to certain constant Ms, vs, and Zs, then the quadratic vertex from (171) takes the particular form

$$\frac{\lambda}{4}v_a^{\bar\Delta}v_b^{\bar\Delta'}g_{\bar\Delta\bar\Delta'}\epsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}B^a_{\mu_1\mu_2\mu_3}B^b_{\mu_4\mu_5\mu_6}. \tag{175}$$

Now, if in (175) we consider the "projection" of the BF 3-forms on the space of null vectors of M^{abc}, $B^{\bar\Delta}_{\mu_1\mu_2\mu_3}\equiv v_a^{\bar\Delta}B^a_{\mu_1\mu_2\mu_3}$, then this becomes

$$\frac{1}{4}\lambda\epsilon_{\mu_1\mu_2\mu_3\mu_1'\mu_2'\mu_3'}g_{\bar\Delta\bar\Delta'}B^{\bar\Delta\mu_1\mu_2\mu_3}B^{\bar\Delta'\mu_1'\mu_2'\mu_3'}, \tag{176}$$

which can be interpreted as a mass-term for the generalized tensor field $B^{\bar\Delta\mu_1\mu_2\mu_3}$. Similarly, related to (174), we can choose to work with solution A, pictured by Formulas (165)–(166) and particularized this time to certain constant Zs, us and Ms, defining the "projection" of the BF 2-forms on the space of null vectors of Z_{ab}

$$u^a_\Delta A_a^{\mu_1\mu_2}\equiv A_\Delta^{\mu_1\mu_2}, \tag{177}$$

and conveniently redenote $(\lambda/3!)f^{\Delta\Delta'\Delta''}$ by $-\lambda^{\Delta\Delta'\Delta''}$; thus, (174) becomes

$$-\frac{\lambda^{\Delta\Delta'\Delta''}}{3!}\varepsilon_{\mu_1\cdots\mu_6}A_\Delta^{\mu_1\mu_2}A_{\Delta'}^{\mu_3\mu_4}A_{\Delta''}^{\mu_5\mu_6} \tag{178}$$

and can be regarded as a generalized "$\lambda\varphi^3$"-type vertex with respect to the projected BF 2-forms $A_{\Delta}^{\mu_1\mu_2}$.

The *stationary surface of the self-interacting BF model* (170) is defined by the equations

$$\bar{\Sigma} : \frac{\delta\bar{S}^L}{\delta\Phi^{\alpha_0}} \equiv \begin{cases} \frac{\delta\bar{S}^L}{\delta\varphi_a} & = -\partial^\rho B_\rho^a + \frac{\lambda}{2}\varepsilon^{\mu_1\cdots\mu_6}\frac{\partial Z_{bc}}{\partial\varphi_a}B^b_{\mu_1\mu_2\mu_3}B^c_{\mu_4\mu_5\mu_6} \\ & \quad +\frac{\lambda}{6}\varepsilon_{\mu_1\cdots\mu_6}\frac{\partial M^{bcd}}{\partial\varphi_a}A_b^{\mu_1\mu_2}A_c^{\mu_3\mu_4}A_d^{\mu_5\mu_6} + \lambda\frac{\partial V}{\partial\varphi_a} \\ \frac{\delta\bar{S}^L}{\delta A_a^{\mu_1\mu_2}} & = -3\partial^\rho B^a_{\rho\mu_1\mu_2} + \frac{\lambda}{2}\varepsilon_{\mu_1\cdots\mu_6}M^{abc}A_b^{\mu_3\mu_4}A_c^{\mu_5\mu_6} \\ \frac{\delta\bar{S}^L}{\delta B^a_{\mu_1\mu_2\mu_3}} & = \partial^{[\mu_1}A_a^{\mu_2\mu_3]} + \lambda\varepsilon^{\mu_1\cdots\mu_6}Z_{ab}B^b_{\mu_4\mu_5\mu_6} \\ \frac{\delta\bar{S}^L}{\delta B^a_{\mu_1}} & = \frac{\delta S^L}{\delta B^a_{\mu_1}} = \partial^{\mu_1}\varphi_a \end{cases} \approx 0. \tag{179}$$

Comparing (179) with (4), we observe that the self-interacting BF theory possesses strongly nonlinear field equations with respect to most of the fields, in contrast to their free limit, meanwhile preserving their differential order as equal to one. It is interesting to remark that the 'most enhanced' field equations with respect to their free limit are those associated with the Euler–Lagrange derivatives $\delta\bar{S}^L/\delta\varphi_a$, in the sense they are the only ones that depend on all of the parameterizing functions.

From the elements of antifield number 1 in (157), provided by (159), we read the *deformed set of generating gauge symmetries* corresponding to the self-coupled action (170) by detaching the antifields and replacing the ghosts with the corresponding gauge parameters from (6)

$$\bar{\delta}_{\Omega^{\alpha_1}}\Phi^{\alpha_0} \equiv \begin{cases} \bar{\delta}_{\Omega^{\alpha_1}}\varphi_a & = 0 \\ \bar{\delta}_{\Omega^{\alpha_1}}A_a^{\mu_1\mu_2} & = \partial^{[\mu_1}\epsilon^{\mu_2]}_{(2,0)a} + \lambda\varepsilon^{\mu_1\cdots\mu_6}Z_{ab}\xi^b_{(3,0)\mu_3\cdots\mu_6} \\ \bar{\delta}_{\Omega^{\alpha_1}}B^a_{\mu_1\mu_2\mu_3} & = -4\partial^\rho\xi^a_{(3,0)\rho\mu_1\mu_2\mu_3} + \frac{2\lambda}{3}\varepsilon_{\mu_1\cdots\mu_6}M^{abc}A_b^{\mu_4\mu_5}\epsilon^{\mu_6}_{(2,0)c} \\ \bar{\delta}_{\Omega^{\alpha_1}}B^a_{\mu_1} & = -2\partial^\rho\xi^a_{(1,0)\rho\mu_1} - 4\lambda\varepsilon^{\rho_1\cdots\rho_6}\frac{\partial Z_{bc}}{\partial\varphi_a}B^b_{\rho_1\rho_2\rho_3}\xi^c_{(3,0)\mu_1\rho_4\rho_5\rho_6} \\ & \quad +\lambda\varepsilon_{\mu_1\cdots\mu_6}\frac{\partial M^{bcd}}{\partial\varphi_a}A_b^{\mu_2\mu_3}A_c^{\mu_4\mu_5}\epsilon^{\mu_6}_{(2,0)d} \end{cases}. \tag{180}$$

