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UV Completeness: From Quantum Field Theory to Quantum Gravity

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UV Completeness: From Quantum Field Theory to Quantum Gravity

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UV Completeness: From Quantum Field Theory to Quantum Gravity

Abstract

One of the main goals of theoretical physics is to describe physics at all energy scales. The usual renormalization scheme of quantum field theory provides a way to investigate the details of physical systems at different energy scales, potentially up to some energy cut-off. When a theory is predictive at arbitrarily high energies and hence the cut-off can be removed, we say the theory is UV complete. The importance of UV completeness lies in its ability to explain physical phenomena at arbitrarily small distances or high energies. However, when standard techniques from quantum field theory fail to provide such a description, new degrees of freedom are expected to become important.

In this thesis, we investigate two types of UV completions; one associated with conformal field theories and the other with quantum gravity. For the former, we study the fixed points of the renormalization group where one needs to integrate-in light solitonic particles/strings in the context of supersymmetric conformal field theories. For the latter, we review the challenges of UV completing quantum gravity and the severe constraints that such a completion is expected to provide for the low-energy physics. The study of such constraints constitutes the idea behind the Swampland program, which aims to identify theories that are inconsistent when coupled to gravity.

A guiding tool for further understanding and constructing both classes of theories is string theory, which provides a framework to analyze strongly coupled systems. In particular, it can be used to provide a classification framework for five dimensional superconformal field theories. However, string theory, being a theory of quantum gravity, can also be used to analyze the possible landscape of supergravity theories. We reconstruct some of the results suggested by string theory from bottom up and in particular focus on questions related to the finiteness of quantum gravity. Our results are suggestive of a possible string universality for supersymmetric gravitational theories.

Contents

This thesis is dedicated to all the women that have fought for my rights and the ability for me to be writing this thesis and for those that are still fighting for their fundamental rights to bodily autonomy, safety, equality, freedom of speech, movement and expression. It is especially dedicated to the strongest woman I know and the one that gave me my life; my mother.

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"The existence of a single structure that unifies such a broad range of physical and mathematical ideas, and many others as well, is unexpected and remarkable. Earlier I declined to define beauty, but one can recognize it when one sees it, and here it is."

Joseph Polchinski [159]

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Introduction

What makes our world tick? That is the question scientists, philosophers and theologists have tried to address for thousands of years. Maybe not all of their approaches are the same but the question is universal. From the perspective of a scientist one would like to analyze their world around them and describe it through some fundamental principles. The laws that underline our cosmos have been studied in different frameworks split into different fields. To make this possible one of the gifts the universe has given us is *scale separation*, which means that short distance effects do not affect long distance physics, and this is why we are able to predict the trajectory of a rolling ball without referring to its molecules and atoms. In high energy physics, scale separation comes in the form of short scale (high energy/UV) processes mediated by heavy or fast particles that can actually decouple from long scale (low energy/IR) physics.

This is the idea behind *Effective Field Theories* (EFT) where physical processes are modelled up to some energy scale, beyond which we need to include processes we had decoupled. A well-known and successful example is the *Standard Model* (SM) of particle physics, which is an EFT that models the interactions of sub-atomic particles up to some cut off. In particular, it gives us a good grasp of the physics between the large scale of the size of the universe ($R^{-1}_{\text{universe}} \sim 10^{-42}$ GeV [1]) and the short scales (~ 1000 GeV), as probed in the LHC[2]. However, the standard model is still not very well understood beyond certain energy limits and in regions the physics becomes strongly coupled.

One particular class of strongly coupled field theories are conformal fields theories (CFT). Such theories have scale invariance and appear in the deep IR/UV limit of quantum field theories[3] making their appearance in condensed matter physics, string theory and holography. Understanding CFTs together with supersymmetry (SCFTs) simplifies the problem, and in particular they are believed to be fully classified in the maximal dimension six[104]. In chapter 4 we will study conformal field theories in five dimensions with minimal supersymmetry. In particular, we will use the conjecture that all lower dimensional conformal field theories come from a higher dimensional theory on various compactifications[112]. Therefore, the five dimensional conformal field theories are expected to be descendants of the six dimensional classification when compactified on some circle with the possible choice of a discrete twist, called *5d Kaluza-Klein* (KK) theories. These twists correspond to automorphisms of the theory for which one could turn on discrete holonomies giving rise to a

[1] Note that GeV is a unit of energy and corresponds to $\hbar c/\text{GeV} \sim 10^{-15}$m in units of distance.

[2] We note that there is physics beyond the Standard Model including Dark matter, Dark Energy, Neutrinos etc.

[3] Including free, empty and strongly coupled CFTs with the latter being most interesting.

bigger class 5d KK theories. The classification of theories without such twists can be found in [29,28]. Such a full list of theories is crucial because it is believed that any 5d SCFT can be reached via a mass deformation of the KK theory [24].

Gravity can also be coupled to the standard model as long as it is weak, and we do not ask questions about high enough energies. But, the notion of shorter scales and the idea of UV/IR decoupling can break down because of the creation of black holes and their thermodynamic properties as will be reviewed in chapter 1. Hence, above some energy scale, a new description is expected to emerge.

Among various approaches to quantum gravity, string theory has been a strong candidate as it unifies gravity with the standard model and predicts various properties of black holes. In fact, gravity is incorporated very naturally within the theory. Despite its various successes, the vast number of possible solutions (vacua) in string theory, called the *Landscape* [4], makes it hard to identify which solution corresponds to our universe. Additionally, the difficulty of computational control also makes it hard to make concrete predictions.

One could then rephrase the question of finding our universe into a more naturalness type question: Why do we live in a specific vacuum if there is a huge number of other possibilities? Why is the standard model gauge group so small ? Why is the cosmological constant so small? Why is there a hierarchy in the masses of subatomic particles?

One could argue that our existence and being here to ask these questions is really the answer. But what if there was a more natural explanation? It could be that gravity, the force that keeps our universe together, is responsible for that. In other worlds, if we were able to have a "Theory of Everything" then we would automatically have an answer to these questions. This is motivated by the old wisdom that gravity has no global symmetries. This really means that there are no free parameters to tune and that gravity does not provide us with superselection sectors. So it is natural to think

[4] of the order $\sim 10^{272000}$ [182]

that a very specific and finely tuned universe is not preferred.

This is a powerful result towards nailing down a fundamental principle of quantum gravity. However, in the search of principles of quantum gravity a natural question arises: What is gravity? This is a questions that we are still not as close to defining but what we can do is understand deeply the basic elements that a gravitational theory should obey. In other words we are looking into describing what gravity really is based on some simple universal properties and rules. But how do we find these rules when we do not have a clear understanding of gravity? We believe that a consistent quantum gravity should also include black holes, a limit of Einstein gravity, the holographic principle, 4 large dimensions, UV/IR mixing, be unitary and more. However, to get inspiration for the fundamental principles we really need a quantum theory of gravity. We might not necessarily have "the" quantum theory of gravity but we have one, *string theory*.

Another way to rephrase this approach is to focus firstly on understanding what is not possible! This is called the Swampland program [186] where one attempts to understand the theories that do not provide consistent quantum gravitational theories.

The important contribution of string theory to the *Swampland program* is that it provides a framework to study such questions and extract some general principles. Then these principle need to be studied outside of string theory and if believed to still hold then provide concrete evidence using UV completion independent methods.

In section 1.1 we will review some of these proposals and the evidence to support them. One fundamental principle is the idea of *duality*. In theories with gravity we expect that at infinite distances in some moduli space we would find new physics described by some dual theory. In chapter 2 we trace such infinite distance limits in theories with 16 supercharges and understand some properties of their dual theories. In particular, in section 2.2 we follow the supersymmetric states and uncover a form of T-duality inspired from string theory and in section 2.1 we are able to follow the path of the non-bps states in 9 dimensions and understand their dual description when they decompactify. Interestingly, the duality scheme we see is very similar to that of string theory. This makes one wonder whether string theory is unique and if all low-energy theories have a UV completion within string theory. This is called *string universality*.

Furthermore, there is another important reason to understand string theory's role in the landscape of low-energy physics. Earlier we described the role string theory plays to inspire and propose general principles of quantum gravity. However, if string theory is only a small part of the quantum gravity landscape then we are being mislead into principles that are only true within the string theory realm. This is called the *String Lamppost Principle* (SLP). Therefore, attempting to understand the universality of string theory is important.

In fact, in theories with 16 supercharges the duality principle can be enough to constraint the massless modes of the low-energy physics. In particular, in subsection 2.2.1, we show that in dimensions $d > 3$ there can only be finitely many massless modes (up to some cut-off) with the rank of the gauge group of these theories bounded by $r_G \leq 26 - d$. For example, $\mathcal{N} = 4$ Super-Yang Mills with $r_G > 22$ are inconsistent, and said to belong to the Swampland.

Interestingly, the idea of finiteness of the quantum gravity landscape [186] can also be tested in six dimensional supergravity theories with 8 supercharges which have chiral anomalies severely constraining the spectrum. In section 3.1 we show that the rank of the possible gauge groups is finite and in section 3.4 we provide strong bounds on the type of matter that can appear. This bound also extends to five dimensions.

However, the finiteness question is significantly harder to address in five dimensions. This is because the theory has no chiral anomalies which could help constraint the spectrum. In fact, even from the string theory point of view it is not clear since a classification of the string landscape is not complete but it is expected that there are only finitely many inequivalent Calabi-Yau threefolds. In section 3.2 we investigate the Calabi-Yau landscape of five dimensional theories with a focus to extract some general principles. In particular, we propose that these hold more generally and we provide evidence to support them from the bottom up.

In chapter 1 we will review details of supergravity theories and superconformal theories in various dimensions with varying amount of supersymmetry which will motivate the more detailed analysis of the next chapters.

This thesis is based on five papers[30,116,120,179,17] completed in collaboration with Cumrun Vafa, Sheldon Katz, Hee-Cheol Kim, Patrick Jefferson, Laksya Bhardwaj, Alek Bedroya and Sanjay Raman.

"Nature isn't classical, dammit, and if you want to make a simulation of nature, you'd better make it quantum mechanical, and by golly it's a wonderful problem, because it doesn't look so easy."

Richard Feynman [184]

1 Quantum Field Theory and Quantum Gravity in a Nutshell

This chapter's main focus is to review certain aspects of supersymmetric field theories and gravity theories that will provide an introduction to important concepts utilized in the subsequent chapters. In particular, section 1.1 will provide a short introduction to key concepts of string theory and the general framework used to construct examples in both gravitational and non-gravitational theories.

In section 1.2 the general ideas of the Swampland program for gravity theories will be reviewed, which provides a way to understand fundamental principles of quantum gravity without referring to a specific UV completion. Additionally, general bottom up principles that quantum field theories need to satisfy to be well-defined in the UV will be discussed.

The rest of the chapter will provide a more detailed introduction to supergravity theories and superconformal theories in various dimensions with various amount of supercharges and their constructions within string theory.

1.1 Top Down

String Theory provides a framework where one can study gravity and field theory as part of a single theory. In particular, string theory is a theory of quantum gravity where the elementary degrees of freedom are strings. For example, the graviton is a closed string excitation and the spacetime is believed to be an emergent phenomenon. The string like nature of the particles solve some of the divergence issues in field theory but most remarkably describes a theory of quantum gravity with finite S-matrix.

The insight in a lot of the UV properties of string theory come from dualities. Meaning that a given theory changes description based on the values of its parameters. This duality web seems to be a fundamental principle of quantum gravity where UV and IR degrees of freedom can be interchangeable. This is a remarkable feature which provides us all the information we need to explore regions that are inaccessible from the traditional descriptions. However, certain non-perturbative corners of string theory are still not as clearly understood.

String theory is best understood together with supersymmetry as it provides a stable vacuum and special computational control. However, non-supersymmetric string also exist, which have no

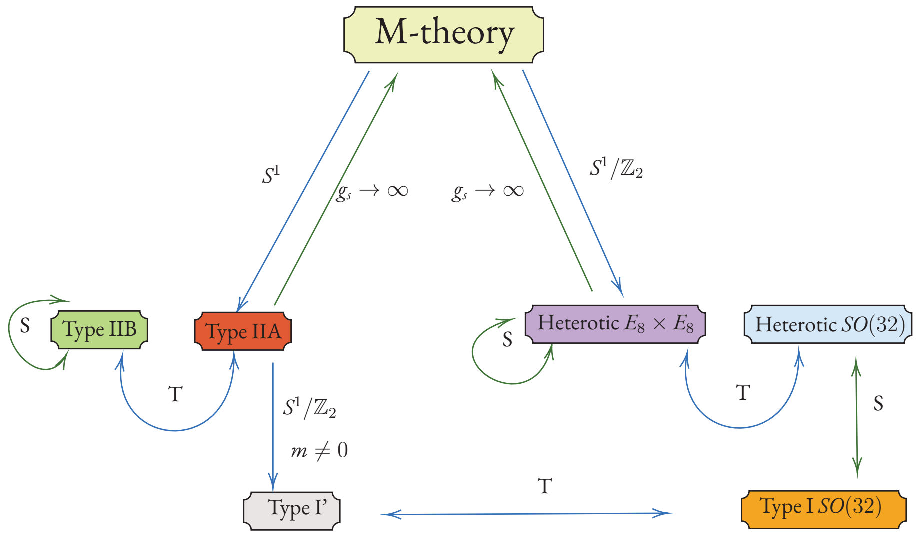

Figure 1.1: Duality web in string theory: T-duality is indicated by T, indicating the equivalence of two theories when put on a small and large circle respectively. M-theory, Type IIA and IIB have 32 supercharges. The heterotic theories, Type I and Type I' have 16 supercharges. These also have gauge groups for which upon compactification on a circle permit the introduction of Wilson lines. In the absence of Wilson lines the heterotic $E_8 \times E_8$ is self T-dual. Certain theories can also enjoy a strong/weak duality (S duality). Type IIA supergravity also permits the introduction of a mass deformation m known as Roman's mass[163] which when non-trivial provides extra possible dualities.

tachyonic degrees of freedom theories but have rolling potentials[57,9].

The maximal string theories live in 10 dimensions with 32 supercharges. In fact there is a choice of chirality which subsequently defines two theories: non-chiral Type IIA and chiral Type IIB. A parent 11 dimensional theory also exists which has no strings but rather membranes called M-theory. This theory can be reached as the strong coupling limit of Type IIA. In 10 dimensions also string theories with 16 supercharges exist which are chiral with anomalies dictating the existence of a gauge group $E_8 \times E_8$ or $SO(32)$. The web of dualities relates them to also orbifolds or orientifolds of M-theory/IIB as seen in Figure 1.1.

String theory can be a powerful tool to understand strongly coupled physics, black hole thermodynamics, holography and more. In the next section compactifications of string theory to lower dimensions will be discussed with an emphasis on Calabi-Yau manifolds. Such manifolds break part of the higher dimensional supersymmetry and hence provide lower dimensional supergravity theories with less supersymmetry. There are other compactifications one can consider beyond Calabi-Yau manifolds like non-geometric worldsheet constructions [150] but they are beyond the scope of this thesis. However, it has been conjectured that all possible geometric or non-geometric compactification are connected through transitions leading to the uniqueness of quantum gravity. An example of such transition is conjectured to exist between Calabi-Yau manifolds and asymmetric orbifolds as studied in [113].

The set of possible low-energy gravity theories in a given dimension and with a given amount of supersymmetry is called the String Landscape. An important open question is to what extend does that cover the full quantum gravity landscape? In an attempt to answer such a question one needs to define what a quantum gravity theory or at least what are the basic properties of a gravity theory. The understanding of the possible landscape of quantum gravity theories starting from low energy physics (bottom up) and how it fits in the string landscape will be the main focus of the next two chapters.

1.2 Bottom Up

1.2.1 Quantum Gravity

As was discussed in the previous section string theory provides a complete framework to study field theories and gravity theories in lower dimensions. However, string theory describes a very particular behavior of gravity in high energies. In particular, the Einstein effective action is believed to break down beyond some energy scale at about M_{pl}. In string that scale is reached before M_{pl} at the string scale M_s where the string like nature of particles and spacetime start to become important. This provides a stringy UV completion. However, one could argue that alternative UV behaviors could also be possible, making string theory not necessarily unique. Although it provides an excellent candidate for a quantum gravity, it would be very beneficial to understand if in fact this behavior is universal of any quantum gravity theory that has the potential to describe our world.

In order to approach such questions one firstly needs to understand the fundamental axioms of gravity. The most well established description of semi-classical gravity is through Einstein's general relativity which provides a general description of the gravitational interactions in terms of the curvature of spacetime and the effect of matter on it. One of the most interesting consequences of general relativity is the prediction of black holes. In fact black holes where recently observed[46] and hence making them fundamental objects in any quantum gravity discussion.

However, this is a course grained description mainly in regions that the physics is well behaved and quantum effects are not as important. Therefore, a Hilbert-Einstein field action can be coupled to the Standard Model as long as one does not ask questions for high enough energies as the lack of a UV completion and the non-renormalizable nature of gravity will provide obstacles.

Perhaps a more intuitive way to understand why quantum gravity should be expected to be hard or different from common field theory is because of black holes. In fact Bekenstein and Hawking[19,18,99] showed that a black hole has a non-trivial temperature and entropy given by:

$$S_{BH} = \frac{A}{4G} \tag{1.1}$$

where A is the area of the horizon of the black hole. This means that black holes have a rich structure with many microstates and degrees of freedom something very surprising from the classical equation of Einstein's gravity which are suggestive of a single state characterised by classical quantities like mass, charge and angular momentum. This fact is particularly interesting because it is indicative of some UV/IR connection and the standard way of separating scales does not work for gravity. This could also be understood in a simple thought experiment where one tries to probe higher and higher energies. The large amount of energy localized in a small region would lead to the creation of a black hole. But the more the energy increases the bigger and more classical the black hole becomes causing trouble to our current understanding of smaller scales [167].

UV/IR Connection

But this is not the only reason to expect gravity to be different but rather the idea of locality falls apart too. For a quantum field theory all observables correspond to local operators. However, a quantum gravity necessarily includes a sum over topologies and geometries making the idea of spacetime itself non-fundamental. In fact the spacetime itself fluctuates and one can think of our particular spacetime as some classical averaged limit of those fluctuations. Locality is expected to be restored at certain "classical" regions. From the string theory point of view dualities also make a clear understanding of locality ambiguous.

Failure of Locality

The black holes also provide a new surprise, the holographic principle implied from the structure of their entropy Equation 1.1. In quantum field theories one would expect that the entropy to satisfy a volume-law but Equation 1.1 seems to imply that the degrees of freedom are encoded on the boundary and hence satisfying an area-law. In fact t'Hooft [176] and Susskind [175] argued that gravity has a holographic nature. A beautiful example of such a principle is the AdS/CFT correspondence which describes a bulk gravitational theory with some conformal field theory living on the boundary. As expected local observables emerge in this description either form the local classical gravity or the locality of the boundary.

Holographic Principle

Therefore, because of the above points one should not have expected to be able to describe gravity using a local field theory description and hence the obstacles of the field theory have a very physical reason to exist.

On the one hand, the above issues may significantly complicate any such description with known tools, but on the other hand they can also be guiding principles to unravel more fundamental concepts of quantum gravity. For example, Equation 1.1 means that one can look for UV signatures to the IR physics.

Furthermore, one can also understand the duality web of the previous section as a consequence of these observations and hence a potential fundamental aspect of gravity. Both the holographic principles and the entropy formula of Equation 1.1 have a description within string theory. In particular, a demonstration of holography can be understood through the AdS/CFT correspondence[135], and the black hole entropy was given a microscopic explanation[174].

However, the large landscape of string theories and the lack of computational control makes it particularly hard to find our universe and make the theory predictive. Therefore, an understanding of more basic rules of quantum gravity are important.

The Swampland program provides a way to study gravity through exclusion. In other words, it can be understood as a way to distinguish low energy theories that can be consistently coupled to gravity versus those that cannot. The goal is to unravel the basic principles of quantum gravity, such that if an apparently consistent low-energy theory does not obey them it is deemed inconsistent when coupled to gravity.

Many of these swampland constraints have origins from string theory and are consistent with the general intuition of gravity. However, an important step is to refine them and provide evidence through more fundamental principles of gravity like black hole physics, scattering amplitudes, causality, unitarity and holography. Some important examples of such constraints are summarized below in their most intuitive form. More details and refinements can be found in [3] and references thereof.

- **No global Symmetries[97,96,13]:** A theory of quantum gravity should have no global symmetries and consequently no free parameters.
- **Cobordism Conjecture[138]:** Quantum gravity is unique and defines an equivalence class of configurations connected through domain walls.
- **Completeness of Spectrum:**[159,97,13] The charge lattice of a quantum gravity theory is complete in the sense that all possible representation appear in the spectrum.
- **Distance Conjecture**[155]: Infinite distance excursions in the moduli space of a given theory leads to a tower of massless states with mass $m \sim M_{pl}\, e^{-\alpha\hat{\varphi}}$.
- **Weak Gravity Conjecture[10]:** For a U(1) gauge field coupled to gravity there must exist an object of charge q and mass m satisfying:

$$\frac{|q|}{m} \geq \frac{|Q|}{M}|_{ext} \tag{1.2}$$

where Q and M are the charge and mass of an extremal black hole.

In fact the first three conjectures are very closely related to each other and the last two can be related (for a review [3]).

A large part of the next chapters will be to understand implications of these conjectures for supersymmetric theories. In particular, the potential finiteness of quantum gravity [186] (see also [89]) will be addressed together with the String Lamppost principle which is concerned with the universality of the string landscape.

1.2.2 Quantum field theory

String theory provides a strong framework to study UV physics of local quantum field theories. However, from the low energy perspective it is not a simple problem to find necessary and sufficient conditions for the existence of a UV completion for a low energy theory. In the previous subsection it was noted that in a gravitational theory that could be possible because of the connected nature of gravity.

However, even in a quantum field theory one can study the necessary conditions for the theory to have a UV completion beyond the requirement to be anomaly free. In fact attempts have been made to study consistency conditions for the existence of UV superconformal field theory (SCFT) points for the low-energy theories. The first attempt for the 5d minimally supersymmetric theories was studied in [166] and a more refined set of conditions was worked out in [112] and references thereof. Additionally, for 6d SCFTs some bottom up details are reviewed in [106] and the E-string global symmetry was attempted to be argued from bottom up in [171]. Moreover, more general properties of the instanton moduli spaces were worked out in [117] where an ADHM-like construction was proposed.

1.3 Review of Supergravity theories with 16 supercharges

Let us start with reviewing the bosonic content of theories with 16 supercharges in d dimensions[1].

- Supergravity multiplet: $g_{\mu\nu}, B_{\mu\nu}, A^i_\mu, \varphi$ where φ is the dilaton and $i \in \{1, \ldots, 10-d\}$.
- Vector multiplet: A^μ and Φ^j where $j \in \{1, \ldots, 10-d\}$.

[1] The only exceptions are the chiral 6d theory which has 21 tensor multiplets fixed by anomalies and the 10d theory which is chiral and completely fixed.

A non-chiral theory with 16-supercharges in d dimensions has matter scalar fields which belong to the coset space[109]

$$\mathcal{M}_V := \frac{SO(r_G, 10-d)}{SO(r_G) \times SO(10-d)} , \tag{1.3}$$

and in addition an $\mathbb{R}^+$ for $d > 4$ and a complex axio-dilaton in $d = 4$ coming from the gravity multiplet. Here r_G denotes the rank of the gauge group G and for generic point of the scalar field the G is abelianized to $U(1)^{r_G}$. Moreover, the gravity multiplets of a non-chiral theory have $(10-d)$ $U(1)$ gauge fields as well as an anti-symmetric two-form field $B_{\mu\nu}$[2]. In theories with 16 supercharges the moduli space does not receive corrections and hence one can follow the trajectory of moduli across different limits.

We can arrange these scalars in a $10 - d + r_G$ dimensional symmetric matrix M, the generalized metric, such that satisfy if satisfies the familiar:

$$M \begin{pmatrix} \mathbb{I}_{10-d} & 0 \\ 0 & -\mathbb{I}_r \end{pmatrix} M = \begin{pmatrix} \mathbb{I}_{10-d} & 0 \\ 0 & -\mathbb{I}_r \end{pmatrix} . \tag{1.4}$$

The kinetic term for the gauge field is

$$-\frac{1}{4} e^{-2\varphi} F^a_{\mu\nu} \left[\begin{pmatrix} \mathbb{I}_{10-d} & 0 \\ 0 & -\mathbb{I}_r \end{pmatrix} M \begin{pmatrix} \mathbb{I}_{10-d} & 0 \\ 0 & -\mathbb{I}_r \end{pmatrix} \right]_{ab} F^b_{\mu\nu}, \tag{1.5}$$

where φ is the dilaton in the gravity multiplet.

[2] Sometimes one considers the dual of this field; for example in 4d the B-field is usually dualized to a scalar.

1.3.1 Type I' Supegravity

The Type I' theory is typically understood as a string theory but we would like to define it as a supergravity. To that end, we will briefly review the massive type IIA supergravity. In particular, the type IIA supergravity has a mass deformation which adds a parameter known as the Romans mass [163]. The case where the Romans mass is set to zero reduces to the ordinary IIA supergravity. There are two important features to keep in mind about the massive type IIA supergravity.

- the Romans mass is not a dynamical field and must be thought as a parameter of the theory[3].
- Due to the coupling of the Romans mass to the dilaton, any non-zero value of the Romans mass creates a linear profile for $e^{-\varphi}$ in space.

We would like to study the supersymmetric backgrounds of massive type IIA theory on an interval. For bottom-up reasons that we will explain in section 2.3the Romans mass needs to change along the interval for that we must allow for jumps in the Romans mass which are mediated by supersymmetric 8-branes. Such branes might or might not exist in the UV theory, but in this section, our analysis is limited to what is or is not allowed in field theory. So we postpone those concerns to future sections.

The supersymmetric 8-branes that mediated the jump must satisfy a BPS condition involving a 9-form charge. In this case, the 9-form gauge field must be dual to the Romans mass. Therefore, the charges of the 8-branes and, consequently, the jumps in the Romans mass must be quantized[4]. We take the tension of the 8-brane with a unit 9-form charge to be μ_8.

[3] Note that the application of no global symmetry conjecture [107] to (-1)-form symmetry implies that no theory of quantum gravity must have a free parameter. For the massive type IIA, this implies that massive type IIA in 10 non-compact dimensions with an arbitrary value of the Romans mass belongs to the Swampland (see [5] for a string theory argument). Therefore, we expect the profile of the Romans mass be set by boundary conditions and defects. As we will shortly see, this is indeed the case in the backgrounds of massive IIA on an interval.

[4] In string theory these are the D8 branes. But here we do not rely on string theory and therefore, so far, we do not have any bottom-up information on the worldvolume theory of one or a stack of the 8-branes.

Type I' supergravity : is a supersymmetric background of massive type IIA on an interval with a configuration of BPS domain walls perpendicular to the interval with the domain walls creating a profile for the dilaton. This is a field theory definition and the existence of such a background depends on the UV properties such as the spectrum of non-perturbative BPS domain walls and end-of-the-universe walls.

Now we are ready to use the equations of the supergravity, to find the profile of the dilaton based on the positions of the supersymmetric 8-branes. The calculations parallel those carried in the context of string theory, however, we ignore any UV input and focus on the supergravity.

Suppose we have 9 large dimensions and a single compact dimension parametrized by x^9 such that $0 \leq x^9 \leq 2\pi$ and $x^9 \sim -x^9$. There are end of the universe walls sitting at the endpoints $x^9 = 0$ and $x^9 = \pi$, which we do not know their microscopic description.

The type I' supergravity action is

$$S_{I'} = \int d^{10}x\sqrt{-g}e^{-2\varphi_{10}}\left(\frac{1}{2}R + 2\partial_M\varphi_{10}\partial^M\varphi_{10}\right) - \frac{1}{2}\int F^*F + \cdots \tag{1.6}$$

Here we have included only the bosonic terms coming from the metric and the top-form gauge field strength F which is dual to the Romans mass. The BPS 8-brane solutions are electrically coupled to the 9-form gauge field A with field strength F. Suppose the 8-branes are positioned at x_i^9. Let the indices $M, N \in \{0, \cdots, 9\}$, and let $\mathfrak{u}, \nu \in \{0, \cdots, 8\}$. Suppose further that there are n_1 branes at x_1^9, n_2 branes at x_2^9, and so on.

The equation of motion for the 9-form field A yields

$$F = \nu_0 dx^0 \wedge \cdots \wedge dx^9, \tag{1.7}$$

where ν_0 is a piecewise constant function which jumps at the positions of the branes.

Since the supergravity equations are identical to the ones appearing in type I' string theory, we can use the solution in [161]. However, due to lack of any bottom-up knowledge about the number of 8-branes at this point, we have to generalize that solution to an arbitrary number of 8-branes. The profile of the type I' dilaton and the type I' metric may be parametrized by a constant C and a piecewise constant function $B(x^9)$ whose value changes at the position of each brane and which is entirely determined by the value $B(0)$. In conformal gauge, $g_{MN} = \Omega^2(x^9)\eta_{MN}$, and we then find

$$e^{\varphi_{10}(x^9)} = z(x^9)^{-5/6}, \quad \Omega(x^9) = Cz(x^9)^{-1/6} \tag{1.8}$$

$$z(x^9) = 3C(B\mu_8 - \nu_0 x^9)/\sqrt{2}. \tag{1.9}$$

Here $\nu_0 = \nu_0(x^9)$ is a piecewise constant function with jump discontinuity $\Delta\nu_0 = n_i\mu_8$ at each stack of n_i 8-branes. See Fig. 1.2 for an illustration of the type I' dilaton profile on the compact interval.

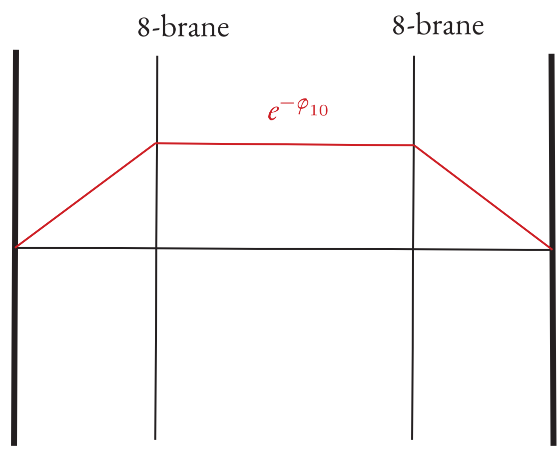

Figure 1.2: A schematic illustration of type I' supergravity with two BPS end of the universe walls and two BPS 8-branes located at the specified positions. The red line describes the profile of the type I' dilaton (in the string frame instead of the Einstein frame) on the compact interval. Note also that the distance between the two 8-branes in the middle can be arbitrary.

1.4 String Theory Examples with 16 Supercharges

1.4.1 Chiral Theories

These theories are chiral and hence subject to strong anomaly cancellation conditions.

- $\mathcal{N} = (1, 0)$ **in 10d**

In particular, the gauge and gravitational anomaly cancellation in 10d implemented by the Green-Schwarz mechanism[79] requires the gauge groups to be limited to: $E_8 \times E_8$, $SO(32)$, $E_8 \times U(1)^{248}$, $U(1)^{496}$. However, only the first two gauge groups lead to consistent supergravities and the latter two belong to the swampland as argued in[2,119].

We can construct the two consistent theories in string theory in various ways: $E_8 \times E_8$ and $SO(32)$ heterotic strings, Type I, or via M-theory on an interval as in Figure 1.3 which are related through various dualities as seen in Figure 1.4 and Figure 1.1.

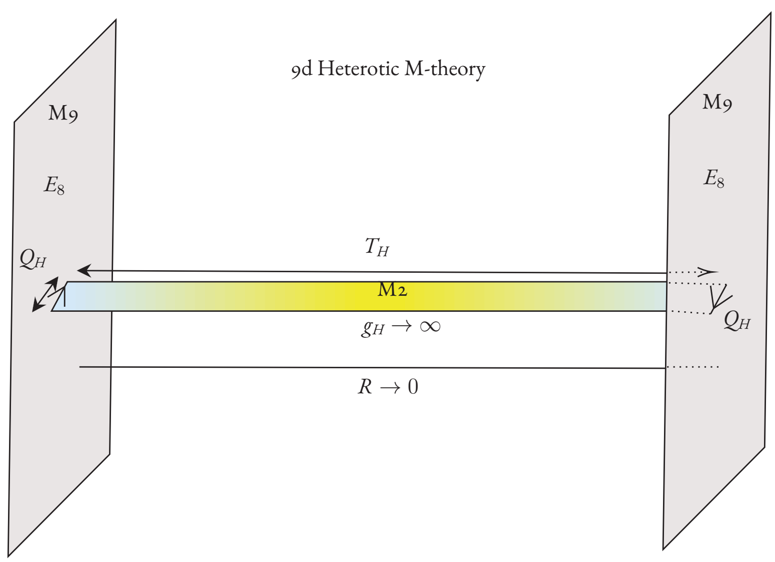

Figure 1.3: M-theory on the Hořava-Witten wall.[12]

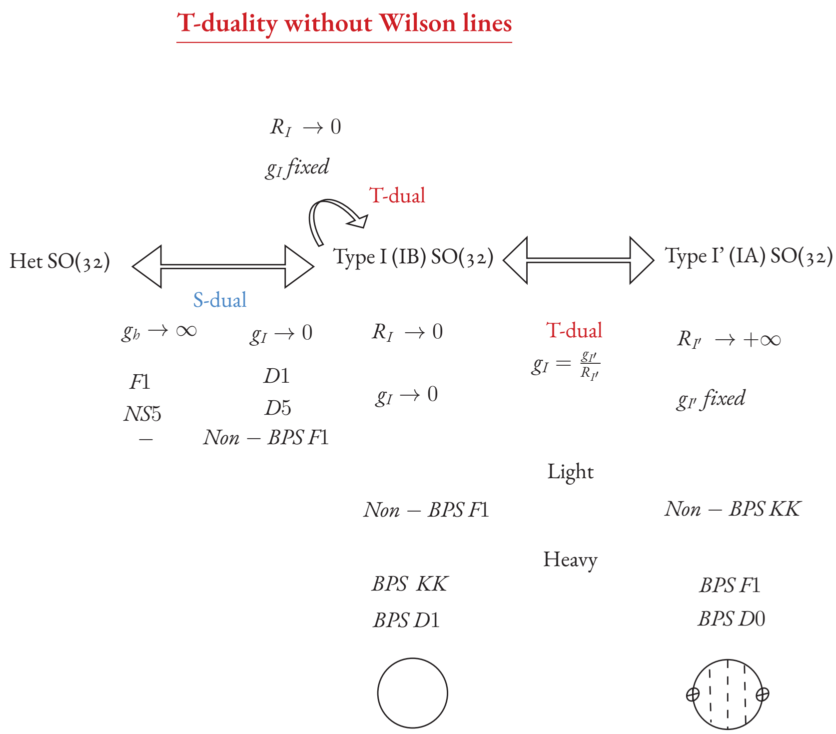

Figure 1.4: A corner of the 10d dualities in string theory with 16 supercharges

- $\mathcal{N} = (2,0)$ **in 6d**

These theories have similar anomaly cancellation condition as the 10 dimensional theories. However, vector multiplets are absent in such a theory and hence only the number of tensor multiplet is restricted for a consistent theory coupled to gravity. In particular, a 6d analogue of the Green-Schwarz mechanism [164] restricts the number of tensor multiplets to be $n = 21$ [183]. The moduli space of scalars in this theory is given by

$$\frac{SO(21,5)}{SO(21)\times SO(5)}\,. \tag{1.10}$$

(where here, and in the following we ignore the duality group quotient for the moduli space and all groups are over the reals).

This theory can be understood as the low-energy limit of Type IIB string theory on a $K3$. A T-dual description of this theory can be found by considering M-theory on the orbifold $T^5/\mathbb{Z}_2$ [190,53].

1.4.2 Non-Chiral Theories

In this subsection, we review various known constructions of non-chiral theories with 16 supercharges in dimensions $3 < d < 10$ giving more detail for the cases $d = 9, 8, 7$ which have been more thoroughly studied.

As we will review below, even though classical supergravity alone allows arbitrary rank r_G, the ones that appear from string theory constructions all satisfy the bound $r_G \leq 26 - d$. This bound is saturated by considering toroidal compactifications of heterotic strings. Below we review some of the string theory constructions leading to various lower ranks as well.

$\mathbf{d} = 9$ Theories

- Rank =17

  The 9 dimensional $\mathcal{N} = 1$ theory with rank 17 can be constructed in many ways. The simplest way is by considering a circle compactification of heterotic strings. This can of course be described in many dual ways, including Type I or M-theory as well.

