Generalised Quantum Double Models from Hopf Monoids

Verallgemeinerte Quanten-Doppel-Modelle basierend auf Hopf Monoiden

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Zusammenfassung

Die vorliegende Dissertation *Generalised Quantum Double Models from Hopf Monoids* konstruiert Verallgemeinerungen der *Kitaevschen Quanten-Doppel-Modelle* auf der Grundlage von involutiven Hopf Monoiden in symmetrisch monoidalen Kategorien. Die ursprünglichen Modelle basierend auf Gruppenalgebren wurden 2002 von Kitaev in [Ki] eingeführt, um ein realistisches festkörperphysikalisches Modell für topologische Quantencomputer zu entwickeln. In [BMCA] wurden die Modelle basierend auf Gruppenalgebren zu Modellen für endlich-dimensionale halbeinfache Hopf-Algebren erweitert.

Der Grundzustand in den ursprünglichen Modellen ist von besonderer Bedeutung, da er eine topologische Invariante orientierter Flächen definiert und insbesondere topologische Quantencomputer vor Fehlern schützt [CDH+]. Darüberhinaus steht er in direktem Bezug zu topologischen Quantenfeldtheorien vom Turaev-Viro-Barrett-Westbury Typ [TV, BaW, BaK].

Die Konstruktion von [MV], die Abbildungsklassengruppenwirkungen aus Hopf Monoiden hervorruft, legt nahe, dass sich Kitaev Modelle weiter von Hopf Algebren zu Hopf Monoiden in symmetrisch monoidalen Kategorien verallgemeinern lassen. Diese Verallgemeinerung steht im Vordergrund der Arbeit, im Speziellen eine Verallgemeinerung des Grundzustandes der Modelle aus [Ki, BMCA]. Um Quanten-Doppel-Modelle für Hopf Monoide zu definieren, kann teilweise auf die Ergebnisse in [Ki, BMCA, MV] zurückgegriffen werden. Ausgehend von einem Schleifengraphen und einem Hopf Monoid wird zunächst jede Kante des Graphen mit einer Kopie des Hopf Monoids versehen. Dem Tensorprodukt dieser Kopien wird dann für jeden Vertex und jede Fläche des Graphen eine Modul- und Komodulstruktur zugeordnet. Durch Anwendung von (Ko)egalisatoren können die (Ko)invarianten dieser (Ko)modulstrukuren bestimmt werden. Mittels eines kategoriellen Bildes werden die Invarianten und Koinvarianten verknüpft, wodurch ein verallgemeinerter Grundzustand definiert wird.

In dieser Arbeit wird gezeigt, dass auch der verallgemeinerte Grundzustand eine topologische Invariante definiert: Er hängt nur vom Geschlecht der Fläche ab, die der Schleifengraph kodiert, nicht von der konkreten Wahl des Graphen. Dieses Resultat bietet einen geeigneten Ausgangspunkt, um den verallgemeinerten Grundzustand für konkret gegebene Hopf Monoide zu bestimmen. Ausgehend von Standardgraphen werden insbesondere die verallgemeinerten Grundzustände für Hopf Monoide in den Kategorien $\mathrm{Set}, \mathrm{Top}, \mathrm{SSet}$ sowie für Gruppenalgebren in der Kategorie $k-\mathrm{Mod}$ für einen kommutativen Ring k beschrieben. Ein besonderer Fokus liegt zudem auf der Bestimmung des Grundzustandes in der Kategorie Cat der kleinen Kategorien, in der es deutlich schwieriger ist Koegalisatoren zu bestimmen. Alle erhaltenen Grundzustände in den erwähnten Kategorien stellen Verallgemeinerungen des Modulraums flacher Zusammenhänge dar.

Unter Benutzung der Ergebnisse in [MV] wird in der Arbeit gezeigt, dass die Grundzustände mit Abbildungsklassengruppenwirkungen versehen sind. Zudem liefert die Arbeit eine konkrete Beschreibung dieser Wirkung für Hopf Monoide in Set, SSet, Cat.

Im letzten Teil der Arbeit wird ein Zusammenhang zwischen den Abbildungsklassengruppenwirkungen in den Modellen und Kuperberg Invarianten von 3-Mannigfaltigkeiten [Ku] hergestellt. Im Speziellen wird gezeigt, dass die verallgemeinerten Kuperberg Invarianten aus [KV] durch eine Folge von Kantenverschiebungen und Dehn Twists beschrieben werden können.

Die Abschnitte 2.1, 2.2, Kapitel 3-5 und 7 sowie die Abschnitte 8.1-8.3, 8.6 und Abschnitt 9.3 sind bis auf kleinere Änderungen und Ergänzungen Teile des Preprints:

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Contents

1 Introduction

Topological invariants of low-dimensional manifolds assign algebraic objects or quantities to them that are invariant under homeomorphisms. Such invariants have attracted strong interest, as they connect monoidal categories, quantum groups, low-dimensional topology, knot theory, conformal field theory, gauge theory and quantum computing. Sometimes the invariants are associated to manifolds with additional structure such as flat bundles. An example is the representation variety $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ or moduli space of flat G-bundles on a surface Σ for a Lie group G and its quantisation [BBJ, BR, AGS].

A Hopf-algebraic generalisation of this space appears in topological quantum computing as the *ground state* or *protected space* of quantum double models on this surface. In this case the (Lie) group G is replaced by a finite-dimensional semisimple Hopf algebra. *Quantum double models* or *Kitaev lattice models* were introduced by Kitaev [Ki] in 2002 to obtain a realistic physics model for topological quantum computing. The protected space or ground state of these models coincides with the vector space a Turaev-Viro-Barrett-Westbury topological quantum field theory (TQFT) [TV, BaW] assigns to the surface [BaK]. This vector space particularly plays an important role for the error correcting process in topological quantum computing.

Kitaev models in [Ki] are based on two ingredients, namely a ribbon graph encoding an oriented surface and the group algebra of a finite group. Buerschaper et al. [BMCA] generalised the latter ingredient to finite-dimensional semisimple complex $*$-Hopf algebras.

This thesis focuses on a further generalisation of this setting. As the work of Meusburger and Voß [MV] suggests, Kitaev models can more generally be based on involutive Hopf monoids in symmetric monoidal categories. Whereas the focus in [MV] is on mapping class group actions constructed from specific graphs, this thesis focuses on a generalisation of the protected space. This is again a topological invariant of oriented surfaces. Moreover, we explicitly determine the generalised protected spaces for examples of Hopf monoids, which yield generalisations of representation varieties. We show that the protected space is independent of the choice of the graph and carries an action of the mapping class group. We also show how it is related to Kuperberg invariants.

More concretely, the main motivations and results of the thesis can be described as follows:

Quantum double models. One main motivation for this thesis is to generalise the algebraic input data of the Kitaev models in [Ki, BMCA].

The models in [BMCA] are built starting from a finite-dimensional semisimple complex $*$-Hopf algebra and a ribbon graph. One assigns a copy of the Hopf algebra to each edge in the graph. The tensor product of these Hopf algebras then forms the *extended Hilbert space*. Additionally, the models are endowed with linear endomorphisms of this Hilbert space assigned to the vertices and faces of the graph, the vertex and face operators. Using normalised Haar integrals of the Hopf algebra and its dual, one can construct commuting projectors from the vertex and face operators. The *ground state* or *protected space* of the model is the intersection of their images.

The important property of the ground state is that it is a topological invariant: it does not depend on the choice of graph used for constructing the models, but only on the homeomorphism class of the oriented surface obtained by gluing disks to the faces of the graph. Moreover, Meusburger showed in [Me] that Kitaev models can be interpreted as Hopf algebra valued lattice gauge theories and are linked to the quantisation of moduli spaces of flat connections.

Quantum double models are a very active topic of research in condensed matter physics and quantum computing. The protected space plays an important role in these contexts, as its topological invariance protects quantum computers from errors. It is rigorously shown by Cui et al. [CDH+] that the ground state of a Kitaev model for a group algebra of a finite group is an error-correcting quantum code. Recent works provide mathematical descriptions and analyses of topological quantum computing [CM] and the error-correcting process [CGR] in these models.

There are various known generalisations of the Kitaev models in [Ki, BMCA], like the formulation of Chang [Ch] in terms of unitary quantum groupoids and the recent generalisation of Jial et al. [JTKC] to a weak Hopf algebra setting. Based on the models for group algebras of finite groups Bombin and Martin-Delgado [BM] described a model with defects and boundaries. Along these lines Koppen [Kop] constructed a model with generalised defects and Voß [Vo] focused on models with topological defects inspired by Turaev-Viro TQFTs.

Hopf monoids are a natural generalisation of Hopf algebras in symmetric monoidal categories. They are obtained by replacing vector spaces by objects and linear maps by morphisms in the category. In the framework of pivotal Hopf monoids in symmetric monoidal categories that are finitely complete and cocomplete, mapping class group actions were constructed in [MV]. For this a copy of the Hopf monoid is associated to each edge of a graph. Then (co)module structures are assigned to the vertices and faces and a notion of (co)invariants and biinvariants of the latter is defined. This formalism resembles the construction of the Kitaev models in [Ki, BMCA] and their known generalisations, but the mathematical data is far less restrictive.

In particular, the categories need not to be abelian, linear, finite or semisimple and need not have duals. While there is a notion of integrals over Hopf monoids and their existence is ensured for left rigid categories in which all idempotents split [BKLT], they are not necessary. Due to the similarity of the formalisms in [Ki, BMCA, MV] and the less restrictive input data in [MV], it is reasonable and desirable to generalise Kitaev models from Hopf algebras to Hopf monoids.

Crossed modules and the category Cat **of small categories.** Considering involutive Hopf monoids in symmetric monoidal categories instead of Hopf algebras allows one to apply the formalism to Hopf monoids in cartesian monoidal categories, in particular in the category Cat of small categories and functors. This defines a generalised protected space consisting of a category. Brown and Spencer [BS] established that Hopf monoids in Cat are given by crossed modules.

Additionally, Cat is of strong interest, as it can be made into a symmetric monoidal bicategory CAT by considering natural transformations as 2-morphisms. It often serves as an essential example when symmetric monoidal bicategories are needed, for instance in the context of operads [MW].

Crossed modules have also been studied intensively to obtain higher gauge theoretical generalisations of quantum double models. In the work of Bullivant et al. [BC+a, BC+b] and Martins and Picken [MPb, MPa] in higher gauge theory settings and the work of Sozer and Virelizier [SV] on 3d homotopy quantum field theories, crossed modules encode additional topological structures and symmetries, especially in higher dimensions. Koppen, Martins and Martin [KMM] define an analogous model for arbitrary dimension based on crossed modules of Hopf algebras. Under certain conditions the latter reduce to crossed modules of groups. Other examples involving crossed modules in such models are due to Chen [Che] and Huxford and Simon [HS]. This differs from their role in the thesis where crossed modules are considered as examples of Hopf monoids, but do not encode geometrical or topological data.

Mapping class group actions. Representations of mapping class groups arise in particular from TQFTs of Turaev-Viro-Barrett-Westbury or Reshetikhin-Turaev type [TV, BaW, RT]. They also show up in the context of conformal field theories, in the quantisation of Chern-Simons theory and are often related to modular functors [BK].

Lyubashenko [Lya, Lyb, Lyc] established a construction of projective mapping class group representations from Hopf algebras in abelian ribbon categories. Related and generalised mapping class group representations are described in the work of Faitg [Fa20, Fa19], in the framework of conformal field theories by Fuchs, Schweigert and Stigner [FS, FSSb] and recently by De Renzi et al. [DGG+]. Mapping class group actions also appear in Kitaev's quantum double model where they can be used as quantum gates [BlW].

Using Gervais' presentation of mapping class groups in terms of generating Dehn twists [Ge], Meusburger and Voß [MV] constructed mapping class group actions from Hopf monoids in symmetric monoidal categories. Remarkably, these actions can be associated to certain graph transformations in a graph with copies of the Hopf monoid assigned to each edge. This suggests that there should be a generalisation of the protected space which carries this mapping class group action.

Kuperberg invariants of 3-manifolds. Kuperberg invariants were introduced in [Ku] using finite-dimensional semisimple Hopf algebras over $\mathbb{C}$. More precisely, by encoding 3-manifolds by Heegaard diagrams Kuperberg associates a scalar to any oriented 3-manifold. The invariants exhibit various relations to other 3-manifold invariants. In particular, they are related to Turaev-Viro-Barrett-Westbury invariants, as shown in [BaW2]. Moreover, Kuperberg invariants based on a finite-dimensional semisimple Hopf algebra H coincide with the Hennings-Kauffmann-Radford invariants [He, KR, Ke, CC] for the Drinfeld double $D(H)$. The invariants of 3-manifolds constructed by Costantino et al. [CGPT] from monoidal categories using modified traces also reduce to Kuperberg invariants in certain cases.

Kashaev and Virelizier [KV] generalise Kuperberg's invariants from Hopf algebras to involutive Hopf monoids H in symmetric monoidal categories. More specifically, to construct the invariant they define an endomorphism of a certain tensor product of H. They then pre- and post-compose generalisations of integrals and cointegrals, called good pairs, to obtain an endomorphism of the tensor unit. This yields the generalised Kuperberg invariants. Both their setting and the construction of the first endomorphism resembles the setting and certain twists in [MV] which give rise to mapping class group actions. This suggests a connection between Kuperberg invariants and mapping class group actions in a generalised Quantum double model based on Hopf monoids.

It also seems plausible that generalisations of the protected space should yield generalisations of the representation variety $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ for groups G and surfaces Σ. The latter is the set of group homomorphisms from the fundamental group of Σ into G up to conjugation. Its cardinality gives the dimension of the ground state in Kitaev models based on the group algebra of finite groups G [CDH+]. Hopf monoids in cartesian monoidal categories are called group objects and often resemble groups. As a consequence it seems reasonable to expect generalised representation varieties as ground states of generalised quantum double models.

Summary of results

The main results of this thesis are:

- A generalisation of the protected space in Kitaev's quantum double models from finite-dimensional semisimple Hopf algebras to Hopf monoids in complete and finitely cocomplete symmetric monoidal categories.
- The proof of their graph independence and hence topological invariance.
- An explicit construction and description of the generalised protected space for Hopf monoids in categories such as $k-$Mod, SSet and Cat.

- The proof that they are equipped with mapping class group actions, namely the ones from [MV] and that these actions are induced by mapping class group actions on representation varieties.
- The proof that these mapping class group actions define (generalised) Kuperberg invariants whenever the Hopf algebras are equipped with integrals.

This is achieved as follows. We consider and investigate a generalisation of Kitaev lattice models based on involutive Hopf monoids in a complete and finitely cocomplete symmetric monoidal category $\mathcal{C}$ along the lines of [BMCA], with focus on a generalisation of the ground state of these models.

We consider a ribbon graph and an involutive Hopf monoid H in $\mathcal{C}$. The ribbon graph encodes an oriented surface obtained by attaching disks to the faces of the graph. Here, the graph is additionally endowed with a choice of marking for each vertex and face. To construct the model we assign a copy of H to each edge in the graph and take the tensor product $H^{\otimes E}$ of the Hopf monoids over the set E of edges of the graph. The tensor product $H^{\otimes E}$ is then equipped with H-module structures assigned to vertices and H-comodule structures assigned to faces in the graph. Their definition involves the markings of vertices and faces and the structure morphisms of the Hopf monoid.

We then coequalise the H-module structures and equalise the H-comodule structures to obtain invariants and coinvariants. Combining invariants and coinvariants via a categorical image yields a categorical counterpart of the ground state or protected space in Kitaev's quantum double models, called *protected object* in the following. By investigating its behaviour under graph transformations we show

Theorem. *(Theorem 5.23) The protected object associated to H and a ribbon graph is a topological invariant of oriented surfaces: its isomorphism class depends only on the homeomorphism class of the surface encoded by the graph.*

As the construction of mapping class group actions in [MV] uses an almost identical setting, it is natural to expect a link to these mapping class group actions. More precisely, the mapping class group actions in [MV] are associated to a specific graph with a single vertex. We show that graph transformations such as edge contractions, edge slides and removing isolated loops induce isomorphisms of the protected object. This allows us to transform any graph that characterises the protected object into one considered in [MV]. In this way we obtain

Theorem. *(Theorem 9.14) The protected object for an involutive Hopf monoid H and a surface Σ of genus $g \geq 1$ carries an action of the mapping class group $\mathrm{Map}(\Sigma)$ by automorphisms.*

Another main focus of the thesis is a concrete construction of examples. In particular, for finite-dimensional semisimple Hopf algebras H as Hopf monoids in $\mathrm{Vect}_{\mathbb{C}}$ we regain the protected space from Kitaev's quantum double models [Ki, BMCA]. The protected object can therefore be viewed as a generalisation of the protected space. In contrast to [Ki, BMCA] the model is also defined in the non-semisimple case and for Hopf monoids in $k-\mathrm{Mod}$ such as group algebras $k[G]$, where k is a commutative ring. The protected object associated to a surface Σ of genus $g \geq 1$ is then the free k-module generated by the representation variety $\mathrm{Hom}(\pi_1(\Sigma), G)/G$.

For a group H as a Hopf monoid in the cartesian monoidal category Set we obtain the representation variety $\mathrm{Hom}(\pi_1(\Sigma), H)/H$. Indeed, the protected object can be interpreted as a generalisation of representation varieties or moduli spaces of flat bundles on surfaces. More specifically, symmetric monoidal functors $F : \mathcal{C} \to \mathrm{Set}$ send Hopf monoids in $\mathcal{C}$ to groups. Whenever F additionally preserves monomorphisms, epimorphisms, equalisers and coequalisers, the protected object in $\mathcal{C}$ is sent to the representation variety of the group.

We also obtain generalisations of representation varieties in the category SSet of simplicial sets. Here, Hopf monoids are simplicial groups $H = (H_n)_{n\in\mathbb{N}_0}$. The associated protected object is a simplicial set given by the representation varieties for the groups H_n and by post-composition with their face maps and degeneracies.

Special emphasis in this thesis is placed on determining protected objects in the cartesian monoidal category Cat of small categories and functors between them. Hopf monoids in this category coincide with crossed modules. The latter consist of two groups A, B with a group homomorphism $\partial : A \to B$ and a group action $\blacktriangleright: B \times A \to A$ by automorphisms which satisfy two compatibility conditions involving conjugation. There is a wide range of examples for crossed modules such as normal subgroups, group extensions and group actions of abelian groups. However, to determine the protected object assigned in Cat to a crossed module and a surface Σ, one has to determine equalisers and coequalisers in Cat.

While equalisers can easily be specified as subcategories with inclusion functors, it is difficult to determine coequalisers in Cat concretely. In this thesis we consider and compare two approaches. The first transports the relevant morphisms in Cat to SSet via the nerve functor. The associated coequalisers in SSet are then transported back to Cat by the left adjoint of the nerve. The second approach describes the coequalisers via generalised congruences on small categories. Both approaches yield an explicit description of protected objects in Cat and their mapping class group actions by automorphisms.

Theorem. *(Theorem 8.30, Corollary 9.16) The protected object for a crossed module* $(B, A, \blacktriangleright, \partial)$ *seen as a Hopf monoid in* Cat *and a surface* Σ *of genus* $g \geq 1$ *is a groupoid* $\mathcal{G}$ *with* $\mathrm{Ob}\mathcal{G} = \mathrm{Hom}(\pi_1(\Sigma), B)/B$ *and with equivalence classes of group homomorphisms* $\tau : \pi_1(\Sigma) \to A \rtimes B$ *as morphisms. The action of the mapping class group* $\mathrm{Map}(\Sigma)$ *is induced by its action on* $\mathrm{Hom}(\pi_1(\Sigma), A \rtimes B)/A \rtimes B$.

Here, two group homomorphisms $\tau, \tau' : \pi_1(\Sigma) \to A \rtimes B$ are equivalent, whenever the associated group homomorphisms $F_{2g} \to A \rtimes B$ can be written as composites $\tau = \tau_1 \circ \tau_2$, $\tau' = \tau_1' \circ \tau_2'$ such that τ_1, τ_1' and τ_2, τ_2' are conjugate. The construction of the protected object and its mapping class group action is illustrated by explicitly computing it for some simple examples of crossed modules.

The last part of the thesis focuses on relations of these mapping class group actions to Kuperberg invariants of 3-manifolds. For this we apply the generalisation of the Kuperberg invariants [Ku] from Hopf algebras to involutive Hopf monoids H in symmetric monoidal categories constructed by Kashaev and Virelizier [KV]. To define the invariants the authors of [KV] encode 3-manifolds by Heegaard diagrams. These consist of an oriented surface of some genus $g \geq 1$ endowed with two systems of g non intersecting oriented curves. Ordering them and equipping them with basepoints yields two orderings on the set of intersection points of the two curve systems. These orderings define an endomorphism of $H^{\otimes g}$ constructed from the structure morphisms of H and the braiding in $\mathcal{C}$. We call this morphism *tensor endomorphism.*

The generalised Kuperberg invariant from [KV] is then obtained by pre- and post-composing the tensor endomorphism with a good pair. The latter generalises pairs of integrals and cointegrals of Hopf algebras.

The tensor endomorphism can be described by a sequence of graph transformations obtained by sliding edges in a suitable ribbon graph. More precisely, each Heegaard diagram D defines a ribbon graph $\Gamma(D)$, in which each curve system in D defines a system of edges in $\Gamma(D)$. Sliding the edges of one edge system along the other system precisely describes Dehn twists. The associated automorphisms of the protected object give rise to the tensor endomorphism.

Theorem. *(Theorem 10.11, Corollary 10.22) Up to pre- and post-composition with (co)units and good pairs, Kuperberg invariants of* 3*-manifolds for involutive Hopf monoids are given by the action of Dehn twists along one edge system in* $\Gamma(D)$ *on the other edge system in* $\Gamma(D)$ *and subsequent multiplication of their contributions.*

Structure of the thesis

The thesis is structured as follows. In Chapter 2 we introduce the relevant algebraic background for the thesis. Section 2.1 contains the background on Hopf monoids in symmetric monoidal categories. In Section 2.2 we then describe (co)modules and Yetter-Drinfeld modules and their (co)invariants over Hopf monoids in symmetric monoidal categories that are complete and finitely cocomplete. The (co)invariants are defined via (co)equalisers in the category. Via a categorical image they can be combined to obtain biinvariants. We conclude the chapter with the necessary background on integrals over Hopf monoids in Section 2.3. A large part of this section focuses on integrals over Hopf algebras in $\mathrm{Vect}_{\mathbb{F}}$, which is mainly relevant for Chapter 6 and Section 10.3.

Chapter 3 summarises the background on ribbon graphs, surfaces and graph transformations required in this thesis.

In Chapter 4 we construct a generalised Kitaev model for a ribbon graph and an involutive Hopf monoid H in a complete and finitely cocomplete symmetric monoidal category. We define a module structure associated to marked vertices and a comodule structure associated to marked faces of the graph on the tensor product $H^{\otimes E}$, where E is the set of edges. The module and comodule structures generalise the vertex and face operators from [Ki, BMCA]. Essentially the same generalisation was used in [MV]. (Co)equalising the (co)module structures and taking the categorical image yields the protected object which is the counterpart of the ground state in Kitaev lattice models.

In Chapter 5 we show that the protected object is a topological invariant of oriented surfaces: It depends only on the homeomorphism class of the surface obtained from the ribbon graph, but not on the choice of the graph. To prove the topological invariance, we first investigate its dependence on edge reversals and shifts of markings at vertices or faces in Section 5.1. We then consider a number of graph transformations and show that they induce isomorphisms on the protected object. This includes edge slides, investigated in Section 5.2, and the deletion of isolated loops in Section 5.3. In Section 5.4 we combine these results to prove the topological invariance.

Sections 5.1 to 5.3 necessarily contain some technical results and computations. The reader primarily interested in results may skip to Section 5.4 for the main theorem. Here, we also treat some examples. This includes the protected objects in Set, Top and for group algebras $k[G]$ and their duals considered as Hopf monoids in $k-\mathrm{Mod}$. The resulting protected objects are all related to representation varieties. We also show that under certain assumptions symmetric monoidal functors into the category Set send protected objects to representation varieties.

In Chapter 6 we compare the construction from Chapter 4 to the formalism in [BMCA]. In particular, we describe how protected spaces for finite-dimensional semisimple involutive Hopf algebras can be described using Haar integrals.

In Chapter 7 we determine the protected object in the category SSet, where Hopf monoids coincide with simplicial groups.

Chapter 8 treats protected objects for Hopf monoids in the cartesian monoidal category Cat. In Section 8.1 we present the relevant background on crossed modules which describe Hopf monoids in Cat. We then describe in Section 8.2 how equalisers are obtained in Cat and explain how coequalisers in Cat can be constructed using the nerve functor $N : \mathrm{Cat} \to \mathrm{SSet}$ and its left adjoint. Section 8.3 applies these results to determine the (co)invariants of the (co)module structures over Hopf monoids in Cat. Section 8.4 provides an alternative way to determine coequalisers in Cat via generalised congruences. In Section 8.5 the invariants of the module structures over Hopf monoids in Cat are described in the formalism of Section 8.4. The results of the previous sections are combined in Section 8.6 to determine protected objects in Cat. We explicitly describe them and investigate two simple examples.

Chapter 9 investigates mapping class group actions on protected objects obtained from the actions in [MV]. We start by presenting the relevant background on mapping class groups, Dehn twists and a presentation of the mapping class group in terms of Dehn twists in Section 9.1. In Section 9.2 we describe in more detail the sequences of edge slides for Dehn twists in the presentation of the mapping class group from Section 9.1. Finally, in Section 9.3 we combine the topological invariance of the protected objects with the results about mapping class group actions from [MV]. This yields mapping class group actions on the protected objects which we concretely describe for Hopf monoids in $\mathrm{Set}, \mathrm{SSet}$ and Cat.

Chapter 10 connects edge slides and Dehn twists in the Kitaev models to Kuperberg invariants of 3-manifolds. First, we introduce in Section 10.1 the necessary background on Heegaard diagrams which encode 3-manifolds. In Section 10.2 we define the tensor endomorphism following [KV]. This endomorphism forms the main part of the construction of Kuperberg invariants. We then describe in Section 10.3 how Kuperberg invariants are obtained from this tensor endomorphism and give some simple examples. In Section 10.4 we assign colored ribbon graphs to Heegaard diagrams. We conclude this chapter by describing how the tensor endomorphism can be expressed in terms of edge slides and Dehn twists in Section 10.5.

Sections 2.1, 2.2, Chapter 3-5 and 7 as well as Sections 8.1-8.3, 8.6 and Section 9.3 are up to minor changes and additions parts of the preprint:

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2 Algebraic background

The aim of this chapter is to give the necessary background on Hopf monoids, their (co)modules and associated (co)invariants as well as biinvariants and integrals over Hopf monoids.

Up to some changes and additions Sections 2.1 and 2.2 are parts of the preprint [HM].

Throughout the thesis $\mathcal{C}$ is a symmetric monoidal category with unit object e and braidings $\tau_{X,Y} : X \otimes Y \to Y \otimes X$. We also suppose that $\mathcal{C}$ is complete and finitely cocomplete. In formulas, we suppress associators and unit constraints and coherence data of monoidal functors.

2.1 Involutive Hopf monoids

Hopf monoids are generalisations of Hopf algebras in symmetric monoidal categories. To derive the generalised definition from Hopf algebras, vector spaces are replaced by objects in the category and linear maps by morphisms.

Definition 2.1.

1. *A* **Hopf monoid** *in $\mathcal{C}$ is an object H in $\mathcal{C}$ together with morphisms $m : H \otimes H \to H$, $\eta : e \to H$, $\Delta : H \to H \otimes H$, $\epsilon : H \to e$ and $S : H \to H$, the multiplication, unit, comultiplication, counit and antipode, such that*
   - *the (co)multiplication satisfies the (co)associativity and (co)unitality conditions*

     $$m \circ (m \otimes 1_H) = m \circ (1_H \otimes m), \qquad m \circ (\eta \otimes 1_H) = m \circ (1_H \otimes \eta) = 1_H, \qquad (1)$$
     $$(\Delta \otimes 1_H) \circ \Delta = (1_H \otimes \Delta) \circ \Delta, \qquad (\epsilon \otimes 1_H) \circ \Delta = (1_H \otimes \epsilon) \circ \Delta = 1_H,$$

   - *comultiplication and counit are monoid morphisms*

     $$\Delta \circ \eta = \eta \otimes \eta, \qquad \Delta \circ m = (m \otimes m) \circ (1_H \otimes \tau_{H,H} \otimes 1_H) \circ (\Delta \otimes \Delta), \qquad (2)$$
     $$\epsilon \circ \eta = 1_e, \qquad \epsilon \circ m = \epsilon \otimes \epsilon,$$

   - *S satisfies the antipode condition*

     $$m \circ (S \otimes 1_H) \circ \Delta = m \circ (1_H \otimes S) \circ \Delta = \eta \circ \epsilon. \qquad (3)$$

   It is called **involutive** *if $S \circ S = 1_H$.*

2. *A* **morphism of Hopf monoids** *in $\mathcal{C}$ is a morphism $f : H \to H'$ in $\mathcal{C}$ with*

   $$f \circ m = m' \circ (f \otimes f), \qquad f \circ \eta = \eta', \qquad (f \otimes f) \circ \Delta = \Delta' \circ f, \qquad \epsilon' \circ f = \epsilon. \qquad (4)$$

We denote by Hopf($\mathcal{C}$) *the category of Hopf monoids and morphisms of Hopf monoids in $\mathcal{C}$.*

The antipode of a Hopf monoid is unique, and it is an anti-monoid and anti-comonoid morphism

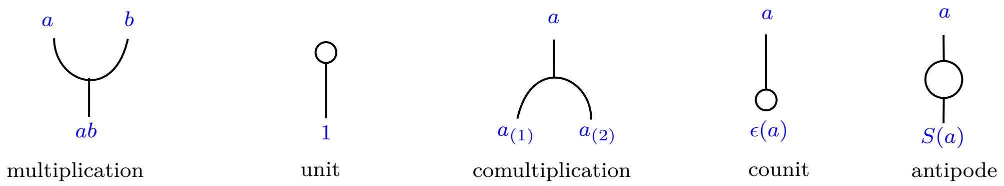
$$S \circ m = m^{op} \circ (S \otimes S), \qquad S \circ \eta = \eta, \qquad (S \otimes S) \circ \Delta = \Delta^{op} \circ S, \qquad \epsilon \circ S = \epsilon, \qquad (5)$$

see for instance Porst [Por, Prop. 36]. If H is involutive, the antipode satisfies the additional identities

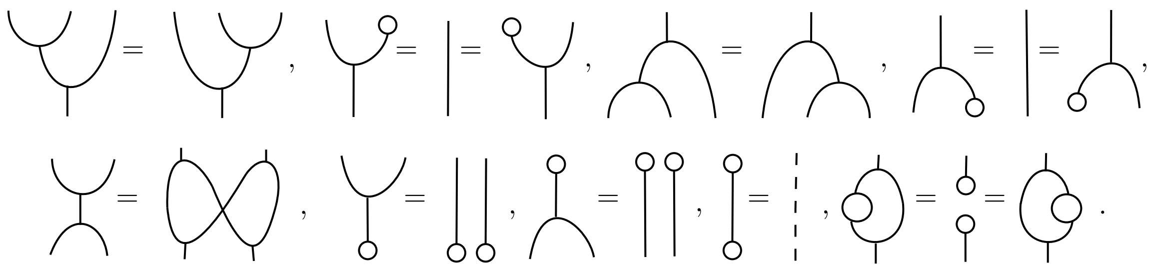
$$m^{op} \circ (S \otimes 1_H) \circ \Delta = m^{op} \circ (1_H \otimes S) \circ \Delta = \eta \circ \epsilon. \qquad (6)$$

Every morphism of Hopf monoids $f : H \to H'$ satisfies $f \circ S = S' \circ f$. This follows as for Hopf algebras by considering the convolution monoid $\mathrm{Hom}_{\mathcal{C}}(H, H)$ with the product $f \star g = m \circ (f \otimes g) \circ \Delta$.

In the following, we use generalised Sweedler notation for the coproduct in a Hopf monoid and write $\Delta(h) = h_{(1)} \otimes h_{(2)}$, $(\Delta \otimes 1_H) \circ \Delta(h) = (1_H \otimes \Delta) \circ \Delta(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$ etc. This is analogous to Sweedler notation for a Hopf algebra. It can be viewed as a shorthand notation for a diagram that describes a morphism in a symmetric monoidal category. For an introduction to the usual diagrammatic notation see for instance Heunen and Vicary [HV]. Throughout the thesis, diagrams are read from left to right for tensor products and from top to bottom for composition. The multiplication, unit, comultiplication, counit and antipode are drawn as



and the conditions (1) - (3) read



Note that we write $m^{(n)} : H^{\otimes(n+1)} \to H$ and $\Delta^{(n)} : H \to H^{\otimes(n+1)}$ for n-fold products and coproducts.

Example 2.2.

1. *For any commutative ring k a Hopf monoid in k-Mod is a Hopf algebra over k. In particular, for any field $\mathbb{F}$ a Hopf monoid in* $\mathrm{Vect}_{\mathbb{F}}$ *is a Hopf algebra over $\mathbb{F}$.*
2. *For any finite group G and commutative ring k, the group algebra $k[G]$ and its dual $k[G]^*$ are Hopf monoids in $k-$*Mod.
3. *The tensor product of two Hopf monoids in $\mathcal{C}$ has a Hopf monoid structure given by the tensor product of (co)units, (co)multiplications and antipodes and the braiding morphisms. Any tensor product of Hopf monoid morphisms is a morphism of Hopf monoids.*
4. *Every Hopf monoid $H = (H, m, \eta, \Delta, \epsilon, S)$ in a symmetric monoidal category $\mathcal{C}$ defines a Hopf monoid $H^* = (H, \Delta, \epsilon, m, \eta, S)$ in the symmetric monoidal category $\mathcal{C}^{op}$. This generalises the dual Hopf algebra in* $\mathrm{Vect}_{\mathbb{F}}$.
5. *For G, H finite groups with a group homomorphism $\Phi : G \to \mathrm{Aut}(H)$ the group algebra $\mathbb{C}[H]$ gives rise to an involutive Hopf monoid in the category $(\mathrm{Rep}_{\mathbb{C}}(G), \otimes, \mathbb{C})$ of finite-dimensional representations of G over $\mathbb{C}$. We denote by $\delta_h : H \to \mathbb{C}$, $\delta_h(k) = 0$ for $k \neq h$ and $\delta_h(h) = 1$ the basis elements of $\mathbb{C}[H]$. Then $\mathbb{C}[H]$ is a $\mathbb{C}[G]$-module with $g \rhd \delta_h = \delta_{\Phi(g)(h)}$ for $g \in G$, $h \in H$. (Co)multiplication, (co)unit and antipode are as usual given by $m(\delta_h \otimes \delta_k) = \delta_{hk}$, $\Delta(\delta_h) = \delta_h \otimes \delta_h$, $\eta(1) = \delta_e$, $\epsilon(\delta_h) = 1$ and $S(\delta_h) = \delta_{h^{-1}}$. It is easy to see that they are $\mathbb{C}[G]$-linear.*

6. *In the situation of 5. the dual vector space* $\mathbb{C}[H]^* = \mathrm{Fun}(H, \mathbb{C})$ *with basis elements* $\rho_h : \mathbb{C}[H] \to \mathbb{C}$ *satisfying* $\rho_h(\delta_k) = 0$ *for* $k \neq h$ *and* $\rho_h(\delta_h) = 1$ *and dual representation*

$$(g \rhd \rho_h)(\delta_l) = \rho_h(g^{-1} \rhd \delta_l) = \rho_{\Phi(g)(h)}(\delta_l) \text{ for } g \in G,\ h, l \in H$$

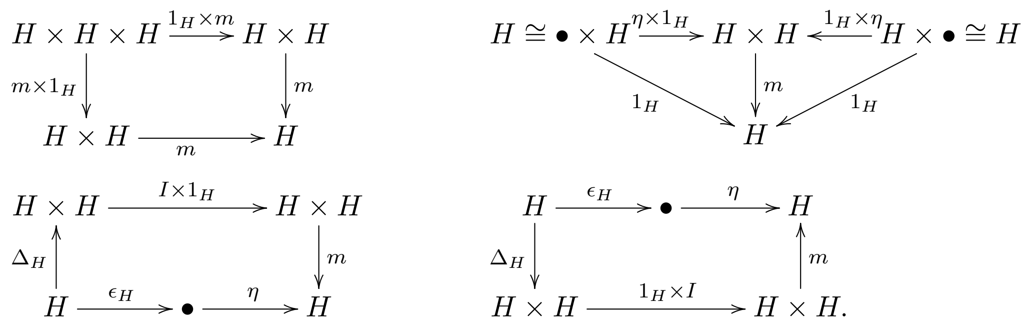
is also an involutive Hopf monoid in $\mathrm{Rep}_{\mathbb{C}}(G)$. *Its structure morphisms are given by*

$$m(\rho_h \otimes \rho_k) = \rho_h(\delta_k)\rho_k,\ \Delta(\rho_h) = \sum_{uv=h} \rho_u \otimes \rho_v,\ \eta(1) = \sum_{h \in H} \rho_h,\ \epsilon(\rho_h) = \rho_h(\delta_e),\ S(\rho_h) = \rho_{h^{-1}}.$$

The following example yields many subexamples, which are a focus in this thesis.

Example 2.3. *Let* $(\mathcal{C}, \times)$ *be a cartesian monoidal category with terminal object* $\bullet$. *Let* $\epsilon_X : X \to \bullet$ *be the terminal morphism and* $\Delta_X : X \to X \times X$ *the diagonal morphism for an object* X.

A Hopf monoid in $\mathcal{C}$ *is a* **group object** *in* $\mathcal{C}$*: an object* H *together with morphisms* $m : H \times H \to H$, $\eta : \bullet \to H$ *and* $I : H \to H$ *such that the following diagrams commute*



(7)

A morphism of Hopf monoids is a **morphism of group objects***: a morphism* $F : H \to H'$ *with*

$$F \circ m = m' \circ (F \times F). \tag{8}$$

Note that this implies $F \circ \eta = \eta'$ *and* $I' \circ F = F \circ I$.

Example 2.4.

1. *A group object in the cartesian monoidal category* $(\mathrm{Set}, \times)$ *is a group.*
2. *A group object in the cartesian monoidal category* $(\mathrm{Top}, \times)$ *is a topological group.*
3. *A group object in the cartesian monoidal category* $(\mathrm{Cat}, \times)$ *of small categories and functors between them is a crossed module (cf. Definition 8.2).*
4. *Let* G *be a group and* $G-\mathrm{Set} = \mathrm{Set}^{BG}$ *the cartesian monoidal category of* G*-sets and* G*-equivariant maps. A group object in* $G-\mathrm{Set}$ *is a group with a* G*-action by automorphisms.*
5. *A group object in the cartesian monoidal category* $\mathrm{SSet} = \mathrm{Set}^{\Delta^{op}}$ *of simplicial sets and simplicial maps is a simplicial group (cf. Definition 7.1).*

The last two examples in Example 2.4 have counterparts for any functor category $\mathcal{C}^{\mathcal{D}}$, where $\mathcal{D}$ is small and $\mathcal{C}$ symmetric monoidal. In this case the functor category $\mathcal{C}^{\mathcal{D}}$ inherits a symmetric monoidal structure from $\mathcal{C}$, and we have

Lemma 2.5. *For any symmetric monoidal category* $\mathcal{C}$ *and a small category* $\mathcal{D}$ *the monoidal categories* $\mathrm{Hopf}(\mathcal{C}^{\mathcal{D}})$ *and* $\mathrm{Hopf}(\mathcal{C})^{\mathcal{D}}$ *are symmetric monoidally equivalent.*

Proof. The equivalence is given by the functor $R : \mathrm{Hopf}(\mathcal{C}^{\mathcal{D}}) \to \mathrm{Hopf}(\mathcal{C})^{\mathcal{D}}$ that sends a Hopf monoid $(H, m, \eta, \Delta, \epsilon, S)$ to the functor $K : \mathcal{D} \to \mathrm{Hopf}(\mathcal{C})$ with $K(D) = H(D)$ and the component morphisms m_D, η_D, Δ_D, ϵ_D, S_D for $D \in \mathrm{Ob}(\mathcal{D})$ and with $K(f) = H(f)$ for a morphism f in $\mathcal{D}$. Hopf monoid morphisms in $\mathcal{C}^{\mathcal{D}}$ are sent to themselves. The functor R has an obvious inverse, and both functors are symmetric monoidal. □

Further examples are obtained by taking the images of Hopf monoids under symmetric monoidal functors. If both of the categories are cartesian monoidal, it is sufficient that the functor preserves finite products, which holds in particular for any right adjoint functor.

Example 2.6.

1. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a symmetric monoidal functor. Then for every Hopf monoid H in $\mathcal{C}$ the image $F(H)$ has a canonical Hopf monoid structure.*
2. *If $\mathcal{C}, \mathcal{C}'$ are cartesian monoidal categories and $F : \mathcal{C} \to \mathcal{C}'$ a functor that preserves finite products, then F is symmetric monoidal, and for every group object H in $\mathcal{C}$ the image $F(H)$ is a group object in $\mathcal{C}'$.*

2.2 (Co)modules and their (co)invariants

As their definitions involve only structure maps, (co)modules over Hopf monoids in symmetric monoidal categories are defined analogously to (co)modules over Hopf algebras. The only difference is that linear maps are replaced by morphisms. In this section, we provide the generalised versions of (co)modules and also generalised versions of invariants, coinvariants and biinvariants.

Definition 2.7. *Let H be a Hopf monoid in $\mathcal{C}$.*

1. *A (left) H-**module** in $\mathcal{C}$ is an object M in $\mathcal{C}$ with a morphism $\rhd : H \otimes M \to M$ satisfying*

$$\rhd \circ (m \otimes 1_M) = \rhd \circ (1_H \otimes \rhd), \qquad \rhd \circ (\eta \otimes 1_M) = 1_M. \tag{9}$$

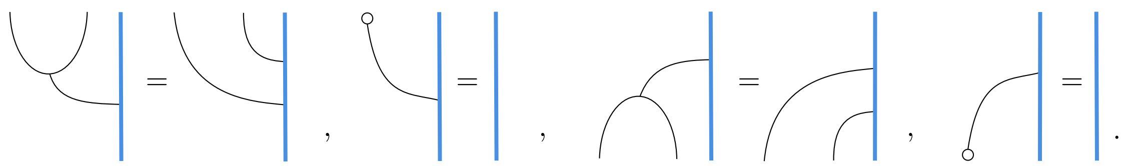
   *A **morphism of (left) H-modules** is a morphism $f : M \to M'$ in $\mathcal{C}$ with $\rhd' \circ (1_H \otimes f) = f \circ \rhd$.*

2. *A (left) H-**comodule** in $\mathcal{C}$ is an object M in $\mathcal{C}$ with a morphism $\delta : M \to H \otimes M$ satisfying*

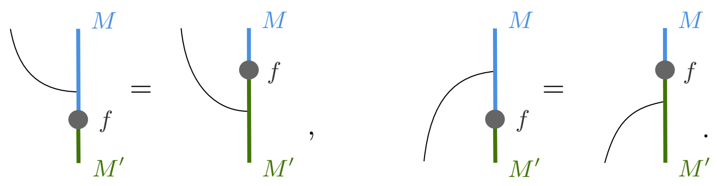
$$(\Delta \otimes 1_M) \circ \delta = (1_H \otimes \delta) \circ \delta, \qquad (\epsilon \otimes 1_M) \circ \delta = 1_M. \tag{10}$$

   *A **morphism of left H-comodules** is a morphism $f : M \to M'$ in $\mathcal{C}$ with $(1_H \otimes f) \circ \delta = \delta' \circ f$.*

In diagrammatic notation we mark (co)modules by vertical lines, which are often coloured, and the action and coaction morphisms are incoming and outgoing lines from the (co)modules, such that the conditions for left modules and left comodules are expressed as



The defining conditions for left module and left comodule morphisms are denoted as



There are analogous notions of right (co)modules and bi(co)modules and morphisms between them. Just as in the case of a Hopf algebra, there are also various compatibility conditions that can be imposed between module and comodule structures. The most important one in the following is the one for Yetter-Drinfeld modules.

Definition 2.8. *Let H be a Hopf monoid in $\mathcal{C}$.*

1. *A* **Yetter-Drinfeld module** *over H is a triple $(M, \triangleright, \delta)$ such that $(M, \triangleright)$ is an H-module, (M, δ) is an H-comodule and*

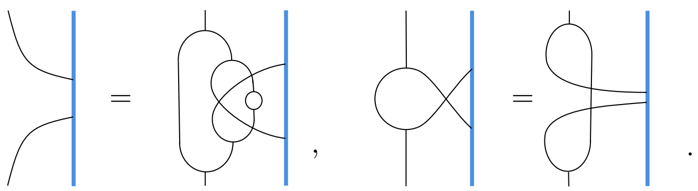
$$\delta \circ \triangleright = (m^{(2)} \otimes \triangleright) \circ (1_{H^{\otimes 2}} \circ \tau_{H,H} \otimes 1_M) \circ (1_{H^{\otimes 3}} \otimes S \otimes 1_M) \circ (1_H \otimes \tau_{H^{\otimes 2},H} \otimes 1_M) \circ (\Delta^{(2)} \otimes \delta). \tag{11}$$

2. *A* **morphism of Yetter-Drinfeld modules** *is a morphism $f : M \to M'$ that is a module and a comodule morphism.*

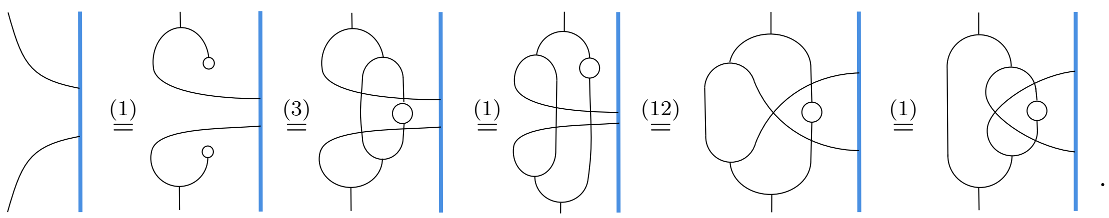
Condition (11) in Definition 2.8 is equivalent to

$$\begin{aligned}&(m \otimes \triangleright) \circ (1_H \otimes \tau_{H,H} \otimes 1_M) \circ (\Delta \otimes \delta) = \\ &(m \otimes 1_M) \circ (\tau_{H,H} \otimes 1_M) \circ (1_H \otimes \delta) \circ (1_H \otimes \triangleright) \circ (\tau_{H,H} \otimes 1_M) \circ (\Delta \otimes 1_M).\end{aligned} \tag{12}$$

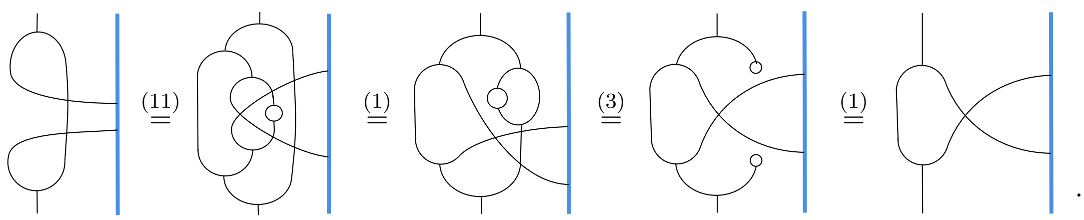
In diagrammatic notation the conditions (11) and (12) are expressed as



The equivalence of (11) and (12) can be verified by a short graphical computation. Assuming (12) we obtain



With (11) this implies



In Sweedler notation with the conventions $\delta(m) = m_{(0)} \otimes m_{(1)}$ and $\Delta(h) = h_{(1)} \otimes h_{(2)}$ the Yetter-Drinfeld module condition in Definition 2.8 reads

$$(h \triangleright m)_{(0)} \otimes (h \triangleright m)_{(1)} = h_{(1)} m_{(0)} S(h_{(3)}) \otimes (h_{(2)} \triangleright m_{(1)}). \tag{13}$$

Yetter-Drinfeld modules over group objects in cartesian monoidal categories are especially simple to describe. In this case, composing the coaction morphism $\delta : M \to H \times M$ with the projection morphism $\pi_1 : H \times M \to H$ yields a morphism $F = \pi_1 \circ \delta : M \to H$ reminiscent of a moment map. The Yetter-Drinfeld module condition states that this morphism intertwines the H-module structure on M and the conjugation action of H on itself.

Example 2.9. *Let H be a group object in a cartesian monoidal category, $(M, \triangleright)$ a module and (M, δ) a comodule over H. Then $(M, \triangleright, \delta)$ is a Yetter-Drinfeld module over H iff the morphism $F := \pi_1 \circ \delta : M \to H$ satisfies*

$$F \circ \triangleright = m^{(2)} \circ (1_H \times \tau_{H,F(M)}) \circ (1_H \times I \times 1_{F(M)}) \circ (\Delta_H \times F). \tag{14}$$

If the objects of $\mathcal{C}$ are sets, condition (14) reads $F(h \triangleright m) = hF(m)h^{-1}$ for all $h \in H$, $m \in M$. By an abuse of notation, we sometimes write such formulas for the general case to keep notation simple.

By Example 2.6 the images of Hopf monoids under symmetric monoidal functors are Hopf monoids. Analogous statements hold for their (co)modules.

Example 2.10.

1. *If $F : \mathcal{C} \to \mathcal{C}'$ is a symmetric monoidal functor and M a (co)module over a Hopf monoid H in $\mathcal{C}$, then $F(M)$ is a (co)module over the Hopf monoid $F(H)$.*
2. *Let $\mathcal{C}, \mathcal{C}'$ be cartesian monoidal categories and $F : \mathcal{C} \to \mathcal{C}'$ a functor that preserves finite products. Then for every (co)module M over a group object H in $\mathcal{C}$ the image $F(M)$ is a (co)module over the group object $F(H)$.*

(Co)invariants of (co)modules cannot be generalised directly from Hopf algebras over fields to Hopf monoids in symmetric monoidal categories. To obtain generalised notions of (co)invariants, we require that the symmetric monoidal category $\mathcal{C}$ has all equalisers and coequalisers.

Definition 2.11. *[MV, Def. 2.6] Suppose that $\mathcal{C}$ has all equalisers and coequalisers and H is a Hopf monoid in $\mathcal{C}$.*

1. *The* **invariants** *of an H-module $(M, \triangleright)$ are defined as the coequaliser (M^H, π) of the morphisms $\epsilon \otimes 1_M$ and $\triangleright$:*

$$H \otimes M \underset{\epsilon \otimes 1_M}{\overset{\triangleright}{\rightrightarrows}} M \xrightarrow{\pi} M^H.$$

2. *The* **coinvariants** *of an H-comodule (M, δ) are defined as the equaliser (M^{coH}, ι) of the morphisms $\eta \otimes 1_M$ and δ:*

$$M^{coH} \xrightarrow{\iota} M \underset{\eta \otimes 1_M}{\overset{\delta}{\rightrightarrows}} H \otimes M.$$

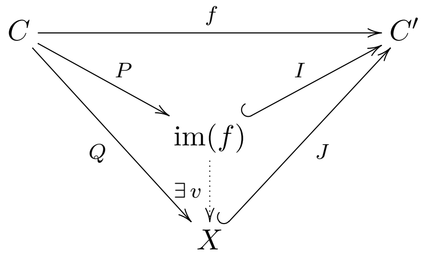
As expected, H-(co)module morphisms induce morphisms between the (co)invariants. This follows directly from the universal properties of the (co)equalisers.

Lemma 2.12. *[MV, Lemma 2.7] Suppose that $\mathcal{C}$ has all equalisers and coequalisers and H is a Hopf monoid in $\mathcal{C}$. Then for every H-module morphism $f:(M,\rhd)\to(M',\rhd')$ there exists a unique morphism $f^H: M^H \to M'^H$ with $f^H\circ\pi=\pi'\circ f$. Likewise, for every H-comodule morphism $f:(M,\delta)\to(M',\delta')$ there exists a unique morphism $f^{coH}: M^{coH}\to M'^{coH}$ with $\iota'\circ f^{coH}=f\circ\iota$.*

Note that all definitions in this section are symmetric with respect to a Hopf monoid H in $\mathcal{C}$ and the dual Hopf monoid H^* in $\mathcal{C}^{op}$ from Example 2.2. Modules and comodules over H in $\mathcal{C}$ correspond to comodules and modules over H^* in $\mathcal{C}^{op}$, respectively, and the same holds for their (co)invariants. It is also directly apparent from the formula in Definition 2.8 that Yetter-Drinfeld modules over H correspond to Yetter-Drinfeld modules over H^*.

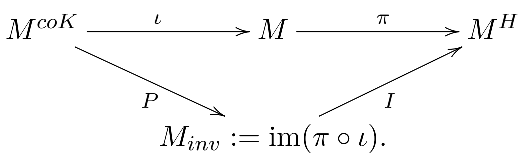
For objects in a symmetric monoidal category $\mathcal{C}$ that are both, modules and comodules over certain Hopf monoids in $\mathcal{C}$, we combine the notion of invariants and coinvariants and impose both conditions. This requires that the category $\mathcal{C}$ is equipped with *images*. We work with a general non-abelian notion of image, see Mitchell [Mi, Sec. I.10] and Pareigis [Pa, Sec. 1.13]. There is an analogous notion of a *coimage*, which is the image of the corresponding morphism in $\mathcal{C}^{op}$, see [Mi, Sec. I.10].

An *image* of a morphism $f: C\to C'$ in $\mathcal{C}$ is an object $\mathrm{im}(f)$ together with a pair (P,I) of a monomorphism $I:\mathrm{im}(f)\to C'$ and a morphism $P: C\to\mathrm{im}(f)$ with $I\circ P=f$ and the following universal property: for any pair (Q,J) of a monomorphism $J: X\to C'$ and a morphism $Q: C\to X$ with $J\circ Q=f$ there is a unique morphism $v:\mathrm{im}(f)\to X$ with $I=J\circ v$.



Images are unique up to unique isomorphism. If $\mathcal{C}$ has all equalisers, then $P: C\to\mathrm{im}(f)$ is an epimorphism [Mi, Prop. 10.1, Sec. I.10]. In an abelian category $\mathcal{C}$ this notion of image coincides with the usual definition of an image as the kernel of the cokernel [Pa, Lemma 3, Sec. 4.2]. If $\mathcal{C}$ is complete, then all images exist, as any complete category has intersections [Mi, Prop. 2.3, Sec. II.2]. This implies the existence of all images [Mi, Sec. I.10].

Definition 2.13. *[MV, Def. 2.8][1] Let H,K be Hopf monoids in a complete and finitely cocomplete symmetric monoidal category $\mathcal{C}$. The* **biinvariants** *of an H-module and K-comodule M are defined as the image of the morphism $\pi\circ\iota: M^{coK}\to M^H$*



(15)

Requiring $\mathcal{C}$ to be complete and finitely cocomplete ensures the existence of invariants, coinvariants and biinvariants. Examples of such categories are Set, Top, Grp, $\mathrm{Vect}_{\mathbb{F}}$, Cat, $k-$Mod and the category $\mathrm{Ch}_{k-\mathrm{Mod}}$ of chain complexes of k-modules. For a small category $\mathcal{D}$ and a complete and finitely cocomplete category $\mathcal{C}$ the category $\mathcal{C}^{\mathcal{D}}$ is also complete and finitely cocomplete, see for instance Pareigis [Pa, Th. 1, Sec. 2.7]. Hence, $G-$Set and SSet also satisfy the requirement.

[1] Def. 2.8 in [MV] considers only the case $H=K$, as that is the only one required there.

Remark 2.14. *1. As discussed in [MV, Rem. 2.9] one could also consider the* coimage *of the morphism $\pi \circ \iota$ instead of its image. This amounts to passing from modules and comodules over the Hopf monoids H, K in $\mathcal{C}$ to comodules and modules over the Hopf monoids H^*, K^* in $\mathcal{C}^{op}$ from Example 2.2. In categories that are balanced, complete and cocomplete images and coimages are isomorphic. This follows from [Pa, Lemma 2 and 3 in Sec. 1.13] and holds in particular for $\mathcal{C} = \mathrm{Set}$ and any abelian category $\mathcal{C}$.*

2. There is another definition of images given by Kashiwara and Schapira in [KaS, Def. 5.1.1]. Here the image of $f : C \to C'$ is defined as the equaliser of the cokernel pair i_1, i_2 of f, which are the pushout of the morphism f with itself such that $i_1 \circ f = i_2 \circ f$. This version of image exists in all finitely complete and cocomplete categories. If for each factorisation of a morphism $f = q \circ j$ into a morphism q and a monomorphism j, the latter is a regular monomorphism, i.e. the equaliser of two parallel morphisms, [KaS, Def. 5.1.1] coincides with the definition of an image from Definition 2.13.

We illustrate (co)invariants and biinvariants with a few simple examples. (Co)invariants of (co)modules over Hopf monoids in SSet and Cat and the associated biinvariants for Yetter-Drinfeld modules are treated in Chapter 7 and Section 8.3, respectively.

Example 2.15.

1. *A Hopf monoid H in $\mathcal{C} = \mathrm{Set}$ (in $\mathcal{C} = \mathrm{Top}$) is a (topological) group H and*
   - *an H-module is a (continuous) H-Set $\rhd : H \times M \to M$,*
   - *$M^H = \{H \rhd m \mid m \in M\}$ with $\pi : M \to M^H$, $m \mapsto H \rhd m$ (and the quotient topology),*
   - *an H-comodule is given by a (continuous) map $F : M \to H$,*
   - *$M^{coH} = F^{-1}(1)$ with the inclusion $\iota : F^{-1}(1) \to M$ (and the subspace topology),*
   - *$M_{inv} = \pi(F^{-1}(1)) = \{H \rhd m \mid F(m) = 1\}$ (with the final topology induced by π).*

   An H-module and H-comodule $(M, \rhd, F)$ is a Yetter-Drinfeld module iff $F(h \rhd m) = hF(m)h^{-1}$ for all $m \in M$, $h \in H$. In Top, the coimage of $\pi \circ \iota$ is the same set $\pi(F^{-1}(1))$, but equipped with the subspace topology of M^H. Determining the image as described by Kashiwara and Schapira (Remark 2.14, 2.) also yields the set $\pi(F^{-1}(1))$ with the subspace topology.