With the help of (5) and (180), it is clear that all the BF form fields of strictly positive form degrees receive deformed gauge transformations, and the 'mostly deformed' transformations with respect to their free limit in the same sense as before are those of the 1-forms $\overset{[1]}{B}{}^a$, which, in addition, involve only the first-order derivatives of both types of parameterizing function sets. This is a general feature of all self-interacting BF models (collections or single-valued), where the gauge transformations of all the 1-forms $\overset{[1]}{B}$ gain new terms depending on the first-order derivatives of all the parameterizing function sets. A superfluous check of the gauge invariance of (170) under (180) produces (up to some irrelevant total space-time derivatives that were discarded) the result

$$\bar{\delta}_{\Omega^{\alpha_1}}\bar{S}^L = \int d^6x\left[-8\lambda^2M^{abd}Z_{dc}\left(A_a^{[\mu_1\mu_2}A_b^{\mu_3\mu_4]}\xi^c_{(3,0)\mu_1\cdots\mu_4} + B^c_{\mu_1\mu_2\mu_3}A_a^{[\mu_1\mu_2}\epsilon^{\mu_3]}_{(2,0)b}\right)\right] = 0, \tag{181}$$

which confirms that identities (153) are essential in restoring the consistency of the self-interacting BF model at all orders of perturbation theory greater or equal to 2.

The components of antifield number strictly greater that 1 from (157), collected in Formulas (160)–(164), provide all the information on the deformed gauge algebra and reducibility of the generating set (180) of gauge transformations.

Initially, we discuss the relevant aspects regarding the gauge algebra induced by the deformed gauge transformations (180). The first-order structure functions of the gauge algebra corresponding to an arbitrary gauge theory $S^L[\Phi^{\alpha_0}]$ with a generating set of gauge transformations $\delta_{\Omega^{\alpha_1}}\Phi^{\alpha_0} = Z^{\alpha_0}_{\alpha_1}([\Phi^{\alpha_0}])\Omega^{\alpha_1}$, where Ω^{α_1} denote the gauge parameters, are of the type $\left\{C^{\alpha_1}_{\beta_1\gamma_1}([\Phi^{\alpha_0}]), M^{\alpha_0\beta_0}_{\beta_1\gamma_1}([\Phi^{\alpha_0}])\right\}$, with $C^{\alpha_1}_{\beta_1\gamma_1}$ and $M^{\alpha_0\beta_0}_{\beta_1\gamma_1}$ antisymmetric in their lower indices for a strictly bosonic spectrum of gauge parameters and $M^{\alpha_0\beta_0}_{\beta_1\gamma_1}$ antisymmetric in their upper indices for a purely bosonic field spectrum Φ^{α_0}. This set of first-order structure functions originates in the commutators among the gauge transformations of the fields, $\left[\delta_{\Omega^{(1)\alpha_1}}, \delta_{\Omega^{(2)\alpha_1}}\right]\Phi^{\alpha_0}$, computed with respect to two different sets of gauge parameters $\Omega^{(1)\alpha_1}$ and $\Omega^{(2)\alpha_1}$. More precisely, as these commutators may only close on-shell (on the stationary surface of the generic gauge theory considered), it follows that any such commutator will reproduce the non-trivial gauge transformations of Φ^{α_0}, now with respect to a newly defined set of gauge parameters, say, $\bar{\Omega}^{\alpha_1}$, modulo some possible trivial gauge transformations, i.e., antisymmetric combinations of the Euler–Lagrange derivatives of the Lagrangian action (in DeWitt condensed notations)

$$\left[\delta_{\Omega^{(1)\alpha_1}}, \delta_{\Omega^{(2)\alpha_1}}\right]\Phi^{\alpha_0} = \delta_{\bar{\Omega}^{\alpha_1}}\Phi^{\alpha_0} + M^{\alpha_0\beta_0}\frac{\delta S^L}{\delta\Phi^{\beta_0}}, \qquad \delta_{\bar{\Omega}^{\alpha_1}}\Phi^{\alpha_0} \equiv Z^{\alpha_0}_{\alpha_1}([\Phi^{\alpha_0}])\bar{\Omega}^{\alpha_1}, \tag{182}$$

$$\bar{\Omega}^{\alpha_1} = C^{\alpha_1}_{\beta_1\gamma_1}([\Phi^{\alpha_0}])\Omega^{(1)\beta_1}\Omega^{(2)\gamma_1}, \qquad M^{\alpha_0\beta_0} = M^{\alpha_0\beta_0}_{\beta_1\gamma_1}([\Phi^{\alpha_0}])\Omega^{(1)\beta_1}\Omega^{(2)\gamma_1}, \tag{183}$$

$$C^{\alpha_1}_{\beta_1\gamma_1} = -C^{\alpha_1}_{\gamma_1\beta_1}, \quad M^{\alpha_0\beta_0}_{\beta_1\gamma_1} = -M^{\alpha_0\beta_0}_{\gamma_1\beta_1} = -M^{\beta_0\alpha_0}_{\beta_1\gamma_1}, \qquad M^{\alpha_0\beta_0} = -M^{\beta_0\alpha_0}. \tag{184}$$