  The moduli space branches for the $E_8 \times E_8$ and $SO(32)$ gauge groups in 9d are actually not distinct but parts of the full moduli space of rank 17 theories [151]. Each of these components can be reached by turning on Wilson lines for the heterotic/Type I theories or equivalently moving D8-branes in Type I'(similar to the dicussion of the previous section). Therefore, for rank 17 there exists one inequivalent 9 dimensional $\mathcal{N} = 1$ theory with moduli space $SO(17,1)/SO(17)$.

- Rank =9

There is also one inequivalent rank 9 theory which in different regions of its moduli space can be described in terms of M-theory on a Mobius strip [158,51], the 9d CHL string [42,43] and IIA with a shift-orientifold $O8^0$ and $O8^- + 7D8$ with an extra D8 [6]. Hence, we obtain 9 dimensional $\mathcal{N} = 1$ theory with gauge group $E_8 \times U(1)$ and moduli space given by $SO(9,1)/SO(9)$.

- Rank =1

  There exist two inequivalent rank 1 theories in 9 dimensions with moduli space given by $SO(1,1)$.

  The first one can be obtained from M-theory on the Klein Bottle [51] resulting in a theory with 16 supercharges and gauge group $U(1)$. The moduli space of this theory has different weakly coupled descriptions at different regions of the space. In particular, the different regions can be characterized by M-theory on the Klein Bottle, IIA with two shift-orientifolds $O8^0$ and the Asymmetric Orbifold of IIA (AOA) [6].

  The second inequivalent theory is given by Type IIB theory on a Dabholkar-Park(DP) background [51]. This latter theory also has various weak coupling descriptions [6]. Upon compactification on a further circle this leads to a T-dual description of M-theory on Klein Bottle mentioned above. Therefore, we have reviewed here two theories in $d = 9$ with $r_G = 1$ which are distinct, since they have disconnected moduli spaces and can only be connected through T-duality by going to 8d.

$\mathbf{d} = 8$ Theories

Discussions for constructions of theories in $d = 8$ can be found in [54,51] as well as in [78].

- Rank =18

This theory can be constructed by considering the circle reduction of the 9d rank 17 theory with an extra $U(1)$ factor coming from the second circle.

- Rank =10

  This theory can be viewed as the 8d CHL string[42] which is dual to IIA orientifold on the Mobius strip[158]. The strong coupling limit of the latter description is M-theory on the Mobius strip which is dual to the 9d CHL string.

- Rank =2

  There are two inequivalent 9d rank 1 theories which describe the same theory in 8 dimensions[6]. Therefore, there is one 8 dimensional rank 2 theory coming from the 9d circle reduction with an extra $U(1)$ factor.

$d = 7$ Theories

The 7 dimensional theories are interesting because they don't all come from a simple circle compactification of the 8d theories[54]. In particular, they find new theories by considering the heterotic string on T^3 with some $\mathbb{Z}_n$ triples of commuting holonomies, IIB orientifolds, and F/M-theory compactifications.

- Rank =19, 11

  These theories are equivalent to the 8d rank 18, 10 theories compactified on a circle respectively.

- Rank =7

  There is one theory with rank 7 obtained from the heterotic string on T^3 with some $\mathbb{Z}_3$ triples. This theory is dual to F-theory on $\frac{K3\times S^1}{\mathbb{Z}_3}$.

- Rank =5

  Similarly, this theory can be constructed from the heterotic string with $\mathbb{Z}_4$ triples and is dual to F-theory on $\frac{K3\times S^1}{\mathbb{Z}_4}$.

- Rank =3

  There are in total 4 inequivalent theories with rank 3. In particular, there are two inequivalent theories obtained from the heterotic string with $\mathbb{Z}_{5,6}$ triples and are dual to F-theory on $\frac{K3\times S^1}{\mathbb{Z}_{5,6}}$ respectively. In addition, there are two inequivalent theories coming from IIA orientifolds $4O6^- + 4O6^+$. They could possibly both be described in M-theory on $K3$ with frozen singularities (assuming two non-isomorphic embeddings of the $(D_4)^4$ weight lattice into the K3 lattice exist) with dual F-theory compactification on $\frac{(T^4\times S^1)}{\mathbb{Z}_2}$.

- Rank =1

  Finally, there are three inequivalent theories coming from F-theory on $\frac{T^4\times S^1}{\mathbb{Z}_{3,4,6}}$. These three theories do not have a heterotic description but have an M-theory description in terms of $K3$ compactification with frozen singularities.

1.5 Review of 6d N=1 Supergravity

In this section we review various features of 6d $(1,0)$ supergravity theories. In addition, we provide a review of the conditions that have been conjectured to be necessary for the consistency of these theories. The set of all such conditions provide Swampland constraints that severely limit the possible low energy theories that could be consistently coupled to gravity.

Anomaly Cancellation consideration : A six-dimensional supergravity with 8 supercharges consists of four types of massless supermultiplets: a gravity multiplet, vector multiplets, tensor multiplets and hypermultiplets. The chiral fields of those multiplets contribute to the anomalies produced in such a theory characterized by an 8-form anomaly polynomial I_8. Such anomalies can be cancelled by the Green-Schwarz-Sagnotti mechanism [164] if the anomaly polynomial I_8 factorizes as

$$I_8(R,F) = \frac{1}{2}\Omega_{\alpha\beta}X_4^\alpha X_4^\beta, \;\; X_4^\alpha = \frac{1}{2}a^\alpha trR^2 + \sum_i b_i^\alpha \frac{2}{\lambda_i} trF_i^2 \tag{1.11}$$

where a^α, b_i^α are vectors in $\mathbb{R}^{1,T}$, $\Omega_{\alpha\beta}$ is the metric on this space and λ_i are normalization factors of the gauge groups G_i. The anomaly factorization conditions for gravitational, gauge and mixed anomalies are summarized as follows:

- $R^4: \;\; H - V = 273 - 29T$ (1.12)
- $F^4: \;\; 0 = B^i_{Adj} - \sum n^i_R B^i_R$ (1.13)
- $(R^2)^2: \;\; a \cdot a = a^\alpha \Omega_{\alpha\beta} a^\beta = 9 - T$ (1.14)
- $F^2R^2: \;\; a \cdot b_i = a^\alpha \Omega_{\alpha\beta} b_i^\beta = \frac{1}{6}\lambda_i (A^i_{Adj} - \sum_R n^i_R A^i_R)$ (1.15)
- $(F^2)^2: \;\; b_i \cdot b_i = b_i^\alpha \Omega_{\alpha\beta} b_i^\beta = \frac{1}{3}\lambda_i^2 (\sum_R n^i_R C^i_R - C^i_{Adj})$ (1.16)
- $F_i^2 F_j^2: \;\; b_i \cdot b_j = b_i^\alpha \Omega_{\alpha\beta} b_j^\beta = \sum_{R,S} \lambda_i \lambda_j n^{ij}_{RS} A^i_R A^j_S \;\; i \neq j$ (1.17)

where H, V, T denote the number of hypermultiplets, vectors multiplets and tensor multiplets in the theory respectively. The number n^i_R represents the number of hypermultiplets in the representation **R** of the gauge group G_i and A^i_R, B^i_R, C^i_R are the following group theory coefficients:

$$tr_{\mathrm{R}} F^2 = A_R trF^2, \quad tr_{\mathrm{R}} F^4 = B_R trF^4 + C_R (trF^2)^2 \tag{1.18}$$

the values of those coefficients for various representations and the normalization factors λ_i are summarized in [127]. In addition as shown in [124] the vectors a^α, $b_i^\alpha \in \mathbb{R}^{1,T}$ are constrained to have integer inner products $a \cdot a, a \cdot b_i, b_i \cdot b_j \in \mathbb{Z}$ with respect to the bilinear form $\Omega_{\alpha\beta}$, we call this the anomaly lattice. The anomaly lattice as described in [168] needs to be embedded in the full string lattice of the 6d supergravity. Moreover, it was shown in [143] that the vector a is a characteristic vector of the lattice Γ, meaning that for any $x \in \Gamma$ we have $a \cdot x + x^2 \in 2\mathbb{Z}$.

Moduli space consideration: The moduli space of the 6d $(1, 0)$ supergravity locally takes the form $SO(1, T)/SO(T)$ which is parameterized by the a vector $j^\alpha \in \mathbb{R}^{1,T}$ with positive norm $j \cdot j > 0$ representing the positivity of the metric on the moduli space. As discussed in [124,164] consistency of the theory requires $j \cdot b_i \geq 0$, $j \cdot a \leq 0$. The first set of conditions are required for the positivity of the gauge kinetic terms and the latter condition is associated to the positivity of the Gauss-Bonnet term in gravity [44,90] which has been conjectured to hold.

1.6 F-theory on a Calabi-Yau threefold

In the previous section we analyzed the structure of minimal 6d supergravity theories. A large class of such theories can be constructed through F-theory on an elliptic Calabi-Yau threefold. An elliptic threefold with a section can be viewed as having base that is $K3$, Enriques, $Bl_p\mathbb{F}_n or Bl_p\mathbb{P}_2$, more details on complex surfaces can be found in Appendix B. The total Calabi-Yau threefold is retrieved by adding non-trivial fibrations over the base as seen in Figure 1.5.

We note that the process of making the fibers smooth described in Figure 1.5 is not possible in 6d because the vector multiplets have no scalars and hence a resolution of this type can only be accomplished by compactifying the theory on a circle which corresponds to M-theory on the resolved threefold. This is because the 6d theory on a circle now can have moduli associated with Wilson lines around the circle. In this picture the KK modes correspond to M2 branes wrapping the fibers. In section 1.8 the 5d supergravities will be the main focus and details of M-theory constructions will be discussed.

The fact that these are the only surfaces can easily be understood because we expect the Calabi-

Yau geometry to be Ricci flat. This means that since the Ricci scalars decompose for the fibers and the base then one needs them to both be zero or cancel each other. This can be accomplished by considering bases that are ruled, rational, or Ricci flat. The classification of minimal compact complex surfaces and their properties is summarized in Table 3.5.

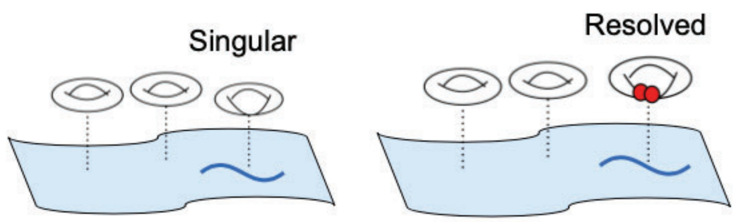

Figure 1.5: This figure depicts the base of an elliptic Calabi-Yau with tori fibered over the base. At special points described by the codimension one locus on the base the fibers may degenerate. One can resolve the singular fiber by replacing the singularity with a collection of spheres $\mathbb{P}^1$. The intersection pattern of these spheres provide us will a Dynkin diagram corresponding to the gauge symmetry.

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The correspondence between the 6d supergravity of the previous section and the geometric data is simple, where strings correspond to D3 branes wrapping various curves in the base:

$$-a \to K_B \tag{1.19}$$

where K_B is the canonical class of the base.

$$b \to C \tag{1.20}$$

where C is a holomorphic curve in the base. The rigid curves have $C^2 = -n < 0$ corresponding to $O(-n) \to \mathbb{P}^1$ rigid bundles. Such curves are shrinkable in the sense that can reach zero size at finite distance in the moduli space. These correspond to instantonic local strings and give rise to 6d SCFTs. In particular, curves with $C^2 < -2$ necessarily require non-trivial fibrations and define the

Non-Higgsable Clusters (NHC)[145]. We will use such curves chapter 4 to understand the structure of 5d superconformal theories(SCFT) as descendants of the 6d SCFTs.

When $Q^2 \geq 0$ they correspond to supergravity strings, which we will define in chapter 3, which exist only in supergravity theories. For curves with $Q^2 = 0$ one can find them in gravity theories corresponding to 5d small black holes or strings in Little string theories in which though gravity is decoupled. Curves associated to the supergravity strings are not shrinkable but they can only be asymptotically massless at infinite distance in the moduli space.

Additionally, in F-theory it is required that the vectors a, b_i satisfy the Kodaira conditon[124]:

$$j \cdot (-12a - \sum_i \nu_i b_i) \geq 0 \tag{1.21}$$

where ν_i is the multiplicity of the respective singularity or equivalently the number of 7 branes needed for the non-abelian gauge group G_i (e.g. $\nu = N$ for $SU(N)$). Additional constraints are imposed from F-theory considerations regarding the irreducibility and effectiveness of divisors. Moreover, for all odd lattices a is primitive and for $b_i^2 \leq 0$ also b_i is primitive in F-theory and in that case the former can also be brought in the form $a = (-3, 1^T)$[124].

The hodge structure of the threefolds is in correspondence with the massless modes of the supergravity theory as follows:

$$h^{1,1}(B) = T + 1 \tag{1.22}$$

$$h^{1,1}(X) = r + h^{1,1}(B) + 1 = r + T + 2 \tag{1.23}$$

$$h^{1,2}(X) = H_{neutral} - 1 = V - H_{charged} + 272 - 29T \tag{1.24}$$

$$\tag{1.25}$$

The geometric origin of the gravitational anomaly $H - V = 273 - 29T$ was discussed in[83].

Interestingly the finiteness of such construction is well established. The only elliptic Calabi-Yau two-fold with SU(2) holonomy is the K3 surface. As for the threefolds, they were also shown to be finite[84] and the largest hodge numbers are given by $(h^{1,1}, h^{2,1}) = (491, 11)$ and $(h^{1,1}, h^{2,1}) = (11, 491)$ which are related by mirror symmetry. In fact both of these theories can be realized with a toric base[180]. In chapter 3 we will show that 6d supergravity theories are finite which matches well with our intuition from F-theory.

1.7 Review of 5d N=1 Supergravity

Consider a 5d $\mathcal{N} = 1$ gravitational theory with gauge group $\mathcal{G}$. We are primarily interested in its effective field theory at low energy on the Coulomb branch of the moduli space. The massless supermultiplets in the spectrum are the gravity multiplet, a number of vector multiplets, and charged and neutral hypermultiplets. The vector multiplets of $\mathcal{G}$ contain the vector fields A_μ and the real scalar fields φ. The scalars φ can take nonzero expectation values in the Cartan subalgebra of the gauge group $\mathcal{G}$. The scalar expectation values, which we denote by $\varphi^a, a = 1, \cdots, \mathbf{r}$, are moduli parametrizing the Coulomb branch. At a generic point on the Coulomb branch, the gauge group $\mathcal{G}$ is broken to its abelian subgroup $U(1)^{\mathbf{r}}$ with $\mathbf{r} = \text{rank}(\mathcal{G})$ and the theory reduces to a supergravity theory coupled to **r** Abelian vector multiplets as well as neutral hypermultiplets.

The bosonic action on the Coulomb branch for the gravity multiplet and vector multiplets is given by[39,33]

$$S = \int \left(*R - G_{IJ} d\varphi^I \wedge *d\varphi^J - G_{IJ} F^I \wedge *F^J - \frac{1}{6} C_{IJK} A^I \wedge F^J \wedge F^K \right) , \qquad (1.26)$$

where R is the Ricci curvature and $F^I = dA^I$ is the field strength of the gauge group. Here we collectively denote the graviphoton field A^0_μ in the gravity multiplet and the **r** gauge fields in the vector multiplets by $A^I_\mu, I = 0, 1, \cdots \mathbf{r}$. G_{IJ} is the metric for the geometry of the scalar moduli space.

C_{IJK} is the level for the cubic Chern-Simons term and it is quantized due to gauge invariance of the Abelian symmetries as $C_{IJK} \in \mathbb{Z}$[191].

The metric on the scalar moduli space in the effective action is determined by the prepotential defined as

$$\mathcal{F} = \frac{1}{6} C_{IJK} \varphi^I \varphi^J \varphi^K \, , \tag{1.27}$$

which is a homogeneous cubic polynomial in the scalar expectation values φ^I obeying the hypersurface constraint,

$$\mathcal{F} = \frac{1}{6} C_{IJK} \varphi^I \varphi^J \varphi^K = 1 \, . \tag{1.28}$$

The geometry parametrized by φ^I under this constraint is called the *very special geometry*. The metric on this hypersurface can be obtained as

$$G_{IJ} = - \left. \frac{1}{2} \frac{\partial^2 \log \mathcal{F}}{\partial \varphi^I \partial \varphi^J} \right|_{\mathcal{F}=1} \, . \tag{1.29}$$

One can also consider hypermultiplets in the low energy theory. Charged hypermultiplets are all massive at a generic point on the Coulomb branch. They are already integrated out in the above effective action. On the other hand, the neutral hypermultiplets remain massless in the low energy theory. They will play some role in our discussion later.

On the Coulomb branch of the moduli space, the low-energy spectrum includes 1/2 BPS extended objects magnetically charged under the Abelian gauge groups. We call them magnetic monopole strings, or monopole strings for short. A monopole string carries the magnetic charge of gauge fields A^I as,

$$q^I = \frac{1}{2\pi} \int_{S^2} F^I \, . \tag{1.30}$$

Here the integration is taken over the two-sphere S^2 surrounding the string. The tension of this string can be exactly computed from the prepotential. For a string carrying unit magnetic charge of

A^I_μ, its tension is given by

$$T_I = \frac{\partial \mathcal{F}}{\partial \varphi^I}\bigg|_{\mathcal{F}=1} . \tag{1.31}$$

The monopole strings and their geometric counterparts will be central ingredients in section 3.2. We will discuss their properties in detail in the next subsections.

The low-energy theory may involve higher derivative corrections to the effective action. Some special higher derivative terms are determined by a combination of topological data and supersymmetry. One is the mixed gauge/gravitational Chern-Simons term of the form,

$$S_{ARR} = \frac{1}{96}\int C_I A^I \wedge \mathrm{tr}(R \wedge R) , \tag{1.32}$$

where $R = d\omega + \omega \wedge \omega$ is the curvature 2-form for the spin connection ω. This term is linear in the gauge field A^I, so we call this term as the linear Chern-Simons term. The supersymmetric completion of this four-derivative correction was obtained in [93] using conformal supergravity techniques.

The level C_I for the linear Chern-Simons term is also quantized as follows [40]. Let us put the theory on a five manifold $\mathcal{M}_5 = S^1 \times \mathcal{M}_4$. Then consider a large gauge transformation of the gauge field A^I,

$$A^I \;\to\; A^I + \frac{n}{R} dx^5 , \tag{1.33}$$

where $n \in \mathbb{Z}$ and R is the radius of the S^1 with coordinate x^5. This gauge transformation varies the linear Chern-Simons term as

$$\delta S_{ARR} = -\frac{n\pi}{24} C_I \int_{\mathcal{M}_4} p_1(T_4) , \tag{1.34}$$

where $p_1(T_4)$ is the first Pontryagin class for the tangent bundle T_4. Note that the integration of p_1

over a spin manifold $\mathcal{M}_4$ gives an integer number:

$$\frac{1}{48}\int_{\mathcal{M}_4} p_1(T_4) \in \mathbb{Z}\,. \tag{1.35}$$

Now demanding that the partition function is invariant under this large gauge transformation quantizes the level as an even integer, therefore $C_I \in \mathbb{Z}$.

The gauge symmetry can enhance to a bigger symmetry at special loci in the moduli space where some charged vector fields become massless. On the special locus, the Abelian gauge groups can enhance to non-Abelian groups provided that the massless charged vector fields form the adjoint representation of the non-Abelian symmetries. The full gauge group is then given by $\mathcal{G} = G \times U(1)^{\mathbf{r}+1-r}$ where G is the product of the enhanced non-Abelian groups with $r = \text{rank}(G)$. There can also be massless hypermultiplets charged under the enhanced gauge symmetry G. The low-energy theory on the special vacua is then described by the gauge theory of the enhanced gauge group $\mathcal{G}$ coupled to the massless charged hypermultiplets.

Two-derivative terms in the gauge theory action for each gauge multiplet Φ_i of a simple non-Abelian group $G_i \subset G$ are determined from the prepotential

$$\mathcal{F}_{G_i} = -\frac{h_i}{2}\text{Tr}(\Phi_i^2) + \frac{\kappa_i}{6}\text{Tr}(\Phi_i^3)\,, \tag{1.36}$$

where h_i is the gauge coupling and κ_i is the classical Chern-Simons level for G_i. The classical Chern-Simons level is an integer and non-zero only for $G_i = SU(N)$ with $N \geq 3$. Here, the gauge coupling h_i is given by a linear sum of the scalar values φ^α in the Abelian part, such as

$$h_i = \sum_{\alpha=1}^{\mathbf{r}+1-r} h_{i,\alpha}\varphi^\alpha\,. \tag{1.37}$$

These scalar moduli φ^α parametrize the special sub-manifold of the moduli space where the gauge symmetry enhancement occurs. There is no four-derivative correction to the non-Abelian action because if it exists, it is linear in Φ_i, but Φ_i is traceless.

One can move away from the special vacua by turning on generic scalar expectation values, say $\varphi_i^a, a = 1, \cdots, \text{rank}(G_i)$, for the Cartan generators of the non-Abelian symmetry G_i. This will bring us back to the Abelian effective theory at low energy. The prepotential of the Abelian theory in the neighborhood of the special loci is determined by a one-loop calculation with charged fermions that become massive with non-zero values of φ_i^a. For a non-Abelian gauge group G_i and matter hypermultiplets in generic representations, the prepotential after the one-loop calculation is[191,166,110]

$$\mathcal{F}_{G_i} = -\frac{b_i}{2} K_{i,ab}\,\varphi_i^a \varphi_i^b + \frac{\kappa_i}{6} d_{i,abc}\varphi_i^a\varphi_i^b\varphi_i^c + \frac{1}{12}\left(\sum_{\mathbf{R}} |\mathbf{R}\cdot\varphi_i|^3 - \sum_f \sum_{\mathbf{w}_f} |\mathbf{w}_f \cdot \varphi_i|^3\right) , \quad (1.38)$$

where $K_{i,ab}$ is the Killing form of G_i and $d_{i,abc} = \frac{1}{2}\text{Tr}(T_i^a\{T_i^b, T_i^c\})$ with the generator T_i^a in the fundamental representation of G_i. $\mathbf{R}$ and $\mathbf{w}_f$ are the roots and the weights for the f-th hypermultiplet of G_i, respectively.

In addition, a mixed gauge/gravitational Chern-Simons term with the level $C_{i,a}$ is induced by integrating out the charged fermions. The result from the one-loop computation is[35]

$$C_{i,a} = -\frac{\partial}{\partial\varphi_i^a}\left(\sum_{\mathbf{R}} |\mathbf{R}\cdot\varphi_i| - \sum_f \sum_{\mathbf{w}_f} |\mathbf{w}_f \cdot \varphi_i|\right) . \quad (1.39)$$

Note that not all supergravity theories have such special sub-manifolds of the moduli space supporting enhanced gauge symmetry. Also, it is possible that a single theory has different special vacua with different non-Abelian gauge theory descriptions, which may lead to interesting dualities. However, at a generic point on the Coulomb branch, the effective theory after integrating out all the massive charged fields always reduces to an Abelian gauge theory description.

A large class of 5d $\mathcal{N} = 1$ supergravities can be constructed from M-theory compactification on compact Calabi-Yau threefolds. Such theories will be discussed in more details in the next section.

1.8 M-theory on Calabi-Yau threefolds

In the previous section we studied general aspects of 5d $\mathcal{N} = 1$ supergravity theories. However, a large class of such supergravity theories can be engineered by compactification of M-theory on compact Calabi-Yau threefolds (CY$_3$'s) with $SU(3)$ holonomy. In M-theory compactifications, the Coulomb branch of the moduli space in the 5d theory is identified with the Kähler moduli space of the CY 3-fold. The Kähler moduli space is parametrized by Kähler parameters φ^I associated to an integral basis ω_I for $H^{1,1}(X)$ in a threefold X. The action and the matter content in the effective five-dimensional theory are specified by topological data of the Kähler moduli space.

Let us expand the three-form potential C_3 in 11d supergravity in terms of the basis two-form classes ω_I of X as

$$C_3 = A^I \wedge \omega_I \,, \tag{1.40}$$

where $A^I = A^I_\mu dx^\mu$ with $I = 0, \cdots, h^{1,1}(X) - 1$ are the 1-form vector fields along the non-compact 5d spacetime. A particular linear combination of the 1-form fields will become the graviphoton field in the gravity multiplet and the remaining $n_V = h^{1,1}(X) - 1$ vector fields will become the $U(1)$ gauge fields in the vector multiplets in the low-energy supergravity theory. The 11d supergravity action integrated on the threefold X reduces to the effective action of the 5d supergravity up to four-derivative terms written in terms of massless supermultiplets.

The reduction of the 11d Chern-Simons term on X leads to the 5d cubic Chern-Simons term [156]:

$$S_{CS} = -\frac{1}{6}\int_{\mathcal{M}_5\times X} C_3 \wedge G_4 \wedge G_4 = -\frac{1}{6}C_{IJK}\int_{\mathcal{M}_5} A^I \wedge F^J \wedge F^J \,, \tag{1.41}$$

where $G_4 = dC_3$ and $F^I = dA^I$. In this expression, the triple intersection number

$$C_{IJK} \equiv \int_X \omega_I \wedge \omega_J \wedge \omega_K \,, \tag{1.42}$$

counts the intersection numbers of 4-cycles dual to ω_I in X. This triple intersection numbers naturally reduce to the cubic Chern-Simons coefficients C_{IJK} in the 5d effective action.

We can now write the Kähler form J in this basis as

$$J = \sum_I \varphi^I \omega_I \,, \quad I = 0, 1, \cdots, h^{1,1}(X) - 1 \,, \tag{1.43}$$

where φ^I are the $h^{1,1}(X) = n_v + 1$ Kähler moduli. Note that one of these moduli controlling the overall volume of X becomes a scalar component in a hypermultiplet and the other moduli are mapped to the scalar expectation values in the vector multiplets in the low-energy theory. The total volume of X measured with respect to J is then given by

$$\mathcal{F} = \frac{1}{6} \int_X J \wedge J \wedge J = \frac{1}{6} C_{IJK} \varphi^I \varphi^J \varphi^K \,. \tag{1.44}$$

We shall fix the value of this volume (so a hypermultiplet scalar) by a constraint $\mathcal{F} = 1$. The remaining scalars φ^I subject to this constraint form an n_V dimensional Kähler moduli space in X. This Kähler moduli space is identified with the Coulomb branch of the moduli space in the 5d supergravity theory.

The metric on the Kähler moduli space is geometrically defined as

$$G_{IJ} = \frac{1}{2} \int_X \omega_I \wedge * \omega_J = -\frac{1}{2} \partial_I \partial_J (\log \mathcal{F})|_{\mathcal{F}=1} \,, \tag{1.45}$$

where the $*$ denotes the Hodge dual taken in the internal Calabi-Yau manifold. This metric agrees with that of the supergravity theory given in (1.29) when we identify the volume $\mathcal{F}$ of X with the prepotential in the 5d gravity theory.

The spectrum of charged objects under the 5d gauge symmetry originate from M2/M5-branes in M-theory wrapped on 2/4-cycles in the 3-fold. The M2-brane has a three-dimensional worldvolume carrying unit electric charge of the 3-form potential C_3. The worldvolume of M2-branes can wrap on holomorphic (and also effective) 2-cycles in the internal 3-fold. The wrapped M2-branes give rise to electrically charged BPS particles coupled to the vector fields A^I for the 2-cycles in the 5d theory. In this case, the mass of the BPS particle is proportional to the volume of the 2-cycle. So the BPS particles coming from the wrapped M2-branes are massive at generic points on the Kähler moduli where all 2- and 4-cycles have finite volume. Thus they can be integrated out and do not appear in the spectrum of the low-energy effective theory.

At certain special values of the Kähler moduli φ^I, some 2-cycles (and also 4-cycles) shrink to zero size and the 3-fold X becomes singular. The singularity with vanishing cycles can support a non-Abelian gauge algebra G and the M2-branes wrapping shrinking 2-cycles give rise to massless charged states in the 5d field theory. In particular, the massless vector states can participate in the gauge symmetry enhancement to the non-Abelian symmetry G supported along the singularity of the 3-fold.

The M5-brane is a magnetically charged object with respect to the 3-form potential. The six-dimensional worldvolume of the M5-brane can wrap around holomorphic 4-cycles labelled by ω_I in the internal threefold. The remaining two-dimensional worldvolume stretches along the 5d non-compact spacetime. Thus the wrapped M5-brane yields a magnetically charged monopole string of the Abelian gauge field A^I_μ in the 5d supergravity theory.

The tension of the BPS monopole string is set by the volumes of 4-cycles in the Calabi-Yau manifold. For the string with unit magnetic charge of A^I, the volume of a basic 4-cycle ω_I, so the string tension, is given by

$$T_I = \partial_I \mathcal{F} = \frac{1}{2}\int_X \omega_I \wedge J \wedge J$$
$$= \frac{1}{2} C_{IJK}\varphi^J \varphi^K \,. \qquad (1.46)$$

The string tension is always positive within the Kähler cone of a smooth 3-fold X.

In the singular limit of X, as stated above, some 4-cycles can collapse to a point or to a collection of 2-cycles. Then the M5-branes wrapping the set of collapsing 4-cycles become tensionless strings. The low-energy theory in the neighborhood of the singular locus when gravity is decoupled reduces to a local 5d SCFT strongly interacting with the tensionless strings [166,147,59]. However, for gravity theories we are not interested such tensionless strings in local 5d SCFTs. But they will become relevant in chapter 4. The wrapped M5-brane states over 4-cycles which never collapse to zero size in the Kähler moduli space of the Calabi-Yau threefold correspond to supergravity strings as will be discussed in section 3.2.

1.9 BPS strings

As noted in earlier sections gravity multiplets with 16 supercharges and those with 8 supercharges in $d > 4$ contain a two-form field. A natural object charged under these fields is a string. If the string is BPS then their tension and charge are determined by supersymmetry. In fact the completeness of spectrum hypothesis summarized in section 1.2 implies the existence of such strings associated to each charge of the two-forms. A stronger version of this conjecture is that any state allowed to be BPS has a BPS representative and hence predicting the existence of such BPS strings.

In this section some of the properties of such strings is summarized including their central charges. It is well known that the gauge symmetries of the bulk theory, if seen by the string, will appear as global symmetries of the string worldsheet in the form of a current algebra with non-zero level. Their levels and central charges can be computed exactly using the anomaly inflow mechanism summarized in Appendix A.1. A necessary condition for all these strings to be BPS with the right R-symmetry is that their levels and central charges are non-negative.

Theory	5d $\mathcal{N}=1$	6d $\mathcal{N}=1$	16 Supercharges
BPS String	(0,4)	(0,4)	(0,8)
Charges	$Q^I = \frac{1}{2\pi}\int_{S^2} F^I \in \mathbb{R}^{n+1}$ $n =$#of the vectors	$Q^I \in \mathbb{R}^{T+1}$ $T =$# of tensors	$Q \in \mathbb{Z}$
R-symmetry	$SU(2)$	$SU(2)_R \times SU(2)_L$	$SO(d-2)$
Central charge	$c_L = C_{IJK}Q^IQ^JQ^K + C_IQ^I - 2$	$c_L = 3Q\cdot Q - 9Q\cdot a + 2$	$c_L = 24\kappa + 2 - d$
	$c_R = C_{IJK}Q^IQ^JQ^K + \frac{1}{2}C_IQ^I - 6$	$c_R = 3Q\cdot Q - Q\cdot a$	$c_R = 12(\kappa - 1)$
Level	$k_{IJ} = C_{IJK}Q^K$	$k_\ell = Q\cdot Q + Q\cdot a + 2, k_G = Q\cdot b_i$	$\kappa = 1,\ k_i = Q$
Tension	$T_I = \frac{\partial \mathcal{F}}{\partial \varphi^I}\Big\vert_{\mathcal{F}=1}$	$T = J\cdot Q$	$T = M_d^2 e^{-\alpha\varphi}$ $(d > 5)$
Supergravity	$C_{IJK}Q^IQ^JQ^K \geq 0$	$Q^2 \geq 0$	Always $(d > 5)$

Table 1.1: This table summarizes the R-symmetry, central charges, levels and tensions of a charge Q string. We present the most general cases that do not have any spontaneous symmetry enhancements in the IR. Some of these theories also have "anti"-BPS strings like local strings which may preserve a different set of supercharges and will be discussed in the next subsection. One exception exists for the 6d chiral theory which has 21 tensor multiplets but the theory is completely fixed by supersymmetry. Additionally for $d = 5$ the vector multiplets can be dualized to two forms and hence one has monopole strings with a moduli space depending on the rank of the theory. In some cases we can also have little string theories but they give rise to null charges.

16 Supercharges BPS string consideration: For dimensions $d > 5$ the charge lattice of the string is 1-dimensional since there exists a single $B_{\mu\nu}$ in the gravity multiplet.

The central charge and hence the tension of these BPS strings are completely fixed by supersymmetry and do not receive any corrections. The form of the tension is

$$T = M_d^2\, e^{-\alpha\,\varphi} \tag{1.47}$$

with a some number fixed by the dimension. The scalar φ is the canonically normalized and dimensionless version of the d-dimensional dilaton. For example, for $d = 10$ one has $\alpha = \frac{1}{\sqrt{2}}$.

6d BPS string consideration: This theory has one self-dual two-form $B^+_{\mu\nu}$ in the gravity multiplet and T tensor multiplets with an anti self-dual two-forms $B^-_{\mu\nu}$. The 6D theory has gravity/gauge dyonic strings with charges $-a$, b_i constrained by the anomaly polynomial. Those charges span the anomaly lattice which is contained in the full string lattice of the 6d theory. Therefore, as discussed in [168] the anomaly lattice is required to have a unimodular embedding into a self-dual lattice and this fact provides a constraint on possible theories. Furthermore, the existence of the two-form fields B_2^α implies the existence of string sources in accordance with the hypothesis that the spectrum of a gravitational theory needs to be complete [13,159]. Therefore, according to [119,116], a supergravity BPS string with charge Q and non-negative tension provides the following constraints:

$$
\begin{gathered}
T \geq 0, Q \cdot Q \geq 0 \\
k_\ell \geq 0,\ k_i \geq 0 \\
\textstyle\sum_i c_{G_i} \leq c_L
\end{gathered}
\tag{1.48}
$$

where k_i is the level of G_i and c_{G_i} the central charge associated with the current algebra of G_i. In addition, k_ℓ is the level of the current algebra associated with $SU(2)_\ell$ which arises from the normal bundle $SO(4) = SU(2)_R \times SU(2)_\ell$ for the transverse $\mathbb{R}^4$, where $SU(2)_R$ is the R-symmetry of the IR (0,4) SCFT and $SU(2)_\ell$ appears as a left current algebra.

5d BPS string consideration: In this case the strings are magnetic monopoles of the various vectors in the theory. To be concrete we have one graviphoton vector and n vector multiplets for a general 5d theory. The constraints are equivalent to the ones discussed in 6d. In section 3.2.2 the supergravity strings will be explicitly defined giving rise to strings that only exist in gravitational theories. This implies that in the gravity decoupling limit also these strings decouple.

The motivation of the conditions on supergravity strings of table 1.1 will be analyzed in more detail in chapter 3. In the language of M-theory on some Calabi-Yau manifold these correspond to M5 branes wrapping Nef divisors which will be explored in 3.2.4. In the F-theory language and hence the 6d theories these would correspond to Nef curves of the base. This positivity condition is necessary because of the entropy formula of black holes. These strings will give rise to BPS black holes in lower dimensions and hence their contribution to the entropy is known [136,185,173]. Therefore, having N multiples of these strings for large N will dominate the entropy $S_{5d} \sim N^3$ and $S_{6d} \sim N^2$ therefore the coefficients need to be non-negative. Equivalently, one could have derived the same condition by looking into extremal BPS states.

This distinction we have made is important because it is informative of when this strings are associated to local theories like SCFTs or gravity theories. In this thesis both classes of BPS strings will be studied. An important distinction between these strings is their R-symmetry, as local strings have an enhanced R-symmetry which will be summarized below.

1.9.1 Potential Symmetry enhancements

These subsection is devoted into the analysis of accidental IR symmetry enhancements beyond the ones presented in Table 1.1. In fact they can be categorized into: supergravity strings with enhanced IR supersymmetry, supergravity strings that come from higher dimensions, SCFT/local strings(including little strings).

Symmetry enhancement in theories with 16 supercharges: In this case there is one type of symmetry enhancement $(0,8) \to (8,8)$. As described in section A.1 one needs to change the anomaly inflow computation of the central charges to account for the extra center of mass degrees of freedom. In fact such strings provide more restrictions as will be demonstrated in chapter 2. Form the string theory point of view they can be thought of as descendants of the type II string.

Symmetry enhancement in 6d: In this case an important type of enhancement occurs when the strings have an accidental $SU(2)_I$ symmetry in the IR which occurs for 6d SCFTs and little string theories in the decoupling limit. In fact this becomes the R-symmetry of the string worldsheet in those cases.

This makes it crucial to identify the correct R-symmetry since the right-moving central charge depends on it.

