Averaging in cosmological models

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Abstract

The averaging problem in cosmology is of considerable importance for the correct interpretation of cosmological data. We review cosmological observations and discuss some of the issues regarding averaging. We present a precise definition of a cosmological model and a rigorous mathematical definition of averaging, based entirely in terms of scalar invariants.

1 Introduction

Cosmological observations [1], based on the assumption of a spatially homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) model plus small perturbations, are usually interpreted as implying that there exists dark energy, the spatial geometry is flat, and that there is currently an accelerated expansion, giving rise to the so-called standard ΛCDM-concordance model. Although the concordance model is quite remarkable, it does not convincingly fit all data (see below). Unfortunately, if the underlying cosmological model is not a perturbation of an exact flat FLRW solution, the conventional data analysis and their interpretation is not necessarily valid.

For example, the standard analysis of type Ia supernovae (SNIa) and cosmic microwave background (CMB) data in FLRW models cannot be applied directly when inhomogeneities or backreaction effects are present. However, supernovae data can be explained without dark energy in inhomogeneous models, where the full effects of general relativity (GR) come into play. In one approach exact inhomogeneous cosmological models can be utilised. Indeed, it has been shown that the Lemaître-Tolman-Bondi (LTB) solution can be used to fit the observed data without the need of dark energy, although it may be necessary to place the observer at a preferred location [2].

A second approach, and the one of interest here, is backreactions through averaging. The averaging problem in cosmology is of considerable importance for the correct interpretation of cosmological data. The correct governing equations on cosmological scales are obtained by averaging the Einstein field equations (EFE) of GR (plus a theory of photon propagation; i.e., information on what trajectories actual particles follow). By assuming spatial homogeneity and isotropy on the largest scales, the inhomogeneities affect the dynamics though correction (backreaction) terms, which can lead to behaviour qualitatively and quantitatively different from the FLRW models; in particular, the expansion rate may be significantly affected.

2 Cosmological observations

From the evidence of the CMB radiation, the universe was very smooth at the time of last scattering. By the Copernican principle, the assumption of global isotropy and spatial homogeneity is then justified at the epoch of last scattering. Thus, the paradigm for our current standard model of the universe assumes the underlying geometry is FLRW, with additional Newtonian perturbations, and in matching the cosmological observables that derive from such a geometry we have been led to the conclusion over the past decade that the present–day universe is dominated by a cosmological constant, Λ, or other fluid–like "dark energy", which violates the strong energy condition. In the case of the ΛCDM paradigm, dark energy only becomes appreciable at late epochs. Dark energy is widely described as the biggest problem in cosmology today.

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There are several problems regarding the ΛCDM model. First, it is difficult to understand the large value for Λ and why the contributions of ordinary matter and the repulsive component are roughly equal today, at around 10 billion years (the coincidence problem). Second, the universe is not perfectly homogeneous and isotropic (or even perturbatively near homogeneity and isotropy). There are non-linear structures in the real universe which are not described by perturbations around a smooth background, with a distribution that is statistically homogeneous and isotropic above a scale of about 100 Mpc (or, more precisely, 100 h^{-1} Mpc, but we shall omit the factor h^{-1} for simplicity here) [3]. Linear perturbations are only valid when both the curvature and the density contrasts remain small, which is certainly not the case in the non-linear regime of structure formation when the SNIe observations are made.

Indeed, the largest structures so far detected are limited only by the size of the surveys that found them [4]. At the present epoch the distribution of matter is far from homogeneous on scales less than 150–300 Mpc. The actual universe has a sponge–like structure, dominated by huge voids surrounded by bubble walls, and threaded by filaments, within which clusters of galaxies are located. Locally there two enormous voids, both 35 to 70 Mpc across, associated with the so-called velocity anomaly [5], a large filament known as the Sloan great wall about 400 Mpc long [6] and the Shapely supercluster with a core diameter of 40 Mpc at a distance of $\sim$ 200 Mpc [7]. In addition, there has been detection of anomolously large local bulk flows [8] and evidence for a significant anisotropy in the local Hubble expansion at distances of $\sim$ 100 Mpc [9]. Recent surveys suggest [10] that some 40–50% of the present volume of the universe is in voids of a characteristic scale 30 Mpc. If smaller minivoids and larger supervoids are included, then our observed universe is presently void–dominated by volume; thus within regions as small as 100 Mpc density contrasts ~ -1 are observed leading to substantial gradients in the (local Ricci) spatial curvature [11]. Therefore, spatial homogeneity is valid only on scales larger than at least 100 Mpc [3], in contradiction with the predictions of the ΛCDM model in which the scale beyond which the distribution should become uniform is about 10 Mpc [4].