If all the coefficients $M^{\alpha_0\beta_0}_{\alpha_1\beta_1}$ are strongly vanishing, then the gauge algebra associated with the chosen generating set of gauge transformations is said to be *closed*, while in the opposite case it is called *open*. The closed situation includes Abelian gauge algebras (all $C^{\gamma_1}_{\alpha_1\beta_1}$ vanish strongly) and Lie gauge algebras (all $C^{\gamma_1}_{\alpha_1\beta_1}$ are constant, with at least one non-vanishing coefficient among them). The relevant terms from the solution to the master equation from which one reads the expressions of $C^{\gamma_1}_{\alpha_1\beta_1}$ and $M^{\alpha_0\beta_0}_{\alpha_1\beta_1}$ lie in antifield number 2, are quadratic in the ghosts η^{α_1} corresponding to the gauge parameters Ω^{α_1}, and may be of two kinds: either linear in the antifields of the ghosts, $\eta^*_{\alpha_1}$, or quadratic in the antifields of the original fields, $\Phi^*_{\alpha_0}$:

$$\mathcal{L}_2 \ni \frac{1}{2}\left(\eta^*_{\gamma_1}C^{\gamma_1}_{\alpha_1\beta_1} - \frac{1}{2}\Phi^*_{\alpha_0}\Phi^*_{\beta_0}M^{\alpha_0\beta_0}_{\alpha_1\beta_1}\right)\eta^{\alpha_1}\eta^{\beta_1}, \tag{185}$$

where $\mathcal{L}_2$ denotes the non-integrated density of antifield number 2 of the solution to the master equation.

Inspecting (160), we remark that there exist non-vanishing analogues of both type of terms from (185) gathered on the last three lines from the right-hand side of (160). Therefore, we can state that the deformed gauge algebra corresponding to the generating set (180) is *open* in general (unless all M^{abc} vanish and all Z_{ab} are constant), in contrast to the free limit, where the gauge algebra of (5) is Abelian. The concrete expressions of the (non-trivial) commutators among the gauge transformations (180), $\left[\bar{\delta}_{\Omega^{(1)\alpha_1}}, \bar{\delta}_{\Omega^{(2)\alpha_1}}\right]\Phi^{\alpha_0}$, where Φ^{α_0} are introduced in (3) and $\Omega^{(1)\alpha_1}$ and $\Omega^{(2)\alpha_1}$, are two different sets of gauge parameters, as in (6); read

$$\left[\bar{\delta}_{\Omega^{(1)\alpha_1}}, \bar{\delta}_{\Omega^{(2)\alpha_1}}\right]\Phi^{\alpha_0}\Big|_{\text{nontriv}} \equiv \left\{\begin{array}{lcl} \left[\bar{\delta}_{\Omega^{(1)\alpha_1}}, \bar{\delta}_{\Omega^{(2)\alpha_1}}\right]A^{\mu_1\mu_2}_a & = & 0 \\ \left[\bar{\delta}_{\Omega^{(1)\alpha_1}}, \bar{\delta}_{\Omega^{(2)\alpha_1}}\right]B^a_{\mu_1\mu_2\mu_3} & = & -4\partial^\rho\bar{\xi}^a_{(3,0)\rho\mu_1\mu_2\mu_3} + \frac{2\lambda}{3}\varepsilon_{\mu_1\cdots\mu_6}M^{abc}(\varphi)A^{\mu_4\mu_5}_b\bar{\varepsilon}^{\mu_6}_{(2,0)c} \\ & & +\lambda\bar{M}^{ab}_{\mu_1\mu_2\mu_3\rho_1}\frac{\delta\bar{S}^L}{\delta B^b_{\rho_1}} \\ \left[\bar{\delta}_{\Omega^{(1)\alpha_1}}, \bar{\delta}_{\Omega^{(2)\alpha_1}}\right]B^a_{\mu_1} & = & -2\partial^\rho\bar{\xi}^a_{(1,0)\rho\mu_1} - 4\lambda\varepsilon^{\rho_1\cdots\rho_6}\frac{\partial Z_{bc}}{\partial\varphi_a}B^b_{\rho_1\rho_2\rho_3}\bar{\xi}^c_{(3,0)\mu_1\rho_4\rho_5\rho_6} \\ & & +\lambda\varepsilon_{\mu_1\rho_1\cdots\rho_5}\frac{\partial M^{bcd}}{\partial\varphi_a}A^{\rho_1\rho_2}_b A^{\rho_3\rho_4}_c\bar{\varepsilon}^{\rho_5}_{(2,0)d} \\ & & +\lambda M^{ab}_{\mu_1\rho_1}\frac{\delta\bar{S}^L}{\delta B^b_{\rho_1}} + \lambda\bar{M}'^{ab}_{\mu_1\rho_1\rho_2\rho_3}\frac{\delta\bar{S}^L}{\delta B^b_{\rho_1\rho_2\rho_3}} \end{array}\right., \tag{186}$$

where the bar gauge parameters that enter (186), defined generically as in (183), take the concrete form (for the self-interacting $D = 6$ BF model under discussion)

$$\bar{\Omega}^{\alpha_1} \equiv \begin{cases} \bar{\epsilon}^{\mu_1}_{(2,0)a} & = 0 \\ \bar{\xi}^{a}_{(3,0)\mu_1\cdots\mu_4} & = \frac{\lambda}{3}\varepsilon_{\mu_1\cdots\mu_6} M^{abc}(\varphi)\epsilon^{(1)\mu_5}_{(2,0)b}\epsilon^{(2)\mu_6}_{(2,0)c} \\ \bar{\xi}^{a}_{(1,0)\mu_1\mu_2} & = 2\lambda\varepsilon_{\mu_1\cdots\mu_6}\left(36\frac{\partial Z_{bc}}{\partial\varphi_a}\tilde{\xi}^{(1)b\mu_3\mu_4}_{(3,0)}\tilde{\xi}^{(2)c\mu_5\mu_6}_{(3,0)} + \frac{\partial M^{bcd}}{\partial\varphi_a}A^{\mu_3\mu_4}_{b}\epsilon^{(1)\mu_5}_{(2,0)c}\epsilon^{(2)\mu_6}_{(2,0)d}\right) \end{cases} \quad (187)$$