Example: Consider the SO(8) Non-Higgsable cluster with $Q^2 = -4$ this means that

$$c_R = -18 < 0 \tag{1.49}$$

However, if one uses the right R-symmetry to compute the central charge as done in [171] a positive central is guaranteed given by $c_I = Qh^v = 6 > 0$. The positivity of the central charge is crucial in order to maintain the unitarity of the string worldsheet.

Symmetry enhancement in 5d: Naively, one expects that the $SU(2)_R$ symmetry would reduce to the IR R-symmetry of the 2d CFTs on monopole strings because this is the only $SU(2)$ symmetry under which the supercharges are charged. However as was demonstrated above this is not always the case and some accidental symmetries may emerges in the IR.

For example, monopole strings living on local 5d SCFTs amount to M5-branes wrapping 4-cycles in local CY_3's corresponding to shrinkable divisors. Such strings become tensionless strings in the CFT limit of the local theory when gravity decouples. The corresponding 4-cycles in a local CY_3 can collapse to zero size in the CFT limit. For those strings, the IR worldsheet CFT acquires an accidental $SU(2)_I$ symmetry inherited from the $SU(2)_I$ R-symmetry of the local 5d SCFT. This emergent $SU(2)_I$ symmetry, instead of $SU(2)_R$, in the IR CFT becomes the R-symmetry of the IR superconformal algebra. Therefore in this case the central charges should be calculated with respect to the $SU(2)_I$ symmetry. These strings can tell us physics of local 5d SCFTs. However since their low-energy physics is not affected by bulk gravitational interactions, we cannot use them to explore consistency of gravity theories. For this reason, we are not interested in these strings embedded in local 5d SCFTs with accidental $SU(2)_I$ symmetry.

Also, the strings arising from 6d self-dual strings by S^1 compactification have a different R-symmetry in their worldsheet CFTs at low-energy. The $SU(2)_R \times U(1)$ symmetry, where the $U(1)$ is for the KK momentum, in the worldsheet theory enhances to $SU(2)_r \times SU(2)_l$ in IR after decoupling the center-of-mass modes and the $SU(2)_r$, instead of $SU(2)_R$, becomes the R-symmetry of the IR $\mathcal{N} = (0, 4)$ superconformal algebra in the interacting sector. Here $SU(2)_R$ is the diagonal subgroup of $SU(2)_l \times SU(2)_r$. The anomaly polynomial and the central charges of self-dual strings in 6d supergravities are computed in [119] by using anomaly inflow mechanism (See also [118,171]). The anomaly polynomial of the 6d self-dual strings reduces to that of 5d monopole strings given in (A.16) by identifying $q^I = Q^I$ and $C_I = -12a_I$, and also $c_2(l) = c_2(r) = c_2(R)$ from the relation $SU(2)_R \subset SU(2)_r \times SU(2)_l$ under S^1 reduction. From this, one can deduce that $C_{IJK}q^I q^J q^K = 0$ and $C_I q^I = -12Q\cdot a$ for the 6d self-dual strings. In M-theory compactified on CY3, the 6d self-dual strings correspond to M5-branes wrapped on elliptic surfaces equipped with elliptic fibration structure which will further be discussed in Section 3.2.5. We note that when a 6d theory is compactified on a circle with automorphism twists, the worldsheet theory on a string that is affected by the twist do not have $SU(2)_l \times SU(2)_r$ symmetry enhancement since the Lorentz symmetry $SU(2)_l \times SU(2)_r$ is broken to $SU(2)_R \times U(1)$ by the twist. In this case, we expect that the $SU(2)_R$ will become the IR R-symmetry of the worldsheet CFTs.

It may also be possible that the IR worldsheet CFT shows supersymmetry enhancement. For example, the worldsheet CFT on self-dual strings in the 6d SCFT of $\mathcal{O}(-2) \to \mathbb{P}^1$ model is realized by a UV $\mathcal{N} = (0, 4)$ gauge theory[85,86], but this theory is expected to flow in the infrared to a CFT with enhanced $\mathcal{N} = (4, 4)$ supersymmetry. Another interesting example of 2d CFTs showing supersymmetry enhancement is the worldsheet theory on strings in the 9d supergravity theory constructed in M-theory on the Klein Bottle [51]. This worldsheet theory naively has only $\mathcal{N} = (0, 8)$ supersymmetry, but the IR SUSY turns out to get enhanced to $\mathcal{N} = (8, 8)$ [8].

Similarly, the $\mathcal{N} = (0, 4)$ supersymmetry on monopole strings in 5d supergravity can also enhance to a larger SUSY in the infrared CFT. Let us first discuss $\mathcal{N} = (4, 4)$ SUSY enhancement. In this case the enhanced superconformal algebra must be the small $\mathcal{N} = (4, 4)$. The large $\mathcal{N} = 4$ algebra in 2d CFTs involves two $SU(2)$ R-symmetries in each chiral sector. However, when coupled to 5d gravity we cannot have such two $SU(2)$ R-symmetries. So the enhanced $(4, 4)$ symmetry can only be the small $\mathcal{N} = (4, 4)$ symmetry. The small $\mathcal{N} = 4$ conformal algebra involves a single (anti-)holomorphic $SU(2)$ R-symmetry which may be identified with the $SU(2)_R \subset SO(1, 4)$ Lorentz symmetry in the 5d theory. However, the small $\mathcal{N} = (4, 4)$ conformal algebra involves two copies of $\mathcal{N} = 4$ conformal algebra referred to as the left-moving and right-moving sectors, and the $SU(2)$ R-symmetries in those two sectors are independent and distinct if the CFT is unitary and the vacuum is normalizable. We expect after removing the center-of-mass degrees of freedom that the interacting sector in the IR CFT on a single monopole string which does not degenerate to monopole strings in local SCFTs is unitary and has normalizable vacua. Thus the small $\mathcal{N} = (4, 4)$ superconformal algebra cannot be realized in the non-trivial CFTs on monopole strings unless there exists an accidental $SU(2)$ symmetry in IR[5].

This argument however cannot rule out the possibility of $\mathcal{N} = (4, 4)$ SUSY enhancement

[5] If a 5d monopole string comes from a 6d self-dual string on S^1 without twist, the worldsheet theory can flow to a non-trivial SCFT with $(4, 4)$ SUSY enhancement. This is because in this case the symmetry $SO(3)_R$ for the transverse $\mathbb{R}^3$ rotation enhances to $SU(2)_l \times SU(2)_r$ in IR, and the $SU(2)_l$ and $SU(2)_r$ can become the left-moving and the right-moving R-symmetries, respectively, of the $(4, 4)$ superconformal algebra. We expect that the 6d supergravity strings studied in [119] with $Q \cdot a = 0$ have a $\mathcal{N} = (4, 4)$ SUSY enhancement on a non-trivial interacting sector. When geometrically realized, such 6d strings have the same number of left-moving bosons and fermions, $N_L^B = N_L^F = 4(g + 1)$ when the pull-back $\widehat{C}$ wrapped by dual M5 brane is a trivial fibration $C \times E$, where E is an elliptic curve and g is the genus of the curve of the string in the base. This is consistent with $(4, 4)$ SUSY (or (8,8) SUSY when $g = 1$) enhancement.

when the interacting sector in the IR CFT is trivial. It is possible that the IR worldsheet CFT consists of only the center-of-mass degrees of freedom so that the interacting CFT sector is trivial. The above argument does not hold for the center-of-mass sector due to the non-compact free bosons parametrizing the transverse motion of the string. The worldsheet theory can flow in the infrared to a free theory consisting of the $\mathcal{N} = (4,4)$ center-of-mass multiplet formed by 3 non-compact bosons $X_{\mu=1,2,3}$ and a compact scalar φ and 4 chiral and anti-chiral fermions $\lambda^{\alpha}_{\pm}$ where α is the doublet index of $SU(2)_R$. In this case, the IR R-symmetry is identified with the $SU(2)_R$ symmetry, and the central charges are $c_L = c_R = 6$. This implies that the $\mathcal{N} = (4,4)$ SUSY enhancement can occur only if $c_R - c_L = 0$ and $k_R = 0$, therefore only if $C_{IJK}q^I q^J q^K = C_I q^I = 0$.

The worldsheet theory can have a further enhancement to $\mathcal{N} = (8,8)$ SUSY. In this case, the worldsheet theory consists of a free $(8,8)$ center-of-mass multiplet and the interacting sector in the IR CFT is again trivial. The central charges from the free $(8,8)$ multiplet are $c_L = c_R = 12$. Thus, this string has $C_{IJK}q^I q^J q^K = C_I q^I = 0$. This string lives in the 5d supergravity theory with 32 supercharges. This string amounts to a M5-brane wrapping an Abelian surface with irregularity $q = 2$ in M-theory compactification.

Lastly, the worldsheet SUSY can enhance to $\mathcal{N} = (0,8)$ supersymmetry. The strings coupled to 5d bulk gravity with such enhancement are those in the 5d supergravity theories with 16 supercharges. It was conjectured in [119] that such strings have central charges $c_L = 24$ and $c_R = 12$ coming from only the $(0,8)$ center-of-mass modes. This indicates that the $\mathcal{N} = (0,8)$ enhancement can occur in the worldsheet theory only when $C_{IJK}q^I q^J q^K = 0$ and $C_I q^I = 24$. An M5-brane wrapping a K3 surface of Table 3.5 leads to such a monopole string with $(0,8)$ supersymmetry.

1.10 Review of Quantum field theory in five and six dimensions

A lot of the structure of quantum field theories is already encoded in the study of supergravities just reviewed if one eliminates the gravitational sector. In fact in many cases a clear distinction between supergravity strings and the local strings was made. In this section the properties of local strings will be reviewed and their relations to SCFTs will be described. Such SCFTs contain tensionless strings making their existence mysterious but as will be discussed below, string theory provides a framework to construct and study such examples.

The most natural starting point to define a quantum field theory would be to consider some gauge theory lagrangian of the form:

$$S = \frac{1}{g^2} \int d^d x F_{\mu\nu} F^{\mu\nu} \tag{1.50}$$

Requiring that the action is dimensionless means that $g \sim [M]^{\frac{4-d}{2}}$. But for $d \geq 5$ it implies that $\frac{4-d}{2} < 0$ and hence the theory is non-renormalizable.

This means that such theories can be defined as effective field theories with some UV cut-off. However, the fact that the theory is non-renormalizable could also be thought of as an indication that new degrees of freedom related to new physics need to be integrated in to make the theory consistent in arbitrarily high energies. This is in similar spirit to the φ^4 theory. A good starting point to study such theories would be adding some amount of supersymmetry. The minimal amount of supersymmetry in 6 and 5 dimensions is 8 supercharges. This corresponds to the 4d $\mathcal{N} = 2$ theory after reduction.

The 6d theory does not have a lagrangian description because the (anti)self-dual tensors make it hard to express it in this way. However, the 4d $\mathcal{N} = 2$ and 5d $\mathcal{N} = 1$ can have a lagrangian description in certain cases expressed in terms of the vector multiplet which includes $(A_\mu, 2\chi, \varphi)$ with one real scalar for 5d and one complex scalar for 4d . We promote the fields to superfields and each superfield contains all the fields in the corresponding multiplet and some of their derivatives. Therefore, the superfield for the vector multiplet will include the fields $\mathcal{A} = (A_\mu, F_{\mu\nu}, \chi, \partial_\mu\chi, \varphi, \partial_\nu\psi)$ and the effective action can be writen as $\mathcal{A}$

$$\mathcal{L} = \int \mathcal{A}^2 + \mathcal{A}^3 + ... \tag{1.51}$$

making the lagrangian a polynomial function in $\mathcal{A}$

$$\mathcal{L} = \int \mathcal{F}(\mathcal{A}) \tag{1.52}$$

This function $\mathcal{F}(\mathcal{A})$ in called a *prepotential* and determines fully the form of the Lagrangian.

In section 1.7 it was discussed that the structure of the supergravity action depends on the (very) special geometry for the (4d) 5d theories. The special geometry is parameterized by the prepotential which exactly fixes the coupling constants and masses of BPS particles as reviewed in Appendix A.

In particular for 5d $\mathcal{N} = 1$ the low-energy abelian theory is given by the prepotential $\mathcal{F}(\varphi_i)$ and takes the form

$$\mathcal{L} = (\partial_i\partial_j\mathcal{F})d\varphi^i \wedge \star d\varphi^j + (\partial_i\partial_j\mathcal{F})F^i \wedge \star F^j + \frac{1}{24\pi^2}(\partial_i\partial_j\partial_k\mathcal{F})A^i \wedge F^j \wedge F^k + \cdots \tag{1.53}$$

The only main difference with the gravity theories is that one does not impose the hypersurface constraint $\mathcal{F} = 1$. This constraint could be thought of as reducing the number of scalars by one to accommodate for the graviphoton that has no scalar parameter.

Moduli Space: The moduli space of this theory is defined to be the space parametrized by the massless scalars which are flat directions of the potential. Assume we have a vector multiplet in the

4d $\mathcal{N} = 2$ theory of some Lie group G. Then, the superymmetry dictates a potential of the form $V(\varphi) = [\varphi^\dagger, \varphi]^2$. This potential is zero exactly when $[\varphi^\dagger, \varphi] = 0$ and the non-trivial $\varphi's$ that satisfy the condition are part of the Cartan subalgebra. Hence, those $\varphi's$ parametrize a non-trivial space, the moduli space.

The scalars in the vector multiplets parametrize the **Coulomb Branch** and the scalar in the hypermultiplets parametrize the **Higgs Branch**. In 5d those branches are completely decoupled and no mixed branches exist. For the 6d theory the Higgs branch is equivalent but there are no moduli associated to the vector multiplets. However, there is something worth to be called a Coulomb branch or a **Tensor Branch** and this is the space parameterized by the scalars in the tensor multiplets.

In fact in 5 and 6 dimensional there are three consistent superconformal groups[149], with 6 being the maximal dimension:

$$d = 6 : \; Osp(6,2|N) = SO(6,2) \times Sp(N)_R \text{ with } N = 1,2 \tag{1.54}$$

and

$$d = 5 : \; F(4) = SO(5,2) \times Sp(1)_R \tag{1.55}$$

This means that in principle if such a superconformal theory does exist and some supersymmetric relevant deformations can be found then they could potentially trigger an RG flow towards a low energy effective action. Superconformal deformations are deformation that will preserve the supersymmetry of the theory. All such deformation where classified in[47] and they correspond to

$$d = 6 : \textbf{Moduli space} \tag{1.56}$$

$$d = 5 : \textbf{Mass deformation } + \mathcal{O}(x) \textbf{ and Moduli space} \tag{1.57}$$

By moduli space it is meant that one can tune the boundary conditions at spatial infinity which will lead to the moduli acquiring non-trivial vacuum expectation values. These expectation values give rise to a non-trivial scale which consequently takes the theory away form the conformal vacuum by breaking the conformal symmetry. Such deformations do not affect the dynamics of the theory as the mass deformation does. So the RG flow can be thought of as connecting the UV CFT to the IR phase when the theory is on its moduli space. Both of the above cases have scalars that parametrize the Tensor and Coulomb branch physics respectively. But in five dimensions relevant deformations also exist associated to flavor masses. These are supersymmetry preserving deformations of the form $\partial\mathcal{L} = \mathcal{O}(x)$.

In fact, string theory provides a framework to realize such SCFTs. The main ingredient is the existence of BPS solitons which provide stable vacuum and help "renormalize" the theory.

The main goal of chapter 3 will be to describe in detail such solitonic objects and differentiate between those in quantum field theories versus those in gravitational theories. Additionally, in chapter 4 a classification framework for 5d superconformal theories will be discussed. These theories can be found in string theory by considering regions where the effective physics is much less than the Planck mass of the system and hence effectively gravity is decoupled.

An important point that will also be discuss in chapter 3 is that the conformal field theory need not be coupled to the same dimensional gravity but rather it could be a lower dimensional localized region in a higher dimensional gravitational theory. This distinction is very important because one of the main points of this thesis is the expectation that only finitely many low energy theories can be coupled to the same dimensional gravity. While on the other hand conformal field theories provide unbounded classes. Therefore, the right way to think about such theories is as theories living on the worldvolume theory of some collection of branes. In fact this is how these theories where first realized in [166]. An implication of this is that the local Calabi-Yau threefolds are not in general expected to be embeddable into compact threefolds. Therefore, thinking of the compactification as a com-

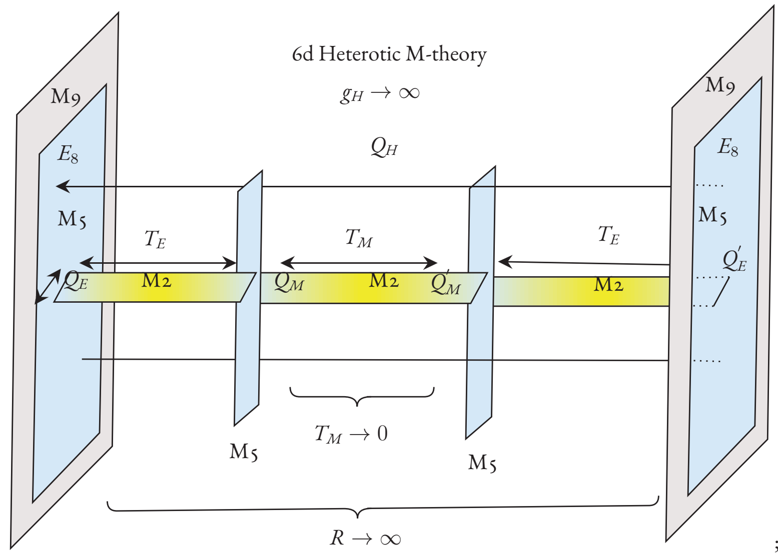


Figure 1.6: This figure depicts M-theory on the interval. The two M9 branes define the boundaries of the interval. Anomaly cancellation of each boundary requires an E_8 gauge symmetry living at each boundary. The addition of M5 branes allows to study the 6 dimensional theory on their worldvolume. When the M5 branes are near the boundaries their worldvolume supersymmetry is spontaneously broken to 8 supercharges. However, the expectation is that in the middle of the interval when the interval is large their supersymmetry is restored. Therefore, the M2 stretched between the two end-of-the-universe walls is the heterotic string Q_H enjoying the $E_8 \times E_8$ global symmetry when the radius is small. The M2 brane stretched between one M5 brane and the boundary M5 brane is the E-string Q_E of the 6d (1,0) theory. The M2 brane stretched between the two central M5 branes is the M-string Q_M associated with the 6d (2,0) SCFTs. For more details on the interplay of these strings[87,85].

pact threefold where the volume has been taken to infinity is not exactly the right way to think of them as this will not be possible for almost all of them.

The simplest examples of a 6d $\mathcal{N} = 1$ SCFT can be constructed in M-theory on the Hořava-Witten wall[12] created by compactifying the theory on $S^1/\mathbb{Z}_2$ together with some M5 and M2 branes as depicted in Figure 1.6.

The 6d $(1,0)$ SCFT in this picture is the E-string which is depicted by an M2 stretched between the boundary and the middle M5. The stretched direction corresponds to the tension of the string and hence when the brane comes close to the boundary the $T_E \to 0$. An equivalent string with the opposite orientation exists on the other side of the interval but in the large radius limit only one stays effectively massless. The E-string has no gauge symmetry associated to it and has an E_8 global symmetry inhereted from the symmetry on the boundary which is forced by anomalies.

On the other hand, the 6d $\mathcal{N} = (2,0)$ SCFT is obtained on the worldvolume of multiple M5 branes which preserve half of the supersymmetry. In Figure 1.6 these can be obtained locally as the worldvolume theory on the overlapping M5 branes in the middle of the interval in the limit that the tension of the M-string is zero and the radius is infinite. The radius being infinite is the effective decoupling of gravity as the 10d Planck mass is given by $M_{10}^8 = M_{11}^9 R$ and hence for a fixed M_{11} the ten dimensional Planck mass goes to infinity. Equivalently, we know that the heterotic string is the one stretched between the two M9 branes with the distance of the branes defining its tension and hence it becomes infinitely massive in that limit and hence the 6 dimensional Planck mass also diverges in the sense that the graviton modes of the heterotic string are decoupled.

In the more general case that the tensionless string caries some gauge symmetry due to the coupling $B \wedge tr(F^2)$, it will also carry instanton number and hence be a tensionless instantonic string.

The next example from string theory is the construction of the 5d minimal SCFTs which is related to this E-string example. In particular, one can take Type IIA on an interval where the boundaries will have two $O8^-$ planes. Consider the case where 7 D8 branes are on the left boundary and 9 on the other. In fact it is known that this is the maximum number of branes since they give the maximal rank. In subsection 1.3.1 the most general form of Type IIA on an interval was described which had non-zero mass deformation called the Roman's mass. The Roman's mass couples to the dilaton and hence gives a non-trivial profile of the dilaton. The case considered here can be thought of as a dilaton having a linear profile at some angle where in the most general case the coupling constants are finite at the boundaries. The strong coupling limit of the worldvolume theory of the D4 branes which is the small instanton of the D8 also corresponds to the strong coupling limit of the bulk coupling. Therefore, the coupling will diverge and the global symmetry of the D4 will enhance to E_8 as depicted in Figure 1.7. The extra U(1) factor that combines with SO(14) to enhance to E_8 is the global symmetry associated to the U(1) current of the 5d SCFT or in other words the instanton $U(1)_I$. In particular, when the coupling is infinite the instantons become massless. Therefore, the SCFT fixed point is characterized by massless instantonic particles and tensionless monopole strings.

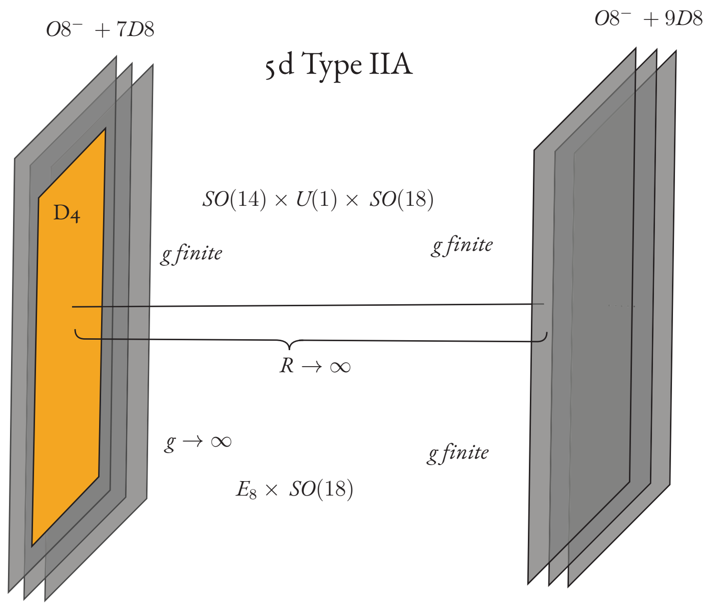


Figure 1.7: This picture shows a way to visualize the 5d SCFT constructed from a D4 brane on the boundary of this theory. The D4 brane has an su(2) gauge symmetry on the boundary since the interval is actually the Coulomb branch of the small instanton which is the D4 itself. Additionally, it has an extra U(1) global symmetry which corresponds to the extra instanton global symmetry on its worldvolume. The coupling of the theory can be finite exactly when the left endpoint has algebra D_7. When the coupling diverges which corresponds to turning the instantonic mass $m_0 = \frac{1}{g^2}$ then the SCFT fixed point is reached. This can be thought of as attaching an extra node to the D_7 Dynkin diagram. From the string theory point of view the gauge instanton on the D4 can be thought of as a D0 brane which has been exchanged with the gauge instanton. The mass of the D0 is exactly the mass of the instanton.

The idea of having massless dyons and monopoles near singularities was already well established in [169]. Reaching the SCFTs from the IR description is not so obvious as one has to integrate in light instantonic strings for the 6d theory and light monopole strings and massless instanton particles for 5d. This makes it clear that such theories are very hard to be studied from the point of view of quantum field theory.

In fact the two pictures of the 6d and 5d SCFTs are connected. If one takes one of the D8 branes

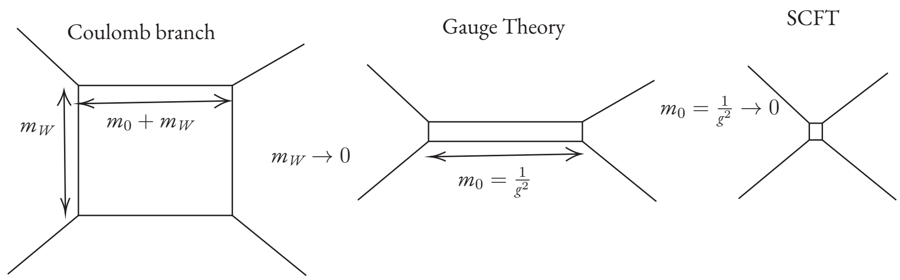


Figure 1.8: This figure shows the gauge theory limit and the SCFT limit of a 5d effective theory with an SU(2) gauge symmetry.

from the right boundary of Figure 1.7 to the left boundary then both sides have 8 D8 branes. Now the coupling does not depend on the Coulomb branch parameter and can be taken to infinity everywhere. This is exactly the M-theory limit of type IIA. From the point of view of the D4 brane the theory also should decompactify because there is no 4 brane in M-theory but it is rather the wrapped M5 brane. Therefore, the worldvolume theory of the D4 decompactifies to M5 and the E-string is recovered. This is exactly what one expects from the SCFT point of view. In the case that the number of flavors is $N_f = 8$ which are the 8 D8 branes, the coupling of the gauge theory does not depend on the any scalars and it is expected to decompactify to the 6d E-string. To see this note that the string stretched between the D4 and the D8 also decompactifies to the M2 brane stretched between the M5 and the boundary which is exactly the E-string. The extra rank can also now be understood as the KK U(1) coming form the reduction.

The 5d SCFTs have an additional visualization from (p,q) 5-brane webs as follows[4]: Consider two (1,0) D5 and two (0,1) NS5 forming a square along one direction. In order for the diagram to be supersymmetric at each edge the forces should cancel and hence a (1,1) brane is needed as seen in Figure 1.8. From this picture the D3 brane wrapping the square is the monopole string which becomes tensionless when the whole square collapses, the W-bosons become massless in the thin and long limit and the instanton particles become massless when the long side collapses too which leads to the fixed point as depicted in Figure 1.8.

One can increase the number of hypermultiplets by bringing in 7 branes from infinity and through Hanany-Witten transitions[94] arrive at $N_f = 7$ corresponding to the theory above. If one adds one more D7 from infinity then the theory has $N_f = 8$ which is expected to have a 6d UV fixed points. As was argued in[121] this 6d limit can be observed from the (p,q) 5-brane web too.

A key observation of the above discussion is that the 5d SCFTs descend from a 6d parent SCFT on a circle. This was conjectured[112] to be a more general aspect of 5d SCFTs where all are believed to come from 6d. In chapter 4 this argument will be reviewed and more evidence will be provided in addition to a full classification of all such parent theories.

One dual way to get such theories is from M-theory on a local Calabi-Yau threefold. In fact as was argued in[95] the (p,q) 5-brane webs can have a geometric interpretation and in fact it matches the toric structure of surfaces in these local threefolds. The rank one theories where classified to be $dP_{n<9}, \mathbb{F}_0$, for a review subsection B.1.3, where dP_9 corresponds to the 6d E-string. These are expected to be surfaces in the local threefold that M5 branes wrap and M2 branes wrap curves inside the surfaces. Then the Figure 1.8 can be thought of as collapsing some curves in the surfaces and the last step as collapsing the whole surface. In fact it is very reasonable that dP_9 would correspond to the E-string both because it can be thought of as $\frac{1}{2}K3$ but also because its triple intersection is zero which as will be reviewed is expected to imply an elliptic structure. It is a well known mathematical fact that dP_9 can be connected to $dP_{n<9}$ through successive blow downs corresponding to integrating out matter. In chapter 4 the classification will be described using F-theory on a local threefold times a circle with some twists. This also predicts the existence of M-theory duals to these twisted compactifications which generalizes the duality between M-theory and F-theory to allow for discrete Wilson lines around the circle.

"The quantum dualities, which are also known as S-duality or U-duality, extend the classical T-duality and lead to a beautiful and coherent picture of stringy dualities."

Nathan Seiberg[167]

2

Supergravity theories with 16 supercharges

As reviewed in the previous chapter the higher the amount of supersymmetries in a given theory, the more constrained the physics is. In fact in theories with 16 supercharges one only needs to specify the gauge group of the theory and the moduli space is known to be given by Equation 1.3 for non-chiral theories while chiral theories are completely fixed as seen in section 1.3. This means that it does not receive corrections and hence it is easier to ask questions about the UV features of gravity theories and their implications to the low energy action. However, from the bottom up perspective there is nothing constraining the rank of the gauge group and naively it seems unbounded while as seen in section 1.3, string theory can only realize theories with rank $r_G \leq 26 - d$. The main focus of this chapter will be to show that this bound is expected to be valid for any theory with 16 supercharges without relying on string theory. Additionally, aspects of dualities for theories with 16 supercharges will be discussed and the String Lamppost Principle(SLP) will be addressed.

2.1 Infinite Distance Limits

The idea of studying infinite distance limits is based on the expectation that in a theory of gravity new physics can be unraveled in these special corners. This is because such limits signal the break down of the current description of the theory and hopefully the emergence of a new one. Interesting limits also exist for quantum field theories which are expected to require the inclusion of new physics but they are expected to be at finite distance. Usually such limits are associated to strong coupling directions and chambers inside the moduli space of the theory which can be crossed at finite distance, such examples are studied in chapter 4. In a theory of quantum gravity we expect that infinite distance limits should still yield a well defined theory and from string theory we expect to uncover some duality scheme of different corners of the moduli space of the theory. For example, in theories with 16 supercharges such duality webs were given in Figure 1.4 and Figure 1.1. Generally, at infinite distance in the moduli space of a gravitational theory we expect, according to the distance conjecture (discussed in section 1.2), an infinite tower of massless states to emerge. Such a tower of massless modes signals the break down of the current description. Gravitational theories are expected to break down at a scale called the species scale[61], which is usually reached before the Plank scale. In string theory that is the string scale M_s, where we expect string effects to become important. The only theory that we expect to break down at M_{pl} is M-theory as it has no other scale in the theory or a similar theory if it exists in other dimensions e.g.[141]. More details on the relation between the species scale and the distance conjecture can be found in[187]. Similarly, one can motivate the expectation of the theory breaking down in that limit from the point of view of global symmetries. At infinite distance limits where the coupling of the theory is weak, approximate global symmetries emerge and they become exact at $g = 0$. However, we know that a theory of quantum gravity should have no global symmetries and hence we expect that the current description of the theory breaks down and some other should becomes relevant. Therefore, we see that in a theory of gravity infinite distance limits are rather special and they can unravel new physics for us.

Usually the only such limits we study are BPS because supersymmetry protects their masses and we can study them across the limit we are considering. In this chapter we will analyse both BPS and Non-BPS such limits.

Definition

A *BPS* limit is an infinite-distance limit in the moduli space in which the leading light tower compared to the Planck scale is BPS.

A *non-BPS* limit is an infinite-distance limit in the moduli space in which there are no light BPS particles compared to the Planck scale.

Sharpened Distance Conjecture

An infinite-distance limit in the moduli space has an emergent light tower of states with masses $m^2 \sim e^{-a\hat{\varphi}}$ where the coefficient $a = \frac{1}{\sqrt{d-2}}$ if the tower is a string tower and $a > \frac{1}{\sqrt{d-2}}$ if the tower is a KK tower.

The sharped distance conjecture[71] tells us that the coefficient for the distance conjecture satisfies $a \geq \frac{1}{\sqrt{d-2}}$. As a stronger observation one sees that in fact when the bound is saturated the leading tower corresponds to a string tower, while in all other cases it is a KK tower signaling a decompactification. The conjecture does not clarify the properties of the string that becomes light. However, the emergent string conjecture[132] predicts a weakly coupled string theory at that limit. In particular, this means that at tree level there is a gravity subsector that is weakly coupled but not all physics of the worldsheet theory need to be weakly coupled. For example, there are F-theory limits that correspond to the heterotic string with multiple NS5 branes[131]. This implies that in these case it is not a simple task to relate the worldsheet fields to the bulk fields. However, crucially as will be discussed in the next subsection for 16 supercharges a perturbative bulk coupling seems to have a simple correspondence with the worldsheet fields.

Emergent String Conjecture

The asymptotic states of the Sharpened Distance Conjecture in the case of the string tower correspond to a weakly coupled string theory. In particular, the graviton is expected to be a mode of the string.

The above behaviors teach us about the fate of these theories in these extreme limits. We expect that each theory should have a well defined description beyond these limits probably in terms of a dual effective description where the modes of the first theory become part of the weak coupling limit of the dual theory.

2.1.1 Moduli representation

In the previous section, we studied the general expectations of infinite distance limits in a theory of gravity. Moreover, in chapter 1 we discussed that in a theory of quantum gravity we expect the charge lattice to be complete. In fact for theories with 16 supercharges we may also assume that when a state can be BPS it does have such a representative. This gives rise to the BPS completeness hypothesis. In general we do expect that at least some sublattice should be BPS and hence would be interesting to also think about relaxing the condition to a sublattice BPS completeness in the future.

In section 1.3 we reviewed basic properties of the supergravity with 16 supercharges and its moduli space. From eq. (1.4), we can see that the gauge couplings in theories with 16 supercharges are controlled by the entries of the matrix M which represents the scalars in the vector multiplets. Since the gauge coupling also affects the gauge charges, the charge lattice depends on M as well. The 16 supercharges almost fix the dependence of the charge lattice on the moduli (see Appendix D in [15]).

The action is invariant under the following transformation of M

$$M \to \Omega M \Omega^T; \; \Omega \in O(10-d, r), \tag{2.1}$$

This transformation also acts on vectors of the charge lattice $Q \in \Lambda$ as

$$Q \to \Omega Q. \tag{2.2}$$

If we identify the first $10 - d$ entries of Q as $\vec{Q}_R$ and the rest with $\vec{Q}_L$, then the quantity $Q^2 = Q_R^2 - Q_L^2$ is invariant such Ω meaning that it acts isometrically on the lattice.

We are interested in infinite distance limits [1] in the moduli space along geodesics. Such limits are characterized by boosts $\lim_{t\to\infty} \Omega(\gamma)M\Omega(\gamma)^T$ where $\Omega(\gamma) = \exp(\gamma U)$. The geodesic we are considering needs to be a global geodesic, which are geodesics that are the shortest path between any two points on them[2]. Note that adding a rotation piece to the boost will lead to spiraling trajectories in the moduli space that are locally geodesic, but not globally. It is easy to show that an infinite boost always takes a null line to the origin.

Therefore, for any infinite distance limit of M, a line of particles that satisfy $|Q_R| = |Q_L|$ and are in the plane of boost will go to the origin of the charge lattice. Let us call that line in the charge lattice, the *line of boost*.

[1] Due to the large number of supercharges we know that the moduli space will not receive corrections.
[2] For more on the definition of the global geodesic see Sec. 2.1.2.

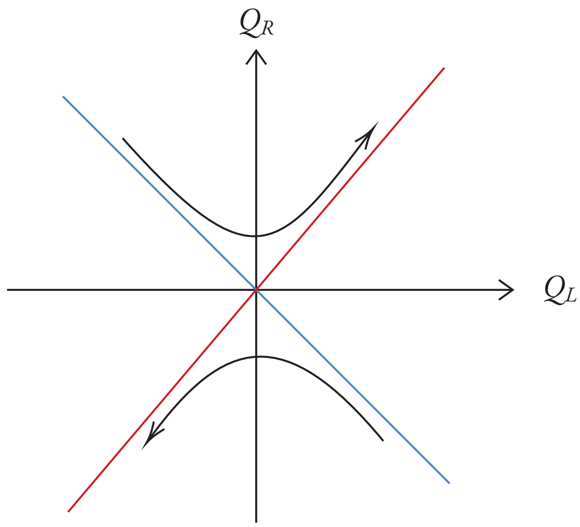


Figure 2.1: In any infinite boost of the charge lattice, a line in the light-cone ($|Q_L| = |Q_R|$) goes to the origin while the other points go to infinity. The line that goes to the origin is colored blue.

We would like to study the BPS states along this limit. The BPS condition in theories with 16 supercharges is [170]

$$m = |Q_R|. \tag{2.3}$$

This is because the mass of the particles is associated with the graviphoton charges as it represents the gravitational charge.