The present distribution of matter is clearly very complex, and since we cannot solve the EFE for this distribution of matter analytically, there is an important question as to how we operationally match the average geometry of this distribution to the simple FLRW models. The mere fact that the universe is presently inhomogeneous means that the assumptions implicit in the FLRW approximation can no longer be justified at the present epoch in the almost exact sense that they were justified at the epoch of last scattering. The situation is further complicated by the fact that most data analysis based on the standard model (FLRW + perturbations; ΛCDM) is model- and prior- dependent [18].

Consequently, spatial homogeneity only applies at the present day in an averaged sense. Given the observed inhomogeneities and that the nonlinear growth of structure appears to be roughly correlated to the epoch when cosmic acceleration is inferred to begin, it has been suggested that the FLRW geometries are inadequate as a description of the universe at late times and the introduction of a smooth dark energy is a mistaken interpretation of the observations. A universe which is homogeneous and isotropic only statistically does not generally expand like an exactly homogeneous and isotropic universe, even on average. It is possible that there are large effects on the observed expansion rate (and hence on other observables) due to the backreaction of inhomogeneities in the universe. Anything that affects the observed expansion history of the universe alters the determination of the parameters of dark energy; in the extreme it may remove the need for dark energy. Indeed, it has been suggested in that inhomogeneities related to structure formation could be responsible for accelerated expansion [15].

The effects of averaging can be signicant. Using perturbation theory, effects of order $\sim 10^{-4}$ are often quoted. However, these affects occur by averaging over the Hubble volumes and not over regions of $\sim 100 - 200$ Mpc. At best this is (only) a self-consistency analysis. In addition, there are highly non-Gaussian inhomogeneities in the late universe, and the coherence of structures causes small deviations in observations to sum to a large deviation, and there can be significant effects on observations from the backreaction of inhomogeneities [12].

In [14] the hierarchy of the critical scales for large scale inhomogeneities (backreaction effects) were calculated, at which 10% effects show up from averaging at different orders over a local domain in space-time. The dominant contribution comes from the averaged spatial curvature, observable up to scales of $\sim$ 200 Mpc. The averaged spatial curvature typically leads to 10% (1%) effects up to $\sim$ 80 (240) Mpc. The cosmic variance of the local Hubble rate is 10% (5%) for spherical regions of radius 40 (60) Mpc. Below $\sim$ 40 Mpc, the cosmic variance of the Hubble rate is larger than 10%. At lower scales the kinematical backreaction, due to second order perturbations caused by local inhomogeneities and anisotropies, are important. The crude estimates are comparable to the actual density variance determined from large scale structure surveys [3, 4, 11]. In addition, it has been found that a matter model with discrete masses (rather than an idealised continuous fluid) leads to corrections for cosmological parameters $\sim 10 - 20\%$ [13]. Indeed, it has been argued that the effects of averaging can theoretically be as large as $\sim 40\%$ when the equivalence principle of GR is properly applied [11].

There are also a number of other potential problems with the standard model. Apart from WMAP date ($z \sim 1100$), the standard model is based on local observations ($z < 2$), and consequently it has been argued that the data does not convincingly imply acceleration [14]. It is noteworthy that the quality of fit of the ΛCDM model has decreased with the introduction of each new SNIa data set, which may hint at inadequacy of the ΛCDM description [15]. Indeed, the standard model does not fit all data; there is tension between different SNIa data sets [16] and tension between different data sets, especially between SNIa data and CMB data [16, 17], but also with nucleosynthesis and other large scale structure data [11].