and the (non-vanishing) coefficients of the type $M^{\alpha_0\beta_0}$ (see (182)) involved in (186) read

$$\bar{M}^{ab}_{\mu_1\mu_2\mu_3\rho_1} = -\frac{4}{3}\varepsilon_{\mu_1\mu_2\mu_3\rho_1\sigma_1\sigma_2}\frac{\partial M^{cda}}{\partial\varphi_b}\epsilon^{(1)\sigma_1}_{(2,0)c}\epsilon^{(2)\sigma_2}_{(2,0)d}, \quad (188)$$

$$M^{ab}_{\mu_1\rho_1} = -4\varepsilon_{\mu_1\rho_1\sigma_1\cdots\sigma_4}\left(36\frac{\partial^2 Z_{cd}}{\partial\varphi_a\partial\varphi_b}\tilde{\xi}^{(1)c\sigma_1\sigma_2}_{(3,0)}\tilde{\xi}^{(2)d\sigma_3\sigma_4}_{(3,0)} + \frac{\partial^2 M^{cde}}{\partial\varphi_a\partial\varphi_b}A^{\sigma_1\sigma_2}_{c}\epsilon^{(1)\sigma_3}_{(2,0)d}\epsilon^{(2)\sigma_4}_{(2,0)e}\right), \quad (189)$$

$$\bar{M}'^{ab}_{\mu_1\rho_1\rho_2\rho_3} = -\frac{4}{3}\varepsilon_{\mu_1\rho_1\rho_2\rho_3\sigma_1\sigma_2}\frac{\partial M^{cdb}}{\partial\varphi_a}\epsilon^{(1)\sigma_1}_{(2,0)c}\epsilon^{(2)\sigma_2}_{(2,0)d}. \quad (190)$$

The notations $\left\{\tilde{\xi}^{(i)a\mu_1\mu_2}_{(3,0)}\right\}_{i=1,2}$ from (187) and (189) respectively denote the Hodge duals of the gauge parameters $\left\{\xi^{(i)a}_{(3,0)\mu_1\cdots\mu_4}\right\}_{i=1,2}$ from the two different sets $\Omega^{(1)\alpha_1}$ and $\Omega^{(2)\alpha_1}$, taken at the evaluation of the commutators among the deformed gauge transformations

$$\tilde{\xi}^{(i)a\mu_1\mu_2}_{(3,0)} \equiv \frac{1}{4!}\varepsilon^{\mu_1\cdots\mu_6}\xi^{(i)a}_{(3,0)\mu_3\cdots\mu_6}. \quad (191)$$

The antisymmetry of the generic functions $M^{\alpha_0\beta_0}$ from (182) is implemented at the level of (186) via the relations

$$\bar{M}^{ab}_{\mu_1\mu_2\mu_3\rho_1} = -\bar{M}'^{ba}_{\rho_1\mu_1\mu_2\mu_3}, \quad M^{ab}_{\mu_1\rho_1} = -M^{ba}_{\rho_1\mu_1}. \quad (192)$$

Next, we analyze the main ingredients connected to the reducibility of the deformed gauge transformations (180). In view of this, we recall that for a given arbitrary gauge theory $S^L[\Phi^{\alpha_0}]$ with a generating set of gauge transformations $\delta_{\Omega^{\alpha_1}}\Phi^{\alpha_0} = Z^{\alpha_0}_{\alpha_1}([\Phi^{\alpha_0}])\Omega^{\alpha_1}$ that is overall reducible of a finite order, say, L; the relevant reducibility formula chains are (again in DeWitt condensed notations):

$$k = \overline{1,L} \quad : \quad \left\{Z^{\alpha_k}_{\alpha_{k+1}}([\Phi^{\alpha_0}]),\ C^{\alpha_{k-1}\beta_0}_{\alpha_{k+1}}([\Phi^{\alpha_0}]),\ C^{\alpha_0\beta_0}_{\alpha_2} = -C^{\beta_0\alpha_0}_{\alpha_2}\right\}, \quad (193)$$

$$k = \overline{1,L} \quad : \quad \left\{Z^{\alpha_{k-1}}_{\alpha_k}([\Phi^{\alpha_0}])Z^{\alpha_k}_{\alpha_{k+1}}([\Phi^{\alpha_0}]) = C^{\alpha_{k-1}\beta_0}_{\alpha_{k+1}}([\Phi^{\alpha_0}])\frac{\delta S^L}{\delta\Phi^{\beta_0}}\right\} \approx 0, \quad (194)$$

$$k = L+1 \quad : \quad Z^{\alpha_L}_{\alpha_{L+1}}([\Phi^{\alpha_0}])\mu^{\alpha_{L+1}}([\Phi^{\alpha_0}]) \approx 0 \Leftrightarrow \mu^{\alpha_{L+1}} \approx 0, \quad (195)$$

Thus, at each reducibility stage $k = \overline{1,L}$ there is assumed to exist a reducibility function set of order k, $\left\{Z^{\alpha_k}_{\alpha_{k+1}}([\Phi^{\alpha_0}])\right\}$ (where the number of functions in each set must be finite at each point in space-time and each set must contain at least one non-trivial function which does not vanish, either strongly or weakly, on the stationary surface for $S^L[\Phi^{\alpha_0}]$) along with the associated reducibility coefficients $\left\{C^{\alpha_{k-1}\beta_0}_{\alpha_{k+1}}([\Phi^{\alpha_0}])\right\}$ (present in the generic case of an on-shell reducible generating set), where, in addition, those corresponding to $k = 1$ must be antisymmetric in order to collapse the reducibility relations of order 1 into trivial gauge symmetries of $S^L[\Phi^{\alpha_0}]$ (see Formula (193)). The accompanying reducibility relations at each stage, $k = \overline{1,L}$, involve two consequent sets of reducibility functions and the attached reducibility coefficients, as in (194). The assumption of a finite value, L, of the overall reducibility order, is translated into the fact that the general solution to Equation (195) involving the L-order reducibility functions is completely trivial (vanish on-shell).