This implies that if there is BPS particle with $|Q_R| = |Q_L|$ on the line of boost, that particle will become light in the infinite distance limit. Of course another set of massless particles are those satisfying $|Q_R| = 0$ which identically zero. These corresponds to $Q^2 = -Q_L^2$ states associated to states like gauge bosons which satisfy $Q^2 = -2$. We note that such states become massless only at special points in the moduli space where $|Q_R| = 0$, such states can be understood as occupying the spacelike directions of 2.1. Infinite towers of these states occur only for conformal field theories at finite distance. From the bottom up perspective this might not be as obvious. However, we do know that only states with $Q^2 \geq 0$ can be black holes because of consistency of the entropy formula section 1.9 or the extremality bound for black holes. This means that any states with $Q^2 < 0$ can be regarded as "non-gravitational" as we saw for the gauge bosons. Now it is clear that an infinite tower of such states signals a local conformal theory. However, in non-chiral theories with 16 supercharges in $d > 4$ we have no such fixed points and therefore such infinite towers should not exist.

This means that the only BPS states that become massless as infinite distance are those with $m = |Q_R|$ and $|Q_R| = |Q_L|$. The BPS completeness hypothesis [119,91] which states that in theories with 16 supercharges, any charge that is allowed to be BPS, is occupied by a BPS state ensures that such a state exists[3]. The black hole extremality condition [170] is given by:

$$m \geq |Q_L|. \tag{2.4}$$

The above inequality follows from the classical extremality condition which can receive higher derivative corrections [115] or quantum corrections [1].

$$m \geq |Q_L| + \mathcal{O}(m_P^2/M)_{\text{corrections}}. \tag{2.5}$$

As long as the corrections on the right side are non-positive, the a BPS particle with $m = |Q_R| = |Q_L|$ is allowed. But the non-positivity of the corrections is guaranteed by the mild version of the Weak Gravity Conjecture (WGC) [10]. Therefore, based on WGC and BPS completeness, we argue that for any point of the charge lattice on the line of boost, at least one BPS particle or black hole of that charge exists. It could be the case that the line of boost does not intersect with the lattice at all. In that case, all the BPS particles become massive and the corresponding limit is non-BPS. However, if the line of boost intersects with a BPS particle, the BPS particles at the intersection will become

[3] In fact if we interested in non-BPS states we do not even need this condition as if the particles are not BPS we are automatically at a non-BPS limit

massless at the infinite distance limit.

Let us make the above discussion more explicit. Consider the following infinite distance limit generated by a boost.

$$\Omega_{a,i}(\gamma) = \begin{bmatrix} e_{aa}\cosh\gamma & e_{ai}\sinh\gamma \\ e_{ia}\sinh\gamma & e_{ii}\cosh\gamma \end{bmatrix}, \tag{2.6}$$

where e_{pq} is a matrix with one non-zero element of 1 in row p and column q. Then the two axes acted upon are $a \in \{1, \cdots, 10-d\}$ axis and $i \in \{1, \cdots, r\}$ axis.

An infinite-distance limit in the moduli space therefore corresponds to $\gamma \to \pm\infty$. Now, take the action of Ω_{ai} on the charge vector $Q = (Q_R, Q_L)$.

Under the action of $\Omega_{a,i}$ on M, a charge vector $Q = (Q_R, Q_L)$ goes to

$$|Q_{R,a}| \mapsto |Q_{R,a}\cosh\gamma + Q_{L,i}\sinh\gamma| = |\frac{1}{2}(Q_{R,a} + Q_{L,i})e^{\gamma} + \frac{1}{2}(Q_{R,a} - Q_{L,i})e^{-\gamma}| \tag{2.7}$$

Depending on the sign of γ in the infinite distance limit, the line of boost is

$$\begin{aligned} Q_{R,b} &\propto \delta_{ba} \\ Q_{L,j} &\propto \delta_{ji} \\ Q_{L,i} &= \pm Q_{R,a}. \end{aligned} \tag{2.8}$$

Suppose we have BPS particles on both lines of boosts. As we take $\gamma \to \pm\infty$, using $m = |Q_R|$ we find that $m \sim e^{|\gamma|}$, unless we have $Q_{R,a} = \mp Q_{L,i}$, in which case the masses scale as $m \sim e^{\mp\gamma}$. Based on the sign of γ such particles become massless in the limit $\gamma \to +\infty$ or $\gamma \to -\infty$. So in the presence of BPS particles on lines of boosts, any infinite distance limit in the moduli space of the vector multiplets could yield light BPS particles.

However, as one can see from (1.5), the gauge couplings also depend on the dilaton. Therefore, the gauge charges (and hence masses) of the BPS particles also depend on the dilaton as $m \propto e^{\varphi}$. However, the expression in (1.5) is in the string frame. After going to the Einstein frame, we find the following mass relation in terms of the canonically normalized dilaton $\hat{\varphi}$.

$$m \simeq e^{\pm\gamma_{a,i}} e^{\frac{d}{2\sqrt{d-2}}\hat{\varphi}} m_p. \tag{2.9}$$

We conclude that even if there are BPS particles on the lines of boost, all of them will become heavy in the limit $\gamma \to \pm\infty$ if we also take $\hat{\varphi}$ to infinity such that $\left|\hat{\varphi}\right| / |\gamma| \geq (2\sqrt{d-2}/d)$. Note that in these limits, there is no light BPS string.

To conclude, we showed that any infinite distance limit $|\gamma|, \hat{\varphi} \to \infty$ that $|\hat{\varphi}|/|\gamma| \geq (2\sqrt{d-2}/d)$ is a non-BPS limit. The complement of these limits are BPS.

2.1.2 Tensionless string or KK tower?

In this section, we will show that most infinite distance limits decompactify. In particular, we will do that by arguing that the are only countably many infinite-distance limits in which the leading light tower is a string tower. We will do that by firstly showing that the number of inequivalent strings is countable and then we will argue that every string limit has a unique inequivalent (up to dualities) direction in the moduli space.

Step 1: countable strings

We will assume that there are uncountably many inequivalent strings each of which has a world-sheet theory which is trustable in some region of the moduli space $\mathcal{M}$. Will argue that this is not possible and using the finiteness of black hole entropy.

Suppose we have uncountably many string limits labelled by $\alpha \in I$ where I is uncountable. Assuming that there is a distinct point p_α in the moduli space such that in a string limit the corresponding string description is valid in a neighborhood of radius ε around p_α. We can write the moduli space as the countable union of $\cup_{n\in N} D_n$ where D_n is a closed disc with radius n/κ with respect to the canonical metric. Assuming the moduli space is completely regular and Hausdorff, the closed discs admit Stone–Čech compactification $\bar{D}_n$. Since $\bar{D}_n$ are compact and have finite volume, we can cover them with a finite number of neighborhoods each with a diameter less than ε. Therefore, we can cover the moduli space $\mathcal{M} \subset \cup_n \bar{D}_n$ with a countable number of neighborhoods of diameters less than ε.

Now that the moduli space is covered by countably many neighborhoods $\bar{D}$, one of them must include uncountably many of the points $\{p_\alpha | \alpha \in I\}$. Let us call that neighborhood U. Therefore, there are uncountably many distinct strings in U. Now we show that there exists a finite cutoff Λ_c, such that uncountably many of the strings have mass scales below Λ_c.

Suppose V_n is the subset of the points p_α in U such that the mass scale of the string α is less than nm_P in U. Since $\cup_{n\in N} V_n$ contains all the uncountably many strings in U, for some ℓ, the set V_ℓ must be uncountable. Therefore, there are uncountably many strings in the neighborhood U that have mass scale below $\Lambda_c = \ell \cdot m_P$. However, this violates the finiteness of the Bekenstein-Hawking entropy, and therefore must not be allowed. Therefore, our original assumption is incorrect, and the number of inequivalent strings must be countable.

Step 2: rigidity of string limits

In this step, we want to show that each one of the countable many string limits forms the leading tower in most one direction in the moduli space. We show that if one makes a small change in the direction of the infinite distance limit away from a string limit, the leading tower is no longer a string tower. But first, let us define a useful notion to study the infinite distance limits. We define a global geodesic in the moduli space to be a geodesic which is the shortest path between any two points on it. We can think of infinite distance limits as global geodesics. This refinement excludes infinite geodesics that are stuck in a compact subspace. Note that not every infinite geodesic that starts at a given point of the moduli space is a global geodesic. For example, in type IIB string theory, for any point p in the moduli space, there is a unique global geodesic that starts at p. This is because, for any two BPS strings, the limits where they become tensionless are mapped to each other via duality. Changing the angle of the initial velocity of the geodesic will create a geodesic which either does not get to infinity or winds around the fundamental domain of $SL(2,\mathbb{Z})$ and is not a global geodesic. Therefore, even if the new path is locally a geodesic, it is not a global geodesic. Due to this, we would say the type IIB theory has a unique inequivalent infinite distance limit. In the rest of this section, we will show that if we make a sufficiently small change in the direction of a string limit, the resulting limit cannot be a string limit. We assume that such a sufficiently small variation can change one global geodesic to another. Otherwise, if the limit is rigid, then we trivially know that there is no nearby string limits.

Sharpened distance conjecture tells us that in a string limit, the tension of the string will go like

$$\sqrt{T} \propto m_p e^{-\frac{\kappa}{\sqrt{d-2}}\hat{\varphi}}, \tag{2.10}$$

where $\hat{\varphi}$ is the canonically normalized distance in the field space. If we make a sufficiently small change in the direction of the infinite distance direction that combines the modulus $\hat{\varphi}$ with another spacetime modulus, the distance travelled in the moduli space per change of $\hat{\varphi}$ will increase[4].

[4] We assume that $\hat{\varphi}$ is a linear spacetime modulus as opposed to something like $\hat{\varphi} = \sqrt{\varphi_1^2+\varphi_2^2}$ which would give the same rate of change of φ in every infinite distance limit in (φ_1, φ_2) plane. By linear modulus, we mean that if a global geodesic at point p, which is in the asymptotics of the moduli space, has an infinitesimal angular separation $\partial\theta$ from $\partial_{\hat{\varphi}}$ then $\Delta\hat{\varphi}/\Delta l$ converges to $\cos(\theta)$ where Δl is the canonical distance. Note that this follows from the emergent string conjecture which conjectures that the string is fundamental. In that case, the modulus φ can be shown to always be a linear spacetime modulus. This follows as a corollary of the argument for the sharpened distance conjecture from the emergent string conjecture in [3].

This would lead to a decrease in the coefficient $1/\sqrt{d-2}$ in the exponent in (2.10). Therefore, the sharpened distance conjecture implies that the string (2.10) can no longer be the leading tower anymore. So we find that any worldsheet theory has at most one infinite distance limit where the string is the leading tower which corresponds to the direction in which the tension decays the fastest.

By putting steps 1 and 2 together, we conclude that there are only countably many infinite distance limits in which the leading tower is a string tower. This is a powerful result because it means that even though the direction we have considered has a tensionless string as the lightest tower, the nearby infinite distance limits must be decompactification limits. For our purposes, this will mean that if there is a non-BPS limit with a string leading tower, any sufficiently close non-BPS limit decompactifies.

2.2 T-duality from bottom up

In the previous section we discussed BPS and Non-BPS infinite distance limits. In fact we made clear that if the limit decompactifies the theory should descend from a higher dimensional theory on some compact space with loosely speaking some S^1 cycles if it is BPS and without any S^1 cycles if it is Non-BPS. An example of a one dimensional decompactification is a circle and an interval respectively.

In this section we will focus on T-duality associated to BPS limits. A simple example is the heterotic shelf T-duality in the absence of any Wilson lines. An example of the T-duality which is not a self-duality is the two inequivalent rank 1 theories we discussed in 9 dimensions, which are T-dual of one another section 1.4.

Summary of steps

- Step 1: Consider a circle compactification with a finite but small coupling.
- Step 2: Consider the limit where $R \to 0$.
- Step 3: Argue the theory decompactifies to a d dimensional theory with 16 supercharges.

We consider a d-dimensional non-chiral theory with 16 supercharges and a gauge group of rank r_G in the low energy effective action on a circle S^1. The moduli space of this theory was written in (1.3) and upon the circle compactification and in the absence of Wilson lines on the circle it only acquires an extra factor $SO(1,1)$ coming from the radius of the circle. This subspace of the moduli space associated with the vector multiplets is then given by

$$\frac{SO(r_G, 10-d)}{SO(r_G) \times SO(10-d)} \times SO(1,1) \,, \tag{2.11}$$

Additionally, the reduction implies the inclusion of:

- Gauge field $B_{\mu\theta}$ and the wound string charged under this gauge field
- Gauge field $G_{\mu\theta}$ and KK momentum modes charged under this gauge field

The fact that a string exists is guaranteed by completeness of spectrum as discussed earlier.

The supersymmetry algebra implies that BPS states charged under the momentum and winding modes have central charges $M_n = n/R$, $M_w = wR$ where (n, w) denote the momentum and winding numbers. Now consider the radius of the circle very small. Then one would expects the dual theory to at least have $(d-1)$ dimensional Lorentzian symmetry. We will now argue that the theory should actually end up being d dimensional in the limit $R \to 0$.

The general statement of the distance conjecture[155] is that at infinite distance in moduli space an infinite tower of light states emerges. A stronger version of this conjecture states that when this tower appears then there exists a dual weakly-coupled description with its basic modes comprised of the states in the light tower. In the case at hand as $R \to 0$ the supersymmetry algebra implies that the winding modes are getting light.

These winding modes should be the dual description of some elementary excitations of a dual theory. On the other hand we have found that the moduli space of the theory whose masses do not depend on R includes

$$\frac{SO(r_G, 10-d)}{SO(r_G)}\,, \tag{2.12}$$

which predicts that the theory must have at least d dimensions, because of the classification of scalar moduli space of theories with 16 supercharges. Thus the light modes which used to be winding modes of the original theory, must now be part of the weak coupling limits of this d-dimensional theory. In other words, they must be the momentum modes of this theory.

Said differently we have argued that every non-chiral theory enjoys T-duality. Of course this argument does not predict whether the T-dual theory is different or the same as the original theory, compatible with the fact that both versions do occur in string theory. Note that this implies that the singly wound string of the original theory in its ground states should carry the same quantum numbers as a graviton multiplet with one unit of momentum around the circle, since it is dual to it. In particular it is a massive state in a $d-1$ dimensional theory with maximum spin 2. We will use this fact in the next subsection.

We note that these arguments depend on supersymmetry very strongly since even in 8 supercharges we know that the moduli space receives corrections which can change the decomposition of the moduli space metric as well as the distances. A simple example is the case of 5d $\mathcal{N}=1$ to 4d $\mathcal{N}=2$ where one sends the radius to zero size, instanton corrections can bring the distance of our limit to finite.

Our only assumption beyond the Distance conjecture to reproduce this arguments is really that a dual theory exists. It could have been that such a limit signals the break down of the theory without any other fundamental description beyond that point. In some sense this could correspond to some cut off.

This discussion can also be rephrased in the language of the emergent string conjecture [132] or sharpened distance conjecture [71] because we consider a simple compactification on S^1 and the leading tower is BPS in both the large and small radius limit. The amount of supersymmetry fixes the masses and tensions of BPS objects and hence the exact coefficient of the distance conjecture can be computed which shows that in fact it is a decompactification.

It would be interesting to study also the T-duality in theories with non-BPS leading towers. In fact we already know from the previous section that most limits decompactify and we will study later on the theory that they decompactify to. So we could start from a theory in d-dimensions on a circle and tune the coupling to infinity so that the BPS states are heavy and the light KK tower is Non-BPS in that case the resulting decompacitifcation involves a Non-BPS tower at some fixed coupling so it should be a manifold with a boundary. This tells us that T-duality is a feature of the theory and it also describes where we land in certain cases. In the Non-BPS case we can see that we end up necessarily to a different theory as we expect that supersymmetry was broken during the compactification. But in decompactifications that the tower is BPS it is not so clear if the theory is self-dual or not. In [15] it was shown that in fact T-duality changes the chirality of the theory in theories with 32 supercharges and hence one expects to land in a different theory. Other bottom up aspects of dualities were also discussed in that work.

2.2.1 Derivation of the bound on the rank

As we discussed in Appendix A the worldsheet theory of the BPS strings is a $(0, 8)$ CFTs. We also argued that the current algebra of the bulk gauge symmetry is in the left sector. In particular, unitarity of the string worldsheet requires that the current algebra of simple non-Abelian group G with level k in the 2d CFT satisfy the following relation (see, e.g.,[56])

$$c_G = \frac{k\,\mathrm{dim}G}{k + h^\vee}\ , \tag{2.13}$$

where dim G is the dimension and $h^\vee$ is the dual-Coxeter number of group G respectively. For a $U(1)$ current algebra, the central charge contribution is $c_{U(1)} = 1$. This leads to the following constraint on the left-moving central charge[119]:

$$\sum_i c_{G_i} \le \tilde{c}_L\ , \tag{2.14}$$

where the sum for i is taken over the left-moving currents.

On the Coulomb branch where the bulk gauge symmetry is broken to Abelian groups including $U(1)^{r_G}$, we can further simplify the bound as

$$r_G \le \tilde{c}_L = 24\kappa + 2 - d\ . \tag{2.15}$$

where κ is associated to the t'Hooft anomaly of the R-symmetry. If this bound is violated, the anomaly inflow from the bulk gravity theory cannot be cancelled by the anomalies of a unitary CFT on the string. Hence, a consistent bulk gravity theory involving BPS strings must satisfy this bound. This is because as we saw in the appendix the addition of the string charge modifies the Bianchi identity and hence requires the anomaly inflow mechanism to cancel those anomalies.

From eq. 2.15 we see that if a bound for $\varkappa$ exists then the rank of the gauge group is bounded. In particular, we will argue that for a gravity theory with 16 supercharges that has a well-defined T-dual theory necessarily has $\varkappa < 2$.

For this we consider a string with $Q = 1$ wrapped around a circle of radius R and study its ground states. Consider first the ground states of the wound string in the Ramond sector of the interacting CFT. As has been argued in [134] the maximum charge for the $U(1)_R$ spectrum in the Ramond sector for this theory is given by $c_R/6$ the central charge of the current, i.e. $2(\varkappa - 1)$ after removing the center-of-mass contribution, which means that the spin is less than or equal to $(\varkappa - 1)$. Moreover as argued there this maximum range is actually realized by the spectral flow of the vacuum state of the NS sector to the R sector[5].

Now we use the fact that the spectrum of the singly wound string already includes spin 2 states arising from the center-of-mass degrees of freedom in the right-moving sector. As we argued, in addition, the internal degrees of freedom from the interacting CFT contains right-movers carrying charges under the $SO(2) \subset SO(d-2)$ rotational symmetry. This simply means that the ground states of the wound string will include a state with a net spin bigger than 2 when $\varkappa - 1 > 0$. This higher spin state is generated by a tensor product of the spin 2 states in the center-of-mass spectrum and the internal right-moving state carrying $SO(2)$ Lorentz charge $\varkappa - 1$. On the other hand, a conseqence of T-duality is that the ground states of the wound string should have the same quantum numbers as the gravity multiplet in the dual theory. So T-duality cannot hold if $\varkappa > 1$ due to the higher spin states. Therefore, we conclude there are only two possibilities

$$\varkappa = 1 \text{ or } 0 \,. \tag{2.16}$$

[5]The argument in [134] is mainly in the context of (2,2) supersymmetric theory, but that can be easily adapted to the $(0, 2) \subset (0, 8)$ being considered here, using the integrality of charges of the $U(1)_R$, which follows from the fact that it is twice the spins in physical space which are integer or half-integer.

When $\kappa = 1$, the central charges of the 2d CFT on a single string are $(c_L, c_R) = (24, 12)$. The unitarity of this string probe when coupled to the bulk gravity imposes a novel constraint on the rank of the bulk gauge groups,

$$r_G \leq 26 - d\,. \tag{2.17}$$

The case $\kappa = 0$ is not allowed for the $(0, 8)$ case as can be seen from the fact that there are always the center of mass modes and thus we cannot have $c_L = c_R = 0$ which would be a consequence of (A.8). However, the $\kappa = 0$ case is in fact forced on us, as we will discuss next, for the case when the supersymmetry is enhanced in the IR to $(8, 8)$.

We now turn to the case with $(8, 8)$ supersymmetry. In this case we still have

$$c_L - c_R = 12\kappa\,,$$

but in addition we have two $U(1)_R$ currents, one left- and one right-moving with anomaly coefficients k_L, k_R, satisfying

$$k_L - k_R = 4\kappa\,,$$

with $c_L = 3k_L, c_R = 3k_R$. By the T-duality argument we just used which implies that we should have no additional spins other than those coming from the center of mass, we learn that $k_L = k_R = 4$ and $\kappa = 0$. Moreover, this implies that $c_L = c_R = 12$ and subtracting the degrees of freedom coming from the center of mass, we get the bound on the rank of the gauge group $r_G \leq 10 - d$. This bound is stronger than the one coming for theories with $(0, 8)$ supersymmetry (2.17). Taking this into account we learn that any gravity theory with 16 supercharges and rank beyond the bound in (2.17) is inconsistent and therefore belongs to the swampland.

Both the $(0, 8)$ and $(8, 8)$ supersymmetric cases on BPS strings are realized in string theory. In particular, the toroidal compactifications of heterotic string realizes the $(0, 8)$ case, consistent with the fact that it has $(c_L, c_R) = (24, 12)$. An interesting example of this is the rank 9 theory in $d = 9$. In this case we can have at the CHL point the $E_8 \times U(1)$ matter gauge symmetry. The central charge of the left-moving degrees of freedom comes from the center of mass contribution of ($c_1 = 7$) plus E_8 at level $k = 2$ ($c_2 = 15\frac{1}{2}$) plus the $U(1)$ ($c_3 = 1$) and a left-over piece which is an Ising model ($c_4 = \frac{1}{2}$) leading to $c_L = 24$.

The case with $(8, 8)$ is also realized: One such example is the IIA limit of M-theory compactified on Klein bottle, called the AOA theory, introduced in[6]. A BPS string in the AOA theory enjoys the $(8, 8)$ symmetry enhancement. The 2d theory on a single string consists of bosonic fields (A_0, A_1, A_2, Y^i) with $i = 1, \cdots, 7$ and a pair of fermionic fields $\psi^\pm$ where A_0, A_1 are 2d gauge fields and A_2 is a compact (but non-chiral) scalar and Y_i are non-compact scalars, and $\pm$ denotes the 2d chirality[6]. The $SO(7)$ Lorentz symmetry acts on both (anti-)chiral fermions $\psi^\pm$ as well as Y_i. These fields form a free $(8, 8)$ multiplet for the center-of-mass degrees of freedom of the string. From the matter content, one can read off the central charges as $c_R = c_L = 12$. We also notice that the 't Hooft anomalies for the $U(1) \times U(1)$ gauge symmetry receive contributions $+1$ and -1, respectively, from the left- and the right-moving components of the compact scalar A_2, which precisely cancel the anomaly inflow for $U(1) \times U(1)$ gauge symmetry. Toroidal compactifications of this theory give rise to other examples with $\mathcal{N} = (8, 8)$ enhanced supersymmetry in $\kappa = 0$ gravity theories.

Note that the upper bound on the rank (2.17) is saturated by the toroidal compactifications of 10d heterotic strings to lower dimensions. Hence the above bound on the rank provides strong evidence for the completeness of the string lamppost principle.

2.2.2 Non-BPS decompactification

In the previous section we analyzed a very particular type of infinite distance by consider the theory on a circle and taking the infinite distance limit of small radius. The amount of supersymmetry was enough to bound the total rank of the theory since the theory enjoyed T-duality.

Moreover, in the first two sections of this chapter we found that infinite distance limits where the leading tower is non-BPS can still have some decompactification limits that we can follow. In the case of nine dimensions we can explicitly study the possible decompactification limits. The 9d theory can only decompactify to either a 10d or an 11d which is the highest dimension for supergravity. Therefore, there are two cases to consider.

11d Backgrounds

We firstly consider the decompactification to 11d supergravity, which is unique at low energies[49,75]. We note first that the compact dimension cannot be disconnected, as if it were, we would find two copies of the supergravity multiplet in lower dimension. The only boundary-less smooth two-dimensional Ricci-flat manifolds are torus, Klein bottle, and the ones with at most one non-trivial cycles are Möbius strip or cylinder. However, all of the above manifolds have an S^1 piece that the KK tower along the S^1 piece will be a light BPS tower.

One could also consider singular combinations of such backgrounds by gluing multiple components along the boundaries with some degrees of freedom living on the boundary (for example as in Figure 2.2). However, there is no evidence of such exotic backgrounds in string theory. This question should be addressed also beyond string theory, since whether such a boundary theory exists is a UV question. We provide a bottom-up argument based on the classification of 5d SCFTs for why such exotic backgrounds do not exist.

Let us consider the small instantons of the 9d theory (for definition see 2.3). Using the BPS completeness hypothesis[119], we assume BPS small instantons exist. Furthermore, the strong version of the cobordism conjecture in[138] implies that the moduli space of these small instantons is connected.

Now, a generic BPS 4-brane in 9d supergravity will carry a 5-form charge[6]. However, the 5-form gauge potential in 9d must be the KK reduction of the dual of the 3-form gauge potential in 11d. Therefore, the 9d small instantons must correspond to wrapped BPS 5-branes in 11d supergravity picture. When the 5-brane approches the boundary, the small instanton will approach the boundary of its moduli space.

Now, from the rank 1 classification of 5d SCFTs, we know that any local piece of the Coulomb branch of an SCFT or a free theory must either look like an open interval which has at most one closed end [166,59,147,24,30]. The one dimensional Coulomb branch of an SCFT has the local structure of $\mathbb{R}/\mathbb{Z}_2$ [166] but not a star shaped singularity. In other words, such exotic compactification are not permited. Thus, the compact dimensions (corresponding to the moduli space of a 5d theory in its IR) can have at most two boundaries, and the only singularities/boundaries allowed in the moduli space of 5d theories (either free or SCFT) are $\mathbb{R}/\mathbb{Z}_2$.

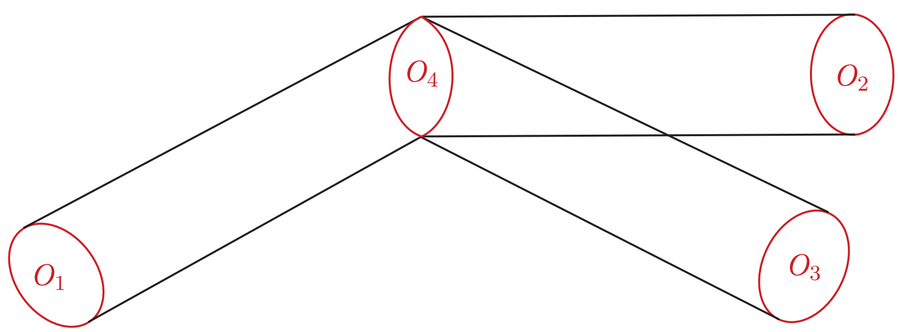


Figure 2.2: An exotic 11d supergravity background on a 2d space resulted from identifying a boundary of three cylinders. The existence of such a background depends on the existence on the appropriate boundary theories O_i, which is a UV question. For example, there is no evidence for the existence of such boundary theories in string theory.

[6]We will give a detailed argument for this point in 2.3, but is a summary of the argument. The number of gauge instantons and branes with 5-brane charge is not independently conserved due to gauge symmetry. Therefore, they must be transformable to each other and share a moduli space.

10d Backgrounds

Suppose our theory decompactifies to a 10d theory. Again we need to exclude the circular compactification since it leads to a BPS tower.

The other possibility is that the local physics is described by a supergravity with 32 supercharges, but the global structure of the compact dimension breaks half the supersymmetry. For example, at the massless level, we can have accidental supersymmetry which is broken by the massive states. Such 10d low-energy effective field theories can type II supergravity theory on a 1d manifold with boundaries. Now let us consider type II supergravities on 1d manifolds with boundaries. Since the type IIB supergravity is chiral, putting it on a manifold with a co-dimension 1 boundary would break all supersymmetry, whether or not they are accidental.

Type IIA supergravity however allows BPS end of the universe walls and BPS domain walls since it is non-chiral. Therefore, the theory can decompactify to a type IIA background on a union of circles and intervals that are joined via some boundary defects and some BPS domain walls on the inside. Even though the supergravity allows for such a background, their existence depends on the spectrum of the non-perturbative 8-branes which is a UV information.

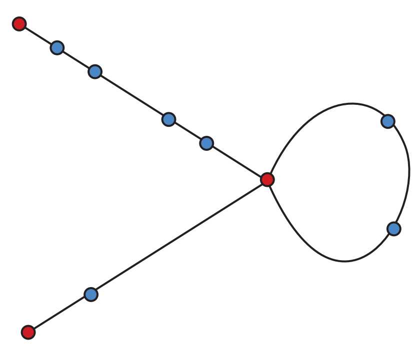


Figure 2.3: The internal geometry for an exotic type IIA background. The red points are boundary theories and the blue points represent BPS domain walls. We provide a bottom-up argument from the classification of 5d SCFTs that rules out such backgrounds.

Similarly as above the one dimensional Coulomb branch of the small instanton can have a local singular structure of $\mathbb{R}/\mathbb{Z}_2$ [166] but not a star-shaped singularity with more than two legs. In other words, star-shaped moduli spaces are not allowed. Therefore, the only candidate for a type IIA background with a boundary is type IIA supergravity on an interval.

A more general background is the massive type IIA supergravity background on an interval which is nothing other than type I' supergravity. In the subsection 1.3.1, we defined and reviewed the general properties of type I' supergravity backgrounds without relying on string theory. In the next subsection, we use the type I' supergravity action to express the 9d moduli discussed in section 2.1.1 in terms of the type I' parameters.

2.2.3 Matching 9d and type I' moduli

Our following study is purely field theoretic and does not rely on string theory. For example, the number of branes are left arbitrary to be determined by the Swampland principles which capture the restrictions imposed by the consistency of a UV completion. Having said that, our calculations must parallel the existing results in string theory [161] when the number of branes match.

The type I' action eq. (1.6) is expected to be exact for BPS configuration where no light field (except the non-trivial profile of dilaton) is excited. Therefore, the dependence of the action on the position of the 8-branes as 9d scalar fields is protected by supersymmetry. This allows us to find the scalar kinetic terms of the 9d action and match them with the moduli in eq. (1.6). Let us first start with the 9d dilaton φ. If we dimensionally reduce the action (2.1.1) we find

$$e^{-2\varphi} = \mu_8^{\frac{7}{2}}(2)^{-\frac{7}{4}}(4\pi)^7 C^{-10/3}\left(\int_0^{2\pi} dx^9 w(x^9)\right)^{-5/2}, \tag{2.18}$$

where

$$w(x^9) = 3^{1/3}2^{-1/6}\left[\mu_8 B(x^9) - x^9\nu_0(x^9)\right]^{1/3}. \tag{2.19}$$

.

Moreover, by comparing the gauge actions we find that

$$M_{MN} = \\ S \begin{bmatrix} (2\pi R)^{-2} & -(8\pi^2 R^2)^{-1} A_k A^k & -\frac{A_i}{(2\pi R)^2} \\ -(8\pi^2 R^2)^{-1} A_k A^k & (2\pi R)^2 + (16\pi^2 R^2)^{-1}(A_k A^k)^2 + A_k A^k & ((8\pi^2 R^2)^{-1} A_k A^k + 1) A_i \\ -\frac{A_i}{(2\pi R)^2} & ((8\pi^2 R^2)^{-1} A_k A^k + 1) A_i & \frac{A_i A_j}{(2\pi R)^2} + \delta_{ij} \end{bmatrix} S, \tag{2.20}$$

where

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_{10-d} & 1_{10-d} & 0 \\ 1_{10-d} & -1_{10-d} & 0 \\ 0 & 0 & 1_{r-(10-d)} \end{pmatrix}, \tag{2.21}$$

and

$$R = 2^{-\frac{3}{4}} \mu_8^{-\frac{1}{2}} \left(\int_0^{2\pi} dx^9 w(x^9) \right)^{1/2} \left(\int_0^{2\pi} dx^9 w(x^9)^{-1} \right)^{-1}, \tag{2.22}$$

and

$$A_i = \frac{1}{2} \left(\int_0^{2\pi} dx^9 w(x^9) \right)^{-1} \left(\int_0^{x_i^9} dx^9 w(x^9)^{-1} \right). \tag{2.23}$$

One can verify that the above results are consistent with those in string theory[7].

[7]For that, one can first express the moduli of type I' in terms of the Heterotic theory [161] and then use eq. 4.7 in [109] to express the Heterotic moduli in terms of the 9d moduli M_{MN}. The parameters R and A_i would be the radius and Wilson lines in the Heterotic picture.

According to (2.9), non-BPS limits correspond to limits where the 9d dilaton goes to infinity sufficiently fast. Keeping M fized and sending φ to infinity corresponds to keeping the relative 8-brane positions fixed while taking C to infinity. If we change the positions of the branes such that M moves on a global geodesic given by (2.6), then non-BPS limits correspond to $\Delta\hat{\varphi} \geq (2\sqrt{d-2}/d)|\Delta\gamma|$ which can *always* be accomplished by taking $C \to \infty$ fast enough. We will see that the $C \to \infty$ limit corresponds to a limit of type I' theory where the length of the interval becomes large.

To conclude this section, we will directly show that $C \to \infty$ yields a decompactification limit, verifying that the type I' description of the non-BPS limits is indeed valid. The size of the type I' interval is given by the square root of the metric in the Einstein frame. To convert between the string frame metric g^s_{MN} and the Einstein frame metric g^{E}_{MN}, we have $g^{\mathrm{E}}_{MN} = e^{-\varphi_{10}/2} g^s_{MN}$. Thus,

$$g^{\mathrm{E}}_{MN} = \tilde{\Omega}^2(x^9)\eta_{MN}, \quad \tilde{\Omega} = Cz^{+1/24} \simeq C^{25/24}. \tag{2.24}$$

The limit where the interval size goes to infinity is thus $C \to \infty$, and in this limit, we have

$$m_{\mathrm{KK}} \simeq C^{-25/24}. \tag{2.25}$$

On the other hand, let us compute the mass scale of the string frame action which we will call the string mass m_s.

$$m_s = e^{\varphi_{10}/4} m_p \simeq C^{-5/24}. \tag{2.26}$$

We therefore see that $m_{\mathrm{KK}} \ll m_s$ in the limit where the type I' interval becomes large. This is a genuine decompactification limit, as required.

2.3 Gauge symmetry enhancements

In the previous section, we showed that the Swampland principles could be used to argue that in certain limits where all BPS states are heavy, the 9d theory must decompactify to a massive IIA background on a compact 1d space with an arrangement of parallel 8-branes. In this section, we will give a bottom-up argument for why the position of the branes uniquely determines the spacetime gauge group. We will show that the gauge groups living on the 8-branes can be determined from Swampland principles.

From string theory, we expect the position of the 8-branes on the interval to encode the information about the gauge theory. This is because the vector multiplets come from open strings ending on the branes, and as two branes approach each other, some massive multiples become massless leading to gauge symmetry enhancement. However, from the bottom-up perspective, the connection between the brane positions and the gauge symmetry is far from clear. Our goal is to provide a bottom-up argument for this connection based on Swampland principles. We provide a bottom-up argument that allows one to read the gauge group directly from the moduli which can be expressed in terms of the brane positions as (2.20). For that, we use the recent progress in demonstrating aspects of the string lamppost principle using the Swampland principles.

In [91,16], it was shown that using the finiteness of black hole entropy and a strong version of the cobordism conjecture, one can classify the possible geometries for the moduli space of the small instantons in minimal supergravities in $d > 6$ with 16 supercharges.

Let us take a moment to explain what small instanton means. For non-Abelian gauge groups, the moduli space of gauge theory instantons can be constructed using the ADHM construction [11]. One of the moduli of the instanton is its size. If we take the size of the instanton to zero. The gauge theory instantons are charged under the $(d-4)$-form gauge symmetry of type IIA and can be instanton. Based on the BPS completeness hypothesis [119], we assume such BPS gauge instantons exist which preserve 8 supercharges. These instantons are known as non-Abelian small instantons.

There is a non-Abelian small instanton for any semisimple piece of the gauge algebra. From the field theory perspective, these can be completely disconnected configurations. However, using the Cobordism conjecture [138], in [16], it was argued that one should be able to continuously deform one non-Abelian small instanton to another without breaking the supersymmetry. Therefore, there must exist a BPS co-dimension four defect with a worldvolume theory preserving 8 supercharges, that has a Coulomb branch that parametrizes the non-Abelian small-instantons as well as the BPS configurations that connect them. We refer to this defect as the small instanton.

The non-Abelian instantons correspond to the point of symmetry enhancement on the Coulomb branch of the worldvolume theory of the BPS defect. The global symmetry of the non-Abelian instanton reflects the action of the spacetime gauge transformation on them. So far, the only gauge instantons we considered were non-Abelian, however, it is natural to have instantons even when the spacetime gauge group is broken to its Abelian subgroup. As opposed to the non-Abelian instantons that can be blown up and given a size, the Abelian instantons only exist as point-like objects. For that reason, they are non-perturbative objects whose existence depends on UV information. The field theory construction for them relies on taking some limit in a regularized solution (e.g. non-commutative instantons [92]). Therefore, the moduli space of the small instanton only has a Higgs branch at specific points of the Coulomb branch where the worldvolume theory has an enhanced non-Abelian global symmetry.