2. *Let G be a group and H a group with a G-action by automorphisms, viewed as a Hopf monoid in $G{-}\mathrm{Set} = \mathrm{Set}^{\mathrm{B}G}$. Then H-modules are $H \rtimes G$-sets, H-comodules are G-sets M with G-equivariant maps $F : M \to H$ and*
   - *$M^H = \{H \rhd m \mid m \in M\}$ is the orbit space for H with the induced G-action and G-equivariant canonical surjection $\pi : M \to M^H$,*
   - *$M^{coH} = F^{-1}(1)$ with the induced G-action and G-equivariant inclusion $\iota : F^{-1}(1) \to M$,*
   - *$M_{inv} = \pi(F^{-1}(1))$ with the induced G-action.*

3. *For a Hopf algebra H over a commutative ring k as a Hopf monoid in k-Mod, H-(co)modules and Yetter-Drinfeld modules are (co)modules and Yetter-Drinfeld modules over H in the usual sense. Their (co)invariants and biinvariants are*
   - *$M^H = M/\langle\{h \rhd m - \epsilon(h)m \mid h \in H, m \in M\}\rangle$,*
   - *$M^{coH} = \{m \in M \mid \delta(m) = 1 \otimes m\}$,*
   - *$M_{inv} = \pi(M^{coH})$.*

While the coinvariants in Example 2.15, 3. coincide with the usual coinvariants for comodules over a Hopf algebra, the invariants form a quotient rather than a subset. This distinction is irrelevant in the case of semisimple Hopf algebras, but not in general. As our definition is symmetric with respect to Hopf monoids in a symmetric monoidal category $\mathcal{C}$ and the dual Hopf monoids in $\mathcal{C}^{op}$, it is more natural in our setting. The following example illustrates this.

Example 2.16. *For a finite group G and a commutative ring k the group algebra $k[G]$ and its dual $k[G]^*$ are Hopf monoids in k-Mod.*

For the group algebra $H = k[G]$

- *the invariants of a H-module $(M, \triangleright)$ are $M^H = M/\langle\{g \triangleright m - m \mid m \in M, g \in G\}\rangle$,*
- *comodules are G-graded k-modules $M = \oplus_{g \in G} M_g$ with $\delta(m) = g \otimes m$ for all $m \in M_g$,*
- *their coinvariants are $M^{coH} = M_1$.*

A $k[G]$-module and comodule $(M, \triangleright, \delta)$ is a Yetter-Drinfeld module iff $g \triangleright M_h = M_{ghg^{-1}}$ for all $g, h \in G$, and in this case $M_{inv} \cong M_1/(M_1 \cap \langle\{g \triangleright m - m \mid m \in M, g \in G\}\rangle) \cong H_0(G, M_1)$.

For the dual Hopf monoid $H = k[G]^*$

- *modules are G-graded k-modules $M = \oplus_{g \in G} M_g$ with $\delta_g \triangleright m = \delta_g(h) m$ for $m \in M_h$,*
- *their invariants are $M^H = M/(\oplus_{g \in G, g \neq 1} M_g) \cong M_1$,*
- *comodules are $k[G]$-right modules $(M, \triangleleft)$ with $\delta(m) = \sum_{g \in G} \delta_g \otimes (m \triangleleft g)$,*
- *their coinvariants are $M^{coH} = \{m \in M \mid m \triangleleft g = m \, \forall g \in G\}$.*

A $k[G]^*$-module and comodule $(M, \triangleright, \delta)$ is a Yetter-Drinfeld module iff $M_h \triangleleft g = M_{ghg^{-1}}$ for all $g, h \in G$, and in this case $M_{inv} \cong H^0(G, M_1)$.

By Lemma 2.12 morphisms of (co)modules over a Hopf monoid H induce morphisms between their (co)invariants. The question if morphisms of both, modules and comodules, induce morphisms between the associated biinvariants is more subtle in general. It is shown in [MV, Lemma 2.10] that this always holds for *isomorphisms.* As a direct generalisation we have in the notation of (15)

Lemma 2.17. *Let $\mathcal{C}$ be complete and finitely cocomplete, H, K Hopf monoids in $\mathcal{C}$ and $\Phi : M \to M'$ an isomorphism of H-modules and K-comodules. There is a unique morphism $\Phi_{inv} : M_{inv} \to M'_{inv}$ with $\pi' \circ \Phi \circ \iota = I' \circ \Phi_{inv} \circ P$, and Φ_{inv} is an isomorphism.*

2.3 (Co)integrals and Hopf algebras

In this section, we summarise the background on (co)integrals in a symmetric monoidal category $\mathcal{C}$, in particular for $\mathcal{C} = \mathrm{Vect}_{\mathbb{F}}$. This is mainly relevant for Chapter 6 and Section 10.3.

Definition 2.18. *[TVi, Sec. 6.3.1] Let H be a Hopf monoid in $\mathcal{C}$.*

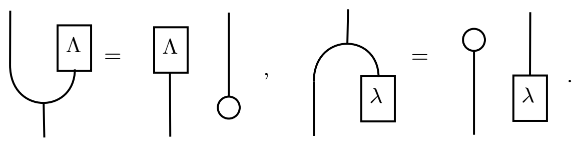
1. A ***left integral*** *of H is a morphism $\Lambda : e \to H$ satisfying*

$$m \circ (1_H \otimes \Lambda) = \Lambda \otimes \epsilon. \tag{16}$$

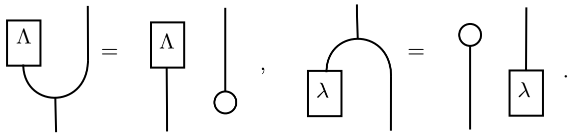
2. A ***left cointegral*** *of H is a morphism $\lambda : H \to e$ satisfying*

$$(1_H \otimes \lambda) \circ \Delta = \eta \otimes \lambda. \tag{17}$$

In diagrammatic notation, the conditions (16) and (17) read



There is an analogous notion of **right integral** $\Lambda : e \to H$ over H, where (16) is replaced by $m \circ (\Lambda \otimes 1_H) = \Lambda \otimes \epsilon$, and **right cointegral** $\lambda : H \to e$ with (17) replaced by $(\lambda \otimes 1_H) \circ \Delta = \eta \otimes \lambda$.



If a morphism is both a left and right (co)integral, it is called **two-sided** (co)integral.

Example 2.19. *1. For the Hopf monoid $\mathbb{C}[H]$ in $\mathrm{Rep}_{\mathbb{C}}(G)$ from Example 2.2, 5. with G, H finite groups and a group homomorphism $\Phi : G \to \mathrm{Aut}(H)$ the G-equivariant map*

$$\Lambda : \mathbb{C} \to \mathbb{C}[H], \quad r \mapsto r \sum_{h \in H} \delta_h$$

is a two-sided integral for $\mathbb{C}[H]$, as

$$m \circ (1_{\mathbb{C}[H]} \otimes \Lambda) \left(\sum_{h \in H} c_h \delta_h \otimes r \right) = r \sum_{h \in H} c_h \sum_{k \in H} \delta_{hk} = r \sum_{h \in H} c_h \sum_{l \in H} \delta_l$$
$$= (\epsilon \otimes \Lambda) \left(\sum_{h \in H} c_h \delta_h \otimes r \right)$$

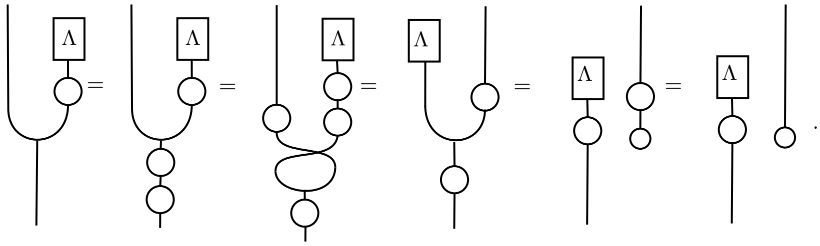
for $c_h, r \in \mathbb{C}$. The right integral condition is verified analogously. The G-equivariant map $\rho_e : \mathbb{C}[H] \to \mathbb{C}$ with $\rho_e(\delta_h) = 0$ for $h \neq e$ and $\rho_e(\delta_e) = 1$ is a two-sided cointegral for $\mathbb{C}[H]$ in $\mathrm{Rep}_{\mathbb{C}}(G)$. It satisfies $\rho_e \circ \Lambda = 1_{\mathbb{C}}$.

2. Dually to 1., the G-equivariant map $\Lambda : \mathbb{C} \to \mathbb{C}[H]^*$, $r \mapsto r\rho_e$ is a two-sided integral of the Hopf monoid $\mathbb{C}[H]^*$ in $\mathrm{Rep}_{\mathbb{C}}(G)$ from Example 2.2, 6.

3. If $F : \mathcal{C} \to \mathcal{C}'$ is a symmetric monoidal functor and Λ a (co)integral over a Hopf monoid H in $\mathcal{C}$, then $F(\Lambda)$ defines a (co)integral over the Hopf monoid $F(H)$.

Note that the existence of (co)integrals is not guaranteed: For instance, a non-trivial group $H \neq \{e\}$ as a Hopf monoid in Set neither has an integral nor a cointegral. An integral would amount to an element $\Lambda \in H$ satisfying $h \cdot \Lambda = \Lambda$ for all $h \in H$, which contradicts $H \neq \{e\}$. The only morphism in Set whose target is given by the tensor unit is the counit. Clearly, the counit does not satisfy the cointegral condition, as (17) yields $h = e_H$ for all $h \in H$.

If a Hopf monoid H in a symmetric monoidal category $\mathcal{C}$ is involutive, then $\Lambda : e \to H$ is a right integral of H iff $S \circ \Lambda$ is a left integral of H. This follows by applying (5), the naturality of the braiding and the involutivity of the antipode:



The other implication is obtained analogously. Analogous statements hold if the roles of left and right integrals are swapped or likewise if the integrals Λ, $S \circ \Lambda$ are replaced by cointegrals λ, $\lambda \circ S$.

Remark 2.20. *There is a more general definition of integrals and cointegrals as for instance used by Bespalov et al. [BKLT, Def. 3.1] and Kashaev and Virelizier [KV, Sec. 5.1]. There, the source of Λ and target of λ is not restricted to the tensor unit, instead any object in $\mathcal{C}$ is allowed. Bespalov et al. [BKLT, Prop. 3.1] also provide an existence statement for integrals and cointegrals over involutive Hopf monoids: If $\mathcal{C}$ is left rigid and all idempotents split, then there exists an object I in $\mathcal{C}$ and left and right integrals with source I as well as left and right cointegrals with target I over H.*

The remainder of this section is concerned with Hopf monoids and their (co)integrals in the category $\mathcal{C} = \mathrm{Vect}_{\mathbb{F}}$ of vector spaces over a field $\mathbb{F}$. Throughout the thesis we write **Hopf algebras** for Hopf monoids in $\mathrm{Vect}_{\mathbb{F}}$.

Note that in $\mathrm{Vect}_{\mathbb{F}}$ (co)integrals form a vector space. We denote the linear subspaces of left and right integrals of a Hopf algebra H by $I_L(H)$ and $I_R(H)$. Integrals over a Hopf algebra H are often defined as elements of H, but this is equivalent to Definition 2.18.

If $(H, m, \eta, \Delta, \epsilon, S)$ is a finite-dimensional Hopf algebra, then its dual H^* has a canonical Hopf algebra structure $(H^*, \Delta^*, \epsilon^*, m^*, \eta^*, S^*)$, where we write f^* for the dual linear map of f. In this case λ is a left (right) integral for H^* if and only if λ is a left (right) cointegral for H. This follows by Hopf algebra duality.

Definition 2.21. *An integral l of a Hopf algebra H is called* ***normalised*** *if $\epsilon(l) = 1$. A normalised two-sided integral is called* ***Haar integral****.*

We give some examples of integrals over Hopf algebras.

Example 2.22. *1. For the group algebra $\mathbb{F}[G]$ of a finite group G the subspaces of integrals and cointegrals are given by*

$$I_L(\mathbb{F}[G]) = I_R(\mathbb{F}[G]) = \mathbb{F} \cdot \sum_{g \in G} \delta_g, \quad I_L(\mathbb{F}[G]^*) = I_R(\mathbb{F}[G]^*) = \mathbb{F} \cdot \rho_e,$$

with $\rho_e(\delta_g) = 1$ if $g = e$ and $\rho_e(\delta_g) = 0$ else. If the characteristic of $\mathbb{F}$ does not divide the group order of G, then $\frac{1}{|G|} \sum_{g \in G} \delta_g$ is a Haar integral of $\mathbb{F}[G]$.

2. Sweedler's four-dimensional Hopf algebra H_4 is as an algebra generated by x, y with relations

$$x^2 = 1, \quad y^2 = 0, \quad xy + yx = 0.$$

The Hopf algebra structure is defined by

$$\begin{aligned} \Delta(x) &= x \otimes x, \quad \epsilon(x) = 1, \quad S(x) = x, \\ \Delta(y) &= 1 \otimes y + y \otimes x, \quad \epsilon(y) = 0, \quad S(y) = xy. \end{aligned}$$

A basis for H_4 is given by $\{1, x, y, xy\}$. Note that its antipode has order 4 and satisfies $S^2(a) = xax^{-1}$ for all $a \in H_4$, see Kassel [Ka, Ex. 2 in Sec. VIII.2]. A simple computation yields

$$I_L(H_4) = \mathbb{F} \cdot (1 + x)y, \quad I_R(H_4) = \mathbb{F} \cdot y(1 + x).$$

3. The tensor product of Haar integrals of Hopf algebras H, K is a Haar integral of the Hopf algebra $H \otimes K$.

Example 2.23. *For an algebraically closed field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$ consider the involutive Hopf algebra H_8, see Radford [Ra, Sec. 16.3]. As an algebra it is generated by x, y, z with relations*

$$x^2 = y^2 = 1, \quad yx = xy, \quad zx = yz, \quad zy = xz, \quad z^2 = \frac{1}{2}(1 + x + y - xy). \tag{18}$$

Comultiplication, counit and antipode are defined by

$$\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = \frac{1}{2}(z \otimes z + yz \otimes z + z \otimes xz - yz \otimes xz),$$
$$\epsilon(x) = \epsilon(y) = \epsilon(z) = 1, \quad S(x) = x, \quad S(y) = y, \quad S(z) = z.$$

A basis for H_8 is given by $\{1, x, y, xy, z, xz, yz, xyz\}$, see [Ra, Sec. 16.3]. Note that H_8 is semisimple, self-dual and neither commutative nor cocommutative [Ma, Th. 2.13, Rem. 2.14]. Applying the identities

$$\begin{aligned} z \cdot xz = xz \cdot xyz = yz \cdot z = xyz \cdot yz &= \frac{1}{2}(y + xy + 1 - x), \\ z \cdot yz = xz \cdot z = yz \cdot xyz = xyz \cdot xz &= \frac{1}{2}(1 + x + xy - y), \\ z \cdot xyz = xz \cdot xz = yz \cdot yz = xyz \cdot z &= \frac{1}{2}(x + y + xy - 1), \end{aligned} \tag{19}$$

it follows that

$$\Lambda = 1 + x + y + xy + z + xz + yz + xyz \in H_8$$

is a two-sided integral for H_8. In particular, $\frac{1}{8}\Lambda$ is a Haar integral and

$$\lambda = \rho_1 : H_8 \to \mathbb{F}, \quad (c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot y + c_4 \cdot xy + c_5 \cdot z + c_6 \cdot xz + c_7 \cdot yz + c_8 \cdot xyz) \mapsto c_1$$

with $c_i \in \mathbb{F}$ is a two-sided cointegral for H_8. As λ is normalised, it is a Haar integral of H_8^*.

Note that the integrals and cointegrals of $\mathbb{C}[H]$ in $\mathcal{C} = \mathrm{Rep}_{\mathbb{C}}(G)$ from Example 2.19, 1. and 2. are a simple extension of Example 2.22, 1. They are related by the forgetful functor $\mathrm{Rep}_{\mathbb{C}}(G) \to \mathrm{Vect}_{\mathbb{C}}$, which sends (co)integrals to (co)integrals. The following Lemma is a consequence of [Ra, Th. 10.2.2, Cor. 10.3.3].

Lemma 2.24. *[Ra, Th. 10.2.2, Cor. 10.3.3] If H is a finite-dimensional Hopf algebra over $\mathbb{F}$, then:*

1. $\dim(I_L(H)) = \dim(I_R(H)) = 1$.
2. *There exists a left integral Λ and a right cointegral λ of H with $\lambda(\Lambda) = 1$.*
3. *H is semisimple if and only if $\epsilon(\Lambda) \neq 0$ for some left or right integral Λ of H. In this case one has $\epsilon(\Lambda) \neq 0$ for all non-zero left or right integrals of H.*

Definition 2.25. *A Hopf algebra H over $\mathbb{F}$ is called* ***unimodular*** *if $I_L(H) = I_R(H)$.*

Lemma 2.26. *[Ra, Cor. 10.3.3] Every finite-dimensional semisimple Hopf algebra over $\mathbb{F}$ is unimodular.*

Lemma 2.24 and 2.26 imply that every finite-dimensional semisimple Hopf algebra H has a Haar integral $l \in H$. In particular, if $S^2 = 1_H$ then it satisfies $S(l) = l$, as $S(l)$ is also an integral of H. Hence, it is $S(l) = k \cdot l$ for some $k \in \mathbb{F}$, as the subspace of integrals is one-dimensional. The identities $1 = \epsilon(l) = \epsilon(S(l)) = k\epsilon(l)$ then yield $k = 1$.

Proposition 2.27. *[LRa, Th. 4.4] Let H be a finite-dimensional Hopf algebra over $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \nmid \dim_{\mathbb{F}}(H)$. If $S^2 = 1_H$, then H and H^* are semisimple.*

In particular, for $\mathrm{char}(\mathbb{F}) = 0$ one has from [LRb, Th. 1] and [LRa, Th. 4.4]:

Theorem 2.28. *[LRa, LRb] Let H be a finite-dimensional Hopf algebra over a field $\mathbb{F}$ with* $\mathrm{char}(\mathbb{F}) = 0$. *Then the following are equivalent:*

i) *H is semisimple*
ii) *H^* is semisimple*
iii) $S^2 = 1_H$.

The following lemma is well-known, but often used implicitly or stated in different versions. For instance, the projection is mentioned in [Skr, Sec. 1] and called *trace* in [CFM]. We formulate it for left modules. There is an analogous statement for right modules.

Lemma 2.29. *Let H be a Hopf algebra with a Haar integral $l \in H$. Then for any H-module $(M, \triangleright)$ the linear map $P_l : M \to M$, $m \mapsto l \triangleright m$ is a projection with*

$$\mathrm{im}(P_l) = M^H := \{m \in M : h \triangleright m = \epsilon(h)m \;\; \forall h \in H\}.$$

The kernel of P_l is the image of the linear map $\triangleright - \epsilon \otimes 1_M : H \otimes M \to M$.

Proof. Applying $hl = \epsilon(h)l$ and $l^2 = l$ yields

$$h \triangleright P_l(m) = \epsilon(h)P_l(m), \quad (P_l \circ P_l)(m) = P_l(m)$$

for $m \in M$, $h \in H$. As $P_l(m) = l \triangleright m = m$ for $m \in M^H$ it follows that P_l is a projection with $\mathrm{im}(P_l) = M^H$. For $h \in H$, $m \in M$, $x \in \ker(P_l)$ one obtains

$$P_l(h \triangleright m - \epsilon(h)m) = (lh) \triangleright m - \epsilon(h)l \triangleright m = \epsilon(h)l \triangleright m - \epsilon(h)l \triangleright m = 0,$$
$$(\triangleright - \epsilon \otimes 1_M)\,(l \otimes (-x)) = \epsilon(l)x = x.$$

Hence, the kernel of P_l coincides with the image of the linear map $\triangleright - \epsilon \otimes 1_M$. □

For a finite-dimensional Hopf algebra H we will consider the quantum double $D(H)$ of H, which combines H and its dual Hopf algebra. The quantum double plays a significant role within the quantum double models from [Ki, BMCA]. More specifically, it provides the mathematical framework for understanding the behaviour of quasiparticles in these models, including the study of their fusion and braiding properties.

Definition 2.30. *[Maj, Sec. 7.1] Let H be a finite-dimensional Hopf algebra. The **Drinfeld double** or **quantum double** $D(H)$ of H is the vector space $H^* \otimes H$ with Hopf algebra structure*

$$(\alpha \otimes h) \cdot (\beta \otimes k) = \beta_{(3)}(h_{(1)})S^{-1}(\beta_{(1)})(h_{(3)})\alpha\beta_{(2)} \otimes h_{(2)}k, \qquad 1 = 1_{H^*} \otimes 1_H,$$
$$\Delta(\alpha \otimes h) = \alpha_{(2)} \otimes h_{(1)} \otimes \alpha_{(1)} \otimes h_{(2)}, \qquad \epsilon(\alpha \otimes h) = \epsilon(\alpha)\epsilon(h),$$
$$S(\alpha \otimes h) = \alpha_{(1)}(h_{(3)})S^{-1}(\alpha_{(3)})(h_{(1)})S(\alpha_{(2)}) \otimes S(h_{(2)}).$$

As explained in [Ra, Cor. 13.2.3], for H finite-dimensional its Drinfeld double $D(H)$ is semisimple if and only if both H and H^* are semisimple. From [Ra, Prop. 13.2.2] follows that $\lambda \otimes l$ is a Haar integral for $D(H)$ if $l \in H$ and $\lambda \in H^*$ are Haar integrals.

3 Ribbon graphs and surfaces

In this chapter, we summarise the background on *ribbon graphs*, also called *fat graphs* or *embedded graphs*, for more details we refer to the textbooks of Lando et. al. [L+] and Ellis-Monaghan and Moffatt [EM].

This chapter is based on Section 3 of the preprint [HM].

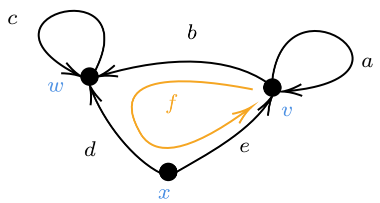
Throughout this thesis, all graphs are *directed* graphs with a finite number of vertices and edges. We do not require that the graphs are connected and allow **isolated vertices** with no incident edges.

Definition 3.1. *A* **ribbon graph** *is a graph with a cyclic ordering of the edge ends at each vertex.*

The cyclic ordering of edge ends at the vertices of a ribbon graph allows one to thicken its edges to strips or ribbons and defines the faces of the ribbon graph. One says that a path in a ribbon graph **turns maximally left at a vertex** if it enters the vertex along an edge end and leaves it along an edge end that comes directly before it with respect to the cyclic ordering. **Face paths** are paths that turn maximally left at each vertex and traverse each edge at most once in each direction. A **face** of a ribbon graph is defined as a cyclic equivalence class of closed face paths. Each isolated vertex is also viewed as a face, and such a face is called an **isolated face**.

In the following we denote by V, E, F the sets of vertices, edges and faces of a ribbon graph and by $s(\alpha), t(\alpha)$ the starting and target vertex of an edge α. The same notation $s(\alpha)$, $t(\alpha)$ is occasionally used for the starting and target end of α. We say that two edge ends incident at a vertex $v \in V$ are **neighbours** or **neighbouring** if one of them comes directly before or after the other with respect to the cyclic ordering at v. An edge α with $s(\alpha) = t(\alpha)$ is called a **loop**. A loop at v whose starting and target end are neighbours is called an **isolated loop**. The starting end of a face path γ is the first edge end traversed by γ. The number of incident edge ends at a vertex is called **valence** of the vertex. When drawing a ribbon graph we take the cyclic ordering of edge ends at vertices as the one in the drawing.

Example 3.2. *In the following graph* Γ



the permutation $(t(a), s(a), s(b), t(e))$ *describes the cyclic ordering at* v. *The closed path* $f = e^+ \circ d^- \circ b^+$ *given by the orange arrow describes a face. Here, we write* x^+ *if* f *traverses the edge* x *parallel to its orientation and* x^- *if* x *is traversed antiparallel to its orientation.*

Ribbon graphs are directly related to embedded graphs on oriented surfaces. Every graph Γ embedded into an oriented surface Σ inherits a cyclic ordering of the edge ends at each vertex and hence a ribbon graph structure. Attaching discs to the faces of the ribbon graph Γ yields an oriented surface Σ_Γ such that the connected components of $\Sigma_\Gamma \setminus \Gamma$ are discs and in bijection with faces of Γ, see Figure 1. If Γ is embedded into an oriented surface Σ, the surface Σ_Γ is homeomorphic to Σ iff each connected component of $\Sigma \setminus \Gamma$ is a disc. In this case, we call Γ **properly embedded** in Σ. Note that this implies a bijection between connected components of Γ and of Σ, and connected

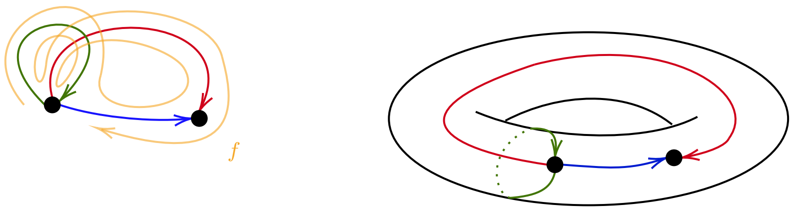
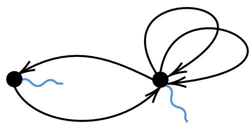


Figure 1: Attaching a disc to the face f yields a torus.

components of Σ containing an isolated vertex are spheres. The genus g of a connected component of Σ is then determined by the Euler characteristic $2-2g = |V| - |E| + |F|$, where $|V|, |E|, |F|$ are the number of vertices, edges and faces of the associated connected component of Γ.

Note that each ribbon graph or embedded graph has a Poincaré dual obtained by replacing each vertex (face) with a face (vertex) and each edge with a dual edge. This transforms the paths that characterise faces into paths that go counterclockwise around a vertex and vice versa. Edge ends correspond to edge sides of the dual graph and their cyclic ordering at a vertex to the cyclic ordering of the edge sides in the dual face.

In the following we sometimes require a *linear* ordering of the edge ends at a vertex or of the edge sides in a face. This is achieved by inserting a marking, the *cilium*, that separates the edge ends or edge sides of minimal and maximal order. For faces this corresponds to the choice of a starting vertex for the associated cyclic equivalence class of paths. We indicate the cilium by a wavy line.



Definition 3.3.

1. *A* **ciliated vertex** *in a ribbon graph is a vertex with a choice of linear ordering of the incident edge ends that is compatible with their cyclic ordering.*
2. *A* **ciliated face** *in a ribbon graph is a closed path that turns maximally left at each vertex, including the starting vertex, and traverses each edge at most once in each direction.*

A **ciliated ribbon graph** *is a ribbon graph in which each face and vertex is assigned a cilium. Isolated vertices and faces are trivially ciliated.*

For a closed surface Σ of genus $g \geq 0$ we often work with a ciliated ribbon graph with a single vertex and a single face that is given by a set of generators of the fundamental group

$$\pi_1(\Sigma) = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \mid [\beta_g^{-1}, \alpha_g] \cdots [\beta_1^{-1}, \alpha_1] = 1 \rangle. \tag{20}$$

The $2g$ generators of the fundamental group of a closed surface of genus g are shown in Figure 2.

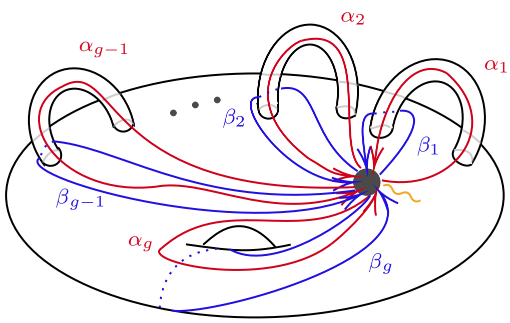
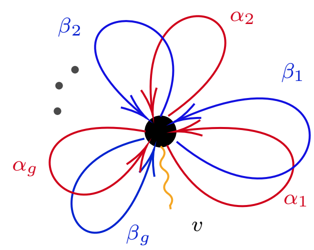


Figure 2: Surface of genus g and the generators of its fundamental group.

Definition 3.4. *The* **standard graph** *of an oriented surface* Σ *of genus* $g \geq 1$ *is the graph*



(21)

with the face $f = [\beta_g^{-1}, \alpha_g] \cdots [\beta_1^{-1}, \alpha_1]$ *and the ordering of edge ends at* v *given by* $s(\alpha_1) < s(\beta_1) < t(\alpha_1) < t(\beta_1) < \ldots < s(\alpha_g) < s(\beta_g) < t(\alpha_g) < t(\beta_g)$. *In particular, the standard graph for* S^2 *consists of a single isolated vertex and the associated isolated face.*

In the following we use certain graph transformations to relate properly embedded ribbon graphs in a connected surface Σ to its standard graph.

Definition 3.5. *Let* Γ *be a ribbon graph with edge set* E *and vertex set* V.

1. *The* **edge reversal** *reverses the orientation of an edge* $\beta \in E$.
2. *The* **contraction** *of an edge* $\alpha \in E$ *that is not a loop removes* $\alpha \in E$ *and fuses the vertices* $s(\alpha)$ *and* $t(\alpha)$.
3. *The* **edge slide** *slides an end of* $\beta \in E$ *that is a neighbour of an end of* $\alpha \in E$ *along* α.
4. *The* **loop deletion** *removes an isolated loop* $\beta \in E$ *from* Γ.
5. *The* **splitting of an edge** *adds a two-valent vertex in the middle of an edge* $\beta \in E$ *such that* β *is split into two edges with induced orientations.*

In all cases except 2. and 5. the resulting ribbon graph inherits all cilia from Γ. *In 2. one erases either the cilium of* $t(\alpha)$ *or of* $s(\alpha)$ *and speaks of contracting* α *towards* $t(\alpha)$ *and* $s(\alpha)$, *respectively. In 5. there are two options to endow the inserted vertex with a cilium.*

These graph transformations are illustrated in Figure 3. Note that they are not independent. Contracting an edge α towards $t(\alpha)$ is the same as first sliding some edge ends along α and then contracting α towards $t(\alpha)$. Contracting an edge α towards $t(\alpha)$ is also the same as first reversing α, then contracting α towards $s(\alpha)$ and then reversing α. By reversing α and β before and after a slide, one can reduce all edge slides to the ones that slide the target end of β along the left of α.

There are of course other possible graph transformations such as deleting edges, which is dual to edge contractions. However, the graph transformations in Definition 3.5 are sufficient to transform any connected ribbon graph into a standard graph. This is well-known and appears implicitly in many publications. We summarise the argument for the convenience of the reader.

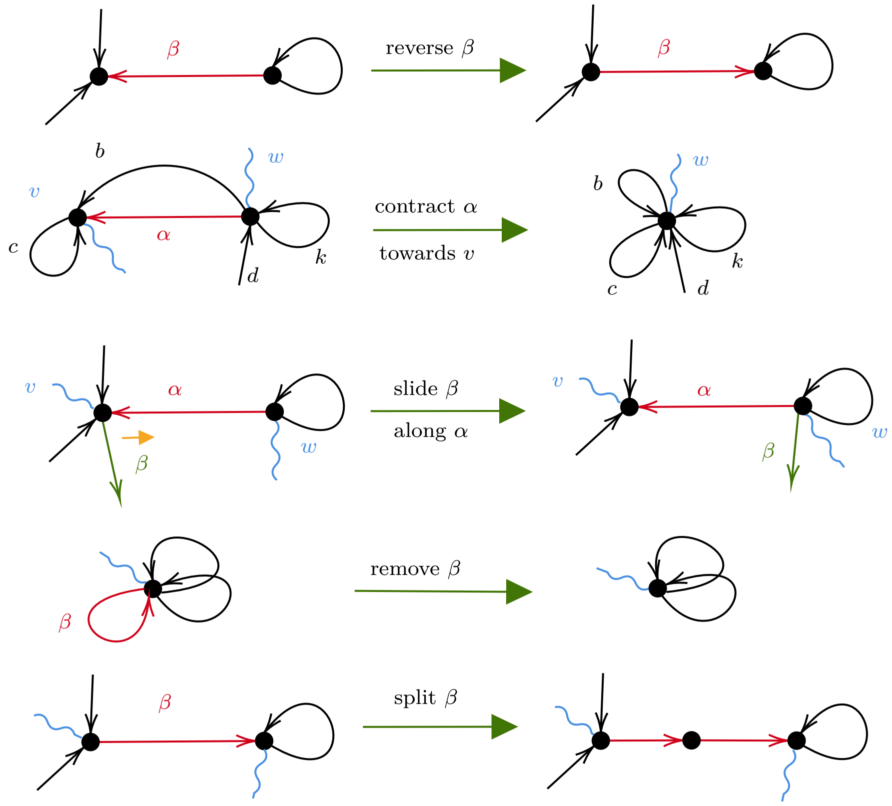


Figure 3: Examples of graph transformations

Proposition 3.6. *Every connected ribbon graph can be transformed into the standard graph* (21) *by edge reversals, edge slides, edge contractions and loop deletions.*

Proof. Selecting a maximal tree in Γ and contracting all edges in the tree transforms Γ into a graph Γ' with a single vertex. By applying edge slides one can transform Γ' into a graph Γ'' that coincides with (21) up to edge orientation and up to the presence a number of isolated loops between the cilium and the starting end of α_1. This follows from an analogous statement for chord diagrams, which correspond to ribbon graphs with a single vertex, see for instance Chmutov, Duzhin and Mostovoy [CDM, Sec. 4.8.5]. Deleting the isolated loops and reversing edges in Γ'' then yields the standard graph (21). □

4 (Co)modules from Hopf monoids and ribbon graphs

In this chapter we use involutive Hopf monoids in symmetric monoidal categories to assign (co)modules over Hopf monoids to ciliated ribbon graphs. In Chapter 5 we then show that their biinvariants are topological invariants: their isomorphism classes depend only on the genus of the surface obtained by attaching discs to the faces of the graph. In Chapters 7 and 8 we determine these biinvariants for simplicial groups as Hopf monoids in SSet and for crossed modules as group objects in Cat.

This chapter is a revised version of Section 4 from the preprint [HM].

The construction generalises Kitaev's quantum double model and the toric code from [Ki], which was first formulated for the group algebra of a finite group over $\mathbb{C}$ and then generalised by Buerschaper et al. in [BMCA] to finite-dimensional semisimple C^*-Hopf algebras. A very similar construction to the one in this thesis is used in [MV] to obtain mapping class group actions from pivotal Hopf monoids in symmetric monoidal categories. The work [MV] considers the biinvariants of a Yetter-Drinfeld module structure assigned to the standard graph (21), but it does not establish that the biinvariants are graph-independent.

The construction of the (co)module structures from an involutive Hopf monoid and a ciliated ribbon graph in this chapter is directly analogous to the one in [MV], which in turn is a straightforward generalisation of [Ki, BMCA]. The main difference is that H^*-modules in [BMCA] are replaced by H-comodules and $D(H)$-modules by Yetter-Drinfeld modules over H.

What differs substantially from [Ki, BMCA] are the notions of (co)invariants, biinvariants and the construction of the topological invariant. The works in [Ki, BMCA] rely on the normalised Haar integral of a finite-dimensional semisimple complex Hopf algebra, which is not available in our setting. Our construction is more general, as the only assumptions are that the underlying symmetric monoidal category is complete and finitely cocomplete and the Hopf monoid involutive. The article [MV] also allows pivotal Hopf monoids. The involutive Hopf monoids in this thesis are examples of pivotal Hopf monoids, with their unit as pivotal structure.

Let H be an involutive Hopf monoid in a complete and finitely cocomplete symmetric monoidal category $\mathcal{C}$ and Γ a ciliated ribbon graph with vertex set V, edge set E and face set F.

We consider the $|E|$-fold tensor product of H with itself, together with an assignment of the copies of H in this tensor product to the edges of Γ. By writing $H^{\otimes E}$ we emphasise this assignment. If $E = \emptyset$, we set $H^{\otimes E} = e$. The object $H^{\otimes E}$ can be viewed as the counterpart of the Hilbert space of Kitaev's quantum double model in [Ki, BMCA].

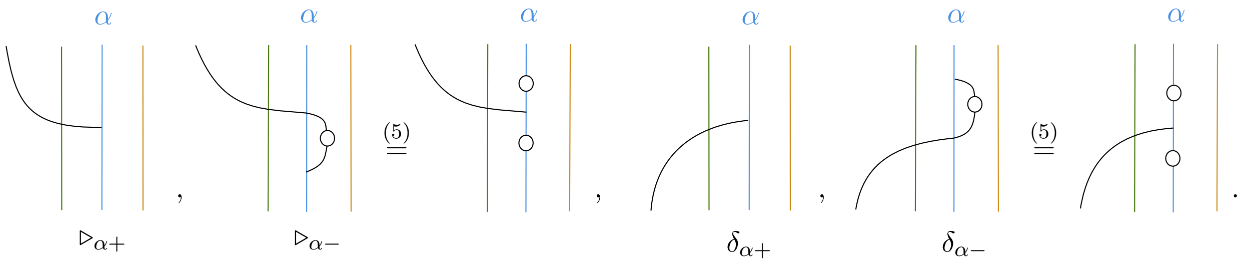
We assign to each edge $\alpha \in E$ two H-module structures $\rhd_{\alpha\pm} : H \otimes H^{\otimes E} \to H^{\otimes E}$ and H-comodule structures $\delta_{\alpha\pm} : H^{\otimes E} \to H \otimes H^{\otimes E}$. The H-module structures $\rhd_{\alpha+}$ and $\rhd_{\alpha-}$ are assigned to the target and starting end of α and the H-comodule structures to its left and right side, respectively. They are induced by the standard H-(co)module structures on H via left (co)multiplication.

This requires some notation. Given a morphism $f : H \to K$ in $\mathcal{C}$ and an edge $\alpha \in E$ we write f_α for the morphism that applies f to the copy of H in $H^{\otimes E}$ that belongs to α and the identity morphism to the other copies. We write $\tau_\alpha : H^{\otimes E} \to H^{\otimes E}$ or $\tau_\alpha : H \otimes H^{\otimes E} \to H \otimes H^{\otimes E}$ for the composite of braidings that moves the copy of H for α to the left. We denote by $m_\alpha : H \otimes H^{\otimes E} \to H^{\otimes E}$ the morphism that moves the first copy of H to the left of the one for α and then applies m to them.

Definition 4.1. *The H-module structures $\rhd_{\alpha\pm} : H \otimes H^{\otimes E} \to H^{\otimes E}$ and H-comodule structures $\delta_{\alpha\pm} : H^{\otimes E} \to H \otimes H^{\otimes E}$ for an edge $\alpha \in E$ are*

$$\rhd_{\alpha+} := m_\alpha, \quad \rhd_{\alpha-} := S_\alpha \circ \rhd_{\alpha+} \circ (1_H \otimes S_\alpha), \quad \delta_{\alpha+} := \tau_\alpha \circ \Delta_\alpha, \quad \delta_{\alpha-} := (1_H \otimes S_\alpha) \circ \delta_{\alpha+} \circ S_\alpha.$$

In diagrammatic notation the module and comodule structures read:



By definition, the (co)module structures assigned to different edges of a graph commute, since they (co)act on different copies of H in the tensor product $H^{\otimes E}$. A direct computation using (1) and (5) shows that the two H-(co)module structures assigned to a given edge commute as well. The proof is directly analogous to the ones for Hopf algebras in [BMCA].

Lemma 4.2. *[MV, Lemma 5.2, 2.] For any edge $\alpha \in E$ the H-module structures $\rhd_{\alpha\pm}$ and the H-comodule structures $\delta_{\alpha\pm}$ commute:*

$$\rhd_{\alpha-} \circ (1_H \otimes \rhd_{\alpha+}) = \rhd_{\alpha+} \circ (1_H \otimes \rhd_{\alpha-}) \circ (\tau_{H,H} \otimes 1_{H^{\otimes E}}),$$
$$(1_H \otimes \delta_{\alpha-}) \circ \delta_{\alpha+} = (\tau_{H,H} \otimes 1_{H^{\otimes E}}) \circ (1_H \otimes \delta_{\alpha+}) \circ \delta_{\alpha-}.$$

The (co)module structures from Definition 4.1 define an H-module structure on $H^{\otimes E}$ for each ciliated vertex v and an H-comodule structure on $H^{\otimes E}$ for each ciliated face f of Γ. The former applies the comultiplication to H, distributes the resulting copies of H to the edge ends at v according to their ordering and acts on them with $\rhd_{\alpha\pm}$ according to their orientation. Dually, the coaction applies the H-coaction $\delta_{\alpha\pm}$ to each edge α in f, depending on its orientation relative to f, and multiplies the resulting copies of H according to the order of the edge sides in f.

Definition 4.3. *[MV, Def. 5.3]*

1. *The H-module structure $\rhd_v : H \otimes H^{\otimes E} \to H^{\otimes E}$ assigned to a ciliated vertex v with incident edge ends $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ is*

$$\rhd_v = \rhd_{\alpha_1} \circ (1_H \otimes \rhd_{\alpha_2}) \circ \ldots \circ (1_{H^{\otimes(n-1)}} \otimes \rhd_{\alpha_n}) \circ (\Delta^{(n-1)} \otimes 1_{H^{\otimes E}}), \tag{22}$$

   where $\rhd_\alpha = \rhd_{e(\alpha)+}$ if α is incoming, $\rhd_\alpha = \rhd_{e(\alpha)-}$ if α is outgoing and $e(\alpha)$ is the edge of α.

2. *The H-comodule structure $\delta_f : H^{\otimes E} \to H \otimes H^{\otimes E}$ assigned to a ciliated face f that traverses the edges $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$ in this order is*

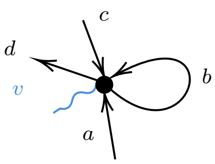
$$\delta_f = (m^{(n-1)} \otimes 1_{H^{\otimes E}}) \circ (1_{H^{\otimes(n-1)}} \otimes \delta_{\alpha_r}) \circ \ldots \circ (1_H \otimes \delta_{\alpha_2}) \circ \delta_{\alpha_1}, \tag{23}$$

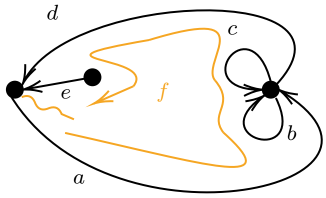
   where $\delta_\alpha = \delta_{\alpha+}$ if α is traversed with, $\delta_\alpha = \delta_{\alpha-}$ if α is traversed against its orientation.

To an isolated vertex and face we assign the (co)module structures $\rhd_v = \epsilon \otimes 1_{H^{\otimes E}}$ and $\delta_f = \eta \otimes 1_{H^{\otimes E}}$.

To avoid heavy notation we use Sweedler notation and describe these (co)module structures by labelling edges of a graph with letters representing the associated copies of H.

Example 4.4.





The H-module structure $\triangleright_v$ for the ciliated vertex v with incident edge ends $t(a) < s(b) < t(b) < t(c) < s(d)$ and the H-comodule structure δ_f for the ciliated face $f = e \circ e^{-1} \circ d \circ c^{-1} \circ b \circ a$ are

$$h \triangleright_v (a \otimes b \otimes c \otimes d) = h_{(1)}a \otimes h_{(3)}bS(h_{(2)}) \otimes h_{(4)}c \otimes dS(h_{(5)}),$$
$$\delta_f (a \otimes b \otimes c \otimes d \otimes e) = e_{(1)}S(e_{(3)})d_{(1)}S(c_{(2)})b_{(1)}a_{(1)} \otimes a_{(2)} \otimes b_{(2)} \otimes c_{(1)} \otimes d_{(2)} \otimes e_{(2)}.$$

The interaction of the H-module and H-comodule structures assigned to ciliated vertices and faces of the graph is investigated in [Ki, BMCA, MV]. They are local in the sense that the H-(co)module structure for a vertex (face) affects only those copies of H that belong to their incident edges. As the action $\triangleright_{\alpha+}$ for an edge $\alpha \in E$ acts by left- and $\triangleright_{\alpha-}$ by right-multiplication, the H-module structures for different vertices commute. The same holds for the H-comodule structures at different faces. Moreover, H-module structures commute with H-comodule structures unless their cilia share a vertex or a face. The H-module and H-comodule structure for each cilium define a Yetter-Drinfeld module structure.

Lemma 4.5. *[MV, Lemma 5.5]*

1. *The H-left module structures for distinct vertices $v \neq v' \in V$ and the H-left comodule structures for distinct faces $f \neq f' \in F$ commute for all choices of cilia:*

$$\triangleright_{v'} \circ (1_H \otimes \triangleright_v) = \triangleright_v \circ (1_H \otimes \triangleright_{v'}) \circ (\tau_{H,H} \otimes 1_{H^{\otimes E}}), \quad (24)$$
$$(1_H \otimes \delta_{f'}) \circ \delta_f = (\tau_{H,H} \otimes 1_{H^{\otimes E}}) \circ (1_H \otimes \delta_f) \circ \delta_{f'}. \quad (25)$$

2. *If two cilia are placed at distinct vertices and distinct faces, the H-module structure assigned to one of them commutes with the H-comodule structure assigned to the other:*

$$\delta_f \circ \triangleright_v = (1_H \otimes \triangleright_v) \circ (\tau_{H,H} \otimes 1_{H^{\otimes E}}) \circ (1_H \otimes \delta_f). \quad (26)$$

3. *If $v \in V$ and $f \in F$ share a cilium, then $(H^{\otimes E}, \triangleright_v, \delta_f)$ is a Yetter-Drinfeld module over H.*

If each vertex and face of Γ is equipped with a cilium, then Definition 4.3 assigns an H-(co)module structure on $H^{\otimes E}$ to each vertex (face) of Γ. By Lemma 4.5 these (co)module structures commute and hence combine into $H^{\otimes E}$-module and $H^{\otimes F}$-comodule structures on $H^{\otimes E}$.

Definition 4.6. *The $H^{\otimes n}$-module structure for a subset $\emptyset \neq \mathcal{V} := \{v_1, \ldots, v_n\} \subset V$ and the $H^{\otimes m}$-comodule structure for a subset $\emptyset \neq \mathcal{F} := \{f_1, \ldots, f_m\} \subset F$ are*

$$\triangleright_{\mathcal{V}} := \triangleright_{v_1} \circ (1_H \otimes \triangleright_{v_2}) \circ \cdots \circ (1_{H^{\otimes(n-2)}} \otimes \triangleright_{v_{n-1}}) \circ (1_{H^{\otimes(n-1)}} \otimes \triangleright_{v_n}) : H^{\otimes n} \otimes H^{\otimes E} \to H^{\otimes E}, \quad (27)$$
$$\delta_{\mathcal{F}} := (1_{H^{\otimes(m-1)}} \otimes \delta_{f_m}) \circ (1_{H^{\otimes(m-2)}} \otimes \delta_{f_{m-1}}) \circ \cdots \circ (1_H \otimes \delta_{f_2}) \circ \delta_{f_1} : H^{\otimes E} \to H^{\otimes m} \otimes H^{\otimes E}.$$

Equations (24) and (25) ensure that the (co)actions do not depend on the numbering of vertices or faces in Definition 4.6. That (27) defines an $H^{\otimes n}$-module structure follows from the identity

$$\triangleright_{\mathcal{V}'} \circ (1_{H^{\otimes|\mathcal{V}'|}} \otimes \triangleright_v) \circ (\tau_{H,H^{\otimes|\mathcal{V}'|}} \otimes 1_{H^{\otimes E}}) = \triangleright_v \circ (1_H \otimes \triangleright_{\mathcal{V}'}),$$

valid for any subset $\emptyset \neq \mathcal{V}' \subset V$, $v \in V \setminus \mathcal{V}'$. The dual statement for $\delta_{\mathcal{F}}$ follows analogously.

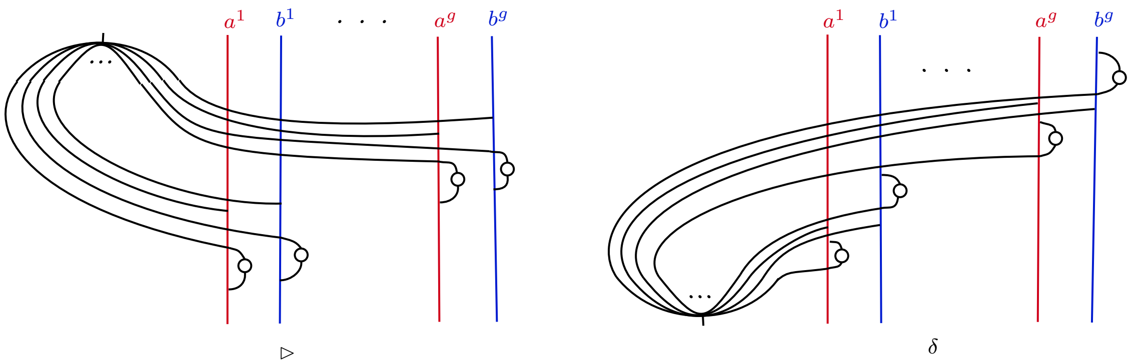
The module and comodule structure from Definition 4.6 define the categorical counterpart of the *protected space* or *ground state* in Kitaev's quantum double model. In the models based on a finite-dimensional semisimple complex Hopf algebra in [Ki, BMCA] the ground state is an eigenspace of a Hamiltonian that combines these H-(co)module structures. The normalised Haar integral defines a projector on the ground state. In our setting these structures are not available. Instead, we consider the biinvariants from Definition 2.13 for the action and coaction from (27).

Definition 4.7. *The* **protected object** *for an involutive Hopf monoid H and a ciliated ribbon graph Γ are the biinvariants $M_{inv} = \mathrm{Im}(\pi \circ \iota)$ of $H^{\otimes E}$ with the module structure $\triangleright_V$ and comodule structure δ_F from* (27)*.*

Overall, we say the *Kitaev model* constructed from a ciliated ribbon graph Γ and H consists of the *extended object* $H^{\otimes E} \in \mathrm{Ob}(\mathcal{C})$, the *triangle actions* $\triangleright_{\alpha\pm}$ and *coactions* $\delta_{\alpha\pm}$ from Definition 4.1 assigned to any edge $\alpha \in E$ and the *vertex actions* $\triangleright_v$ and *face coactions* δ_f from Definition 4.3 assigned to any ciliated vertex v and ciliated face f as well as the *protected object* M_{inv} from Definition 4.7.

Example 4.8. *Let H be an involutive Hopf monoid in $\mathcal{C}$ and Γ the standard graph* (21) *on a surface Σ of genus $g \geq 1$. Then the associated Yetter-Drinfeld module structure on $H^{\otimes E}$ is*

$$\begin{aligned} &h \triangleright (a^1 \otimes b^1 \otimes \ldots \otimes a^g \otimes b^g) \\ &\quad = h_{(3)} a^1 S(h_{(1)}) \otimes h_{(4)} b^1 S(h_{(2)}) \otimes \ldots \otimes h_{(4g-1)} a^g S(h_{(4g-3)}) \otimes h_{(4g)} b^g S(h_{(4g-2)}) \\ &\delta(a^1 \otimes b^1 \otimes \ldots \otimes a^g \otimes b^g) \\ &\quad = S(b^g_{(3)}) a^g_{(1)} b^g_{(1)} S(a^g_{(3)}) \cdots S(b^1_{(3)}) a^1_{(1)} b^1_{(1)} S(a^1_{(3)}) \otimes a^1_{(2)} \otimes b^1_{(2)} \otimes \ldots \otimes a^g_{(2)} \otimes b^g_{(2)}. \end{aligned} \tag{28}$$



If H is a group object in a cartesian monoidal category, this reduces to

$$\begin{aligned} h \triangleright (a_1, b_1, \ldots, a_g, b_g) &= (h a_1 h^{-1}, h b_1 h^{-1}, \ldots, h a_g h^{-1}, h b_g h^{-1}) \\ \delta(a_1, b_1, \ldots, a_g, b_g) &= ([b_g^{-1}, a_g] \cdots [b_1^{-1}, a_1], a_1, b_1, \ldots, a_g, b_g). \end{aligned} \tag{29}$$

The following remark shows how one can reformulate the coinvariance condition for the standard graph of the torus. It can be verified by a direct Hopf monoid computation.

Remark 4.9. *For the torus, the coinvariance under the comodule structure from Example 4.8 can be expressed as follows. Let H be an involutive Hopf monoid in $\mathcal{C}$ and $\delta : H^{\otimes 2} \to H \otimes H^{\otimes 2}$ with*

$$\delta(a \otimes b) = S(b_{(3)}) a_{(1)} b_{(1)} S(a_{(3)}) \otimes a_{(2)} \otimes b_{(2)}$$

the comodule structure associated to $H^{\otimes 2}$ and the standard graph of the torus as in Example 4.8. Then

$$\delta(a \otimes b) = 1 \otimes a \otimes b \quad \textit{if and only if} \quad b_{(2)}a_{(2)} \otimes a_{(1)} \otimes b_{(1)} = a_{(1)}b_{(1)} \otimes a_{(2)} \otimes b_{(2)}.$$

In the quantum double models for a finite-dimensional semisimple complex Hopf algebra it is directly apparent that imposing (co)invariance under all individual (co)actions at the vertices (faces) of a graph is the same as imposing (co)invariance under the combined action in Definition 4.6. In this setting the (co)invariants for the individual (co)actions are linear subspaces of $H^{\otimes E}$ and the (co)invariants of the combined (co)actions their intersections. In our setting an analogous statement follows from the universal properties of the coequaliser $\pi_{\mathcal{V}} : H^{\otimes E} \to M^H_{\mathcal{V}}$ for the action $\rhd_{\mathcal{V}}$ and the equaliser $\iota_{\mathcal{F}} : M^{coH}_{\mathcal{F}} \to H^{\otimes E}$ for the coaction $\delta_{\mathcal{F}}$, as given in Definition 2.11.

Lemma 4.10. *Let $\emptyset \neq \mathcal{V} \subset V$, $\emptyset \neq \mathcal{F} \subset F$ be subsets.*

1. *For any subset $\emptyset \neq \mathcal{V}' \subset \mathcal{V}$ the morphism $\pi_{\mathcal{V}} : H^{\otimes E} \to M^H_{\mathcal{V}}$ satisfies*

$$\pi_{\mathcal{V}} \circ \rhd_{\mathcal{V}'} = \pi_{\mathcal{V}} \circ (\epsilon^{|\mathcal{V}'|} \otimes 1_{H^{\otimes E}}). \tag{30}$$

There is a unique morphism $\chi_{\mathcal{V}',\mathcal{V}} : M^H_{\mathcal{V}'} \to M^H_{\mathcal{V}}$ with $\chi_{\mathcal{V}',\mathcal{V}} \circ \pi_{\mathcal{V}'} = \pi_{\mathcal{V}}$. It is an epimorphism.

2. *For any subset $\emptyset \neq \mathcal{F}' \subset \mathcal{F}$ the morphism $\iota_{\mathcal{F}} : M^{coH}_{\mathcal{F}} \to H^{\otimes E}$ satisfies*

$$\delta_{\mathcal{F}'} \circ \iota_{\mathcal{F}} = (\eta^{|\mathcal{F}'|} \otimes 1_{H^{\otimes E}}) \circ \iota_{\mathcal{F}}. \tag{31}$$

There is a unique morphism $\xi_{\mathcal{F}',\mathcal{F}} : M^{coH}_{\mathcal{F}} \to M^{coH}_{\mathcal{F}'}$ with $\iota_{\mathcal{F}'} \circ \xi_{\mathcal{F}',\mathcal{F}} = \iota_{\mathcal{F}}$. It is a monomorphism.

Proof. We prove 1., as 2. is the dual statement. It suffices to verify (30) for $\mathcal{V} = \{v_1, \ldots, v_n\}$, $\mathcal{V}' = \{v_j\}$, and the claim follows by induction over $|\mathcal{V}'|$. For this note first that Definition 4.6 implies

$$\rhd_{\mathcal{V}} \circ (\eta^{\otimes (j-1)} \otimes 1_H \otimes \eta^{\otimes (n-j)} \otimes 1_{H^{\otimes E}}) = \rhd_{v_j} \qquad \forall j \in \{1, \ldots, n\}. \tag{32}$$

As $\pi_{\mathcal{V}}$ is the coequaliser of $\rhd_{\mathcal{V}}$ and $\epsilon^{\otimes n} \otimes 1_{H^{\otimes E}}$ one obtains

$$\begin{aligned}\pi_{\mathcal{V}} \circ \rhd_{v_j} &\overset{(32)}{=} \pi_{\mathcal{V}} \circ \rhd_{\mathcal{V}} \circ (\eta^{\otimes (j-1)} \otimes 1_H \otimes \eta^{\otimes (n-j)} \otimes 1_{H^{\otimes E}}) \\ &= \pi_{\mathcal{V}} \circ (\epsilon^{\otimes n} \otimes 1_{H^{\otimes E}}) \circ (\eta^{\otimes (j-1)} \otimes 1_H \otimes \eta^{\otimes (n-j)} \otimes 1_{H^{\otimes E}}) \;=\; \pi_{\mathcal{V}} \circ (\epsilon \otimes 1_{H^{\otimes E}}).\end{aligned}$$

Equation (30) and the universal property of the coequaliser $\pi_{\mathcal{V}'}$ imply the existence of a unique morphism $\chi_{\mathcal{V}',\mathcal{V}} : M^H_{\mathcal{V}'} \to M^H_{\mathcal{V}}$ with $\chi_{\mathcal{V}',\mathcal{V}} \circ \pi_{\mathcal{V}'} = \pi_{\mathcal{V}}$. For any two morphisms $q_1, q_2 : M^H_{\mathcal{V}} \to X$ with $q_1 \circ \chi_{\mathcal{V}',\mathcal{V}} = q_2 \circ \chi_{\mathcal{V}',\mathcal{V}}$ one has $q_1 \circ \chi_{\mathcal{V}',\mathcal{V}} \circ \pi_{\mathcal{V}'} = q_1 \circ \pi_{\mathcal{V}} = q_2 \circ \pi_{\mathcal{V}} = q_2 \circ \chi_{\mathcal{V}',\mathcal{V}} \circ \pi_{\mathcal{V}'}$. As $\pi_{\mathcal{V}}$ is a coequaliser and hence an epimorphism, this implies $q_1 = q_2$, and $\chi_{\mathcal{V}',\mathcal{V}}$ is an epimorphism. □

It is also directly apparent from Definition 4.6 that (co)module morphisms with respect to all individual (co)module structures at vertices and faces in $\mathcal{V}$ and $\mathcal{F}$ are also (co)module morphisms with respect to the (co)actions $\rhd_{\mathcal{V}}$ and $\delta_{\mathcal{F}}$. More precisely, for ciliated ribbon graphs Γ, Γ', subsets $\emptyset \neq \mathcal{V} \subset V$, $\emptyset \neq \mathcal{V}' \subset V'$ and a bijection $\varphi : \mathcal{V} \to \mathcal{V}'$, $v \mapsto v'$, any morphism $g : H^{\otimes E} \to H^{\otimes E'}$ that is a module morphism with respect to $\rhd_v$ and $\rhd_{v'}$ for all $v \in V$ is also a module morphism with respect to $\rhd_{\mathcal{V}}$ and $\rhd_{\mathcal{V}'}$. An analogous statement holds for $\delta_{\mathcal{F}}$ and comodule morphisms.