An equivalent manner of understanding the reducibility is via introducing the reducibility parameters $\{\Omega^{\alpha_{k+1}}\}$ at each stage $k = \overline{1,L}$, as in (196), and transforming all the gauge parameters $\{\Omega^{\alpha_1}\}$ and reducibility parameters at each stage $k = \overline{1,L-1}$, $\{\Omega^{\alpha_{k+1}}\}$, as in (196) below, in terms of the reducibility parameters corresponding to the next consequent stage by means of the appropriate reducibility functions from (193).

$$\{\Omega^{\alpha_{k+1}}\}_{k=\overline{1,L}}, \qquad \{\Omega^{\alpha_k} \to \Omega^{\alpha_k}(\Omega^{\alpha_{k+1}}) = Z^{\alpha_k}_{\alpha_{k+1}}([\Phi^{\alpha_0}])\Omega^{\alpha_{k+1}}\}_{k=\overline{1,L}} \Leftrightarrow \tag{196}$$

$$\Leftrightarrow \left\{ \delta_{\Omega^{\alpha_1}(\Omega^{\alpha_2})}\Phi^{\alpha_0} = C^{\alpha_0\beta_0}\frac{\delta S^L}{\delta\Phi^{\beta_0}}, \right. \tag{197}$$

$$\left. \left\{ \Omega^{\alpha_k}(\Omega^{\alpha_{k+1}}(\Omega^{\alpha_{k+2}})) = C^{\alpha_k\beta_0}\frac{\delta S^L}{\delta\Phi^{\beta_0}} \right\}_{k+1=\overline{2,L}} \right\} \approx 0, \tag{198}$$

$$\Omega^{\alpha_L}(\Omega^{\alpha_{L+1}}) \equiv Z^{\alpha_L}_{\alpha_{L+1}}([\Phi^{\alpha_0}])\Omega^{\alpha_{L+1}} \approx 0 \Leftrightarrow \Omega^{\alpha_{L+1}} \approx 0, \tag{199}$$

where

$$C^{\alpha_0\beta_0} \equiv C^{\alpha_0\beta_0}_{\alpha_2}\Omega^{\alpha_2}, \qquad C^{\alpha_0\beta_0} = -C^{\beta_0\alpha_0}, \qquad C^{\alpha_k\beta_0} \equiv C^{\alpha_k\beta_0}_{\alpha_{k+2}}\Omega^{\alpha_{k+2}}, \quad k = \overline{1,L-1}. \tag{200}$$

In this alternative approach, the above transformations are completely equivalent to the reducibility relations at each stage $k = \overline{1,L}$ expressed as in (197) (at order 1) and (198) (at orders $k+1 = \overline{2,L}$). The finite value, L, of the overall reducibility order in this alternative language is expressed by (199), meaning that there are no non-trivial transformations of the reducibility parameters of maximum order (L) that render trivial the expressions of those corresponding to the before last stage ($L-1$). We recall that the usual terminology is that several of the *reducibility relations* associated with a current stage $k = \overline{1,L}$ *hold on-shell* whenever the corresponding coefficients $\left\{C^{\alpha_{k-1}\beta_0}_{\alpha_{k+1}}\right\}$ (or, equivalently, $C^{\alpha_{k-1}\beta_0}$ defined in (200)) *are non-vanishing*, while in the opposite situation, with a *vanishing coefficient*, $C^{\alpha_{k-1}\beta_0}_{\alpha_{k+1}} = 0$, *the corresponding reducibility relation is said to hold off-shell*.

The relevant terms from the solution to the master equation that enable us to read the reducibility functions and coefficients, restore the reducibility relations and their type (on-shell or off-shell) at each stage, and obtain the overall reducibility order whenever finite, *lie in antifield numbers* $\overline{2,L+1}$, are always linear in the ghosts $\{\eta^{\alpha_{k+1}}\}_{k=\overline{1,L}}$ associated with the reducibility parameters (also known as ghosts for ghosts), and fall under two categories: either *linear in the antifields of the ghosts* and containing precisely *the reducibility functions*, or *simultaneously quadratic in the antifields* Φ^*_A *and including at least one antifield of the original fields*, $\Phi^*_{\beta_0}$ (actually quadratic in $\Phi^*_{\beta_0}$'s at antifield number 2 and quadratic in Φ^*_A's, though including a single $\Phi^*_{\beta_0}$ at higher antifield numbers). For a generic gauge theory with purely bosonic fields and gauge parameters, such terms belong to the non-integrated densities of the solution to the classical master equation of antifield number j, denoted by $\mathcal{L}_j$, as follows:

$$\text{red.stage} = k = \overline{1,L} \to \text{ant} = k+1 = \overline{2,L+1} : \mathcal{L}_{k+1} \ni \left(\eta^*_{\alpha_k}Z^{\alpha_k}_{\alpha_{k+1}} + (-)^{k-1}d(k-1)\eta^*_{\alpha_{k-1}}\Phi^*_{\beta_0}C^{\alpha_{k-1}\beta_0}_{\alpha_{k+1}}\right)\eta^{\alpha_{k+1}}, \tag{201}$$

with

$$d(k-1) = \left\{ \begin{array}{ll} \frac{1}{2}, & k=1 \\ 1, & k \neq 1 \end{array} \right., \tag{202}$$