2.0.1 Discussion: spatial curvature

Clearly, backreaction (averaging) effects are real, but their relative importance still need to be determined. Within perturbation theory, the value of the normalized spatial curvature, Ω_k, is expected to be small. However, different authors have argued that Ω_k can be as large as $5 - 10\%$ [11, 13, 15]. In particular, CMB data does not necessarily imply flatness [15]; the position of the CMB peaks is consistent with significant spatial curvature provided that the expansion history is sufficiently close to the spatially flat ΛCDM model. Indeed, conclusions drawn about spatial curvature from the CMB are model- and prior-dependent; in a clumpy universe, the usual expression is inapplicable due to the non-trivial evolution of the spatial curvature as well as the fact that clumping contributes to the expansion rate, and there is no simple argument for obtaining the position of the CMB peaks. In addition, if the spatial curvature (parameter k) is allowed to be a function of position, then considerable spatial curvature (locally) is permissable (consistent with CMB observations) [13, 15], since curvature can affect different observations at different scales in different ways (e.g., large scale structure, $z < 2$, and CMB, $z \sim 1100$).

Observational data perhaps suggests a normalized spatial curvature $|\Omega_k| \approx 0.01 - 0.02$ (i.e., of about a percent). Combining these observations with large scale structure observations then puts stringent limits on the curvature parameter in the context of adiabatic ΛCDM models; however, these data analyses are very model- and prior-dependent, and care is needed in the proper interpretation of the data. There is a heuristic argument that $\Omega_k \sim 10^{-3} - 10^{-2}$ ($\Omega_k \sim 1\%$) [20], which is consistent with CMB observations [1] and agrees with estimates for intrinsic curvature fluctuations using realistically modelled clusters and voids in a Swiss-cheese model. In particular, the MG equations (see below) were explicitly solved in a FLRW background geometry and it was found that the correlation tensor (backreaction) is of the form of a spatial curvature [19]. Thus, the averaged EFE for a flat spatially homogeneous, isotropic macroscopic space-time geometry has the form of the EFE of GR for a non-flat spatially homogeneous, isotropic space-time geometry.

It must be appreciated that such a value for Ω_k, at the 1% level, is relatively large and may have a significant dynamical effect on the evolution of the universe and the interpretation of cosmological observations. Indeed, in such a scenario the current contribution from the spatial curvature is comparable to the energy density in luminous matter. In addition, such a value cannot be naturally explained by inflation. From standard analysis, depending on the initial conditions and the details of a specific model of inflation, $|\Omega - 1|$ would be extremely small. Therefore, any value for Ω_k at the 1 % level can only be naturally explained in terms of an averaging effect. In addition, such an effect would compromise any efforts to use data to constrain dark energy models (within the standard paradigm) with a variable equation of state [21].

3 The averaging problem in cosmology

3.1 General Approaches

The Universe is not isotropic or spatially homogeneous on local scales. The gravitational FE on large scales are obtained by averaging the EFE of GR. It is necessary to use an exact covariant approach which gives a prescription for the correlation functions that emerge in an averaging of the full tensorial EFE.

There are a number of approaches to the averaging problem. In the approach of Buchert a 3+1 cosmological space-time splitting is employed (i.e., this procedure is not generally covariant) and only scalar quantities are averaged (and thus the governing equations are not closed) [22]. The perturbative approach (backreaction about an FLRW background [12]) involves averaging the perturbed EFE. However, a perturbation analysis cannot provide any information about an averaged geometry; thus perturbation theory cannot be conclusive and provide a complete solution.

To date the macroscopic gravity (MG) approach is the only approach to the averaging problem in GR which gives a prescription for the correlation functions which emerge in an averaging of the non-linear FE (without which the averaging of the EFE simply amount to definitions of the new averaged terms) [23]. The MG space-time averaging approach is a fully covariant, gauge independent and exact method, in which the averaged EFE are written in the form of the EFE for the macroscopic metric tensor when the correlation terms are moved to the right-hand side of the averaged EFE to serve as the geometric modification to the averaged (macroscopic) matter energy-momentum tensor. For the cosmological problem additional assumptions are required: with reasonable cosmological assumptions, the correlation tensor in Zalaletdinov's scheme takes the form of a spatial curvature [19], and Buchert's scheme can be realized as a consistent limit [24].