5 Graph independence

In this chapter, we show that the protected object from Definition 4.7 is a topological invariant: Although its definition requires a ciliated ribbon graph Γ, its isomorphism class depends only on the homeomorphism class of the surface obtained by attaching discs to the faces of Γ.

To prove this, we show first in Section 5.1 that the (co)invariants associated to the (co)module structures at the vertices (faces) of Γ depend neither on the edge orientation nor on the choices of the cilia. Reversing the orientation of edges and different choices of cilia yield isomorphisms between these (co)invariants and hence also between the biinvariants. We then show in Section 5.2 and 5.3 that edge slides, edge contractions and removals of isolated loops induce isomorphisms between the protected objects, although not necessarily between the (co)invariants. In Section 5.4 we combine these results to obtain topological invariance and treat some simple examples.

Up to some additions and minor changes this chapter is part of the preprint [HM].

Note that in the following we use the fact that one can assume an arbitrary order of the different copies of H in $H^{\otimes E}$, as $\mathcal{C}$ is symmetric monoidal. We occasionally suppress factors in $H^{\otimes E}$ that are not affected by certain morphisms. Additionally, we sometimes denote by $a, b, c, \ldots$ both edge ends and their corresponding edges.

As in Chapter 4 we consider a complete and finitely cocomplete symmetric monoidal category $\mathcal{C}$, an involutive Hopf monoid H in $\mathcal{C}$ and a ciliated ribbon graph Γ.

5.1 Edge orientation reversal and moving the cilium

As edge orientation reversal switches the start and target and the left and right side of an edge $\alpha \in E$, it exchanges the associated actions $\triangleright_{\alpha\pm}$ and coactions $\delta_{\alpha\pm}$ from Definition 4.1. It is directly apparent from their definitions that this is achieved by applying the antipode.

Definition 5.1. *The automorphism of $H^{\otimes E}$ associated to the* **reversal** *of an edge $\alpha \in E$ is $S_\alpha : H^{\otimes E} \to H^{\otimes E}$.*

Lemma 5.2. *For any ciliated vertex $v \in V$, ciliated face $f \in F$ and edge $\beta \in E$ the edge reversal S_β is an isomorphism of H-modules and H-comodules with respect to $\triangleright_v$ and δ_f.*

Proof. We denote by $\triangleright'_v$ and δ'_f the module and comodule structure in the graph where the orientation of β is reversed and verify that $\triangleright'_v \circ (1_H \otimes S_\beta) = S_\beta \circ \triangleright_v$ and $\delta'_f \circ S_\beta = (1_H \otimes S_\beta) \circ \delta_f$. If β is not incident at v and f, the copy of H in $H^{\otimes E}$ assigned to β is not affected by $\triangleright_v, \triangleright'_v$ and δ_f, δ'_f, and the identity follows directly. If β is incident at v or f, it follows from the expressions for the (co)actions in Definitions 4.1 and 4.3. □

As a direct consequence of Lemma 5.2, Lemma 2.12 and Lemma 2.17 one has

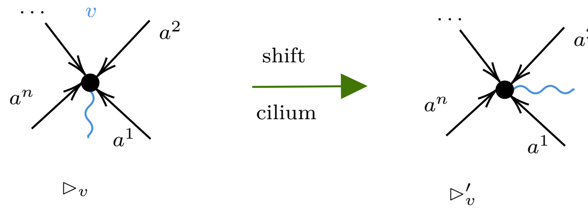
Corollary 5.3. *Reversing the orientation of an edge in Γ to obtain Γ' induces isomorphisms between the invariants, coinvariants and protected objects of Γ and Γ'.*

Lemma 5.4. *The (co)invariants for the H-(co)module structure at a given vertex (face) do not depend on the choice of cilia: moving the position of the cilium yields isomorphic (co)invariants. This induces isomorphisms of the protected objects.*

Proof. We focus on the H-module structure and its invariants. We consider a fixed position of the cilium at a vertex v with associated vertex action $\triangleright_v$ and coequaliser $\pi_v : H^{\otimes E} \to M_v^H$ and compare it to the action $\triangleright'_v$ and coequaliser $\pi'_v : H^{\otimes E} \to M_v'^H$ obtained by rotating the cilium counterclockwise by one position. We first show that the coequaliser $\pi_v : H^{\otimes E} \to M_v^H$ satisfies

$$\pi_v \circ \triangleright'_v = \pi_v \circ (\epsilon \otimes 1_{H^{\otimes E}}). \tag{33}$$

By definition of the H-module structure $\triangleright_v$ and by Lemma 5.2 it is sufficient to prove this for a vertex with n incoming edges. The computations for vertices with incident loops are analogous.



For a vertex with n incoming edges we have

$$\begin{aligned}
&\pi_v \circ \triangleright'_v (h \otimes a^1 \otimes a^2 \otimes \ldots \otimes a^n) \;=\; \pi_v\,(h_{(n)}a^1 \otimes h_{(1)}a^2 \otimes \ldots \otimes h_{(n-1)}a^n) \\
&= \pi_v\,(h_{(2)_{(1)}} S(h_{(1)}) h_{(3)} a^1 \otimes h_{(2)_{(2)}} a^2 \otimes \ldots \otimes h_{(2)_{(n)}} a^n) \\
&= \pi_v \circ \triangleright_v (h_{(2)} \otimes S(h_{(1)}) h_{(3)} a^1 \otimes a^2 \otimes \ldots \otimes a^n) = \pi_v\,(\epsilon(h_{(2)}) \otimes S(h_{(1)}) h_{(3)} a^1 \otimes a^2 \otimes \ldots \otimes a^n) \\
&= \pi_v \circ (\epsilon \otimes 1_{H^{\otimes n}})\,(h \otimes a^1 \otimes a^2 \otimes \ldots \otimes a^n),
\end{aligned}$$

where we used first the definition of $\triangleright'_v$, then the defining property of the antipode and that $S \circ S = 1_H$, then the definition of $\triangleright_v$, the fact that π_v coequalises $\triangleright_v$ and $\epsilon \otimes 1_{H^{\otimes E}}$ and then again the defining properties of the antipode and the counitality of H.

Inductively, we obtain (33) for all positions of the cilium at v and the same identity with $\pi_v, \triangleright_v$ and $\pi'_v, \triangleright'_v$ swapped. With the universal property of the coequalisers π_v, π'_v this yields unique morphisms $\phi : M_v^H \to M_v'^H$, $\phi' : M_v'^H \to M_v^H$ with $\phi \circ \pi_v = \pi'_v$ and $\phi' \circ \pi'_v = \pi_v$. As π_v, π'_v are epimorphisms, this implies $\phi' = \phi^{-1}$.

The dual claim for the comodule structure and its coinvariants follows analogously. For all positions of the cilium at f with associated coaction δ'_f, there is a unique morphism $\psi : M^{coH} \to M'^{coH}$ with $\iota'_f \circ \psi = \iota_f$, and ψ is an isomorphism. Combining these statements for the (co)invariants of all vertices (faces) and using Lemmas 2.17 and 4.10 yields isomorphisms of the protected objects. □

5.2 Edge slides and edge contractions

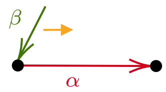
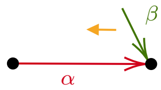
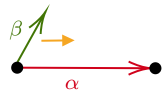
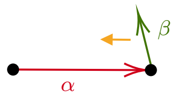
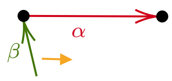
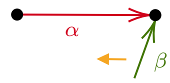
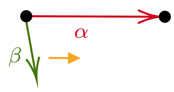
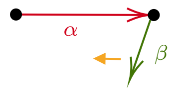
We now consider the edge slides and edge contractions from Definition 3.5. Edge slides were already investigated in [MV], where it was shown that they define mapping class group actions. They yield automorphisms of the object $H^{\otimes E}$ that are morphisms of H-modules and H-comodules as long as no edge ends slide over cilia.

Definition 5.5. *[MV, Def. 6.1]*
Let $\alpha \neq \beta$ be edges of Γ with the starting end of α directly before the target end of β in the ordering at $s(\alpha) = t(\beta)$. The **edge slide** *of the target end of β along α corresponds to the isomorphism*

$$S_{\alpha,\beta} := \triangleright_{\beta+} \circ \delta_{\alpha+} : H^{\otimes E} \to H^{\otimes E} \;\; \textit{with} \;\; S^{-1}_{\alpha,\beta} = \triangleright_{\beta+} \circ (S \otimes 1_{H^{\otimes E}}) \circ \delta_{\alpha+} : H^{\otimes E} \to H^{\otimes E}.$$

Edge slides for other edge orientations are defined by reversing edge orientations with the antipode.

Remark 5.6. *There are eight possible cases of edge slides. The end of β can either slide along the left or right of α, from $s(\alpha)$ to $t(\alpha)$ or from $t(\alpha)$ to $s(\alpha)$. The end of β can either be $s(\beta)$ or $t(\beta)$. This yields the following cases:*

a) left: $t(\beta)$ to $t(\alpha)$  $S_{\alpha,\beta}(\alpha \otimes \beta) = \alpha_{(2)} \otimes \alpha_{(1)}\beta$	b) left: $t(\beta)$ to $s(\alpha)$  $S_{\alpha,\beta}(\alpha \otimes \beta) = \alpha_{(2)} \otimes S(\alpha_{(1)})\beta$
c) left: $s(\beta)$ to $t(\alpha)$  $S_{\alpha,\beta}(\alpha \otimes \beta) = \alpha_{(2)} \otimes \beta S(\alpha_{(1)})$	d) left: $s(\beta)$ to $s(\alpha)$  $S_{\alpha,\beta}(\alpha \otimes \beta) = \alpha_{(2)} \otimes \beta\alpha_{(1)}$
e) right: $t(\beta)$ to $t(\alpha)$  $S_{\alpha,\beta}(\alpha \otimes \beta) = \alpha_{(1)} \otimes \alpha_{(2)}\beta$	f) right: $t(\beta)$ to $s(\alpha)$  $S_{\alpha,\beta}(\alpha \otimes \beta) = \alpha_{(1)} \otimes S(\alpha_{(2)})\beta$
g) right: $s(\beta)$ to $t(\alpha)$  $S_{\alpha,\beta}(\alpha \otimes \beta) = \alpha_{(1)} \otimes \beta S(\alpha_{(2)})$	h) right: $s(\beta)$ to $s(\alpha)$  $S_{\alpha,\beta}(\alpha \otimes \beta) = \alpha_{(1)} \otimes \beta\alpha_{(2)}$

Situation a) and its inverse b) are given in Definition 5.5. Case c) is obtained by first reversing the edge orientation of β, applying the edge slide from a) and then reversing the orientation of β. In the same way its inverse d) arises from b). Reversing the orientation of α before and after applying b) yields e). Its inverse f) is derived from a) in an analogous way. By additionally reversing the orientation of β, one obtains g) and its inverse h).

By construction, edge slides affect only the two copies of H in $H^{\otimes E}$ of the edges involved in the slide and commute with edge orientation reversals. Moreover, they respect the module and comodule structures at vertices and faces and hence induce isomorphisms between the protected objects.

Proposition 5.7. *[MV, Prop. 6.2]*
Let v and f be a ciliated vertex and face in a ribbon graph Γ with associated H-module structure $\rhd_v$ and H-comodule structure δ_f. Any edge slide that does not slide edge ends over their cilia is an isomorphism of H-left modules and H-left comodules with respect to $\rhd_v$ and δ_f.

Corollary 5.8.
Edge slides from a ribbon graph Γ to a ribbon graph Γ' induce isomorphisms between the invariants, coinvariants and protected objects of Γ and Γ'.

Proof. For edge slides that do not slide edge ends over cilia, this follows directly from Lemmas 2.12, 2.17 and Proposition 5.7. If an edge end slides over a cilium, we can apply Lemma 5.4 to move the cilium and obtain the same result. □

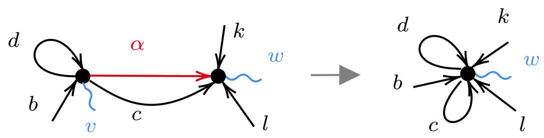
We now consider edge contractions. Recall from Definition 3.5 that an edge $\alpha \in E$ may only be contracted if its starting and target vertex differ and that contracting α towards $v \in \{s(\alpha), t(\alpha)\}$ erases the cilium at v, while the cilium at the other vertex is preserved.

Definition 5.9. *The morphism $c_{\alpha,v} : H^{\otimes E} \to H^{\otimes(E-1)}$ induced by an* **edge contraction** *of an edge α towards $v \in \{s(\alpha), t(\alpha)\}$ is*

$$c_{\alpha,v} = \begin{cases} \rhd_{v,\alpha} \circ \tau_\alpha \circ S_\alpha & \text{if } v = t(\alpha) \\ \rhd_{v,\alpha} \circ \tau_\alpha & \text{if } v = s(\alpha) \end{cases}$$

where $\rhd_{v,\alpha} : H^{\otimes E} \to H^{\otimes(E-1)}$ denotes the H-module structure from Definition 4.3 at v, where α is replaced by a cilium and τ_α is given before Definition 4.1. If v is univalent, then $c_{\alpha,v} = \epsilon_\alpha$.

Example 5.10. *Contracting the edge α towards v in*



gives the morphism $c_{\alpha,v}$ with

$$c_{\alpha,v}\,(\alpha \otimes b \otimes c \otimes d \otimes k \otimes l) = \alpha_{(3)} b \otimes c S(\alpha_{(4)}) \otimes \alpha_{(1)} d S(\alpha_{(2)}) \otimes k \otimes l.$$

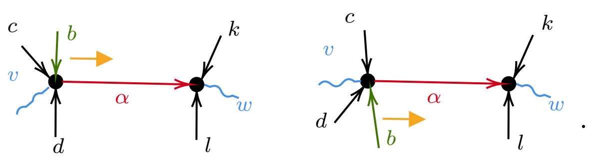
It follows directly from Definition 5.9 that first reversing the orientation of an edge β and then contracting it is the same as just contracting β. It also follows from Definitions 4.1 and 4.3 that reversing the orientation of an edge β commutes with contractions of all edges $\alpha \neq \beta$. The contraction of an edge α also commutes with edge slides along α, which allows one to express any edge contraction as a composite of edge slides and an edge contraction towards a univalent vertex.

Lemma 5.11. *Let Γ' be obtained by reversing an edge β in Γ. Then*

$$c'_{\beta,v} \circ S_\beta = c_{\beta,v} \qquad\qquad c'_{\alpha,v} \circ S_\beta = S_\beta \circ c_{\alpha,v} \quad \text{for } \alpha \neq \beta. \tag{34}$$

Lemma 5.12. *Contracting an edge α gives the same morphism as first sliding edge ends along α and then contracting α.*

Proof. It suffices to slide a single edge end along α, as the statement follows inductively. We denote by $c_{\alpha,v}$ the contraction of α in Γ and by $c'_{\alpha,v}$ the contraction of α in the graph Γ' obtained by sliding an edge b along α. Suppose that there are no loops incident at $s(\alpha)$ and $t(\alpha)$ in Γ and Γ'. As edge slides and edge contractions commute with edge reversals by Definition 5.5 and Lemma 5.11, respectively, we can assume $v = s(\alpha)$ and all other edge ends at v and $w = t(\alpha)$ are incoming. It is then sufficient to consider an edge slide of b along the left and right of α:



(35)

Omitting the copies of H for edges not incident at v, w we compute for the edge slides in (35)

$$\begin{aligned} c'_{\alpha,v} \circ S_{\alpha,b}(\alpha \otimes b \otimes c \otimes d \otimes k \otimes l) &= c'_{\alpha,v}(\alpha_{(2)} \otimes \alpha_{(1)} b \otimes c \otimes d \otimes k \otimes l) \\ &= \alpha_{(1)} b \otimes \alpha_{(2)} c \otimes \alpha_{(3)} d \otimes k \otimes l \;=\; c_{\alpha,v}(\alpha \otimes b \otimes c \otimes d \otimes k \otimes l), \\ c'_{\alpha,v} \circ S_{\alpha,b}(\alpha \otimes b \otimes c \otimes d \otimes k \otimes l) &= c'_{\alpha,v}(\alpha_{(1)} \otimes \alpha_{(2)} b \otimes c \otimes d \otimes k \otimes l) \\ &= \alpha_{(3)} b \otimes \alpha_{(1)} c \otimes \alpha_{(2)} d \otimes k \otimes l \;=\; c_{\alpha,v}(\alpha \otimes b \otimes c \otimes d \otimes k \otimes l). \end{aligned}$$

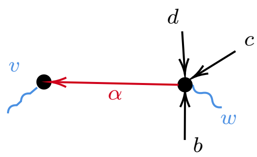
As edge slides from w to v are the inverses of edge slides from v to w, the corresponding identities for those follow by pre-composing with the inverses. The proof for vertices with different numbers of incident edge ends or incident loops is analogous. □

Next, we consider the interaction of edge contractions with the (co)module structures for the vertices (faces) of the graph. For this, note that the contraction of an edge α towards $v \in \{s(\alpha), t(\alpha)\}$ defines a bijection between the sets F, F' of faces before and after the contraction and likewise a bijection between the sets $V \setminus \{v\}$ and V'. If faces and vertices are identified via these bijections, the edge contraction becomes a (co)module morphism. In contrast, the module structure $\rhd_v$ is coequalised.

Lemma 5.13. *The contraction of an edge α towards a ciliated vertex v coequalises $\rhd_v$ and $\epsilon \otimes 1_{H\otimes E}$ and is a (co)module morphism with respect to the (co)actions $\rhd_z$ and δ_f for all ciliated vertices $z \neq v$ and ciliated faces $f \in F$ that do not start at v:*

$$c_{\alpha,v} \circ \rhd_v = c_{\alpha,v} \circ (\epsilon \otimes 1_{H\otimes E}), \tag{36}$$
$$c_{\alpha,v} \circ \rhd_z = \rhd'_z \circ (1_H \otimes c_{\alpha,v}), \tag{37}$$
$$\delta'_f \circ c_{\alpha,v} = (1_H \otimes c_{\alpha,v}) \circ \delta_f. \tag{38}$$

Proof. As edge slides along α are module and comodule isomorphisms by Proposition 5.7 and commute with the contraction of α by Lemma 5.12, we can assume that v is univalent. With Lemma 5.11 we can assume that $v = t(\alpha)$ and that all edge ends at $w = s(\alpha)$ are incoming:



. (39)

For the vertices v and w in (39) we compute

$$\begin{aligned} c_{\alpha,v} \circ \rhd_v(h \otimes \alpha \otimes b \otimes c \otimes d) &= c_{\alpha,v}(h\alpha \otimes b \otimes c \otimes d) = \epsilon(h\alpha) b \otimes c \otimes d = c_{\alpha,v}(\epsilon(h)\alpha \otimes b \otimes c \otimes d) \\ &= c_{\alpha,v} \circ (\epsilon \otimes 1_{H\otimes E})(h \otimes \alpha \otimes b \otimes c \otimes d) \\ c_{\alpha,v} \circ \rhd_w(h \otimes \alpha \otimes b \otimes c \otimes d) &= c_{\alpha,v}(\alpha S(h_{(3)}) \otimes h_{(4)} b \otimes h_{(1)} c \otimes h_{(2)} d) \\ &= \epsilon(\alpha) h_{(3)} b \otimes h_{(1)} c \otimes h_{(2)} d = \rhd'_w \circ (1_H \otimes c_{\alpha,v})(h \otimes \alpha \otimes b \otimes c \otimes d). \end{aligned}$$

The computations for graphs with a different number of edge ends or loops incident at w are analogous. For vertices $z \in V \setminus \{v, w\}$ the action $\rhd_z$ does not affect the copy of H for α and commutes with $\rhd_{v,\alpha}$ and hence with $c_{v,\alpha}$. This proves (36) and (37).

If f is a face that contains α, but does not start at v, then the associated coaction is of the form

$$\delta_f(\alpha \otimes b \otimes c \otimes d \otimes \ldots) = (\cdots S(d_{(2)}) S(\alpha_{(3)}) \alpha_{(1)} b_{(1)} \cdots) \otimes \alpha_{(2)} \otimes b_{(2)} \otimes c \otimes d_{(1)} \otimes \ldots,$$

where the dots stand for contributions of parts of Γ that are not drawn in (39). This yields

$$\begin{aligned} &(1_H \otimes c_{\alpha,v}) \circ \delta_f(\alpha \otimes b \otimes c \otimes d \otimes \ldots) = \epsilon(\alpha_{(2)})(\cdots S(d_{(2)}) S(\alpha_{(3)}) \alpha_{(1)} b_{(1)} \cdots) \otimes b_{(2)} \otimes c \otimes d_{(1)} \otimes \ldots \\ &\overset{(6)}{=} \epsilon(\alpha)(\cdots S(d_{(2)}) b_{(1)} \cdots) \otimes b_{(2)} \otimes c \otimes d_{(1)} \otimes \ldots = \delta'_f \circ c_{\alpha,v}(\alpha \otimes b \otimes c \otimes d \otimes \ldots). \end{aligned}$$

If f does not contain α, the edge α does not contribute to the coaction δ_f, which proves (38). □

With these results, we investigate how edge contractions interact with the (co)invariants of the H-(co)module structures at ciliated vertices and faces of Γ. For subsets $\emptyset \neq \mathcal{V} \subset V$ and $\emptyset \neq \mathcal{F} \subset F$ we denote by $\rhd_{\mathcal{V}}$ and $\delta_{\mathcal{F}}$ the associated $H^{\otimes \mathcal{V}}$-module structure and $H^{\otimes \mathcal{F}}$-comodule structure from (27) and by $\pi_{\mathcal{V}}$ and $\iota_{\mathcal{F}}$ their invariants and coinvariants from Definition 2.11.

We then find that edge contractions send coinvariants for $\delta_{\mathcal{F}}$ to coinvariants for the corresponding face set in the contracted graph. The same holds for the invariants of the action $\rhd_{\mathcal{V}}$, as long as $\mathcal{V}$ contains the starting and target vertex of the contracted edge. The morphism η_α that creates a copy of H assigned to α by applying the unit of H is right inverse to the edge contraction $c_{v,\alpha}$ and a left inverse on the coinvariants. This corresponds to the following technical lemma.

Lemma 5.14. *Let Γ' be obtained from Γ by contracting an edge α incident at $v, w \in V$. Then $\eta_\alpha : H^{\otimes(E-1)} \to H^{\otimes E}$ is right inverse to the edge contraction $c_{\alpha,v} : H^{\otimes E} \to H^{\otimes(E-1)}$, and for all subsets $\{v, w\} \subset \mathcal{V} \subset V$, $\emptyset \neq \mathcal{F} \subset F$ one has*

$$\delta'_{\mathcal{F}} \circ c_{\alpha,v} \circ \iota_{\mathcal{F}} = (\eta^{\otimes|\mathcal{F}|} \otimes c_{\alpha,v}) \circ \iota_{\mathcal{F}} \tag{40}$$

$$\delta_{\mathcal{F}} \circ \eta_\alpha \circ \iota'_{\mathcal{F}} = (\eta^{\otimes|\mathcal{F}|} \otimes \eta_\alpha) \circ \iota'_{\mathcal{F}} \tag{41}$$

$$\pi_{\mathcal{V}} \circ \eta_\alpha \circ c_{\alpha,v} = \pi_{\mathcal{V}} \tag{42}$$

$$\pi'_{\mathcal{V}} \circ c_{\alpha,v} \circ \rhd_{\mathcal{V}} = \pi'_{\mathcal{V}} \circ (\epsilon^{\otimes|\mathcal{V}|} \otimes c_{\alpha,v}) \tag{43}$$

$$\pi_{\mathcal{V}} \circ \eta_\alpha \circ \rhd'_{\mathcal{V}} = \pi_{\mathcal{V}} \circ (\epsilon^{\otimes|\mathcal{V}|-1} \otimes \eta_\alpha). \tag{44}$$

Proof. 1. It follows directly from Definition 5.9 that the morphism η_α is a right inverse to $c_{\alpha,v}$. From the formula for the (co)action in Definition 4.3 it is apparent that η_α is a comodule morphism for the coactions δ_f at all ciliated faces and a module morphism with respect to the actions $\rhd_z$ at all vertices $z \in V \setminus \{v, w\}$. Moreover, it is clear from Definition 5.5 that sliding edge ends over α after applying η_α yields a morphism η''_α which splits the vertex w in a different way. Thus, we have

$$c_{\alpha,v} \circ \eta_\alpha = 1_{H^{\otimes E}}, \quad \delta_f \circ \eta_\alpha = \eta_\alpha \circ \delta'_f, \quad \eta_\alpha \circ \rhd'_z = \rhd_z \circ \eta_\alpha, \quad S_{\alpha,\beta} \circ \eta_\alpha = \eta''_\alpha \tag{45}$$

for all vertices $z \in V \setminus \{v, w\}$ and faces $f \in F$ and edge slides $S_{\alpha,\beta}$ along α. We can therefore assume that the vertex $v = t(\alpha)$ is univalent, all edge ends at $w = s(\alpha)$ are incoming, the graph Γ is locally given by (39) and the edge contraction by $c_{\alpha,v} = \epsilon_\alpha$, as in the proof of Lemma 5.13.

2. We prove the auxiliary identities

$$\pi_v \circ \eta_\alpha \circ c_{\alpha,v} = \pi_v, \tag{46}$$

$$\pi_{\{v,w\}} \circ \eta_\alpha \circ (\epsilon \otimes 1_{H^{\otimes(E-1)}}) = \pi_{\{v,w\}} \circ \eta_\alpha \circ \rhd'_w, \tag{47}$$

$$\delta'_f \circ c_{\alpha,v} \circ \iota_f = (\eta \otimes 1_{H^{\otimes(E-1)}}) \circ c_{\alpha,v} \circ \iota_f \qquad \forall f \in F. \tag{48}$$

Omitting all copies of H in $H^{\otimes E}$ except the one for α, we verify (46)

$$\pi_v(\alpha \otimes \ldots) = \pi_v \circ (\alpha \rhd_v)(1 \otimes \ldots) = \pi_v(\epsilon(\alpha)\, 1 \otimes \ldots) = \pi_v \circ \eta_\alpha \circ \epsilon_\alpha(1 \otimes \ldots) = \pi_v \circ \eta_\alpha \circ c_{\alpha,v}(1 \otimes \ldots).$$

To show (47), we consider the graph (39) and compute with Lemma 4.10

$$\begin{aligned}
&\pi_{\{v,w\}} \circ \eta_\alpha \circ \rhd'_w(h \otimes b \otimes c \otimes d) = \pi_{\{v,w\}} \circ \eta_\alpha(h_{(3)}b \otimes h_{(1)}c \otimes h_{(2)}d) = \pi_{\{v,w\}}(1 \otimes h_{(3)}b \otimes h_{(1)}c \otimes h_{(2)}d) \\
&= \pi_{\{v,w\}}(h_{(3)}S(h_{(4)}) \otimes h_{(5)}b \otimes h_{(1)}c \otimes h_{(2)}d) = \pi_{\{v,w\}} \circ h_{(3)} \rhd_v (S(h_{(4)}) \otimes h_{(5)}b \otimes h_{(1)}c \otimes h_{(2)}d) \\
&= \pi_{\{v,w\}}(\epsilon(h_{(3)})\, S(h_{(4)}) \otimes h_{(5)}b \otimes h_{(1)}c \otimes h_{(2)}d) = \pi_{\{v,w\}}(S(h_{(3)}) \otimes h_{(4)}b \otimes h_{(1)}c \otimes h_{(2)}d) \\
&= \pi_{\{v,w\}} \circ h \rhd_w (1 \otimes b \otimes c \otimes d) = \pi_{\{v,w\}}(\epsilon(h)\, 1 \otimes b \otimes c \otimes d) = \pi_{\{v,w\}} \circ \eta_\alpha \circ (\epsilon \otimes 1_{H^{\otimes(E-1)}})(h \otimes b \otimes c \otimes d).
\end{aligned}$$

Identity (48) follows from identity (38) in Lemma 5.13 for all faces $f \in F$ that do not start at v. If f starts at v one has for the graph in (39)

$$\begin{aligned}
&\delta'_f \circ c_{\alpha,v}(\alpha \otimes b \otimes c \otimes d) = \epsilon(\alpha)\delta'_f(b \otimes c \otimes d) = \epsilon(\alpha) b_{(1)} \cdots S(d_{(2)}) \otimes b_{(2)} \otimes c \otimes d_{(1)} \\
&= S(\alpha_{(2)})\alpha_{(1)} b_{(1)} \cdots S(d_{(2)}) S(\alpha_{(4)})\alpha_{(3)} \otimes b_{(2)} \otimes c \otimes d_{(1)} \\
&= (\lhd_{ad} \otimes 1_{H^{\otimes(E-1)}}) \circ (1_H \otimes \tau_\alpha) \circ \delta_f(\alpha \otimes b \otimes c \otimes d),
\end{aligned}$$

where $\lhd_{ad} : H \otimes H \to H$, $h \otimes \alpha \mapsto S(\alpha_{(1)}) h \alpha_{(2)}$. In this case, contracting α deletes the cilium of f, but Lemma 5.4 allows one to place a new cilium for f in any position. As $\lhd_{ad} \circ (\eta \otimes 1_H) = \eta \circ \epsilon : H \to H$ this yields

$$\begin{aligned}
&\delta'_f \circ c_{\alpha,v} \circ \iota_f = (\lhd_{ad} \otimes 1_{H^{\otimes(E-1)}}) \circ (1_H \otimes \tau_\alpha) \circ \delta_f \circ \iota_f = (\lhd_{ad} \otimes 1_{H^{\otimes(E-1)}}) \circ (1_H \otimes \tau_\alpha) \circ (\eta \otimes 1_{H^{\otimes E}}) \circ \iota_f \\
&= ((\eta \circ \epsilon) \otimes 1_{H^{\otimes(E-1)}}) \circ \tau_\alpha \circ \iota_f = (\eta \otimes 1_{H^{\otimes(E-1)}}) \circ \epsilon_\alpha \circ \iota_f = (\eta \otimes 1_{H^{\otimes(E-1)}}) \circ c_{\alpha,v} \circ \iota_f.
\end{aligned}$$

3. We prove the identities in the Lemma. Identity (40) follows by pre-composing (48) with the morphism $\xi_{f,\mathcal{F}} := \xi_{\{f\},\mathcal{F}}$ from Lemma 4.10 and inductively applying this equation for all $f \in \mathcal{F}$. Likewise, identity (41) follows by applying the identity $\delta_f \circ \eta_\alpha \circ \iota'_f = (\eta \otimes 1_{H^{\otimes E}}) \circ \eta_\alpha \circ \iota'_f$ obtained from the second identity in (45) and pre-composing it with $\xi'_{f,\mathcal{F}}$. Post-composing (46) with the morphism $\chi_{v,\mathcal{V}} := \chi_{\{v\},\mathcal{V}}$ from Lemma 4.10 yields (42). From (37), we obtain for all $z \in V \setminus \{v, w\}$

$$\pi'_{\mathcal{V}} \circ c_{\alpha,v} \circ \rhd_z = \chi'_{z,\mathcal{V}} \circ \pi'_z \circ c_{\alpha,v} \circ \rhd_z = \chi'_{z,\mathcal{V}} \circ \pi'_z \circ \rhd'_z \circ (1_H \otimes c_{\alpha,v}) = \pi'_{\mathcal{V}} \circ (\epsilon \otimes c_{\alpha,v}). \tag{49}$$

Together with the identity $\pi'_{\mathcal{V}} \circ c_{\alpha,v} \circ \rhd_w \circ (1_H \otimes \rhd_v) = \pi'_{\mathcal{V}} \circ (\epsilon^{\otimes 2} \otimes c_{\alpha,v})$, which follows from (36) and (37) with $z = w$ and the identity $\pi'_{\mathcal{V}} = \chi'_{w,\mathcal{V}} \circ \pi'_w$, this yields (43). Identity (44) follows by post-composing (47) with $\chi_{\{v,w\},\mathcal{V}}$ and the third identity in (45) with $\pi_{\mathcal{V}} = \chi_{z,\mathcal{V}} \circ \pi_z$. □

We now apply Lemma 5.14 to show that edge contractions induce morphisms between the coinvariants for $\emptyset \neq \mathcal{F} \subset F$. If $\mathcal{V}$ contains the starting and target vertex of the contracted edge, they also induce isomorphisms between the invariants and isomorphisms between the protected objects.

For this, we consider a ciliated ribbon graph Γ and the graph Γ' obtained by contracting an edge α in Γ. We denote by $\mathcal{M}^{coH}$, $\mathcal{M}^H$, $\mathcal{M}_{inv}$ the coinvariants, invariants and biinvariants of $\delta_{\mathcal{F}}$, $\rhd_{\mathcal{V}}$ for Γ and by $\mathcal{M}'^{coH}$, $\mathcal{M}'^H$, $\mathcal{M}'_{inv}$ the corresponding quantities for Γ'. As in Lemma 4.10 we write $\iota_{\mathcal{F}}$ and $\pi_{\mathcal{V}}$ for the associated equaliser and coequaliser and $I : \mathcal{M}_{inv} \to \mathcal{M}^H$ and $P : \mathcal{M}^{coH} \to \mathcal{M}_{inv}$ for the monomorphism and epimorphism that characterise $\mathcal{M}_{inv}$ as the image of $\pi_{\mathcal{V}} \circ \iota_{\mathcal{F}}$. The corresponding morphisms for Γ' are denoted $\iota'_{\mathcal{F}}$, $\pi'_{\mathcal{V}}$, I' and P'.

Proposition 5.15. *Let Γ' be obtained from a ciliated ribbon graph Γ by contracting an edge α incident at v, w towards v. Then for all $\{v, w\} \subset \mathcal{V} \subset V$, $\emptyset \neq \mathcal{F} \subset F$ the contraction of α induces*

- *a morphism $u : M^{coH}_{\mathcal{F}} \to M'^{coH}_{\mathcal{F}}$ with a right inverse that satisfies $\iota'_{\mathcal{F}} \circ u = c_{\alpha,v} \circ \iota_{\mathcal{F}}$,*
- *an isomorphism $r : M^H_{\mathcal{V}} \to M'^H_{\mathcal{V}}$ that satisfies $r \circ \pi_{\mathcal{V}} = \pi'_{\mathcal{V}} \circ c_{\alpha,v}$,*
- *an isomorphism $\phi_{inv} : M_{inv} \to M'_{inv}$ with $I = r^{-1} \circ I' \circ \phi_{inv}$.*

Proof. Using equation (40) together with the universal property of the equaliser $\iota'_{\mathcal{F}}$ yields a unique morphism $u : M^{coH}_{\mathcal{F}} \to M'^{coH}_{\mathcal{F}}$ with $\iota'_{\mathcal{F}} \circ u = c_{\alpha,v} \circ \iota_{\mathcal{F}}$. Equation (41) and the equaliser $\iota_{\mathcal{F}}$ yield a unique morphism $u^{-1} : M'^{coH}_{\mathcal{F}} \to M^{coH}_{\mathcal{F}}$ with $\iota_{\mathcal{F}} \circ u^{-1} = \eta_\alpha \circ \iota'_{\mathcal{F}}$. To show that u^{-1} is a right inverse of u note that $\iota'_{\mathcal{F}} \circ u \circ u^{-1} = c_{\alpha,v} \circ \iota_{\mathcal{F}} \circ u^{-1} = c_{\alpha,v} \circ \eta_\alpha \circ \iota'_{\mathcal{F}} = \iota'_{\mathcal{F}}$, since η_α is right inverse to $c_{\alpha,v}$. As $\iota'_{\mathcal{F}}$ is a monomorphism, this implies $u \circ u^{-1} = 1_{M'^{coH}_{\mathcal{F}}}$.

Analogously, (43) and the universal property of the coequaliser $\pi_{\mathcal{V}}$ define a unique morphism $r : M^H_{\mathcal{V}} \to M'^H_{\mathcal{V}}$ with $r \circ \pi_{\mathcal{V}} = \pi'_{\mathcal{V}} \circ c_{\alpha,v}$. The coequaliser $\pi'_{\mathcal{V}}$ together with (44) yields a unique morphism $r^{-1} : M'^H_{\mathcal{V}} \to M^H_{\mathcal{V}}$ with $r^{-1} \circ \pi'_{\mathcal{V}} = \pi_{\mathcal{V}} \circ \eta_\alpha$. The morphisms r and r^{-1} are mutually inverse isomorphisms, since $\pi'_{\mathcal{V}}$, $\pi_{\mathcal{V}}$ are epimorphisms with

$$\begin{aligned} r \circ r^{-1} \circ \pi'_{\mathcal{V}} &= r \circ \pi_{\mathcal{V}} \circ \eta_\alpha = \pi'_{\mathcal{V}} \circ c_{\alpha,v} \circ \eta_\alpha = \pi'_{\mathcal{V}}, \\ r^{-1} \circ r \circ \pi_{\mathcal{V}} &= r^{-1} \circ \pi'_{\mathcal{V}} \circ c_{\alpha,v} = \pi_{\mathcal{V}} \circ \eta_\alpha \circ c_{\alpha,v} \overset{(42)}{=} \pi_{\mathcal{V}}. \end{aligned}$$

Hence, we constructed commuting diagrams

$$\begin{array}{ccccc} M^{coH}_{\mathcal{F}} & \xrightarrow{\iota_{\mathcal{F}}} & M & \xrightarrow{\pi_{\mathcal{V}}} & M^H_{\mathcal{V}} \\ \downarrow u & & \downarrow c_{\alpha,v} & & \downarrow r \\ M'^{coH}_{\mathcal{F}} & \xrightarrow[\iota'_{\mathcal{F}}]{} & M' & \xrightarrow[\pi'_{\mathcal{V}}]{} & M'^H_{\mathcal{V}} \end{array} \qquad \begin{array}{ccccc} M'^{coH}_{\mathcal{F}} & \xrightarrow{\iota'_{\mathcal{F}}} & M' & \xrightarrow{\pi'_{\mathcal{V}}} & M'^H_{\mathcal{V}} \\ \downarrow u^{-1} & & \downarrow \eta_\alpha & & \downarrow r^{-1} \\ M^{coH}_{\mathcal{F}} & \xrightarrow[\iota_{\mathcal{F}}]{} & M & \xrightarrow[\pi_{\mathcal{V}}]{} & M^H_{\mathcal{V}}. \end{array}$$

To construct the isomorphism ϕ_{inv}, we set $j := r^{-1} \circ I' : M'_{inv} \to M^H_{\mathcal{V}}$ and $q := P' \circ u : M^{coH}_{\mathcal{F}} \to M'_{inv}$. As r^{-1} is an isomorphism and I' a monomorphism, the morphism j is a monomorphism. The composite $j \circ q$ satisfies

$$j \circ q = r^{-1} \circ I' \circ P' \circ u = r^{-1} \circ \pi'_{\mathcal{V}} \circ \iota'_{\mathcal{F}} \circ u = \pi_{\mathcal{V}} \circ \eta_\alpha \circ \iota'_{\mathcal{F}} \circ u = \pi_{\mathcal{V}} \circ \eta_\alpha \circ c_{\alpha,v} \circ \iota_{\mathcal{F}} \overset{(42)}{=} \pi_{\mathcal{V}} \circ \iota_{\mathcal{F}}.$$

The universal property of the image M_{inv} then yields a unique morphism $\phi_{inv} : M_{inv} \to M'_{inv}$ with $I = j \circ \phi_{inv} = r^{-1} \circ I' \circ \phi_{inv}$. To construct its inverse we set $j' := r \circ I : M_{inv} \to M'^H_{\mathcal{V}}$ and $q' := P \circ u^{-1} : M'^{coH}_{\mathcal{F}} \to M_{inv}$. As r is an isomorphism and I a monomorphism, j' is a monomorphism, and we have

$$j' \circ q' = r \circ I \circ P \circ u^{-1} = r \circ \pi_{\mathcal{V}} \circ \iota_{\mathcal{F}} \circ u^{-1} = \pi'_{\mathcal{V}} \circ c_{\alpha,v} \circ \iota_{\mathcal{F}} \circ u^{-1} = \pi'_{\mathcal{V}} \circ c_{\alpha,v} \circ \eta_\alpha \circ \iota'_{\mathcal{F}} = \pi'_{\mathcal{V}} \circ \iota'_{\mathcal{F}},$$

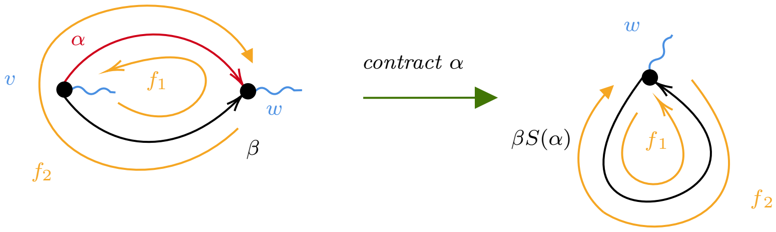
where we applied that η_α is right inverse to $c_{\alpha,v}$ in the last step. By the universal property of the image M'_{inv} there is a unique morphism $\phi^{-1}_{inv} : M'_{inv} \to M_{inv}$ with $I' = j' \circ \phi^{-1}_{inv} = r \circ I \circ \phi^{-1}_{inv}$ and

$$\begin{aligned} I \circ \phi^{-1}_{inv} \circ \phi_{inv} &= r^{-1} \circ r \circ I \circ \phi^{-1}_{inv} \circ \phi_{inv} = r^{-1} \circ I' \circ \phi_{inv} = I, \\ I' \circ \phi_{inv} \circ \phi^{-1}_{inv} &= r \circ r^{-1} \circ I' \circ \phi_{inv} \circ \phi^{-1}_{inv} = r \circ I \circ \phi^{-1}_{inv} = I'. \end{aligned}$$

As I, I' are monomorphisms, it follows that ϕ_{inv} and ϕ^{-1}_{inv} are mutually inverse isomorphisms. □

Corollary 5.16. *Edge contractions induce isomorphisms between protected objects.*

Remark 5.17. *In general, edge contractions do not induce isomorphisms on the coinvariants. Consider for example a group $H \neq \{e\}$ as a Hopf monoid in $\mathcal{C} = \mathrm{Set}$ and the following graph, where the edge α is contracted towards v.*



The comodule structure of the left graph is given by

$$\delta_F := (1_H \times \delta_{f_2}) \circ \delta_{f_1} : H^{\times 2} \to H^{\times 4},$$
$$(a,b) \mapsto (a^{-1}b, ab^{-1}, a, b),$$

hence the coinvariants are the set $M^{coH} = \{(a,b) \in H^{\times 2} : a = b\} \cong H$. *After the contraction it is* $\delta'_{F'}(b) = (b, b^{-1}, b)$ *and the coinvariants are trivial:*

$$M'^{coH} = \{b \in H : b = b^{-1} = e\} = \{e\}.$$

5.3 Deleting isolated loops

We now consider the next graph transformation from Definition 3.5, the deletion of isolated loops. The morphism associated to the deletion of an isolated loop α applies the counit to the corresponding copy of the Hopf monoid H. Just as edge contractions, this is in general not an isomorphism in $\mathcal{C}$. The morphism η_α that creates a copy of H for α by applying the unit is a right inverse and corresponds to inserting a loop.

Definition 5.18. *The morphism induced by* **deleting an isolated loop** α *is* $\epsilon_\alpha : H^{\otimes E} \to H^{\otimes E \setminus \{\alpha\}}$.

As for edge contractions we investigate how these morphisms interact with the coinvariants for the $H^{\otimes \mathcal{F}}$-comodule structure $\delta_{\mathcal{F}}$ and the $H^{\otimes \mathcal{V}}$-module structure $\rhd_{\mathcal{V}}$ from Definition 4.6 for subsets $\emptyset \neq \mathcal{F} \subset F$ and $\emptyset \neq \mathcal{V} \subset V$. We find that loop deletions send the invariants for $\rhd_{\mathcal{V}}$ to invariants for the corresponding vertex set of the graph with the loop removed. The same holds for coinvariants of $\delta_{\mathcal{F}}$, as long as the two faces incident to the loop are contained in $\mathcal{F}$. Analogous statements hold for the right inverse η_α, and on the coinvariants η_α is also a left inverse. This is a consequence of the following technical lemma.

Lemma 5.19. *Let* Γ^+ *be obtained from a ciliated ribbon graph* Γ *by removing an isolated loop* α *with adjacent faces* f_1, f_2 *at a vertex* v. *Then for all subsets* $\emptyset \neq \mathcal{V} \subset V$ *and* $\{f_1, f_2\} \subset \mathcal{F} \subset F$

$$\pi^+_{\mathcal{V}} \circ \epsilon_\alpha \circ \rhd_{\mathcal{V}} = \pi^+_{\mathcal{V}} \circ (\epsilon^{\otimes |\mathcal{V}|} \otimes \epsilon_\alpha) \tag{50}$$
$$\delta^+_{\mathcal{F}} \circ \epsilon_\alpha \circ \iota_{\mathcal{F}} = (\eta^{\otimes |\mathcal{F}|-1} \otimes \epsilon_\alpha) \circ \iota_{\mathcal{F}} \tag{51}$$
$$\eta_\alpha \circ \epsilon_\alpha \circ \iota_{\mathcal{F}} = \iota_{\mathcal{F}} \tag{52}$$
$$\pi_{\mathcal{V}} \circ \eta_\alpha \circ \rhd^+_{\mathcal{V}} = \pi_{\mathcal{V}} \circ (\epsilon^{\otimes |\mathcal{V}|} \otimes \eta_\alpha) \tag{53}$$
$$\delta_{\mathcal{F}} \circ \eta_\alpha \circ \iota^+_{\mathcal{F}} = (\eta^{\otimes |\mathcal{F}|} \otimes \eta_\alpha) \circ \iota^+_{\mathcal{F}}. \tag{54}$$

Proof. 1. We first prove some auxiliary identities for the interaction of the morphisms ϵ_α and η_α with the module and comodules structures at the vertices and faces.

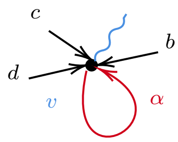
1.(a) As η_α and ϵ_α affect only the copy of H for α, we have for any vertex $z \neq v$ and any ciliated face f that does not contain α

$$\epsilon_\alpha \circ \rhd_z = \rhd^+_z \circ (1_H \otimes \epsilon_\alpha) \qquad \eta_\alpha \circ \rhd^+_z = \rhd_z \circ (1_H \otimes \eta_\alpha) \tag{55}$$
$$(1_H \otimes \epsilon_\alpha) \circ \delta_f = \delta^+_f \circ \epsilon_\alpha \qquad (1_H \otimes \eta_\alpha) \circ \delta^+_f = \delta_f \circ \eta_\alpha. \tag{56}$$

1.(b) For the H-module structure at the vertex v, we show that

$$\pi^+_v \circ \epsilon_\alpha \circ \rhd_v = \pi^+_v \circ (\epsilon \otimes \epsilon_\alpha) \tag{57}$$
$$\pi_v \circ \eta_\alpha \circ \rhd^+_v = \pi_v \circ (\epsilon \otimes \eta_\alpha). \tag{58}$$

As reversing edge orientations commutes with $\epsilon_\alpha, \eta_\alpha$ and $\triangleright_v$, we can assume that all edges $\beta \neq \alpha$ at v are incoming and that $s(\alpha)$ is directly before $t(\alpha)$ with respect to the cyclic ordering at v:



For this graph we compute

$$\begin{aligned}
&\pi_v^+ \circ \epsilon_\alpha \circ \triangleright_v (h \otimes \alpha \otimes b \otimes c \otimes d) = \pi_v^+ \circ \epsilon_\alpha (h_{(4)} \alpha S(h_{(3)}) \otimes h_{(5)} b \otimes h_{(1)} c \otimes h_{(2)} d) \\
&= \pi_v^+ (\epsilon(\alpha) \otimes h_{(3)} b \otimes h_{(1)} c \otimes h_{(2)} d) = \pi_v^+ \circ (\epsilon \otimes \triangleright_v^+) (\alpha \otimes h \otimes b \otimes c \otimes d) \\
&= \pi_v^+ \circ (\epsilon \otimes \epsilon \otimes 1_{H^{\otimes 3}}) (h \otimes \alpha \otimes b \otimes c \otimes d) = \pi_v^+ \circ (\epsilon \otimes \epsilon_\alpha) (h \otimes \alpha \otimes b \otimes c \otimes d), \\
&\pi_v \circ \eta_\alpha \circ \triangleright_v^+ (h \otimes b \otimes c \otimes d) = \pi_v \circ \eta_\alpha (h_{(3)} b \otimes h_{(1)} c \otimes h_{(2)} d) \\
&= \pi_v (h_{(4)} S(h_{(3)}) \otimes h_{(5)} b \otimes h_{(1)} c \otimes h_{(2)} d) = \pi_v \circ \triangleright_v (h \otimes 1 \otimes b \otimes c \otimes d) \\
&= \pi_v \circ (\epsilon \otimes 1_{H^{\otimes 4}}) (h \otimes 1 \otimes b \otimes c \otimes d) = \pi_v \circ (\epsilon \otimes \eta_\alpha) (h \otimes b \otimes c \otimes d),
\end{aligned}$$

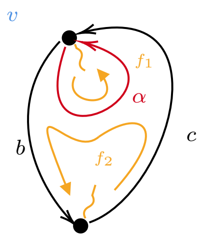
which proves (57) and (58). The computation for graphs with a different number of edge ends at v are analogous. The claim for the case where the cilium is between the edge ends of α follows, because the invariants of the H-module structure at v do not depend on the choice of the cilium by Lemma 5.4. It can also be verified directly by analogous computations.

1.(c) We consider the H-comodule structures at the faces f_1, f_2. Under the assumption that the starting end of α comes directly before its target end with respect to the cyclic ordering at v, one of these faces coincides with α, and we assume it is $f_1 = \alpha$. We then have

$$\delta_{f_1} \circ \eta_\alpha = \eta \otimes \eta_\alpha \tag{59}$$

$$\delta_{f_2} \circ \eta_\alpha = (1_H \otimes \eta_\alpha) \circ \delta_{f_2}^+ \qquad \delta_{f_2} \circ \eta_\alpha \circ \iota_{f_2}^+ = (\eta \otimes \eta_\alpha) \circ \iota_{f_2}^+. \tag{60}$$

Equation (59) is obvious, and to prove (60), we can assume that Γ is locally given by



(61)

as edge reversals commute with the module and comodule structures at the vertices and faces and with the morphisms η_α and ϵ_α. We then compute

$$\delta_{f_2} \circ \eta_\alpha(b \otimes c) = \delta_{f_2}(1 \otimes b \otimes c) = b_{(1)} c_{(1)} \otimes 1 \otimes b_{(2)} \otimes c_{(2)} = (1_H \otimes \eta_\alpha) \circ \delta_{f_2}^+(b \otimes c).$$

The computations for graphs with different numbers of edges in f_2 are analogous, and with the identity $\delta_{f_2}^+ \circ \iota_{f_2}^+ = (\eta \otimes 1_{H^{\otimes(E-1)}}) \circ \iota_{f_2}^+$ we obtain the second identity in (60).

2. We prove the identities (50) to (54). To show (52) it is sufficient to consider the graph (61) with

$$\begin{aligned}
\delta_{f_1}(\alpha \otimes b \otimes c) &= \alpha_{(1)} \otimes \alpha_{(2)} \otimes b \otimes c \\
\delta_{f_2}(\alpha \otimes b \otimes c) &= b_{(1)} S(\alpha_{(2)}) c_{(1)} \otimes \alpha_{(1)} \otimes b_{(2)} \otimes c_{(2)}.
\end{aligned}$$

As $f_1, f_2 \in \mathcal{F}$, this yields

$$\iota_\mathcal{F} = ((\epsilon \circ \eta) \otimes 1_{H\otimes E}) \circ \iota_\mathcal{F} = (\epsilon \otimes 1_{H\otimes E}) \circ \delta_{f_2} \circ \iota_\mathcal{F} \overset{(*)}{=} \eta_\alpha \circ \epsilon_\alpha \circ \iota_\mathcal{F},$$

where we apply in $(*)$ the coinvariance under δ_{f_1}.

Identity (50) follows inductively from the identity $\pi^+_\mathcal{V} \circ \epsilon_\alpha \circ \rhd_z = \pi^+_\mathcal{V} \circ (\epsilon \otimes \epsilon_\alpha)$ for all vertices $z \in V$, which is obtained for $z \neq v$ by post-composing the first identity in (55) with $\pi^+_\mathcal{V} = \chi^+_{z,\mathcal{V}} \circ \pi^+_z$ and for $z = v$ by post-composing (57) with $\chi_{v,\mathcal{V}}$.

Identity (51) follows from (52) and the first identities in (56), (60), which yield for all $f \in \mathcal{F} \setminus \{f_1\}$

$$\begin{aligned}\delta^+_f \circ \epsilon_\alpha \circ \iota_\mathcal{F} = (1_H \otimes (\epsilon_\alpha \circ \eta_\alpha)) \circ \delta^+_f \circ \epsilon_\alpha \circ \iota_\mathcal{F} &\overset{(56),(60)}{=} (1_H \otimes \epsilon_\alpha) \circ \delta_f \circ \eta_\alpha \circ \epsilon_\alpha \circ \iota_\mathcal{F}\\ &\overset{(52)}{=} (1_H \otimes \epsilon_\alpha) \circ \delta_f \circ \iota_\mathcal{F} = (1_H \otimes \epsilon_\alpha) \circ \delta_f \circ \iota_f \circ \xi_{f,\mathcal{F}} = (\eta \otimes \epsilon_\alpha) \circ \iota_\mathcal{F}.\end{aligned}$$

Identity (53) follows inductively by applying the identity $\pi_\mathcal{V} \circ \eta_\alpha \circ \rhd^+_z = \pi_\mathcal{V} \circ (\epsilon \otimes \eta_\alpha)$ for $z \in V$, obtained by post-composing (55) with $\pi_\mathcal{V} = \chi_{z,\mathcal{V}} \circ \pi_z$ for $z \neq v$ and (58) with $\chi_{v,\mathcal{V}}$ for $z = v$.

Analogously, (54) follows from the identity $\delta_f \circ \eta_\alpha \circ \iota^+_\mathcal{F} = (\eta \otimes \eta_\alpha) \circ \iota^+_\mathcal{F}$ for $f \in F$, which is obtained for $f = f_1$ by pre-composing (59) with $\iota^+_\mathcal{F}$, for $f = f_2$ by pre-composing (60) with $\xi^+_{f_2,\mathcal{F}}$ and for $f \notin \{f_1, f_2\}$ by pre-composing (56) with $\iota^+_\mathcal{F} = \iota^+_f \circ \xi^+_{f,\mathcal{F}}$. □

We now apply Lemma 5.19 to show that loop deletions induce morphisms between the invariants of $\rhd_\mathcal{V}$ for subsets $\emptyset \neq \mathcal{V} \subset V$. If $\mathcal{F}$ contains the two faces adjacent to the loop, they also induce isomorphisms between the coinvariants of $\delta_\mathcal{F}$ and isomorphisms between the protected objects.

For this we denote by Γ^+ the graph obtained by deleting a loop α in Γ. For Γ we use the notation from Proposition 5.15. For Γ^+ we denote by $\mathcal{M}^{+coH}$, $\mathcal{M}^{+H}$, $\mathcal{M}^+_{inv}$ the coinvariants, invariants and biinvariants of $\delta^+_\mathcal{F}$, $\rhd^+_\mathcal{V}$, by $\iota^+_\mathcal{F}$ and $\pi^+_\mathcal{V}$ the associated equaliser and coequaliser and by $I^+ : \mathcal{M}^+_{inv} \to \mathcal{M}^{+H}$ and $P^+ : \mathcal{M}^{+coH} \to \mathcal{M}^+_{inv}$ the monomorphism and epimorphism that characterise $\mathcal{M}^+_{inv}$ as the image of $\pi^+_\mathcal{V} \circ \iota^+_\mathcal{F}$.