Collecting all the terms of the form (201) that are present in (160)–(163) and recalling notations (6), (9), (12), (15), and (18), we infer all the *non-trivial* transformations (196) and reducibility relations (197) and (198). Due to the fact that both kinds of terms from (201) are included in components (160)–(163), we can already conclude that *certain reducibility functions are modified at the level of the self-interacting $D = 6$ BF model with respect to their free limit* (by terms of order one in the coupling constant), while *other deformed reducibility relations now hold only on-shell, in contrast to the starting free $D = 6$ BF model*, the reducibility relations of which were purely off-shell (see relations (10), (13), (16), and (19)). The analogue of (*non-trivial*) transformations (196) reads

$$\Omega^{\alpha_1}(\Omega^{\alpha_2}) \equiv \begin{cases} \epsilon^{\mu_1}_{(2,0)a}(\Omega^{\alpha_2}) & = \partial^{\mu_1}\epsilon_{(2,1)a} - \lambda\varepsilon^{\mu_1\cdots\mu_6}Z_{ab}\xi^b_{(3,1)\mu_2\cdots\mu_6} \\ \xi^a_{(3,0)\mu_1\cdots\mu_4}(\Omega^{\alpha_2}) & = -5\partial^\rho\xi^a_{(3,1)\rho\mu_1\cdots\mu_4} - \frac{\lambda}{6}\varepsilon_{\mu_1\cdots\mu_6}M^{abc}A^{\mu_5\mu_6}_b\epsilon_{(2,1)c} \\ \xi^a_{(1,0)\mu_1\mu_2}(\Omega^{\alpha_2}) & = -3\partial^\rho\xi^a_{(1,1)\rho\mu_1\mu_2} + 10\lambda\varepsilon^{\rho_1\cdots\rho_6}\frac{\partial Z_{bc}}{\partial\varphi_a}B^b_{\rho_1\rho_2\rho_3}\xi^c_{(3,1)\mu_1\mu_2\rho_4\rho_5\rho_6} \\ & \quad -\frac{\lambda}{2}\varepsilon_{\mu_1\cdots\mu_6}\frac{\partial M^{bcd}}{\partial\varphi_a}A^{\mu_3\mu_4}_b A^{\mu_5\mu_6}_c\epsilon_{(2,1)cd} \end{cases}, \tag{203}$$

$$\Omega^{\alpha_2}(\Omega^{\alpha_3}) \equiv \begin{cases} \epsilon_{(2,1)a}(\Omega^{\alpha_3}) & = -\lambda\varepsilon^{\mu_1\cdots\mu_6}Z_{ab}\xi^b_{(3,2)\mu_1\cdots\mu_6} \\ \xi^a_{(3,1)\mu_1\cdots\mu_5}(\Omega^{\alpha_3}) & = -6\partial^\rho\xi^a_{(3,2)\rho\mu_1\cdots\mu_5} \\ \xi^a_{(1,1)\mu_1\mu_2\mu_3}(\Omega^{\alpha_3}) & = -4\partial^\rho\xi^a_{(1,2)\rho\mu_1\mu_2\mu_3} - 20\lambda\varepsilon^{\rho_1\cdots\rho_6}\frac{\partial Z_{bc}}{\partial\varphi_a}B^b_{\rho_1\rho_2\rho_3}\xi^c_{(3,2)\mu_1\mu_2\mu_3\rho_4\rho_5\rho_6} \end{cases}, \tag{204}$$

$$\Omega^{\alpha_3}(\Omega^{\alpha_4})\big|_{\text{nontriv}} \equiv \xi^a_{(1,2)\mu_1\cdots\mu_4}(\Omega^{\alpha_4}) = -5\partial^\rho\xi^a_{(1,3)\rho\mu_1\cdots\mu_4}, \tag{205}$$

$$\Omega^{\alpha_4}(\Omega^{\alpha_5}) \equiv \xi^a_{(1,3)\mu_1\cdots\mu_5}(\Omega^{\alpha_5}) = -6\partial^\rho\xi^a_{(1,4)\rho\mu_1\cdots\mu_5}. \tag{206}$$

Finally, the generic reducibility relations (197) and (198) are provided by the (non-trivial) relations

$$\bar{\delta}_{\Omega^{\alpha_1}(\Omega^{\alpha_2})}\Phi^{\alpha_0}\Big|_{\text{nontriv}} \equiv \begin{cases} \bar{\delta}_{\Omega^{\alpha_1}(\Omega^{\alpha_2})}A^{\mu_1\mu_2}_a & = \lambda\bar{C}^{b\mu_1\mu_2}_{a\rho_1}\frac{\delta\bar{S}^L}{\delta B^b_{\rho_1}} \\ \bar{\delta}_{\Omega^{\alpha_1}(\Omega^{\alpha_2})}B^a_{\mu_1\mu_2\mu_3} & = \lambda C^{ab}_{\mu_1\mu_2\mu_3\rho_1\rho_2\rho_3}\frac{\delta\bar{S}^L}{\delta B^b_{\rho_1\rho_2\rho_3}} + \lambda\bar{C}^{ab}_{\mu_1\mu_2\mu_3\rho_1}\frac{\delta\bar{S}^L}{\delta B^b_{\rho_1}} \\ \bar{\delta}_{\Omega^{\alpha_1}(\Omega^{\alpha_2})}B^a_{\mu_1} & = \lambda C^{ab}_{\mu_1\rho_1}\frac{\delta\bar{S}^L}{\delta B^b_{\rho_1}} + \lambda\bar{C}'^{a\rho_1\rho_2}_{b\mu_1}\frac{\delta\bar{S}^L}{\delta A^{\rho_1\rho_2}_b} + \lambda\bar{C}'^{ab}_{\mu_1\rho_1\rho_2\rho_3}\frac{\delta\bar{S}^L}{\delta B^b_{\rho_1\rho_2\rho_3}} \end{cases} \approx 0, \tag{207}$$