To get a better intuition about the small instantons that connect the gauge instantons, it is helpful to see what they correspond to in string theory. The gauge instantons are typically confined to a stack of branes on which the gauge theory lives. Moreover, different semi-simple pieces of the gauge symmetry correspond to distinct stacks of branes. The BPS configurations that connect the two non-Abelian instantons are D-branes that have four dimensions less than the original branes. The connectedness of this space follows from the fact that an instanton on a stack of brane can be shrunk to zero size and pinched off into a bulk D-brane with four fewer dimensions [192,58]. Therefore, the small instanton at a generic point of its Coulomb branch is usually a D-brane that can be absorbed into branes with four higher dimensions and create a gauge theory instanton on them. The only conserved quantity that is protected by higher form gauge symmetry is the instanton number plus the corresponding D-brane charge. If they could not transform into each other, we would have conservation of each brane separately, which is not protected by gauge symmetry and leads to a global $(d-4)$-form global symmetry which is forbidden in quantum gravity [13,138]. This implies that in supergravities, the D-brane solutions must be connected to gauge instantons. This implies that co-dimension four D-branes are indeed the small instanton.

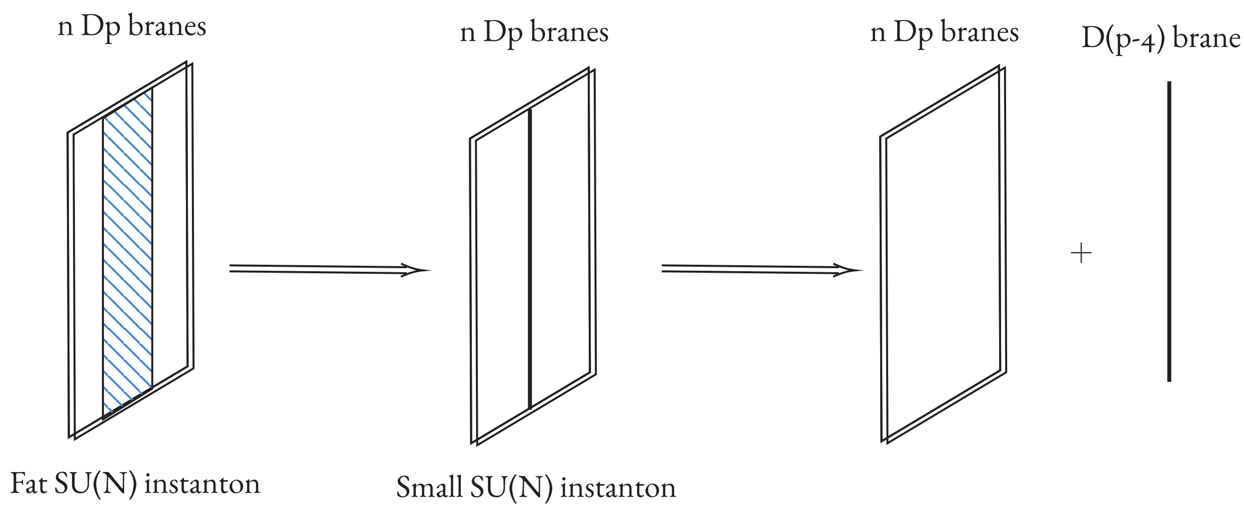


Figure 2.4: The process of shrinking a gauge instanton on the worldvolume of Dp branes to zero size and pinching it off into a spacetime $D(p-4)$ brane.

Now we go back to summarizing the results of [16]. Using the cobordism conjecture and finiteness of black hole entropy, the authors showed that the moduli space of small instantons in various dimensions is given as follows[8]:

[8] In all the examples, the resulting manifold matches the internal geometry in a string theory construction, i.e. IIA on interval, F-theory on elliptic K3, and M theory on K3. This is no surprise, given that from string theory we know that the Coulomb branch has a geometric meaning.

- 9d: An interval with a piece-wise linear profile of $1/g^2$, where g is the coupling constant of the worldvolume theory of the small instanton. The derivative of the coupling constant becomes singular at some points.
- 8d: A sphere, with an $SL(2,\mathbb{Z})$-equivariant profile of τ, where τ is the complexified coupling constant of the worldvolume theory of the small instanton. We can view this configuration as an elliptic K3 surface with some singularities.
- 7d: A K3 surface with frozen singularities.

Moving in the Coulomb branch of the small instanton corresponds to moving a brane around in the internal geometry. The global symmetries correspond to different singular behaviors in the Coulomb branch of the worldvolume theory of the small instanton. Therefore, one can read off the global symmetries of the small instantons from the geometry of the Coulomb branch. The matching of global symmetries on the small instantons and the gauge symmetries in the spacetime then was used to find a bottom-up classification of ranks and gauge symmetry enhancements in supergravities in $d > 6$.

This picture classifies the gauge symmetry enhancements based on the geometry of the moduli space of the small instantons. However, it does not tell us how that geometry depends on the spacetime moduli. This missing piece of information is necessary to understand at which points of the spacetime moduli space the gauge symmetry enhances.

In this section, we show that our results from the previous section easily fill this gap. In the previous section, we showed that the 9d theory admits a massive type IIA background in specific corners of its moduli space. In these limits, it is easy to see that the compact dimension must represent the Coulomb branch of the 9d small instanton. Above, we used the no-global symmetry conjecture to show that a generic small instanton carries a $(d-5)$-brane charge. Therefore, we are interested in the moduli space of BPS configurations with a BPS 4-brane. Supergravity tells us that placing a

4-brane parallel to the 8-branes and perpendicular to the compact dimension does not break the supersymmetry. Therefore, the position of the 4-brane in the compact dimension is part of the small instanton moduli space in 9d. On the other hand, based on [16] we know that the moduli space of small instantons has one real dimension. Therefore, it must correspond to the compact dimension.

Now we can match the singular point in the moduli space of small instantons to the position of the 8-branes. Then we can use the results of[16] to read off the spacetime gauge symmetry. Therefore, we can find the gauge symmetry in the spacetime by finding the position of the 8-branes in type I' supergravity. For example, in a generic point of the moduli space where the gauge group is $U(1)^r$, there must be $r+1$ 8-branes along the interval whose positions are constrained to make sure that dilaton goes to infinity at the two endpoints of the interval. As 8-branes approach each other and coincide, the gauge symmetry enhances. We can see the enhanced gauge symmetry, by reading off the global symmetry of the small instanton that has the corresponding singular structure in [16]. For example, in the maximal rank case of $r=17$, if we move 8 8-branes to each endpoint and leave two at the center, the gauge algebra will be $e_8+e_8+su(2)$.

Note that if the gauge group has an E_n piece, the worldvolume theory of the small instanton at the endpoint of its Coulomb branch will have E_n global symmetry. From the classification of 5d SCFTs we know that the coupling must diverge at that point. Therefore, following the above mapping between the internal geometry and the moduli of the small instanton, we conclude that in such cases, the dilaton must diverge at the endpoints of the interval.

This approach also provides a Swampland argument for the fact that the world volume theory of a stack of 8-branes has a $U(n)$ gauge group. Moreover, we can read off the gauge group living on the end of the universe wall, and the top-form charge that brane must have to cancel the flux of 8-branes. By fixing the number of the 8-branes, this argument even fixes the top-form charge of the end of the universe wall and we can even read off the gauge group living on the stack of 8-branes on top of an end of the universe branes, which is non-trivial.

Our argument maps the gauge symmetry to the position of the branes, and the brane positions can be expressed in terms of the 9d moduli. Therefore, we know the location of the gauge symmetry enhancements in terms of the 9d moduli. On the other hand, the dependence of the gauge symmetry enhancement on the 9d moduli is controlled by the charge lattice. The dependence of the charge lattice on the moduli is controlled by supersymmetry[9] and whenever the sublattice of zero charges under the graviphoton has an ADE sublattice, we have a gauge symmetry enhancement. Therefore, our argument uniquely fixes the charge lattice. Since the charge lattice is even self-dual in string theory constructions in 9d, we find that the choice is unique. In [16], all gauge symmetry enhancements of the lattice were classified and shown to belong in the same moduli space. This strongly suggests that the charge lattice which gives rise to those symmetry enhancements is unique. Our work provides a concrete explanation for the uniqueness of the charge lattice but also gives a bottom-up explanation for the strong coupling behavior of the infinite distance limit.

2.4 Summary and Discussion

In this chapter important properties of theories with 16 supercharges were studied with a focus in the finiteness of massless modes and the String Lamppost Principle. In particular, it was shown that following infinite distance limits of BPS states, which receive no corrections due to supersymmetry, gives rise to dualities which constraint the rank of any gauge group to be $r_G \leq 26 - d$. Interestingly, consequent works[144] extended that result and used Swampland conjectures to prove a refined statement that in 9 and 8 dimensions, the possible gauge group ranks satisfy $r_G = 1 \mod 8$ and $r_G = 2 \mod 8$, respectively, which all have some string theory realization[52,7]. In subsequent works[50,91], the bottom-up argument for such features of string theory were taken a step further in[16], the authors used Swampland conjectures to derive a list of gauge algebras in $d > 6$ supergrav-

[9]See Appendix D in[15]

ities[10] and showed that they identically match the algebras realized in string theory. In this chapter we also considered this classification from a different perspective. In fact for theories in 9d it was shown how these enhancements can associated to BPS 8-branes. In fact the infinite distance limit of Non-BPS states was studied and it was shown that the theory must decompactify to IIA supergravity on a one-dimensional manifold. Then, we used the classification of 5d SCFTs and no-global symmetry conjecture [13] to show that the internal geometry must be an interval with BPS 8-branes placed along the interval. We also showed that the internal geometry must match the moduli space of the 9d small instantons. This allowed us to read the gauge group from the positions of the branes which are directly expressed in terms of the 9d moduli using supersymmetry alone. Therefore, we find a bottom-up argument for the low-energy worldvolume theory of an arbitrary stack of non-perturbative 8-branes. This provides strong evidence for the completeness of the SLP in such theories.

Most of the results discussed depend on the amount of supersymmetry and hence it is crucial to study such questions in theories with less supersymmetry which will be the main topic of the next section. Lastly, it is important to note that even in theories with 16 supercharges many open questions still exist especially related to the possible landscape in dimensions less than 7, where not all possible ranks and gauge groups are known even in string theory. Therefore, it is particularly important to understand the string landscape and subsequently understand if what appears from string theory is universal.

[10] The $d = 7$ case assumed a classification of 3d SCFTs which lacks a bottom-up argument outside string theory.

"People started studying string theory and they say; Oh there is gravity in this; I didn't put it in; where did that come from ? So it gives more credibility to the theory; it's kind of brings out things we want to have without actually putting it in so we believe that's one of the most beautiful aspects of string theory, this magical property that things that you get out of it are more than what you put in."

Cumrun Vafa[62]

3

Supergravity theories with 8 supercharges

The previous section was focused on supegravity theories with 16 supercharges and their bottom up properties. In particular it was evident that certain features of string theory are reproducible using Swampland idea from bottom up and hence heavily constraining the low energy physics. The most important result was that for supersymmetric theories with 16 supercharges there is an upper bound on their rank $r_G \leq 26 - d$, a bound also first suggested by string theory. This therefore completes the finiteness arguments for the number of massless modes in supergravity theories with 16 supercharges. A natural next question is whether similar results also apply to theories with 8 supercharges. The highest dimension with 8 supercharges is 6d where chiral anomalies are also present. Over the past decades a lot of effort on analyzing the landscape of such theories [124,181,127,126,145,146,125,162,119,133,157] has led to a better understanding of the possible consistent theories, although not yet complete. The chiral anomalies heavily constraint the possible spectrum but not enough to address finiteness type questions. However, if combined with Swampland ideas many of the infinite families can be shown to be finite. This will be the focus of section 3.1. On the other hand, in 5d chiral anomalies are absent and hence the spectrum is generally not easily constrained. However, section 3.2 is devoted on investigating such questions and finding possible ways to study the Swampland in 5d. Additionally, in contrast to the theories with 16 supercharges, in 8 supercharges one can also have hypermultiplets in a given theory. In section 3.4 also constraints on the type of matter will be analysed as enforced by the existence of BPS strings for both 5d and 6d.

3.1 Towards a 6d finite Landscape

In [127,124] it was shown that a large subset of all possible distinct combinations of non-abelian gauge groups and matter representations that can appear in a 6d $\mathcal{N} = 1$ supergravity is finite for $T < 9$, where T is the number of tensor multiplets.

However, their arguments in some cases do not generalize to $T \geq 9$. In particular, [165,127] provide five potentially infinite families with two simple gauge group factors that are not constrained to have an upper bound in the number of massless modes and three with three simple gauge factors for $T \geq 9$. In this section it will be shown that these theories are in fact restricted to a finite subset, and similar finiteness conditions will extend to more classes of non-abelian theories.

As discussed in section 1.5 a 6d $\mathcal{N} = 1$ supergravity contains two-form fields which could imply the existence of string sources. In particular, the completeness of spectrum hypothesis will require that all charges compatible with the Dirac quantization condition appear in the theory[159] and form the string lattice Γ of the 6d theory. Moreover,in the case of supersymmetric theories the lattice of all states should be generated by BPS states because any black hole in the theory could eventually decay to a collection of BPS/anti-BPS states and hence these charges should be in the lattice too. Even though this is a heuristic argument, it is a motivation for this assumption. Therefore, each charge in Γ is a $\mathbb{Z}$-linear combination of the BPS charges and hence they generate the lattice. In fact the belief is that this assumption is more general than the setup in this chapter. Namely the lattice of allowed BPS charges are generated by BPS generators in all cases which seems to be the case in all string theory examples. This can be summarized as follows:

> The string charge lattice Γ always has a basis of BPS charges that spans the entire lattice.

Secondly, a second assumption is:

> There are only finitely many inequivalent theories with a given gauge group G and matter M.

Although, no proof of this statement is provided it constitutes a reasonable physical assumption. It would be rather strange to have a fixed low energy matter content be represented by infinitely many inequivalent theories.

Lastly,when a particular theory has enough matter to be Higgsed then the string lattice Γ of that supergravity does not get affected by the process. This is because the Higgsing process only involves the hypermultiplets and vector multiplets of the theory and does not affect the tensor multiplets and hence the dyonic string charge lattice Γ should remain unaffected. In addition, one should note that the vectors a, b_i provide the coupling of the two form B fields to the spacetime curvature $B \cdot a trR^2$ and the coupling to the field strengths $B \cdot b_i trF_i^2$. Therefore, since the 6d theory contains strings of charges b_i associated to gauge instantons[60] we know that b_i should belong to Γ, similarly it has been argued that the vector a should also belong to the lattice corresponding to a gravitational instanton and hence should also be unaffected by the Higgsing process.

The next step is to construct potential infinite families which will be argued to be inconsistent. In order to construct infinite families of unbounded size one can start by identifying gauge groups that satisfy the trF^4 anomalies for arbitrarily large size. The simple gauge groups that can have unbounded dimension are $SU(N), SO(N), Sp(N/2)$. For example, a theory with an $SU(N)$ factor should satisfy:

$$B_{adj} = 2N = \sum_R n_R B_R \tag{3.1}$$

As discussed in[127,124] for large N the only representations that can appear have B_R at most linear in N. Those are the fundamental, adjoint, two-index antisymmetric and symmetric representations. The set of possible such theories including the groups $SO(N), Sp(N/2)$ is summarized in Table 3.1.

In particular, from Table 3.1 it can be noted that the only theories that satisfy the gravitational anomaly for arbitrary N are $SU(N)$ with Adj$/1\square\!\square + 1\begin{smallmatrix}\square\\\square\end{smallmatrix}$ and $SO(N)/Sp(N/2)$ with $\begin{smallmatrix}\square\\\square\end{smallmatrix}/\square\!\square$ with $T \leq 9$. As discussed in[124] $T < 9$ are excluded since there is no solution for the vectors a, b satisfying $a^2 > 0, b^2 = 0, a \cdot b = 0$. However, for $T = 9$ both a, b are null vectors with $a \cdot b = 0$ and hence parallel, i.e. $b = \lambda a$ with $\lambda < 0$ (such that $j \cdot a < 0$ and $j \cdot b > 0$). Specifically, in this case it is simple to find solutions a, b and in fact such an example is constructed later in this section. Therefore, for $T = 9$ this theory constitutes a potentially infinite family with unbounded size.

Next consider theories with gauge groups of the form $G_1 \times G_2$ with G_i drawn from Table 3.1. In[128,127,165] they identify 5 potentially infinite families with arbitrarily large dimension given in Table 3.2 and composed of two simple gauge factors from Table 3.1. The expectation is that even

Group	Matter	$H-V$
$SU(N)$	1 Adj	0
	$1\,\Box\Box + 1\,\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	1
	$2N\,\Box$	N^2+1
	$(N+8)\,\Box + 1\,\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	$\frac{1}{2}N^2 + \frac{15}{2}N + 1$
	$(N-8)\,\Box + 1\,\Box\Box$	$\frac{1}{2}N^2 - \frac{15}{2}N + 1$
	$16\,\Box + 2\,\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	$15N+1$
$SO(N)$	$(N-8)\,\Box$	$\frac{1}{2}N^2 - \frac{7}{2}N$
	$1\,\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	0
$Sp(N/2)$	$(N+8)\,\Box$	$\frac{1}{2}N^2 + \frac{7}{2}N$
	$16\,\Box + 1\,\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	$15N-1$
	$1\,\Box\Box$	0

Table 3.1: Most theories have $H-V \to \infty$ as $N \to \infty$ except those with $H-V = 0, 1$ for which $T \leq 9$ and there is no obstruction to having an infinitely large gauge group from anomalies alone.

$SU(N) \times SU(N)$	$2\,(\Box,\Box)$
$SO(2N+8) \times Sp(N)$	$(\Box,\Box)$
$SU(N) \times SO(N+8)$	$(\Box,\Box) + (\begin{smallmatrix}\Box\\\Box\end{smallmatrix},1)$
$SU(N) \times SU(N+8)$	$(\Box,\Box) + (\begin{smallmatrix}\Box\\\Box\end{smallmatrix},1) + (1,\Box\Box)$
$Sp(N) \times SU(2N+8)$	$(\Box,\Box) + (1,\Box\Box)$

Table 3.2: Potentially infinite families with two simple gauge group factors.

though each individual factor may not satisfy the gravitational anomaly, it can be arranged so that introducing matter charged under both gauge groups reduces $H-V$ enough to make it possible. Furthermore, it is important to note that the only matter charged under two gauge groups is bifundamental matter. This can be justified by considering the fact that for 6d $\mathcal{N}=1$ gauge theories we know that all theories are Higgsable until one reaches the Non-Higgsable Clusters(NHC)[145] or the gauge group gets completely Higgsed away and hence we expect that any family of theories should be Higgsable to some minimal gauge group. As discussed earlier Higgsing does not affect the string lattice and consequently the vectors b_i of the instantonic strings of the gauge theory, which implies that their inner products defined through the anomaly cancellations condition (1.17) should be independent of the size N of the gauge group which gets reduced by the process.

In particular, for two vectors b_1, b_2 their inner product is given by $b_1 \cdot b_2 = \sum_{R,S} \lambda_i \lambda_j n^{ij}_{RS} A^i_R A^j_S$ which as noted in [127,124] can only be independent of N if both R and S are the fundamental representations. But more specific to the theories from Table 3.1, one can see that no theory has enough matter to gauge any of the Adj, $\Box\Box$, $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ because for example a theory of the form $SU(N) \times G_2(N)$ with (Adj, R) would require that $SU(N)$ has $dim(R)$ number of Adj representations but any theory has at most one. Therefore, no G_i factor can be $SU(N)$ with Adj$/1\Box\Box + 1\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ or $SO(N)/Sp(N/2)$ with $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}/\Box\Box$.

One could consider k gauge groups from Table 3.1 with matter charged under only one factor and constant $H - V$. But the gravitational anomaly would then become $(H_{ch} - V)k \leq 273 - 29T$ with $(H_{ch} - V) \geq 0$ and hence restricting the number of terms. Therefore, we only need to focus on excluding the theories of Table 3.2[1]. The first theory is valid for $T \leq 9$ and the rest for $T \leq 10$. For $T < 9$ it was shown in [124,127] that no solution exists for a, b_i for any of the theories that satisfy all the consistency conditions studied earlier and specifically all b_i's be associated with positive kinetic terms. Similarly, for all theories except the first one, there is also no solution for vectors a, b_i when $T = 9$. This is easy to verify for example for $SO(2N + 8) \times Sp(N)$, which has vectors $a, b_i \in \mathbb{R}^{1,T}$ that satisfy:

$$a \cdot b_1 = 2,\ a \cdot b_2 = -1,\ b_1^2 = -4,\ b_2^2 = -1,\ b_1 \cdot b_2 = 2 \tag{3.2}$$

There are two null vectors a, $(b_1 + 2b_2)$ that satisfy $a \cdot (b_1 + 2b_2) = 0$ and hence need to be parallel $b_1 + 2b_2 = \lambda a \implies b_1 = \lambda a - 2b_2$ for some $\lambda \in \mathbb{R}$. However, since $b_1 \cdot b_2 = 2$ then $\lambda = 0$ implying that $j \cdot b_1 = -j \cdot 2b_2$, meaning that we can not find vector j ensuring the positivity of both

[1] We note that we do not present the full set of theories since exchanging matter with its conjugate when gauging it provides distinct theories but this does not affect our calculations.

kinetic terms. In an identical fashion one can show the same result by considering the null vectors a and $2b_1 + b_2$(for the third) or $b_1 + b_2$ (for the fourth and fifth).

$SU(N-8) \times SU(N) \times SU(N+8)$	$(\square \otimes \square \otimes 1) + (1 \otimes \square \otimes \square)$ $+(\begin{smallmatrix}\square\\\square\end{smallmatrix} \otimes 1 \otimes 1) + (1 \otimes 1 \otimes \square\square)$
$Sp((N-8)/2) \times SU(N) \times SO(N+8)$	$(\square \otimes \square \otimes 1) + (1 \otimes \square \otimes \square)$
$SU(N-8) \times SU(N) \times SO(N+8)$	$(\square \otimes \square \otimes 1) + (1 \otimes \square \otimes \square) + (\begin{smallmatrix}\square\\\square\end{smallmatrix} \otimes 1 \otimes 1)$
$Sp((N-8)/2) \times SU(N) \times SU(N+8)$	$(\square \otimes \square \otimes 1) + (1 \otimes \square \otimes \square) + (1 \otimes 1 \otimes \square\square)$

Table 3.3: Potential infinite families with three simple gauge factors.

Furthermore, as described in the Appendix anomalies permit classes of infinite families with more than two simple gauge factors. For example, the gauge group theories described in Table 3.3 where first introduced in [124]. More generally, one can construct theories which satisfy all the anomalies with arbitrarily large number of gauge factors. Specifically, linear chains of such theories are presented in Table 3.4. In addition, in Appendix C.2 we discuss theories that have gauge groups connected in a non-linear fashion.

For example, one can construct theories where one gauge group is connected to multiple others. Specifically, in C.2 we find that a large class of these theories have inner products $b_i \cdot b_j$'s corresponding to the affine ADE, where each b_i represents a node on the Dynkin diagram. Each b_i is associated with a gauge group $SU(a_i^\vee N)$ where $a^\vee$ is the dual coxeter label and the matter is bifundamentals according to the links of the affine Dynkin diagram.

However, an interesting observation is that even though anomalies permit the A, D type inner products to have arbitrarily many factors, a more careful analysis shows that this is not possible. This is because the vectors a, b_i form the anomaly lattice which needs to be embedded in the string lattice Γ. The anomaly lattice is generated by at most $k+1$ vectors a, b_i and the full lattice Γ is of signature $(1, (-1)^T)$ and generated by $T+1$ vectors. This implies, that $k \leq T$ and since the gravitational anomaly cancellation requires $T \leq 9$ then $k \leq 9$.

Moreover, one can show that $V = \sum_i^k b_i$ is a null vector which satisfies $a \cdot V = 0$. Hence for $T < 9$ we have $a^2 > 0$ which implies that $V = 0$ but now one can notice that not all $j \cdot b_i > 0$ can be satisfied simultaneously. Therefore, the only case for which solutions could be found is for $T = 9$.

Similarly, for the theories of Table 3.4 with k factors the anomaly lattice has signature $(1, (-1)^{k-1})$ for $T < 8 + k$ which means that $k \leq T + 1$. Therefore, using the last column of Table 3.4 one can show that all the theories have a finite number of gauge factors and more specifically the second theory has $T \leq 138$ while the rest $T \leq 137$.

Furthermore, we note that the theories of Table 3.4 with $T < 8 + k$ have no consistent solutions. In particular, the first and last theories have the same anomaly lattice and hence can be considered together, same holds for the second and third. For the first and last theory one can consider $T < 9$ and note that $a^2 > 0$ and $(b_1 + \cdots + b_k)^2 = 0$ with $a \cdot (b_1 + \cdots b_k) = 0$ from which it follows that $b_1 + \cdots b_k = 0$ and hence cannot satisfy $j \cdot b_i > 0$ simultaneously for all i. We can extend this for $8 + k > T \geq 9$ by considering the following vectors:

$$V_1 = a + \sum_{i=k-T+9}^{k-1} b_i(i - k + T - 8) + (T - 9)b_k, \; V_2 = \sum_{i=1}^{k} b_i \tag{3.3}$$

It is simple to verify that $V_1^2 = V_2^2 = V_1 \cdot V_2 = 0$ from which it follows that $V_2 = \lambda V_1$. Now consider the product $\underbrace{b_k \cdot V_2}_{=0} = \underbrace{b_k \cdot \lambda V_1}_{=\lambda} \implies \lambda = 0$. Therefore, $V_2 = 0$ and hence not all $j \cdot b_i > 0$ conditions can be satisfied. This method though does not constrain the theories that have $T = 8 + k$ which arise when $k \leq 6$ for the first theory and when $k \leq 7$ for the last and solutions can be found as we will see later. Similarly, one can note that also for the second and third cases there are no solutions for $T < 8 + k$.

We will now provide a general argument that restricts all the theories presented above to a finite set. The argument is based on [119] where they use completeness of spectrum as evidence for the

Gauge group	Matter	Tensors
$SU(N-8)\times SU(N)\times SU(N+8)\times\cdots\times SU(N+8(k-2))$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}\otimes 1\cdots\otimes 1+1\otimes 1\cdots\otimes\square\square$	$T\leq\frac{27k}{29}+\frac{245}{29}$
$Sp((N-8)/2)\times SU(N)\times SU(N+8)\times\cdots\times SO(N+8(k-2))$		$T\leq\frac{27k}{29}+\frac{247}{29}$
$SU(N-8)\times SU(N)\times SU(N+8)\times\cdots\times SO(N+8(k-2))$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}\otimes 1\cdots\otimes 1$	$T\leq\frac{27k}{29}+\frac{246}{29}$
$Sp((N-8)/2)\times SU(N)\times SU(N+8)\times\cdots\times SU(N+8(k-2))$	$1\otimes 1\cdots\otimes\square\square$	$T\leq\frac{27k}{29}+\frac{246}{29}$

Table 3.4: Each theory has bifundamental matter between any adjacent groups and the matter indicated in the table is matter charged under only one gauge group. The last column indicates the upper bound on T that the gravitational anomaly imposes.

existence of BPS strings with some charge $Q=(q_1,\cdots q_{10})$ and $q_i\in\mathbb{Z}$ satisfying consistency conditions (1.48). Those consistency conditions will then provide us with an upper bound on the size N of the gauge group.

All the theories above have a gauge group with a finite number of non-abelian simple gauge groups and their size is controlled by the parameter N which is not bounded by the arguments already presented. However, one can notice that each family of theories labelled by N is connected through Higgsing. For example, $SU(N)+1Adj$ can be Higgsed to $SU(N-1)+1Adj$ by making $2N-1$ full hypermultiplets massive. However, as discussed earlier the Higgsing process does not affect the string lattice which implies that any vectors in the lattice is independent of the size N. Specifically, by considering $\{Q_i\}$ as the BPS string states that generate Γ and satisfy the conditions (1.48) then one has that these charges are also independent of N. This therefore implies that there should exist a minimal choice of BPS charge $Q\in\{Q_i\}$ that is also independent of N. Therefore, for an infinite family drawn from the examples above and using that $Q^2+Q\cdot a\geq -2$ the unitarity bound becomes (for at least one non-zero k_j):

$$k_j\frac{\dim G_j}{k_j+h_j^\vee}\leq c_\ell+\sum_i k_i\frac{\dim G_i}{k_i+h_i^\vee}\leq c_L=3Q^2-9Q\cdot a+2\leq 12Q^2+20 \tag{3.4}$$

where k_i is the level of the G_i current algebra and $c_\ell=\frac{3k_\ell}{2+k_\ell}$ the central charge of $SU(2)_\ell$. Note that if $k_i\neq 0$ then for $G_i=SU(N_i)$ one has that $N_i-1=\frac{\dim G_i}{1+h_i^\vee}\leq k_i\frac{\dim G_i}{k_i+h_i^\vee}$, for $G_i=Sp(N_i)$ one has that $2N_i - 3 \leq \frac{\dim G_i}{1+h_i^\vee} \leq k_i \frac{\dim G_i}{k_i+h_i^\vee}$, for $G_i = SO(N_i)$ one has that $\frac{N_i}{2} = \frac{\dim G_i}{1+h_i^\vee} \leq k_i \frac{\dim G_i}{k_i+h_i^\vee}$, where all N_i are a linear function of N. This implies that left-hand side of the inequality is always a linear function of N's. Moreover, since Q^2 is independent of N then this provides a finite upper bound for the size N. This is clear if there is one chain of theories related by Higgsing for arbitrarily large N. However, there is a slight loophole in this argument: it may be that there is no such infinite chain, but that there are infinitely many finite Higgs chains each of which start from a maximal N_{max}. Then by Higgsing them down to a given N it can be seen that for a fixed N one has to have infinitely many inequivalent theories with the same massless matter content which was assumed can never happen.

In the above argument it was assumed that at least one of the levels k_i can be chosen to be non-zero. One can now argue that some Q can always be chosen to have at least one non-zero k_i. Let us assume that there is no charge Q such that $b_i \cdot Q = k_i > 0$. In this case $b_i \cdot Q = 0$ for any of the Q's. But for any b_i there exists a vector in the lattice which has a non-vanishing inner product with it, by the requirement of the self-duality of the charge lattice. However, since Q generates the lattice this leads to a contradiction. And so there are some BPS states Q with non-vanishing k_i.

Even though our argument above does restrict the infinite families to only a finite consistent set under reasonable assumptions, it does not provide us with a concrete upper bound of the size of the gauge groups for each theory. Therefore, the remaining of the section will be devoted to go through some of the theories presented above and find particular solutions for a, b_i such that we can illustrate using unitarity the exact upper bound for the size N in those cases.

Considering the single gauge group infinite families: $SU(N) + 1\text{Adj}$ or $1\,\square\!\square + 1\,\begin{smallmatrix}\square\\ \square\end{smallmatrix}$ with $T = 9$. In order to ensure that the theory is unitary the following inequality needs to hold:

$$c_\ell + \frac{k(N^2-1)}{k+N} \leq c_L \tag{3.5}$$

where k is the level of the $SU(N)$ current algebra and $c_\ell = \frac{3k_\ell}{2+k_\ell}$ being the central charge of $SU(2)_\ell$. The inequality would be strongest for a minimal choice of c_L which depends on the choice of charge Q. Let us consider a representation of the a, b vectors. In particular in the integral basis let us take $\Omega = \text{diag}(1, (-1)^9)$, and $a = (-3, 1^9)$, and choose a string with minimal charge $Q = (1, -1, -1, 0 \ldots, 0)$ which gives $c_L = 8$ and $Q^2 = -1, k = -\lambda, k_\ell = 0$. Therefore with this realization, one can easily check that the only possible k, N that satisfy the unitarity bound are:

$$(k \geq 1, N = 0, 1, 2, 3), (4 \geq k \geq 1, N = 4) \tag{3.6}$$

$$(2 \geq k \geq 1, N = 5), (k = 1, N = 6, 7, 8, 9) \tag{3.7}$$

Therefore, the size of the gauge group for this theory is bounded by $N \leq 9$ at least for this realization of vectors. In the next section it will be shown that the theories of the second class with $k = 1$ belong to the Swampland and hence the size is bounded by $N \leq 5$. To show that these are general Swampland bounds one needs to show that these results hold independently of possible inequivalent realizations of the (a, b) vectors in the lattice. One potential issue is that for $N \leq 3$ there are infinitely many potential solutions for the vector b but the number of massless modes is still finite. However, in this work, as was discussed earlier, it will assumed that there are only finitely many theories with a fixed gauge group and matter, and therefore such issues are avoided. Furthermore, according to [124] both vectors a, b need to be primitive in F-theory and hence theories with $\lambda > 1$ can not have an F-theory construction. Combining this with our conjecture of the next section one expects that no F-theory construction should be possible also for $\lambda = 1$.

A more general worry is that the above result was deduced with the assumption that $\Omega = \text{diag}(1, (-1)^9), a = (-3, 1^9)$ while one could imagine other inequivalent choices for these. In fact, since $T \equiv 1 \pmod 8$ in this case one could either have the lattice be odd and isomorphic to $\mathbb{Z}^{T+1}$

or it can be an even lattice isomorphic to $U \otimes E_8(-1)$ with $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. One needs to ensure that these as well as other choices for a provide finite size too. Therefore, our previous general argument ensures the finiteness of these theories independently of the type of lattice or particular solution.

Next step is to move to theories with two simple gauge group factors summarized in Table 3.2.

- $SU(N) \times SU(N)$

For $T = 9$ in [119] it was shown that for a particular choice of Ω, a, b_i's all theories with $N > 9$ belong to the Swampland because they contain non-unitary strings.

More general solutions can be found by noticing that $a, b_1 + b_2$ are null with $a \cdot (b_1 + b_2) = 0$ and hence satisfy $-a = m(b_1 + b_2)$ with $m > 0$. In this case the general argument translates into the equations $N \leq 12Q^2 + 20$ with Q^2 some constant.

- $SO(2N + 8) \times Sp(N)$

The anomaly cancelation conditions dictate the following inner products between the vectors $a, b_i \in \mathbb{R}^{1,T}$:

$$a \cdot b_1 = 2,\ a \cdot b_2 = -1,\ b_1^2 = -4,\ b_2^2 = -1,\ b_1 \cdot b_2 = 2 \tag{3.8}$$

For $T = 10$ solutions a, b_i exist but one can show that unitarity ensures the finiteness of the theory. We may choose a presentation of these such that the bilinear form Ω and the vectors a, b_1, b_2 are given as follows:

$$\begin{aligned} \Omega = \text{diag}(1, (-1)^{10}), \qquad & a = (-3, 1^{10}) \\ b_1 = -2a, \qquad & b_2 = (1, -1, -1, 0^8) \end{aligned} \tag{3.9}$$

In this presentation one can choose $j = (1, 0^{10})$ which satisfies $j \cdot a < 0$ and $j \cdot b_i > 0$ as desired. Considering a BPS string with charge $Q = (q_1, \cdots q_{11})$ satisfying conditions (1.48) then unitarity of the string worldsheet requires that:

$$\frac{k_1((2N+8)(2N+7)/2)}{k_1 + 2N + 6} + \frac{k_2(2N(2N+1)/2)}{k_2 + (N+1)} \leq c_L \tag{3.10}$$

One can easily check that a minimal string charge solution can be $Q = (1, -1, 0^8, -1)$, which has levels $k_1 = 2, k_2 = 0$ and central charge $c_L = 8$. The unitarity bound for this string configuration reduces to:

$$\frac{2((2N+8)(2N+7)/2)}{2 + 2N + 6} \leq 8 \Longrightarrow N \leq 1/2$$

This seems to be reassuring because it does not rule out the theories at $N = 0$ with a single $SO(8)$ which do have known string theory realizations [146,137,180]. As for the case of $N = 1/2$ one has a single $SO(9)$ with 1 fundamental hypermultiplet which is the unHiggsed version of the $SO(8)$ theory and if it exists could have the same base.

- $SU(N) \times SO(N+8)$

We note that the gravitational anomaly restricts $T \leq 10$ and hence we need to ensure finiteness of the theory for $T = 10$ as before. In this case the charge lattice is given by

$$a \cdot b_1 = -1,\ a \cdot b_2 = 2,\ b_1^2 = -1,\ b_2^2 = -4,\ b_1 \cdot b_2 = 2 \tag{3.11}$$

It seems that the anomaly charge lattice is identical to the one before and hence we can use those results. In other words, for $T = 10$ the vectors are identical as in the previous example but with $b_1 \leftrightarrow b_2$. Therefore this string configuration with $k_2 = 2, k_1 = 0$ implies that

$$\frac{k_2((N+8)(N+7)/2)}{k_2+N+6} \leq 8 \Longrightarrow N \leq 1 \tag{3.12}$$

Therefore, as expected this bound does not rule out the single $SO(8)$ or $SO(9)$ theories as discussed above.