Proposition 5.20. *Let Γ^+ be obtained from Γ by removing an isolated loop α with incident faces f_1, f_2. Then for all subsets $\emptyset \neq \mathcal{V} \subset V$, $\{f_1, f_2\} \subset \mathcal{F} \subset F$ the loop removal induces*

- *an isomorphism $y : M^{coH}_\mathcal{F} \to M^{+coH}_\mathcal{F}$ with $\iota^+_\mathcal{F} \circ y = \epsilon_\alpha \circ \iota_\mathcal{F}$,*
- *a morphism $t : M^H_\mathcal{V} \to M^{+H}_\mathcal{V}$ with a right inverse and $t \circ \pi_\mathcal{V} = \pi^+_\mathcal{V} \circ \epsilon_\alpha$*
- *an isomorphism $\psi_{inv} : M_{inv} \to M^+_{inv}$ with $I = t^{-1} \circ I^+ \circ \psi_{inv}$.*

Proof. Equation (51) and the universal property of the equaliser $\iota^+_\mathcal{F}$ yield a unique morphism $y : M^{coH}_\mathcal{F} \to M^{+coH}_\mathcal{F}$ with $\iota^+_\mathcal{F} \circ y = \epsilon_\alpha \circ \iota_\mathcal{F}$. The universal property of the equaliser $\iota_\mathcal{F}$ and (54) provide a unique morphism $y^{-1} : M^{+coH}_\mathcal{F} \to M^{coH}_\mathcal{F}$ with $\iota_\mathcal{F} \circ y^{-1} = \eta_\alpha \circ \iota^+_\mathcal{F}$. The two morphisms are inverse to each other, as $\iota_\mathcal{F}$, $\iota^+_\mathcal{F}$ are monomorphisms and

$$\begin{aligned}\iota^+_\mathcal{F} \circ y \circ y^{-1} &= \epsilon_\alpha \circ \iota_\mathcal{F} \circ y^{-1} = \epsilon_\alpha \circ \eta_\alpha \circ \iota^+_\mathcal{F} = \iota^+_\mathcal{F}\\ \iota_\mathcal{F} \circ y^{-1} \circ y &= \eta_\alpha \circ \iota^+_\mathcal{F} \circ y = \eta_\alpha \circ \epsilon_\alpha \circ \iota_\mathcal{F} \overset{(52)}{=} \iota_\mathcal{F}.\end{aligned}$$

Similarly, equation (50) and the universal property of the coequaliser $\pi_\mathcal{V}$ yield a unique morphism $t : M^H_\mathcal{V} \to M^{+H}_\mathcal{V}$ with $t \circ \pi_\mathcal{V} = \pi^+_\mathcal{V} \circ \epsilon_\alpha$ and equation (53) with the universal property of the coequaliser $\pi^+_\mathcal{V}$ a unique morphism $t^{-1} : M^{+H}_\mathcal{V} \to M^H_\mathcal{V}$ with $t^{-1} \circ \pi^+_\mathcal{V} = \pi_\mathcal{V} \circ \eta_\alpha$. The morphism t^{-1} is a right inverse of t, since $\pi^+_\mathcal{V}$ is an epimorphism and

$$t \circ t^{-1} \circ \pi^+_\mathcal{V} = t \circ \pi_\mathcal{V} \circ \eta_\alpha = \pi^+_\mathcal{V} \circ \epsilon_\alpha \circ \eta_\alpha = \pi^+_\mathcal{V}.$$

We have constructed commuting diagrams

$$\begin{array}{ccccc} M_{\mathcal{F}}^{coH} & \xrightarrow{\iota_{\mathcal{F}}} & M & \xrightarrow{\pi_{\mathcal{V}}} & M_{\mathcal{V}}^{H} \\ \downarrow y & & \downarrow \epsilon_\alpha & & \downarrow t \\ M_{\mathcal{F}}^{+coH} & \xrightarrow[\iota_{\mathcal{F}}^{+}]{} & M^{+} & \xrightarrow[\pi_{\mathcal{V}}^{+}]{} & M_{\mathcal{V}}^{+H} \end{array} \qquad \begin{array}{ccccc} M_{\mathcal{F}}^{+coH} & \xrightarrow{\iota_{\mathcal{F}}^{+}} & M^{+} & \xrightarrow{\pi_{\mathcal{V}}^{+}} & M_{\mathcal{V}}^{+H} \\ \downarrow y^{-1} & & \downarrow \eta_\alpha & & \downarrow t^{-1} \\ M_{\mathcal{F}}^{coH} & \xrightarrow[\iota_{\mathcal{F}}]{} & M & \xrightarrow[\pi_{\mathcal{V}}]{} & M_{\mathcal{V}}^{H}. \end{array}$$

Up to minor changes the construction of $\psi_{inv} : M_{inv} \to M_{inv}^{+}$ and its inverse is analogous to the construction of ϕ_{inv} in the proof of Proposition 5.15. More concretely, from the monomorphisms $j = t^{-1} \circ I^{+}$, $j^{+} = t \circ I$ and epimorphisms $q = P^{+} \circ y$, $q^{+} = P \circ y^{-1}$ the morphisms $\psi_{inv} : M_{inv} \to M_{inv}^{+}$ and $\psi_{inv}^{-1} : M_{inv}^{+} \to M_{inv}$ are constructed. They satisfy $I = t^{-1} \circ I^{+} \circ \psi_{inv}$ and $I^{+} = t \circ I \circ \psi_{inv}^{-1}$. In contrast to the morphism r from Proposition 5.15 the morphism t is no isomorphism. Thus, we use a slightly different argument to show that j^{+} is a monomorphism: For $x_1, x_2 \in \mathrm{Hom}(X, M_{inv})$ with $j^{+} \circ x_1 = j^{+} \circ x_2$ we can post-compose this equation with t^{-1} and replace $t^{-1} \circ j^{+}$ by

$$t^{-1} \circ j^{+} = t^{-1} \circ t \circ I = t^{-1} \circ t \circ t^{-1} \circ I^{+} \circ \psi^{-1} = t^{-1} \circ I^{+} \circ \psi_{inv} = I.$$

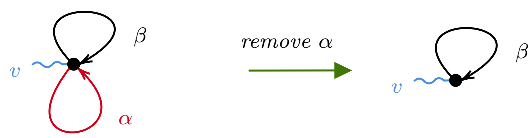
The equation $x_1 = x_2$ follows, as I is a monomorphism. After verifying $\psi_{inv} \circ \psi_{inv}^{-1} = 1_{M_{inv}^{+}}$ as in the proof of Proposition 5.15, one can use this equation to show $\psi_{inv}^{-1} \circ \psi_{inv} = 1_{M_{inv}}$:

$$I \circ \psi_{inv}^{-1} \circ \psi_{inv} = t^{-1} \circ I^{+} \circ \psi_{inv} \circ \psi_{inv}^{-1} \circ \psi_{inv} = t^{-1} \circ I^{+} \circ \psi_{inv} = I.$$

□

Corollary 5.21. *Deletions of isolated loops induce isomorphisms between the protected objects.*

Remark 5.22. *Deletions of isolated loops do in general not induce isomorphism on the invariants. One can for example consider a group H as a Hopf monoid in $\mathcal{C} = \mathrm{Set}$ and the following graph, where the isolated loop α is deleted.*



The module structures are given by $\triangleright : H^{\times 3} \to H^{\times 2}$, $h \triangleright (a, b) = (hah^{-1}, hbh^{-1})$ and by $h \triangleright' b = hbh^{-1}$ after the deletion. Hence the invariants are the orbit sets $M^{H} = \{[(a, b)] : (a, b) \in H \times H\}$ with $[(a, b)] = \{(hah^{-1}, hbh^{-1}) : h \in H\}$ and $M'^{H} = \{[b] : b \in H\}$ with $[b] = \{hbh^{-1} : h \in H\}$. For $H = S_3$ the first set has 11 elements, whereas the second has only 3 elements.

5.4 Protected objects

Combining the results from Section 5.1 to 5.3 one has that ciliated ribbon graphs related by moving cilia, edge reversals, edge contractions and deletions of isolated loops have isomorphic protected objects. As these are sufficient to relate any connected ribbon graph to the standard graph from (21), the protected object of a ciliated ribbon graph is determined up to isomorphisms by the genera of the connected components of the associated surface.

Theorem 5.23. *The isomorphism class of the protected object for an involutive Hopf monoid H and a ciliated ribbon graph Γ depends only on H and the homeomorphism class of the surface for Γ.*

Proof. By Lemma 5.4 the invariants, coinvariants and hence the protected object of a ciliated ribbon graph are independent of the choice of the cilia. By Proposition 3.6 every ribbon graph can be transformed into a disjoint union of standard graphs by edge reversals, edge contractions, edge slides and removing isolated loops. In each step the cilia can be arranged in such a way that no edge ends slide over cilia. By Corollaries 5.3, 5.8, 5.16 and 5.21 these graph transformations induce isomorphisms between the protected objects. □

As the protected object is a topological invariant, one can use any embedded graph whose complement is a disjoint union of discs to compute the protected object. For a sphere, the simplest such graph consists of a single isolated vertex. This is associated with the trivial H-(co)module structure on e given by the (co)unit of H and yields the tensor unit as protected object.

Example 5.24. *The protected object for a sphere S^2 is the tensor unit of H: $\mathcal{M}_{inv} = e$.*

We now focus on oriented surfaces Σ of genus $g \geq 1$ and use the standard graphs (21) to determine their protected objects. The associated module and comodule structures are given in Example 4.8 and form a Yetter-Drinfeld module.

Example 5.25. *For a group H as a Hopf monoid in $\mathcal{C} = \mathrm{Set}$ the coinvariants are the set of group homomorphisms from $\pi_1(\Sigma)$ to H*

$$M^{coH} = \{(a_1, b_1, \ldots, a_g, b_g) \in H^{\times 2g} : [b_g^{-1}, a_g] \cdot \ldots \cdot [b_1^{-1}, a_1] = 1\} \cong \mathrm{Hom}(\pi_1(\Sigma), H). \tag{62}$$

The invariants are the set of orbits for the conjugation action $\triangleright$ from (29) *on $H^{\times 2g}$, and the protected object is the* representation variety *or* moduli space of flat H-bundles $M_{inv} \cong \mathrm{Hom}(\pi_1(\Sigma), H)/H$.

Example 5.26. *For a topological group H as a Hopf monoid in $\mathcal{C} = \mathrm{Top}$ the protected object is $M_{inv} \cong \mathrm{Hom}(\pi_1(\Sigma), H)/H$ as a set by Example 2.15. It is equipped with the quotient topology induced by the canonical surjection $\pi : \mathrm{Hom}(\pi_1(\Sigma), H) \to \mathrm{Hom}(\pi_1(\Sigma), H)/H$ and the compact-open topology on $\mathrm{Hom}_{\mathrm{Top}}(\pi_1(\Sigma), H)$ for the discrete topology on $\pi_1(\Sigma)$.*

Example 5.27. *For a Hopf monoid H in $\mathcal{C} = G{-}\mathrm{Set} = \mathrm{Set}^{\mathrm{B}G}$ the coinvariants for the comodule structure δ from Example 4.8 are the set* (62) *with the diagonal G-action. The invariants for the module structure $\triangleright$ are the associated orbit space. By Example 2.15, 2. the protected object is the representation variety $M_{inv} \cong \mathrm{Hom}(\pi_1(\Sigma), H)/H$ with the induced G-set structure.*

Example 5.28. *Let k be a commutative ring, $\mathcal{C} = k$-Mod and G a finite group.*

For the group algebra $H = k[G]$ as a Hopf monoid in $\mathcal{C}$ and the standard graph in (21) *one has $M = k[G]^{2g} \cong k[G^{\times 2g}]$. The Yetter-Drinfeld module structure of M is given by* (29) *on a basis. The coinvariants and invariants are*

$$\begin{aligned} M^{coH} &= \langle\{(a_1, b_1, \ldots, a_g, b_g) \mid [b_g^{-1}, a_g] \cdots [b_1^{-1}, a_1] = 1\}\rangle_k \cong \langle \mathrm{Hom}(\pi_1(\Sigma), G)\rangle_k \\ M^H &= k[G^{\times 2g}]/\langle\{(a_1, \ldots, b_g) - (ha_1h^{-1}, \ldots, hb_gh^{-1}) \mid a_1, b_1, \ldots, a_g, b_g, h \in G\}\rangle, \end{aligned}$$

and the protected object is the free k-module generated by the representation variety $\mathrm{Hom}(\pi_1(\Sigma), G)/G$

$$M_{inv} = \langle \mathrm{Hom}(\pi_1(\Sigma), G)/G\rangle_k. \tag{63}$$

For the dual Hopf monoid $H = k[G]^* = \mathrm{Map}(G, k)$ of maps from G to k with Hopf monoid structure

$$\delta_g \cdot \delta_h = \delta_g(h)\delta_g, \quad 1 = \sum_{g \in G} \delta_g, \quad \Delta(\delta_g) = \sum_{x,y \in G, xy=g} \delta_x \otimes \delta_y, \quad \epsilon(\delta_g) = \delta_g(e), \quad S(\delta_g) = \delta_{g^{-1}} \tag{64}$$

one has $M = \mathrm{Map}(G,k)^{\otimes 2g} \cong \mathrm{Map}(G^{\times 2g},k)$ *with the Yetter-Drinfeld module structure*

$$\begin{aligned} \delta_h \rhd (\delta_{a_1} \otimes \delta_{b_1} \otimes \ldots \otimes \delta_{a_g} \otimes \delta_{b_g}) &= \delta_h([b_g^{-1},a_g]\cdots[b_1^{-1},a_1])\, \delta_{a_1} \otimes \delta_{b_1} \otimes \ldots \otimes \delta_{a_g} \otimes \delta_{b_g} \\ \delta(\delta_{a_1} \otimes \delta_{b_1} \otimes \ldots \otimes \delta_{a_g} \otimes \delta_{b_g}) &= \Sigma_{h\in G} \delta_{h^{-1}} \otimes \delta_{ha_1h^{-1}} \otimes \delta_{hb_1h^{-1}} \otimes \ldots \otimes \delta_{ha_gh^{-1}} \otimes \delta_{hb_gh^{-1}} \end{aligned} \tag{65}$$

computed from (28) *and* (64)*. It follows that the coinvariants and invariants are given by*

$$\begin{aligned} M^{coH} &= \mathrm{Map}(G^{\times 2g},k)^G \\ M^H &= \{f : G^{\times 2g} \to k \mid \mathrm{supp}(f) \subseteq \{(a_1,\ldots,b_g) \mid [b_g^{-1},a_g]\cdots[b_1^{-1},a_1] = 1\}\}, \end{aligned} \tag{66}$$

and the protected object is the set of functions

$$M_{inv} = \mathrm{Map}(\mathrm{Hom}(\pi_1(\Sigma),G)/G,k). \tag{67}$$

Example 5.28 shows that the protected object in this thesis indeed generalises the protected space of Kitaev's quantum double models. If one sets $k = \mathbb{C}$ in Example 5.28 one obtains precisely the protected space for Kitaev's quantum double model for the group algebra $\mathbb{C}[G]$ and its dual, see [Ki, Sec. 4, 5]. However, Example 5.28 also yields an analogous result for any commutative ring k, for which the usual quantum double models are not defined.

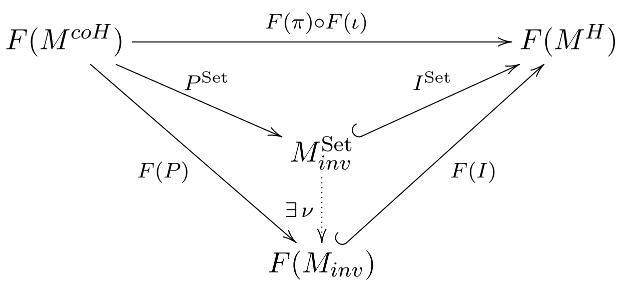
The question arises whether symmetric monoidal functors can transport protected objects to protected objects. The following Lemma provides cases in which protected objects are sent to the representation variety in Set from Example 5.25.

Lemma 5.29. *Let* $F : \mathcal{C} \to \mathrm{Set}$ *be a symmetric monoidal functor that preserves equalisers, coequalisers, monomorphisms and epimorphisms. Suppose* Σ *is an oriented surface of genus* $g \geq 1$*. Then the protected object satisfies*

$$F(M_{inv}) \cong \mathrm{Hom}(\pi_1(\Sigma),F(H))/F(H).$$

Proof. By Example 2.6, 1. the set $F(H)$ has a canonical Hopf monoid structure. The vertex action $\rhd$ and face coaction δ from Example 4.8 associated to the standard graph (21) are built from (co)multiplications, braidings and the antipode. Thus, $F(\rhd)$ and $F(\delta)$ are the vertex action and face coaction for the Hopf monoid $F(H)$ and the standard graph. As the functor F preserves (co)equalisers, $(F(M^{coH}),F(\iota))$ is the equaliser of $F(\delta)$ and $F(\eta \otimes 1_{H^{\otimes E}})$ and $(F(M^H),F(\pi))$ the coequaliser of $F(\rhd)$ and $F(\epsilon \otimes 1_{H^{\otimes E}})$. Thus, the protected object associated to $F(H)$ and Σ is given as the image $(M_{inv}^{\mathrm{Set}}, P^{\mathrm{Set}}, I^{\mathrm{Set}})$ of $F(\pi) \circ F(\iota)$.

Denote by $P : M^{coH} \to M_{inv}$ and $I : M_{inv} \to M^H$ the morphisms with $I \circ P = \pi \circ \iota$ that characterise the protected object M_{inv} in $\mathcal{C}$. As $\mathcal{C}$ has equalisers and F preserves monomorphisms and epimorphisms, P and $F(P)$ are epimorphisms and $F(I)$ is a monomorphism. The universal property of M_{inv}^{Set} implies that there exists a unique injective map $\nu : M_{inv}^{\mathrm{Set}} \to F(M_{inv})$ with $F(I) \circ \nu = I^{\mathrm{Set}}$.



In Set images and coimages coincide, compare Remark 2.14, 1. Thus, the universal property of the coimage M_{inv}^{Set} implies that there exists a unique surjective map $\nu^{-1} : F(M_{inv}) \to M_{inv}^{\mathrm{Set}}$ with $\nu^{-1} \circ F(P) = P^{\mathrm{Set}}$. As $F(P)$ is an epimorphism and $F(I)$ a monomorphism and

$$F(I) \circ \nu \circ \nu^{-1} \circ F(P) = I^{\mathrm{Set}} \circ P^{\mathrm{Set}} = F(I) \circ F(P),$$

we obtain $\nu \circ \nu^{-1} = 1_{F(M_{inv})}$. This implies $\nu \circ \nu^{-1} \circ \nu \circ \nu^{-1} = \nu \circ \nu^{-1}$. Applying that ν is injective and ν^{-1} surjective yields $\nu^{-1} \circ \nu = 1_{M_{inv}^{\mathrm{Set}}}$. By Example 5.25 the protected object M_{inv}^{Set} is given by

$$M_{inv}^{\mathrm{Set}} = \mathrm{Hom}(\pi_1(\Sigma), F(H))/F(H).$$

□

Note that in particular right adjoints preserve limits and monomorphisms, compare Riehl [R, Th. 4.5.2, Exercise 4.5.v]. Left adjoints preserve colimits and epimorphisms, see [R, Th. 4.5.3, Exercise 4.5.v]. This allows one to relate the protected objects in Top and Set.

Example 5.30. *1. The symmetric monoidal forgetful functor U : Top $\to$ Set is a left and right adjoint [R, Ex. 4.1.6]. By Example 5.26 the functor U applied on the protected object in* Top *gives the representation variety from Example 5.25.*

2. The object functor ob : Cat $\to$ Set *that sends a small category to its set of objects and a functor to its object map is a left and right adjoint, compare [R, Exercise 4.1.ii]. This implies that the protected object associated to an oriented surface of genus $g \geq 1$ and a group object in* Cat *is given by a representation variety on the objects. We will determine the protected objects in* Cat *in Section 8.6.*

6 Kitaev model and protected objects in Vect

The goal of this chapter is to compare the construction of the Kitaev models from Chapter 4 applied to $\mathcal{C} = \mathrm{Vect}_{\mathbb{F}}$ for a field $\mathbb{F}$ to the Kitaev model in [BMCA]. We particularly focus on the protected object and also outline how it can be described in $\mathrm{Vect}_{\mathbb{F}}$ using Haar integrals.

We consider a surface Σ of genus $g \geq 1$ and use the standard graph (21) to characterise the protected object.

Lemma 6.1. *Let H be a finite-dimensional semisimple involutive Hopf algebra with normalised Haar integrals $l \in H$, $\lambda \in H^*$ and $\triangleright : H \otimes H^{\otimes E} \to H^{\otimes E}$, $\delta : H^{\otimes E} \to H \otimes H^{\otimes E}$ the module and comodule structure from Example 4.8 for the standard graph (21). Then the protected object is given by*

$$\begin{aligned} M_{inv} &\cong \{m \in H^{\otimes E} : h \triangleright m = \epsilon(h)m \ \ \forall h \in H\} \cap \{m \in H^{\otimes E} : \delta(m) = 1 \otimes m\} \\ &= P_l(H^{\otimes E}) \cap P_\lambda(H^{\otimes E}), \end{aligned}$$

where

$$\begin{aligned} &P_l : H^{\otimes E} \to H^{\otimes E},\ m \mapsto l \triangleright m \ \ \text{and} \\ &P_\lambda : H^{\otimes E} \to H^{\otimes E},\ m \mapsto (\lambda \otimes 1_{H^{\otimes E}}) \circ \delta(m). \end{aligned}$$

Proof. The invariants M^H and coinvariants $M^{coH} = \{m \in H^{\otimes E} : \delta(m) = 1 \otimes m\}$ in $\mathrm{Vect}_{\mathbb{F}}$ are characterised in Example 2.15, 3. Combining this with Lemma 2.29 we obtain for the invariants

$$\begin{aligned} M^H &= H^{\otimes E}/\langle\{h \triangleright m - \epsilon(h)m : h \in H, m \in H^{\otimes E}\}\rangle \cong \{m \in H^{\otimes E} : h \triangleright m = \epsilon(h)m \forall h \in H\} \\ &= P_l(H^{\otimes E}), \end{aligned}$$

as H is semisimple. The biinvariants can be described as the intersection

$$M_{inv} \cong \{m \in H^{\otimes E} : h \triangleright m = \epsilon(h)m \ \ \forall h \in H\} \cap \{m \in H^{\otimes E} : \delta(m) = 1 \otimes m\}$$

with linear maps $P : M^{coH} \to M_{inv}$, $m \mapsto P_l(m)$ and $I : M_{inv} \to M^H$, $m \mapsto m$. For any linear map $Q : M^{coH} \to X$ and an injective linear map $J : X \to M^H$ with $J \circ Q(m) = P_l(m)$ for all $m \in M^{coH}$ there is an induced linear map $\nu : M_{inv} \to X$, $m \mapsto Q(m)$. For $m \in P_l(H^{\otimes E}) \cap M^{coH}$ one obtains

$$J \circ \nu(m) = J \circ Q(m) = P_l(m) = m.$$

If there is another linear map $\nu' : M_{inv} \to X$ with $J \circ \nu'(m) = m = J \circ \nu(m)$ for all $m \in P_l(H^{\otimes E}) \cap M^{coH}$, it directly follows $\nu' = \nu$, as J is injective. Hence, the universal property of the image is satisfied.

The left H-comodule structure $\delta : H^{\otimes E} \to H \otimes H^{\otimes E}$, $m \mapsto m_{(0)} \otimes m_{(1)}$ corresponds to a right H^*-module structure via

$$\triangleleft : H^{\otimes E} \otimes H^* \to H^{\otimes E}, \quad m \triangleleft \alpha = \alpha(m_{(0)})m_{(1)}.$$

This follows for instance from [Maj, Sec. 1.6.2] or [DNR, Sec. 2.2]. The corresponding invariants and coinvariants coincide:

$$\begin{aligned} M^{H^*} &= \{m\ H^{\otimes E} : m \triangleleft \alpha = \alpha(1)m \ \ \forall \alpha \in H^*\} = \{m \in H^{\otimes E} : \alpha(m_{(0)})m_{(1)} = \alpha(1)m \ \ \forall \alpha \in H^*\} \\ &= M^{coH}. \end{aligned}$$

With Lemma 2.29 we thus obtain a projector on the coinvariants by $M^{coH} = P_\lambda(H^{\otimes E})$, where $P_\lambda : H^{\otimes E} \to H^{\otimes E}$, $P_\lambda(m) = \lambda(m_{(0)})m_{(1)}$. □

Next, we explicitly compare the construction from Chapter 4 for $\mathcal{C} = \mathrm{Vect}_{\mathbb{F}}$ to the Kitaev models given in [Ki, BMCA]. We follow [BMCA], but use the notation provided by Meusburger [Me, Sec. 3]. For this we consider a finite-dimensional semisimple Hopf algebra H with $S^2 = 1_H$ in $\mathrm{Vect}_{\mathbb{F}}$. Note that in contrast to [BMCA], where they set $\mathbb{F} = \mathbb{C}$ and require that H is a $*$-Hopf algebra, we allow any field $\mathbb{F}$, but restrict to involutive Hopf algebras. The $*$-structure on H in [BMCA] is required for the unitarity of certain operators on $H^{\otimes E}$, which is irrelevant to this thesis.

Analogously to our construction, the starting point in [Ki, BMCA, Me] is the vector space $H^{\otimes E}$, called *extended space*, obtained by assigning a copy of H to each edge in a ciliated ribbon graph Γ. For $h \in H$, $\beta \in H^*$ they define linear maps $L^h_\pm, T^\beta_\pm : H \to H$ given by

$$L^h_+(x) = hx, \quad L^h_-(x) = xh, \quad L^\beta_+(x) = \beta(x_{(2)})x_{(1)}, \quad L^\beta_-(x) = \beta(x_{(1)})x_{(2)}.$$

To each edge $\alpha \in E$ they then associate the linear maps $L^h_\pm, T^\beta_\pm : H^{\otimes E} \to H^{\otimes E}$. These are obtained by extending $L^h_\pm, T^\beta_\pm : H \to H$ to linear maps $H^{\otimes E} \to H^{\otimes E}$ that are equal to $L^h_\pm, T^\beta_\pm$ on the copy of H assigned to α and act as the identity on all other copies of H. The linear maps are called *triangle operators*. They correspond to the H-(co)module structures from Definition 4.1 via

$$L^h_{\alpha+} = h \rhd_{\alpha+} -, \;\; L^h_{\alpha-} = S(h) \rhd_{\alpha-} -, \;\; T^\beta_{\alpha+} = ((\beta \circ S) \otimes 1_{H^{\otimes E}}) \circ \delta_{\alpha-}, \;\; T^\beta_{\alpha-} = (\beta \otimes 1_{H^{\otimes E}}) \circ \delta_{\alpha+}.$$

Similarly, the *vertex operator* $A^h_v : H^{\otimes E} \to H^{\otimes E}$ for a ciliated vertex v with incoming edge ends $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ and $h \in H$ is defined by

$$A^h_v = L^{h_{(1)}}_{\alpha_1+} \circ L^{h_{(2)}}_{\alpha_2+} \circ \ldots \circ L^{h_{(n)}}_{\alpha_n+}.$$

Here, $L^{h_{(i)}}_{\alpha_i+}$ is replaced by $L^{S(h_{(i)})}_{\alpha_i-}$ if α_i is outgoing at v. The operator corresponds to the vertex action from Definition 4.3 via $A^h_v = h \rhd_v -$. Likewise, *face operators* $B^\beta_f : H^{\otimes E} \to H^{\otimes E}$ for $\beta \in H^*$ and ciliated faces f that traverse the edges $\alpha_n, \ldots, \alpha_1$ in this order and parallel to their orientations are defined as

$$B^\beta_f = T^{S(\beta_{(1)})}_{\alpha_n-} \circ T^{S(\beta_{(2)})}_{\alpha_{n-1}-} \circ \ldots \circ T^{S(\beta_{(n)})}_{\alpha_1}.$$

In this expression $T^{S(\beta_{(n-i)})}_{\alpha_i-}$ is replaced by $T^{\beta_{(n-i)}}_{\alpha_i+}$ if the edge α_i is traversed antiparallel to its orientation. Note that as in [Me] faces turn maximally right instead of left, the above formula correlates to the formula for the face that traverses $\alpha_1^{-1}, \ldots, \alpha_n^{-1}$ in [Me]. The face operator corresponds to the face coaction from Definition 4.3 via $B^\beta_f = (S(\beta) \otimes 1_{H^{\otimes E}}) \circ \delta_f$. In analogy to Lemma 4.5, which recalls [MV, Lemma 5.5], the vertex and face operators satisfy several relations. In particular, they define a representation of the Drinfeld double $D(H)$ for each site of the graph. A site consists of a ciliated vertex v and a face $f(v)$ based at the cilium at v. The assigned representation of $D(H)$ on $H^{\otimes E}$ is given by

$$\rhd : D(H) \otimes H^{\otimes E} \to H^{\otimes E}, \quad (\alpha \otimes h) \otimes m \mapsto B^\beta_{f(v)} \circ A^h_v(m),$$

compare [BMCA, Me]. Denote by $l \in H$, $\lambda \in H^*$ the Haar integrals. Then the associated vertex and face operators A^l_v, B^λ_f no longer depend on the choice of cilia due to cyclic invariance of coproducts of Haar integrals, compare [BaK, Sec. 2.4] or [Me, Sec. 3]. The *protected space* or *ground state* is defined as

$$\mathcal{L} = \{m \in H^{\otimes E} : A^l_v(m) = B^\lambda_f(m) = m \;\; \forall v \in V, \; \forall f \in F\}, \tag{68}$$

see [BMCA, Me]. In [BaK] it is interpreted as *vacuum state*. It is shown in [Ki, BMCA] that the protected space depends only on the homeomorphism class of the surface Σ encoded by Γ.

Clearly, for the standard graph (21) which has a single vertex and face, $\mathcal{L}$ coincides with the protected object M_{inv} as described in Lemma 6.1. They also coincide for general graphs, as for the (co)module structures $\triangleright := \triangleright_V$ and $\delta := \delta_F$ from Definition 4.6 in $\mathrm{Vect}_{\mathbb{F}}$ the (co)invariants can be split into an intersection of the (co)invariants for the single vertex and face (co)actions:

$$\{m \in H^{\otimes E} : h \triangleright m = \epsilon^{(|V|)}(h)m \ \forall h \in H^{\otimes |V|}\} = \bigcap_{v \in V} \{m \in H^{\otimes E} : h \triangleright_v m = \epsilon(h)m \ \forall h \in H\},$$
$$\{m \in H^{\otimes E} : \delta(m) = 1^{(|F|)} \otimes m\} = \bigcap_{f \in F} \{m \in H^{\otimes E} : \delta_f(m) = 1 \otimes m\}.$$

Balsam and Kirillov [BaK, Th. 4.1] show that the protected space for H and Σ coincides with the vector space $Z_{TV}(\Sigma)$ the Turaev-Viro TQFT based on the representation category $H-$Mod assigns to the surface Σ.

An essential motivation for the consideration of the Kitaev models in [Ki] is their role in topological quantum computing. One considers a finite-dimensional Hilbert space $\mathcal{H}$, where information is encoded as a state in $\mathcal{H}$. Quantum codes can then be realised as linear subspaces of the Hilbert space, see Gottesman [Go].

It is a crucial task to protect the quantum computer from various errors. Cui, Galindo and Romero [CGR] describe the creation of errors by linear operators on $\mathcal{H}$. The correction of such errors involves both their detection and recovery. That is, returning the quantum system to the original state of the code, see [NC, Sec. 10.3]. It follows that any quantum code that corrects a set of errors also corrects any linear combination of them, compare [Go, Th. 2]. For a formal definition of error-correcting quantum codes we refer to [NC, Sec. 10.3]. Based on this Cui et al. [CDH+, Sec. 3] show that the protected space $\mathcal{L}$ of the quantum double model associated to the group algebra $\mathbb{C}[G]$ of a finite group G is an error-correcting quantum code. Specifically, the protected space of the Kitaev model associated to $\mathbb{C}[\mathbb{Z}_2]$ and the torus is known as *toric code*, compare [CDH+]. A detailed analysis of the error-correcting process in quantum double models for a group algebra of a finite abelian group is provided in [CGR].

As explained in [BaK, Sec. 5], *topological excitations* can be modelled in quantum double models. They are interpreted as "quasiparticles", also called *anyons* in [Ki], sitting on disjoint sites of the ciliated ribbon graph Γ. Sites in Γ are said to be disjoint if none of their faces traverses the vertices of other sites. When defining a space of n particles localised at disjoint sites $s_1, \ldots, s_n$, the invariance under vertex and face operators associated to these sites is not imposed. Consequently, the associated subspace is defined as

$$\mathcal{L}(s_1, \ldots, s_n) := \{m \in H^{\otimes E} : A_v^l(m) = m \ \forall v \notin S, \ B_f^\lambda(m) = m \ \forall f \notin S\},$$

where $l \in H$, $\lambda \in H^*$ are Haar integrals, compare [BaK, Sec. 5]. This subspace can be decomposed into irreducible representations of the quantum double $D(H)$. Quasiparticles are then labelled by these irreducible representations, compare Cowtan and Majid [CM]. Moreover, the quasiparticles can be transported from one site of the graph to another site. This is modelled by defining so-called ribbon operators, see [CM]. It specifically plays an important role for topological quantum computing, as less formally quantum computation is performed by creating excitation pairs, moving them around and annihilating them, compare [Ki]. For a detailed description of the ribbon operators we refer to [CM].

7 Protected objects in SSet

In this chapter, we investigate protected objects for group objects in the category SSet. This is essentially identical to Section 6 from the preprint [HM].

We denote by Δ the simplex category with finite ordinals $[n] = \{0, 1, \dots, n\}$ for $n \in \mathbb{N}_0$ as objects and weakly monotonic maps $\alpha : [m] \to [n]$ as morphisms from $[m]$ to $[n]$.

Objects in $\text{SSet} = \text{Set}^{\Delta^{op}}$ are simplicial sets, functors $X : \Delta^{op} \to \text{Set}$ that are specified by sets X_n, face maps $d_i : X_{n+1} \to X_n$ and degeneracies $s_i : X_n \to X_{n+1}$ for $n \in \mathbb{N}_0$ and $i \in \{0, \dots, n\}$ that satisfy the simplicial relations

$$\begin{aligned} &d_j \circ d_i = d_i \circ d_{j+1} \text{ if } i \leq j, \quad s_i \circ s_j = s_{j+1} \circ s_i \text{ if } i \leq j, \\ &d_i \circ s_j = s_{j-1} \circ d_i \text{ if } i < j, \quad d_i \circ s_j = \text{id if } i \in \{j, j+1\}, \quad d_i \circ s_j = s_j \circ d_{i-1} \text{ if } i > j+1. \end{aligned} \tag{69}$$

Morphisms in SSet are simplicial maps, natural transformations $f : X \to Y$ specified by component maps $f_n : X_n \to Y_n$ satisfying $f_{n-1} \circ d_i = d_i \circ f_n$ and $f_{n+1} \circ s_i = s_i \circ f_n$ for $n \in \mathbb{N}_0$ and admissible i. The category SSet is cartesian monoidal with the objectwise product induced by the product in Set.

Unpacking the definition of a group object in a cartesian monoidal category from Example 2.3 yields

Definition 7.1.

1. *A group object in* SSet *is a* **simplicial group***: a simplicial set $H : \Delta^{op} \to \text{Set}$ with group structures on the sets H_n such that all face maps and degeneracies are group homomorphisms.*
2. *A morphism of group objects in* SSet *is a* **morphism of simplicial groups***: a simplicial map $f : H \to H'$ such that all maps $f_n : H_n \to H'_n$ are group homomorphisms.*

For examples of simplicial groups, see Section 8.2, in particular Corollary 8.10 and Example 8.11. Modules, comodules and Yetter-Drinfeld modules over simplicial groups are given by Example 2.9.

Lemma 7.2. *Let $H : \Delta^{op} \to \text{Set}$ be a simplicial group.*

1. *A* **module** *over H is a simplicial set $M : \Delta^{op} \to \text{Set}$ together with a collection of H_n-actions $\rhd_n : H_n \times M_n \to M_n$ that define a simplicial map $\rhd : H \times M \to M$.*
2. *A* **comodule** *over H is a simplicial set $M : \Delta^{op} \to \text{Set}$ with a simplicial map $F : M \to H$.*
3. *If $(M, \rhd)$ is a module and (M, F) a comodule over H, then $(M, \rhd, F)$ is a* **Yetter-Drinfeld module** *over H iff $F_n(g \rhd_n m) = g \cdot F_n(m) \cdot g^{-1}$ for all $m \in M_n$, $g \in H_n$ and $n \in \mathbb{N}_0$.*

(Co)limits in SSet are objectwise, see for instance Riehl [R, Prop. 3.3.9] or Leinster [L, Th. 6.2.5]. Thus, (co)invariants of a (co)module over a group object in SSet are obtained from (co)equalisers in Set. It is also straightforward to compute the biinvariants of a Yetter-Drinfeld module.

Proposition 7.3. *Let H be a simplicial group.*

1. *The coinvariants $\mathcal{M}^{coH}$ of a H-comodule M defined by a simplicial map $F : M \to H$ are given by the sets $M_n^{coH} = \{m \in M_n \mid F_n(m) = e\}$ and the induced face maps and degeneracies.*
2. *The invariants M^H of a H-module $(M, \rhd)$ are given by the sets $M_n^H = \{H_n \rhd_n m \mid m \in M_n\}$ and the induced face maps and degeneracies.*
3. *The biinvariants M_{inv} of a Yetter-Drinfeld module $(M, \rhd, F)$ over H are given by the sets $(M_{inv})_n = \{H_n \rhd_n m \mid m \in M_n, F_n(m) = e\}$ and the induced face maps and degeneracies.*

Proof. 1. The coinvariant object of a H-comodule (M, F) is the equaliser of the simplicial maps $\delta = F \times \text{id} : M \to H \times M$ and $\eta \times \text{id} : M \to H \times M$. As limits in SSet are objectwise, this is the simplicial set $M^{coH} : \Delta^{op} \to \mathrm{Set}$ that assigns to an ordinal $[n]$ the equaliser in Set of the maps $F_n \times \mathrm{id} : M_n \to H_n \times M_n$ and $\eta_n \times \mathrm{id} : M_n \to H_n \times M_n$, which is $M_n^{coH} = \{m \in M_n \mid F_n(m) = e\}$. The face maps and degeneracies are induced by the ones of M, and the simplicial map $\iota : M^{coH} \to M$ is given by the maps $\iota_n : M_n^{coH} \to M_n$, $m \mapsto m$.

2. Analogously to 1., the invariant object of $(M, \rhd)$ is the simplicial set $M^H : \Delta^{op} \to \mathrm{Set}$ that assigns to the ordinal $[n]$ the coequaliser in Set of the maps $\rhd_n : H_n \times M_n \to M_n$ and $\epsilon_n \times \mathrm{id} : H_n \times M_n \to M_n$. This is the set $M_n^H = M_n/\sim_n$ with $m \sim_n m'$ iff there is a $g \in H_n$ with $m' = g \rhd_n m$. The simplicial map $\pi : M \to M^H$ is given by the maps $\pi_n : M_n \to M_n^H$, $m \mapsto H_n \rhd_n m$.

3. The simplicial maps $I : M_{inv} \to M^H$ and $P : M^{coH} \to M_{inv}$ with $\pi \circ \iota = I \circ P$ that characterise M_{inv} with $(M_{inv})_n = \{H_n \rhd_n m \mid m \in M_n^{coH}\}$ as the image of $\pi \circ \iota$ are given by

$$I_n : (M_{inv})_n \to M_n^H, H_n \rhd_n m \mapsto H_n \rhd_n m, \qquad P_n : M_n^{coH} \to (M_{inv})_n, m \mapsto H_n \rhd_n m.$$

As monomorphisms and epimorphisms in SSet are those simplicial maps whose component morphisms are injective and surjective, see for instance [L, Ex. 6.2.20], it follows directly that I is a monomorphism and P an epimorphism in SSet. Every pair (J, Q) of a monomorphism $J : X \to M^H$ and morphism $Q : M^{coH} \to X$ in SSet with $J \circ Q = \pi \circ \iota$ defines injective maps $J_n : X_n \to M_n^H$ and thus identifies $Q(M_n^{coH})$ with a subset of M_n^H. As J_n is a monomorphism and due to the identity $J_n \circ Q_n(g \rhd_n m) = \pi_n \circ \iota_n(g \rhd_n m) = \pi_n \circ \iota_n(m) = J_n \circ Q_n(m)$, we have $Q_n(g \rhd_n m) = Q_n(m)$ for all $m \in M_n^{coH}$ and $g \in H_n$. The maps $V_n : (M_{inv})_n \to X_n$, $H_n \rhd_n m \mapsto Q_n(m)$ define a simplicial map $V : M_{inv} \to X$ with $I = J \circ V$. □

We now determine the coinvariants, invariants and the protected objects for Kitaev models on oriented surfaces Σ of genus $g \geq 1$ and for a simplicial group H as a Hopf monoid in SSet.

Proposition 7.4. *Let H be a simplicial group and Σ an oriented surface of genus $g \geq 1$. The associated protected object is the simplicial set $X : \Delta^{op} \to \mathrm{Set}$ with $X_n = \mathrm{Hom}(\pi_1(\Sigma), H_n)/H_n$, where the quotient is with respect to conjugation by H_n, and face maps and degeneracies given by*

$$d_i : X_n \to X_{n-1},\ [\rho] \mapsto [d_i \circ \rho], \qquad s_i : X_n \to X_{n+1},\ [\rho] \mapsto [s_i \circ \rho].$$

Proof. By Theorem 5.23 the protected object of Σ can be computed from the standard graph in (21). This yields a Yetter-Drinfeld module $(M, \rhd, F)$ over H given by formula (29) in Example 4.8. Hence, we have $M_n = H_n^{\times 2g}$ for all $n \in \mathbb{N}_0$ with the face maps and degeneracies of H applied to each component simultaneously. The Yetter-Drinfeld module structure is given by

$$F_n : H_n^{\times 2g} \to H_n,\ (a_1, b_1, \ldots, a_g, b_g) \mapsto [b_g^{-1}, a_g] \cdots [b_1^{-1}, a_1]$$
$$\rhd_n : H_n \times H_n^{\times 2g} \to H_n^{\times 2g},\ (h, a_1, b_1, \ldots, a_g, b_g) \mapsto (ha_1h^{-1}, hb_1h^{-1}, \ldots, ha_gh^{-1}, hb_gh^{-1}).$$

By Proposition 7.3 the associated protected object is the simplicial set $\mathcal{M}_{inv}$ with

$$(M_{inv})_n = \{H_n \rhd_n (a_1, b_1, \ldots, a_g, b_g) \in H_n^{\times 2g} \mid [b_g^{-1}, a_g] \cdots [b_1^{-1}, a_1] = e\} \cong \mathrm{Hom}(\pi_1(\Sigma), H_n)/H_n.$$

Face maps and degeneracies are given by post-composing group homomorphisms $\rho : \pi_1(\Sigma) \to H_n$ with the face maps $d_i : H_n \to H_{n-1}$ and degeneracies $s_i : H_n \to H_{n+1}$. □

8 Protected objects in Cat

This chapter is dedicated to the investigation of protected objects in Cat. To determine the protected objects we require a description of both equalisers and coequalisers in Cat. Whereas determining equalisers in Cat is standard, it requires more effort to determine coequalisers in Cat. We describe two ways to obtain coequalisers in Cat. One approach uses the nerve and homotopy functor, while the other approach deals with generalised congruences. We then apply both concepts to determine the invariants of a module over a Hopf monoid in Cat and hence protected objects. Their characterisation relies on crossed modules, as they amount to group objects in Cat.

For this, the category Cat of small categories and functors between them is considered as a cartesian monoidal category with terminal object $\{\cdot\}$. For a finite product $\mathcal{C}_1 \times \ldots \times \mathcal{C}_n$ of small categories, we denote by $\pi_i : \mathcal{C}_1 \times \ldots \times \mathcal{C}_n \to \mathcal{C}_i$ the associated projection functors. For a small category $\mathcal{C}$ we denote by $\mathrm{Ob}(\mathcal{C})$ the set of objects and by $\mathcal{C}^{(1)} = \bigcup_{X,Y \in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X,Y)$ the set of all morphisms in $\mathcal{C}$.

In Section 8.1 we describe group objects in Cat in terms of crossed modules. Both in Section 8.2 and 8.4 we explain how coequalisers in Cat can be determined. While the approach in Section 8.2 relies on the nerve and homotopy functor, Section 8.4 describes them in terms of generalised congruences. The concepts are applied in Section 8.3 and Section 8.5 to describe invariants of modules over group objects, before we determine the protected objects in Section 8.6.

Up to some additions and minor changes Sections 8.1, 8.2, 8.3 and 8.6 are parts of the preprint [HM].

8.1 Crossed modules as group objects in Cat

In this section we describe how group objects in Cat can be specified by crossed modules.

Definition 8.1.

1. *A* ***group object*** *in Cat is a small category H together with functors $m : H \times H \to H$, $\eta : \{\cdot\} \to H$ and $I : H \to H$ such that the diagrams* (7) *commute.*
2. *A* ***morphism*** *$F : (H, m, \eta, I) \to (H', m', \eta', I')$* ***of group objects*** *is a functor $F : H \to H'$ that satisfies* (8).

We denote by $\mathcal{G}(\mathrm{Cat})$ the category of group objects and morphisms of group objects in Cat and write $e := \eta(\cdot)$, $f^{-1} = I(f)$, $g \cdot f = m(g, f)$ and likewise for multiple products.

Brown and Spencer [BS] showed that group objects in Cat correspond to crossed modules. We summarise this correspondence for the convenience of the reader.

Definition 8.2.
A ***crossed module*** *is a quadruple $(B, A, \blacktriangleright, \partial)$ of groups A and B, a group homomorphism $\partial : A \to B$ and a group action $\blacktriangleright : B \times A \to A$ by automorphisms that satisfy the Peiffer identities*

$$\partial(b \blacktriangleright a) = b\partial(a)b^{-1}, \qquad \partial(a) \blacktriangleright a' = aa'a^{-1} \qquad \forall a, a' \in A, b \in B. \tag{70}$$

A ***morphism of crossed modules*** *$f = (f_1, f_2) : (B, A, \blacktriangleright, \partial) \to (B', A', \blacktriangleright', \partial')$ is a pair of group homomorphisms $f_1 : B \to B'$, $f_2 : A \to A'$ such that*

$$\partial' \circ f_2 = f_1 \circ \partial, \qquad \blacktriangleright' \circ (f_1 \times f_2) = f_2 \circ \blacktriangleright .$$

We denote by $\mathcal{CM}$ the category of crossed modules and morphisms between them.

Example 8.3.

1. *A normal subgroup* $A \subset B$ *defines a crossed module with the inclusion* $\partial : A \to B$, $a \mapsto a$ *and the conjugation action* $\blacktriangleright: B \times A \to A$, $b \blacktriangleright a = bab^{-1}$.
2. *Any crossed module* $(A, B, \blacktriangleright, \partial)$ *yields a crossed module* $(A/\ker\partial, B, \blacktriangleright', \partial')$ *with injective* ∂'. *This identifies* $A/\ker\partial$ *with a normal subgroup of* B *and hence yields 1.*
3. *Any group action* $\blacktriangleright: B \times A \to A$ *by automorphisms of an abelian group* A *yields a crossed module with* $\partial \equiv e_B$.
4. *Any group* A *defines a crossed module with* $B = \mathrm{Aut}(A)$, $\blacktriangleright: \mathrm{Aut}(A) \times A \to A$, $\phi \blacktriangleright a = \phi(a)$ *and* $\partial : A \to \mathrm{Aut}(A)$, $g \mapsto C_g$, *where* $C_g(x) = gxg^{-1}$.
5. *Every extension of a group* G *by a group* X

$$1 \longrightarrow X \overset{\iota}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} G \longrightarrow 1$$

defines a crossed module with $A := X$, $B := E$, $\partial := \iota$ *and* $b \blacktriangleright a := \iota^{-1}(b\iota(a)b^{-1})$ *for all* $b \in B$, $a \in A$. *The group extension is central iff* $\iota(X) \subset Z(E)$, *which is equivalent to* $\blacktriangleright$ *trivial.*

6. *Conversely, any crossed module* $(B, A, \blacktriangleright, \partial)$ *gives an extension of* $B/\partial(A)$ *by* $A/\ker(\partial)$:

$$1 \longrightarrow A/\ker(\partial) \overset{\partial'}{\hookrightarrow} B \overset{\pi}{\twoheadrightarrow} B/\partial(A) \longrightarrow 1$$

with $\partial'(a\ker(\partial)) := \partial(a)$. *For crossed modules with surjective* ∂, *the group* A *is also an extension of* B *by* $\ker(\partial)$:

$$1 \longrightarrow \ker(\partial) \overset{\iota}{\hookrightarrow} A \overset{\partial}{\twoheadrightarrow} B \longrightarrow 1.$$

Theorem 8.4. *[BS, Th. 1]*
The following functors $\triangle : \mathcal{G}(\mathrm{Cat}) \to \mathcal{CM}$ *and* $\triangledown : \mathcal{CM} \to \mathcal{G}(\mathrm{Cat})$ *form an equivalence of categories.*

The functor $\triangle$ *sends a group object* (H, m, η, I) *to the crossed module* $(B, A, \blacktriangleright, \partial)$ *with*

- $A = \mathrm{Cost}_e := \bigcup_{X \in \mathrm{Ob}(H)} \mathrm{Hom}_H(X, e)$ *with multiplication* $m_A : A \times A \to A$, $(a, a') \mapsto m(a, a')$,
- $B = \mathrm{Ob}(H)$ *with multiplication* $m_B : B \times B \to B$, $(b, b') \mapsto m(b, b')$,
- $\partial : A \to B$, $(f : X \to e) \mapsto X$,
- $\blacktriangleright: B \times A \to A$, $b \blacktriangleright a = 1_b \cdot a \cdot 1_b^{-1}$,

and a morphism of group objects $F : (H, m, \eta, I) \to (H', m', \eta', I')$ *to the pair of group homomorphisms* $f_1 := F : \mathrm{Ob}(H) \to \mathrm{Ob}(H')$ *and* $f_2 := F : \mathrm{Cost}_e \to \mathrm{Cost}'_{e'}$.

The functor $\triangledown$ *sends a crossed module* $(B, A, \blacktriangleright, \partial)$ *to the group object* (H, m, η, I) *with*

- $\mathrm{Ob}(H) = B$,
- $\mathrm{Hom}_H(b, b') = \{(a, b) \in A \times B : \partial(a)b = b'\}$ *with composition* $(a', \partial(a)b) \circ (a, b) = (a'a, b)$,
- $m : H \times H \to H$ *with* $m(b, b') = bb'$ *and* $m((a', b'), (a, b)) = (a'(b' \blacktriangleright a), b'b)$,
- $\eta : \{\cdot\} \to H$, $\eta(\cdot) = e_B$, $\eta(1.) = (e_A, e_B)$,
- $I : H \to H$, $I(b) = b^{-1}$, $I((a, b)) = (b^{-1} \blacktriangleright a^{-1}, b^{-1})$,

and a morphism $(f_1, f_2) : (B, A, \blacktriangleright, \partial) \to (B', A', \blacktriangleright', \partial')$ *of crossed modules to the functor* $F : H \to H'$ *with* $F(b) = f_1(b)$ *for all* $b \in B = \mathrm{Ob}(H)$ *and* $F((a, b)) = (f_2(a), f_1(b))$.

By Theorem 8.4 the *group structure* on the set $H^{(1)}$ of morphisms of a group object H is the semidirect product $A \rtimes B$ for the group action $\blacktriangleright: B \times A \to A$. As a *category* H is the action groupoid for the group action $\blacktriangleright': A \times B \to B$, $a \blacktriangleright' b = \partial(a)b$.

There are also other ways of constructing group objects in Cat, see for instance the following example.

Example 8.5. *For any small category $\mathcal{M}$ the category* $\mathrm{Aut}(\mathcal{M})$ *of invertible endofunctors $F : \mathcal{M} \to \mathcal{M}$ as objects and natural isomorphisms as morphisms is a group object in* Cat *with multiplication, unit and inverse given by:*

- *m on objects: $(F, F') \mapsto FF' : \mathcal{M} \to \mathcal{M}$,*
- *m on morphisms: $(\sigma : F \to F', \sigma' : K \to K')$ is sent to $\tau : FK \to F'K'$ with component morphisms $\tau_X := \sigma_{K'(X)} \circ F(\sigma'_X)$,*
- *$\eta : \bullet \mapsto \mathrm{id}_{\mathcal{M}}$, $1_\bullet$ is sent to the natural transformation $\mathrm{id}_{\mathrm{id}_{\mathcal{M}}} : \mathrm{id}_{\mathcal{M}} \to \mathrm{id}_{\mathcal{M}}$,*
- *I on objects: $F \mapsto F^{-1} : \mathcal{M} \to \mathcal{M}$,*
- *I on morphisms: $\sigma : F \to F'$ is sent to τ with component morphisms $\tau_X := F^{-1}(\sigma^{-1}_{(F')^{-1}(X)})$.*

The category $\mathrm{Aut}(\mathcal{M})$ *is small, as functor categories $\mathcal{D}^{\mathcal{C}}$ are small for $\mathcal{D}, \mathcal{C}$ small categories, see Riehl [R, Remark 1.7.3].*

Modules $(\mathcal{M}, \rhd)$ over group objects H in Cat *are small categories $\mathcal{M}$ with a morphism $\rho : H \to$* $\mathrm{Aut}(\mathcal{M})$ *of group objects. For a module $(\mathcal{M}, \rhd)$ the functor ρ is given by*

- *on objects: $h \in Ob(H)$ is sent to $\rho(h)$ with $\rho(h)(X) = h \rhd X$ for $X \in Ob(\mathcal{M})$ and $\rho(h)(X) = \mathrm{id}_h \rhd X$ for $X \in \mathcal{M}^{(1)}$.*
- *on morphisms: $f \in H^{(1)}$ is sent to $\rho(f)$ with component morphisms $\rho(f)_X := f \rhd \mathrm{id}_X$.*

Conversely, a morphism $\rho : H \to \mathrm{Aut}(\mathcal{M})$ defines a module structure $\rhd : H \times \mathcal{M} \to \mathcal{M}$ by $g \rhd X := \rho(g)(X)$ on objects and $f \rhd t := \rho(g')(t) \circ \rho(f)_X$ for morphisms $f : g \to g'$, $t : X \to X'$.

8.2 Equalisers and coequalisers in Cat

To determine the coinvariants, invariants and the protected object for a group object in Cat, we require equalisers, coequalisers and images in Cat. It is well-known that Cat is complete and cocomplete, see for instance [R, Prop. 3.5.6, Cor. 4.5.16]. The following result on equalisers is standard, see for example Schubert [Sch, Sec. 7.2].

Lemma 8.6. *The equaliser of two functors $F, K : \mathcal{C} \to \mathcal{D}$ between small categories is the subcategory $\mathcal{E} \subset \mathcal{C}$ with*

- $\mathrm{Ob}(\mathcal{E}) = \{C \in \mathrm{Ob}(\mathcal{C}) \mid F(C) = K(C)\}$,
- $\mathrm{Hom}_{\mathcal{E}}(C, C') = \{f \in \mathrm{Hom}_{\mathcal{C}}(C, C') \mid F(f) = K(f)\}$.

To describe coequalisers in Cat we use that Cat is a reflective subcategory of SSet with the inclusion given by the nerve functor $N : \mathrm{Cat} \to \mathrm{SSet}$, which is full and faithful. Its left adjoint is the homotopy functor $h : \mathrm{SSet} \to \mathrm{Cat}$, and the composite $hN : \mathrm{Cat} \to \mathrm{Cat}$ is naturally isomorphic to the identity functor via the counit of the adjunction, see Riehl [R, Ex. 4.5.14 (vi)] or Lurie [Lu, Sec. 1.2]. As a left adjoint, h preserves colimits. This allows one to compute colimits in Cat by applying the homotopy functor h to the associated colimits in SSet, see for instance [R, Prop. 4.5.15].

Lemma 8.7. *The coequaliser of two functors $F, K : \mathcal{C} \to \mathcal{D}$ between small categories is the functor $h(\pi) : hN(\mathcal{D}) \to h(X)$, where $\pi : N(\mathcal{D}) \to X$ is the coequaliser of $N(F), N(K)$ in* SSet.

To compute such coequalisers, we require an explicit description of the nerve and the homotopy functor. We summarise the details from [R, Ex. 4.5.14 (vi)] and [Lu, Sec. 1.2]. For $n \in \mathbb{N}_0$ we denote by $[n]$ the ordinals in Δ as well as the associated categories with objects $0, 1, \ldots, n$ and a single morphism from i to j if $i \leq j$. As every weakly monotonic map $\alpha : [m] \to [n]$ defines a functor $\alpha : [m] \to [n]$, this defines an embedding $\iota : \Delta \to \mathrm{Cat}$.

Definition 8.8. *The* **nerve** $N : \mathrm{Cat} \to \mathrm{SSet}$ *is the functor that sends a small category* $\mathcal{C}$ *to the simplicial set* $N(\mathcal{C}) : \Delta^{op} \to \mathrm{Set}$ *with*

- $N(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C})$,
- $N(\mathcal{C})(\alpha) : N(\mathcal{C})_n \to N(\mathcal{C})_m$, $F \mapsto F \circ \alpha$ *for every weakly monotonic* $\alpha : [m] \to [n]$,

and a functor $F : \mathcal{C} \to \mathcal{D}$ *to the simplicial map* $N(F) : N(\mathcal{C}) \to N(\mathcal{D})$ *that post-composes with* F.

By definition, $N(\mathcal{C})_0 = \mathrm{Ob}\,\mathcal{C}$ and $N(\mathcal{C})_n$ is the set of sequences $(f_1, \ldots, f_n) : C_0 \xrightarrow{f_1} \ldots \xrightarrow{f_n} C_n$ of composable morphisms in $\mathcal{C}$ for $n \in \mathbb{N}$. The simplicial set structure is given by the face maps $d_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n-1}$ and degeneracies $s_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n+1}$ for $i \in \{0, ..., n\}$. The face maps act on a sequence $(f_1, \ldots, f_n)$ by removing f_1 and f_n for $i = 0$ and $i = n$, respectively, and by replacing $(\ldots, f_i, f_{i+1}, \ldots)$ with $(\ldots, f_{i+1} \circ f_i, \ldots)$ for $1 \leq i \leq n-1$. For $n = 1$ and $f_1 : C_0 \to C_1$ one has $d_0(f_1) = C_1$ and $d_1(f_1) = C_0$. The degeneracies act on $(f_1, \ldots, f_n)$ by inserting the identity morphism 1_{C_i}. In particular, for $n = 0$ one has $s_0(C) = 1_C$ for every $C \in \mathrm{Ob}\,\mathcal{C}$. The simplicial map $N(F)$ for a functor $F : \mathcal{C} \to \mathcal{D}$ applies F to all morphisms in $(f_1, \ldots, f_n)$.

The left adjoint of the nerve $N : \mathrm{Cat} \to \mathrm{SSet}$ is the homotopy functor $h : \mathrm{SSet} \to \mathrm{Cat}$. It is the left Kan extension along the Yoneda embedding $y : \Delta \to \mathrm{SSet}$ of the embedding functor $\iota : \Delta \to \mathrm{Cat}$. Concretely, it is given as follows.

Definition 8.9. *The* **homotopy functor** $h : \mathrm{SSet} \to \mathrm{Cat}$ *sends a simplicial set* X *to the category* hX *with* $\mathrm{Ob}\, hX = X_0$, *generating morphisms* $\sigma : d_1(\sigma) \to d_0(\sigma)$ *for* $\sigma \in X_1$ *and relations*

$$s_0(x) = 1_x \text{ for } x \in X_0, \qquad\qquad d_1(\sigma) = d_0(\sigma) \circ d_2(\sigma) \text{ for } \sigma \in X_2. \tag{71}$$

It sends a simplicial map $f : X \to Y$ *to the functor* $hf : hX \to hY$ *given by* f *on the generators.*

The simplicial relations imply that for elements of X_2 that are in the image of a degeneracy map, the second relation in (71) is satisfied trivially. In this case one of the two morphisms on the right is an identity and the other coincides with the morphism on the left. Only non-degenerate elements of X_2 give rise to non-trivial relations in hX.