There are other approaches to averaging. The formal mathematical issues of averaging tensors on a differential manifold has recently been revisted. We note that integrating scalars on spacetime regions is always well-defined and it may be possible to avoid several of the technical problems of averaging by adopting an approach based on scalar curvature invariants.

4 Cosmological models

A cosmological model is a *mixed* model, in that the matter is already assumed to be averaged but the geometry is not (necessarily). Therefore, we need a consistent model for the matter, represented on the characteristic averaging scale, and its appropriate (averaged) physical properties. It is known that the separation between the gravitational field and the matter is not scale invariant and the notion of a perfect fluid is not scale invariant; averaging (in the presence of a gravitational field) modifies the equation of state of the matter. In addition, since averaging does not conserve geodesics, we need further assumptions in order to be able to compare the models with observational data.

A precise definition of a cosmological model is necessary; i.e., a framework in which to do averaging. The definition we shall adopt is given by the following conditions C1 – C5 [25]: **C1. Spacetime Geometry:** The spacetime geometry $(\mathbf{M}, \mathbf{g})$ is defined by a smooth Lorentzian metric $\mathbf{g}$ (characterizing the macroscopic gravitational field) defined on a smooth differentiable manifold $\mathbf{M}$. The macroscopic metric geometry is obtained by an appropriate spacetime averaging of the microgeometry; thus part of the definition of a cosmological model consists of specifying *the averaging scheme* (which must be consistent with the physical assumptions of the model encapsulated in the conditions C3 and C4 below) and *the cosmological scale* over which averaging or the smoothing occurs (i.e., we must specify the averaging scale ℓ or averaging region).

C2. Timelike Congruence: There exists a timelike congruence $(\mathbf{u})$ (in principle locally), representing a family of fundamental observers. Mathematically this means that the spacetime is $\mathcal{I}$*-non-degenerate* and hence the spacetime is uniquely characterized by its scalar curvature invariants [26]. In addition to the formal parts C1 and C2 of the definition of a cosmological model $(\mathbf{M}, \mathbf{g}, \ell, \mathbf{u})$, we must also specify the physical relationship (interaction) between the macroscopic geometry and the matter fields, including how the matter responds to the macroscopic geometry.

C3. Macroscopic FE: There exists an appropriate set of macroscopic FE relating the averaged matter and appropriately averaged (or macroscopic) geometry. This is based on an underlying microscopic theory of gravity (such as, for example, GR), and an appropriate formalism to average the geometry and find corrections (correlations) due to averaging the Einstein tensor in the resulting FE:

$$\tilde{G}^a{}_b + C^a{}_b = T^a_b, \tag{4.1}$$

where $\tilde{G}^a{}_b \equiv \tilde{R}^a{}_b - \frac{1}{2}\delta^a{}_b\tilde{R}$ and $\tilde{R}^a{}_b$ is the Ricci tensor of the averaged macrogeometry, $C^a{}_b$ is the correlation tensor, and T^a_b is the energy momentum tensor (already assumed averaged).

C4. Equations of motion: We also need to know the trajectories along which the cosmological matter moves (and also the light trajectories, which determine observational relations). In principle, the average motion of a photon need not be a null geodesic in the averaged geometry [20]. **C5. Observations:** Finally, we need to be able to relate averaged quantities with physical observables, which ultimately must be consistent with cosmological data.

In the standard FLRW model there are a number of simplifications and assumptions. The past approaches to averaging have been ideally suited to the FLRW models (with small, vanishing in the limit, perturbations). In these models, the macrometric $\mathbf{g}$ is the FLRW metric (C1) and $\mathbf{u}$ also has a geometric meaning (C2). In the usual point of view there are no correlations due to averaging (i.e., $C^a{}_b = 0$) or, more precisely, they are negligible (C3). In this case it follows from the contracted Bianchi identities that energy-momentum is conserved: $T^c_{b;c} = 0$, which relates the matter to the averaged geometry. All other effects are assumed negligible (C4). However, there is no formal argument that such assumptions arise from a rigorous averaging scheme of some appropriate (physically motivated) microgeometry. In addition, there are some important effects in the standard model which are not necessarily small perturbations.