- $SU(N) \times SU(N+8)$

This family has charge lattice vectors satisfying:

$$a \cdot b_1 = -1,\ a \cdot b_2 = 1,\ b_1^2 = -1,\ b_2^2 = -1,\ b_1 \cdot b_2 = 1 \tag{3.13}$$

Similarly to the previous example for $T = 10$ such vectors exist but there are finitely many consistent unitary solutions. One such representation is given by the choice

$$\begin{aligned} &\Omega = \text{diag}(1, (-1)^{10}), \quad a = (-3, 1^{10}) \\ &b_1 = (1, -1, -1, 0^8), \qquad b_2 = -a \end{aligned} \tag{3.14}$$

One can easily check that $Q = (1, -1, 0^8, -1)$ is a minimal string charge which satisfies eq.(1.48) with levels $k_1 = 0, k_2 = 1$ and $c_L = 8$. For the string configuration to be unitary we need to satisfy:

$$\frac{((N+8)^2-1)}{1+(N+8)} \leq 8 \Longrightarrow N \leq 1 \tag{3.15}$$

This bound potentially allows for $N = 0, 1$ corresponding to $SU(8) + \square\square$ and $SU(9)) + \square + \square\square$. However, such string theory realizations are not known and as we will argue in the next section these theories belong to the Swampland.

- $Sp(N) \times SU(2N+8)$

This theory has the same anomaly lattice as (3.11) and hence we can reuse those results. For $T = 10$ vectors a, b_i can be found as in (3.14). Therefore, for $k_1 = 0,\ k_2 = 1$ we see that unitarity implies

$$\frac{k_2((2N+8)^2-1)}{k_2+2N+8} \leq 8 \Longrightarrow N \leq 1/2 \tag{3.16}$$

This bound allows for $N = 0, 1/2$ corresponding to $SU(8) + \square\square$ and $SU(9) + \square + \square\square$ but as was discussed these theories will be ruled out in the next section.

More generally, since for the last two examples $(a + b_2)$, $(b_1 + b_2)$ are null and orthogonal, the most general vectors needed are given by the family of solutions $a = \lambda\, b_1 + (\lambda - 1)b_2$ with $\lambda \leq 0$ in order to ensure positivity of the kinetic terms(for the first two examples one can replace $b_1 \to 2b_1$). Similarly to the first example, since the above theories can be Higgsed from N to $N-1$ then unitarity would imply the finiteness of each family of theories as was discussed earlier.

Next we move on to theories with three simple gauge factors. For example, the set of theories from Appendix C.2 have b_i's form inner products according to the affine ADE algebras. For example, the $\hat{A}_2$ type theory with $SU(N)^3$ and $T = 9$ has the anomaly lattice:

$$\Lambda = \begin{pmatrix} a^2 & -a\cdot b_1 & -a\cdot b_2 & -a\cdot b_3 \\ -a\cdot b_1 & b_1^2 & b_1\cdot b_2 & b_1\cdot b_3 \\ -a\cdot b_2 & b_1\cdot b_2 & b_2^2 & b_2\cdot b_3 \\ -a\cdot b_3 & b_1\cdot b_2 & b_2\cdot b_2 & b_3^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix} \tag{3.17}$$

These inner products can be solved for vectors satisfying the linear relation $a = \lambda(b_1 + b_2 + b_3)$ for

$\lambda < 0$. For example, a solution to the anomaly lattice (3.17) is given by:

$$\begin{aligned} &\Omega = \text{diag}(1, (-1)^9), \qquad && a = (-3, 1^9) \\ &b_1 = (1, -1, -1, -1, 0^6), \qquad b_2 = (1, 0^3, -1, -1, -1, 0^3), \qquad && b_3 = (1, 0^6, -1, -1, -, 1) \end{aligned} \tag{3.18}$$

One can choose $j = (1, 0^9)$ and charge $Q = (1, -1, 0, 0, -1, 0^5)$ which gives $k_1 = k_2 = 0$, $k_3 = 1$ and $c_L = 8$. Therefore, worldsheet unitarity implies:

$$\frac{(N^2 - 1)}{1 + N} \leq 8 \Longrightarrow N \leq 9 \tag{3.19}$$

Therefore, for the particular choice of anomaly vectors this theory is finite and theories with $N > 9$ belong to the Swampland. For more general possible representations of the vectors, the argument works exactly as discussed earlier.

Furthermore, other types of theories with three gauge groups can be found in Table 3.3 but they are all particular cases of those in Table 3.4 for $k = 3$ and hence can be handled together.

- $SU(N-8) \times SU(N) \times SU(N+8) \times \cdots \times SU(N+8(k-2))$

For this theory the maximum number of tensor multiplets arises for $k \leq 6$ and has $T_{max} = 8 + k$ and these constitute the only case we need to consider as the other values of T were ruled out earlier. The anomaly charge lattice of the strings determined by the type of gauge group and matter is given by :

$$\Lambda = \begin{pmatrix} a^2 & -a\cdot b_1 & -a\cdot b_2 & -a\cdot b_3 & \cdots & -a\cdot b_k \\ -a\cdot b_1 & b_1^2 & b_1\cdot b_2 & b_1\cdot b_3 & \dots & b_1\cdot b_k \\ -a\cdot b_2 & b_1\cdot b_2 & b_2^2 & b_2\cdot b_3 & \dots & b_2\cdot b_k \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a\cdot b_k & b_1\cdot b_k & b_2\cdot b_k & b_3\cdot b_k & \dots & b_k\cdot b_k \end{pmatrix} = \begin{pmatrix} 9-T & 1 & 0 & 0 & \cdots & -1 \\ 1 & -1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ 0 & 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \tag{3.20}$$

We may consider a particular solution for the vectors a, b_i given by:

$$\begin{gathered} a = (-3,1,1,1\cdots,1), \quad b_1 = (1,-1,-1,0\cdots,0), \quad b_2 = (0,0,1,-1\cdots,0) \\ b_3 = (0,0,0,1,-1\cdots,0), \quad b_i = (0\cdots,1,-1,\cdots,0), \quad b_k = -a + \textstyle\sum_{i=1}^{k-2} b_i(-i+k-1) \end{gathered} \tag{3.21}$$

Moreover, we also need to identify a consistent Kähler form j and we would like to make a minimal choice of string charge Q for each k.

For $k = 3$ we may choose: $j = (2,0,0,1,0^8), Q = (1,-1,0^9,-1)$ for which $k_1 = 0, k_2 = 0, k_3 = 1, c_L = 8$. Therefore, string unitarity can be expressed as:

$$\frac{k_3((N+8)^2-1)}{k_3+N+8} \leq 8 \Longrightarrow N <= 1 \tag{3.22}$$

For $k = 4$ we may choose: $j = (3,0,0,1,2,0^8), Q = (1,-1,0^{10},-1)$ with $k_1 = 0, k_2 = 0, k_3 = 0, k_4 = 1, c_L = 8$

$$\frac{k_4((N+8(4-2))^2-1)}{k_4+N+8(4-2)} \leq 8 \Longrightarrow N \leq -7 \tag{3.23}$$

For $k = 5$ we may choose: $j = (4, 0, 0, 1, 2, 3, 0^8)$, $Q = (1, -1, 0^{10}, -1)$ with $k_1 = 0, k_2 = 0, k_3 = 0, k_4 = 0, k_5 = 1, c_L = 8$

$$\frac{k_5((N+8(5-2))^2-1)}{k_5+N+8(5-2)} \leq 8 \Longrightarrow N \leq -15 \tag{3.24}$$

For $k = 6$ we may choose: $j = (6, 0, 0, 1, 2, 3, 4, 0^8)$, $Q = (1, -1, 0^{12}, -1)$ with $k_1 = 0, k_2 = 0, k_3 = 0, k_4 = 0, k_5 = 0, k_6 = 1, c_L = 8$.

$$\frac{k_5((N+8(6-2))^2-1)}{k_5+N+8(6-2)} \leq 8 \Longrightarrow N \leq -23 \tag{3.25}$$

We therefore, conclude that the above inequalities suggest that only $SU(9) + 1\square + 1\square\square$ and $SU(8) + 1\square\square$ are allowed which as was discussed earlier will be ruled out in the next section.

- $Sp((N-8)/2) \times SU(N) \times SU(N+8) \times \cdots \times SO(N+8(k-2))$

The anomaly lattice is the same as in the previous theory except with $b_k \to 2b_k$ and the maximum $T_{max} = k+8$ attained for $k \leq 7$.

For $k = 3$ we have $k_1 = 0, k_2 = 0, k_3 = 2, c_L = 8$ giving us:

$$\frac{2((N+8)(N-7)/2)}{2+N+6} \leq 8 \Longrightarrow N \leq 1 \tag{3.26}$$

For $k = 4$ we have $k_1 = 0, k_2 = 0, k_3 = 0, k_4 = 2, c_L = 8$ giving us:

$$\frac{2((N+8(4-2))((N+8(4-2)-1)/2)}{2+N+8(4-2)-2} \leq 8 \Longrightarrow N \leq -7 \tag{3.27}$$

For $k = 5$ we have $k_1 = 0, k_2 = 0, k_3 = 0, k_4 = 2, k_5 = 2, c_L = 8$ giving us:

$$\frac{k_5((N+8(5-2))^2-1)}{k_5+N+8(5-2)} \leq 8 \Longrightarrow N \leq -15 \tag{3.28}$$

For $k = 6$ we have $k_1 = 0, k_2 = 0, k_3 = 0, k_4 = 0, k_5 = 0, k_6 = 2, c_L = 8$ giving us:

$$\frac{k_6((N+8(6-2))^2-1)}{k_6+N+8(6-2)} \leq 8 \Longrightarrow N \leq -23 \tag{3.29}$$

For $k = 7$ we $j = (8, 0, 0, 1, 2, 3, 4, 5, 0^8)$, $Q = (1, -1, 0^{13} - 1)$ with

$$k_1 = 0, k_2 = 0, k_3 = 0, k_4 = 0, k_5 = 0, k_6 = 0, k_7 = 2, c_L = 8 \tag{3.30}$$

$$\frac{k_7((N+8(7-2))^2-1)}{k_7+N+8(7-2)} \leq 8 \Longrightarrow N \leq -31 \tag{3.31}$$

Therefore, unitarity implies that the only theories that survive are $SO(9) + 1\square$ or $SO(8)$ which have been discussed earlier in this section.

Finally, there are two more infinite families of this type that can be found by replacing $Sp \to SU$ or $SO \to SU$ giving us identical results to those above.

- $SU(N)^k$

Earlier in this section the cases $k = 2, 3$ where shown to be finite and hence we need to focus on $k > 3$ for $T = 9$. The anomaly lattice of this theory is determined by the inner products:

$$a^2 = 0, a \cdot b_i = 0, b_i \cdot b_{i+1} = 1, b_i^2 = -2, b_1 \cdot b_k = 1 \tag{3.32}$$

Consider the quadratic form to be $\Omega = \text{diag}(1, (-1)^9)$ then a solution to the anomaly lattice for $k \leq 9$ (the upper bound was determined by requiring the anomaly lattice to embed into Γ) is:

$$\begin{array}{lll} a = (-3, 1, 1, 1 \cdots, 1), & b_1 = (0, 1, -1, 0 \cdots, 0), & b_2 = (0, 0, 1, -1 \cdots, 0) \\ b_3 = (0, 0, 0, 1, -1 \cdots, 0), & b_i = (0 \cdots, 1, -1, \cdots, 0), & b_k = -a - \sum_{i=1}^{k-1} b_i \end{array} \tag{3.33}$$

A compatible Kähler form can be found for example $j = (4, 1, 2, \cdots, 9)$. For $k < 9$ a minimal choice of BPS string charge is $Q = (1, -1, 0 \cdots, 0, -1)$ which has $Q^2 = -1$ and $Q \cdot a = -1$ and $k_1 = 1, k_i = 0, k_k = 0$ and hence

$$\frac{(N^2-1)}{1+N} \leq c_L = 8 \Longrightarrow N \leq 9 \tag{3.34}$$

which is the same result we found previously for $k = 3$.

For $k = 9$ a minimal choice of BPS string charge is $Q = (1, -1, 0 \cdots, 0, 0)$ which has $Q^2 = 0$ and $Q \cdot a = -2$ and $k_1 = 1, k_i = 0, k_k = 1$ and hence

$$\frac{2(N^2-1)}{1+N} \leq c_L = 20 \Longrightarrow N \leq 11 \tag{3.35}$$

As presented earlier in the section and in the Appendix one can see that there are more theories that we could analyze but the methods are parallel to those already discussed. Therefore, the general argument in the beginning of the section applies to those infinite families too and similar choice of solutions as to those already made would reveal potential upper bounds for the sizes of the gauge groups.

We note that our general argument resticted the dimension of each gauge group to be finite. Additionally, we were able to show that a number of theories with $SU(N), SO(N), Sp(N/2)$ type gauge groups may only have finitely many simple gauge groups by studying the lattice embedding of the anomaly lattice to the full 6d string lattice. However, more theories can be constructed with bounded dimension and unbounded number of tensor multiplets allowed by anomalies.

Recall that the gravitational anomaly is given by

$$H_{ch} - V \leq 273 - 29T \tag{3.36}$$

As we have seen before constructing theories with arbitrarily many gauge factors not restricted by anomalies requires $H - V < 0$ so that eq. (3.36) is always satisfied. Therefore, if one could choose more theories of finite dimension and minimal matter that satisfy the anomaly conditions but have negative $H - V$ then it could be possible to have an unbounded number of those. Additionally, assuming that $H_{ch} - V < 0$ for a given simple gauge group we can rearrange eq. (3.36) to write it as

$$T \leq \frac{273}{29} - \frac{(H_{ch} - V)k}{29} \tag{3.37}$$

where k is the number of simple gauge factors. However, as was discussed earlier in this section one needs to be able to embed the anomaly lattice in the full string lattice of the 6d theory and hence satisfy $k \leq T$. This is possible only if $(H_{ch} - V) \leq -29$.

Examples of theories with minimal matter include the NHC's and more found in [145,106]. For example, pure $SO(8)$ has $H_{ch} - V = -28$ and $SO(9) + 1\square$ has $H_{ch} - V = -27$. However, neither satisfy $H_{ch} - V \leq -29$ and hence one can not have an infinite number of those. Likewise, also $SU(3)^k$ is bounded because $H_{ch} - V = -8$ and hence $k \leq 17$. Also, for $(g_2 \times SU(2))^k$ one has $H_{ch} - V = -9$ and for $(SU(2) \times SO(7) \times SU(2))^k$ one has $H_{ch} - V = -11$. Therefore, from the NHC's the following are compatible with $(H_{ch} - V) \leq -29$:

- f_4 with $b_i \cdot b_i = -5$.

The gravitational anomaly determines that $T \leq \frac{52k}{29} + \frac{273}{29}$. For example when $T = k + 9$ we can find solutions of the form:

$$a = (-3, 1^T) \tag{3.38}$$

$$b_1 = (-1, -1, -1, 2, 0^{T-3}) \tag{3.39}$$

$$b_2 = (0, 0, -2, -1, 0^{T-3}) \tag{3.40}$$

$$\vdots \tag{3.41}$$

$$b_i = (-1, -1, 0^{2(i-1)}, -1, 2, 0^{T-1-2i}) \tag{3.42}$$

$$b_{i+1} = (0^{2i}, -2, -1, 0^{T-1-2i}) \tag{3.43}$$

$$\vdots \tag{3.44}$$

$$b_{k-1} = (-1, -1, 0^{2(k/2-1)}, -1, 2, 0^{T-1-2k/2}) \tag{3.45}$$

$$b_k = (0^{2k/2}, -2, -1, 0^{T-1-2k/2}) \tag{3.46}$$

If k is odd just replace $k/2 \rightarrow \lfloor k/2 \rfloor$ and $k \rightarrow k-1$ and add as the last vector: $b_k^{odd} = (-1, -1, 0^{2(\lfloor k/2 \rfloor)}, -1, 2, 0^{T-3-2\lfloor k/2 \rfloor})$. As for Kähler class we can choose:

$$j = (-j_0, 1^T) \text{ for } \frac{T}{3} \geq j_0 > \sqrt{T} e.g\, j_0 = \lfloor k/3 \rfloor - 1 \text{ and } k \geq 21 \tag{3.47}$$

where the upper bound is chosen such that $-j \cdot a > 0$ and the lower bound to ensure $j^2 > 0$. Moreover, it is also simple to check that $j \cdot b_i > 0$. One could find more solutions for small k but we are only interested in this work for large k and hence we will not attempt to enumerate those. This choice of vectors shows that anomalies permit to have unbounded many such gauge groups.

However, one could consider a string with minimal charge $Q = (-q, 0^T)$. This choice of charge has: $k_{i=odd} = q, k_{i=even} = 0, k_\ell = q^2 + 3q + 2 \geq 0, c_R = 3q^2 - 9q \geq 0$ true for

$q \geq 3$. However imposing worldsheet unitarity

$$\frac{3(q^2+3q+2)}{2+(q^2+3q+2)} + \lceil \frac{k}{2} \rceil \frac{52}{q+9} \leq 3q(q-9)+2 \tag{3.48}$$

one can note that the inequality cannot be satisfied when $3 \leq q \leq 9$ for any k(with lower bound as discussed above).

In particular, more generally these solutions are valid for any $k+2 \leq T \leq \frac{52k}{29} + \frac{273}{29}$ and hence similarly restrict k just as we saw above.

- e_6 with $b_i \cdot b_i = -6$

The gravitational anomaly imposes that $T \leq \frac{78k}{29} + \frac{273}{29}$. For example a solution can be found when $T = 2k+9$:

$$a = (-3, 1^T) \tag{3.49}$$

$$b_1 = (-1, -1, 1, -2, 1, 0^{T-4}) \tag{3.50}$$

$$b_2 = (0^3, -1, -2, -1, 0^{T-5}) \tag{3.51}$$

$$\vdots \tag{3.52}$$

$$b_i = (-1, -1, 0^{4(i-1)}, 1, -2, 1, 0^{T-4i}) \tag{3.53}$$

$$b_{i+1} = (0^3, 0^{4(i-1)}, -1, -2, -1, 0^{T-4i-1}) \tag{3.54}$$

$$\vdots \tag{3.55}$$

$$b_{k-1} = (-1, -1, 0^{4(k/2-1)}, 1, -2, 1, 0^{T-4k/2}) \tag{3.56}$$

$$b_k = (0^3, 0^{4(k/2-1)}, -1, -2, -1, 0^{T-4k/2-1}) \tag{3.57}$$

If k is odd as before we can replace $k/2 \to \lfloor k/2 \rfloor$ and $k \to k-1$ and add as the last vector: $b_k^{odd} = (-1, -1, 0^{4(\lfloor k/2 \rfloor)}, -1, -2, -1, 0^{T-4(\lfloor k/2 \rfloor + 1)})$. As for Kähler class we can choose:

$j = (-j_0, 1^T)$ for $\frac{T}{3} \geq j_0 \geq \sqrt{T}$.

However, just as we saw above strings with charge $Q = (-q, 0^T)$ and $3 \leq q < 10$ satisfy $c_R \geq 0, k_\ell \geq 0$ but are none unitary because unitarity relation cannot be satisfied:

$$\frac{3(q^2+3q+2)}{2+(q^2+3q+2)} + \lceil \frac{k}{2} \rceil \frac{78}{q+12} \leq 3q(q-9)+2 \tag{3.58}$$

More, generally these solutions can be adjusted and used for any $2k+2 \leq T$. Apart from the NHC one can note that also e_6 with 1 fundamental hypermultiplet is possible[106]. This theory has $b_i^2 = -5, a \cdot b_i = 3$ and $-51k \leq 273 - 29T$. The analysis of this is very similar to f_4 above so we will not repeat it.

- e_7 with $b_i \cdot b_i = -7$ with $\frac{1}{2}56$ matter

  The gravitational anomaly imposes that $T \leq \frac{105k}{29} + \frac{273}{29}$. For example a solution can be found when $T = 3k+9$:

$$a = (-3, 1^T) \tag{3.59}$$

$$b_1 = ((-1)^2, -1, -2, (1)^2, 0^{T-5}), \tag{3.60}$$

$$b_2 = (0^2, (-1)^2, -2, -1, 0^{T-5}) \tag{3.61}$$

$$\vdots \tag{3.62}$$

$$b_i = ((-1)^2, 0^{5(i-1)}, -1, -2, (1)^2, 0^{T-5i}), \tag{3.63}$$

$$b_{i+1} = (0^2, 0^{5(i-1)}, (-1)^2, -2, -1, 0^{T-5i}) \tag{3.64}$$

$$\vdots \tag{3.65}$$

$$b_{k-1} = ((-1)^2, 0^{5(k/2-1)}, -1, -2, (1)^2, 0^{T-5k/2}), \tag{3.66}$$

$$b_k = (0^2, 0^{5(k/2-1)}, (-1)^2, -2, -1, 0^{T-5k/2}) \tag{3.67}$$

Similarly, as above strings with charge $Q = (-q, 0^T)$ where $3 \leq q < 10$ are none unitary because they do not satisfy unitarity relation:

$$\frac{3(q^2+3q+2)}{2+(q^2+3q+2)} + \lceil\frac{k}{2}\rceil\frac{133}{q+18} \leq 3q(q-9)+2 \tag{3.68}$$

These solutions can be used for any T such that $3k+1 \leq T$ giving the same result.

- e_7 with $b_i \cdot b_i = -8$

The gravitational anomaly imposes that $T \leq \frac{133k}{29} + \frac{273}{29}$. For example the following solutions can be found when $T = 4k+9$:

$$a = (-3, 1^T) \tag{3.69}$$

$$b_1 = ((-1)^2, (-1)^2, -2, (1)^2, 0^{T-6}) \tag{3.70}$$

$$b_2 = (0^3, (-1)^3, -2, -1, 0^{T-7}) \tag{3.71}$$

$$\vdots \tag{3.72}$$

$$b_i = ((-1)^2, 0^{6(i-1)}, (-1)^2, -2, (1)^2, 0^{T-6i}) \tag{3.73}$$

$$b_{i+1} = (0^3, 0^{6(i-1)}, (-1)^3, -2, -1, 0^{T-6i-1}) \tag{3.74}$$

$$\vdots \tag{3.75}$$

$$b_{k-1} = ((-1)^2, 0^{6(k/2-1)}, (-1)^2, -2, (1)^2, 0^{T-6k/2}) \tag{3.76}$$

$$b_k = (0^3, (-1)^{6(k/2-1)}, -2, -1, 0^{T-6k/2-1}) \tag{3.77}$$

For the stings of charge $Q = (-q, 0^T)$ one has $k_{i=odd} = q, k_{i=even} = 0, k_\ell \geq 0, c_R \geq 0$ true for $q \geq 3$ but the unitarity bound:

$$\frac{3(q^2+3q+2)}{2+(q^2+3q+2)} + \lceil \frac{k}{2} \rceil \frac{133}{q+18} \leq 3q(q-9)+2 \tag{3.78}$$

shows that strings with $3 \leq q \leq 10$ are non-unitary. Generically, we can find such solutions for all $3k+3 \leq T$.

Apart from the two NHCs we studied one can note that also e_7 with $1, \frac{3}{2}$ fundamental hypermultiplet are possible. These theories have $b_i^2 = -6/-5, a \cdot b_i = 4/3$ and $-77k/-49k \leq 273-29T$ respectively. The analysis of this is very similar to f_4, e_6 as above and hence we will not repeat.

- e_8 with $b_i \cdot b_i = -12$

  A specific solution for this theory for large T is discussed in [124,119] where in the latter work they show that k can not be arbitrarily large for that solution.

Even though in the last four cases we do not have a more general way to show that there can only be finitely many terms, the solutions above seem to suggest so.

To sum up, in this section we have shown that certain theories which could potentially be allowed to have arbitrarily large size or arbitrarily many gauge factors, have in fact an upper bound or a more careful analysis reveals that they do not exist.

This gives a positive answer to the assumption of the Lampost principle that there should be an upper bound on the number of massless modes in a theory of quantum gravity at least for the majority of the proposed infinite families of anomaly free matter content.

3.2 Five dimensional supergravity

The previous section was devoted on addressing the finiteness of 6d landscape, which already was heavily constrained due to chiral anomalies. The next step would be to study similar questions in five dimensions. However, the problem is significantly more complicated as there are not as many conditions constraining the low energy physics as in 6d. One should note that even from the string geometry perspective a classification of Calabi-Yau threefolds is not complete and neither is a proof of their finiteness. However, as mentioned in chapter 1 swampland ideas suggest that indeed we expect the landscape to be finite. A good starting point would be to understand general swampland constraints for the 5d minimally supersymmetric theories. This section is devoted on the study of such conditions from both the bottom up and top down.

In chapter 1 the salient features of the $\mathcal{N} = 1$ supergravity theory in five dimensions were reviewed. In particular, a gauge theoretic perspective of the 5d supergravity theory was presented which is described by the effective theory of gravity coupled to vector multiplets for the gauge group $\mathcal{G}$ and hypermultiplets carrying the gauge charges. Additionally, basic concepts of the monopole strings of the 5d theory were reviewed and their existence was argued using the completeness hypothesis. In this section a special class of those strings will defined called *supergravity strings* which exist only in a supergravity theory.

3.2.1 Monopole strings in 5d supergravities

In section 1.9 and section 1.2 it was suggested that completeness of spectrum implies the existence of BPS monopole strings in 5d and their various properties were discussed.

Similarly to the 6d case and as reviewed in section A.1 the anomaly inflow for the gauge and the Lorentz transformations must be cancelled by the anomalies developed by the worldsheet degrees of freedom living on the monopole strings as. This fact allows us to compute the quantum anomaly of the 2d CFT on the string worldsheet from the anomaly inflow. Collecting those results the following relations are expected to hold:

- The monopole string has $(0,4)$ superconformal algebra in the IR CFT with $SU(2)_R$ as the R-symmetry
- $c_L = C_{IJK}q^I q^J q^K + C_I q^I$ and $c_R = C_{IJK}q^I q^J q^K + \frac{1}{2}C_I q^I$
- Center of mass modes: $c_L^{\text{com}} = 3\,, \quad c_R^{\text{com}} = 6$
- $\hat{c}_L = C_{IJK}q^I q^J q^K + C_I q^I - 3$ and $\hat{c}_R = C_{IJK}q^I q^J q^K + \frac{1}{2}C_I q^I - 6$
- 't Hooft anomaly coefficient $k_R = \frac{c_R}{3}$ for the $SU(2)_R$ is quantized to be an integer which means $c_R \in 6\mathbb{Z}$.
- The worldsheet theory can carry the current algebras for the bulk gauge symmetry. As noted in Equation A.19 't Hooft anomaly k_{IJ} was computed to be

$$k_{IJ} = C_{IJK}q^K\,, \tag{3.79}$$

  for the mixed anomaly between two Abelian currents J^I and J^J. In our convension, the right- (or left-) moving charged fields add positive (or negative) contributions to the anomaly coefficient k_{IJ}.
- According to Equation A.20 bulk Abelian gauge symmetry can enhance to non-Abelian symmetry, say G_i, at some special points of the Coulomb branch. Then:

$$-\frac{1}{4}k_i \text{Tr}F_i^2 \quad \text{with} \quad k_i = -h_{i,\alpha}q^\alpha\,, \tag{3.80}$$

  where F_i is the field strength of G_i and $h_{i,\alpha}$ is the coefficient in the gauge coupling h_i for G_i in the bulk effective action. The 't Hooft anomaly coefficient k_i for the non-Abelian symmetry is quantized as an integer number. The k_i is related to the level for the current algebra of the symmetry G_i. The level k current algebra of G_i realized by right-movers (or left-movers) provides $+k$ (or $-k$) contribution to the anomaly coefficient k_i.

As a simple example, let us consider the M-theory compactification on the quintic Calabi-Yau 3-fold discussed in the next section. This engineers a 5d $\mathcal{N} = 1$ supergravity theory with a single $U(1)$ gauge symmetry at low energy. The effective action is characterized by the cubic and the linear Chern-Simons levels given by

$$C_{000} = 5 \quad \text{and} \quad C_0 = 50 \,. \tag{3.81}$$

Now consider a monopole string with positive magnetic charge q for the $U(1)$ gauge symmetry. Using (A.18) and (A.19), one can easily compute the central charges of the 2d CFT on the string,

$$c_L = 5q^3 + 50q \,, \quad c_R = 5q^3 + 25q \,, \tag{3.82}$$

and the 't Hooft anomaly of the $U(1)$ current,

$$k_{U(1)} = 5q \,. \tag{3.83}$$

This implies that the worldsheet theory has a $U(1)$ current algebra with level $5q$ in the right-moving sector.

3.2.2 Supergravity strings

We will now introduce a special class of monopole strings called *supergravity strings*. The supergravity strings are 1/2 BPS objects that appear only in gravity theories and not in local CFTs. In this section, we will make this distinction clear and investigate consistency conditions on supergravity strings together with their implications for 5d supergravity theories. If the supergravity is geometrically engineered then these strings can be understood as M5 branes wrapping *semi-ample* 4-cycles in the geometry. In particular, the distinction between supergravity strings and local strings can be understood through the different properties that the 4-cycles need to satisfy. We will also investigate these properties in detail which will lead us to a geometric definition of supergravity strings studied in more detail in Section 3.2.4.

The BPS states of the 5d supergravity are electrically charged particles and the dual magnetically charged monopole strings. All BPS states are expected to have non-negative masses and non-negative tensions on the Coulomb branch, which essentially defines the Coulomb branch of the scalar vevs in the vector multiplets. We first propose that the Coulomb branch $\mathcal{C}$ is the space of the scalar moduli φ^I bounded by the set of hyperplanes where some BPS particle states become massless:

$$\mathcal{C} = \{\varphi^I \,, I = 1, \cdots, \mathbf{r} \,|\, m^2(\varphi^I) \geq 0\} \,. \tag{3.84}$$

Here $m^2(\varphi^I) \geq 0$ denotes that mass squared of all BPS particles are non-negative at the point labelled by φ^I. We also conjecture that if all BPS particles have non-negative mass squared, then the monopole string tensions are also non-negative at the point. This follows from the fact that if the volumes of all 2-cycles are non-negative, the volumes of 4-cycles are also non-negative in Calabi-Yau threefolds.

Definition: A 1/2 BPS magnetic monopole string on the Coulomb branch $\mathcal{C}$ in a 5d supergravity theory defines a *supergravity string* if all supersymmetrically compatible BPS particle states in the theory carry non-negative electric charge $e(A_\mu)$ under the dual Abelian gauge field A_μ:

$$e_i(A_\mu) \geq 0 \,, \tag{3.85}$$

where i runs over all particles obeying the BPS mass formula $|m| = \sum_I e_I \varphi^I$.

The dual gauge field A_μ in this definition precisely means the Abelian gauge field whose positive minimal magnetic charge is carried by the supergravity string with tension $T = \partial\mathcal{F} > 0$. We remark that the supergravity string here is defined with respect to a particular pair of supercharges shared by the supergravity string and the BPS particles taken into account in (3.85). For a given supergravity string preserving these supercharges, the electric charge condition (3.85) holds for all BPS particles preserving the same supercharges, but does not need to hold for anti-BPS particles satisfying $|m| = -\sum_I e_I \varphi^I$ that preserve another set of supercharges. This distinction between BPS and anti-BPS states allows one to clearly distinguish the supergravity string from other strings. In the followings, BPS states we will use refer to the states preserving this pair of supercharges.

We now claim that the supergravity strings exist only in supergravity theories, while local theories such as 5d SCFTs cannot have any supergravity strings. This property will enable us to explore some distinguished features of gravitational theories by using the supergravity strings. We note that the BPS W-boson of $U(1) \subset SU(2)$ gauge symmetry in a local gauge theory, called E_1 theory, in this convention has negative charge -2 under the $U(1)$ symmetry. Similarly, on the Coulomb branch the BPS W-bosons of a non-Abelian gauge group G carry electric gauge charges $e_a = -(\mathcal{K}_{ab})$ of the $U(1)^r \subset G$ gauge fields A_μ^b, where $\mathcal{K}_{ab}$ is the Cartan matrix of G with rank r. This implies that in local 5d gauge theories, the BPS monopole strings with magnetic charge $q^a > 0$ of the dual gauge fields A_μ^a cannot satisfy (3.85) and thus cannot be supergravity strings. More generally, we expect that supergravity strings cannot reside in any local theories. This can be proven for the cases admitting geometric constructions using geometric properties of special 4-cycles related to supergravity strings as we will see in the next section.

Conjectures

1. Supergravity strings exist only in supergravity theories.
2. The worldsheet theory on a supergravity string with magnetic charge q^I flows to a $(0,4)$ SCFT with $SU(2)_R$ R-symmetry if $C_{IJK}q^Iq^Jq^K > 0$.

The condition $C_{IJK}q^Iq^Jq^K > 0$ for the second conjecture rules out emergence of accidental symmetries in the worldsheet theories. For the strings with this condition, the R-symmetry in the infrared worldsheet CFT will be the $SU(2)_R$ symmetry. We are interested in only this type of supergravity strings in this section.

As presented above, on the other hand, it is possible that the worldsheet theories on monopole strings with $C_{IJK}q^Iq^Jq^K = 0$ have the SUSY enhancements or come from the strings in the 6d supergravity theories. The 6d supergravity strings were already studied in [119] by employing the same idea we use in this section. The monopole strings hosting interacting CFTs with $\mathcal{N} = (4,4)$ SUSY are also a part of the 6d supergravity strings. The supergravity strings with $\mathcal{N} = (0,8)$ SUSY were studied in [119] and the strings with $\mathcal{N} = (8,8)$ SUSY come from Type II strings compactified on T^5.

Supergravity strings are magnetic sources for the gauge fields in a supergravity theory and they should exist due to the completeness assumption for the spectrum in the gravity theory [159,13]. We will be assuming a stronger version of completeness assumption because we will in addition assume that the corresponding states in the spectrum are represented by BPS objects. This is obvious in geometry because any semi-ample divisor $\mathcal{D}$ with $\mathcal{D}^3 > 0$, which is defined in the next section, is effective and thus can be wrapped by an M5-brane. The wrapped M5-brane always leads to a supergravity string in the 5d gravity theory.

We expect that the worldsheet degrees of freedom living on a supergravity string satisfy unitary conditions. Basic unitary conditions are the following:

1. The central charges of the interacting 2d SCFT on a supergravity string with charge q^I are given by (A.18) and they have to be non-negative

$$\hat{c}_L \geq 0\,, \quad \hat{c}_R \geq 0\,. \tag{3.86}$$

2. The right-moving central charge is quantized as

$$c_R = C_{IJK}q^I q^J q^K + \frac{1}{2}C_I q^I \in 6\mathbb{Z}\,. \tag{3.87}$$

3. The tension of the supergravity string eq. (1.31) is always non-negative, i.e. $T_I \geq 0$, on the Coulomb branch $\mathcal{C}$.

From the condition (3.85) for supergravity strings, one can find an interesting relation between supergravity strings and Coulomb branch in supergravity theories. The condition (3.85) tells us that the positive scalar vev in the vector multiplet[2] dual to a supergravity string by itself without turning on other scalar vevs parametrizes a direction of moduli space where all particle states have non-negative mass squared. Therefore, this positive scalar vev necessarily lies within the Coulomb branch in the supergravity theory. Conversely, one may be able to find associated supergravity strings for any line on the Coulomb branch parametrized by a positive real number.

3.2.3 General supergravity conditions

In this subsection, we will discuss unitarity conditions on supergravity strings without assuming geometric embeddings of the gravity theory.

[2] This scalar vev can also be in a hypermultiplet if the dual gauge field is the graviphoton.

We will also argue that some geometric conditions discussed in the next section can be interpreted as physical constraints on supergravity strings.

Consider a BPS monopole string with magnetic charge q^I in the 5d gravity theory and suppose that this string is a supergravity string. Then the central charges $\hat{c}_R$ and $\hat{c}_L$ in (A.18) for the 2d IR CFT should be non-negative. This is an obvious condition for unitarity of the supergravity string.

More conditions can be found by using the 't Hooft anomalies given in (A.19) and in (A.20) for the Abelian and the non-Abelian symmetries, respectively. For this, let us first clarify the relation between the 't Hooft anomaly coefficients and the levels of Kac-Moody current algebras of G in the 2d CFTs. For a non-Abelian group G_i, the 't Hooft anomaly coefficient k_i given in (A.20) is required to be an integer by quantization condition. The 't Hooft anomaly receives positive contributions from right-moving modes while receives negative contributions from left-moving modes. Thus the level k current algebra adds $+k$ or $-k$ to the associated anomaly coefficient according to its chirality. It then follows that a non-zero coefficient k_i implies that the 2d CFT contains at least a Kac-Moody current algebra for G_i with level $k = |k_i|$ in the right-moving sector when $k_i > 0$ or in the left-moving sector when $k_i < 0$.