In general, morphisms in the homotopy category of a simplicial set X are finite sequences of composable elements of X_1. However, if the simplicial set X is an ∞-category, which is always the case if $X = N(\mathcal{C})$ for some category $\mathcal{C}$, every morphism in hX is represented by a single element in X_1, see for instance [Lu, Sec. 1.2.5]. Most of the simplicial sets considered in the following are even Kan complexes, as they are nerves of groupoids.

As a right adjoint, the nerve preserves limits, and as a left adjoint, the homotopy functor preserves colimits. It follows directly from its definition that the nerve also preserves coproducts, and the homotopy functor preserves finite products, see for instance Joyal [Jo, Prop. 1.3]. This implies with Examples 2.6 and 2.10

Corollary 8.10. *The nerve* $N : \mathrm{Cat} \to \mathrm{SSet}$ *and the homotopy functor* $h : \mathrm{SSet} \to \mathrm{Cat}$ *are symmetric monoidal with respect to the cartesian monoidal category structures of* Cat *and* SSet. *In particular:*

1. *The nerve of a crossed module is a simplicial group.*
2. *The homotopy category of a simplicial group is a crossed module.*
3. *The nerve of a (co)module over a crossed module is a (co)module over its nerve.*

4. The homotopy category of a (co)module over a simplicial group is a (co)module over its homotopy category.

Concretely, the nerve of a crossed module $(B, A, \blacktriangleright, \partial)$ is the simplicial group H with $H_n = A^{\times n} \times B$ for $n \in \mathbb{N}_0$ with group multiplication

$$(a_1, ..., a_n, b) \cdot (a_1', ..., a_n', b') = (a_1(b \blacktriangleright a_1'), a_2(\partial(a_1)b \blacktriangleright a_2'), ..., a_n(\partial(a_{n-1} \cdots a_1)b \blacktriangleright a_n'), bb') \quad (72)$$

and face maps and degeneracies

$$d_i : H_n \to H_{n-1}, \quad (a_1, ..., a_n, b) \mapsto \begin{cases} (a_2, ..., a_n, \partial(a_1)b) & i = 0 \\ (a_1, ..., a_{i+1}a_i, ..., a_n, b) & 1 \leq i \leq n-1 \\ (a_1, ..., a_{n-1}, b) & i = n \end{cases} \quad (73)$$

$$s_i : H_n \to H_{n+1}, \quad (a_1, ..., a_n, b) \mapsto (a_1, \ldots, a_i, 1, a_{i+1}, ..., a_n, b) \quad 0 \leq i \leq n.$$

Example 8.11.

1. *A group action $\blacktriangleright : B \times A \to A$ by automorphisms on an abelian group A yields a simplicial group with $H_n = A^{\times n} \rtimes' B$, where B acts diagonally via $\blacktriangleright$, and with the face maps and degeneracies (73) for $\partial \equiv 1$.*
2. *Every injective group homomorphism $\partial : A \to B$ from an abelian group A into the centre of a group B yields a simplicial group, where $H_n = A^{\times n} \times B$ is the direct product, and the face maps and degeneracies are given by (73).*
3. *Every abelian group A is a simplicial group with $H_n = A^{\times n}$, the group multiplication of $A^{\times n}$, the face maps and degeneracies (73) for $B = \{e\}$ and $\partial \equiv 1$.*
4. *Any normal subgroup $A \subset B$ determines a simplicial group with $H_n = A^{\times n} \times B$ and group multiplication (72), face maps and degeneracies (73), where $\partial : A \to B$ is the inclusion and $\blacktriangleright : B \times A \to A$ the conjugation action.*

8.3 (Co)invariants of (co)modules over group objects in Cat

The coinvariants of a comodule $(\mathcal{M}, \delta)$ over a group object (H, m, η, I) in Cat are given as the equaliser of $\delta = (F \times 1_{\mathcal{M}}) \circ \Delta : \mathcal{M} \to H \times \mathcal{M}$ and $\eta \times 1_{\mathcal{M}} : \mathcal{M} \to H \times \mathcal{M}$. This is the subcategory on which δ and $\eta \times 1_{\mathcal{M}}$ coincide, together with its inclusion functor, see Lemma 8.6. In terms of the associated functor $F : \mathcal{M} \to H$ from Example 2.9 we have

Lemma 8.12.
Let $(\mathcal{M}, \delta)$ be a comodule over a group object (H, m, η, I) in Cat*. Then the coinvariants are given by the subcategory $\mathcal{M}^{coH} \subset \mathcal{M}$ with*

- $\mathrm{Ob}(\mathcal{M}^{coH}) = \{A \in \mathrm{Ob}(\mathcal{M}) \mid F(A) = e\}$,
- $\mathrm{Hom}_{\mathcal{M}^{coH}}(A, A') = \{f \in \mathrm{Hom}_{\mathcal{M}}(A, A') \mid F(f) = 1_e\}$,

and the inclusion functor $\iota : \mathcal{M}^{coH} \to \mathcal{M}$.

The invariants of a module $(\mathcal{M}, \rhd)$ over a group object H in Cat are the coequaliser of the functors $\rhd, \pi_2 : H \times \mathcal{M} \to \mathcal{M}$. They are computed with Lemma 8.7.

Proposition 8.13. *Let $(\mathcal{M}, \rhd)$ be a module over a group object $H = \bigtriangledown(B, A, \blacktriangleright, \delta)$ in* Cat*. Then its invariants are the category $\mathcal{M}^H$, whose*

- *objects are orbits of the B-action on* $\mathrm{Ob}(\mathcal{M})$*,*
- *morphisms are generated by orbits of the $A \rtimes B$-action on $\mathcal{M}^{(1)}$ subject to the relations $[f_2] \circ [f_1] = [f_2 \circ f_1]$ for all $A \rtimes B$-orbits $[f_1]$, $[f_2]$ of composable morphisms f_1, f_2 in $\mathcal{M}$.*

We denote by $\pi : \mathcal{M} \to \mathcal{M}^H$ the projection functor that sends each object of $\mathcal{M}$ to its B-orbit and each morphism in $\mathcal{M}$ to the equivalence class of its $A \rtimes B$-orbit.

Proof. By Corollary 8.10, applying the nerve to the group object H in Cat and to a module $(\mathcal{M}, \rhd)$ over H yields a simplicial group $N(H)$ and a module $N(\mathcal{M})$ over $N(H)$ in SSet. By Lemma 8.7 the coequaliser of the morphisms $\rhd, \pi_2 : H \times \mathcal{M} \to \mathcal{M}$ in Cat is obtained by applying the homotopy functor to the coequaliser of $\rhd' = N(\rhd), \pi_2' = N(\pi_2) : N(H) \times N(\mathcal{M}) \to N(\mathcal{M})$ in SSet.

As colimits in SSet are computed objectwise, see for instance [R, Prop. 3.3.9], the coequaliser of $\rhd', \pi_2'$ is the simplicial set $N(\mathcal{M})^H$ with $N(\mathcal{M})^H_n = N(\mathcal{M})_n / \sim_n$, where $\sim_n$ is the equivalence relation on $N(\mathcal{M})_n$ defined by the $N(H)$-action: $m \sim_n m'$ iff there is a $g \in N(H)_n$ with $m' = g \rhd' m$. The face maps and degeneracies of $N(\mathcal{M})^H$ are induced by the ones of $N(\mathcal{M})$.

As $N(H)_0 = \mathrm{Ob}\, H = B$ and $N(\mathcal{M})_0 = \mathrm{Ob}\, \mathcal{M}$, the elements of $N(\mathcal{M})^H_0$ are the orbits of the B-action on $\mathrm{Ob}\, \mathcal{M}$. As $N(H)_1 = H^{(1)} = A \rtimes B$, the set $N(\mathcal{M})^H_1$ contains the orbits of the $A \rtimes B$-action on $\mathcal{M}^{(1)}$. Elements of $N(\mathcal{M})_2$ and $N(H)_2$ are pairs of composable morphisms in $\mathcal{M}$ and H. Thus, the set $N(\mathcal{M})^H_2$ consists of equivalence classes of pairs (f_1, f_2) of composable morphisms in $\mathcal{M}$ with $(f_1, f_2) \sim (f_1', f_2')$ iff there are $(a_1, b_1), (a_2, b_2) \in A \rtimes B$ with $\partial(a_1) b_1 = b_2$ such that $f_1' = (a_1, b_1) \rhd' f_1$ and $f_2' = (a_2, b_2) \rhd' f_2$.

For any composable pair $(f_1, f_2) \in N(\mathcal{M})_2$, one has $d_0(f_1, f_2) = f_2$, $d_1(f_1, f_2) = f_2 \circ f_1$ and $d_2(f_1, f_2) = f_1$. This implies $d_0[(f_1, f_2)] = [f_2]$, $d_1[(f_1, f_2)] = [f_2 \circ f_1]$ and $d_2[(f_1, f_2)] = [f_1]$ for their equivalence classes in $N(\mathcal{M})^H_2$ and $N(\mathcal{M})^H_1$.

Applying the homotopy functor from Definition 8.9 thus yields a category $\mathcal{M}^H$ with objects $\mathrm{Ob}\, \mathcal{M}^H = N(\mathcal{M})^H_0 = \mathrm{Ob}\, \mathcal{M}/B$. Its generating morphisms are $A \rtimes B$-orbits of morphisms in $\mathcal{M}$, and the second relation in (71) translates into the relation $[f_2] \circ [f_1] = [f_2 \circ f_1]$ for the $A \rtimes B$-orbits of composable pairs (f_1, f_2) of morphisms in $\mathcal{M}$. □

We now restrict attention to Yetter-Drinfeld modules $(\mathcal{M}, \rhd, \delta)$ over group objects H in Cat and determine their biinvariants. We denote again by $F : \mathcal{M} \to H$ the functor defined by δ from Example 2.9, by $\iota : \mathcal{M}^{coH} \to \mathcal{M}$ the inclusion functor from Lemma 8.12 and by $\pi : \mathcal{M} \to \mathcal{M}^H$ the projection functor from Proposition 8.13.

Proposition 8.14.
Let $(\mathcal{M}, \rhd, F)$ be a Yetter-Drinfeld module over a group object H in Cat*. Then $\mathcal{M}_{inv}$ is given by*

$$\mathrm{Ob}\mathcal{M}_{inv} = \{\pi(M) \mid M \in \mathrm{Ob}\mathcal{M} \text{ with } F(M) = e\},$$
$$\mathrm{Hom}_{\mathcal{M}_{inv}}(\pi(M_1), \pi(M_2)) = \{\pi(f) \mid f \in \mathcal{M}^{(1)} \text{ with } \pi(s(f)) = \pi(M_1), \pi(t(f)) = \pi(M_2), F(f) = 1_e\}.$$

Proof. 1. We verify that $\mathcal{M}_{inv}$ is a category. If $F(M) = e$ for an object M in $\mathcal{M}$, then $F(g \rhd M) = g \cdot F(M) \cdot g^{-1} = e$ for all objects g in H by the Yetter-Drinfeld module condition in Example 2.9. Likewise, if f is a morphism in $\mathcal{M}$ with $F(f) = 1_e$, then $F(g \rhd f) = g \cdot F(f) \cdot g^{-1} = 1_e$ for all $g \in H^{(1)}$. This shows that for every object M and morphism f of $\mathcal{M}^{coH}$ the entire $\mathrm{Ob}\, H$-orbit of M and $H^{(1)}$-orbit of f is contained in $\mathcal{M}^{coH}$. Any identity morphism on an object $M \in \mathrm{Ob}\, \mathcal{M}^{coH}$

satisfies $F(1_M) = 1_e$ and hence is contained in $\mathcal{M}^{coH}$. If (f_1, f_2) is a pair of composable morphisms in $\mathcal{M}^{coH}$, then $F(f_2 \circ f_1) = F(f_2) \circ F(f_1) = 1_e$ and hence $f_2 \circ f_1 \in \mathcal{M}^{coH}$ as well.

Suppose now that $f_1 : M_0 \to M_1$ and $f_2 : M_1' \to M_2$ are morphisms in $\mathcal{M}^{coH}$ such that $\pi(f_1)$ and $\pi(f_2)$ are composable in $\mathcal{M}^H$. Then there is a $g \in \mathrm{Ob}\, H$ with $M_1' = g \rhd M_1$, and the morphisms f_1 and $g^{-1} \rhd f_2$ are composable in $\mathcal{M}^{coH}$. With the relations of $\mathcal{M}_{inv}$ one obtains $\pi(f_2) \circ \pi(f_1) = \pi(g^{-1} \rhd f_2) \circ \pi(f_1) = \pi((g^{-1} \rhd f_2) \circ f_1)$ with $(g^{-1} \rhd f_2) \circ f_1 \in \mathcal{M}^{coH}$.

2. We show that $\mathcal{M}_{inv}$ has the universal property of the image in Cat. The inclusion functor $I : \mathcal{M}_{inv} \to \mathcal{M}^H$ is a monomorphism in Cat and satisfies $IP = \pi\iota$, where $P : \mathcal{M}^{coH} \to \mathcal{M}_{inv}$ is the functor that sends an object M in $\mathcal{M}^{coH}$ to $\pi(M)$ and a morphism f in $\mathcal{M}^{coH}$ to $\pi(f)$. If (J, Q) is a pair of a monomorphism $J : \mathcal{C} \to \mathcal{M}^H$ and a functor $Q : \mathcal{M}^{coH} \to \mathcal{C}$ with $JQ = \pi\iota$, then J is a monomorphism in Cat. This allows one to identify $\mathcal{C}$ with a subcategory of $\mathcal{M}^H$ and J with its inclusion functor. As $JQ = \pi\iota$, the subcategory $\mathcal{C} \subset \mathcal{M}^H$ contains $\mathcal{M}_{inv}$ as a subcategory $\mathcal{M}_{inv} \subset \mathcal{C}$. Hence, there is a unique functor, the inclusion $V : \mathcal{M}_{inv} \to \mathcal{C}$, with $I = JV$. □

8.4 Alternative description of coequalisers in Cat in terms of generalised congruences

In this section we describe an alternative way to characterise coequalisers in Cat using the construction of Bednarczyk, Borzyszkowski and Pawlowski [BBP] via *generalised congruences* and associated *quotient categories*. For a summary of this construction, see also Bruckner [Br], Haucourt [Ha] and for a more detailed version Yau [Ya]. With this approach one can determine the invariants of a module over a group object in Cat via quotient categories associated to a generated generalised congruence. We will provide the details of this application in the next section.

For a morphism $f \in \mathrm{Hom}_{\mathcal{D}}(D, D')$ we denote by $s(f) = D$ and $t(f) = D'$ its source and target. For an equivalence relation $\sim_0$ on $\mathrm{Ob}(\mathcal{D})$ we say a sequence $(f_0, f_1, \ldots, f_n)$ of morphisms in $\mathcal{D}$ is $\sim_0$*-composable* if $t(f_i) \sim_0 s(f_{i+1})$ for all $i = 0, \ldots, n-1$.

Definition 8.15. *[Br, Def. 2.6] A* ***generalised congruence*** *on a small category $\mathcal{D}$ is a pair $(\sim_0, \sim_m)$ of equivalence relations $\sim_0$ on $\mathrm{Ob}(\mathcal{D})$ and $\sim_m$ on the set of non-empty, $\sim_0$-composable sequences of morphisms in $\mathcal{D}$ such that*

(G1) $X \sim_0 Y$ *implies* $(\mathrm{id}_X) \sim_m (\mathrm{id}_Y)$,
(G2) $(f_0, f_1, \ldots, f_n) \sim_m (h_0, h_1, \ldots, h_m)$ *implies* $t(f_n) \sim_0 t(h_m)$ *and* $s(f_0) \sim_0 s(h_0)$,
(G3) $s(h) = t(f)$ *implies* $(f, h) \sim_m (h \circ f)$,
(G4) *if* $(f_0, f_1, \ldots, f_n) \sim_m (f_0', f_1', \ldots, f_{n'}')$, $(h_0, \ldots, h_m) \sim_m (h_0', \ldots, h_{m'}')$ *and* $t(f_n) \sim_0 s(h_0)$, *then* $(f_0, f_1, \ldots, f_n, h_0, \ldots, h_m) \sim_m (f_0', f_1', \ldots, f_{n'}', h_0', \ldots, h_{m'}')$.

Lemma 8.16. *[BBP, 3.9 and Proposition 3.10] A generalised congruence $\sim = (\sim_0, \sim_m)$ on $\mathcal{D}$ defines a* **quotient category** *$\mathcal{D}/\sim$ with*

- $\mathrm{Ob}(\mathcal{D}/\sim) = \mathrm{Ob}(\mathcal{D})/\sim_0$,
- *$\sim_m$-classes of $\sim_0$-composable sequences $(f_0, \ldots, f_n)$ as morphisms from $[s(f_0)]$ to $[t(f_n)]$,*
- *composition of morphisms* $[(h_0, \ldots, h_m)] \circ [(f_0, f_1, \ldots, f_n)] = [(f_0, f_1, \ldots, f_n, h_0, \ldots, h_m)]$,

and an essentially surjective **quotient functor** *$Q : \mathcal{D} \to \mathcal{D}/\sim$ that sends an object $X \in \mathrm{Ob}(\mathcal{D})$ to $[X]$ and a morphism f in $\mathcal{D}$ to $[(f)]$.*

To describe coequalisers in Cat one considers generalised congruences that are generated by relations on small categories. More specifically, the relations relevant for determining coequalisers are defined by functors.

Definition 8.17. *Let $\mathcal{C}, \mathcal{D}$ be small categories.*

1. *A **relation** on $\mathcal{D}$ is a pair (R_0, R_m) of a relation R_0 on $\mathrm{Ob}(\mathcal{D})$ and a relation R_m on the set of finite, non-empty sequences of morphisms in $\mathcal{D}$.*
2. *For functors $F, K : \mathcal{C} \to \mathcal{D}$ the relation $R_{F=K} := (R_0, R_m)$ on $\mathcal{D}$ is given by*
   - $R_0 = \{(F(X), K(X)) : X \in \mathrm{Ob}(\mathcal{C})\} \subset \mathrm{Ob}(\mathcal{D}) \times \mathrm{Ob}(\mathcal{D})$,
   - $F(f) \sim_{R_m} K(f)$ *for all morphisms f in $\mathcal{C}$.*

Note that the total relation that identifies all objects and all sequences of morphisms in a small category $\mathcal{D}$ is a generalised congruence [BBP, Br]. Hence, for every relation (R_0, R_m) on $\mathcal{D}$ there is at least one generalised congruence $(\sim_0, \sim_m)$ on $\mathcal{D}$ with $R_0 \subset \sim_0$ and $R_m \subset \sim_m$. It is also straightforward to check that for any family $(\sim_{0,i}, \sim_{m,i})_{i\in I}$ of generalised congruences on $\mathcal{D}$, the intersection $(\cap_{i\in I} \sim_{0,i}, \cap_{i\in I} \sim_{m,i})$ is a generalised congruence on $\mathcal{D}$. One can thus define the minimal generalised congruence on $\mathcal{D}$ generated by a relation (R_0, R_m) as the intersection of all generalised congruences $(\sim_0, \sim_m)$ with $R_0 \subset \sim_0$ and $R_m \subset \sim_m$.

Definition 8.18. *[Br, Sec. 2] Let (R_0, R_m) be a relation on a small category $\mathcal{D}$. The **principal congruence generated by** (R_0, R_m) is the minimal generalised congruence $(\sim_0, \sim_m)$ on $\mathcal{D}$ with $R_0 \subset \sim_0$, $R_m \subset \sim_m$.*

Proposition 8.19. *[BBP, Proposition 4.1] The* **coequaliser** *of morphisms $F, K : \mathcal{C} \to \mathcal{D}$ in Cat is the quotient $\mathcal{D}/\sim$ for the principal congruence $\sim$ generated by $R_{F=K}$, together with the quotient functor $Q : \mathcal{D} \to \mathcal{D}/\sim$.*

The quotient with respect to the principal congruence generated by a relation is in general difficult to describe. The situation simplifies to some degree, if the underlying relation (R_0, R_m) on $\mathcal{D}$ consists of an *equivalence relation* R_0 and a relation R_m that identifies only singleton sequences of morphisms with equivalent sources and targets.

Lemma 8.20. *[BBP2, Lemma 3.6] Let (R_0, R_m) be a relation on $\mathcal{D}$ such that R_0 is an equivalence relation, R_m identifies only singleton sequences of morphisms and $(f) \sim_{R_m} (f')$ implies $s(f) \sim_{R_0} s(f')$ and $t(f) \sim_{R_0} t(f')$. Then the objects of $\mathcal{D}/\sim$ are R_0-equivalence classes of objects in $\mathcal{D}$.*

Proof. 1. We show that the following defines a generalised congruence $(R_0, \sim^s_m)$ on $\mathcal{D}$

$$(f_0, \ldots, f_n) \sim^s_m (h_0, \ldots, h_m) \Leftrightarrow s(f_0) \sim_{R_0} s(h_0) \text{ and } t(f_n) \sim_{R_0} t(h_m). \tag{74}$$

As R_0 is an equivalence relation on $\mathrm{Ob}(\mathcal{D})$, $\sim^s_m$ is an equivalence relation on the set of non-empty, R_0-composable sequences of morphisms in $\mathcal{D}$. Clearly, $(\sim_{R_0}, \sim^s_m)$ satisfies (G1)-(G4).

2. We show that the principal congruence $\sim = (\sim_0, \sim_m)$ generated by (R_0, R_m) satisfies $\sim_0 = R_0$ and $\sim_m \subset \sim^s_m$: By definition, it is the intersection of all generalised congruences $(\sim^+_0, \sim^+_m)$ on $\mathcal{D}$ satisfying $R_0 \subset \sim^+_0$, $R_m \subset \sim^+_m$. As $(f) \sim_{R_m} (f')$ implies $s(f) \sim_{R_0} s(f')$, $t(f) \sim_{R_0} t(f')$ and hence $(f) \sim^s_m (f')$, one has $\sim_{R_m} \subset \sim^s_m$. Thus, $(\sim_{R_0}, \sim^s_m)$ participates in the intersection, which yields $\sim_0 = \sim_{R_0}$. Both $\sim_m$ and $\sim^s_m$ are defined on the set of non-empty, $\sim_{R_0}$-composable sequences of morphisms, thus $\sim_m \subset \sim^s_m$. □

8.5 Alternative description of invariants in Cat via generalised congruences

Now we apply the description of coequalisers in Cat in terms of quotient categories from the last section to determine the invariants of a module $(\mathcal{M}, \rhd)$ over a group object H in Cat. This leads to an alternative proof of Proposition 8.13 and to an alternative characterisation of the biinvariants of a Yetter-Drinfeld module over a group object in Cat.

As the invariants of $(\mathcal{M}, \rhd)$ are the coequaliser of the functors $\rhd, \pi_2 : H \times \mathcal{M} \to \mathcal{M}$, the associated relation $R_{\rhd=\pi_2} = (R_0, R_m)$ on $\mathcal{M}$ is given by

- $X \sim_{R_0} g \rhd X$ for all $(g, X) \in \mathrm{Ob}(H \times \mathcal{M})$,
- $t \sim_{R_m} f \rhd t$ for all $(f, t) \in (H \times \mathcal{M})^{(1)}$.

It is easy to see that R_0 defines an equivalence relation on $\mathrm{Ob}(\mathcal{M})$ and R_m an equivalence relation on $\mathcal{M}^{(1)}$ and that the conditions of Lemma 8.20 are satisfied.

This guarantees that the objects of the quotient $\mathcal{M}^H = \mathcal{M}/\sim$ are R_0-equivalence classes of objects in $\mathcal{M}$. Its morphisms are represented by morphisms in $\mathcal{M}$.

Definition 8.21. *Let $(\mathcal{M}, \rhd)$ be a module over a group object H in* Cat*, and $R_{\rhd=\pi_2} = (R_0, R_m)$ the relation characterising its invariants. We say that $\Upsilon \in \mathcal{M}^{(1)}$ is a* ***representing singleton*** *of an R_0-composable sequence $(f_0, \ldots, f_n)$ if $\Upsilon = f_n \circ (\mathrm{id}_{c_n} \rhd f_{n-1}) \circ \ldots \circ (\mathrm{id}_{c_n \cdots c_1} \rhd f_0)$ for some $c_1, \ldots, c_n \in \mathrm{Ob}(H)$ with $s(f_i) = c_i \rhd t(f_{i-1})$ for $i = 1, \ldots, n$.*

Lemma 8.22. *Let $(\mathcal{M}, \rhd)$ be a module over a group object H in* Cat *and $(\sim_0, \sim_m)$ the principal congruence generated by $R_{\rhd=\pi_2} = (R_0, R_m)$. Then every non-empty R_0-composable sequence $f := (f_0, \ldots, f_n)$ of morphisms in $\mathcal{M}$ has a representing singleton. Every representing singleton Υ of f satisfies $\Upsilon \sim_m f$.*

Proof. As $(f_0, \ldots, f_n)$ is R_0-composable, there are $c_1, ..., c_n \in \mathrm{Ob}(H)$ with $s(f_i) = c_i \rhd t(f_{i-1})$ for $i = 1, ..., n$. The morphisms $g_i := \mathrm{id}_{c_n \cdots c_{i+1}} \rhd f_i$ satisfy

$$s(g_i) = c_n \cdots c_{i+1} \rhd s(f_i) = c_n \cdots c_{i+1} \rhd (c_i \rhd t(f_{i-1})) = c_n \cdots c_i \rhd t(f_{i-1}) = t(g_{i-1})$$

as well as $g_i \sim_m f_i$ because $R_m \subset \sim_m$. With (G4) in Definition 8.15 this implies $(g_0, \ldots, g_n) \sim_m (f_0, \ldots, f_n)$, and with (G3) and the transitivity of $\sim_m$ one obtains $g_n \circ \ldots \circ g_0 \sim_m (f_0, \ldots, f_n)$. □

Representing singletons are in general not unique. Recall that for a module $(\mathcal{M}, \rhd)$ over a group object $H = \triangledown(B, A \blacktriangleright, \partial)$ the action on the objects and hence R_0 are characterised by the B-action on $\mathrm{Ob}(\mathcal{M})$. The action on morphisms and R_m are given by an $A \rtimes B$-action on $\mathcal{M}^{(1)}$. By Lemma 8.22 any R_0-composable sequence has an equivalent representing singleton. As a consequence, one can reduce the description of $\sim_m$ to only singleton sequences of morphisms. This leads to an alternative proof of Proposition 8.13.

*Proof of Proposition 8.13 (**alternative version**).*
We show that the principal congruence $(\sim_0, \sim_m)$ generated by $R_{\rhd=\pi_2} = (R_0, R_m)$ is given by $\sim_0 = R_0$ and $f := (f_0, \ldots, f_n) \sim_m h := (h_0, \ldots, h_m)$ iff their representative singletons Υ, Θ are equivalent: $\Upsilon \sim \Theta$. Here, $\sim$ is the equivalence relation generated by orbits of the $A \rtimes B$-action on $\mathcal{M}^{(1)}$ subject to the relations $[\tau_2] \circ [\tau_1] = [\tau_2 \circ \tau_1]$ for all $A \rtimes B$-orbits $[\tau_1]$, $[\tau_2]$ of composable morphisms τ_1, τ_2 in $\mathcal{M}$.

1. We first show that this is well-defined: If Υ, Υ' are representing singletons of $(f_0, \ldots, f_n)$, then $\Upsilon \sim \Upsilon'$.
Let $(f_0, \ldots, f_n)$ be R_0-composable with $s(f_i) = c_i \rhd t(f_{i-1}) = c_i' \rhd t(f_{i-1})$ for some $c_1, \ldots, c_n \in \mathrm{Ob}(H)$, $c_1', \ldots, c_n' \in \mathrm{Ob}(H)$ for $i = 1, \ldots, n$ and associated representing singletons

$$\Upsilon = f_n \circ (\mathrm{id}_{c_n} \rhd f_{n-1}) \circ \ldots \circ (\mathrm{id}_{c_n \ldots c_1} \rhd f_0), \quad \Upsilon' = f_n \circ (\mathrm{id}_{c_n'} \rhd f_{n-1}) \circ \ldots \circ (\mathrm{id}_{c_n' \ldots c_1'} \rhd f_0).$$

From the identity

$$\mathrm{id}_{c'_n\ldots c'_{k+1}I(c_n\ldots c_{k+1})} \rhd \mathrm{id}_{c_n\ldots c_{k+1}} \rhd f_k = id_{c'_n\ldots c'_{k+1}} \rhd f_k,$$

we have $\mathrm{id}_{c'_n\ldots c'_{k+1}} \rhd f_k \sim_{R_m} \mathrm{id}_{c_n\ldots c_{k+1}} \rhd f_k$ for all $k = 0,\ldots,n-1$. The relation $[\tau_2]\circ[\tau_1] = [\tau_2\circ\tau_1]$ for composable morphisms in $\mathcal{M}$ then implies $\Upsilon \sim \Upsilon'$.

2. By Lemma 8.20 the principal congruence generated by $R_{\rhd=\pi_2}$ is given by R_0 on the objects.

2.1. As R_m is an equivalence relation on $\mathcal{M}^{(1)}$, it follows that $\sim_m$ is an equivalence relation on the set of non-empty R_0-composable sequences of morphisms.

2.2. We show that $(R_0, \sim_m)$ is a generalised congruence by verifying (G1)-(G4) in Definition 8.15:

(G1) If $X \sim_{R_0} Y$ for $X, Y \in \mathrm{Ob}(\mathcal{M})$, then there exists a $g \in \mathrm{Ob}(H)$ with $X = g \rhd Y$, which implies $\mathrm{id}_X = \mathrm{id}_g \rhd \mathrm{id}_Y$ and hence $\mathrm{id}_X \sim_m \mathrm{id}_Y$.

(G2) If $(f_0,\ldots,f_n) \sim_m (h_0,\ldots,h_m)$ for sequences of R_0-composable morphisms with representing singletons Υ, Θ, then we obtain $\Upsilon \sim \Theta$ and

$$t(f_n) \sim_{R_0} t(\Upsilon), \quad s(f_0) \sim_{R_0} s(\Upsilon), \quad t(h_m) \sim_{R_0} t(\Theta), \quad s(h_0) \sim_{R_0} s(\Theta).$$

This is derived from the definition of representing singletons. As morphisms $f \sim_{R_m} f'$ and composable $\tau_1, \tau_2 \in \mathcal{M}^{(1)}$ satisfy

$$\begin{aligned}&s(f) \sim_{R_0} s(f'), \quad t(f) \sim_{R_0} t(f'), \quad s([\tau_2]\circ[\tau_1]) = [s(\tau_1)] = s([\tau_2\circ\tau_1]),\\ &t([\tau_2]\circ[\tau_1]) = [t(\tau_2)] = t([\tau_2\circ\tau_1]),\end{aligned}$$

the representing singletons Υ and Θ satisfy $s(\Upsilon) \sim_{R_0} s(\Theta)$ and $t(\Upsilon) \sim_{R_0} t(\Theta)$. The claim then follows from the transitivity of R_0.

(G3) For (f, h) with $s(h) = t(f)$ a representing singleton of (f, h) is $h \circ f$, hence $(f, h) \sim_m h \circ f$.

(G4) For $(f_0,\ldots,f_n) \sim_m (f'_0,\ldots,f'_{n'})$, $(h_0,\ldots,h_m) \sim_m (h'_0,\ldots,h'_{m'})$ R_0-composable sequences with $t(f_n) \sim_{R_0} s(h_0)$ and representing singletons

$$\begin{aligned}&\Upsilon = f_n \circ (\mathrm{id}_{c_n} \rhd f_{n-1}) \circ \ldots \circ (\mathrm{id}_{c_n\ldots c_1} \rhd f_0), \qquad \Upsilon',\\ &\Theta = h_m \circ (\mathrm{id}_{d_m} \rhd h_{m-1}) \circ \ldots \circ (\mathrm{id}_{d_m\ldots d_1} \rhd h_0), \quad \Theta' = h'_{m'} \circ (\mathrm{id}_{q_{m'}} \rhd h'_{m'-1}) \circ \ldots \circ (\mathrm{id}_{q_{m'}\ldots q_1} \rhd h'_0),\end{aligned}$$

it is $\Upsilon \sim \Upsilon'$, $\Theta \sim \Theta'$. Additionally, there exist $x, x' \in \mathrm{Ob}(H)$ with $x \rhd t(f_n) = s(h_0)$, $x' \rhd t(f_{n'}) = s(h'_0)$. Then

$$\begin{aligned}\wp &:= \Theta \circ (\mathrm{id}_{d_m\ldots d_1 x} \rhd \Upsilon)\\ &= \Theta \circ (\mathrm{id}_{d_m\ldots d_1 x} \rhd f_n) \circ (\mathrm{id}_{d_m\ldots d_1 x c_n} \rhd f_{n-1}) \circ \ldots \circ (\mathrm{id}_{d_m\ldots d_1 x c_n \ldots c_1} \rhd f_0),\\ \wp' &:= \Theta' \circ (\mathrm{id}_{q_{m'}\ldots q_1 x'} \rhd \Upsilon')\end{aligned}$$

are representing singletons of $(f_0,\ldots,f_n,h_0,\ldots,h_m)$, $(f'_0,\ldots,f'_{n'},h'_0,\ldots,h'_{m'})$. From

$$[\Theta] = [\Theta'], \quad [\mathrm{id}_{d_m\ldots d_1 x} \rhd \Upsilon] = [\Upsilon] = [\Upsilon'] = [\mathrm{id}_{q_{m'}\ldots q_1 x'} \rhd \Upsilon']$$

we obtain $[\wp] = [\Theta]\circ[\Upsilon] = [\Theta']\circ[\Upsilon'] = [\wp']$ and the claim follows.

2.3. It remains to show that $(R_0, \sim_m)$ is minimal. Suppose $(\sim_0^*, \sim_m^*)$ is the principal congruence generated by (R_0, R_m), hence $R_m \subseteq \sim_m^*$. By Lemma 8.20 it is $\sim_0^* = R_0$. Let $f, h \in \mathcal{M}^{(1)}$ be composable. Then we have $h' \circ f' \sim_m^* h \circ f$ for all composable $h', f' \in \mathcal{M}^{(1)}$ with $h' \sim_m^* h$, $f' \sim_m^* f$ by (G4). This implies $[h]\circ[f] \subseteq [h\circ f]$. As $\sim_m^*$ is an equivalence relation on the set of non-empty, R_0-composable sequences of morphisms in $\mathcal{M}^{(1)}$, we obtain $[h]\circ[f] = [h\circ f]$. Hence, $\sim_m \subseteq \sim_m^*$. $\square$

Corollary 8.23. *Let $(\mathcal{M}, \rhd)$ be a module over a group object $H = \triangledown(B, A, \blacktriangleright, \partial)$ in* Cat *and $R_{\rhd=\pi_2} = (R_0, R_m)$ such that $h \circ f \sim_{R_m} h' \circ f'$ for all $f, f', h, h' \in \mathcal{M}^{(1)}$ with $s(h) = t(f)$, $s(h') = t(f')$ and $f \sim_{R_m} f'$, $h \sim_{R_m} h'$. Then the principal congruence generated by $R_{\rhd=\pi_2}$ is given by*

$$(f_0, \ldots, f_n) \sim_m (h_0, \ldots, h_m)$$

iff there are representing singletons Υ of $(f_0, \ldots, f_n)$ and Θ of $(h_0, \ldots, h_m)$ with $\Upsilon \sim_{R_m} \Theta$.

Proof. It directly follows $[h]_{R_m} \circ [f]_{R_m} = [h \circ f]_{R_m}$ for all composable $h, f \in \mathcal{M}^{(1)}$ and R_m is an equivalence relation on $\mathcal{M}^{(1)}$. Hence, the claim follows from Proposition 8.13. □

We also obtain an alternative description of the biinvariants of a Yetter-Drinfeld module over a group object in Cat. This is already described in Proposition 8.14, but can also be formulated in terms of generalised congruences.

Proposition 8.24. *Let $(\mathcal{M}, \rhd, F)$ be a Yetter-Drinfeld module over a group object H in* Cat*. Then the principal congruence $\sim = (\sim_0, \sim_m)$ generated by $R_{\rhd=\pi_2}$ restricts to a generalised congruence $\sim' = (\sim'_0, \sim'_m)$ on $\mathcal{M}^{coH}$, and the biinvariants of $(\mathcal{M}, \rhd, F)$ are the quotient $\mathcal{M}_{inv} = \mathcal{M}^{coH}/\sim'$.*

Proof. Due to the conditions on a Yetter-Drinfeld module in Example 2.9, the equivalence relation $\sim_0 = R_0$ restricts to an equivalence relation $\sim'_0$ on the set of objects A with $F(A) = e$. Likewise, the equivalence relation $\sim_m$ restricts to an equivalence relation $\sim'_m$ on the set of non-empty R_0-composable sequences of morphisms f with $F(f) = 1_e$. That $\sim'_0$ and $\sim'_m$ define a generalised congruence on $\mathcal{M}^{coH}$ follows by a simple verification of (G1)-(G4). Denote by $\mathrm{im}(\mathcal{M}) = \mathcal{M}^{coH}/\sim'$ the associated quotient and by $\beta : \mathcal{M}^{coH} \to \mathrm{im}(\mathcal{M})$ the associated quotient functor.

We define a functor $\alpha : \mathrm{im}(\mathcal{M}) \to \mathcal{M}^H$ by setting

- on objects: $\alpha : [N]_{\sim'_0} \mapsto [N]_{\sim_0}$,
- on morphisms: $\alpha : [(f_0, f_1, \ldots, f_n)]_{\sim'_m} \mapsto [(f_0, f_1, \ldots, f_n)]_{\sim_m}$

and show that α is a monomorphism in Cat with $\alpha \circ \beta = \pi \circ \iota$, where $\iota : \mathcal{M}^{coH} \to \mathcal{M}$ and $\pi : \mathcal{M} \to \mathcal{M}^H$ denote the inclusion and projection functor.

That α is a functor with $\alpha \circ \beta = \pi \circ \iota$ follows directly. To see that α is a monomorphism, let $q_1, q_2 : \mathcal{Y} \to \mathrm{im}(\mathcal{M})$ be functors with $\alpha \circ q_1 = \alpha \circ q_2$ for some small category $\mathcal{Y}$. For all objects Y in $\mathcal{Y}$ and $M_i \in \mathrm{Ob}(\mathcal{M})$ with $q_i(Y) = [M_i]_{\sim'_0}$ the Yetter-Drinfeld module condition then implies

$$q_1(Y) = [M_1]_{\sim'_0} = [M_1]_{\sim_0} = \alpha \circ q_1(X) = \alpha \circ q_2(X) = [M_2]_{\sim_0} = [M_2]_{\sim'_0} = q_2(Y).$$

Likewise, if g is a morphism in $\mathcal{Y}$ with $(f_0, \ldots, f_n) \in q_1(g)$ and $(h_0, \ldots, h_m) \in q_2(g)$, the identity $\alpha \circ q_1(g) = \alpha \circ q_2(g)$ yields $(f_0, \ldots, f_n) \sim_m (h_0, \ldots, h_m)$. As $(f_0, \ldots, f_n)$, $(h_0, \ldots, h_m)$ are sequences of morphisms in $\mathcal{M}^{coH}$, this implies $(f_0, \ldots, f_n) \in [(h_0, \ldots, h_m)]_{\sim'_m}$ and hence $q_1(g) = q_2(g)$.

To show the universal property of $(\mathrm{im}(\mathcal{M}), \alpha, \beta)$, let $j : X \to \mathcal{M}^H$ be a monomorphism and $q : \mathcal{M}^{coH} \to X$ a morphism in Cat with $j \circ q = \pi \circ \iota$.

We define a functor $\nu : \mathrm{im}(\mathcal{M}) \to X$ by setting

- on objects: $\nu : [N]_{\sim'_0} \mapsto q(N)$,
- on morphisms: $\nu : [(f_0, \ldots, f_n)]_{\sim'_m} \mapsto q(f)$, where f is a morphism in $\mathcal{M}^{coH}$ with $(f) \sim'_m (f_0, \ldots, f_n)$. It exists by Lemma 8.22.

It is well-defined, since $N \sim_0' N'$ implies $N \sim_0 N'$ and hence $j \circ q(N) = \pi \circ \iota(N) = \pi \circ \iota(N') = j \circ q(N')$. As j is a monomorphism, it follows that $q(N) = q(N')$. Analogously, $(f) \sim_m' (f')$ implies $(f) \sim_m (f')$ and $j \circ q(f) = \pi \circ \iota(f) = \pi \circ \iota(f') = j \circ q(f')$ and hence $q(f) = q(f')$.

For the functoriality of ν note that $\nu([\mathrm{id}_N]_{\sim_m'}) = q(\mathrm{id}_N) = \mathrm{id}_{q(N)}$. If $(f_0, \ldots, f_n)$, $(h_0, \ldots, h_m)$ are $\sim_0'$-composable sequences of morphisms in $\mathcal{M}^{coH}$ with $s(f_0) \sim_0' t(h_m)$, then by Lemma 8.22 they have representing singletons $f \sim_m' (f_0, \ldots, f_n)$ and $h \sim_m' (h_0, \ldots, h_m)$ with $s(h) = x \rhd t(f)$ for an $x \in \mathrm{Ob}(H)$. This implies $(f_0, \ldots, f_n, h_0, \ldots, h_m) \sim_m' h \circ (\mathrm{id}_x \rhd f)$ and

$$\begin{aligned}
&\nu([(f_0, \ldots, f_n)]_{\sim_m'} \circ [(h_0, \ldots, h_m)]_{\sim_m'}) = \nu([(f_0, \ldots, f_n, h_0, \ldots, h_m)]) = \nu(h \circ (\mathrm{id}_x \rhd f)) \\
&= \nu(h) \circ \nu(\mathrm{id}_x \rhd f) = \nu(h) \circ \nu(f) = \nu([(f_0, \ldots, f_n)]_{\sim_m'}) \circ \nu([(h_0, \ldots, h_m)]_{\sim_m'}).
\end{aligned}$$

Hence, ν is a functor. A simple computation using $j \circ q = \pi \circ \iota$ shows that ν satisfies $\alpha = j \circ \nu$. The fact that j is a monomorphism then yields that $\nu : \mathrm{im}(\mathcal{M}) \to X$ is the unique functor with $\alpha = j \circ \nu$. Thus, $\mathrm{im}(\mathcal{M})$ has the universal property of the image. $\square$

Proposition 8.24 means that one only considers R_0-composable sequences of morphisms in $\mathcal{M}^{coH}$. Nevertheless, composites $\tau_2 \circ \tau_1$, $\tau_2' \circ \tau_1' \in (\mathcal{M}^{coH})^{(1)}$ with $\tau_1 \sim_{R_m} \tau_1'$, $\tau_2 \sim_{R_m} \tau_2'$ can be identified $\tau_2 \circ \tau_1 \sim_m' \tau_2' \circ \tau_1'$ in $(\mathcal{M}^{coH})^{(1)}$, even if $\tau_1, \tau_2, \tau_1', \tau_2'$ are morphisms in $\mathcal{M}$ but not in $\mathcal{M}^{coH}$.

Proposition 8.24 leads to the question whether $\mathcal{M}_{inv}$ is the coequaliser of the restricted action

$$\rhd_{|\mathcal{M}^{coH}} := \rhd \circ (1_H \times \iota) : H \times \mathcal{M}^{coH} \to \mathcal{M}^{coH}$$

and $\pi_2 : H \times \mathcal{M}^{coH} \to \mathcal{M}^{coH}$. This is true iff $\sim' = (\sim_0', \sim_m')$ is minimal, i.e. the principal congruence generated by $R_{\rhd_{|\mathcal{M}^{coH}} = \pi_2}$. In general this is not clear, but the following Lemma provides a sufficient condition applicable to the Kitaev model.

Lemma 8.25. *Let Σ be a surface of genus $g \geq 1$ and $H = \triangledown(B, A, \blacktriangleright, \partial)$ with $\partial \equiv e_B$ a group object in* Cat*, such that the coinvariants $\mathcal{M}^{coH}$ are a full subcategory of $\mathcal{M} = H^{\times E}$. Then the protected object $\mathcal{M}_{inv}$ is the coequaliser of the restricted action $\rhd_{|\mathcal{M}^{coH}}$ and $\pi_2 : H \times \mathcal{M}^{coH} \to \mathcal{M}^{coH}$.*

Proof. 1. We first show that for R_0-composable sequences $(f_0, \ldots, f_n)$ of morphisms in $\mathcal{M}$ the following two equivalences hold:

$$\begin{aligned}
s(f_0) \in \mathrm{Ob}(\mathcal{M}^{coH}) &\Leftrightarrow \; f_0, \ldots, f_n \text{ are all in } \mathcal{M}^{coH}, \\
s(f_0) \notin \mathrm{Ob}(\mathcal{M}^{coH}) &\Leftrightarrow \; \text{none of the morphisms } f_0, \ldots f_n \text{ is in } \mathcal{M}^{coH}.
\end{aligned}$$

To verify this recall that $H^{\times E}$ is a Yetter-Drinfeld module, which implies $g_1 \rhd X \in \mathrm{Ob}(\mathcal{M}^{coH})$ and $g_2 \rhd x \in (\mathcal{M}^{coH})^{(1)}$ for all $g_1 \in \mathrm{Ob}(H)$, $g_2 \in H^{(1)}$, $X \in \mathrm{Ob}(\mathcal{M}^{coH})$, $x \in (\mathcal{M}^{coH})^{(1)}$. For a morphism $f = (a, b) \in H^{(1)} = A \rtimes B$, the source and target of f are given by $s(f) = b$ and $t(f) = \partial(a)b = b$.

Hence, for an R_0-composable sequence $(f_0, \ldots, f_n)$ of morphisms in $\mathcal{M}$ with $s(f_0) \in \mathrm{Ob}(\mathcal{M}^{coH})$ inductively follows that $s(f_i) = t(f_i) \in \mathrm{Ob}(\mathcal{M}^{coH})$ for all $i = 0, \ldots, n$. Thus, we obtain $f_i \in (\mathcal{M}^{coH})^{(1)}$ for $i = 0, \ldots, n$, as $\mathcal{M}^{coH}$ is a full subcategory. Clearly, $f_0 \in (\mathcal{M}^{coH})^{(1)}$ implies $s(f_0) \in \mathrm{Ob}(\mathcal{M}^{coH})$. Analogously, $f_0 \notin (\mathcal{M}^{coH})^{(1)}$ yields $s(f_0) = t(f_0) \notin \mathrm{Ob}(\mathcal{M}^{coH})$, as the latter is a full subcategory. If $s(f_0) \notin \mathrm{Ob}(\mathcal{M}^{coH})$, then $s(f_0) = t(f_0) \sim_{R_0} s(f_1)$. It follows inductively that $s(f_i)$ and f_i are not in $\mathcal{M}^{coH}$ for $i = 0, \ldots, n$.

2. We now show that the restriction $\sim' = (\sim_0', \sim_m')$ of the principal congruence generated by $R_{\rhd = \pi_2}$ on $\mathcal{M}^{coH}$ is the principal congruence generated by $R_{\rhd_{|\mathcal{M}^{coH}} = \pi_2}$. Suppose there exists a generalised

congruence $(\sim_0', \sim_m^\#)$ on $\mathcal{M}^{coH}$ with $R_{m|\mathcal{M}^{coH}} \subseteq \sim_m^\# \subsetneq \sim_m'$. We define $(\sim_0, \sim_m^k)$ on $\mathcal{M}$ via

$$(f_0, \ldots, f_n) \sim_m^k (h_0, \ldots, h_m) :\Leftrightarrow \begin{cases} (f_0, \ldots, f_n) \sim_m^\# (h_0, \ldots, h_m) & \text{if } s(f_0), s(h_0) \in \mathrm{Ob}(\mathcal{M}^{coH}), \\ (f_0, \ldots, f_n) \sim_m^s (h_0, \ldots, h_m) & \text{if } s(f_0) \notin \mathrm{Ob}(\mathcal{M}^{coH}) \\ & \quad \text{or } s(h_0) \notin \mathrm{Ob}(\mathcal{M}^{coH}) \end{cases}$$

on the set of R_0-composable sequences of morphisms in $\mathcal{M}$, with $\sim_m^s$ from (74). We show that $(\sim_0, \sim_m^k)$ is a generalised congruence on $\mathcal{M}$. By 1. it is well-defined and as $(\sim_0, \sim_m^s)$ is a generalised congruence, $\sim_m^k$ is an equivalence relation. Likewise (G1)-(G3) follow. For $(f_0, \ldots, f_n) \sim_m^k (f_0', \ldots, f_{n'}')$, $(h_0, \ldots, h_m) \sim_m^k (h_0', \ldots, h_{m'}')$ with $t(f_n) \sim_0 s(h_0)$ either both pairs of sequences are related via $\sim_m^\#$ and hence also $(f_0, \ldots, f_n, h_0, \ldots, h_m)$ and $(f_0', \ldots, f_{n'}', h_0', \ldots, h_{m'}')$ by (G4) or they are both related via $\sim_m^s$.

The intersection of $(\sim_0, \sim_m)$ and $(\sim_0, \sim_m^k)$ is a generalised congruence on $\mathcal{M}$, which is a proper subset of the principal congruence $(\sim_0, \sim_m)$, as $\sim_m^\# \subsetneq \sim_m'$. This contradicts the minimality of $(\sim_0, \sim_m)$. □

Both approaches, the one in Section 8.2 and the one in Section 8.4 can be applied to determine coequalisers and invariants of modules over group objects in Cat. Here, the approach via nerve and homotopy functor from Section 8.2 appears to be more elegant. In particular, instead of dealing with sequences of morphisms of any finite length, by applying nerve and homotopy functor one can restrict to composable pairs of morphisms. Consequently, the essential relation determining the equivalence classes of morphisms in the invariants $\mathcal{M}^H$ is more transparent.

Although the approach involving generalised congruences is more technical, it also provides some insights about the invariants and protected objects. Specifically, it is helpful in determining when the protected object is the coequaliser of a restricted action. Moreover, the generalised congruences from (74) and Corollary 8.23 offer some intuition and examples for the equivalence classes of morphisms in $\mathcal{M}_{inv}$.

8.6 Protected objects for group objects in Cat

We now give a concrete description of the coinvariants and the protected objects for oriented surfaces Σ of genus $g \geq 1$ and group objects $H = \triangledown(B, A, \blacktriangleright, \partial)$ in Cat. We start by considering the Yetter-Drinfeld module and the coinvariants for the standard graph from (21) and show that they are given by group homomorphisms $\rho : F_{2g} \to A \rtimes B$ and $\rho : \pi_1(\Sigma) \to A \rtimes B$, respectively. To describe their category structure, we consider group-valued 1-cocycles.

Definition 8.26.
Let K, A be groups and $\blacktriangleright : K \times A \to A$ a group action of K on A by automorphisms.

1. *A 1-**cocycle** is a map $\phi : K \to A$ with $\phi(\lambda\mu) = \phi(\lambda) \cdot (\lambda \blacktriangleright \phi(\mu))$ for all $\lambda, \mu \in K$.*
2. *A 1-**coboundary** is a map $\eta_a : K \to A$, $\lambda \mapsto a(\lambda \blacktriangleright a^{-1})$ for some $a \in A$.*
3. *$\phi, \psi : K \to A$ are **related by a coboundary** if $\psi(\lambda) = a \cdot \phi(\lambda) \cdot (\lambda \blacktriangleright a^{-1})$ for some $a \in A$.*

If A is abelian, 1-cocycles form a group $Z^1(K, A, \blacktriangleright)$ with pointwise multiplication and coboundaries form a subgroup $B^1(K, A, \blacktriangleright)$. The factor group is the first cohomology group $H^1(K, A, \blacktriangleright)$. More generally, 1-cocycles with values in a (not necessarily abelian) group A arise from group homomorphisms into a semidirect product $A \rtimes B$.

Lemma 8.27.
Let $\triangleright\colon B\times A\to A$ *a group action by automorphisms and* $A\rtimes B$ *the associated semidirect product.*

1. *Group homomorphisms* $\sigma: K\to A\rtimes B$ *correspond to pairs* (ϕ,ρ) *of a group homomorphism* $\rho: K\to B$ *and a 1-cocycle* $\phi: K\to A$ *for the action* $\rho^*\triangleright\colon K\times A\to A$, $(\lambda,a)\mapsto\rho(\lambda)\triangleright a$.
2. *Two 1-cocycles* $\phi,\phi': K\to A$ *for* $\rho^*\triangleright$ *are related by a coboundary iff the group homomorphisms* $(\phi,\rho),(\phi',\rho): K\to A\rtimes B$ *are related by conjugation with* $A\subset A\rtimes B$.

If the semidirect product in Lemma 8.27 arises from a crossed module $(B,A,\triangleright,\partial)$, the group homomorphism $\partial: A\to B$ allows one to organise the group homomorphisms $\rho: K\to B$ and 1-cocycles $\phi: K\to A$ from Lemma 8.27 into a groupoid. Denoting by $\phi\cdot\psi$ and ϕ^{-1} the pointwise product and inverse of maps $\phi,\psi: K\to A$ we have

Lemma 8.28.
Any group K *and crossed module* $(B,A,\triangleright,\partial)$ *defines a groupoid* $\mathrm{Hom}(K,B\triangleright A)$ *with*

- *group homomorphisms* $\rho: K\to B$ *as objects,*
- $\mathrm{Hom}(\rho,\rho')=\{(\phi,\rho)\mid\phi: K\to A$ *1-cocycle for* $\rho^*\triangleright$ *with* $(\partial\circ\phi)\cdot\rho=\rho'\}$,
- *composition of morphisms:* $(\psi,(\partial\circ\phi)\cdot\rho)\circ(\phi,\rho)=(\psi\cdot\phi,\rho)$,
- *inverse morphisms:* $(\phi,\rho)^{-1}=(\phi^{-1},(\partial\circ\phi)\cdot\rho)$.

Proof. A direct computation using (70) shows that for any pair (ϕ,ρ) of a group homomorphism $\rho: K\to B$ and a 1-cocycle $\phi: K\to A$ for $\rho^*\triangleright$, the map $(\partial\circ\phi)\cdot\rho: K\to B$ is another group homomorphism. Similarly, if ϕ is a 1-cocycle for $\rho^*\triangleright$ and ψ a 1-cocycle for $((\partial\circ\phi)\cdot\rho)^*\triangleright$, then $\psi\cdot\phi$ is another 1-cocycle for $\rho^*\triangleright$. The formula for the inverse morphism follows directly. □

By applying this lemma to Example 4.8, we obtain a groupoid that describes the Yetter-Drinfeld module for a group object $H=\triangledown(B,A,\triangleright,\partial)$ in Cat and the standard graph (21), if we set $K=F_{2g}$ and identify the generators of F_{2g} with the edges of the graph. An analogous result holds for the associated coinvariants for $K=\pi_1(\Sigma)$ and any properly embedded graph with a single vertex.

Proposition 8.29.
Let Γ *be a properly embedded graph with a single vertex on a surface* Σ *of genus* $g\geq 1$ *and* $H=\triangledown(B,A,\triangleright,\partial)$ *a group object in* Cat*. Then the associated coinvariants are the groupoid from Lemma 8.28 for* $K=\pi_1(\Sigma)$.

Proof. By Theorem 5.23 it suffices to consider the graph in (21).

By Example 4.8 the coinvariants are the equaliser of the morphisms $\eta\,\epsilon, F: H^{\times 2g}\to H$ in Cat, where $\epsilon: H^{\times 2g}\to\{\cdot\}$ is the terminal morphism, $\eta:\{\cdot\}\to H$ is as in Definition 8.1 and $F: H^{\times 2g}\to H$ is given by $F(a_1,b_1,\ldots,a_g,b_g)=[b_g^{-1},a_g]\cdots[b_1^{-1},a_1]$. By Lemma 8.6, this equaliser is the subcategory $\mathcal{E}\subset H^{\times 2g}$ consisting of objects C and morphisms f with $F(C)=e$ and $F(f)=1_e$. For $H=\triangledown(B,A,\triangleright,\partial)$, this yields with Theorem 8.4

$$\mathrm{Ob}(\mathcal{E})=\{(a_1,b_1,\ldots,a_g,b_g)\in B^{\times 2g}\mid[b_g^{-1},a_g]\cdots[b_1^{-1},a_1]=1\},$$
$$\mathcal{E}^{(1)}=\{(a_1,b_1,\ldots,a_g,b_g)\in(A\rtimes B)^{2g}\mid[b_g^{-1},a_g]\cdots[b_1^{-1},a_1]=1\}.$$

Thus, every object $\rho\in\mathrm{Ob}(\mathcal{E})$ corresponds to a group homomorphism $\rho:\pi_1(\Sigma)\to B$ and every morphism $\sigma\in\mathcal{E}^{(1)}$ to a group homomorphism $\sigma:\pi_1(\Sigma)\to A\rtimes B$. By Lemma 8.27 the latter defines a pair $\sigma=(\phi,\rho)$ of a group homomorphism $\rho:\pi_1(\Sigma)\to B$ and a 1-cocycle ϕ for $\rho^*\triangleright$. □

We now use the description of the coinvariants in Proposition 8.29 and the description of the image object in Cat from Proposition 8.14 to compute the protected object for a surface Σ of genus $g \geq 1$ and a crossed module $(B, A, \blacktriangleright, \partial)$.

Theorem 8.30. *The protected object for a group object $H = \triangledown(B, A, \blacktriangleright, \partial)$ in* Cat *and a surface Σ of genus $g \geq 1$ is a groupoid $\mathcal{M}_{H,\Sigma}$ with*

- *conjugacy classes of group homomorphisms $\rho : \pi_1(\Sigma) \to B$ as objects,*
- *equivalence classes of group homomorphisms $\tau = (\phi, \rho) : \pi_1(\Sigma) \to A \rtimes B$ as morphisms from $[\rho]$ to $[(\partial \circ \phi) \cdot \rho]$.*

The equivalence relation is given by $\tau_2 \circ \tau_1 \sim \tau_2' \circ \tau_1'$ for all composable pairs (τ_1, τ_2) and (τ_1', τ_2') of group homomorphisms $\tau_i, \tau_i' : F_{2g} \to A \rtimes B$ such that τ_i, τ_i' are conjugate and $\tau_2 \circ \tau_1, \tau_2' \circ \tau_1'$ define group homomorphisms $\pi_1(\Sigma) \to A \rtimes B$.