$$\Omega^{\alpha_1}(\Omega^{\alpha_2}(\Omega^{\alpha_3})) \equiv \begin{cases} \epsilon^{\mu_1}_{(2,0)a}(\Omega^{\alpha_2}(\Omega^{\alpha_3})) & = \lambda\hat{C}^{b\mu_1}_{a\rho_1}\frac{\delta\bar{S}^L}{\delta B^b_{\rho_1}} \approx 0 \\ \xi^a_{(3,0)\mu_1\cdots\mu_4}(\Omega^{\alpha_2}(\Omega^{\alpha_3})) & = 0 \\ \xi^a_{(1,0)\mu_1\mu_2}(\Omega^{\alpha_2}(\Omega^{\alpha_3})) & = \lambda\hat{C}^{ab}_{\mu_1\mu_2\rho_1}\frac{\delta\bar{S}^L}{\delta B^b_{\rho_1}} + \lambda\hat{C}^{a\rho_1\rho_2}_{b\mu_1\mu_2}\frac{\delta\bar{S}^L}{\delta A^{\rho_1\rho_2}_b} \approx 0 \end{cases}, \tag{208}$$

$$\Omega^{\alpha_2}(\Omega^{\alpha_3}(\Omega^{\alpha_4}))\big|_{\text{nontriv}} \equiv \xi^a_{(1,1)\mu_1\mu_2\mu_3}(\Omega^{\alpha_3}(\Omega^{\alpha_4})|_{\text{nontriv}}) = 0, \tag{209}$$

$$\Omega^{\alpha_3}(\Omega^{\alpha_4}(\Omega^{\alpha_5}))\big|_{\text{nontriv}} \equiv \xi^a_{(1,2)\mu_1\cdots\mu_4}(\Omega^{\alpha_4}(\Omega^{\alpha_5})) = 0. \tag{210}$$

The various (non-vanishing) coefficients from (207)–(210), defined as in (200), are expressed by

$$C^{\alpha_0\beta_0}\Big|_{\text{nonvan}} \equiv \begin{cases} \bar{C}^{b\mu_1\mu_2}_{a\rho_1} & = 5\varepsilon^{\mu_1\cdots\mu_6}\frac{\partial Z_{ac}}{\partial\varphi_b}\xi^c_{(3,1)\rho_1\mu_3\cdots\mu_6} \\ C^{ab}_{\mu_1\mu_2\mu_3\rho_1\rho_2\rho_3} & = -2\varepsilon_{\mu_1\mu_2\mu_3\rho_1\rho_2\rho_3}M^{abc}\epsilon_{(2,1)c} \\ \bar{C}^{ab}_{\mu_1\mu_2\mu_3\rho_1} & = -\frac{2}{3}\frac{\partial M^{acd}}{\partial\varphi_b}\varepsilon_{\mu_1\mu_2\mu_3\rho_1\rho_2\rho_3}A^{\rho_2\rho_3}_c\epsilon_{(2,1)d} \\ C^{ab}_{\mu_1\rho_1} & = 20\varepsilon^{\sigma_1\cdots\sigma_6}\frac{\partial^2 Z_{cd}}{\partial\varphi_a\partial\varphi_b}B^c_{\sigma_1\sigma_2\sigma_3}\xi^d_{(3,1)\mu_1\rho_1\sigma_4\sigma_5\sigma_6} \\ & \quad -\varepsilon_{\mu_1\rho_1\cdots\rho_5}\frac{\partial^2 M^{cde}}{\partial\varphi_a\partial\varphi_b}A^{\rho_2\rho_3}_c A^{\rho_4\rho_5}_d\epsilon_{(2,1)e} \\ \bar{C}'^{a\rho_1\rho_2}_{b\mu_1} & = -5\varepsilon^{\rho_1\cdots\rho_6}\frac{\partial Z_{ac}}{\partial\varphi_b}\xi^c_{(3,1)\mu_1\rho_3\cdots\rho_6} \\ \bar{C}'^{ab}_{\mu_1\rho_1\rho_2\rho_3} & = -\frac{2}{3}\frac{\partial M^{bcd}}{\partial\varphi_a}\varepsilon_{\mu_1\rho_1\cdots\rho_5}A^{\rho_4\rho_5}_c\epsilon_{(2,1)d} \end{cases}, \tag{211}$$

$$C^{\alpha_1\beta_0}\Big|_{\text{nonvan}} \equiv \begin{cases} \hat{C}^{b\mu_1}_{a\rho_1} & = -6\varepsilon^{\mu_1\cdots\mu_6}\frac{\partial Z_{ac}}{\partial\varphi_b}\xi^c_{(3,2)\rho_1\mu_2\cdots\mu_6} \\ \hat{C}^{ab}_{\mu_1\mu_2\rho_1} & = 60\varepsilon^{\sigma_1\cdots\sigma_6}\frac{\partial^2 Z_{cd}}{\partial\varphi_a\partial\varphi_b}B^c_{\sigma_1\sigma_2\sigma_3}\xi^d_{(3,2)\mu_1\mu_2\rho_1\sigma_4\sigma_5\sigma_6} \\ \hat{C}^{a\rho_1\rho_2}_{b\mu_1\mu_2} & = -15\varepsilon^{\rho_1\cdots\rho_6}\frac{\partial Z_{bc}}{\partial\varphi_a}\xi^c_{(3,2)\mu_1\mu_2\rho_3\cdots\rho_6} \end{cases}, \tag{212}$$

while the required antisymmetry of (211) is ensured via the relations

$$C^{\alpha_0\beta_0}\Big|_{\text{nonvan}} = -C^{\beta_0\alpha_0}\Big|_{\text{nonvan}} \Leftrightarrow \begin{cases} \bar{C}^{b\mu_1\mu_2}_{a\rho_1} & = -\bar{C}'^{b\mu_1\mu_2}_{a\rho_1} \\ C^{ab}_{\mu_1\mu_2\mu_3\rho_1\rho_2\rho_3} & = -C^{ba}_{\rho_1\rho_2\rho_3\mu_1\mu_2\mu_3} \\ \bar{C}^{ab}_{\mu_1\mu_2\mu_3\rho_1} & = -\bar{C}'^{ba}_{\rho_1\mu_1\mu_2\mu_3} \\ C^{ab}_{\mu_1\rho_1} & = -C^{ba}_{\rho_1\mu_1} \end{cases}. \tag{213}$$