The 't Hooft anomalies for Abelian groups can mix each other. The anomaly coefficient k_{IJ} in (A.19) is a symmetric matrix and the eigenvalues are identified with the levels of Abelian current algebras. The precise values of the levels are not important in our discussions. However, the signature of the anomaly coefficient k_{IJ} is of some significance. It encodes the lower bounds on the number of representations of Abelian current algebras in the left- or right-moving sector. If the anomaly coefficient k_{IJ} has signature (n_+, n_-), where n_+ and n_- denote the number of positive and negative eigenvalues respectively, the worldsheet CFT necessarily involves at least n_+ Abelian current algebras in the right-moving sector and n_- Abelian current algebras in the left-moving sector.

We shall now argue that the signatures of 't Hooft anomaly coefficients in the 2d CFT on a supergravity string are restricted. The current algebras of a worldsheet CFT are realized by zero modes of the bulk fields on the magnetic monopole string background. The right-movers come from the Goldstone modes of broken symmetry generators including three position zero modes and a compact bosonic mode for the broken $U(1)$ gauge symmetry as well as their fermionic partners. On the other hand the other charged matter fields coupled to the supergravity string leave only fermionic zero modes that are left-movers on the worldsheet[36]. This means that the current algebras of gauge groups, but a single Abelian group, necessarily sit in the left-moving sector.

In the next section we define supergravity strings geometrically as M5-brane states wrapping semi-ample divisors by definition 3.93. Note that a semi-ample divisor has a non-negative intersection with every divisor in the threefold. This implies that a semi-ample divisor at the intersection with the gauge divisors for gauge group G can provide only hypermultiplets charge under the gauge symmetry. These hypermultiplets give fermionic zero modes in the left-moving sector on the string background.

We can rephrase this as the following condition on the signatures of the 't Hooft anomaly coefficients of the gauge groups under which the string is charged:

$$\mathrm{sig}(k_{IJ}) = (1, r-1) , \tag{3.88}$$

for Abelian groups where r is the rank of Abelian groups, and

$$k_i < 0 , \tag{3.89}$$

for all non-Abelian groups. This condition on the 't Hooft anomalies and thus on the levels of current algebras is one of the special features of supergravity strings. Moreover, this provides a field theory interpretation of the Hodge index theorem B.1.32 for the signature of the intersection pairing of semi-ample divisors in geometry.

Note that one of sig(k_{IJ}) for a supergravity string is always positive. This positive level is the center-of-mass contribution. The corresponding $U(1)$ current algebra is generated by the compact right-moving scalar field φ in the $\mathcal{N} = (0, 4)$ center-of-mass free hypermultiplet. In the geometric setting, this current algebra is generated by the divisor class itself for the monopole string[136]. After subtracting this center-of-mass contribution, all other levels in the interacting worldsheet CFT are negative meaning that the current algebras of gauge symmetries, but that of the $SU(2)_R$ symmetry, are realized in the left-moving sector.

We remark here that the supergravity strings in the 6d supergravity theories share the same property. As studied in[119], the worldsheet CFTs on 6d supergravity strings contain only left-moving current algebras. This property was used to distinguish supergravity strings from the instanton strings in 6d local SCFTs or little string theories.

A unitary realization of a current algebra contributes to the central charges in the 2d CFT. The central charge contribution from an Abelian current algebra is $c_{U(1)} = 1$. For a non-Abelian current algebra of G_i at level k, the central charge contribution is

$$c_{G_i} = \frac{k \cdot \text{dim}G_i}{k + h_i^\vee} \,, \tag{3.90}$$

where dimG_i is the dimension and $h_i^\vee$ is the dual Coxeter number of group G_i. Unitarity requires the level k to be positive definite. As discussed, all the current algebras are sitting in the left-moving sector in the interacting CFTs on supergravity strings. From this we find an inequality between the levels and the left-moving central charge

$$n + \sum_i c_{G_i} \leq \hat{c}_L \,, \tag{3.91}$$

where n is the number of Abelian gauge groups and G_i's are the non-Abelian groups whose current algebras are realized in the interacting CFT.

The unitary 2d $(0,4)$ CFT on a supergravity string must satisfy this inequality together with the conditions $\hat{c}_L, \hat{c}_R \geq 0$. If these conditions are violated by a supergravity string, then it means the string cannot host a unitary CFT that cancels the anomaly inflow arising from the bulk 5d supergravity theory. As a result the supergravity string cannot consistently couple to the 5d supergravity. By the completeness of string spectrum, the supergravity theory is therefore in the Swampland.

One can find more conditions on the supergravity strings from the properties of black holes. For example, let us consider a cone of monopole strings given by a linear combination of supergravity strings as $\mathcal{D} = \sum_I n_I q^I$ with positive coefficients n_I. This defines a Kähler cone of the low-energy theory. Choose now an arbitrary string of D inside the Kähler cone with positive coefficients $n_I > 0$. Then the large multiple of the chosen string, i.e. mD with $m \gg 1$, is expected to form a black string state with the entropy $S \propto \sqrt{\hat{c}_L}$. At large m, the cubic terms in the $\hat{c}_L$ scale as m^3 and will dominate the other terms. The central charge of the black string should be positive by unitarity. This immediately restricts the cubic term for any supergravity string to be positive semi-definite,

$$C_{IJK} q^I q^J q^K \geq 0 \,. \tag{3.92}$$

In geometry such string mD at large m amounts to a very ample divisor.

Indeed, the wrapped M5-brane on a very ample divisor with large c_L form a black string[136].

3.2.4 Supergravity strings from wrapped M5 branes

In Section 3.2.2 we defined a certain class of 5d monopole strings called *supergravity strings* which appear only in supergravity theories. In fact the definition of supergravity strings was motivated by geometric considerations of monopole strings and associated 4-cycles in Calabi-Yau geometry. In the case that the supegravity is geometrically engineered through M-theory on a Calabi-Yau threefold, the BPS states can be understood as M2- and M5-branes wrapping holomorphic 2- and 4-cycles respectively. Therefore, the 5d monopole strings arise from M5-branes wrapping 4-cycles represented by effective divisor classes $[\mathcal{D}] \in Pic(X) \cong H^2(X, \mathbb{Z})$ of the Calabi-Yau 3-fold X. The divisor class being effective means that it can be represented by an effective divisor $\mathcal{D}$, i.e. $\mathcal{D}$ is a non-negative linear combination of surfaces D_i (possibly singular): $\mathcal{D} = \sum_i n_i D_i$ (Definition B.1.1).

In order for a monopole string to be a *supergravity string* as defined in (3.85), we require the electric charges of all BPS particles for the dual gauge field to be non-negative. A BPS particle is the M2-brane wrapping a curve C and its electric charge for the gauge field dual to a divisor class $\mathcal{D}$ is given by the intersection of C and $\mathcal{D}$. We thus claim that

The monopole string wrapping an effective divisor $\mathcal{D}$ is a *supergravity string* precisely when

$$\mathcal{D} \cdot C \geq 0 \ \text{ for all curves } \ C \subset X. \tag{3.93}$$

The above condition on a divisor $\mathcal{D}$ is called *nef* in the algebraic geometry literature (Definition B.1.4). So our supergravity strings arise from M5-branes wrapping nef and effective divisors. Assuming a conjecture which we will formulate and motivate below, this is equivalent to just nef, or just semi-ample (Definition B.1.5).

The nef condition is closely related to the condition of a divisor being ample, or equivalently that its cohomology class is a Kähler class. Like nef divisors, ample divisors are characterized by its intersections, but the condition is more complicated: for $\mathcal{D}$ to be ample we require $\mathcal{D}^3 > 0$, $\mathcal{D}^2 \cdot S > 0$ for all surfaces $S \subset X$, and $\mathcal{D} \cdot C > 0$ for all curves $C \subset X$. This is the Nakai-Moishezon criterion for ampleness (Theorem B.1.16). In particular, from the point of view of divisors, the Kähler cone $\mathcal{K}(X)$ is generated by ample divisor classes.

As explained immediately after Theorem B.1.17, the closure of the Kähler cone $\overline{\mathcal{K}}(X)$ is simply the *nef cone* spanned by the classes of nef divisors. Therefore, the BPS states can be understood as M2-branes wrapping curves in the Mori cone $\mathcal{M}(X)$ and the supergravity strings are the dual M5-branes wrapping surfaces in the dual cone $\overline{\mathcal{K}}(X)$.

The effectiveness of the divisor class that the M5-brane wraps is required in order for it to be represented by a surface. However, the basic criterion of distinguishing supergravity strings from other monopole strings wrapping surfaces is the condition that the 4-cycle is nef.

> **Conjecture:** *any nef divisor is linearly equivalent to an effective divisor, and so can be represented by a surface.*

Assuming the conjecture, we only need the nef condition in order to get a supergravity string in the geometric setting. This conjecture is an open question of mathematics which has been discussed in the mathematics literature for several decades, e.g. [129,152].

However, we are making this conjecture based on considerations of physics, specifically the completeness of spectrum hypothesis. But first, we explain a bit more of the mathematical background, referring to the appendices for more complete definitions, proofs, and references. Assuming that the divisor $\mathcal{D}$ is nef, we have the following:

- If $\mathcal{D}^3 > 0$ we know that $\mathcal{D}$ is in fact big (Definition B.1.8) by Theorem B.1.21, which implies that is linearly equivalent to an effective divisor (Corollary B.1.29).

- If $\mathcal{D}^3 = 0$ but $\mathcal{D}^2 \neq 0$, and $c_2(X) \cdot \mathcal{D} > 0$ then by [152] we know that the divisor is also linearly equivalent to an effective divisor.

- If we only require $c_2(X) \cdot \mathcal{D} > 0$ then by [152] we know that there is a multiple of the divisor $m\mathcal{D}$ that is linearly equivalent to an effective divisor for $m \gg 1$.

Mathematically, there is no known proof that we can take $m = 1$ in the last case. However, since $m\mathcal{D}$ is an effective nef divisor then an M5 brane wrapping it gives rise to a supergravity string. In addition, by the completeness of spectrum hypothesis we know that the charge lattice should include all minimal charge states for a given state. Therefore this supports the claim that if $m\mathcal{D}$ gives rise to a supergravity string then the class of $\mathcal{D}$ should too. This implies that $\mathcal{D}$ should be effective as well in order to be represented by a physical surface.

Moreover, if we consider the cases where $\mathcal{D}^3 = c_2 \cdot \mathcal{D} = 0$, by Section 3.2.1 we expect that those cases are presenting supersymmetry enhancement of the monopole string worldsheet. In particular, these divisors lead to (4,4) and (8,8) supersymmetry enhancement. As we will see in the next section, (hyper-)elliptic and abelian surfaces of this type can be understood as giving rise to exactly that amount of supersymmetry respectively. Therefore, we expect that the divisors associated to $\mathcal{D}^3 = c_2 \cdot \mathcal{D} = 0$ are also effective.

(Semi-)Ample divisors in a CY_3 have several distinguished features. Let us consider a subspace of the Kähler moduli space parametrized solely by the Kähler parameter for a given ample divisor $\mathcal{D}$. The volume of a 2-cycle C in X is determined with respect to the Kähler form J as $\text{vol}(C) = J \cdot C$. On this subspace, the Kähler form can be written as $J = \varphi\mathcal{D}$ with the positive Kähler parameter φ for $\mathcal{D}$. Then the volume of a curve C is given by $vol(C) = \varphi\,\mathcal{D} \cdot C$. Since the ample divisors positively intersect any C in the Mori cone, the volume of every C in the Mori cone is positive with the positive Kähler parameter φ, i.e. $vol(C) > 0$ for all curves $C \subset$ Mori cone. This implies that the 3-fold X is smooth with positive volume on this subspace. Similarly, on a subspace of the Coulomb branch parametrized by a single positive Kähler parameter for a semi-ample divisor, the volume of every curve in the 3-fold is non-negative, thus $vol(C) \geq 0$ for all curves $C \subset$ Mori cone.

Remarks

1. If $\mathcal{D}$ is nef and effective, then it is semi-ample (Definition B.1.5), which implies that the class

of $m\mathcal{D}$ can be represented by a smooth surface for $m \gg 1$ (see Bertini's Theorem B.1.14 and the remark immediately following). With our main conjecture, we only need to assume $\mathcal{D}$ is nef.

2. It can be seen using the results in Appendix B.1.4 that the conjecture "nef implies effective" is mathematically equivalent to the juxtaposition of the two conjectures "nef implies semi-ample" and "semi-ample implies effective". If the conjecture "nef implies effective" is true, then every nef divisor is nef and effective, hence semi-ample (Theorem B.1.22). Furthermore, if a divisor is semi-ample, then it is nef (Proposition B.1.15), hence semi-ample by the assumed conjecture. The other direction of the equivalence is trivial. In particular, our conjecture implies that *the monopole strings which are supergravity strings are precisely the ones which wrap semi-ample divisor classes*.

3. The statement that all nef divisors are semi-ample was conjectured in [152].

In the meantime we would like to distinguish the supergravity strings from the monopole strings in local theories from a geometric viewpoint. Local field theories such as 5d SCFTs and 6d SCFTs on S^1 can be engineered by M-theory compactified on local non-compact Calabi-Yau 3-folds. We illustrate this with local $\mathbb{P}^2$. Let $\mathcal{D}$ be $\mathbb{P}^2$, thought of as a divisor in the local threefold. Let C be a curve of degree d in this $\mathbb{P}^2$. Then $\mathcal{D} \cdot C = -3d$. So $\mathcal{D}$ is not nef and the associated monopole string cannot be a supergravity string, even after embedding our local geometry in a compact geometry. On the other hand, $(-\mathcal{D}) \cdot C = 3d > 0$. This is precisely the condition needed to get an SCFT! Returning to mathematical terminology, $-\mathcal{D}$ is nef, if we understand nef on a noncompact threefold to be a condition on intersections with all compact curves.[3] But $-\mathcal{D}$ is not effective. Rather, it is *anti-effective*, meaning that its negative is effective. So on local $\mathbb{P}^2$, we have an anti-effective nef

[3] See the remark following Theorem B.1.17 for further discussion of this point.

divisor, which guarantees that we have an SCFT.[4]

Recall that the geometries for local 5d (and also 6d) SCFTs are constructed by gluing ruled or rational surfaces and their blowups [111,30]. The monopole strings in these theories are constructed from M5 branes wrapping those surfaces, but they are not supergravity strings as the wrapped divisors are never nef. By simply changing some signs in [111], we summarize the local case by saying that we get a 5d or 6d SCFT when the local geometry supports an anti-effective nef divisor which includes each of the glued surfaces in its support. Thus we claim that nef divisors exist only in compact CY 3-folds. This supports our claim that supergravity strings exist only in the 5d supergravity theories.

The supergravity strings are closely related to black strings (or MSW strings) studied in [136,139]. One can find a black string solution with a smooth horizon when the corresponding M5-brane wraps on a *very ample* divisor with a large central charge in the compact CY_3 [136]. Theorem (B.1.24) tells us that a very ample divisor can be constructed by considering a multiple of an ample divisor, i.e. $10\mathcal{D}$ is very ample if $\mathcal{D}$ is ample. In the black string solution, the attractor mechanism forces the Kähler class at the horizon to be the divisor class wrapped by the M5-brane giving rise to the black string. Then the requirement for the volume of every 2-cycle near the horizon to be positive restricts divisors forming black strings to be ample. So the black string should come from a wrapped M5-brane over an ample divisor. Furthermore, for the black hole solution being weakly curved, the ample divisor necessarily has large triple intersection number, implying that the divisor for the black string solution has to be very ample. This therefore means that a black string can always be written as a positive linear combination of Kähler cone $\overline{\mathcal{K}}(X)$ generators which are semi-ample divisors and thus is related to supergravity strings.

[4] If we embed the divisor $\mathcal{D} = \mathbb{P}^2$ inside a compact Calabi-Yau, then the anti-effective divisor $-\mathcal{D}$ cannot be nef. To see this, take any curve C which meets $\mathcal{D}$ in a finite nonzero number of points, for example the intersection of two general very ample divisors. Then $(-\mathcal{D}) \cdot C < 0$ and consequently $-\mathcal{D}$ is not nef.

3.2.5 Geometric conditions

We can extrapolate several geometric conditions by considering the various properties of semi-ample, ample and very ample divisors in a Calabi-Yau threefold. Ideally, our supergravity strings would arise from wrapping smooth surfaces. Any very ample divisor class has a smooth representative by Bertini's Theorem (Theorem B.1.14), but this is not necessarily the case for semi-ample or even ample divisors. An example of an ample divisor class that does not have a smooth representative is given in Appendix B.2.1. All we know is that some multiple of a semi-ample class has a smooth representative. For ample divisor classes $\mathcal{D}$, we can bound the multiple needed as the class $5\mathcal{D}$ has a smooth representative as observed before.

If a semi-ample divisor class can be represented by a smooth surface P, we can say more. Using tools from algebraic geometry one can show that smooth semi-ample divisors are minimal surfaces[5] with Kodaira dimension $\kappa \geq 0$, as shown in Proposition B.1.27. If P is an ample divisor, or more generally if we have a divisor P satisfying $P^3 > 0$ and $P \cdot C \geq 0$ for all curves C, then $b_1(P) = 0$ and $\kappa = 2$, also by Proposition B.1.27.

A complete classification of minimal projective surfaces is provided in Table 3.5 in terms of κ. The first row of Table 3.5 represents surfaces with $\kappa = -\infty$ which can give monopole strings in local field theories, not relevant for our current study.

For this reason, we only consider surfaces with $\kappa \geq 0$ in the rest of this section.

[5] These surfaces are shown to be minimal in Appendix B.1.4, meaning that they are not isomorphic to the blow up of any other smooth surface.

[6] Consider an elliptic threefold with a section and a curve C in base of the fibration wrapped by a D3 brane. The M/F-theory duality implies that the pull-back $\widehat{C}$ is wrapped by an M5 brane but such an elliptic surface always has a section. Therefore, an elliptic surface with no section does not correspond to a 6d string.

κ	Kähler surfaces S	Type
$-\infty$	$\mathbb{P}^2$, ruled surface $\mathbb{F}_n^g$	Local Theories
0	$K3$	(0,8) susy enhancement.
0	Enriques surface	5d supergravity string
0	hyperelliptic surface	(4,4) susy enhancement
0	abelian surface	(8,8) susy enhancement.
1	minimal elliptic surface	6d supergravity string on a circle when it has a section (5d otherwise)[6].
2	minimal surface of general type	5d (0,4) supergravity string when $b_1 = 0$

Table 3.5: The first column is the Kodaira dimension κ of the surface S. The second column presents the Enriques-Kodaira classification of minimal Kähler surfaces. The third column describes the surfaces of the second column that could be 5d supergravity strings as defined in section 3.2.2.

Computing the central charges c_R, c_L from geomerty

The central charges of the 2d SCFT on monopole strings in the 5d supergravity are related to invariants of the associated surfaces. In particular, the degrees of freedom contributing to the central charges come from the moduli of the surface P, the two-form tensor fields and fermions on the worldvolume of the M5-branes on P. The computation of the central charges c_R, c_L, which we will review here, was done originally by [136], assuming very ampleness of the divisor P. However, the same arguments hold for an ample smooth divisor which we will assume for the next computation. Here P is a 4-cycle and its cohomology class is $[P] \in H^2(X, \mathbb{Z})$, which we will also write as P for simplicity. In particular, P can be expressed with respect to a basis ω_I of $H^2(X, \mathbb{Z})$ and charges $q^I \geq 0$ as $P = \sum q^I \omega_I$, meaning that the divisor is effective.

The left-moving central charge c_L has no contributions coming from fermion zero modes since $b_1(P) = 0$ by (B.1.31)[7] and the contributions from the bosonic degrees of freedom in terms of $P^3 : H^2(X, \mathbb{Z}) \to \mathbb{Z}$ and $c_2(X) \in H^4(X, \mathbb{Z})$ the second chern class of the threefold X, are the $m_p = \frac{1}{3}P^3 + \frac{1}{6}c_2(X) \cdot P - 2$ real moduli of P and the $b_2^- = \frac{2}{3}P^3 + \frac{5}{6}c_2(X) \cdot P - 1$ dimensional space

[7] Assuming the Calabi-Yau threefold has an $SU(3)$ holonomy.

of anti-self dual two-forms on P. Therefore,

$$c_L = m_p + b_2^- + 3\,, \tag{3.94}$$

where the last contribution represents the 3 translation zero modes. Similarly, the bosonic contribution to c_R is given by the moduli m_p and the $b_2^+ = \frac{1}{3}P^3 + \frac{1}{6}c_2(X)\cdot P - 1$ dimensional space of self-dual two-forms on P, together with the 3 translational zero modes. In addition, c_R also has $f_p = \frac{1}{3}P^3 + \frac{1}{6}c_2(X)\cdot P$ fermion contributions as required by supersymmetry which come from $(0,2)$ forms on P. Hence,

$$c_R = m_p + b_2^+ + 3 + f_p \tag{3.95}$$

The c_L, c_R central charges can be expressed in terms of P as

$$\boxed{c_L = P^3 + c_2\cdot P \text{ and } c_R = P^3 + \frac{1}{2}c_2(X)\cdot P}\,, \tag{3.96}$$

In particular, one can note that the right-moving central charge c_R is mapped to the geometric genus $p_g = h^{2,0}(P)$ of the divisor P. This can by seen by considering the holomorphic Euler characteristic $\chi(\mathcal{O}_P)$ which by Theorem B.1.18 is given by

$$\chi(\mathcal{O}_P) = \frac{1}{6}P^3 + \frac{1}{12}P\cdot c_2(X) = p_g + 1 \quad \Longrightarrow \quad \boxed{c_R = 6(p_g+1)}\,, \tag{3.97}$$

Note that we used the fact that $\chi(\mathcal{O}_P) = p_a + 1 = p_g - q + 1$ where p_a is the arithmetic genus of P and that the irregularity $q = h^{1,0}(P) = 0$, as we saw previously by (B.1.31). The left-moving central charge c_L including the center-of-mass contribution can be understood as the topological

Euler characteristic $\chi(P)$:

$$\chi(P) = c_L = P^3 + P \cdot c_2(X) \,. \tag{3.98}$$

In the case of semi-ample divisors with $P^3 = 0$ inside a Calabi-Yau threefold, technically the irregularity $q = h^{1,0}(P)$ might not be zero.

The cases of surfaces with a non-zero irregularity either lead to SUSY enhancement of the 2d worldsheet CFT or describe 6d supergravity strings.

Claim 3.2.1 *Any surface with $q > 0$ corresponds either to a string of a 6d supergravity compactified on a circle or susy enhancement of the worldsheet CFT with the exception of $\kappa = 1$ surfaces without a section* [8]. *In particular, smooth Kähler irregular surfaces fall in the following classes:*

- *$\kappa = 0$ hyperelliptic surfaces: which have $q = 1$, $c_2 \cdot P = 0$ with $c_R = c_L = 6$. This surface gives rise to $\mathcal{N} = (4,4)$ supersymmetry enhancement.*

- *$\kappa = 0$ abelian surfaces: which have $q = 2$, $c_2 \cdot P = 0$ with $c_R = c_L = 12$. This surface gives rise to $\mathcal{N} = (8,8)$ supersymmetry enhancement.*

- *$\kappa = 1$ elliptic surfaces (with a section) over genus $g > 0$ curves: which have $q = g$* [9]*, $c_2 \cdot P \geq 0$ with $c_R = 6g + \frac{1}{2}c_2 \cdot P$, $c_L = 6g + c_2 \cdot P$. These surfaces give rise to 6d supergravity strings*[119] *on circle.*

The central charge of the first two surfaces can be computed to be $c_R = c_L = 6q$ by similar methods as we did in Section 3.2.5 but with $h^{1,0}(P) \neq 0$. They both have the same number of left-moving bosons and fermions, $N_L^B = N_L^F = 4q$ where the left-moving fermions are induced by the

[8] As described previously this class of surfaces does not describe 6d strings but they can be irregular with $q \neq 0$. In particular, consider the bundle L isomorphic to the Hodge bundle (whose fiber over $p \in C$ is the 1-dimensional vector space of holomorphic 1-forms on the elliptic fiber E_p over p). The Hodge bundle is trivial if and only if you can find (globally over C) a family of nonvanishing holomorphic 1-forms on the fibers E_p which varies holomorphically in p. In the case that it is trivial $q = g + 1$, otherwise $q = g$[76].

[9] When the surface is a trivial fibration of the form $C \times E$ for C a genus g curve and E an elliptic curve, $q = g + 1$ and hence $c_R = 6(g+1)$, $c_L = 6(g+1)$ since $c_2 \cdot (C \times E) = 0$.

non-zero $h^{1,0}(P) = q$[136]. This is compatible with (4,4) and (8,8) supersymmetry enhancements of the worldsheet CFT. However, the last case of surfaces gives rise to 6d supergravity strings on a circle. This is because by[188,152] we know that the existence of $\kappa = 1$ elliptic surfaces with $c_2 \cdot P \geq 0$ inside a Calabi-Yau threefold means that the threefold always has an elliptic fiber structure. Since the threefold is elliptic we can invoke the M-theory/F-theory duality and realize the string coming from M5-brane wrapping the elliptic surface over genus g curve C as a D3-brane wrapping the curve C. As was discussed in[88,48] the central charges for these surfaces can be computed as in Section 3.2.5 with non-vanishing $h^{1,0}$ with the addition of an emergent $SU(2)_R$ flavor symmetry in the IR. Therefore, the central charges will be the same as in the formula (3.96) as expected.

An important note is that the geometric genus p_g of the surface P is an integer number which is in accordance with the quantization condition of $c_R \in 6\mathbb{Z}$ as seen in (3.97) .

The formulas for the c_R, c_L central charges match the one found from field theories in (A.18), which are given by a combination of the cubic and linear Chern-Simons terms evaluated on the charge q^I string. These Chern-Simons terms are the Chern classes of the divisor P which can be express in terms of a bases ω_I of $H^2(X, \mathbb{Z})$ and charges q^I as $P = \sum q^I \omega_I$:

$$P^3 \equiv \int_P c_1^2(P) = C_{IJK} q^I q^J q^K \,, \quad P \cdot c_2(X) \equiv \int_P c_2(X) = C_I q^I \,. \tag{3.99}$$

Until now we only considered smooth divisors, but there is no reason to assume that the divisor is smooth and in fact there is no simple algebraic criterion to determine smoothness. However, even though the geometric procedure of computing the central charges of Section 3.2.5 is no longer well-defined, we can still compute the central charge from physics as we did in Section 3.2 through anomaly inflows. This procedure shows that the central charges are still given by (A.18) and hence by the equation (3.99), so they can still be expressed geometrically as (3.96). Unfortunately, there is no classification for singular Kähler surfaces as we saw for smooth ones in Table 3.5. However, if we have a singular nef surface P then it is also semi-ample as we saw previously and hence some multiple mP for $m \geq 0$ is smooth. Therefore, mP is smooth and semi-ample and hence is one of the smooth surfaces described in Table 3.5. For example, consider mP to be a smooth K3 surface for $m > 1$. In that case we have that $(mP)^3 = 0$, hence $P^3 = 0$. From eq. (3.97) we can see that $2P^3 + c_2 \cdot P$ is a multiple of 12, we conclude that $c_2 \cdot P \in 12\mathbb{Z}$ since $P^3 = 0$. We also know that $c_2 \cdot mP = 24$ for K3 , and therefore we conclude that $m = 2$. Similarly, we note that mP cannot be a smooth Enriques because in that case $c_2 \cdot mP = 12$, hence $m = 1$. But we assumed that P is not smooth.

3.2.6 Conditions on 4-cycles and strings

In this section, we will analyze the geometric condition that arise by considering semi-ample, ample or very ample divisors in a compact threefold together with their implications as Swampland conditions for the associated supergravity theory. In particular, we will relate the conditions on the divisors of supergravity strings with the constraints on the gauge group and the matter content in the bulk supergravity theory. Interestingly, some of the geometric conditions can be interpreted as unitarity constraints on 2d worldsheet CFTs of supergravity strings.

Geometric conditions on the supergravity strings

The various properties that the divisors will need to satisfy will lead us to various consistency conditions for the supergravity when geometrically engineered. Some of the conditions which we will discuss below can also be derived from a field theory analysis of the 5d supegravity theory, and some are new ingredients that do not have obvious origin in the physics. Supergravity strings amount to semi-ample (or, equivalently, nef) divisors in a Calabi-Yau threefold.

As we have discussed above, a smooth semi-ample divisor is a minimal surface with Kodaira dimension $\kappa \geq 0$ (See Table 3.5). Some bounds on the invariants of these minimal surfaces are listed in Table 3.6. These bounds are discussed in some detail in appendix B. We conjecture that these bounds hold for singular semi-ample divisors as well.

1. $P^3 \geq 0$, in particular $\begin{cases} P^3 = 0 & \text{for } \kappa = 0, 1 \\ P^3 > 0 & \text{for } \kappa = 2 \end{cases}$

2. $P \cdot c_2(X) \geq 0$, (Theorem B.1.28) (with strict inequality if ample) and $P \cdot c_2(X) \in 2\mathbb{Z}$ (Theorem B.1.18).

3. $4P^3 - P \cdot c_2(X) + C \geq 0$ with $C = \begin{cases} 36 & P^3 \text{ even} \\ 30 & P^3 \text{ odd} \end{cases}$
   for smooth surfaces with $\kappa = 0, 2$ (Theorem B.1.33)

4. $h^{1,1}(X) \leq P^3 + P \cdot c_2(X) - 2$ for P smooth and ample (Theorem B.1.34)

5. $N_{-2} \leq \frac{1}{6}(4P^3 + 5P \cdot c_2(X)) - 1$ for P big and nef (Theorem B.1.35)
   Here, N_{-2} denotes the number of rational (-2) curves on P.

Table 3.6: Conditions that the surface P needs to satisfy depending on its general properties.

In the first inequality, $P^3 \geq 0$ follows from Theorem B.1.17 while the subcases follow from Proposition B.1.27. The first two inequalities imply that the supergravity strings arising from M-theory on a threefold X have non-negative cubic and linear Chern-Simons terms found in eq.(3.99). The fourth bound can be rewritten as $h^{1,1}(X) \leq c_L - 2$ and it will precisely match the constraint on the rank of the gauge group of the low-energy 5d theory coming from a constraint on the unitary 2d CFTs living on supergravity strings. The fifth bound, as we will see in the next section, will turn out to be very useful in constraining the rank of the non-Abelian gauge groups in the low-energy theory engineered by geometry. This inequality does not seem to have an obvious origin from physics.

Another important property is given by the *Hodge index theorem* (Theorem B.1.32). This theorem tells us that on any smooth divisor P, the intersection product on $H^2(P)$ has signature $(1, h^{1,1}(P) - 1)$. This mathematical theorem is interpreted in physics as the condition (3.88) on the signature of levels of the current algebras in the worldsheet CFT on the supergravity string.

Lastly, there exists an interesting inequality on the Hodge numbers of a Calabi-Yau threefold X:

$$-36P^3 - 80 \leq \frac{c_3(X)}{2} = h^{1,1}(X) - h^{2,1}(X) \leq 6P^3 + 40 \quad \text{for } P \text{ very ample} \tag{3.100}$$

Moreover, the inequality can be sharpened by replacing the left hand side by -80, -180 and the right hand side by 28, 54 when $P^3 = 1, 3$ respectively. The proof of this inequality can be found in [114].

In particular, this inequality does not seem to have an obvious origin from physics but it provides a strong bound on the dimension of the Higgs branch, therefore on the representations of matter hypermultiplets in the 5d effective theory.

Suppose for example that a threefold X leads to an effective theory with gauge group G, and this theory can Higgs to another threefold X' with $h^{1,1}(X') = 1$. The gauge group G of the original theory will be broken to $U(1)$ under this Higgsing. The charged hypermultiplets in the original theory parametrize the Higgs branch of the moduli space [172] which is a subspace of the complex moduli parametrized by cohomology classes in $h^{2,1}(X')$. This implies the relation

$$\sum_i \dim(R_i) - \dim(G) + 1 \leq h^{2,1}(X') \,, \tag{3.101}$$

where i runs over all hypermultiplets and $\dim(R_i)$ is the dimension of the representation R_i of the i-th hypermultiplet. The resulting 3-fold X' has a single Kähler class represented by an ample divisor P. According to (B.1.24), the divisor classes nP are very ample when $n \geq 10$. Using the inequality (3.100) for $h^{2,1}(X')$, we find a bound on the representations of charged hypermultiplets in the original theory of the 3-fold X:

$$\sum_i \dim(R_i) - \dim(G) \leq 36n^3P^3 + 80, \text{ where } P \text{ is ample and } n \geq 10\,. \tag{3.102}$$

3.3 Constraints on supergravity theories

In this section, we will constrain 5d supergravity theories by using the geometric conditions on semi-ample divisors and the unitary conditions on worldsheet CFTs of supergravity strings presented in the previous sections.

3.3.1 $U(1) \times G$ theories

The first example is the supergravity theory with $U(1) \times G$ gauge group where G is a product of non-Abelian groups $G = \prod_i G_i$. Without loss of generality, we can choose a basis for the $U(1)$ divisor H such that both its triple intersection and Kähler parameter φ^0 are positive. In addition the gauge couplings h_i for non-Abelian symmetries are required to be positive. The effective theory in this basis has

$$H^3 = C_{000} > 0\,,\ \varphi^0 > 0 \quad \text{and} \quad h_i = h_{i,0}\varphi^0 > 0 \text{ for all } G_i\,. \tag{3.103}$$

The perturbative hypermultiplets carrying the $U(1)$ charge have masses proportional to φ^0 and integrating them out leads to shifts in the Chern-Simons levels. We assume that all such $U(1)$ hypermultiplets are already integrated out. Then the remaining perturbative states are charged only under the non-Abelian group.

There can also be non-perturbative states carrying the $U(1)$ charge: for example, the instanton particles of the non-Abelian gauge group G. When the Coulomb branch parameters for G are small enough compared to h_i, the BPS instanton state has mass proportional to the gauge coupling, $|m_{\text{inst}}| = h_i$ up to a constant factor. This implies that all the BPS particles carrying non-zero $U(1)$ charges have positive electric charge under the $U(1)$ gauge symmetry. The divisor H is the dual to this $U(1)$ gauge symmetry. Therefore the magnetic monopole string with positive charge q on this divisor H is a supergravity string and the divisor H is thus semi-ample. This should be true even if the supergravity theory is not geometrically realized. Since the string on H is a supergravity string, we can analyze consistency of this string by using the conditions presented in the previous sections and can examine if the bulk gravity theory with the string is consistent or not.

The worldsheet theory on the monopole string of the divisor H should be a $\mathcal{N} = (0,4)$ CFT. Since the H positively intersects the gauge divisors, the worldsheet theory should contain unitary representations of current algebras for G. We find that the CFT on a single string with unit magnetic charge $q = 1$ contains the current algebras for the bulk gauge group G_i at level $k_i = h_{i,0}$. Then the unitary condition in (3.91) puts a bound on the total rank of the non-Abelian gauge group as

$$\sum_i c_{G_i} = \sum_i \frac{k_i \cdot \text{dim} G_i}{k_i + h_i^\vee} \leq \hat{c}_L \quad \rightarrow \quad \sum_i r_i \leq C_{000} + C_0 - 3 \,, \tag{3.104}$$

where $r_i = \text{rank}(G_i)$. Here we have used the fact that c_{G_i} takes the minimum value r_i when $k_i = 1$. In particular, this shows that the rank of the non-Abelian gauge group G in the bulk 5d supergravity theory is bounded from above by the Chern-Simons coefficients C_{000} and C_0. We remark that this bound generically holds for any 5d supergravity theory with gauge group $U(1) \times G$ regardless of whether it admits geometric construction or not.

For example, suppose that a supergravity theory with $U(1) \times G$ gauge group is Higgsed to a quintic threefold. The Higgsing does not change the Chern-Simons coefficients C_{000} and C_0. This implies that the Chern-Simons levels of the original theory before being Higgsed are fixed to be those of the quintic hypersurface H, i.e. $C_{000} = H^3 = 5$ and $C_0 = H \cdot c_2 = 50$. From this, we find a strict bound on the rank of the non-Abelian gauge group G

$$\sum_i r_i \leq 52\,, \tag{3.105}$$

in any supergravity theory with a single $U(1)$ symmetry which Higgses to a quintic threefold.

When the supergravity theory before being Higgsed has a geometric construction, we can find a stronger bound by using the geometric bound in (B.1.35). Note that the non-Abelian symmetry G can remain unbroken, when the Kähler parameter φ^0 is turned on, only if the $U(1)$ divisor H and the gauge divisors E_i are glued along rational (-2) curves in H and the fibers in E_i. Also H must be glued to all E_i divisors in order that the low-energy theory has gauge couplings h_i with proper signs. This imposes a bound $\sum_i r_i \ \leq\ N_{-2}$ on the rank of the non-Abelian group with respect to the number of (-2) curves in H. Therefore, the bound (B.1.35) on N_{-2} tells us that

$$\sum_i r_i \leq \frac{2}{3}H^3 + \frac{5}{6}H \cdot c_2(X) - 1 = 44\,. \tag{3.106}$$

for supergravity theories admitting M-theory construction on Calabi-Yau 3-fold that reduce to a quintic threefold after Higgsing.