Since there are no scales explicitly specified in the model, in a sense the model is incomplete. Indeed, the model does not even have the ability to determine whether there is a scale above which the geometry is exactly FLRW or whether at all scales the geometry is only approximately FLRW (with a given perturbation scale). Furthermore, regarding C5, we can ask whether the model agrees with observations? If it does not, then even if the model agrees in some approximate sense with most observations, there is no structure within which to discuss the potential small discrepancies with observed data, which is a deficiency of the model. If the model does, then it would be remarkable, although there is still the need for a physical explanation for the dark energy. Finally, if observations indicate that $\Omega_k \approx 1-2\%$, then there is no physical mechanism within the model (particularly if there is an inflationary period) to produce an intrinsic curvature parameter k of this magnitude, whereas an effective curvature parameter $\hat{k}$ of about a percent arises naturally from averaging.

5 An approach to averaging using scalars

For any given spacetime $(\mathbf{M}, \mathbf{g})$ we define the set of all scalar polynomial curvature invariants

$$\mathcal{I} \equiv \{R, R1, R2, R3, R^2, R^\mu{}_\nu R^\nu{}_\mu, \dots, C^2, \dots\} \tag{5.2}$$

(where the Ri are eigenvalues of the Ricci tensor, and $C^2 \equiv C^{\mu\nu\alpha\beta}C_{\mu\nu\alpha\beta}$). Consider a spacetime $(\mathbf{M}, \mathbf{g})$ with a set of invariants $\mathcal{I}$. Then, if there does not exist a continuous metric deformation of g having the same set of invariants as g, the set of invariants will be called *non-degenerate*, and the spacetime metric, g, *$\mathcal{I}$-non-degenerate*. This implies that for a metric which is $\mathcal{I}$-non-degenerate the invariants characterize the spacetime uniquely, at least locally. It was proven [26] that a 4D spacetime is either $\mathcal{I}$-non-degenerate or the metric is a degenerate Kundt metric. This is a striking result because it tells us that the only metrics not locally determined by their scalar invariants must be of Kundt form.

Hence, in general, since we know how to average scalar quantities, we can average all of the scalar curvature invariants that then represents an averaged spacetime (with that set of averaged scalar invariants). In particular, we note that cosmological models (as defined above) belong to the set of spacetimes completely characterized by their scalar invariants, suggesting that we can average a cosmological model using scalar invariants. Therefore, we have a microgeometry completely characterized by its set of scalar curvature invariants $\mathcal{I}$. We then average these microgeometry scalar curvature invariants to obtain a new set of macrogeometry scalar curvature invariants $\tilde{\mathcal{I}}$, which now completely characterizes the macrogeometry [25].

5.1 Averaging the geometry

In the general mathematical context we want to describe the averaged geometry (represented by the Riemann tensor and its covariant derivatives) and interpret the results. Let us consider, $\mathcal{I}$, defined by (5.2), which is an ordered set of functions on $\mathbf{M}$. Let us we write $\tilde{\mathcal{I}} \equiv \{\tilde{R}, \dots, \widetilde{R^{\mu}{}_{\nu} R^{\nu}{}_{\mu}}, \dots\}$, which is also an ordered set of functions. The question is then: does the ordered set of functions $\tilde{\mathcal{I}}$ correspond to the associated scalar curvature invariants for some metric $\tilde{g}$ (which could then serve to *define* the macrometric $\tilde{g}$).

It is certainly plausible that (some appropriately defined subset of) the ordered set of functions $\tilde{\mathcal{I}}$ correspond to the associated scalar curvature invariants for some macrometric $\tilde{g}$ for the class of $\mathcal{I}$-non-degenerate geometries that constitute the class of cosmological models defined. Since the geometries are $\mathcal{I}$-non-degenerate and in 4D the properties of the geometry can be represented in terms of scalars, and since relations between different terms (functions) in the set $\mathcal{I}$ (e.g., R and R^2 are functionally dependent) and the corresponding terms in the set $\tilde{\mathcal{I}}$ (e.g., $\tilde{R}$ and $\tilde{R}^2$) are functionally related in exactly the same way and syzygies (e.g., describing the algebraic type) are maintained under averaging, it follows that in general the set $\tilde{\mathcal{I}}$ gives rise to a macrometric $\tilde{g}$ (which will have similar algebraic properties to the micrometric g).