Proof. By Theorem 5.23 the protected object of Σ is a topological invariant and can be computed from the standard graph in (21). This yields a Yetter-Drinfeld module $(\mathcal{M}, \rhd, \delta)$ over $\triangledown(B, A, \blacktriangleright, \partial)$ given by formula (29). Hence, we have $\mathcal{M}^{(1)} = (A \rtimes B)^{2g} \cong \mathrm{Hom}(F_{2g}, A \rtimes B)$ with the module structure given by conjugation and the comodule structure by the defining relation of $\pi_1(\Sigma)$.

By Proposition 8.29 the associated coinvariants form a groupoid $\mathcal{M}^{coH}$ with group homomorphisms $\rho : \pi_1(\Sigma) \to B$ as objects and group homomorphisms $\tau = (\phi, \rho) : \pi_1(\Sigma) \to A \rtimes B$ as morphisms from ρ to $(\partial \circ \phi) \cdot \rho$. By Propositions 8.13 and 8.14 the associated image object is the groupoid, whose objects are orbits of group homomorphisms $\rho : \pi_1(\Sigma) \to B$ under the conjugation action of B and whose morphisms are the images of group homomorphisms $\tau : \pi_1(\Sigma) \to A \rtimes B$ under the projection functor $\pi : \mathcal{M} \to \mathcal{M}^H$. The latter is given by the equivalence relation in the theorem. □

There are a number of cases in which the protected object has a particularly simple form. They correspond to crossed modules in which part of the data is trivial. The first corresponds to the case, where the Moore complex of the crossed module has trivial non-abelian homologies, namely $\ker(\partial) = \{1\}$ and $B/\partial(A) = 1$. The second is the case where the action of B on A is trivial.

Example 8.31. *Let Σ be a surface of genus $g \geq 1$ and $(B, A, \blacktriangleright, \partial)$ a crossed module, where ∂ is an isomorphism. Then the protected object has*

- *conjugacy classes of group homomorphisms $\rho : \pi_1(\Sigma) \to B$ as objects,*
- *exactly one morphism between any two objects.*

Proof. All morphism sets in the groupoid from Lemma 8.28 contain exactly one morphism, since $\mathrm{Hom}(\rho, \sigma) = \{(\partial^{-1}(\sigma \cdot \rho^{-1}), \rho)\}$ for all group homomorphisms $\rho, \sigma : \pi_1(\Sigma) \to B$. Conjugating a morphism in $\mathrm{Hom}(\rho, \sigma)$ with an element of $(a, b) \in A \rtimes B$ yields the unique morphism from $b\rho b^{-1}$ to $(\partial(a)b)\, \sigma\, (\partial(a)b)^{-1}$. This shows that all morphisms from conjugates of a group homomorphism $\rho : \pi_1(\Sigma) \to B$ to a conjugate of a group homomorphism $\sigma : \pi_1(\Sigma) \to B$ are conjugated and hence identified in $\mathcal{M}^H$ and in $\mathcal{M}_{inv}$. □

Example 8.32. *Let Σ be a surface of genus $g \geq 1$ and $(B, A, \blacktriangleright, \partial)$ a crossed module with a trivial group action $\blacktriangleright$. Then the protected object is* $\mathrm{Hom}(\pi_1(\Sigma), A \times B)/A \times B$ *with*

- *conjugacy classes of group homomorphisms $\rho : \pi_1(\Sigma) \to B$ as objects,*
- *group homomorphisms $\phi : \pi_1(\Sigma) \to A$ as morphisms from $[\rho]$ to $[(\partial \circ \phi) \cdot \rho]$.*

Proof. If $\blacktriangleright : B \times A \to A$ is trivial, then conditions (70) imply that A is abelian with $\partial(A) \subset Z(B)$. As A is abelian and $\blacktriangleright$ trivial, the 1-cocycles from Definition 8.26 are simply group homomorphisms

$\phi : \pi_1(\Sigma) \to A$ and any 1-coboundary is trivial. The groupoid $\mathcal{M}^{coH}$ from Lemma 8.28 thus has as objects group homomorphisms $\rho : \pi_1(\Sigma) \to B$ and as morphisms $\tau = (\phi, \rho) : \rho \to (\partial \circ \phi) \cdot \rho$ group homomorphisms $\tau = (\phi, \rho) : \pi_1(\Sigma) \to A \times B$.

As A is abelian and $\blacktriangleright$ trivial, two group homomorphisms $\tau = (\phi, \rho), \tau' = (\phi', \rho') : \pi_1(\Sigma) \to A \rtimes B$ are conjugate iff $\phi' = \phi$ and $\rho' = b\rho b^{-1}$ for some $b \in B$. Thus, the relation on morphisms in Theorem 8.30 identifies τ and τ' iff $\phi = \phi'$ and $[\rho] = [\rho']$. □

In the case of a trivial group homomorphism $\partial : A \to B$ all morphisms in $\mathcal{M}$, $\mathcal{M}^{coH}$, $\mathcal{M}^H$ and $\mathcal{M}_{inv}$ are automorphisms. This yields

Example 8.33. *Let Σ be a surface of genus $g \geq 1$ and $H = \bigtriangledown(B, A, \blacktriangleright, \partial)$ with A abelian and a trivial group homomorphism $\partial \equiv 1$. Then the associated protected object is*

$$\mathcal{M}_{H,\Sigma} = \amalg_{[\rho] \in \mathrm{Hom}(\pi_1(\Sigma), B)/B}\, G_{[\rho]},$$

where $G_{[\rho]}$ is a factor group of $H^1(\pi_1(\Sigma), A, \rho^* \blacktriangleright)$.

Proof. If ∂ is trivial and A abelian, then every 1-cocycle $\phi : \pi_1(\Sigma) \to A$ for $\rho^* \blacktriangleright$ defines an automorphism of ρ in $\mathcal{M}^{coH}$, which implies $\mathcal{M}^{coH} = \amalg_{\rho \in \mathrm{Hom}(\pi_1(\Sigma), B)} Z^1(\pi_1(\Sigma), A, \rho^* \blacktriangleright)$.

As all morphisms in $\mathcal{M}^{coH}$ are automorphisms, two morphisms given by group homomorphisms $\tau = (\phi, \rho) : \pi_1(\Sigma) \to A \rtimes B$ and $\tau' = (\phi', \rho') : \pi_1(\Sigma) \to A \rtimes B$ are composable iff $\rho = \rho'$. By Lemma 8.27, 2. two group homomorphisms $(\phi, \rho) : \pi_1(\Sigma) \to A \rtimes B$ and $(\phi', \rho) : \pi_1(\Sigma) \to A \rtimes B$ are related by conjugation with $A \subset A \rtimes B$ iff ϕ, ϕ' are related by a 1-coboundary. Thus for a group homomorphism $\rho : \pi_1(\Sigma) \to B$, the automorphism group of $[\rho]$ in $\mathcal{M}_{inv}$ is a factor group of $H^1(\pi_1(\Sigma), A, \rho^* \blacktriangleright)$. □

By Theorem 8.30 group homomorphisms $\tau, \tau' : \pi_1(\Sigma) \to A \rtimes B$ that are conjugated define the same morphism in $\mathcal{M}_{inv}$. This implies in particular that the morphism in $\mathcal{M}_{H,\Sigma}$ defined by a group homomorphism $\sigma = (\phi, \rho) : \pi_1(\Sigma) \to A \rtimes B$ depends on ϕ only up to coboundaries. Modifying ϕ with a coboundary yields a group homomorphism $\sigma' = (\phi', \rho)$ conjugated to σ by Lemma 8.27, 2.

However, except for the situation in Examples 8.31 and 8.32, it is difficult to describe the category $\mathcal{M}_{H,\Sigma}$ explicitly, even for genus $g = 1$ and crossed modules given by normal subgroups. This is due to the fact that the equivalence relation in Theorem 8.30 also identifies morphisms in $\mathcal{M}^{coH}$ in different $A \rtimes B$-orbits. This is illustrated by the following two examples.

Example 8.34. *Let Σ be the torus with $\pi_1(\Sigma) = \mathbb{Z} \times \mathbb{Z}$ and consider the crossed module $(S_3, A_3, \blacktriangleright, \iota)$, where $\iota : A_3 \to S_3$ is the inclusion and $\blacktriangleright : S_3 \times A_3 \to A_3$, $b \blacktriangleright a = bab^{-1}$ the conjugation action.*

We specify group homomorphisms $\rho : \mathbb{Z} \times \mathbb{Z} \to S_3$ and 1-cocycles $\phi : \mathbb{Z} \times \mathbb{Z} \to A_3$ by the images of $(1,0)$ and $(0,1)$ and write $\rho = (\rho(1,0), \rho(0,1))$ for the former and $\phi = \langle \phi(1,0), \phi(0,1) \rangle$ for the latter. We determine the conjugacy classes of group homomorphisms $\rho : \mathbb{Z} \times \mathbb{Z} \to S_3$. For instance, as

$$((12)\,\mathrm{id}\,(12), (12)(123)(12)) = (\mathrm{id}, (132)),$$

the group homomorphisms $(\mathrm{id}, (123))$ and $(\mathrm{id}, (132))$ are in the same conjugacy class. Overall, the conjugacy classes of group homomorphisms $\rho : \mathbb{Z} \times \mathbb{Z} \to S_3$ are given by

$C_1 = \{(\mathrm{id}, \mathrm{id})\}$	
$C_2 = \{(\mathrm{id}, c) \mid c \in A_3 \setminus \{\mathrm{id}\}\}$	$C_2' = \{(c, \mathrm{id}) \mid c \in A_3 \setminus \{\mathrm{id}\}\}$
$C_3 = \{(c, c) \mid c \in A_3 \setminus \{\mathrm{id}\}\}$	
$C_4 = \{(c, c') \mid c \neq c' \in A_3 \setminus \{\mathrm{id}\}\}$	
$C_5 = \{(\mathrm{id}, \sigma) \mid \sigma \in S_3 \setminus A_3\}$	$C_5' = \{(\sigma, \mathrm{id}) \mid \sigma \in S_3 \setminus A_3\}$
$C_6 = \{(\sigma, \sigma) \mid \sigma \in S_3 \setminus A_3\}$	

If $\rho(\mathbb{Z} \times \mathbb{Z}) \subset A_3$, then 1-cocycles for $\rho^* \blacktriangleright$ are simply group homomorphisms $\phi : \mathbb{Z} \times \mathbb{Z} \to A_3$. The 1-cocycles for $\rho^* \blacktriangleright$ for $\rho \in C_5$ $\rho \in C_5'$ and $\rho \in C_6$ on the contrary have a more restrictive form. If $\rho \in C_5$ the group homomorphism has the form $\rho(k, l) = \mathrm{id}$ if l is even and $\rho(k, l) = \sigma \in S_3 \setminus A_3$ if l is odd, where $k, l \in \mathbb{Z}$. A 1-cocycle $\phi : \pi_1(\Sigma) \to A_3$ for $\rho^* \blacktriangleright$ has to satisfy

$$\phi(k + r,\, l + q) = \phi(k, l)\, \rho(k, l)\, \phi(r, q)\, \rho(k, l)^{-1} \tag{75}$$

for all $k, l, r, q \in \mathbb{Z}$, which is equivalent to

$$\phi(k, l) = \begin{cases} \phi(k + r,\, l + q)\, \phi(r, q)^{-1} & l \text{ even} \\ \phi(k + r,\, l + q)\, \sigma\, \phi(r, q)^{-1}\, \sigma = \phi(k + r,\, l + q)\, \phi(r, q) & l \text{ odd.} \end{cases} \tag{76}$$

Swapping roles of k, r and l, q yields

$$\phi(r, q) = \begin{cases} \phi(k + r,\, l + q)\, \phi(k, l)^{-1} & q \text{ even} \\ \phi(k + r,\, l + q)\, \phi(k, l) & q \text{ odd.} \end{cases}$$

Combining the last two equations for l odd implies

$$\phi(k, l) = \begin{cases} \phi(k + r,\, l + q)^2\, \phi(k, l)^{-1} & l \text{ odd, } q \text{ even} \\ \phi(k + r,\, l + q)^2\, \phi(k, l) & l \text{ odd, } q \text{ odd.} \end{cases}$$

Hence, for l, q odd we obtain $\phi(k + r,\, l + q)^2 = \mathrm{id}$. Together with $\phi(k + r,\, l + q) \in A_3$ this yields $\phi(k + r, l + q) = \mathrm{id}$, which implies $\phi(k, l) = \mathrm{id}$ for all even $l \in \mathbb{Z}$ and all $k \in \mathbb{Z}$. Moreover, combined with (76) for all odd $l, l' \in \mathbb{Z}$ and all $k \in \mathbb{Z}$ it follows that

$$\mathrm{id} = \phi(k + k,\, l + l') = \phi(k, l)\, \phi(k, l')^{-1},$$

hence $\phi(k, l) = \phi(k, l')$. Overall, the 1-cocycles for $\rho^* \blacktriangleright$ with $\rho \in C_5$ are thus given by

$$\langle \mathrm{id}, c \rangle(k, l) = \begin{cases} c & l \text{ odd} \\ \mathrm{id} & l \text{ even,} \end{cases}$$

where $c \in A_3$ and $k, l \in \mathbb{Z}$. Analogous computations yield that the 1-cocycles for $\rho \in C_5'$ are given by $\langle c, \mathrm{id} \rangle$ and for $\rho \in C_6$ given by $\langle c, c \rangle$ with

$$\langle c, \mathrm{id} \rangle(k, l) = \begin{cases} c & k \text{ odd} \\ \mathrm{id} & k \text{ even} \end{cases}, \quad \langle c, c \rangle(k, l) = \begin{cases} c & k + l \text{ odd} \\ \mathrm{id} & k + l \text{ even} \end{cases},$$

where $c \in A_3$, $k, l \in \mathbb{Z}$.

The protected object $\mathcal{M}_{inv}$ has the objects C_1, C_2, C_2', C_3, C_4, C_5, C_5', C_6. Its morphisms are equivalence classes of morphisms in $\mathcal{M}^{coH}$, that is, of 1-cocycles. The morphisms in $\mathcal{M}^{coH}$ starting in $\rho \in C_1$ are the trivial 1-cocycle $\phi \equiv \mathrm{id}$ as identity morphism and the cocycles $\phi; \langle c, c' \rangle : (\mathrm{id}, \mathrm{id}) \to$

(c, c') for $c, c' \in A_3$. A pair $(\phi, \rho \equiv \mathrm{id})$ is conjugate to a pair $(\phi', \rho \equiv \mathrm{id})$ iff there exist $a \in A_3$, $b \in S_3$ such that

$$(a, b) \cdot (\phi(\lambda), \mathrm{id}) \cdot (a, b)^{-1} = (\phi'(\lambda), \mathrm{id})$$

for all $\lambda \in \mathbb{Z} \times \mathbb{Z}$. This is equivalent to

$$(ab\phi(\lambda)b^{-1}a^{-1}, \mathrm{id}) = (\phi'(\lambda), \mathrm{id}).$$

For $\phi = \langle \mathrm{id}, (123)\rangle$, $\phi' = \langle \mathrm{id}, (132)\rangle$ we find that with $a = \mathrm{id}$, $b = (12)$ we have $(\phi, \mathrm{id}) \sim (\phi', \mathrm{id})$. Overall, we obtain the conjugate pairs

$$\begin{aligned}
&\langle \mathrm{id}, (123)\rangle : (\mathrm{id}, \mathrm{id}) \to (\mathrm{id}, (123)) &&\sim&& \langle \mathrm{id}, (132)\rangle : (\mathrm{id}, \mathrm{id}) \to (\mathrm{id}, (132))\\
&\langle (123), \mathrm{id}\rangle : (\mathrm{id}, \mathrm{id}) \to ((123), \mathrm{id}) &&\sim&& \langle (132), \mathrm{id}\rangle : (\mathrm{id}, \mathrm{id}) \to ((132), \mathrm{id})\\
&\langle (123), (123)\rangle : (\mathrm{id}, \mathrm{id}) \to ((123), (123)) &&\sim&& \langle (132), (132)\rangle : (\mathrm{id}, \mathrm{id}) \to ((132), (132))\\
&\langle (123), (132)\rangle : (\mathrm{id}, \mathrm{id}) \to ((123), (132)) &&\sim&& \langle (132), (123)\rangle : (\mathrm{id}, \mathrm{id}) \to ((132), (123)),
\end{aligned}$$

where we again use cycle notation for elements of S_3. As each of these pairs defines a single morphism in $\mathcal{M}_{inv}$, there is exactly one morphism from C_1 to each of the conjugacy classes C_2, C_2', C_3, C_4. As $\mathcal{M}_{inv}$ is a groupoid, there is exactly one morphism between any two of these conjugacy classes.

Each of the 1-cocycles $\langle \mathrm{id}, c\rangle$, $\langle c, \mathrm{id}\rangle$, $\langle c, c\rangle$ with $c \in A_3$ defines a morphism in $\mathcal{M}^{coH}$ within the conjugacy classes C_5, C_5', C_6. The morphisms between objects in C_5 in $\mathcal{M}^{coH}$ are

$$\begin{aligned}
&\langle \mathrm{id}, \mathrm{id}\rangle : (\mathrm{id}, (12)) \to (\mathrm{id}, (12)) && \langle \mathrm{id}, \mathrm{id}\rangle : (\mathrm{id}, (13)) \to (\mathrm{id}, (13)) && \langle \mathrm{id}, \mathrm{id}\rangle : (\mathrm{id}, (23)) \to (\mathrm{id}, (23))\\
&\langle \mathrm{id}, (123)\rangle : (\mathrm{id}, (12)) \to (\mathrm{id}, (13)) && \langle \mathrm{id}, (123)\rangle : (\mathrm{id}, (13)) \to (\mathrm{id}, (23)) && \langle \mathrm{id}, (123)\rangle : (\mathrm{id}, (23)) \to (\mathrm{id}, (12))\\
&\langle \mathrm{id}, (132)\rangle : (\mathrm{id}, (12)) \to (\mathrm{id}, (23)) && \langle \mathrm{id}, (132)\rangle : (\mathrm{id}, (13)) \to (\mathrm{id}, (12)) && \langle \mathrm{id}, (132)\rangle : (\mathrm{id}, (23)) \to (\mathrm{id}, (13)).
\end{aligned}$$

All morphisms in the first line are conjugate. The first morphism in the second line is conjugate to the morphisms in the second and third line via cyclic permutations and transpositions. As

$$\begin{aligned}
\langle \mathrm{id}, \mathrm{id}\rangle &= \langle \mathrm{id}, (123)\rangle \circ \langle \mathrm{id}, (132)\rangle : (\mathrm{id}, (12)) \to (\mathrm{id}, (12))\\
\langle \mathrm{id}, (123)\rangle &= \langle \mathrm{id}, (132)\rangle \circ \langle \mathrm{id}, (132)\rangle : (\mathrm{id}, (12)) \to (\mathrm{id}, (13))
\end{aligned}$$

with $\langle \mathrm{id}, (132)\rangle \sim \langle \mathrm{id}, (123)\rangle$, all morphisms are identified by the relation in Theorem 8.30 and define a single morphism in $\mathcal{M}_{inv}$. Hence, the identity morphism is the only automorphism of C_5 in $\mathcal{M}_{inv}$ and likewise for C_5' and C_6. Thus $\mathcal{M}_{inv}$ is the groupoid in Figure 4.

Example 8.35. *Let Σ be the torus and consider the crossed module $(S_3, A_3, \blacktriangleright, \partial)$ with the trivial group homomorphism $\partial : A_3 \to S_3$, $a \mapsto \mathrm{id}$ and $\blacktriangleright : S_3 \times A_3 \to A_3$, $b \blacktriangleright a = bab^{-1}$.*

Then the protected object $\mathcal{M}_{inv}$ has the same objects as in Example 8.34. As ∂ is trivial, all morphisms in $\mathcal{M}^{coH}$ and $\mathcal{M}_{inv}$ are automorphisms. The object $(\mathrm{id}, \mathrm{id})$ in $\mathcal{M}^{coH}$ has the identity morphism $\langle \mathrm{id}, \mathrm{id}\rangle$ and the following four conjugate pairs of automorphisms

$$\begin{aligned}
&\langle \mathrm{id}, (123)\rangle \sim \langle \mathrm{id}, (132)\rangle, && \langle (123), \mathrm{id}\rangle \sim \langle (132), \mathrm{id}\rangle,\\
&\langle (123), (123)\rangle \sim \langle (132), (132)\rangle, && \langle (123), (132)\rangle \sim \langle (132), (123)\rangle.
\end{aligned}$$

The relation for morphisms in Theorem 8.30 then implies

$$\begin{aligned}
&\langle \mathrm{id}, \mathrm{id}\rangle = \langle \mathrm{id}, (123)\rangle \circ \langle \mathrm{id}, (132)\rangle \sim \langle \mathrm{id}, (132)\rangle \circ \langle \mathrm{id}, (132)\rangle = \langle \mathrm{id}, (123)\rangle\\
&\langle \mathrm{id}, \mathrm{id}\rangle = \langle (123), \mathrm{id}\rangle \circ \langle (132), \mathrm{id}\rangle \sim \langle (132), \mathrm{id}\rangle \circ \langle (132), \mathrm{id}\rangle = \langle (123), \mathrm{id}\rangle\\
&\langle \mathrm{id}, \mathrm{id}\rangle = \langle (123), (123)\rangle \circ \langle (132), (132)\rangle \sim \langle (132), (132)\rangle \circ \langle (132), (132)\rangle = \langle (123), (123)\rangle\\
&\langle \mathrm{id}, \mathrm{id}\rangle = \langle (123), (132)\rangle \circ \langle (132), (123)\rangle \sim \langle (132), (123)\rangle \circ \langle (132), (123)\rangle = \langle (123), (132)\rangle.
\end{aligned}$$

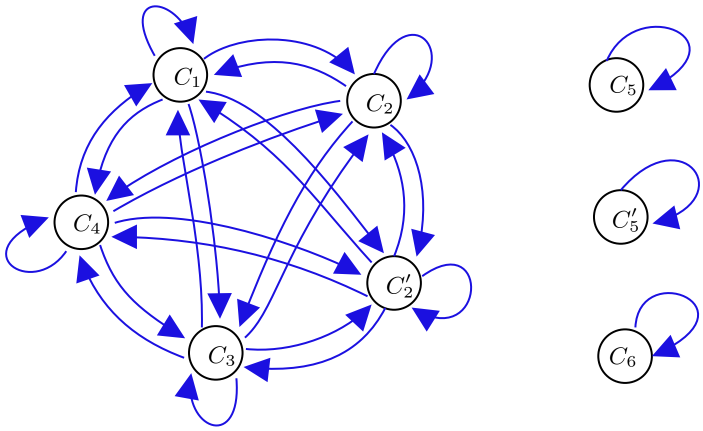


Figure 4: The groupoid $\mathcal{M}_{inv}$ from Example 8.34

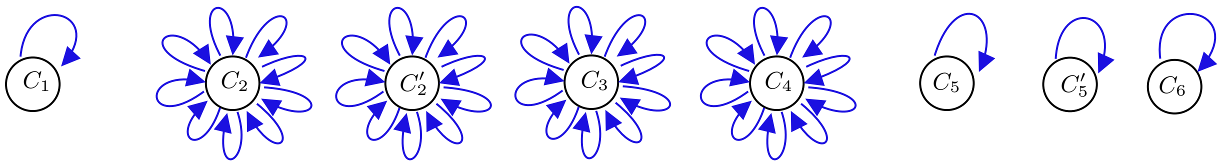


Figure 5: The groupoid $\mathcal{M}_{inv}$ from Example 8.35

As all automorphisms of $(\mathrm{id}, \mathrm{id})$ *in* $\mathcal{M}^{coH}$ *are identified,* C_1 *has a single automorphism in* $\mathcal{M}_{inv}$*. As in Example 8.34, all morphisms between objects in* C_5 *are identified and likewise for* C_5'*,* C_6*.*

In contrast, the automorphism group of each element of C_2*,* C_2'*,* C_3*,* C_4 *in* $\mathcal{M}$ *and* $\mathcal{M}^{coH}$ *is* $A_3 \times A_3$*. Automorphisms of these objects in* $\mathcal{M}$ *coincide with their automorphisms in* $\mathcal{M}^{coH}$*. Each automorphism of an object in one of these conjugacy classes is conjugate only to itself and to automorphisms of different objects in the same conjugacy class. As any composable sequence of morphisms in* $\mathcal{M}$ *involves only automorphisms of the same object, the automorphism groups of these conjugacy classes in* $\mathcal{M}_{inv}$ *are given by* $A_3 \times A_3$*. Thus, the groupoid* $\mathcal{M}_{inv}$ *is as in Figure 5.*

Remark 8.36. *Examples 8.34 and 8.35 show that for a group object in* Cat *the equivalence classes of morphisms in the protected object from Theorem 8.30 do in general not correspond to conjugacy classes. Moreover, they reveal two further aspects:*

1. Example 8.34 shows that there are cases where the generalised congruence $(R_0, \sim_m^s)$ *from (74) is the principal congruence. Recall that this corresponds to the case where two morphisms are equivalent if and only if their sources as well as targets are in the same conjugacy class. Clearly, by Example 8.35 this is not the general case.*

2. In the proof of Proposition 8.14 for the standard graph, the whole conjugacy class of a morphism f *is contained in* $\mathcal{M}^{coH}$ *iff* f *is contained in* $\mathcal{M}^{coH}$*. This does not hold for the equivalence classes of* f *from Proposition 8.13. A counterexample can be found for the torus and the group object* $H = \triangledown(S_3, A_3, \blacktriangleright, \partial)$ *from Example 8.35. Morphisms in* $H^{\times 2}$ *can be written as* (c_1, d_1, c_2, d_2) *with* $c_1, c_2 \in A_3$*,* $d_1, d_2 \in S_3$*. We consider the morphisms*

$$f := (\mathrm{id}, (23), (123), \mathrm{id}) : ((23), \mathrm{id}) \to ((23), \mathrm{id}),$$
$$h := (\mathrm{id}, (23), (132), \mathrm{id}) : ((23), \mathrm{id}) \to ((23), \mathrm{id}).$$

Conjugating $(\mathrm{id},(23))$ *and* $((123),\mathrm{id})$ *by* $(\mathrm{id},(23))$ *in* $A \rtimes B$ *yields* $(\mathrm{id},(23))$ *and* $((132),\mathrm{id})$*. Hence, we obtain*

$$(\mathrm{id},(23)) \triangleright f = (\mathrm{id},(23),(132,\mathrm{id}) : ((23),\mathrm{id}) \to ((23),\mathrm{id}).$$

Both f *and* $(\mathrm{id},(23)) \triangleright f$ *can be post-composed by* h*, which yields*

$$h \circ f = (\mathrm{id},(23),\mathrm{id},\mathrm{id}) \sim h \circ ((\mathrm{id},(23)) \triangleright f) = (\mathrm{id},(23),(123),\mathrm{id}) = f.$$

Thus, f *and* $h \circ f$ *are in the same equivalence class, but* $h \circ f \in (\mathcal{M}^{coH})^{(1)}$*, whereas* $f \notin (\mathcal{M}^{coH})^{(1)}$*.*

9 Mapping class group actions

In [MV] Meusburger and Voß constructed mapping class group actions associated to pivotal Hopf monoids in symmetric monoidal categories. In the notation and conventions of this thesis, these act on the Yetter-Drinfeld module from Example 4.8 and on protected objects for graphs with a single vertex and face. The mapping class group actions are obtained by assigning sequences of edge slides to a set of generating Dehn twists in a presentation of the mapping class group. In this chapter, we recall the mapping class group actions from [MV] and then combine them with our results to mapping class group actions on protected objects for any connected ribbon graph. We show that these mapping class group actions for $\mathcal{C} = \mathrm{Set}, \mathrm{SSet}, \mathrm{Cat}$ correspond to mapping class group actions induced by representation varieties.

We start with some background on mapping class groups and Dehn twists and recall Gervais' presentation of the mapping class group in Section 9.1. Section 9.2 summarises the construction from [MV]. We then show in Section 9.3 how these actions induce mapping class group actions on protected objects and explicitly describe them for $\mathcal{C} = \mathrm{Set}, \mathrm{SSet}, \mathrm{Cat}$.

Unless stated otherwise, throughout this chapter Σ is an oriented surface of genus $g \geq 1$ and $\Sigma \setminus D$ the associated surface with a disc removed and fundamental group $\pi_1(\Sigma \setminus D) = F_{2g}$.

9.1 Background on mapping class groups and Dehn twists

In this section, we summarise the background on mapping class groups and Dehn twists. In particular, we consider a finite presentation of the mapping class group in terms of Dehn twists by Gervais [Ge]. For the general definitions we follow Farb and Margalit [FM].

Definition 9.1. *[FM, Sec. 2.1] The* ***mapping class group*** $\mathrm{Map}(\Sigma)$ *of* Σ *is the quotient of the group* $\mathrm{Homeo}_+(\Sigma)$ *of orientation preserving homeomorphisms of* Σ *by the normal subgroup* $\mathrm{Homeo}_0(\Sigma)$ *of homeomorphisms homotopic to the identity.*

The mapping class group is isomorphic to the group of outer automorphisms of the fundamental group $\pi_1(\Sigma)$

$$\mathrm{Map}(\Sigma) = \mathrm{Homeo}_+(\Sigma)/\mathrm{Homeo}_0(\Sigma) \cong \mathrm{Out}(\pi_1(\Sigma)) = \mathrm{Aut}(\pi_1(\Sigma))/\mathrm{Inn}(\pi_1(\Sigma)).$$

As the homeomorphisms in $\mathrm{Homeo}_+(\Sigma)$ are orientation preserving, one can replace homotopic by isotopic in the definition of $\mathrm{Map}(\Sigma)$, compare Farb and Margalit [FM, Sec. 2.1]. The mapping class group of $\Sigma \setminus D$ is defined analogously with the additional condition that all homeomorphisms fix the boundary of D pointwise, see Farb and Margalit [FM, Sec. 2.1]. The mapping class groups $\mathrm{Map}(\Sigma)$ and $\mathrm{Map}(\Sigma \setminus D)$ can be presented with the same generators but with additional relations for $\mathrm{Map}(\Sigma)$, see for instance [Ge].

Informally, a Dehn twist along a simple closed curve c on an oriented surface is the orientation preserving homeomorphism of the surface obtained by cutting it along c, twisting one of the ends by 2π and gluing it back, see Saveliev [Sav, Sec. 1.4]. For a more formal definition we mainly follow Farb and Margalit [FM, Sec. 3.1.1], but also Birman [Bi, Sec. 4.3], Morita [Mo, Sec. 5] and Korkmaz [Kor, Sec. 2.1].

First, we consider the surface of a cylinder $C := S^1 \times [0, 1]$ with central curve $\gamma := S^1 \times \{\frac{1}{2}\}$. To endow C with an orientation, one can embed C into $\mathbb{R}^2$ via polar coordinates $(\theta, t) \mapsto (\theta, t + 1)$ and equip it with the inherited standard orientation. On the surface of the cylinder we define the map

$$D'_\gamma : C \to C, \quad D'_\gamma(\theta, t) = (\theta + 2\pi t, t). \tag{77}$$

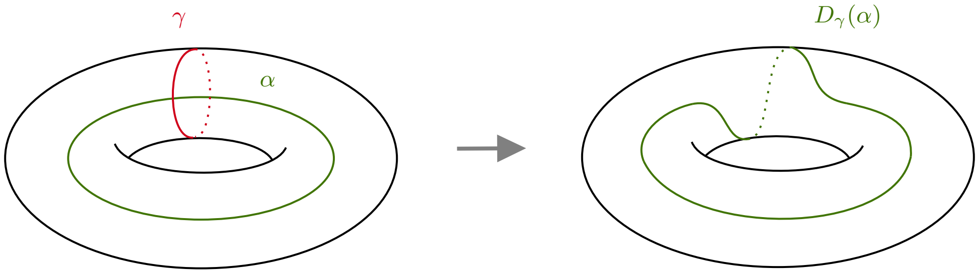
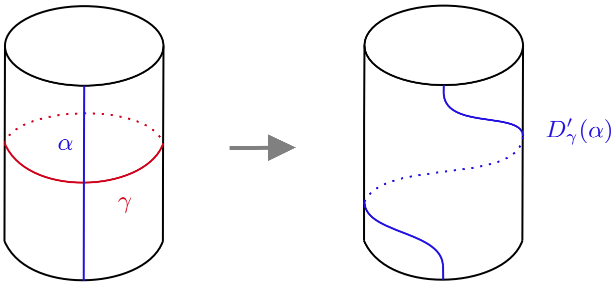


Figure 6: Right Dehn twist along γ.

Clearly, D'_γ is an orientation preserving homeomorphism that fixes ∂C. It can be illustrated as follows:



The image $D'_\gamma(\alpha)$ of a segment α as depicted above turns left into γ, follows γ and then goes back to the endpoint of α.

To define the Dehn twist along a simple closed curve c on an oriented surface Σ, we choose an orientation preserving embedding $I_c : C \to \mathcal{N}(c) \subseteq \Sigma$ onto a neighborhood $\mathcal{N}(c)$ of c such that $I_c(\gamma) = c$.

Definition 9.2. *[FM, Sec. 3.1.1] The* ***Dehn twist*** *along the simple closed curve c is the mapping class $D_c : \Sigma \to \Sigma$ defined by $I_c \circ D'_\gamma \circ I_c^{-1}$ on $\mathcal{N}(c)$ with D'_γ from (77) and extended to Σ by the identity map outside $\mathcal{N}(c)$.*

Different choices of the neighborhood $\mathcal{N}(c)$ or replacing c by an isotopic curve yield isotopic Dehn twists, compare [Sav, Sec. 1.4]. The described Dehn twists D_c are *left* Dehn twists along c. Replacing (77) by

$$D'_\gamma(\theta, t) = (\theta - 2\pi t, t)$$

in Definition 9.2 yields Dehn twists in the opposite direction, called *right* Dehn twists. Right Dehn twists are inverse to left Dehn twists, see [FM, Sec. 3.1.1]. Figure 6 illustrates a right Dehn twist on the torus.

There are many different presentations of mapping class groups. For instance, in [Be, Th. 7.1] Bene gave a presentation of $\mathrm{Map}(\Sigma \setminus D)$ in terms of chord slides. Chord slides can be seen as edges in a connected graph with a single vertex. It is shown in [MV, Prop.] that the edge slides from Definition 5.5 satisfy the relations in Bene's presentation. As a consequence, the slides give rise to actions of $\mathrm{Map}(\Sigma \setminus D)$ by automorphisms of Yetter-Drinfeld modules.

Another finite presentation of mapping class groups is provided by Gervais [Ge] in terms of generating Dehn twists building on the work of Wajnryb [Wa]. Based on Gervais' result actions of the mapping

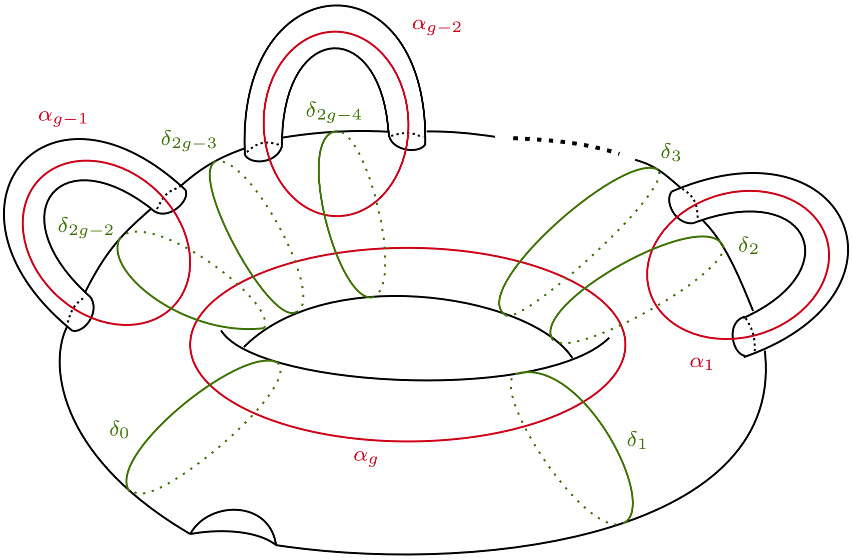


Figure 7: The curves δ_j and α_i from Theorem 9.3.

class group $\mathrm{Map}(\Sigma \setminus D)$ on the Yetter-Drinfeld module in Example 4.8 are obtained in [MV]. In analogy to Gervais' presentation the actions can be concretely described in terms of generating Dehn twists. Moreover, this result induces an action of $\mathrm{Map}(\Sigma)$ by automorphisms of the biinvariants.

We recall Gervais' presentation in the notation of [MV, Th. 4.1, Sec. 9]. This presentation is for oriented surfaces of any genus $g \geq 1$ and any number n of boundary components as long as $g+n \geq 2$. We restrict attention to the cases $n = 0, 1$. We say that simple closed curves α, β are disjoint if their homotopy classes have representatives that are disjoint. Likewise, we write $|\alpha \cap \beta| = 1$ if they intersect in a single point.

Theorem 9.3. *[Ge, Th. 1] Let*

$$\Omega := \{\alpha_i \mid i = 1, \ldots, g\} \cup \{\delta_j \mid j = 0, \ldots, 2g-2\} \cup \{\gamma_{k,l} \mid k, l = 0, \ldots, 2g-2, k \neq l\}$$

be the set of the simple closed curves depicted in Figures 7 and 8. Then the mapping class group $\mathrm{Map}(\Sigma \setminus D)$ *is generated by Dehn twists along the curves in* Ω*, subject to the following relations*

a) $D_\alpha \circ D_\beta = D_\beta \circ D_\alpha$ *for all disjoint* $\alpha, \beta \in \Omega$*.*
b) $D_\alpha \circ D_\beta \circ D_\alpha = D_\beta \circ D_\alpha \circ D_\beta$ *for all* $\alpha, \beta \in \Omega$ *with* $|\alpha \cap \beta| = 1$*.*
c) $D_{\gamma_{2j+1,2j}} = D_{\gamma_{2j,2j-1}}$ *for* $j = 0, \ldots, g-2$ *and* $D_{\gamma_{1,2g-2}} = D_{\gamma_{2g-2,2g-3}}$*.*
d) $(D_{\delta_k} \circ D_{\delta_i} \circ D_{\delta_j} \circ D_{\alpha_g})^3 = D_{\gamma_{i,j}} \circ D_{\gamma_{j,k}} \circ D_{\gamma_{k,i}}$ *for* i, j, k *not all equal and such that the triple* (i, k, j) *or a cyclic permutation satisfies* $i \leq k \leq j$ *with* $D_{\gamma_{i,i}} := \mathrm{id}$ *for all* i*.*

9.2 Twists assigned to paths in ribbon graphs

Throughout this section, let H be an involutive Hopf monoid in $\mathcal{C}$. Every curve in the presentation of the mapping class group $\mathrm{Map}(\Sigma \setminus D)$ from Theorem 9.3 defines an element of $\pi_1(\Sigma \setminus D)$. In [MV] endomorphisms of $H^{\otimes E}$, called twists, are associated to these curves. They are given as sequences of edge slides and satisfy the relations of Theorem 9.3. Hence, the endomorphisms correspond to

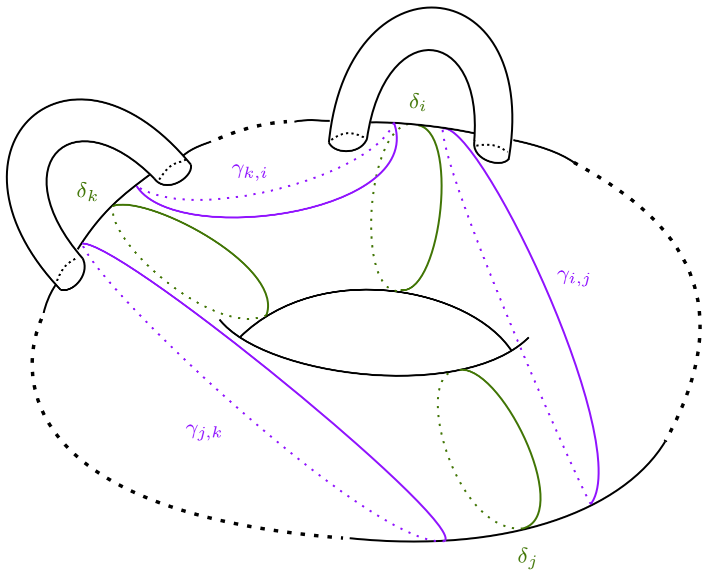


Figure 8: The curves $\gamma_{i,j}, \gamma_{j,k}, \gamma_{k,i}$ and $\delta_i, \delta_j, \delta_k$ for $j < i < k$ from Theorem 9.3.

Dehn twists and give rise to actions of the mapping class group $\mathrm{Map}(\Sigma \setminus D)$ on $H^{\otimes E}$. They also induce actions of $\mathrm{Map}(\Sigma)$ on the protected objects considered in the next section. Recall that the fundamental group of $\Sigma \setminus D$ is given by

$$\pi_1(\Sigma \setminus D) = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \rangle = F_{2g}.$$

The edges of the standard graph from (21) correspond to the generators of $\pi_1(\Sigma \setminus D)$, compare Chapter 3. By comparing Figure 2 with Figures 7 and 8 one can express the curves from Theorem 9.3 in terms of the generators $\alpha_1, \ldots, \beta_g$, see Table 1 and [MV, Sec. 9]. Note that in contrast to [MV, Sec. 9] where the surface can have several boundary components, we restrict to the situation of $\Sigma \setminus D$.

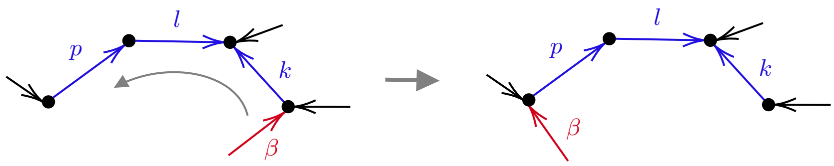
index	curve	representing element in $\pi_1(\Sigma \setminus D)$
$i = 1, \ldots, g$	α_i	α_i
	δ_0	β_g^{-1}
$1 \leq j \leq g-1$	δ_{2j-1}	$\alpha_j^{-1} \circ \beta_j \circ \alpha_j \circ \beta_j^{-1} \circ \ldots \circ \alpha_{g-1}^{-1} \circ \beta_{g-1} \circ \alpha_{g-1} \circ \beta_{g-1}^{-1} \circ \alpha_g^{-1} \circ \beta_g \circ \alpha_g$
$1 \leq j \leq g-1$	δ_{2j}	$\beta_j^{-1} \circ \alpha_{j+1}^{-1} \circ \beta_{j+1} \circ \alpha_{j+1} \circ \beta_{j+1}^{-1} \circ \ldots \circ \alpha_{g-1}^{-1} \circ \beta_{g-1} \circ \alpha_{g-1} \circ \beta_{g-1}^{-1}$ $\circ \alpha_g^{-1} \circ \beta_g \circ \alpha_g$
$1 \leq i \leq 2g-2$	$\gamma_{i,0}$	$\delta_i \circ \beta_g^{-1}$
$1 \leq j < i \leq 2g-1$	$\gamma_{i,j}$	$\delta_j \circ \delta_i^{-1}$
$1 \leq j \leq 2g-2$	$\gamma_{0,j}$	$\delta_j \circ \alpha_g^{-1} \circ \beta_g^{-1} \circ \alpha_g$
$1 \leq i < j \leq 2g-2$	$\gamma_{i,j}$	$\delta_i \circ \alpha_g \circ \delta_j^{-1} \circ \alpha_g^{-1}$

Table 1: The curves from Theorem 9.3 as composites of the generators of $\pi_1(\Sigma \setminus D)$ from [MV, Sec. 9].

All paths in Table 1 except for the paths $\gamma_{k,l}$ for $1 \leq k < l \leq 2g-2$ are face paths. Recall from Chapter 3 that face paths are paths that traverse each edge at most once in each direction and turn maximally left at each vertex. In order to define twists along all paths from Table 1 we first define slides along face paths following [MV].

Definition 9.4. *[MV, Def. 8.1] Let $\gamma : v \to w$ be a face path that traverses the edges $\gamma_n, \gamma_{n-1}, \ldots, \gamma_1$ in this order and β the edge end that comes directly after the starting end of γ in the cyclic ordering at v. If β is not traversed by γ, the **slide** S^{fp}_β of β **along the face path** γ is given as follows: Successively slide β along $\gamma_n, \gamma_{n-1}, \ldots, \gamma_1$ such that β is slid on the left of γ_i towards $t(\gamma_i)$ if γ_i is traversed by γ parallel to its orientation and on the right of γ_i towards $s(\gamma_i)$ else.*

Example 9.5. *Sliding β along the face path $\gamma = p^{-1} \circ l^{-1} \circ k$ yields*



$$\begin{aligned} S^{fp}_\beta(\beta \otimes k \otimes l \otimes p) &= S_{p,\beta} \circ S_{l,\beta} \circ S_{k,\beta}(\beta \otimes k \otimes l \otimes p) = S_{p,\beta} \circ S_{l,\beta}(k_{(1)}\beta \otimes k_{(2)} \otimes l \otimes p) \\ &= S_{p,\beta}(S(l_{(2)})k_{(1)}\beta \otimes k_{(2)} \otimes l_{(1)} \otimes p) = S(p_{(2)})S(l_{(2)})k_{(1)}\beta \otimes k_{(2)} \otimes l_{(1)} \otimes p_{(1)}. \end{aligned}$$

Next, we extend the list of graph transformations from Definition 3.5 by the adding of an edge to a face path and extend the loop deletion to a deletion of any edge in a ribbon graph Γ with edge set E.

Definition 9.6. *[MV, Sec. 8.2]*

*a) The **edge deletion** removes an edge $\beta \in E$ from Γ.*
*b) Let $\gamma : v \to w$ be a face path in Γ. The **adding of an edge to the face path** γ is obtained by:*
   1. Adding a loop γ' at v such that the target end of γ' comes directly after the starting end of γ and directly before $s(\gamma')$ in the cyclic or linear ordering at v.
   2. Sliding $t(\gamma')$ along the face path γ.

Definition 9.7. *[MV, Def. 8.7]*

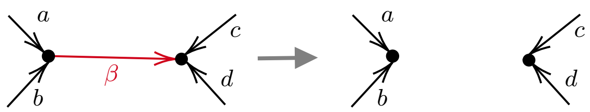
*a) The **removal of an edge** $\beta \in E$ is associated with the morphism*

$$\epsilon_\beta : H^{\otimes E} \to H^{\otimes(E-1)}.$$

*b) **Adding an edge** γ' **to a face path** γ in Γ is associated with the morphism*

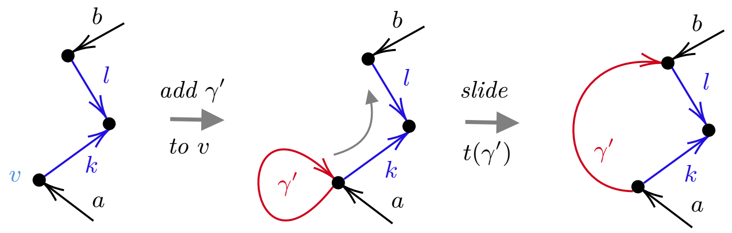
$$C_\gamma := S^{fp}_\gamma \circ \eta_{\gamma'} : H^{\otimes E} \to H^{\otimes(E+1)}.$$

Example 9.8. *a) Removing the edge β in the following graph yields*



$$\epsilon_\beta(a \otimes b \otimes c \otimes d \otimes \beta) = \epsilon(\beta)a \otimes b \otimes c \otimes d.$$

b) Adding an edge to the face path $\gamma = l^{-1} \circ k$ yields the graph



and the corresponding morphism

$$C_\gamma (a \otimes b \otimes k \otimes l) = S_\gamma^{fp} \circ \eta_{\gamma'} (a \otimes b \otimes k \otimes l) = S_{l,\gamma'} \circ S_{k,\gamma'} (1 \otimes a \otimes b \otimes k \otimes l)$$
$$= S(l_{(2)})k_{(1)} \otimes a \otimes b \otimes k_{(2)} \otimes l_{(1)}.$$

Applying the last definitions we now define twists along loops, face paths and along the paths $\gamma_{i,j}$ for $1 \leq i < j \leq 2g-2$ from Theorem 9.3 following [MV].

Definition 9.9. *[MV, Def. 8.10, 8.11]*

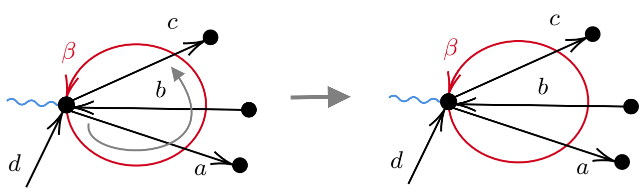
a) *Let β be a loop at a ciliated vertex in the ribbon graph Γ. The* ***twist*** *D_β* ***along the loop*** *β is the morphism obtained by successively sliding all edge ends to the left of β towards $t(\beta)$ if $s(\beta) < t(\beta)$. If $t(\beta) < s(\beta)$, the twist D_β is obtained by successively sliding all edge ends to the right of β towards $s(\beta)$.*
b) *Let v be a ciliated vertex and $\gamma : v \to v$ a closed face path in a ribbon graph Γ. The* ***twist*** *D_γ* ***along the face path*** *γ is the morphism*

$$D_\gamma = \epsilon_{\gamma'} \circ D_{\gamma'} \circ C_\gamma : H^{\otimes E} \to H^{\otimes E}$$

given as follows:
1. Start with adding an edge γ' to the face path γ.
2. Carry out a twist along the loop γ' as in a) and erase the loop γ' afterwards.

Note that a) and b) of Definition 9.9 are consistent with each other. If a face path γ only consists of a loop, the twist along the face path γ from b) reduces to the twist from a), compare [MV, Rem. 8.12].

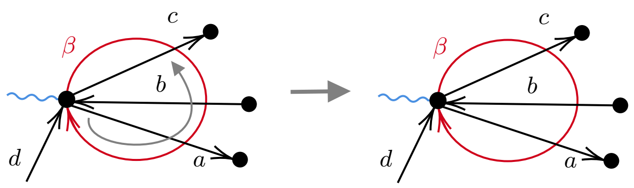
Example 9.10. *a) The loop β in the following graph satisfies $s(\beta) < t(\beta)$. Hence, the twist along β is given by successively sliding a, b, c to the left of β towards $t(\beta)$*



The corresponding morphism is given by

$$\begin{aligned} D_\beta (\beta \otimes a \otimes b \otimes c \otimes d) &= S_{\beta,c} \circ S_{\beta,b} \circ S_{\beta,a} (\beta \otimes a \otimes b \otimes c \otimes d) \\ &= S_{\beta,c} \circ S_{\beta,b} (\beta_{(2)} \otimes aS(\beta_{(1)}) \otimes b \otimes c \otimes d) \\ &= S_{\beta,c} (\beta_{(3)} \otimes aS(\beta_{(1)}) \otimes \beta_{(2)} b \otimes c \otimes d) \\ &= \beta_{(4)} \otimes aS(\beta_{(1)}) \otimes \beta_{(2)} b \otimes cS(\beta_{(3)}) \otimes d. \end{aligned}$$

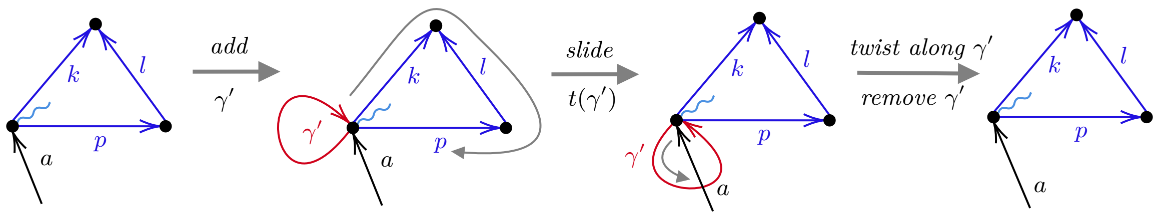
If $t(\beta) < s(\beta)$ as in the following graph



the twist along β is obtained by successively sliding a, b, c to the right of β towards $s(\beta)$. The corresponding morphism is given by

$$D_\beta\,(\beta\otimes a\otimes b\otimes c\otimes d) = S_{\beta,c}\circ S_{\beta,b}\circ S_{\beta,a}\,(\beta\otimes a\otimes b\otimes c\otimes d)$$
$$= \beta_{(1)}\otimes a\beta_{(4)}\otimes S(\beta_{(3)})b\otimes c\beta_{(2)}\otimes d.$$

b) The twist along the face path $\gamma = p^{-1}\circ l^{-1}\circ k$ is given by



with corresponding morphism

$$D_\gamma\,(a\otimes k\otimes l\otimes p) = \epsilon_{\gamma'}\circ D_{\gamma'}\circ S_{p,\gamma'}\circ S_{l,\gamma'}\circ S_{k,\gamma'}\,(1\otimes a\otimes k\otimes l\otimes p)$$
$$= \epsilon_{\gamma'}\circ S_{\gamma',a}\,(S(p_{(2)})S(l_{(2)})k_{(1)}\otimes a\otimes k_{(2)}\otimes l_{(1)}\otimes p_{(1)})$$
$$= S(p_{(2)})S(l_{(2)})k_{(1)}a\otimes k_{(2)}\otimes l_{(1)}\otimes p_{(1)}.$$

Remark 9.11. *The inverse of the twist D_β from Definition 9.9, a) along a loop β is given by successively sliding all edge ends to the left of β towards $s(\beta)$ if $s(\beta) < t(\beta)$. This follows from the formulas for edge slides from Remark 5.6. If $t(\beta) < s(\beta)$, the inverse D_β^{-1} is obtained by successively sliding all edge ends to the right of β towards $t(\beta)$.*

Definition 9.12. *[MV, Def. 9.1] For $1\le i<j\le 2g-2$ the* ***twist along the path*** $\gamma_{\mathbf{i,j}}$ *is the morphism $D_{\gamma_{i,j}}: H^{\otimes E}\to H^{\otimes E}$ obtained as follows:*

1. *Start with adding edges δ'_i and δ'_j to the face paths δ_i and δ_j as in Definition 9.6.*
2. *Slide all edge ends between $s(\alpha_g)$ and $t(\delta'_j)$ along α_g towards the target end of α_g. Then slide the edge end $s(\beta_g)$ along the edge δ'_i.*
3. *Carry out a twist along the face path $\gamma^*_{i,j} := \delta'_i\circ\alpha_g\circ\delta'^{-1}_j\circ\alpha_g^{-1}$.*
4. *Reverse all edge slides from 2. by returning all edge ends to their initial positions. Then erase the edges δ'_i and δ'_j.*

This twist is illustrated in Figure 9. Here, the vertex of the standard graph from (21) is cut open at the cilium and corresponds to the baseline of the pictures. Steps i) to v) in this Figure can be specified as follows.

- Step i): A loop δ'_j is inserted.
- Step ii): Its target end is slid along δ_j. Analogously, a loop δ'_i is inserted and its target end slid along δ_i.

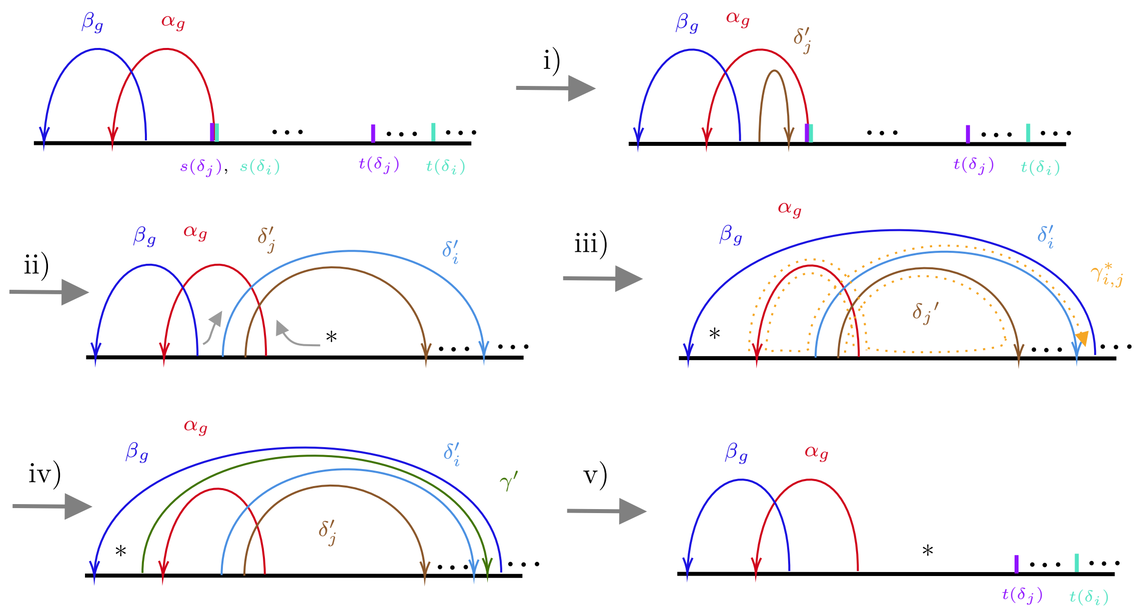


Figure 9: The twist along $\gamma_{i,j}$ from Definition 9.12.

- Step iii): All edge ends between $s(\alpha_g)$ and $t(\delta'_j)$, which are indicated with $*$, are slid along α_g and $s(\beta_g)$ is slid along δ'_i.
- Step iv): A loop γ' is inserted and its target end slid along $\gamma^*_{i,j}$.
- Step v): The edge γ' is removed. The edge ends $s(\beta_g)$ and $*$ are slid back and the edges δ'_i, δ'_j are removed.

In [MV] it is shown that the twists from Definition 9.9 and 9.12 along the paths from Table 1 satisfy the relations in Gervais' presentation of the mapping class group $\mathrm{Map}(\Sigma \setminus D)$. This induces an action of $\mathrm{Map}(\Sigma \setminus D)$ on the Yetter-Drinfeld module from Example 4.8. We recall this result, which we again restrict to the situation of $\Sigma \setminus D$ and to involutive Hopf monoids H instead of pivotal Hopf monoids.