On behalf of (203)–(210), we can synthesize the particularities of the reducibility of the deformed generating set of gauge transformations (180) corresponding to the self-interacting $D = 6$ BF model into: *(i)* all the first-order and most of the second-order reducibility functions (see (203) and (204)) gain new terms at order one in the coupling constant, while the third- and fourth-order reducibility functions (see (205) and (206)) coincide with those from the free limit; *(ii)* all the first-order and most of the second-order reducibility relations (see (207) and (208)) only hold on-shell, i.e., on the deformed stationary surface (179), in contrast to what happens at the level of the original free BF model, while all the third- and fourth-order reducibility relations coincide with those from the free limit and thus hold off-shell; *(iii)* a major surprise is the gauge transformations with respect to the parameters $\epsilon^{\mu_1}_{(2,0)a}$, which are reducible of order *two* in the presence of self-interactions (more precisely, due to the existence of the non-vanishing parameterizing functions denoted by Z, as can be observed from the first relation in (204)), instead of order one in the free limit. The last feature is induced by the presence of new terms in the expression of the transformed gauge parameters $\epsilon^{\mu_1}_{(2,0)a}(\Omega^{\alpha_2})$ (the latter quantities from the first line of (203)), which depend on the Zs and involve the first-order reducibility parameters $\xi^{b}_{(3,1)\mu_2\cdots\mu_6}$ that were previously associated (in the free limit) strictly with the reducibility chain of the BF 3-forms $\overset{[3]}{B}{}^{a}$. This does not impact upon the overall reducibility order of the self-coupled model (4), the number of independent gauge symmetries from (180), the total number of physical degrees of freedom corresponding to the self-interacting BF model described by action (170) (0), or the field/ghost/antifield spectra, which are all preserved with respect to the free limit. Actually, there appears a 'redistribution' of the physical degrees of freedom at the level of each type of BF form fields of strictly positive form degrees when new gauge symmetries are allowed to enter the deformed generating set of gauge transformations (180) other than its free limit (5), which is due to the respective differences between the features of the Hamiltonian formulation for the deformed and the free BF theory.

Finally, the higher-order structure functions corresponding to the open, deformed gauge algebra generated by (180) as well as to the reducibility of the deformed generating set of gauge transformations (180) can be deduced from the remaining terms in (161)–(164), which have not been analyzed here.

5. Conclusions

In conclusion, there exist consistent, non-trivial self-interactions that can be added to a special collection of free topological BF models in $D = 6$ space-time dimensions, the field spectrum of which comprises four sets of form fields, of form degree ranging between 0 and 3. The couplings are deduced within the cohomological framework of the antifield-BRST deformation method and in the presence of several usual selection rules employed in gauge field theory, namely, analyticity in the coupling constant, space-time locality, Lorentz covariance, Poincaré invariance, and conservation of the differential order of each interacting field equation with respect to its free limit.

The fully deformed solution to the classical master equation stops at order one in the coupling constant, comprises components of antifield number valued between 0 and 6, and is parameterized by a 'potential' of the undifferentiated scalar fields, $V(\varphi)$, and two sets of functions depending only on the undifferentiated BF 0-forms, among which one is fully symmetric $\{M^{abc}(\varphi)\}$ and the other antisymmetric $\{Z_{ab}(\varphi)\}$, both with purely internal collection indices. The consistency of deformations at all orders of perturbation theory greater or equal to 2 is completely equivalent to certain specific algebraic identities that restrict only the two parameterizing function sets, expressed via the annihilation of their simple contraction. The resulting self-coupled Lagrangian density adds to the free Lagrangian two kinds of vertices that break the PT-invariance (which was not set as a selection rule), among which one is cubic in the 2-forms $\overset{[2]}{A}_{a}$ and has the set $M^{abc}(\varphi)$ as background and the other is quadratic in the 3-forms $\overset{[3]}{B}{}^{a}$, with the complementary set,

$Z_{ab}(\varphi)$, as background. The third class of vertices is contained in the 'potential', $V(\varphi)$, which remains an arbitrary smooth function that is not affected by the consistency of deformations in higher orders of perturbation theory; it collects all possible self-interactions that can be introduced among a set of real scalar fields, and can be constrained from various arguments, such as the renormalizability of the emerging gauge theory or the inclusion of a specific phenomenological ansatz (such as the enforcement of certain limits on the scalar field masses or the introduction of particular potentials required by some definite symmetry-breaking scenarios). The structure of the deformed solution to the classical master equation emphasizes a self-coupled $D = 6$ topological BF theory with several new features compared to the starting free limit: all the gauge transformations of the BF-forms of strictly positive form degrees are modified, the associated gauge algebra becomes open (in contrast to the original Abelian one), and the reducibility functions and relations are deformed in general, with several of the latter holding only on-shell (on the deformed stationary surface of self-coupled field equations), in opposition to the original ones, which take place purely off-shell. A very interesting feature of the interacting model is the appearance of the quadratic vertex in $\overset{[3]}{B}{}^{a}$s mentioned in the above, which generalizes the famous vertex present in the BF formulation of $D = 2$ gravity. Nevertheless, the similar, generalized vertex is less restricted here, in the sense that the Zs remain antisymmetric while no longer satisfying the $D = 2$ Jacobi identities, which allows these functions to be interpreted as the Poisson two-tensor of a certain Poisson manifold. The $D = 2$ Jacobi identities were implemented there as the equations ensuring the consistency of deformations to all orders of perturbation theory greater or equal to 2. Here, their role is taken over by the algebraic equations mentioned in the above, which involve the two sets of parameterizing functions, the Zs and the Ms. The novelty of our results include, among others, the identification of three special classes of elements from the local homology of the Koszul–Tate differential at strictly positive values of the antifield number and at pure ghost number 0, which are not gauge-invariant under the free-limit generating set of gauge symmetries of the action and the expressions of which are crucial in future generalizations of our results to an arbitrarily high finite-dimensional space-time.

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Abbreviations

The following abbreviations are used in this manuscript:

BRST	Becchi–Rouet–Stora–Tyutin

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