5.1.1 Proposal: Scalar Averaging Procedure

Let us consider the ordered set of functions $\mathcal{I}$ in the form of (5.2). First, let us omit any scalars from this set that are not algebraically independent (e.g., $\{R^2, R^{\mu}{}_{\nu} R^{\nu}{}_{\mu} \dots\}$) to obtain an (appropriate 'independent') subset $\mathcal{I}_A$. Second, for a particular spacetime, we omit any scalars from $\mathcal{I}_A$ that can be obtained from syzygies defining that particular spacetime (e.g., defining the algebraic type of the spacetime, such as the Segre type or the Petrov type). For example, for a Ricci tensor corresponding to the algebraic form of a perfect fluid we could omit $\{R2, R3\}$ (relative to $\{R, R1\}$). We consequently obtain the subset $\mathcal{I}_{SA}$: e.g., $\mathcal{I}_{SA} \equiv \{R, R1, \dots, C^2, \dots\}$. For the spacetimes under consideration the microgeometry is then completely characterized by the (sub)set of scalar curvature invariants $\mathcal{I}_{SA}$.

We now construct the new ordered set of functions $\tilde{\mathcal{I}}_{SA}$ by averaging the various scalar invariants of $\mathcal{I}_{SA}$: $\tilde{\mathcal{I}}_{SA} \equiv \{\tilde{R}, \widetilde{R1}, \dots, \widetilde{C^2}, \dots\}$, where all of the original scalar invariants omitted from the original set $\mathcal{I}$ are replaced by a new set of functions obeying exactly the same algebraic properties (or syzygies) as $\mathcal{I}_{SA}$. Therefore, it is assumed that $\tilde{\mathcal{I}}_{SA}$ comes equiped with these syzygies, so that we could construct the corresponding set $\tilde{\mathcal{I}}$ consisting of the members of $\tilde{\mathcal{I}}_{SA}$ and all of the corresponding syzygies. Consequently, the set $\mathcal{I}_{SA}$ is an ordered set of functions (scalar curvature invariants) on $\mathbf{M}$ which uniquely determines the macrogeometry with exactly the same algebraic properties as the original microgeometry.

5.1.2 Cosmological models

In the case of a cosmological model, from C3 we have an effective set of FE and we only need to consider the macrogeometric Ricci tensor $\tilde{R}^a{}_b$ (the correlation tensor is obtained from the averaging procedure). The microgeometric Ricci tensor $R^a{}_b$ is completely characterized by a set of scalar curvature invariants $\mathcal{I}_R$. Averaging these scalar curvature invariants we obtain the set $\tilde{\mathcal{I}}_{\tilde{R}}$, which completely characterizes the macrogeometric Ricci tensor $\tilde{R}^a{}_b$. Since constructing the Ricci tensor from a set of scalar curvature invariants $\mathcal{I}_R$ is relatively simple compared to the corresponding problem for the Riemann tensor, and since the reduced set of scalar curvature invariants $\mathcal{I}_R$ is considerably smaller than $\mathcal{I}$, we have reduced the complexity of the problem in this new averaging approach. Indeed, for a Ricci tensor of the algebraic form of a perfect fluid, there are effectively (only) two independent zeroth order scalar invariants, the Ricci scalar and a single Ricci eigenvalue (corresponding to the effective energy density, ρ, and pressure, p, of the perfect fluid). Therefore, in the context of the scalar averaging procedure, we have the set $\{\tilde{R}, \widetilde{R1}\}$.

It is necessary to determine whether the correlations due to averaging alter the geometry or affect the effective energy-momentum tensor. This is partly a question of interpretation, which must be done within the context of the underlying cosmological model. In particular, in the cosmological application it may be appropriate to reinterpret the averaging correlations as corrections to the matter fields (and hence the effective equation of state) through the EFE.

In [26] the specific example of a static spherically symmetric perfect fluid spacetime was considered. This is a simple and appropriate model for illustration since it can include an arbitrary function of one variable, there is a non-vanishing pressure, the averaging region does not change with time and there are no gravitational waves. The average correlations can be interpreted as contributing a small constant curvature term, arising from the averaging of local inhomogeneities in the micro-Ricci tensor to the smooth macro-Ricci tensor (consistent with the results of [19]).

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