Theorem 9.13. *[MV, Th. 9.2] Let Γ be the standard graph from (21) and α_i, δ_j, $\gamma_{k,l}$ for $i \in \{1, \dots, g\}$, $j, k \neq l \in \{0, \dots, 2g-2\}$ the associated paths from Table 1. Then the twists along α_i, δ_j, $\gamma_{k,l}$ satisfy the relations in Theorem 9.3. They define a group homomorphism*

$$\phi : \mathrm{Map}(\Sigma \setminus D) \to \mathrm{Aut}_{YD}(H^{\otimes 2g}).$$

9.3 Mapping class group actions on protected objects

In this section, we combine the mapping class group actions from [MV] for graphs with a single vertex and face with the topological invariance of protected objects from Chapter 5. In particular, we explicitly describe the actions in $\mathcal{C} = \mathrm{Set}, \mathrm{SSet}, \mathrm{Cat}$ as induced actions of representation varieties.

This section essentially coincides with Section 8 of the preprint [HM]. Only minor changes and additions have been made.

The action of $\mathrm{Map}(\Sigma)$ from [MV] on protected objects associated to certain graphs is induced by the action of $\mathrm{Map}(\Sigma \setminus D)$ on the Yetter-Drinfeld module from Example 4.8. This was recalled in Section 9.2. More specifically, a presentation of $\mathrm{Map}(\Sigma)$ can be derived from the presentation of

$\mathrm{Map}(\Sigma \setminus D)$ in Theorem 9.3 by imposing the additional conditions

$$D_{\gamma_{1,0}} = 1, \quad D_{\delta_0} = D_{\delta_1}, \quad D_{\gamma_{0,1}} = (D_{\delta_0}^3 D_{\alpha_g})^3, \quad D_{\gamma_{0,k}} = D_{\gamma_{1,k}}, \quad D_{\gamma_{k,0}} = D_{\gamma_{k,1}}$$

for $1 < k \leq 2g-1$, see Gervais [Ge] and [MV, Sec. 9]. The automorphisms in Theorem 9.13 induce automorphisms of the protected objects, which satisfy the additional relations of $\mathrm{Map}(\Sigma)$ given above. Thus, they induce actions of $\mathrm{Map}(\Sigma)$ on the protected objects, see [MV, Th. 9.5].

As we established in Theorem 5.23 that the protected object is independent of the choice of the underlying graph, we can reformulate [MV, Th. 9.5] as follows.

Theorem 9.14. *Let H be an involutive Hopf monoid in $\mathcal{C}$ and Σ an oriented surface of genus $g \geq 1$. Then the edge slides from Definition 5.5 induce an action of the mapping class group* $\mathrm{Map}(\Sigma)$ *by automorphisms of the protected object.*

For group objects in cartesian monoidal categories such as simplicial groups and crossed modules this mapping class group action admits a concrete description in terms of mapping class group actions on representation varieties. For this, recall that for any group G the group $\mathrm{Aut}(\pi_1(\Sigma))$ acts on the set of group homomorphisms $\rho : \pi_1(\Sigma) \to G$ via $(\phi \rhd \rho)(\lambda) = \rho(\phi^{-1}(\lambda))$ for all $\lambda \in \pi_1(\Sigma)$ and $\phi \in \mathrm{Aut}(\pi_1(\Sigma))$. This induces an action of $\mathrm{Map}(\Sigma) = \mathrm{Out}(\pi_1(\Sigma)) = \mathrm{Aut}(\pi_1(\Sigma))/\mathrm{Inn}(\pi_1(\Sigma))$ on the representation variety $\mathrm{Hom}(\pi_1(\Sigma), G)/G$.

To relate this to the mapping class group actions from [MV, Th. 9.5] note that for a group object H in a cartesian monoidal category the formulas for the edge slides in Definition 5.5 and Remark 5.6 reduce to left and right multiplication with H, sometimes composed with inversions.

It follows that any finite sequence of edge slides from the standard graph to itself induces an automorphism of $H^{\otimes 2g}$ that arises from an automorphism of $F_{2g} = \pi_1(\Sigma \setminus D)$. As it preserves the Yetter-Drinfeld module structure in Example 4.8, it induces automorphisms of $\mathcal{M}^{coH}$, $\mathcal{M}^H$ and $\mathcal{M}_{inv}$. Inner automorphism of $\pi_1(\Sigma)$ induce trivial automorphisms of $\mathcal{M}_{inv}$. For a group H as a group object in Set it is then directly apparent that the induced action of $\mathrm{Map}(\Sigma)$ on $\mathcal{M}_{inv}$ is the one on the representation variety $\mathrm{Hom}(\pi_1(\Sigma), H)/H$, see also Examples 9.6 and 9.7 in [MV]. This result can be applied to determine the mapping class group action for a simplicial group.

Corollary 9.15. *Let $H = (H_n)_{n \in \mathbb{N}_0}$ be a simplicial group as a Hopf monoid in* SSet*. Then the action of* $\mathrm{Map}(\Sigma)$ *on the representation varieties* $\mathrm{Hom}(\pi_1(\Sigma), H_n)/H_n$ *induces an action of* $\mathrm{Map}(\Sigma)$ *on $\mathcal{M}_{inv}$ by simplicial maps, and this coincides with the action in [MV, Th. 9.5].*

Proof. The induced $\mathrm{Map}(\Sigma)$-action on $\mathcal{M}_{inv}$ is by simplicial maps, because the face maps and degeneracies of $\mathcal{M}_{inv}$ act elements of the representation varieties $\mathrm{Hom}(\pi_1(\Sigma), H_n)/H_n$ by post-composition with the face maps and degeneracies $d_i : H_n \to H_{n-1}$ and $s_i : H_n \to H_{n+1}$, whereas $\mathrm{Map}(\Sigma)$ acts by pre-composition. This coincides with the action from [MV, Th. 9.5], because the latter reduces to the $\mathrm{Map}(\Sigma)$-action on $\mathrm{Hom}(\pi_1(\Sigma), H_n)/H_n$ for the group H_n as an involutive Hopf monoid in Set, and all (co)limits, images and (co)actions in SSet are degreewise. □

In the case of a crossed module as a Hopf monoid in Cat, the mapping class group action on the protected object is induced by the mapping class group action on the representation variety for the associated semidirect product group.

Corollary 9.16. *Let $H = (B, A, \blacktriangleright, \partial)$ be a crossed module. Then the* $\mathrm{Map}(\Sigma)$*-action on* $\mathcal{M}_{inv}$ *from Theorem 9.14 is induced by the* $\mathrm{Map}(\Sigma)$*-action on* $\mathrm{Hom}(\pi_1(\Sigma), A \rtimes B)/A \rtimes B$.

Proof. As the group structure of H as a group object in Cat is the one of the semidirect product $A \rtimes B$, the $\mathrm{Map}(\Sigma \setminus D)$-action on $\mathcal{M} = H^{\times 2g}$ for the standard graph (21) can be identified with the $\mathrm{Map}(\Sigma \setminus D)$-action on $\mathcal{M} = (A \rtimes B)^{\times 2g}$ one for the group $A \rtimes B$ as a group object in Set. The crossed module structure ensures that this $\mathrm{Map}(\Sigma \setminus D)$-action respects the category structure of $(A \rtimes B)^{\times 2g}$ and defines a $\mathrm{Map}(\Sigma \setminus D)$-action by invertible endofunctors.

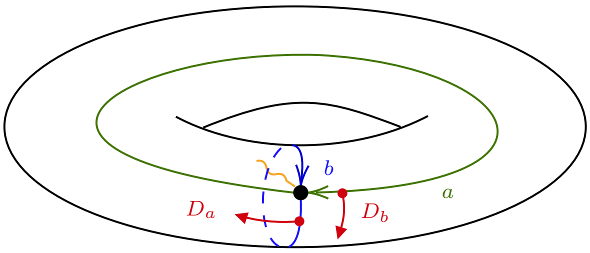
The $\mathrm{Map}(\Sigma \setminus D)$-action on $\mathcal{M}$ induces the $\mathrm{Map}(\Sigma)$-action on the protected object $\mathcal{M}_{inv}$ for both, the group $A \rtimes B$ as a group object in Set and for H as a group object in Cat. The former is the action on the representation variety $\mathrm{Hom}(\pi_1(\Sigma), A \rtimes B)/A \rtimes B$. As the protected object $\mathcal{M}_{inv}$ is a quotient of this representation variety by Theorem 8.30, its $\mathrm{Map}(\Sigma)$-action is induced by the $\mathrm{Map}(\Sigma)$-action on the representation variety. $\square$

Example 9.17. *We consider the mapping class group action on the groupoids* $\mathcal{M}_{inv}$ *from Example 8.34, 8.35 for the crossed module* $(S_3, A_3, \blacktriangleright, \partial)$ *and the torus.*

The mapping class group of the torus T is the group

$$\mathrm{Map}(T) = \mathrm{SL}(2, \mathbb{Z}) = \langle D_a, D_b \mid D_a D_b D_a = D_b D_a D_b, (D_a D_b D_a)^4 = 1 \rangle. \tag{78}$$

It is generated by the Dehn twists D_a, D_b along the a- and b-cycle, which can be depicted as follows



The Dehn twists act on $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$ by

$$D_a : a \mapsto a, b \mapsto b - a \qquad\qquad D_b : a \mapsto a + b, b \mapsto b. \tag{79}$$

In both, Example 8.34 and 8.35, the $\mathrm{SL}(2, \mathbb{Z})$*-action on the objects of* $\mathcal{M}_{inv}$ *is the* $\mathrm{SL}(2, \mathbb{Z})$*-action on the representation variety* $\mathrm{Hom}(\mathbb{Z} \times \mathbb{Z}, S_3)/S_3$ *with orbits* $\{C_1\}$, $\{C_2, C_2', C_3, C_4\}$ *and* $\{C_5, C_5', C_6\}$.

In Example 8.34 the $\mathrm{SL}(2, \mathbb{Z})$*-action on* $\mathcal{M}_{inv}$ *is determined uniquely by the action on the objects. This follows, because for all choices of objects* $s, t \in \mathrm{Ob}\,\mathcal{M}_{inv}$ *the groupoid* $\mathcal{M}_{inv}$ *has at most one morphism $f : s \to t$. In Example 8.35 an analogous statement holds for morphisms between the objects* C_1, C_5, C_5', C_6*, since all of them are identity morphisms.*

In contrast, the $\mathrm{SL}(2, \mathbb{Z})$*-action on the automorphisms of* C_2, C_2', C_3, C_4 *in Example 8.35 is non-trivial and can be identified with an orbit of the* $\mathrm{SL}(2, \mathbb{Z})$*-action on* $\mathrm{Mat}(2 \times 2, \mathbb{Z}_3)$ *by left multiplication. In this action, D_a and D_b correspond to left-multiplication with the generators*

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Automorphisms of C_2, C_2', C_3, C_4 *in* $\mathcal{M}_{inv}$ *are given by group homomorphisms* $\tau : \mathbb{Z} \times \mathbb{Z} \to A_3^{\times 2} \cong \mathbb{Z}_3^{\times 2}$*, which are determined by the images* $\tau(1, 0), \tau(0, 1) \in \mathbb{Z}_3 \times \mathbb{Z}_3$*. Interpreting an element* $(c, d) \in \mathbb{Z}_3 \times \mathbb{Z}_3$

as an automorphism $c : d \to d$ and taking $\tau(0,1)$ as the first and $\tau(1,0)$ as the second row of a matrix, we find that the $\mathrm{SL}(2,\mathbb{Z})$*-action induced by* (79) *coincides with the* $\mathrm{SL}(2,\mathbb{Z})$*-orbit containing those matrices whose second column is non-trivial.*

As our construction yields objects equipped with mapping class group actions and assigns the tensor unit to the sphere S^2, it is natural to ask if the protected objects satisfy the axioms of a modular functor from [BK, Def 5.1.1]. Although the latter are formulated for categories of vector spaces, they have obvious generalisations to other symmetric monoidal categories.

However, the assignment of protected objects to surfaces can in general not be expected to satisfy these axioms. The problem is axiom (iii) in [BK, Def 5.1.1], which requires that the object assigned to a disjoint union $\Sigma_1 \amalg \Sigma_2$ of surfaces is the tensor product of the objects assigned to Σ_1, Σ_2. This does in general not hold for the protected objects from Definition 4.7, as they are constructed by taking equalisers and coequalisers. The tensor product of two (co)equalisers in a symmetric monoidal category $\mathcal{C}$ is in general not a (co)equaliser of their tensor product. This is already apparent in the symmetric monoidal category $\mathrm{Ab} = \mathbb{Z}{-}\mathrm{Mod}$ with the usual tensor product that does not preserve equalisers. Nevertheless, the construction satisfies this axiom, if the underlying symmetric monoidal category is Set, SSet or Cat.

Proposition 9.18. *Let H be a group object in $\mathcal{C} = \mathrm{Set}$, SSet or Cat and Σ an oriented surface with connected components $\Sigma_1, \ldots, \Sigma_k$. Then the protected object for Σ is the product of the protected objects for $\Sigma_1, \ldots, \Sigma_k$.*

Proof. The claim follows by induction over k, and it is sufficient to consider $k = 2$. By Theorem 5.23 we can compute the protected object for Σ by choosing a standard graph Γ_i from (21) on each connected component Σ_i. We denote by E_i the edge set of Γ_i and by $E = E_1 \cup E_2$ the edge set of Γ.

The (co)actions $\rhd$ and δ for Γ are then given by formula (27), and it follows directly that they are the products of the (co)actions $\rhd_i$, δ_i for Γ_i, up to braidings. Lemma 4.5 and some simple computations imply that $(H^{\times E}, \rhd, \delta)$ is a Yetter-Drinfeld module over $H \times H$.

We denote by $F_\Gamma : H^{\times E} \to H \times H$ the morphism from Example 2.9 for Γ and by $F_{\Gamma_i} : H^{\times E_i} \to H$ the corresponding morphisms for Γ_i.

- $\mathcal{C} = \mathrm{Set}$: As in Example 2.15 we obtain

$$\begin{aligned} M^{coH} &= F_\Gamma^{-1}(1) = \{(x,y) \in H^{\times E_1} \times H^{\times E_2} : (F_{\Gamma_1}(x), F_{\Gamma_2}(y)) = (1,1)\} \;=\; M_1^{coH} \times M_2^{coH}, \\ M^H &= \{H^{\times 2} \rhd (m_1, m_2) : m_1 \in H^{\times E_1}, m_2 \in H^{\times E_2}\} = M_1^H \times M_2^H \end{aligned}$$

with inclusions $\iota = (\iota_1, \iota_2) : M^{coH} \to H^{\times E}$ and canonical surjections $\pi = (\pi_1, \pi_2) : H^{\times E} \to M^H$. As the image of a morphism $f : A \to B$ in Set is the usual image of a map, we have $\mathrm{im}((f_1, f_2)) = (\mathrm{im}(f_1), \mathrm{im}(f_2))$ and $M_{inv} = M_{inv,1} \times M_{inv,2}$.

- $\mathcal{C} = \mathrm{SSet}$: By Proposition 7.3 the coinvariants are given by the sets

$$M_n^{coH} = \{(m_1, m_2) \in (H^{\times E_1} \times H^{\times E_2})_n : (F_{\Gamma_1,n}(m_1), F_{\Gamma_2,n}(m_2)) = (1,1)\} \;=\; M_{1,n}^{coH} \times M_{2,n}^{coH}.$$

Face maps and degeneracies are induced by the ones of $H^{\times E_1} \times H^{\times E_2}$ and the simplicial map $\iota : M^{coH} \to H^{\times E}$ is given by the maps $\iota_n = (\iota_{1,n}, \iota_{2,n})$. As the product in SSet is objectwise, this yields $M^{coH} = M_1^{coH} \times M_2^{coH}$. An analogous argument shows that the sets M_n^H, $(M_{inv})_n$ from Proposition 7.3 are given by $M_n^H = M_{1,n}^H \times M_{2,n}^H$ and $(M_{inv})_n = (M_{inv,1})_n \times (M_{inv,2})_n$.

- $\mathcal{C} = \mathrm{Cat}$: By Lemma 8.12 the coinvariants M^{coH} for the comodule $M = H^{\times E} \cong H^{E_1} \times H^{\times E_2}$ are the subcategory with objects

$$\mathrm{Ob}(M^{coH}) = \{(A_1, A_2) \mid A_i \in \mathrm{Ob}(H^{\times E_i}), F_{\Gamma_i}(A_i) = e\}$$
$$\mathrm{Hom}_{M^{coH}}((A_1, A_2), (A'_1, A'_2)) = \{(f_1, f_2) \mid f_i \in \mathrm{Hom}_M(A_i, A'_i), F_{\Gamma_i}(f_i) = 1_e\}$$

and hence M^{coH} is the product category $M_1^{coH} \times M_2^{coH}$. For the invariants we can apply Lemma 8.7 and the results from SSet. As a right adjoint, the nerve N preserves products, and hence $N(\rhd) = N(\rhd_1) \times N(\rhd_2)$, up to braidings, and the same holds for the trivial actions $\epsilon \otimes 1_{H^{\otimes E}}$ and $\epsilon \otimes 1_{H^{\otimes E_i}}$. It follows that the coequaliser of $N(\rhd)$ and $N(\epsilon^{\times 2} \times 1_{H^{\times E}})$ is the product of the coequalisers of $N(\rhd_i)$ and $N(\epsilon \times 1_{H^{\times E_i}})$. As the homotopy functor preserves finite products, see for instance [Jo, Prop. 1.3], this yields $M^H = M_1^H \times M_2^H$ and $M_{inv} = M_{inv,1} \times M_{inv,2}$. □

Note that besides [BK, Def. 5.1.1] there are various other definitions of modular functors. Schweigert and Woike [SW] for instance construct a category whose objects are surfaces where boundary components are equipped with marked points and labelled by elements of a given set. Its morphisms are generated by mapping classes and sewings of surfaces. A modular functor is then defined as a symmetric monoidal functor into the category of chain complexes [SW, Def. 3.2], or of vector spaces [SW, Rem. 3.5], such that an excision property formulated via homotopy coends is satisfied.

The starting point of Brochier's and Woike's definition in [BrW] is a symmetric monoidal functor Surf : Graphs → Cat, called modular surface operad. Modular functors are then defined in terms of extensions over Surf and modular algebras over them, see [BrW, Def. 3.5]. The objects of the category Graphs are finite unions of corollas. That is, a single vertex with a finite number of attached edges. The edges are only incident to this vertex, their other end is not attached to a vertex.

Morphisms in Graphs are defined as equivalence classes of graphs that are consistent with decomposing graphs into corollas by cutting or contracting edges. Brochier and Woike define $\mathrm{Surf}(T)$ for any corolla T and extend it to unions of corollas T, T' via $\mathrm{Surf}(T \cup T') = \mathrm{Surf}(T) \times \mathrm{Surf}(T')$. With a suitable extension of our model to corollas, this could provide a way to treat unions of graphs and to link protected objects to other versions of modular functors. However, this question is beyond the scope of the thesis.

10 Kuperberg invariants and mapping class group actions

The aim of this chapter is to relate Kuperberg invariants of 3-manifolds to edge slides and Dehn twists that induce mapping class group actions. Kuperberg invariants were introduced by Kuperberg in [Ku]. Given a finite-dimensional semisimple Hopf algebra over $\mathbb{C}$ they associate a scalar to any oriented 3-manifold represented by a Heegaard diagram. Kashaev and Virelizier [KV] generalised this setting from Hopf algebras to involutive Hopf monoids H in symmetric monoidal categories with certain additional structures, the good pairs. The resulting 3-manifold invariant is given as an endomorphism of the tensor unit. We will follow Kashaev's and Virelizier's presentation.

An essential part of their construction is the assignment of an endomorphism of $H^{\otimes g}$ for a fixed $g \in \mathbb{N}$ to a Heegaard diagram. The 3-manifold invariant is obtained from this endomorphism via the good pairs. In the thesis the considered endomorphism is called tensor endomorphism. We show that it can be constructed from edges slides in a ribbon graph associated to the Heegaard diagram. For the case of a finite-dimensional semisimple Hopf algebra in $\mathrm{Vect}_{\mathbb{C}}$ our formalism yields the Kuperberg invariants from [Ku].

In Section 10.1 we summarise the background on Heegaard diagrams and their relation to oriented 3-manifolds. Following [KV] we define the tensor endomorphism in Section 10.2 and explain in Section 10.3 how Kuperberg invariants are defined using the tensor endomorphism. In Section 10.4 we assign colored ribbon graphs to Heegaard diagrams, before we show in Section 10.5 how edge slides in this graph are related to the tensor endomorphism.

10.1 Heegaard diagrams

In this section, we summarise the background on Heegaard diagrams and explain how Heegaard diagrams are related to Heegaard splitting and oriented 3-manifolds. In particular, we provide some basic notation, following Kashaev and Virelizier [KV]. By a circle on a surface Σ we mean the image of an embedding $S^1 \hookrightarrow \Sigma$.

Definition 10.1. *[Sav, Sec. 1.7] A* ***Heegaard diagram*** *is a triple $D = (\Sigma, \mathcal{U}, \mathcal{L})$ where*

- *Σ is a closed, connected and oriented surface of genus $g \geq 1$,*
- *$\mathcal{U}$ is a set of g pairwise disjoint circles on Σ such that $\Sigma \setminus \bigcup_{u \in \mathcal{U}} u$ is connected,*
- *$\mathcal{L}$ is a set of g pairwise disjoint circles on Σ such that $\Sigma \setminus \bigcup_{l \in \mathcal{L}} l$ is connected,*
- *each $u \in \mathcal{U}$ is transverse to each $l \in \mathcal{L}$.*

We call the circles in $\mathcal{U}$ **upper circles**, the ones in $\mathcal{L}$ **lower circles**. In pictures we indicate the upper circles in red and the lower circles in blue, see for instance Example 10.4. As each upper circle is transverse to each lower circle, the set $\mathcal{I} := (\bigcup_{u \in \mathcal{U}} u) \cap (\bigcup_{l \in \mathcal{L}} l)$ is finite. We call its elements **intersection points** of the diagram and set $N := |\mathcal{I}|$. For $x \in \mathcal{U} \cup \mathcal{L}$ the number of intersection points contained in x is denoted $|x|$.

Definition 10.2. *[KV, Sec. 3.3] A Heegaard diagram $D = (\Sigma, \mathcal{U}, \mathcal{L})$ is called*

- ***oriented*** *if each upper and lower circle is oriented,*
- ***based*** *if each upper and lower circle is endowed with a basepoint that does not coincide with an intersection point,*
- ***ordered*** *if $\mathcal{U}$ and $\mathcal{L}$ are both totally ordered.*

If D is based we denote by B_D the set of its basepoints. In an oriented Heegaard diagram we call an intersection point c between an upper circle u and a lower circle l **positive** if l traverses u from

Figure 10: Positive and negative intersection point.

the right to the left viewed in the direction of u. Otherwise, c is negative. This is illustrated in Figure 10.

If D is ordered, oriented and based, we can endow the set $\mathcal{I}$ of intersection points with two total orders, one obtained from $\mathcal{U}$ and the other one from $\mathcal{L}$. For this, let $\mathcal{I}_u$ be the set of intersection points in u for $u \in \mathcal{U}$. One obtains a total order on $\mathcal{I}_u$ by enumerating the intersection points in u following the orientation of u starting at its basepoint. Then $\mathcal{I} = \bigcup_{u\in\mathcal{U}} \mathcal{I}_u$ is equipped with the lexicographic order induced by the orders of $\mathcal{I}_u$ and $\mathcal{U}$: For $c \in \mathcal{I}_u$, $c' \in \mathcal{I}_{u'}$ with $u, u' \in \mathcal{U}$ it is

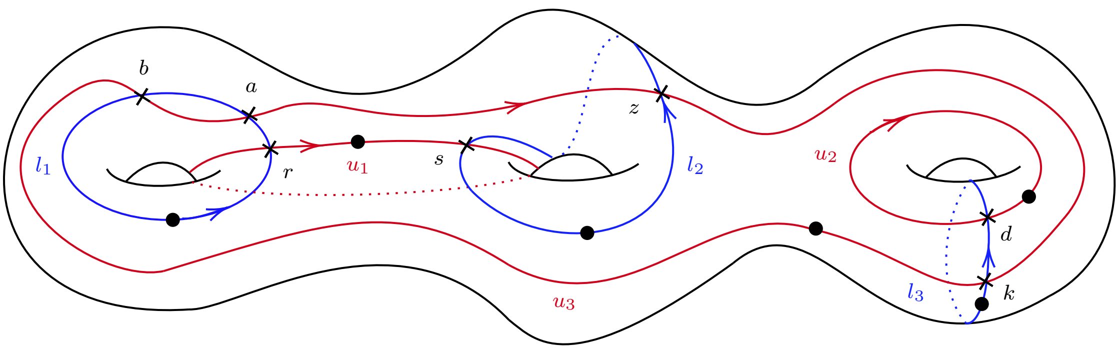
$$c \leq c' \;:\Leftrightarrow\; (u < u' \text{ in } \mathcal{U}) \;\text{ or }\; (u = u' \text{ and } c \leq c' \text{ in } \mathcal{I}_u).$$

The total order on $\mathcal{I}$ obtained from $\mathcal{L}$ is defined analogously by replacing $\mathcal{U}$ by $\mathcal{L}$.

Definition 10.3. *For $u \in \mathcal{U}$ a pair (a, b) with $a, b \in \mathcal{I}_u$ is called **inversion pair** in u if a is negative, b positive and either b is the successor of a or a is the element of highest order and b of lowest order in $\mathcal{I}_u$.*

We denote the number of inversion pairs in u by N^*_u and set $N^* := \sum_{u\in\mathcal{U}} N^*_u$.

Example 10.4. *The following oriented based Heegaard diagram*



is based on the Heegaard diagram shown by Lipshitz, Ozsváth and Thurston in [LOT, Figure 1.1], but equipped with orientations and basepoints. We endow $\mathcal{U}$ and $\mathcal{L}$ with orders given by $u_1 < u_2 < u_3$, $l_1 < l_2 < l_3$. Then the order of the intersection points obtained from $\mathcal{U}$ is given by

$$s < r < d < b < a < z < k,$$

while the order induced by $\mathcal{L}$ is given by

$$r < a < b < z < s < k < d.$$

The intersection points a, r, z are positive, whereas b, s, d, k are negative. The pair (s, r) is an inversion pair in u_1 and (b, a) is an inversion pair in u_3.

Connection of Heegaard diagrams to 3-manifolds

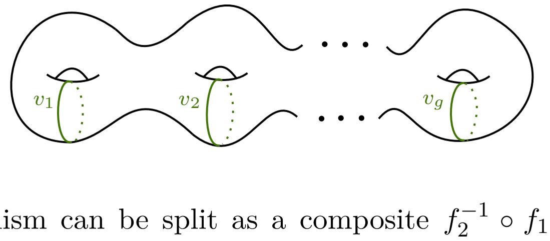
Heegaard diagrams describe Heegaard splittings of closed oriented 3-manifolds into two handlebodies. A handlebody of genus g is an oriented 3-manifold that is obtained by attaching g copies of handles $D^2 \times [-1, 1]$ to the 3-ball D^3. The boundary of a handlebody is an oriented surface of genus g.

Given two handlebodies N, N' of genus g and a homeomorphism $f : \partial N \to \partial N'$ of their boundaries, the handlebodies can be attached to each other by this homeomorphism. More precisely, in the disjoint union of N and N' identify each point $x \in \partial N$ with $f(x) \in \partial N'$: $x \sim f(x)$. The obtained space $N \amalg N' / \sim$ is a 3-manifold, compare [PS, Sec. 8]. The described presentation of an oriented 3-manifold M as the union of two handlebodies N, N' with same genus and common boundary $N \cap N' = \partial N = \partial N'$ is called *Heegaard splitting* of M. Figure 11 schematically illustrates a Heegaard splitting of S^3 into two handlebodies of genus 1.

As it is possible to orient the handlebodies ∂N, $\partial N'$ in such a way that the gluing homeomorphism $f : \partial N \to \partial N'$ is orientation preserving, Heegaard splittings are naturally linked to mapping class groups. For more details we refer to [Sav, Sec. 1.4]. More specifically, if ∂N, $\partial N'$ are glued together by isotopic homeomorphisms, the resulting 3-manifolds are homeomorphic, compare [Sav, Sec. 1.4].

Indeed, any oriented 3-manifold M admits a Heegaard splitting. In particular, a Heegaard splitting is obtained from a triangulation of M. The first handlebody is obtained from the triangulation by replacing the vertices by balls and edges by cylinders. The second handlebody arises from its complement, see [Sav, Th. 1.1] or [OSa, Th. 2.1]. Note that Heegaard splittings are not unique: Every 3-manifold has many different Heegaard splittings, compare [PS, Sec. 8].

Heegaard diagrams encode Heegaard splittings of 3-manifolds. A Heegaard splitting of M into two handlebodies N, N' of genus g defines a Heegaard diagram as follows: Consider the handlebodies N, N' and draw a system of meridians $v_1, \ldots, v_g$ on their boundaries:



The attaching homeomorphism can be split as a composite $f_2^{-1} \circ f_1$ of two homeomorphisms $f_1 : \partial N \to \Sigma$, $f_2 : \partial N' \to \Sigma$ into an oriented surface Σ of genus g. The surface Σ together with the curves $f_1(v_1), \ldots, f_1(v_g)$ and $f_2(v_1), \ldots, f_2(v_g)$ forms a Heegaard diagram, compare [PS, Sec. 10].

Conversely, from a Heegaard diagram $(\Sigma, \mathcal{U}, \mathcal{L})$ one can construct a Heegaard splitting: Start with two copies of the surface Σ and cut the first copy along the circles in $\mathcal{U}$, the second copy along the circles of $\mathcal{L}$. This yields two spheres, each with $2g$ holes. The boundary components at the first sphere are labelled by the circles in $\mathcal{U}$, the boundary components of the second space by the ones in $\mathcal{L}$. Then one glues $2g$ disks to each space to obtain spheres and attaches a 3-ball to each of the two spheres. This is illustrated in Figure 12.

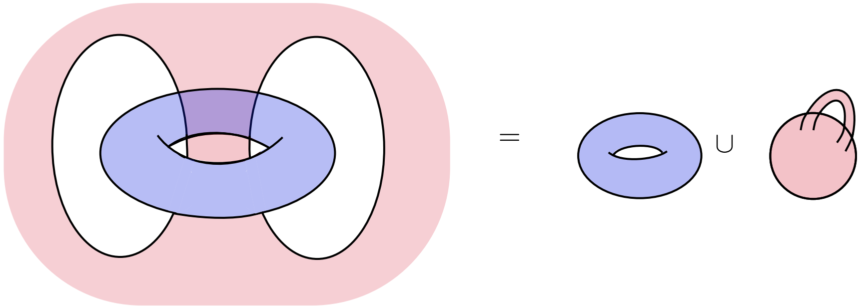


Figure 11: A Heegaard splitting of S^3, here viewed as the one-point compactification $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$ of $\mathbb{R}^3$, into two handlebodies of genus 1. One handlebody is schematically shown in blue, the complementary handlebody, which in particular includes the compactification point ∞, is indicated in red. For a better visibility some space between the handlebodies is left. A similar schematic illustration can be found in [CEZ, Fig. 5] and [PS, Fig. 8.7].

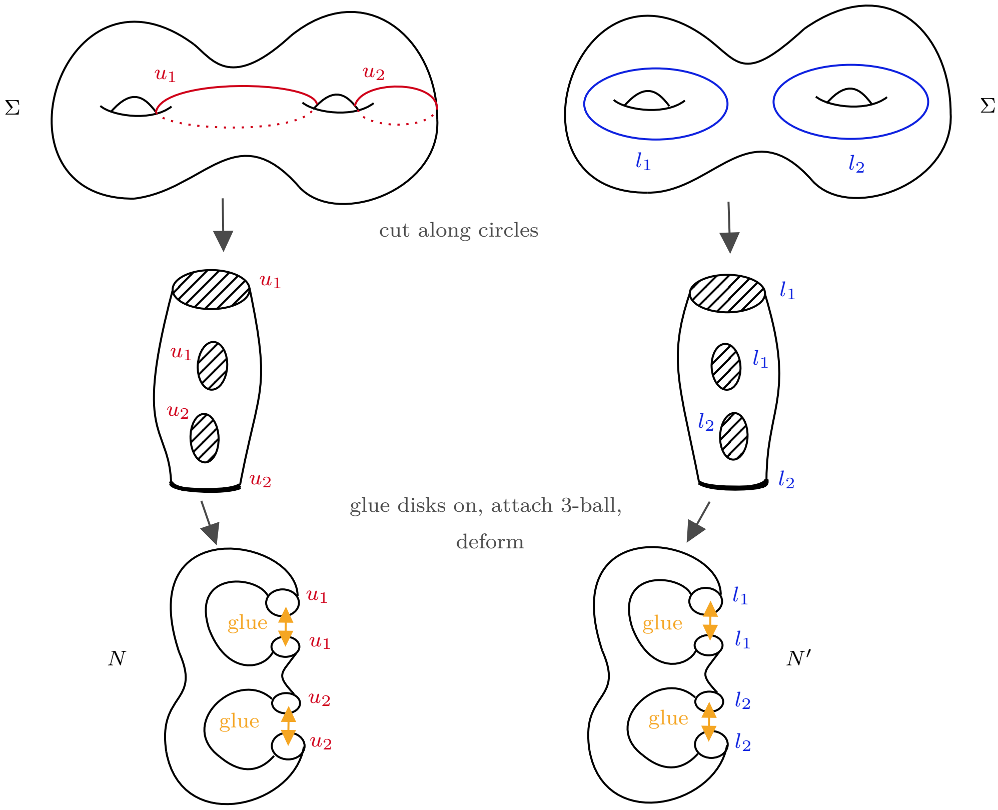


Figure 12: Illustration of the construction of a Heegaard splitting obtained from a Heegaard diagram, compare [PS, Fig. 10.4].

One can homeomorphically deform the spaces such that each pair of glued on disks with the same labelling corresponds to a meridian of the handlebodies. Finally identify each two disks with the same labelling to actually obtain the handlebodies N, N' of genus g. This procedure is illustrated in Figure 12. For more details see [PS, Sec. 10] and [Sad, Sec. 2.1]. The attaching homeomorphism of the Heegaard splitting is obtained by mapping the meridians of N to the circles in $\mathcal{U}$, the meridians of N' to the circles of $\mathcal{L}$ in Σ.

There is a well-known theorem, the *Reidemeister-Singer theorem*, that clarifies the question which Heegaard diagrams encode homeomorphic 3-manifolds: Two Heegaard diagrams describe homeomorphic oriented 3-manifolds iff they are related by a finite sequence of the following moves and their inverses: a homeomorphism of the surface, an isotopy of the diagram, stabilization and sliding one circle past another. We describe the moves in Section 10.3, when we summarise the proof of topological invariance of Kuperberg invariants from Kashaev and Virelizier [KV]. For more details on the Reidemeister-Singer theorem see for instance Kuperberg [Ku, Th. 4.1] or Singer [Si, Sec. III-IV].

10.2 Tensor endomorphism associated to Heegaard diagrams using Hopf monoids

Let again H be an involutive Hopf monoid in a symmetric monoidal category $\mathcal{C}$ and $D = (\Sigma, \mathcal{U}, \mathcal{L})$ an ordered oriented based Heegaard diagram with Σ an oriented surface of genus $g \geq 1$. In this section, we explain how to assign to this data an endomorphism $K_H(D) : H^{\otimes g} \to H^{\otimes g}$ that defines the Kuperberg invariants. We follow the presentation in [KV].

To define $K_H(D)$ we denote by $\mathcal{I}_\mathcal{U}$ the set $\mathcal{I}$ of intersection points equipped with the order obtained from $\mathcal{U}$ as described in Section 10.1. Analogously, $\mathcal{I}_\mathcal{L}$ is the set $\mathcal{I}$ equipped with the order induced by $\mathcal{L}$. Let $\sigma \in S_N$ be the permutation that relates the two orders on $\mathcal{I}$: For an intersection point that is the i-th element in $\mathcal{I}_\mathcal{L}$ and the j-th element in $\mathcal{I}_\mathcal{U}$ we set $\sigma(i) := j$. The permutation $\sigma \in S_N$ defines an automorphism $P_\sigma : H^{\otimes N} \to H^{\otimes N}$ obtained as follows: Decompose σ as a composite of elementary transpositions $(i,\, i+1)$. To $(i,\, i+1)$ we associate the morphism

$$P_{(i,\,i+1)} := 1_{H^{\otimes(i-1)}} \otimes \tau_{H,H} \otimes 1_{H^{\otimes(N-i-1)}}$$

and to σ the composite of those morphisms. For $N = 1$ we set $P_\sigma = 1_H$. We set

$$S_\mathcal{L} := \bigotimes_{c \in \mathcal{I}_\mathcal{L}} S^{(\kappa_c)} : H^{\otimes N} \to H^{\otimes N}, \qquad S_\mathcal{U} := \bigotimes_{c \in \mathcal{I}_\mathcal{U}} S^{(\kappa_c)} : H^{\otimes N} \to H^{\otimes N}$$

with

$$\kappa_c = \begin{cases} 0 & \text{if } c \text{ is positive,} \\ 1 & \text{if } c \text{ is negative.} \end{cases} \tag{80}$$

Here, the tensor products indexed by $\mathcal{L}$ and $\mathcal{U}$ denote the tensor product of the morphisms $S^{(\kappa_c)}$ in the order induced by the ordering of $\mathcal{I}_\mathcal{L}$ and $\mathcal{I}_\mathcal{U}$. Recall from Section 2.1 that we write $m^{(n)}$ and $\Delta^{(n)}$ for n-fold products and coproducts. As mentioned in Section 10.1 we denote $|x|$ the number of intersection points in $x \in \mathcal{U} \cup \mathcal{L}$. Applying this notation we set

$$\Delta_\mathcal{L} := \bigotimes_{l \in \mathcal{L}} \Delta^{(|l|)} : H^{\otimes g} \to H^{\otimes N}, \qquad m_\mathcal{U} := \bigotimes_{u \in \mathcal{U}} m^{(|u|)} : H^{\otimes N} \to H^{\otimes g}.$$

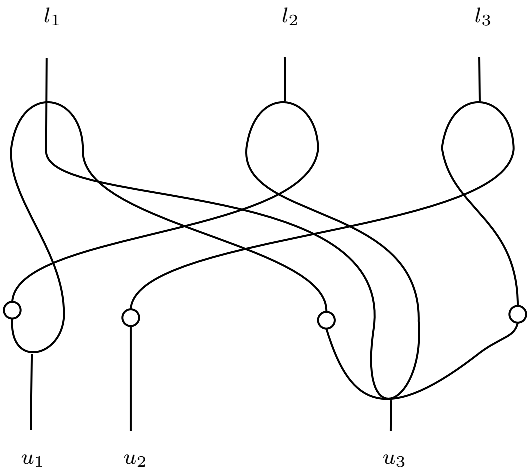
Note that the naturality of the braiding morphisms in $\mathcal{C}$ implies $P_\sigma \circ S_\mathcal{L} = S_\mathcal{U} \circ P_\sigma$. With the above morphisms we can define $K_H(D)$.

Definition 10.5. *[KV, Sec. 3.4] The* ***tensor endomorphism*** *associated to the ordered oriented based Heegaard diagram $D = (\Sigma, \mathcal{U}, \mathcal{L})$ is given by*

$$K_H(D) := m_{\mathcal{U}} \circ P_\sigma \circ S_{\mathcal{L}} \circ \Delta_{\mathcal{L}} = m_{\mathcal{U}} \circ S_{\mathcal{U}} \circ P_\sigma \circ \Delta_{\mathcal{L}} : H^{\otimes g} \to H^{\otimes g}. \tag{81}$$

We give some examples of Heegaard diagrams and their associated tensor endomorphisms. Note that we use the opposite convention of [KV] and read morphisms in diagrams from top to bottom.

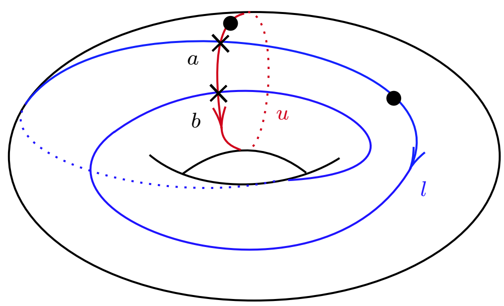
Example 10.6.
1. The tensor endomorphism for the ordered oriented based Heegaard diagram from Example 10.4 is given by



$$\begin{aligned} K_H(D)\,(l^1 \otimes l^2 \otimes l^3) &= S({l^2}_{(2)}){l^1}_{(1)} \otimes S({l^3}_{(2)}) \otimes S({l^1}_{(3)}){l^1}_{(2)}{l^2}_{(1)}S({l^3}_{(1)}) \\ &= S({l^2}_{(2)})l^1 \otimes S({l^3}_{(2)}) \otimes {l^2}_{(1)}S({l^3}_{(1)}). \end{aligned} \tag{82}$$

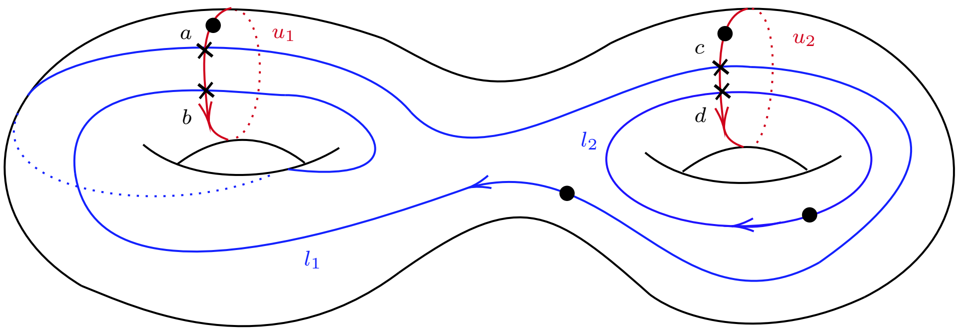
Here, (6) is applied to simplify $K_H(D)$.

2. In [Les, Figure 1] Lescop shows two oriented Heegaard diagrams for $\mathbb{RP}^3$, which we equip with basepoints and orders. To the ordered oriented based Heegaard diagram of genus 1



we assign the morphism $K_H(D)\,(l) = l_{(2)}l_{(1)}$, as both intersection points are positive and the order in $\mathcal{I}_{\mathcal{L}}$ is given by $b < a$, the order in $\mathcal{I}_{\mathcal{U}}$ by $a < b$.

To the ordered oriented based Heegaard diagram of genus 2

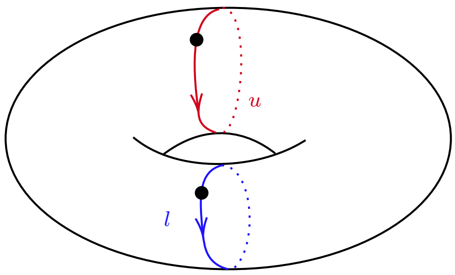


with the ordering $l_1 < l_2$, $u_1 < u_2$ in $\mathcal{L}$ and $\mathcal{U}$ we assign the morphism

$$K_H(D)\,(l^1 \otimes l^2) = {l^1}_{(2)}{l^1}_{(1)} \otimes {l^1}_{(3)}l^2$$

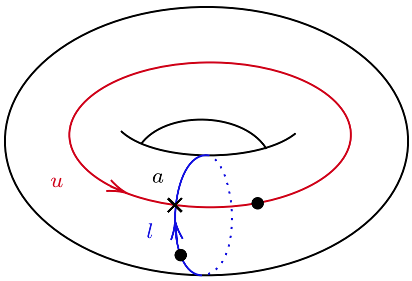
All intersection points are positive, the order in $\mathcal{I}_{\mathcal{L}}$ is given by $b < a < c < d$, the order in $\mathcal{I}_{\mathcal{U}}$ is given by $a < b < c < d$.

3. The ordered oriented based Heegaard diagram



represents $S^1 \times S^2$, compare for example Ozsváth and Szabó [OSa, Sec. 2.5]. As the diagram has no intersection point, one has $K_H(D) = \eta \circ \epsilon$.

4. The ordered oriented based Heegaard diagram



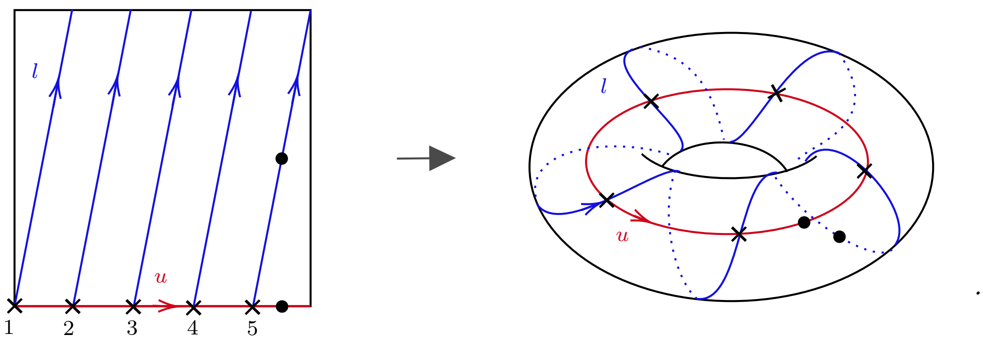
represents S^3, compare for example [OSa, Sec. 2.5]. As there is only a single intersection point, which is positive, we obtain $K_H(D) = \mathrm{id}_H$.

5. A special class of 3-manifolds are lens spaces $L(p, q)$ with p and q coprime positive integers, $p \geq 3$. They are obtained by considering a group action of $\mathbb{Z}/p\mathbb{Z}$ on the 3-sphere $S^3 \subseteq \mathbb{C}^2$. For σ a generator of $\mathbb{Z}/p\mathbb{Z}$ this fix point-free action is given by

$$\sigma \triangleright (z, w) = (e^{2\pi i/p}z, e^{2\pi iq/p}w).$$

The lens space $L(p,q)$ is the quotient of S^3 obtained by identifying each point $x \in S^3$ with the points $\sigma \triangleright x, \ldots, \sigma^{p-1} \triangleright x$, see [PS, Sec. 11]. The lens spaces $L(p,1)$ can be described by a standard Heegaard diagram $D = (\Sigma, \mathcal{U}, \mathcal{L})$ on the torus, compare [WY, Sec. 4.2]. For this we take the square $[0,1] \times [0,1]$ and identify its opposite sides to obtain a torus. The single element in $\mathcal{U}$ is represented by $[0,1] \times \{0\}$, its intersection points with the single element in $\mathcal{L}$ are given by the p points $\{\frac{k}{p}\} \times \{0\}$, $k = 0, \ldots, p-1$ in the square. The element in $\mathcal{L}$ is represented by the p straight lines which connect the points $\{\frac{k}{p}\} \times \{0\}$ and $\{\frac{k+1}{p}\} \times \{1\}$ for $k = 0, \ldots, p-1$. Suppose that the lines are oriented in the x, y-direction and equip $u \in \mathcal{U}$ and $l \in \mathcal{L}$ with induced orientations. Additionally, we add the basepoints $\{\frac{2p-1}{2p}\} \times \{0\}$ and $\{\frac{2p-1}{2p}\} \times \{\frac{1}{2}\}$.

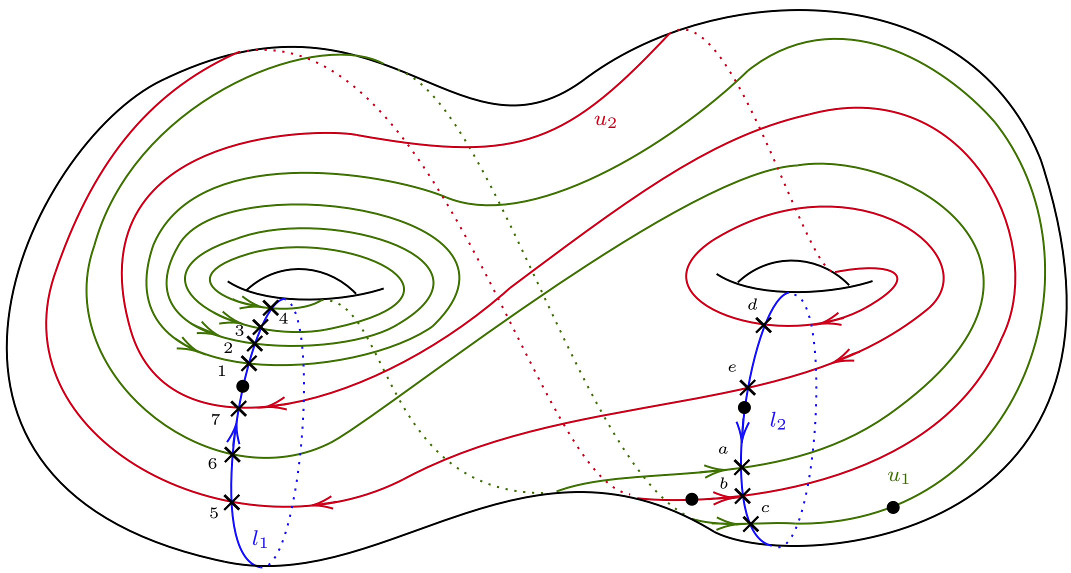
For the lens space $L(5,1)$ this yields the following Heegaard diagram:



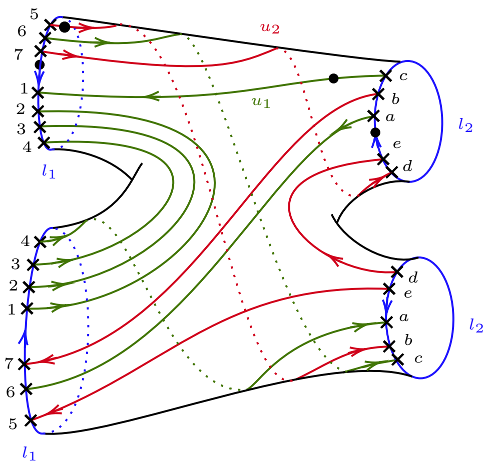
All intersection points in the Heegaard diagram for $L(p,1)$ are positive and the ordering of the intersection points induced by $\mathcal{L}$ and $\mathcal{U}$ coincides. Hence, the associated tensor endomorphism is given by

$$K_H(D)(l) = l_{(1)} l_{(2)} \ldots l_{(p)}.$$

6. The following oriented based Heegaard diagram

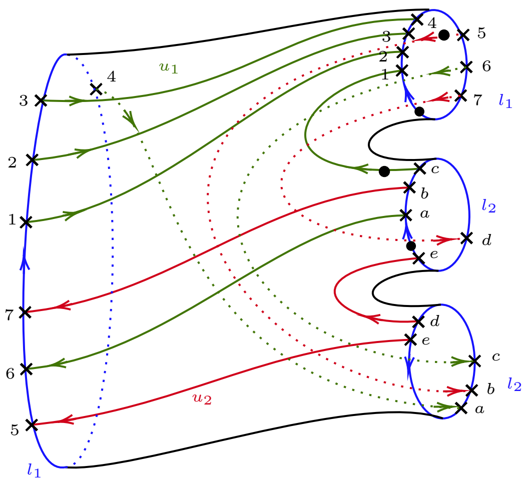


represents Poincaré's homology 3-sphere. It can be constructed by identifying opposite faces of a dodecahedron after a twist of $\frac{\pi}{5}$, see Kirby and Scharlemann [KS, Description 5] for more details. For better viewing the circle u_1 is drawn in green contrary to our usual colour convention. This Heegaard diagram is often depicted in a different way: The surface Σ is cut along the circles l_1 and l_2:

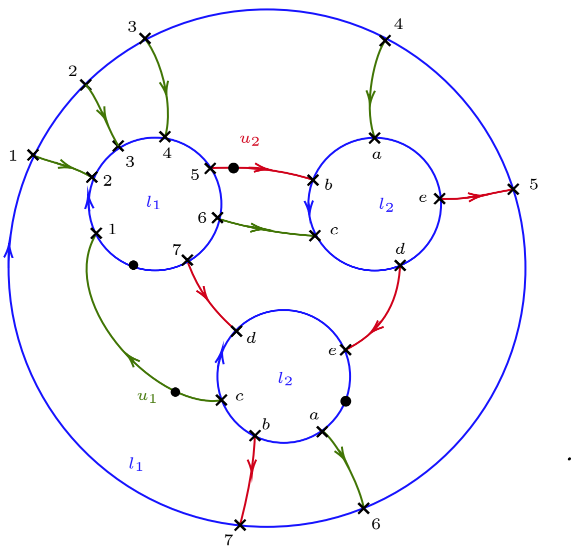


.

Then it is deformed such that there is one big circle l_1 on the left-hand side and one small circle l_1 as well as two small circles l_2 on the right-hand side.



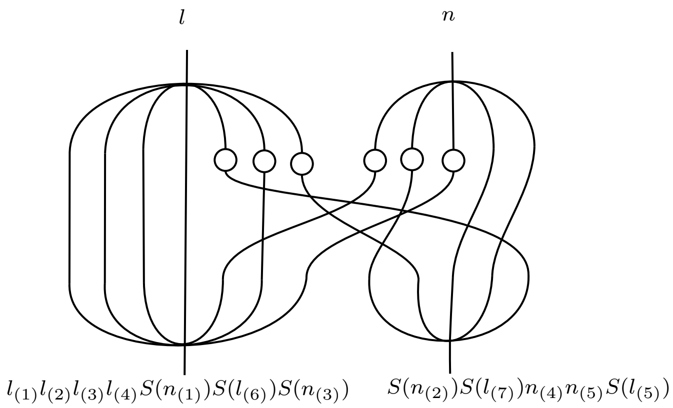
Looking at this diagram from the right gives the diagram:



The last diagram coincides with the one from [KV, Sec. 3.5,(3)], which is the one Poincaré provided in [Poi, Fig. 4] with slightly different notation. Equipping $\mathcal{U}$ and $\mathcal{L}$ with the orders $u_1 < u_2$, $l_1 < l_2$ induces the orders

$$1 < 2 < 3 < 4 < a < 6 < c < b < 7 < d < e < 5,$$
$$1 < 2 < 3 < 4 < 5 < 6 < 7 < a < b < c < d < e$$

of intersections points by $\mathcal{U}$ and $\mathcal{L}$. As the negative intersection points are $5, 6, 7, a, b, c$, the tensor endomorphism $K_H(D)$ is given by



$$K_H(D)\,(l \otimes n) = l_{(1)}l_{(2)}l_{(3)}l_{(4)}S(n_{(1)})S(l_{(6)})S(n_{(3)}) \otimes S(n_{(2)})S(l_{(7)})n_{(4)}n_{(5)}S(l_{(5)}).$$

10.3 Kuperberg invariants of 3-manifolds

In this section, we summarise the construction of Kuperberg invariants of 3-manifolds by Kashaev and Virelizier [KV] using involutive Hopf monoids in a symmetric monoidal category $\mathcal{C}$ and good pairs. Their construction is based on the work of Kuperberg [Ku] who introduced the invariants in terms of involutive Hopf algebras over $\mathbb{C}$ and their (co)integrals.

The construction assigns to an oriented 3-manifold, which we require to be closed and connected in this section, a morphism

$$\tau_{(\lambda,\Lambda)}(M) \in \mathrm{End}_{\mathcal{C}}(e),$$

called *Kuperberg invariant*. It is a topological invariant of M, but defined via an ordered oriented based Heegaard diagram representing M. A main ingredient of the definition is the tensor endomorphism from Section 10.2. Before we summarise the proof that $\tau_{(\lambda,\Lambda)}(M)$ is a topological invariant of M given by Kashaev and Virelizier [KV] and provide some examples, we outline the definition of good pairs and the construction of the Kuperberg invariant. In contrast to [KV] we restrict attention to good pairs involving the tensor unit instead of more general invertible objects as source or target of morphisms.

Throughout this section let H be an involutive Hopf monoid in $\mathcal{C}$. Recall that $\mathrm{End}_{\mathcal{C}}(e)$ is a commutative monoid with unit 1_e. Its elements are often called *scalars*. For any $k \in \mathbb{N}$ the map

$$\mathrm{End}_{\mathcal{C}}(e) \to \mathrm{End}_{\mathcal{C}}(e^{\otimes k}),\ \alpha \mapsto \alpha \otimes 1_{e^{\otimes k}}$$

is an isomorphism of monoids, see [KV, Sec. 2.5]. For $f \in \mathrm{End}_{\mathcal{C}}(e^{\otimes k})$ we denote by $\langle f\rangle_{e^{\otimes k}}$ its inverse, i.e. the unique morphism $\langle f\rangle_{e^{\otimes k}} \in \mathrm{End}_{\mathcal{C}}(e)$ satisfying

$$f = \langle f\rangle_{e^{\otimes k}} \otimes 1_{e^{\otimes k}}. \tag{83}$$

Applying this notation, we can define *good pairs*, as in [KV, Sec. 4.1].

Definition 10.7. *[KV, Sec. 4.1] Let H be an involutive Hopf monoid in $\mathcal{C}$. A* ***good pair*** *for H is a pair $(\lambda : H \to e,\ \Lambda : e \to H)$ of morphisms in $\mathcal{C}$ such that*

a) $\lambda \circ \Lambda = 1_e$,
b)

$$(\lambda \otimes \lambda) \circ (m \otimes 1_H) \circ (1_H \otimes \Delta) = \lambda \otimes \lambda, \quad (m \otimes 1_H) \circ (1_H \otimes \Delta) \circ (\Lambda \otimes \Lambda) = \Lambda \otimes \Lambda, \tag{84}$$

c) setting $\nu_{(\lambda,\Lambda)} := \langle \lambda \circ S \circ \Lambda\rangle_e \in \mathrm{End}_{\mathcal{C}}(e)$ one has

$$\lambda \circ S = \nu_{(\lambda,\Lambda)} \otimes \lambda \ \text{ and } \ S \circ \Lambda = \nu_{(\lambda,\Lambda)} \otimes \Lambda, \tag{85}$$

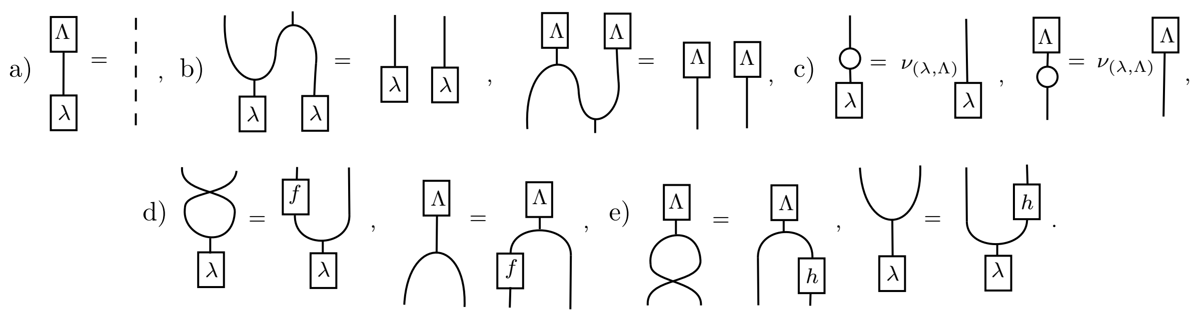
d) there exists a morphism $f : H \to H$ in $\mathcal{C}$ satisfying

$$\lambda \circ m \circ \tau_{H,H} = \lambda \circ m \circ (f \otimes 1_H) \ \text{ and } \ \Delta \circ \Lambda = (f \otimes 1_H) \circ \Delta \circ \Lambda, \tag{86}$$

e) there exists a morphism $h : H \to H$ in $\mathcal{C}$ satisfying

$$\tau_{H,H} \circ \Delta \circ \Lambda = (1_H \otimes h) \circ \Delta \circ \Lambda \ \text{ and } \lambda \circ m = \lambda \circ m \circ (1_H \otimes h). \tag{87}$$

Graphically, the conditions in Definition 10.7 are given by



Example 10.8. *1. Any finite-dimensional involutive Hopf algebra H over a field $\mathbb{F}$ with* $\mathrm{char}(\mathbb{F}) \nmid \dim_{\mathbb{F}}(H)$ *has a two-sided integral $\Lambda \in H$ and a two-sided cointegral $\lambda \in \mathrm{Hom}(\mathbb{F}, H) = H^*$ such that $\lambda(\Lambda) = 1$. This is derived from the semisimplicity and unimodularity of H via Proposition 2.27 and Lemma 2.26. From [KV, Sec. 4.4,(1)] follows that (λ, Λ) is a good pair for H. In particular, it satisfies $\nu_{(\lambda,\Lambda)} = \langle \lambda \circ S \circ \Lambda \rangle_e = 1_e$, as $S \circ \Lambda = \Lambda$. This is obtained as the space of left integrals of H is one-dimensional, hence $\Lambda = k \cdot l$ for some $k \in \mathbb{F}$ and l the Haar integral of H. Thus, one has $S(\Lambda) = k \cdot S(l) = k \cdot l = \Lambda$.*

2. *For the Hopf monoid $\mathbb{C}[H]$ in $\mathrm{Rep}_{\mathbb{C}}(G)$ from Example 2.2, 5. the integral $\Lambda : \mathbb{C} \to \mathbb{C}[H]$, $r \mapsto r \sum_{h \in H} \delta_h$ and cointegral $\lambda = \rho_e : \mathbb{C}[H] \to \mathbb{C}$ from Example 2.19, 1. form a good pair. The conditions in Definition 10.7 are satisfied with $\nu_{(\lambda,\Lambda)} = 1_{\mathbb{C}}$ and $f = h = 1_{\mathbb{C}[H]}$. This example in particular includes the special case $G = \{\bullet\}$, where* $\mathrm{Rep}_{\mathbb{C}}(G)$ *coincides with the category* $\mathrm{Vect}_{\mathbb{C}}^{fd}$ *of finite-dimensional $\mathbb{C}$-vector spaces.*
3. *For any Hopf monoid H in a symmetric monoidal category the counit $\lambda := \epsilon$ and unit $\Lambda := \eta$ form a good pair for H. The conditions in Definition 10.7 are satisfied with $\nu_{(\lambda,\Lambda)} = 1_e$ and $f = h = 1_H$.*
4. *A pair (λ, Λ) is a good pair for H in $\mathcal{C}$ if and only if (Λ, λ) is a good pair for H^* in $\mathcal{C}^{op}$.*

For a good pair (λ, Λ) for an involutive Hopf monoid H in $\mathcal{C}$ we set

$$\gamma_{(\lambda,\Lambda)} := \begin{cases} 1 & \text{if } \nu_{(\lambda,\Lambda)} = 1_e \\ 2 & \text{else,} \end{cases} \tag{88}$$

where $\nu_{(\lambda,\Lambda)}$ is given in Definition 10.7, c). To an oriented 3-manifold M represented by an ordered oriented based Heegaard diagram D of genus g we assign the morphism

$$\lambda^{(g)} \circ K_H(D) \circ \Lambda^{(g)} \in \mathrm{End}_{\mathcal{C}}(e^{\otimes g})$$

with the tensor endomorphism $K_H(D) : H^{\otimes g} \to H^{\otimes g}$ from (81). Passing to the corresponding morphism in the monoid $\mathrm{End}_{\mathcal{C}}(e)$ and using (88) yields the following definition of the Kuperberg invariant.

Definition 10.9. *[KV, Sec. 4.3] Let (λ, Λ) be a good pair for an involutive Hopf monoid H in $\mathcal{C}$. Suppose M is an oriented* 3*-manifold represented by an ordered oriented based Heegaard diagram D of genus g. Then the* ***Kuperberg invariant*** *assigned to M is the endomorphism*

$$\tau_{(\lambda,\Lambda)}(M) := \langle \lambda^{(g)} \circ K_H(D) \circ \Lambda^{(g)} \rangle_{e^{\otimes g}}^{\gamma_{(\lambda,\Lambda)}} \in \mathrm{End}_{\mathcal{C}}(e) \tag{89}$$

with $K_H(D)$ from (81) and $\gamma_{(\lambda,\Lambda)}$ from (88).

Example 10.10. *For a finite-dimensional involutive Hopf algebra H over a field $\mathbb{F}$ with* $\mathrm{char}(\mathbb{F}) \nmid \dim_{\mathbb{F}}(H)$ *and the good pair (λ, Λ) from Example 10.8, 1. one has*

$$\tau_{(\lambda,\Lambda)}(M) = \langle \lambda^{(g)} \circ K_H(D) \circ \Lambda^{(g)} \rangle_{e^{\otimes g}}.$$

In particular, $\tau_{(\lambda,\Lambda)}(M)$ coincides with the Kuperberg invariant as introduced by Kuperberg in [Ku], compare [KV, Sec. 4.4,(1)].

Theorem 10.11. *[KV, Th. 1] The Kuperberg invariant $\tau_{(\lambda,\Lambda)}(M)$ from (89) assigned to an oriented* 3*-manifold M is a topological invariant of M.*

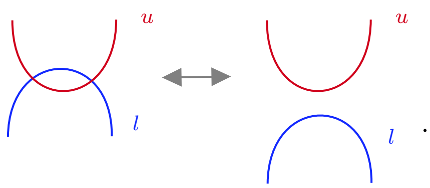
Summary of the proof of Theorem 10.11 from [KV, Sec. 4.6]. As the Kuperberg invariant is defined based on an ordered oriented based Heegaard diagram D, first one has to show that it does not depend on orientations, basepoints and orders of $\mathcal{U}$ and $\mathcal{L}$. Each change of orientation, basepoint or order transforms D into another diagram D' with corresponding Kuperberg invariant denoted $\tau'_{(\lambda,\Lambda)}(M)$. If the orientation of an upper or lower circle is reversed, applying the antimultiplicativity of S from (5), the involutivity of S and (85) transforms $\tau'_{(\lambda,\Lambda)}(M)$ into $\tau_{(\lambda,\Lambda)}(M)$. Hence, $\tau_{(\lambda,\Lambda)}(M) = \tau'_{(\lambda,\Lambda)}(M)$.

If the basepoint of a circle $a \in \mathcal{U} \cup \mathcal{L}$ is changed, the order of $\mathcal{I}_a$ is cyclically permuted. Compared to $\tau_{(\lambda,\Lambda)}(M)$ the morphism $\tau'_{(\lambda,\Lambda)}(M)$ defined by (89) and (81) has additional braidings. By the associativity of m, naturality of the braiding and (86), or (87) respectively, the morphisms can be transformed into each other. Likewise, changing the order of $\mathcal{U}$ or $\mathcal{L}$ involves additional braiding morphisms. The identity $\tau'_{(\lambda,\Lambda)}(M) = \tau_{(\lambda,\Lambda)}(M)$ then follows from the naturality of the braiding.

It remains to show that $\tau_{(\lambda,\Lambda)}(M)$ is independent of the choice of the Heegaard diagram itself. For this we can apply the *Reidemeister-Singer theorem*, see for instance Kuperberg [Ku, Th. 4.1] or Singer [Si, Sec. III-IV]. By this theorem, two Heegaard diagrams describe homeomorphic oriented 3-manifolds if and only if the diagrams can be transformed into each other by a finite sequence of moves and their inverses of the following four types: a) homeomorphism of the surface, b) isotopy of the diagram, c) stabilization and d) sliding one circle past another. We describe the four moves in the following.

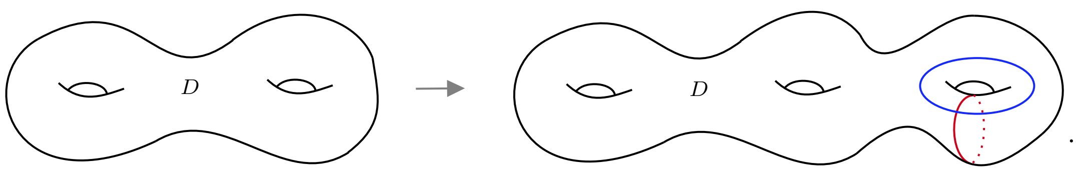
a) homeomorphism of the surface: An orientation-preserving homeomorphism from Σ into an oriented surface Σ' carries D to a Heegaard diagram D' on Σ', where the upper and lower circles of D are brought to the upper and lower circles of D'. Basepoints, orientations of the circles and orders of $\mathcal{U}'$ and $\mathcal{L}'$ in D' can be chosen such that they are preserved by the homeomorphism, compare [KV, Sec. 4.6]. Hence, $K_H(D) = K_H(D')$ and thus $\tau'_{(\lambda,\Lambda)}(M) = \tau_{(\lambda,\Lambda)}(M)$.

b) isotopy of the diagram: An upper (lower) circle is replaced by an isotopic circle that is disjoint from the other upper (lower) circles, compare [OSb, Sec. 2.1]. The isotopy can be decomposed into a sequence of the following two-point moves with $u \in \mathcal{U}$ and $l \in \mathcal{L}$, see [Ku, Th. 4.1] and [KV, Sec. 4.6]:



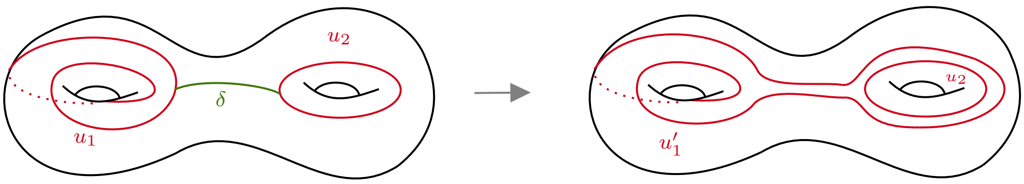
As one of the above intersection points of u and l is positive and the other one negative, the invariance of $\tau_{(\lambda,\lambda)}(M)$ under the two-point move follows from (3).

c) stabilization: A disk from Σ which is disjoint from all circles is removed and replaced by a punctured torus equipped with one upper and one lower circle. One of the new circles can be chosen to be the standard meridian and the other one as the standard longitude of the torus:



For more details on stabilizations we refer to [Ku, Th. 4.1], [KV, Sec. 4.6] and [Sav, Sec. 1.3]. The invariance of $\tau_{(\lambda,\Lambda)}(M)$ under stabilizations can be seen as a consequence of Definition 10.7, a).

d) sliding one circle past another: Here, the two considered circles are both upper or both lower. A circle $u_1 \in \mathcal{U}$ is slid past a circle $u_2 \in \mathcal{U}$ if u_1 is replaced by a circle u_1' disjoint from the upper circles such that u_1, u_1' and u_2 bound an embedded pair of pants in $\Sigma \setminus \bigcup_{u \in \mathcal{U}\setminus\{u_1,u_2\}} u$ [OSa, Sec. 2.6] and analogously for lower circles. Another way to view the handleslide is to connect u_1, u_2 by an arc δ in $\Sigma \setminus \bigcup_{u \in \mathcal{U}\setminus\{u_1,u_2\}} u$. Then u_1 is replaced by u_1' which is the connected sum of u_1 with a parallel copy of u_2, where the connected sum is taken along a neighborhood of δ, see [Hom, Sec. 1.2]:



The invariance of $\tau_{(\lambda,\Lambda)}(M)$ under handleslides follows by computing $\tau'_{(\lambda,\Lambda)}(M)$ from (89) for the slided diagram. By applying the coassociativity of Δ, anticomultiplicativity of S, multiplicativity of Δ and (84) one can show that $\tau'_{(\lambda,\lambda)}(M) = \tau_{(\lambda,\Lambda)}(M)$.

For the detailed computations associated with these steps we again refer to [KV, Sec. 4.6]. $\Diamond$

Example 10.12. *1. We consider examples of Kuperberg invariants based on the Hopf monoid* $\mathbb{C}[H]$ *in* $\mathrm{Rep}_{\mathbb{C}}(G)$ *from Example 2.2, 5. and its good pair consisting of* $\lambda = \rho_e$ *and* $\Lambda : r \mapsto r \sum_{h \in H} \delta_h$ *from Example 10.8, 2. As for* $n \geq 3$ *there is a group homomorphism*

$$\Phi : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}), \quad \Phi_{\bar{0}}(\bar{k}) = \bar{k}, \quad \Phi_{\bar{1}}(\bar{k}) = -\bar{k},$$

we can consider $G = \mathbb{Z}/2\mathbb{Z}$, $H = \mathbb{Z}/4\mathbb{Z}$*. For* $\mathbb{RP}^3$ *described by the Heegaard diagram from Example 10.6, 2. with tensor endomorphism* $K_H(D)(l) = l_{(2)} l_{(1)}$ *and* $r \in \mathbb{C}$ *we obtain*

$$\lambda \circ K_H(D) \circ \Lambda(r) = \rho_{\bar{0}} \circ K_H(D)(r \cdot (\delta_{\bar{0}} + \delta_{\bar{1}} + \delta_{\bar{2}} + \delta_{\bar{3}})) = r \cdot \rho_{\bar{0}}(\delta_{\bar{0}} + \delta_{\bar{2}} + \delta_{\bar{0}} + \delta_{\bar{2}}) = 2r.$$

Hence, the associated Kuperberg invariant, which is a scalar in $\mathbb{C}$*, is* $\tau_{(\lambda,\Lambda)}(\mathbb{RP}^3) = 2$*. Analogous computations for the other* 3*-manifolds in Example 10.6, 1. to 6. yield:*

manifold M	*reference*	$\tau_{(\lambda,\Lambda)}(M)$
described by Example 10.4	*Example 10.4 and 10.6, 1.*	1
$\mathbb{RP}^3$	*Example 10.6, 2.*	2
$S^1 \times S^2$	*Example 10.6, 3.*	4
S^3	*Example 10.6, 4.*	1
$L(5,1)$	*Example 10.6, 5.*	1
Poincaré's homology 3*-sphere*	*Example 10.6, 6.*	1

Here, we often obtain 1 *as Kuperberg invariant. By replacing for instance* $\mathbb{C}[\mathbb{Z}/4\mathbb{Z}]$ *by* $\mathbb{C}[\mathbb{Z}/5\mathbb{Z}]$ *in* $\mathrm{Rep}_{\mathbb{C}}(\mathbb{Z}/2\mathbb{Z})$ *for example in the case of the lens space one obtains* $\tau_{(\lambda,\Lambda)}(L(5,1)) = 5$.

2. We consider the Hopf algebra H_8 *in* $\mathbb{F} = \mathbb{C}$ *with integral and cointegral*

$$\Lambda = 1 + x + y + xy + z + xz + yz + xyz, \quad \lambda = \rho_1$$

from Example 2.23. For $\mathbb{RP}^3$ from Example 10.6, 2. with $K_H(D)(l) = l_{(2)}l_{(1)}$ and $r \in \mathbb{C}$ we obtain

$$\begin{aligned}\lambda \circ K_H(D) \circ \Lambda(r) &= r \cdot \rho_1(K_H(D)(1 + x + y + xy + z + xz + yz + xyz)) \\ &= r \cdot \rho_1(1 + 1 + 1 + 1 + 4 \cdot \frac{1}{2}(1 + x + xy - y) + 4 \cdot \frac{1}{2}(y + xy + 1 - x) = 8r\end{aligned}$$

by applying the identities (18), hence $\tau_{(\lambda,\Lambda)}(\mathbb{RP}^3) = 8$.

For the manifold M represented by the Heegaard diagram from Example 10.4 with tensor endomorphism (82) we obtain $\tau_{(\lambda,\Lambda)}(M) = 1$. This follows from the identity

$$\rho_1^{(3)}(K_H(D)(a \otimes b \otimes c)) = 0$$

if a has the form

$$a = a_1 \cdot 1 + a_2 \cdot x + a_3 \cdot y + a_4 \cdot xy + a_5 \cdot z + a_6 \cdot xz + a_7 \cdot yz + a_8 \cdot xyz$$

with $a_1 = 0$ and likewise for b, c.

In contrast to mapping class group actions, Kuperberg invariants for Hopf monoids in cartesian monoidal categories are trivial. Although the construction of Kuperberg invariants works, all Kuperberg invariants are given by the single element in $\mathrm{End}_{\mathcal{C}}(e)$ in these categories.

Remark 10.13. *1. In comparison to [KV] we restrict the source of Λ and target of λ in a good pair (λ, Λ) to the tensor unit, as it is the most relevant case for us. In [KV] the tensor unit e is replaced by any invertible object. This is an object I such that there exists an object J in $\mathcal{C}$ with $I \otimes J$ isomorphic to e. Defining the Kuperberg invariant for good pairs based on an invertible object I is accompanied by taking the morphism $\langle \tau_{I,I} \rangle_{I \otimes I}$ into account in the definition of $\gamma_{(\lambda,\Lambda)}$ in (88). In the case $I = e$ it reduces to $\langle \tau_{I,I} \rangle_{I \otimes I} = 1_e$ and can be neglected.*

2. One can use integrals and cointegrals over Hopf monoids H in symmetric monoidal categories $\mathcal{C}$ to construct good pairs of H. In detail this is described in [KV, Sec. 5.3 and 5.4] and in particular in [KV, Th. 2].

For this one needs a left integral $\Lambda : I \to H$ and a right cointegral $\lambda : H \to I$ of H with $\lambda \circ \Lambda = 1_I$. Here, in contrast to Definition 2.18 the tensor unit is replaced by an invertible object I. Both Λ and λ have to be universal, meaning that for any left integral $\Lambda' : X \to H$ and any right cointegral $\lambda' : H \to Y$ there exist unique morphisms $f_1 : X \to I$ and $f_2 : I \to Y$ with $\Lambda \circ f_1 = \Lambda'$ and $f_2 \circ \lambda = \lambda'$. The universality of Λ and λ in particular implies the existence of unique morphisms $g : e \to H$ and $\alpha : H \to e$ with

$$g \otimes \lambda = (1_H \otimes \lambda) \circ \Delta, \quad m \circ (\Lambda \otimes 1_H) = \Lambda \otimes \alpha.$$

The existence of such a pair $(\lambda : H \to I, \Lambda : I \to H)$ of universal (co)integrals with I an invertible object is guaranteed if $\mathcal{C}$ is left rigid and all idempotents split, see Bespalov et al. [BKLT, Prop. 3.1]. If $\mathcal{C}$ is also preadditive and additional assumptions on the morphisms g and α are satisfied, compare [KV, Sec. 5.3], then [KV, Th. 2] provides a construction of a good pair from (λ, Λ) via sums of (co)integrals, (co)multiplication and the distinguished morphisms.

Overall, in comparison to the main part of the thesis, the assumptions on the category $\mathcal{C}$ mentioned in Remark 10.13 are far more restrictive. Apart from the existence result in [BKLT, Prop. 3.1], there appears to be no efficient way to check universality of given (co)integrals. It is also worth pointing out that not all good pairs arise from integrals and cointegrals. As mentioned earlier,

(ϵ, η) is a good pair for any Hopf monoid H in Set, but there are no integrals and cointegrals over $H \neq \{e\}$ in Set.

Besides Kashaev's and Virelizier's generalisation, there are various other extensions of Kuperberg invariants. For instance, Kuperberg [Ku2] himself generalised his original setting to invariants of framed 3-manifolds based on finite-dimensional Hopf (super)algebras, which are not necessarily involutive. The framing equips the manifold with linearly independent vector fields with suitable orientations. Neumann [Ne] worked in the setting of involutive Hopf superalgebras and sutured manifolds. That is, a 3-manifold M with a collection of pairwise disjoint annuli in ∂M that divide ∂M into two subsurfaces. Whereas Virelizier [Vi] extended the Kuperberg invariants to invariants of flat G-bundles over 3-manifolds based on involutive Hopf G-coalgebras for a group G. If G is the trivial group, involutive Hopf algebras and Kuperberg's invariants of 3-manifolds are recovered.

Likewise, Costantino et al. [CGPT] regained the Kuperberg invariants for finite-dimensional involutive Hopf algebras as a special case when constructing invariants of 3-manifolds using pivotal $\mathbb{F}$-categories equipped with traces for a field $\mathbb{F}$.

Kuperberg invariants are also related to other 3-manifold invariants. For example, in the case of finite-dimensional semisimple Hopf algebras they are linked to Turaev-Viro-Barrett-Westbury invariants, see Barrett and Westbury [BaW, BaW2] and Turaev and Viro [TV].

10.4 Colored ribbon graphs assigned to Heegaard diagrams

Heegaard diagrams are related to mapping class group actions via gluing homeomorphisms in Heegaard splittings as described in Section 10.1. This raises the question how they are related to the mapping class group actions by edge slides in Section 9.3. We will address this question in the next section by expressing the tensor endomorphism as a sequence of edge slides. The first step towards this result is to assign a 2-colored ribbon graph $\Gamma(D)$ as well as g paths $P_1(D), \ldots, P_g(D)$ in $\Gamma(D)$ to each ordered oriented based Heegaard diagram D.

Definition 10.14. *A **coloring** of a graph by a set X is an assignment of an element of X to each edge. If $|X| = n \in \mathbb{N}$ we call the graph **n-colored**.*

In the following, we consider 2-colored graphs and assign the colors red and blue to the edges of a graph. To construct the desired graph from a Heegaard diagram D we use an auxiliary ribbon graph $\Gamma'(D)$, obtained by taking the upper and lower circles of D together and inserting vertices at each intersection point and basepoint. More formally, we characterise $\Gamma'(D)$ as follows.

Definition 10.15. *The 2-colored ribbon graph $\Gamma'(D)$ for an ordered oriented based Heegaard diagram $D = (\Sigma, \mathcal{U}, \mathcal{L})$ has:*

- *vertex set $V'(D) = \mathcal{I} \cup B_D$,*
- *edge set $E'(D) = E'_{\mathcal{L}}(D) \,\dot\cup\, E'_{\mathcal{U}}(D) = (\bigcup_{l \in \mathcal{L}} E'_l(D)) \,\dot\cup\, (\bigcup_{u \in \mathcal{U}} E'_u(D))$, where the set $E'_l(D)$ for $l \in \mathcal{L}$ consists of $|l| + 1$ oriented blue edges, from each intersection point in $\mathcal{I}_l$ to its successor, from the base point to the first intersection point and from the last intersection point to the basepoint. The set $E'_u(D)$ of red edges for each upper circle $u \in \mathcal{U}$ is defined analogously.*
- *cyclic orderings at an intersection point $c \in \mathcal{I}$ between incoming edges $r \in E'_u(D)$, $b \in E'_l(D)$ and outgoing edges $r' \in E'_u(D)$, $b' \in E'_l(D)$ given by $(t(r), t(b), s(r'), s(b'))$ if c is positive and by $(t(r), s(b'), s(r'), t(b))$ if c is negative.*

This construction is illustrated in Example 10.17. Note that in particular if a circle in the Heegaard diagram has no intersection points, there is only a loop attached at the basepoint.

For any $a \in \mathcal{U} \cup \mathcal{L}$ the set of target vertices of edges in $E'_a(D)$ is the union of $\mathcal{I}_a$ and the basepoint of a. Assigning to each edge its target vertex defines a total order on $E'_a(D)$. Here, the basepoint is chosen to be the last element.

At positive intersection points we call the incoming red edge end $t(r)$ the *inner red edge end*. At negative intersection points the outgoing red edge end $s(r')$ is called inner red edge end. Note that a red edge can have two inner red edge ends. For instance, in the graph $\Gamma'(D)$ in Example 10.17 the edge from s to r has two inner red edge ends.

From the ribbon graph $\Gamma'(D)$ associated to a Heegaard diagram, we can construct another ribbon graph $\Gamma(D)$, which has exactly one based loop for each lower circle. This is achieved by contracting edges, adding vertices, reversing edge orientations and splitting edges, cf. Chapter 3.

Definition 10.16. *The* 2*-colored ribbon graph* $\Gamma(D)$ *associated to an ordered oriented based Heegaard diagram* $D = (\Sigma, \mathcal{U}, \mathcal{L})$ *arises from the graph* $\Gamma'(D)$ *from Definition 10.15 by successively iterating 1. over* $u \in \mathcal{U}$ *in the order given by* $\mathcal{U}$*, then carrying out 2. and iterating 3. over* $l \in \mathcal{L}$ *in the order given by* $\mathcal{L}$*:*

1. *If* $|E'_u(D)| > 1$ *an edge in* $E'_u(D)$ *is contracted towards the basepoint: If the first intersection point in* $\mathcal{I}_u$ *is positive (negative), then the last (first) edge in* $E'_u(D)$ *is contracted. For the remaining edges* a *in* $E'_u(D)$ *the following is iterated in the order of* $E'_u(D)$*: If* $s(a)$ *is negative and* $t(a)$ *positive,* a *is split into two edges.*
2. *The orientations of all red edges are reversed.*
3. *If* $|E'_l(D)| > 1$ *each edge in* $E'_l(D)$ *except for the last one is contracted such that all intersection points on* l *are removed and the basepoint is kept.*

If the condition in 3. is not satisfied, nothing is contracted. As reversing the orientations of two different edges commutes, one can choose any order in which the orientations of the red edges in step 2. are reversed.

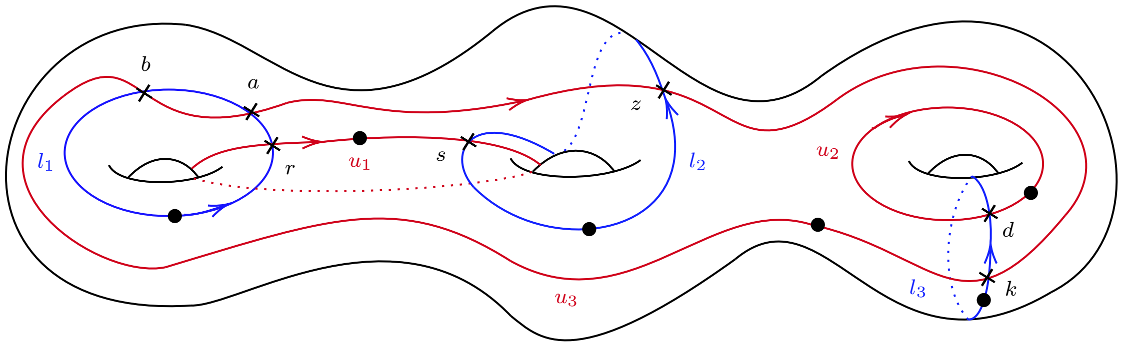
We denote the edge sets arising from $E'_u(D)$, $E'_l(D)$ for $u \in \mathcal{U}$, $l \in \mathcal{L}$ in steps 1., 3. by $E_u(D)$, $E_l(D)$. The latter are singletons, and we label the unique edge in $E_l(D)$ by l. The sets $E_u(D)$ are endowed with the induced order of $E'_u(D)$, such that contracted edges are omitted in the order. Splitting edges is taken into account as follows: If an edge a with possible predecessor b' and successor b'' in $E'_u(D)$ is split into an edge a' with $s(a') = s(a)$ and an edge a'' with $t(a'') = t(a)$, then in the order of $E_u(D)$ the edge b' is succeeded by a', then a'' and b''. If $s(a) = s(a')$ is the last element in $\mathcal{I}_u$, then a' is the last element and a'' the first element with successor b'' in $E_u(D)$. This is illustrated in Example 10.17.

Note that in Definition 10.16 the splitting of an edge a happens if and only if $(s(a), t(a))$ is an inversion pair. The splitting of edges ensures that each red edge has at most one inner red edge end. Conversely, as each intersection point has an inner red edge end, we can label the corresponding red edge by the intersection point.

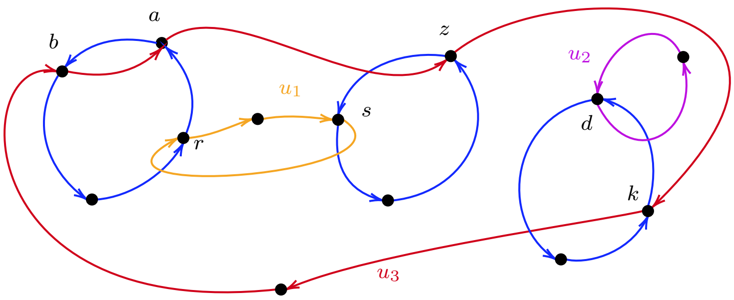
Reversing the orientations of all red edges in Definition 10.16, 2. amounts to reversing the order of $E_u(D)$. This ordered set $E_u(D)$ defines a closed path in $\Gamma(D)$. We denote by $P_i(D)$ the *path described by* $E_{u_i}(D)$ for the i-th element u_i in $\mathcal{U}$.

The following example illustrates the described procedures.

Example 10.17. *From the Heegaard diagram*

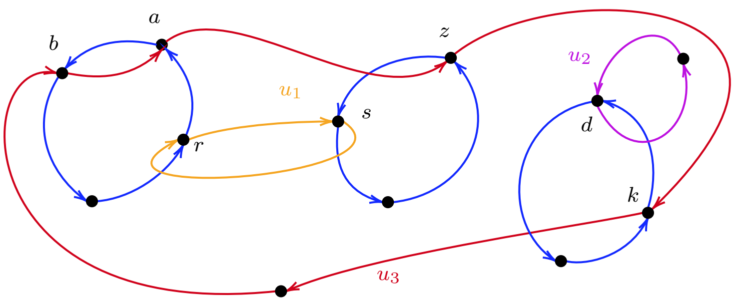


from Example 10.4 we first construct the 2*-colored ribbon graph* $\Gamma'(D)$ *given by*

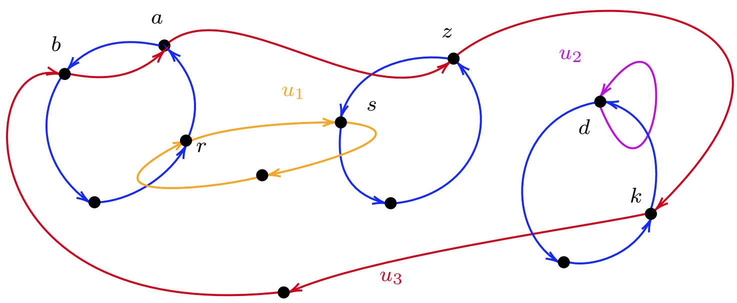


The edges arising from u_1 *are indicated in orange, the ones arising from* u_3 *in magenta.*

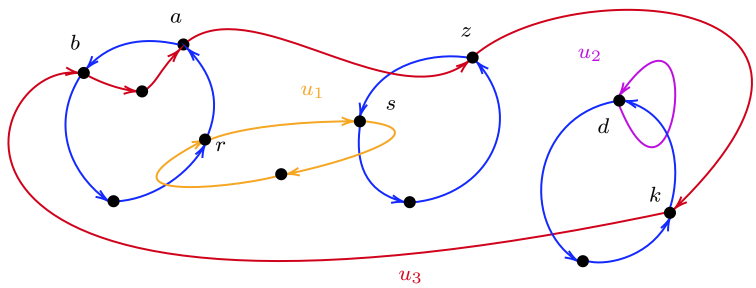
As the first intersection point s *in* $\mathcal{I}_{u_1}$ *is negative, the first edge in* $E'_{u_1}(D)$ *is contracted:*



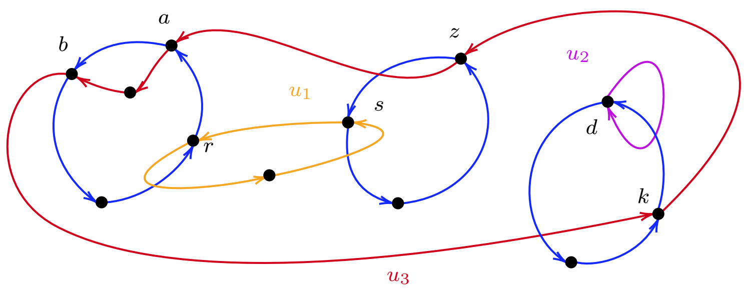
As (s, r) *is an inversion pair, the edge that is outgoing at* s *is split. As the first intersection point in* $\mathcal{I}_{u_2}$ *is negative, the first edge in* $E'_{u_2}(D)$ *is contracted.*



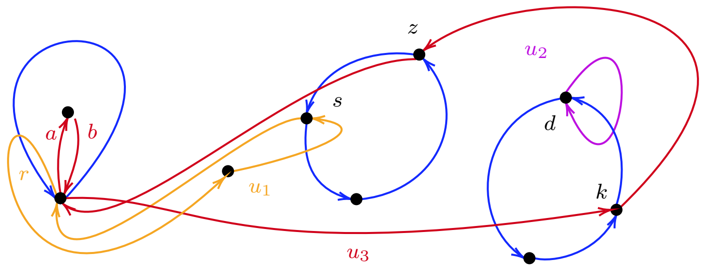
Likewise, the first intersection point b in $\mathcal{I}_{u_3}$ is negative and thus the first edge in $E'_{u_3}(D)$ contracted. As (b, a) is an inversion pair, the red edge outgoing at b is split:



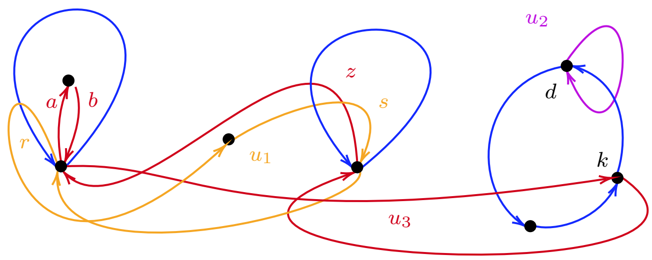
In the next step the orientations of all red edges are reversed:



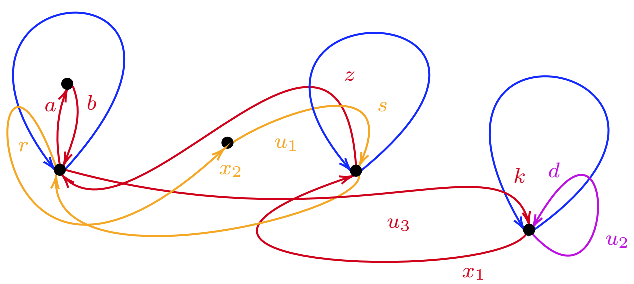
After that all blue edges in $E'_{l_1}(D)$ except for the last one are contracted, the contained intersection points vanish. The red edges with inner red edge ends at the disappearing intersection points are labelled by these points. The single element in $E_{l_1}(D)$ is labelled by l_1.



Then all edges in $E'_{l_2}(D)$ except for the last one are contracted



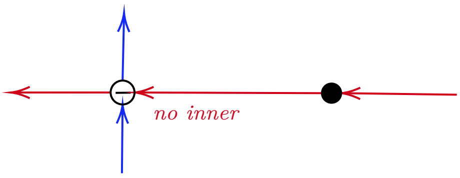
and in the last step all edges in $E'_{l_3}(D)$ except for the last one are contracted. This yields the graph $\Gamma(D)$:



The edges x_1, x_2 have no inner edge end. The paths associated to the three red circles are

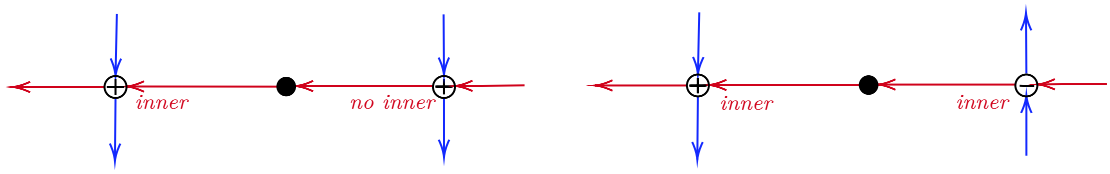
$$P_1(D) = s \circ r \circ x_2, \quad P_2(D) = d, \quad P_3(D) = b \circ a \circ z \circ x_1 \circ k.$$

Remark 10.18. *The distinction between contractions of first or last edges in Definition 10.16, 1. has technical reasons. The general aim is to remove the basepoint and one edge for $u \in \mathcal{U}$, hence contract this edge. But inner red edge ends should not be contracted, as contracting the corresponding edge would possibly lead to red edges with two inner edge ends, both labelled by an intersection point. More specifically, if the first intersection point in $\mathcal{I}_u$ is negative, the first element in $E'_u(D)$ has no inner edge end and is contracted.*



If the first intersection point in $\mathcal{I}_u$ is positive and additionally the last element in $\mathcal{I}_u$ positive, the last edge in $E'_u(D)$ has no inner red edge end and is contracted.

If the last intersection point in $\mathcal{I}_u$ is negative, both the first and the last edge in $E'_u(D)$ have an inner edge end. After contracting the last element in $E'_u(D)$, the first edge in $E'_u(D)$ has two inner edge ends. But as its source vertex is negative and its target positive, Definition 10.16 ensures that it is split into two edges, each having one inner edge end.



The 2-colored ribbon graphs $\Gamma(D)$ assigned to the Heegaard diagrams D from Example 10.6 are collected in Appendix A.

10.5 Relation between Kuperberg invariants and mapping class group actions

After constructing the ribbon graph associated to a Heegaard diagram in the last section, we now show how the tensor endomorphism from Definition 10.5 can be obtained via edge slides in the associated graph. This relates it to mapping class group actions considered in the thesis.

For this, let $D = (\Sigma, \mathcal{U}, \mathcal{L})$ again be an ordered oriented based Heegaard diagram and H an involutive Hopf monoid in a symmetric monoidal category $\mathcal{C}$. Recall that $N^* = \sum_{u \in \mathcal{U}} N^*_u$ is the number of inversion pairs in all upper circles, N the number of intersection points. Let $\Gamma(D)$ be the 2-colored ribbon graph associated to D from Definition 10.16.

Roughly speaking, we take the ribbon graph $\Gamma(D)$ and assign to each blue edge a copy of H and to each red edge a copy of the tensor unit. We then perform a Dehn twist along each lower circle, that is, slide all red edge ends to the left of the blue edge towards its starting end. After applying the counit to each copy of H associated with a lower circle, and multiplying the contributions of each red edge for each lower circle, this yields the tensor endomorphism.

Definition 10.19. *Let $D = (\Sigma, \mathcal{U}, \mathcal{L})$ be an ordered, oriented based Heegaard diagram of genus g and $\Gamma(D)$ the associated 2-colored ribbon graph with paths $P_1(D), \ldots, P_g(D)$. The **sliding endomorphism** $K'_H(\Gamma(D)) : H^{\otimes g} \to H^{\otimes g}$ associated to $\Gamma(D)$ is obtained as follows:*

1. *Start with $H^{\otimes g}$, with each copy of H assigned to a blue edge of $\Gamma(D)$.*
2. *Apply a unit for each red edge.*
3. *Successively slide all red edge ends to the left of a blue edge towards its starting end.*
4. *Apply a counit for each blue edge.*
5. *For $i = 1, \ldots, g$ multiply the contributions of the edges contained in $P_i(D)$ in reversed order.*

As edge slides only affect the copies of H associated to edges involved in the slide, sliding along different blue edges in $\Gamma(D)$ commutes. Thus, one can choose any order of blue edges for the slides in Definition 10.19, 3., in particular the order given by $\mathcal{L}$. We denote the composite of all slides in 3. of Definition 10.19 by

$$S_{s(\mathcal{L})^L} : H^{\otimes(g+N+N^*)} \to H^{\otimes(g+N+N^*)}$$

and the tensor product of all multiplications in 5. by

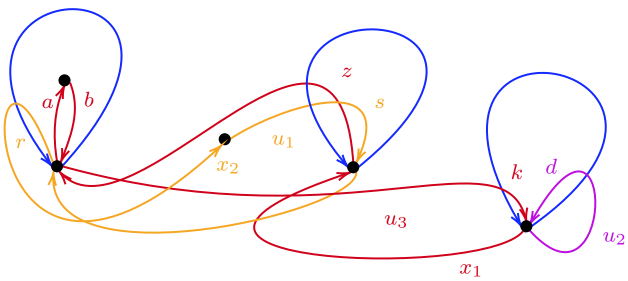
$$m_{P^{-1}} : H^{\otimes(N+N^*)} \to H^{\otimes g}.$$

Then we obtain the sliding endomorphism as

$$K'_H(\Gamma(D)) = (\epsilon^{(g)} \otimes m_{P^{-1}}) \circ S_{s(\mathcal{L})^L} \circ (1_{H^{\otimes g}} \otimes \eta^{(N+N^*)}) : H^{\otimes g} \to H^{\otimes g}. \tag{90}$$

We show that the sliding endomorphism coincides with the tensor endomorphism. This is illustrated in an example before we state and prove the theorem.

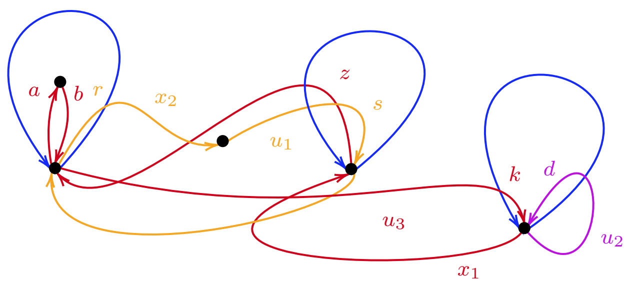
Example 10.20. *We consider the graph $\Gamma(D)$*



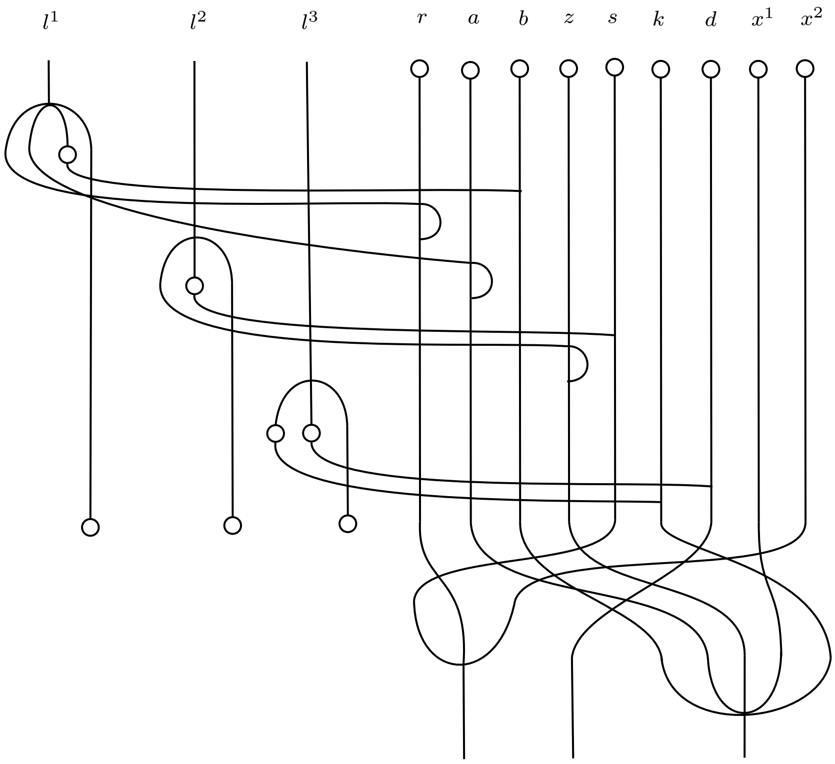
associated to the Heegaard diagram D from Examples 10.4, 10.17 with paths

$$P_1(D) = s \circ r \circ x_2, \quad P_2(D) = d, \quad P_3(D) = b \circ a \circ z \circ x_1 \circ k.$$

Successively sliding all red edge ends to the left of a blue edge towards its starting end means that first the outgoing edge end of r is slid towards the starting end of l_1 such that:



Then the edge ends of a, b to the left of l_1, then the edge ends of z, s to the left of l_2 and afterwards the edge ends of k, d to the left of l_3 are slid towards the starting end of the corresponding blue loop. The associated sliding endomorphism $K'_H(\Gamma(D)) = (\epsilon^{(3)} \otimes m_{P-1}) \circ S_{s(\mathcal{L})^L} \circ (1_{H^{\otimes 3}} \otimes \eta^{(9)})$ is given by



This coincides with the morphism $K_H(D)$ assigned to D in Example 10.6, 1.

Other examples of sliding endomorphisms based on the Heegaard diagrams from Example 10.6 are given in Appendix A.

We now prove that the sliding endomorphism always coincides with the tensor endomorphism.

Theorem 10.21. *Let $D = (\Sigma, \mathcal{U}, \mathcal{L})$ be an ordered oriented based Heegaard diagram, $\Gamma(D)$ the associated 2-colored ribbon graph and $P_1(D), \ldots, P_g(D)$ the paths corresponding to the upper circles in $\mathcal{U}$ from Section 10.4.*

Then the sliding endomorphism from Definition 10.19 coincides with the tensor endomorphism from (81): $K'_H(\Gamma(D)) = K_H(D)$.

Proof. By construction of $\Gamma(D)$ each $l \in \mathcal{L}$ is a loop at its basepoint. In the cyclic ordering of the basepoint there are exactly $|l|$ edge ends between the starting and the target end of l. The $|l|$ edge ends are all inner red edge ends, hence labelled by the associated intersection point with l. The cyclic ordering at the basepoint is inherited from $\Gamma'(D)$ and thus given by the opposite order in $\mathcal{I}_l$: The successor of $s(l)$ is the last element of $\mathcal{I}_l$, followed by the predecessors in $\mathcal{I}_l$ such that the first element in $\mathcal{I}_l$ comes directly before $t(l)$. The sliding procedure in Definition 10.19, 3. successively slides the $|l|$ red edge ends towards $s(l)$, starting with the first element in $\mathcal{I}_l$.

The associated sliding morphims are given in Definition 5.5 and Remark 5.6. All edge slides are left edge slides, but one has to distinguish slides of the target or the starting end of an edge c along a blue loop l. If c is incoming at the basepoint the morphism $l \otimes c \mapsto l_{(2)} \otimes S(l_{(1)})c$ is applied. If c is outgoing one applies $l \otimes c \mapsto l_{(2)} \otimes c\, l_{(1)}$, see Remark 5.6. Pre-composing this with the unit for the edge c yields $l \otimes 1 \mapsto l_{(2)} \otimes S^{(\kappa_c)}(l_{(1)})$, where κ_c from (80) is given by $\kappa_c = 1$ if the red edge c is incoming at the basepoint and $\kappa_c = 0$ else.

After sliding all edge ends as in Definition 10.19, 3. there are $|l| + 1$ copies of H assigned to $l \in \mathcal{L}$. Applying the counit to the blue loop associated to $l \in \mathcal{L}$ removes the $(|l| + 1)$-th copy. The order in which the other $|l|$ copies are assigned to intersection points is given by $\mathcal{I}_l$. Edge slides along distinct elements of $\mathcal{L}$ commute, as they do not share intersection points. Thus, up to braiding morphisms $(\epsilon^{(g)} \otimes 1_{H^{\otimes N}}) \circ S_{s(\mathcal{L})^L} \circ (1_{H^{\otimes g}} \otimes \eta^{(N)})$ coincides with $S_{\mathcal{L}} \circ \Delta_{\mathcal{L}}$. The units associated to red edges that are not labelled by intersection points are not affected by edge slides.

Applying $m_{P^{-1}}$ after the edge slides gives the same morphism as post-composing $S_{\mathcal{L}} \circ \Delta_{\mathcal{L}}$ with $m_{\mathcal{U}} \circ P_\sigma$, which describes the tensor endomorphism from (81): the i-th path $P_i(D)$ contains the edges labelled by intersection points in $\mathcal{I}_u$ in reversed order, where u is the i-th element of $\mathcal{U}$. Possibly, edges that are not labelled by intersection points are contained in $P_i(D)$. As these edges are not affected by edge slides, their contribution in the multiplication along $P_i(D)$ is the unit. Hence, they can be neglected.

As applying the morphism $m_{P^{-1}}$ multiplies for each $i = 1, \ldots, g$ the edges in $P_i(D)$ in reversed order, the i-th component coincides with multiplication along the i-th element of $\mathcal{U}$. Hence, the assertion follows. ☐

The sliding procedure in step 3. of Definition 10.19 exactly describes the Dehn twist from Remark 9.11. This is the inverse Dehn twist to the twist in Definition 9.9, a) with an imaginary cilium at blue edges β such that $s(\beta) < t(\beta)$. Thus, for an ordered oriented based Heegaard diagram D with associated 2-colored ribbon graph $\Gamma(D)$, where a copy of H is assigned to each blue curve in $\Gamma(D)$ and the tensor unit to each red edge, Theorem 10.21 can be reformulated in terms of Dehn twists:

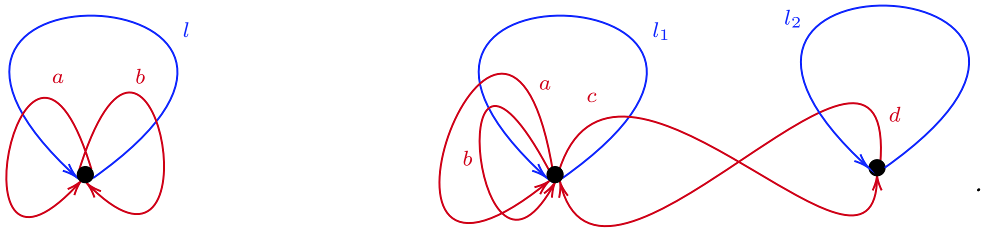
Corollary 10.22. *The tensor endomorphism is the composition of Dehn twists along the blue curves in $\Gamma(D)$, applying the counit to each blue curve and multiplying together the contributions of red edges for each upper circle.*

Appendix

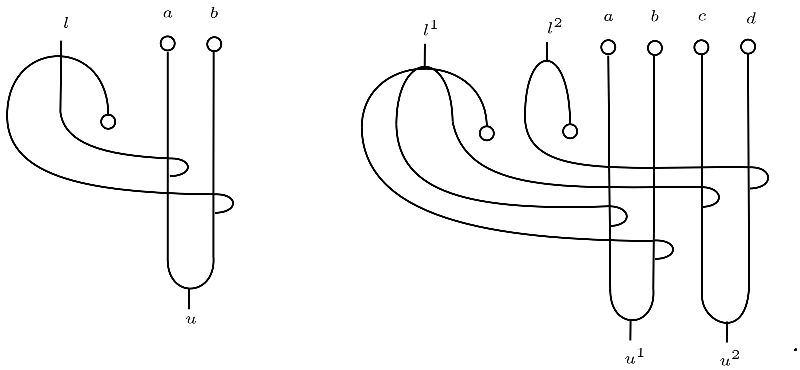
A Examples of assigned ribbon graphs and sliding endomorphisms

In this appendix, we collect the 2-colored ribbon graphs $\Gamma(D)$ assigned to the Heegaard diagrams in Example 10.6, 2.-6. and the corresponding sliding endomorphisms. For the definition of $\Gamma(D)$ we refer to Section 10.4. The sliding endomorphism is defined in Section 10.5.

Example A.1. *The* 2*-colored ribbon graphs* $\Gamma(D)$ *assigned to the two Heegaard diagrams in Example 10.6, 2. are given by*

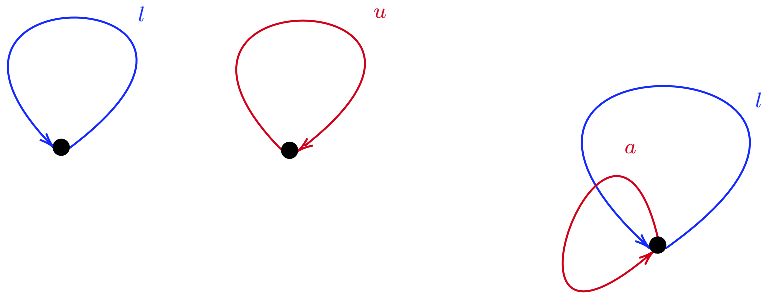


The graph on the left has a single path $P_1(D) = a \circ b$*, the graph on the right the paths* $P_1(D) = a \circ b$ *and* $P_2(D) = c \circ d$*. This yields the following sliding endomorphisms:*

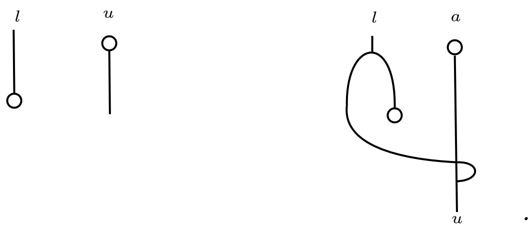


As shown in Theorem 10.21, the sliding endomorphisms coincide with the tensor endomorphisms from Example 10.6, 2.

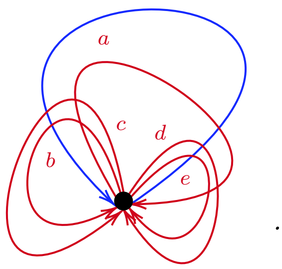
The graphs assigned to the Heegaard diagrams from Example 10.6, 3. and 4. are



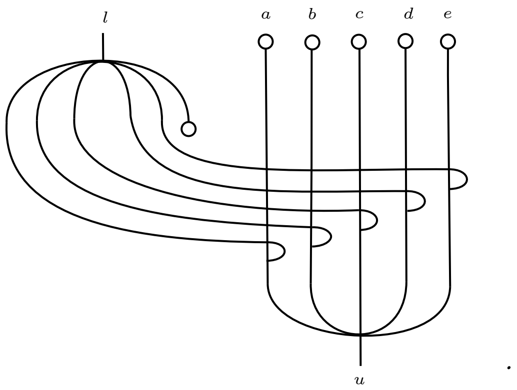
with paths $P_1(D) = u$ for the graph on the left and $P_1(D) = a$ for the graph on the right. The associated sliding endomorphisms are given by



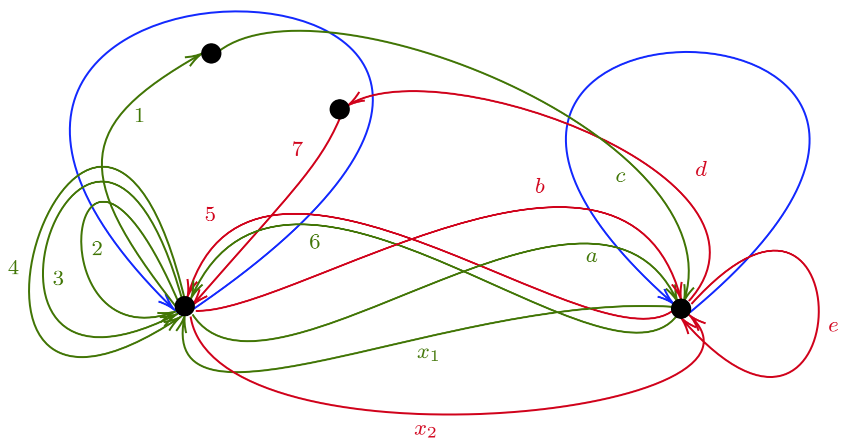
The 2-colored ribbon graph $\Gamma(D)$ associated to Example 10.6, 5. with $p = 5$ is depicted as



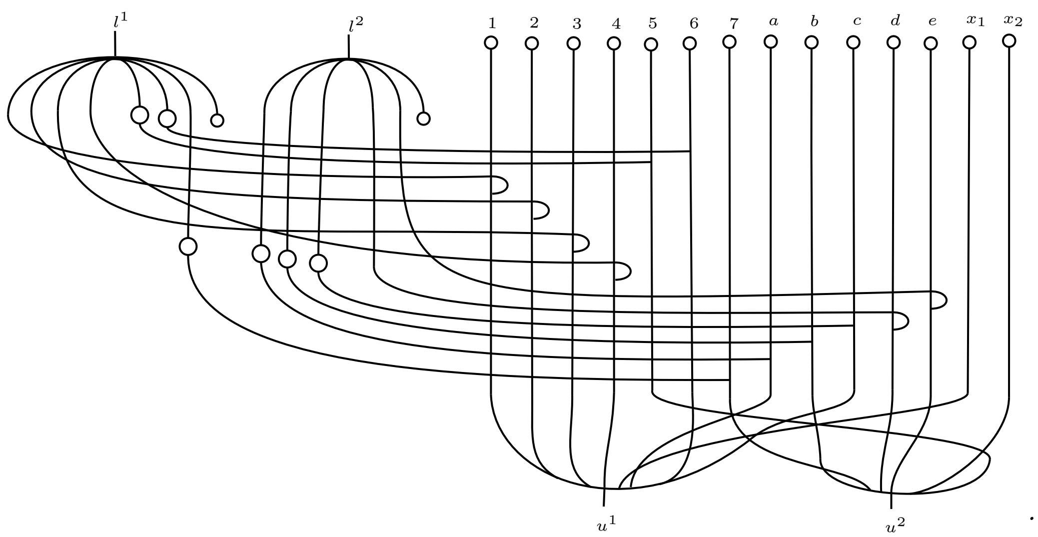
The single path is given by $P_1(D) = a \circ b \circ c \circ d \circ e$. Together they yield the following sliding endomorphism:



The ribbon graph $\Gamma(D)$ *assigned to Example 10.6, 6. is*



with paths $P_1(D) = 1 \circ 2 \circ 3 \circ 4 \circ x_1 \circ a \circ 6 \circ c$ *and* $P_2(D) = b \circ 7 \circ d \circ e \circ x_2 \circ 5$*. Together they yield the sliding endomorphism:*



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