In Appendix B.2, we present a number of concrete constructions of compact Calabi-Yau threefolds that Higgs to a quintic threefold. These geometries engineer the supergravity theories with gauge group $U(1) \times SU(2)$ or $U(1) \times SU(3)$. It is obvious that the above bound (3.106) is consistent with these examples. One can check that divisors contained in these geometries and the corresponding monopole strings satisfy all the conditions we listed in the previous sections.

The first example is for the supergravity theory with $U(1) \times SU(2)$ gauge group coupled to $N_f = 9\, SU(2)$ fundamental hypermultiplets. The threefold consists of two divisor classes H, the proper transform of the hyperplane class of the quintic, and E, the exceptional divisor of the blowup. The triple intersections of H and E classes are given in (B.69). Two divisors are glued along a (-2) curve in H and the fiber class r in E. In this example, the Käher cone is generated by H itself and $H - E$. The supergravity strings are then the M5-branes wrapping any 4-cycles D which can be written as

$$D = mH + n(H - E) , \quad m, n \geq 0 . \tag{3.107}$$

The central charges of the supergravity strings are

$$\begin{aligned} \hat{c}_L &= m(5m^2 + 15mn + 9n^2) + 50m + 36n - 3 , \\ \hat{c}_R &= m(5m^2 + 15mn + 9n^2) + 25m + 18n - 6 . \end{aligned} \tag{3.108}$$

One can easily see that the central charges are positive and $\hat{c}_R \in 6\mathbb{Z}$ for the non-trivial supergravity strings as expected. On generic points of the Käher moduli space, the gauge symmetry is broken to $U(1) \times U(1)$. We checked that the signature of the levels of $U(1) \times U(1)$ current algebras, which include the center-of-mass sector, in the supergravity strings is always $\text{sig}(k_{IJ}) = (1, 1)$. This is consistent with the condition (3.88) on the signature. This result is guaranteed by the Hodge index theorem B.1.32 for the semi-ample divisors in this geometry. All other geometric conditions are surely satisfied.

When we turn on only the Kähler parameter of the H class, then the $SU(2)$ gauge symmetry remains unbroken. In this case, the 2d CFT on the monopole string of the H class carries a $U(1)$ current algebra at level 5 in the right-moving sector and a $SU(2)$ current algebra at level 1 in the left-moving sector. The central charge of the $SU(2)$ current algebra in the worldsheet is $c_{SU(2)} = 1$. So the inequality $c_{SU(2)} \leq \hat{c}_L$ in eqn. (3.91) is satisfied.

More examples of supergravity theories with $U(1) \times SU(2)$ gauge symmetry are given in Appendix B.2.2.

The second example is the supergravity theory of $U(1) \times SU(3)$ gauge symmetry with $N_f = 11$ fundamental hypermultiplets. The CY_3 geometries for this theory are constructed in Appendix B.2.3. Each threefold is labelled by an integer $0 \leq n \leq 3$ and corresponds to the $SU(3)$ gauge group at Chern-Simons level $\kappa = -\frac{3}{2} + n$. The threefold consists of three surfaces, H, E_1 and E_2. The divisor H is the proper transform of the hyperplane class of the quintic and E_1 is a Hirzebruch surface $\mathbb{F}_{5+n}$ with 11 blowups[10] and E_2 is a Hirzebruch surface. Two (-2) curves in H are each glued to a fiber in E_1 and another fiber in E_2. Two surfaces E_1 and E_2 are glued along the section $e^2 = -(n+5)$ in E_1 and a rational curve $C^2 = n+3$ in E_2. The triple intersections of three surfaces are given in eqn. (B.100).

Let us consider the case with $n = 0$. The Kähler cone is generated by $H, H - E_1 - E_2$ and $2H - 2E_1 - E_2$. The supergravity strings come from the M5-branes wrapping linear combinations of these generators with non-negative integer coefficients. The central charges $(\hat{c}_L, \hat{c}_R)$ for these three generators are $(52, 24)$, $(33, 12)$ and $(79, 42)$ respectively. As expected since the (self-)triple intersections of the generators are non-negative, all supergravity strings have positive central charges. We also checked that the signature of the level of the current algebras is $\text{sig}(k_{IJ}) = (1, 2)$ for all three generators which is in accordance with the condition (3.88).

The low-energy theory has the $SU(3)$ gauge symmetry enhancement along the Kähler moduli space of the Kähler parameter for H. The M5-brane wrapping the divisor H gives rise to a monopole string hosting in the left-moving sector a level 1 current algebra for the $SU(3)$ symmetry. The unitary condition $c_{SU(3)} \leq \hat{c}_L$ for this monopole string is therefore satisfied with $c_{SU(3)} = 2$ and $\hat{c}_L = 52$.

[10]Our construction exhibits this surface as an 11-fold blowup of $\mathbb{F}_0$ or $\mathbb{F}_1$ depending on the parity of n, but by blowing down the half-fibers disjoint from the section where the two components are glued, we get $\mathbb{F}_{5+n}$.

3.3.2 Abelian gauge theories

Now consider a generic point on the Coulomb branch of the moduli space in a supergravity theory engineered in M-theory on a CY3. The gauge symmetry G in the 5d supergravity is fully broken to its Cartan subgroup $U(1)^{\mathbf{r}+1}$. There are a set of basis 4-cycles P_I with $I = 0, 1, \cdots, \mathbf{r}$ for the $U(1)^{\mathbf{r}+1}$ gauge group. We claim that the holomorphic surfaces P_I can always be chosen to be semi-ample divisors in the 3-fold. In other words, all Abelian gauge groups in the low energy effective theory can be represented by a set of $\mathbf{r} + 1$ semi-ample divisors that are part of the Kähler cone generators. Since they are semi-ample, the corresponding strings are all supergravity strings.

The effective Abelian theory is characterized by the triple intersections $C_{IJK} = \int_X P_I \cdot P_J \cdot P_K$ and the second Chern classes $C_I = \int_X P_I \cdot c_2(X)$. From the fact that all P_I are *nef* and semi-ample divisors, one finds that the triple intersections and the second Chern classes are all non-negative. We propose

$$C_{IJK} \geq 0\,, \quad C_I \geq 0 \quad \text{for all } I, J, K\,, \tag{3.109}$$

from the properties of semi-ample divisors. The inequalities for C_{III} and C_I are obvious by the definition of semi-ample divisors. Also, C_{IIJ} with $I \neq J$ is the sum of intersection numbers $P_I \cdot C$ between a semi-ample divisor P_I and curves C at the intersection $P_I \cap P_J$, which tells us that $C_{IIJ} \geq 0$. Similary, C_{IJK} is the sum of intersection numbers between the divisor P_I and curves at the intersection $P_J \cap P_K$, and it needs to be non-negative for the semi-ample divisors.

In addition, the surfaces P_I must satisfy the Hodge index theorem. For $\mathbf{r} = 1$ cases, for instance, the Hodge index theorem or the relation (3.88) says that the signature of the levels k_{IJ} with $I, J = 0, 1$ should be $(1, 1)$ for the worldsheet CFT on wrapped M5-brane over each P_I. One can then deduce the following two conditions on the triple intersections from the M5-branes wrapping once on P_0 and P_1 respectively :

$$C_{000}C_{001} \leq C_{011}^2 \,, \quad C_{111}C_{011} \leq C_{001}^2 \,. \tag{3.110}$$

3.3.3 Generic gauge theories

We will now turn to supergravity theories coupled to generic gauge groups. If these theories can be geometrically engineered, then we can constrain them by using conditions on divisors in the 3-fold as follows.

Let us consider a 3-fold X and the low-energy theory at a special submanifold on the Kähler moduli space of X where some Abelian symmetries enhance to non-Abelian symmetries $G = \prod_i G_i$. More precisely, we are interested in the effective theory in the moduli space where all the Kähler parameters φ_α for $U(1)$ symmetries are taken to be large, while the Kähler parameters φ_i for some non-Abelian symmetries G are turned off. If the non-Abelian symmetry G in the low-energy theory remains unbroken even after integrating out all matters charged under the Abelian symmetries, then we say that M-theory compactified on X at low-energy is described by the supergravity theory with gauge group G times multiple Abelian factors. We shall now assume this and constrain such effective theories.

We first conjecture that all the 4-cycles P_α for Abelian gauge groups in X can be chosen to be semi-ample divisors. So there exists a basis where all P_α's for $U(1)$ gauge groups are semi-ample. In this basis we find the following conditions on the Chern-Simons levels,

$$C_{\alpha\beta\gamma} \geq 0 \,, \quad C_\alpha \geq 0 \quad \text{for all } \alpha, \beta, \gamma \,, \tag{3.111}$$

where α, β, γ denote the indices for the Abelian gauge groups. These conditions again follow from the fact that divisors P_α are semi-ample and they non-negatively intersect all effective 2- and 4-cycles in X.

The requirement for the non-Abelian symmetry G_i unbroken imposes non-trivial constraints on the intersection structure between the Abelian divisors P_α and non-Abelian gauge divisors E_i. First, since we want to preserve the non-Abelian symmetry G on the moduli space of the Kähler parameter φ^α for P_α, the triple intersections $C_{\alpha\beta i}$ necessarily vanish. Otherwise the corresponding Chern-Simons interaction (partially) breaks the symmetry G. This condition $C_{\alpha\beta i} = 0$ should be true even after we turn on small Kähler parameters φ^i for the non-Abelian gauge divisors E_i because the massive states sitting in some representations of the non-Abelian symmetry that is weakly broken by φ^i after integrated out cannot induce Chern-Simons terms with coefficient $C_{\alpha\beta i}$.

Also the gauge couplings of the non-Abelian groups need to be positive. This forces

$$h_{i,\alpha} \geq 0\,, \tag{3.112}$$

for all i and α. From this, one can deduce more conditions on the Chern-Simons levels when X is fully resolved. Let us turn on small Kähler parameters φ^i for non-Abelian gauge divisors E_i and assume $\varphi^i \ll \varphi^\alpha$. The positivity of the gauge couplings (3.112) is then translated into

$$C_{\alpha ii} \leq 0 \quad \text{for all } i\,, \qquad C_{\alpha ij} \geq 0 \quad \text{for all } i \neq j\,. \tag{3.113}$$

3.4 A bound on the matter representations

In this section a further consistency condition that needs to be imposed for a consistent 6d supergravity theory is proposed. In the previous sections it was summarized how the existence of BPS strings strongly constrains the bulk theory. In particular, the bulk gauge groups emerge as current algebras on the 2d worldsheet and together with unitarity on the worldsheet, one can impose constraints on the rank of the gauge groups. Similarly, we argue that the 2d worldsheet also has information about the bulk matter. In particular, we now argue that massless matter hypermultiplets in the bulk correspond to relevant/marginal vertex operators on the string. Evidence to support this claim comes from the fact that at least, when giving a vev to a charged massless hypermultiplet it can Higgs the bulk gauge group, the worldsheet theory of the BPS string for which there is a flavor current associated to the group should get deformed. This is because the gauge symmetry in the bulk induces the flavor symmetry on the BPS string and consequently the Higgsing process also reduces the flavor symmetry on the BPS string. This means that there must exist a relevant/marginal deformation of the BPS worldsheet associated to a primary field in representation **R** of the matter field on the worldsheet (note that non-primary fields except from the current itself will always have dimension bigger than 1). Since the current is on the left-moving sector of the string which is non-supersymmetric, this means that there is an operator of left-moving dimension less than or equal to 1 associated to a primary field of representation **R**. This argument can be extended to all massless representations regardless of whether they can Higgs the gauge group: Having massless fields in the representation **R** of a gauge group should lead to at most marginally irrelevant deformations. In other words giving a vev to them is obstructed by more than quadratic terms in the bulk theory. So at the quadratic/leading level they behave as if they are Higgsing the bulk theory and so should be at most marginally irrelevant, i.e. dimension no more than 1.

A simple example of this condition is realized in the heterotic string on $K3$, where the massless charged fields are represented by primary fields with (left,right) dimension $(1, 1/2)$ of the $(0, 4)$ supersymmetric theory on the worldsheet.

To summarize we have argued that the hypermultiplets transforming in a particular representation **R** need to satisfy the following conditions:

1. The vertex operator of the massless modes with representation $\mathbf{R}$ of G with conformal weight $\Delta_{\mathbf{R}} = \frac{C_2(\mathbf{R})}{2(k+h^v)}$ where $C_2(\mathbf{R})$ is the second Casimir of the $\mathbf{R}$ must obey:

$$\Delta_{\mathbf{R}} \leq 1 \tag{3.114}$$

2. The representation $\mathbf{R}$ of a primary[11] with highest weight $\mathbf{\Lambda} = (\Lambda_i, \cdots, \Lambda_r)$ where r is the rank of the Lie algebra must satisfy :

$$\sum_i^r \Lambda_i \leq k \tag{3.115}$$

where k is the level of the current algebra of G on the worldsheet.

The first condition as discussed above requires the hypermultiplet states of the spacetime theory to appear as vertex operators in the WZW model and in particular they need to be relevant/marginal primary fields. Therefore, the conformal dimension associated to the hypermultiplets can be at most 1. The second condition is a standard result of the highest-weight representation in Kac-Moody algebras[56].

In addition, these inequalities are independent of the dimension of spacetime and can also be extended to BPS strings in 5d and 4d. For example, in 5d $N = 1$ we have monopole strings which need to satisfy the above consistency conditions in the presence of bulk matter and hence constraining the possible representations that can appear.

For example, consider the 5d $N = 1, SU(2) \times U(1)$ theory constructed in[116] and subsection B.2.2 with the geometry being the singular quintic with A_1 singularity along a curve of degree d

[11] With the exception of the adjoint representation which need not be a primary.

and genus g. Assuming that H is the proper transform of the hyperplane class of the quintic, and E the exceptional divisor of the blowup then the following relations are true:

$$H^3 = 5, H^2E = 0, HE^2 = -2d, E^3 = 4 - 4g - 5d \quad (3.116)$$

In this case the t'Hooft anomaly of the non-abelian gauge symmetry is given by:

$$\frac{-1}{4} k_i tr F_i^2 \quad (3.117)$$

with $k_i = -h_{i,a}q^a$, where $h_{i,a}$ are the the coefficients in the gauge coupling h_i for G_i in the bulk effective action and q^a the string charges. Therefore, the levels of U(1) , SU(2) with divisors H, E respectively and $q = (1, 0)$ are:

$$k_0 = C_{000} = H^3, k_1 = -\frac{3}{6}C_{011} = \frac{-3}{6}HE^2 = d \quad (3.118)$$

which implies that condition (3.115) is given by:

$$\sum_i \Lambda_i \leq d \quad (3.119)$$

Therefore, for a degree $d = 1$ curve we can only have fundamental matter in **2** of $SU(2)$. This is in accordance with the fact that

$$E^3 = 4 - 4g - 5d = -1 \text{ for } d = 1, g = 0 \quad (3.120)$$

was interpreted as having $N = 9$ fundamental hypers rather than 1 adjoint and 1 fundamental since the genus was zero. Geometrically, this is the fact that there is no degree 1 genus 1 curve.

However, if $d = 2$ we have

$$\sum_i \Lambda_i \leq 2 \tag{3.121}$$

and say $E^3 = -6$ could be either $N = 14$ fundamental hypers or $N = 6$ fundamental hypers and 1 adjoint. In other words our inequality does not restrict which case it is. From geometry we know that the first case is correct in this example because $E^3 = 4 - 4g - 5d = -10$ for a genus 1 and degree 2 curve.

Returning to 6d we are interested in seeing how these inequalities can help us as Swampland conditions. Let us start by considering the 6d supergravity theory coupled to $SU(N)$ with $(N - 8)\,\square + 1\,\square\square$. The gravitational anomaly restricts these theories to exist up to $T = 10$ and the gauge/gravitational anomalies are cancelled for $a \cdot b = 1, b \cdot b = -1$. We can choose a basis such that the bilinear form and the vectors a, b are given by:

$$\Omega = \text{diag}(1, (-1)^T),\ a = (-3, 1^T),\ b = (0^T, -1) \tag{3.122}$$

In this particular basis we can choose the Kähler form to be $J = (n, 0^{T-1}, 1)$ which satisfies $J^2 \geq 1$ for $n \geq 1$ and $J \cdot a < 0, J \cdot b > 0$. Now we consider a BPS string with charge $Q = (q_0, \cdots q_T)$ which must satisfy eq.(1.48):

$$q_0^2 - \sum_{i=1}^T q_i^2 \geq -1,\ q_0^2 - \sum_{i=1}^T q_i^2 - 3q_0 - \sum_{i=1}^T q_i \geq -2, k = Q \cdot b \geq 0 \tag{3.123}$$

A string charge consistent with these inequalities is $Q = (3, 0^{T-1}, 1)$ which gives level $k = 1$ for any

T. We can now use eq.(3.115) which states that every representation should satisfy:

$$\sum_i \Lambda_i \leq k = 1 \tag{3.124}$$

However, the symmetric representation has highest weight $\Lambda = (2, 0^{N-2})$ and therefore does not satisfy this inequality. We conclude that this theory belongs to the Swampland. This is consistent with the observation in [124] that for $T = 1$ this theory has no F-theory realization.

Another example, that was also discussed in the previous section is: $1\,\square\!\square + 1\,\begin{smallmatrix}\square\\ \square\end{smallmatrix}$ with $T = 9$. We found that the following choices of (k, N) are consistent by using unitarity considerations:

$$(k \geq 1, N = 0, 1, 2, 3), (4 \geq k \geq 1, N = 4) \tag{3.125}$$

$$(2 \geq k \geq 1, N = 5), (k = 1, N = 6, 7, 8, 9) \tag{3.126}$$

However, if we apply condition (3.115) we see that $k = 1$ is not a consistent choice because both the symmetric representation have $\sum_i \Lambda_i = 2$. Therefore in particular all theories with $N > 5$ belong to the Swampland.

Consequently, the second condition has helped us rule out theories that do not have string theory realizations but methods such as unitarity bounds of the previous section did not exclude them. However, the first condition even though non-trivial we did not find useful in these examples. The issues are that for simple representations that we consider here this is automatically satisfied (for example $\Delta_{\square} = \frac{(N^2-1)}{2N(k+N)}$, $\Delta_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} = \frac{(N-2)(N+1)}{N(k+N)}$, $\Delta_{\mathbf{Adj}} = \frac{N}{N+k} \leq 1$). Therefore, this condition could have a chance to be useful for higher index symmetric and antisymmetric representations and exotic ones. However, most such examples constructed are for $T = 0$ [126], but those theories tend to have a very large level k since a, b are scalars. Therefore, we would expect this to be more useful if a full 6d supergravity classification is considered and more exotic representation are considered for large T.

3.5 Summary and Discussion

In this chapter the finiteness of massless modes was addressed for 6d theories with 8 supercharges showing that the rank of a given theory is bounded. We also studied general swampland principles for 5d and 6d theories including constraints on the matter of the theory. It would be interesting to refine the results in 6d and understand where do they fit with regard to the string landscape. An important direction to understand this, is to look into constructions outside F-theory compactifications and perhaps to non-geometric families as studied in [113] were such constructions were shown to give rise to more exotic models which may be related to the usual F-theory through stringy transitions. Such studies will give a more general understanding of the string landscape and hence a good guide to understand the swampland. With regards to the 5d swampland similar results are harder to obtain but it would be beneficial to address the boundaries of the possible landscape including finiteness and potential new exotic models e.g. [141].

"Turning on quantum mechanics makes the behavior much gentler; in particular, the effective dimension of space-time is not changed. Perhaps a better understanding of the singular behavior of the conformal field theory would enable one to understand in a more a priori fashion what happens quantum mechanically."

Edward Witten[189]

4

Quantum Field Theory in Five Dimensions

In the previous chapters we discussed extensively conditions that low energy theories need to satisfy in order to have a gravitational UV completion. However, a similar UV question is interesting in quantum field theory too in the absence of gravity. As discussed in subsection 1.2.2 one can also study necessary conditions for a quantum field theory to have a UV completion. In fact as was seen in section 1.10 five dimensional gauge theories traditionally where thought to be not interesting as they are non-renormalizable. However, in[166] it was shown that under some conditions, 5d UV fixed point do exist for such theories and string theory provides a framework to construct them.

In some interesting cases that did not satisfy the consistency conditions to have a 5d UV fixed point it was shown to have a 6d [166]. All known examples of such theories are characterized by the emergence of an intrinsic length scale that is interpreted as the size of a compactification circle, and it has been argued that each of these theories is a circle compactification of a 6*d* SCFT possibly twisted by the action of a discrete global symmetry[1]; see for example [111,101,102,194,103,121,29,28,100].

As reviewed in section 1.10 it is believed that all 5*d* SCFTs can be obtained via RG flows starting from the 5*d* Kaluza-Klein (KK) theories and hence every 6*d* SCFT compactified on a circle provides a natural starting point for the systematic identification of a large family of 5*d* SCFTs.

In this chapter, we focus on the geometric approach in which one realizes a 5*d* KK theory via a compactification of M-theory on a genus one fibered Calabi-Yau threefold. The set of holomorphic curves in the threefold completely encode the information about the spectrum of BPS particles required to track all RG flows down to 5*d* SCFTs. Therefore, a precursor to classifying RG flows from 5d KK theories to 5d SCFTs is to geometrically classify all 5d KK theories themselves in terms of Calabi-Yau threefolds.

It is believed that all 6*d* SCFTs can be constructed by compactifying F-theory on singular elliptically fibered Calabi-Yau threefolds admitting certain singular limits characterized by the contraction of holomorphic curves in the base of the fibration. Here we include constructions that may also include $O7^+$ planes from the point of view of type IIB string theory, which correspond to the *frozen* phase [193,54,177] of F-theory. The unfrozen 6*d* SCFTs were classified in [104,105] (see also [23]) while the ones coming from the frozen phase of F-theory were classified in [25].

A 5*d* KK theory corresponding to the untwisted compactification of a 6*d* SCFT arising in the unfrozen phase can be constructed by compactifying M-theory on a Calabi-Yau threefold which is

[1] Twisting the theory around the circle means that we introduce a holonomy for the background gauge fields associated to discrete global symmetries of the theory.

a resolution of the Calabi-Yau threefold arising in the F-theory construction. This fact is a special case of the duality between M-theory and (unfrozen phase of) F-theory compactified on a circle (without any twist). Explicit resolution of all Calabi-Yau threefolds associated to $6d$ SCFTs was performed by [28,29], and hence the Calabi-Yau threefolds associated to corresponding $5d$ KK theories was determined. These threefolds are elliptically fibered since the threefolds associated to $6d$ SCFTs are elliptically fibered to begin with.

In this chapter, we extend the work of this to include potential twists along the circle and determine a resolved local Calabi-Yau threefold describing every $5d$ KK theory. We find that these Calabi-Yau threefolds are in general only genus one fibered [2] which is associated to the twist we have considered.

Our analysis can be divided into two parts. In the first part of the analysis, which is purely field theoretic, we determine the prepotential for each $5d$ KK theory by using the following observations: Each $6d$ SCFT admits a $6d$ gauge theory description which can be reduced on a circle with an appropriate twist to obtain a canonical $5d$ gauge theory description of the associated $5d$ KK theory. The Green-Schwarz term in $6d$ reduces to a Chern-Simons term in the $5d$ gauge theory, which induces a tree-level contribution to the prepotential. Combining this contribution with the one-loop contribution coming from the $5d$ gauge theory produces the full prepotential for the $5d$ KK theory.

In the second part of the analysis, we interpret the prepotential as describing the triple intersection numbers of 4-cycles inside a yet to be determined Calabi-Yau threefold. Using the data of these triple intersection numbers, along with some other consistency conditions, we are able to determine a description of the Calabi-Yau threefold as a neighborhood of intersecting Kähler surfaces along the lines of the discussion in [111,29,28], and we verify that each threefold admits the structure of genus one fibration.

[2] See for example [148] for a discussion of F-theory compactifications on genus one fibered, in contrast to elliptically fibered, Calabi-Yau varieties.

One can view these Calabi-Yau threefolds as providing previously unknown M-theory duals of general unfrozen and frozen F-theory configurations compactified on a circle possibly with a discrete twist. Even though we have provided explicit results only for F-theory configurations realizing $6d$ SCFTs, our methods should in principle apply to any general F-theory configuration.

This chapter will be a summary of[30] where we will discuss in detail only some parts of this classification in order to demonstrate our methods with few examples with most of the details left to the original work.

4.1 Structure of 6d SCFTs

The 6d SCFTs can only be reached through the tensor branch and hence that is enough to define their RG flow. The low-energy theory on the tensor branch of a $6d$ SCFT $\mathfrak{T}$ can be organized in terms of tensor multiplets B_i. There is a gauge algebra $\mathfrak{g}_i$ associated to each i where $\mathfrak{g}_i$ can either be a simple or a trivial algebra. Each tensor multiplet B_i is also associated to a "fundamental" BPS string excitation S^i such that the charge of S^i under B_j is the Kronecker delta δ^i_j. The Dirac pairing Ω^{ij} between S^i and S^j appears in the Green-Schwarz term in the Lagrangian

$$\Omega^{ij} B_i \wedge \mathrm{tr}(F_j^2) \tag{4.1}$$

where F_j is the field strength for $\mathfrak{g}_j$ if $\mathfrak{g}_j$ is simple and $F_j = 0$ if $\mathfrak{g}_j$ is trivial.

$[\Omega^{ij}]$ is a symmetric, positive definite matrix with all of its entries valued in integers. Thus, it is analogous to the Cartan matrix for a simply laced Lie algebra. The only possible values for off-diagonal entries are $\Omega^{ij} = 0, -1, -2$. We note that $\Omega^{ij} = -2$ is only possible for $6d$ SCFTs arising from the frozen phase of F-theory[25,31].

We can thus display the data of a $6d$ SCFT in terms of an associated graph $\Sigma_{\mathfrak{T}}$ that is constructed as follows:

$\mathfrak{g}_i$ Ω^{ii}	Comments	Hypermultiplet content
$\mathfrak{sp}(n)_\theta$ 1	$\theta = 0, \pi$	$(2n+8)\mathsf{F}$
$\mathfrak{su}(n)$ 1	$n \geq 3$	$(n+8)\mathsf{F} + \Lambda^2$
$\mathfrak{su}(\widehat{n})$ 1	$n \geq 8$; frozen; non-geometric	$(n-8)\mathsf{F} + \mathsf{S}^2$
$\mathfrak{su}(\tilde{6})$ 1		$15\mathsf{F} + \frac{1}{2}\Lambda^3$
$\mathfrak{su}(n)$ 2		$2n\mathsf{F}$
$\mathfrak{su}(3)$ 3		
$\mathfrak{so}(n)$ 4	$n \geq 8$	$(n-8)\mathsf{F}$
$\mathfrak{so}(8)$ k	$1 \leq k \leq 3$	$(4-k)\mathsf{F} + (4-k)\mathsf{S} + (4-k)\mathsf{C}$
$\mathfrak{so}(n)$ k	$1 \leq k \leq 3; 7 \leq n \leq 12, n \neq 8$	$(n-4-k)\mathsf{F} + 2^{\lceil \frac{9-n}{2} \rceil}(4-k)\mathsf{S}$
$\mathfrak{so}(\widehat{12})$ k	$k = 1, 2$	$(8-k)\mathsf{F} + \frac{1}{2}(3-k)\mathsf{S} + \frac{1}{2}\mathsf{C}$
$\mathfrak{so}(13)$ 2		$7\mathsf{F} + \frac{1}{2}\mathsf{S}$
$\mathfrak{g}_2$ k	$1 \leq k \leq 3$	$(10-3k)\mathsf{F}$
$\mathfrak{f}_4$ k	$1 \leq k \leq 5$	$(5-k)\mathsf{F}$
$\mathfrak{e}_6$ k	$1 \leq k \leq 6$	$(6-k)\mathsf{F}$
$\mathfrak{e}_7$ k	$1 \leq k \leq 8$	$\frac{1}{2}(8-k)\mathsf{F}$
$\mathfrak{e}_8$ 12		

Table 4.1: List of all the possible nodes with non-trivial $\mathfrak{g}_i$ appearing in graphs associated to $6d$ SCFTs. A hat or a tilde distinguishes different nodes having same values of Ω^{ii} and $\mathfrak{g}_i$.

$\overset{\mathfrak{g}_i}{\Omega^{ii}}$	Comments	Flavor symmetry algebra, $\mathfrak{f}$
$\overset{\mathfrak{sp}(0)_\theta}{1}$	$\theta = 0, \pi$	$\mathfrak{e}_8$
$\overset{\mathfrak{su}(1)}{2}$		$\mathfrak{su}(2)$

Table 4.2: List of all the possible nodes with trivial $\mathfrak{g}_i$ that can appear in graphs associated to $6d$ SCFTs. If $\Omega^{ii} = 2$, we refer to the trivial gauge algebra as $\mathfrak{su}(1)$ and if $\Omega^{ii} = 1$, we refer to the trivial gauge algebra as $\mathfrak{sp}(0)$. In the latter case, sometimes a $\mathbb{Z}_2$ valued theta angle is physically relevant. We also list the flavor symmetry algebra $\mathfrak{f}$ for each case. The sum of gauge algebras neighboring each such node must be contained inside the corresponding $\mathfrak{f}$.

- **Nodes**: For each tensor multiplet B_i, we place a node i with value $\overset{\mathfrak{g}_i}{\Omega^{ii}}$. All such possibilities are listed in Table 4.1 when $\mathfrak{g}_i$ is non-trivial, and in Table 4.2 when $\mathfrak{g}_i$ is trivial. In the former case, each node contributes hypers charged under a representation $\mathcal{R}_i$ of $\mathfrak{g}_i$ where $\mathcal{R}_i$ is shown in Table 4.1.

- **Edges**: Each edge corresponds to a hyper transforming in a mixed representation $\mathcal{R}_{ij} = \mathcal{R}_i \otimes \mathcal{R}_j$ of $\mathfrak{g}_i \oplus \mathfrak{g}_j$ where $\mathcal{R}_i$ is a representation of $\mathfrak{g}_i$ and $\mathcal{R}_j$ is a representation of $\mathfrak{g}_j$.

An example of such a gauging procedure is demonstrated in Table 4.3 where one joins the two nodes with an edge corresponding to bifundamental matter. A full list of all possible edges between nodes of table 4.1 and 4.2 can be found in [30].

$\overset{\mathfrak{g}_i}{\Omega^{ii}}$ —— $\overset{\mathfrak{g}_j}{\Omega^{jj}}$	Condition	Mixed hyper content
$\overset{\mathfrak{su}(n)}{2}$ —— $\overset{\mathfrak{su}(m)}{2}$	$m \leq 2n; n \leq 2m$	$\mathsf{F} \otimes \mathsf{F}$

Table 4.3: Example of an edge joining two nodes corresponding to gauging of bifundamental matter.

4.2 Structure of 5d KK theories

The most general way to compactify the 6d SCFTs to 5d is by permitting the gauging of some discrete symmetry around the circle. In this section we will present two main classes of such discrete symmetries and discuss their action on the theory. In particular, we can understand them as two classes of automorphisms acting on the gauge symmetry of the fibers of the elliptic fibration or automorphisms associated to exchanged tensor multiplets in the base.

4.2.1 Twists

Consider a QFT $\mathfrak{T}$ that admits a discrete global symmetry group Γ. When we compactify $\mathfrak{T}$ on a circle, we have the option of "twisting" $\mathfrak{T}$ around the circle. This means that we introduce a holonomy $\gamma \in \Gamma$ for the background gauge field corresponding to Γ. The number of inequivalent twists is given by the number of conjugacy classes in Γ. This is because two holonomies that are conjugate in Γ are physically equivalent and thus lead to the same twist.

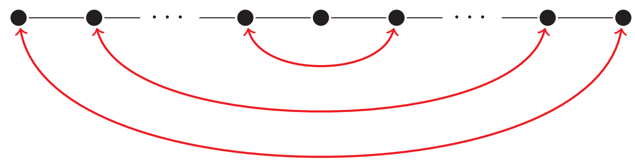
4.2.2 Discrete symmetries from outer automorphisms

Let us consider the most basic discrete symmetries that arise from outer automorphisms of gauge algebras $\mathfrak{g}_i$. This automorphism can be thought of as a symmetry of the Dynkin diagram. Turning on a holonomy for such symmetries leads to the twisted affine algebras.

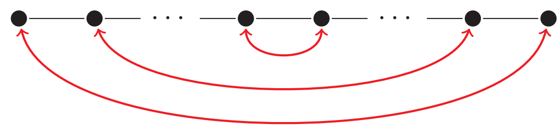
The algebras that admit an order two outer automorphism $\mathcal{O}^{(2)}$, are $\mathfrak{su}(n)$ for $n \geq 3$, $\mathfrak{so}(2m)$

for $m \geq 4$ and $\mathfrak{e}_6$ which exchanges the roots in the following fashion

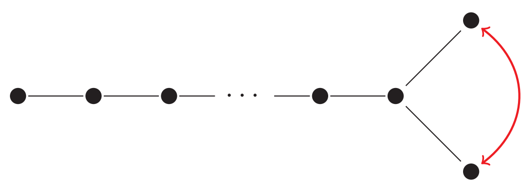
$\underline{\mathfrak{su}(2n), \mathcal{O}^{(2)}}$:



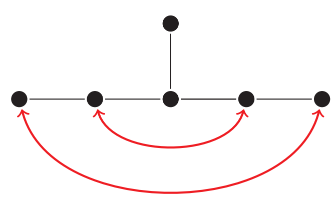
$\underline{\mathfrak{su}(2n+1), \mathcal{O}^{(2)}}$:



$\underline{\mathfrak{so}(2n), \mathcal{O}^{(2)}}$:

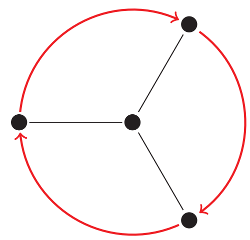


$\underline{\mathfrak{e}_6, \mathcal{O}^{(2)}}$:



The $\mathfrak{so}(8)$ Dynkin diagram also admits an order three outer automorphism $\mathcal{O}^{(3)}$. It cyclically permutes the roots as shown below

$\underline{\mathfrak{so}(8), \mathcal{O}^{(3)}:}$



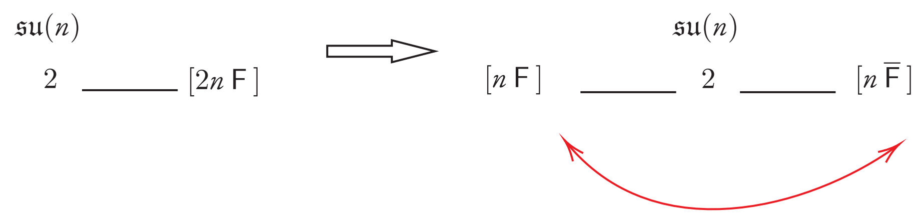
The full group of outer automorphisms of $\mathfrak{so}(8)$ is the symmetric group S_3 which can be generated by combining $\mathcal{O}^{(2)}$ and $\mathcal{O}^{(3)}$.

The above action of an outer automorphism $\mathcal{O}^{(q)}$ (for $q = 2, 3$) on the roots of $\mathfrak{g}$ translates to an action on the Dynkin coefficients of the weights for representations of $\mathfrak{g}$. In other words, the action of $\mathcal{O}^{(q)}$ can be viewed as an action on representations of $\mathfrak{g}$—see Table 4.4.

$\mathfrak{g}$	$\mathcal{O}^{(q)}$	$\mathcal{O}^{(q)} \cdot \mathcal{R}_{\mathfrak{g}}$
$\mathfrak{su}(m)$	$\mathcal{O}^{(2)}$	$\mathsf{F} \longleftrightarrow \bar{\mathsf{F}}, \Lambda^n \longleftrightarrow \bar{\Lambda}^n, \mathsf{S}^2 \longleftrightarrow \bar{\mathsf{S}}^2$
$\mathfrak{so}(2m)$	$\mathcal{O}^{(2)}$	$\mathsf{F} \longrightarrow \mathsf{F}, \mathsf{S} \longleftrightarrow \mathsf{C}$
$\mathfrak{e}_6$	$\mathcal{O}^{(2)}$	$\mathsf{F} \longleftrightarrow \bar{\mathsf{F}}$
$\mathfrak{so}(8)$	$\mathcal{O}^{(3)}$	$\mathsf{F} \longrightarrow \mathsf{S}, \mathsf{S} \longrightarrow \mathsf{C}, \mathsf{C} \longrightarrow \mathsf{F}$

Table 4.4: List of non-trivial outer automorphisms $\mathcal{O}^{(q)}$ of $\mathfrak{g}$ and their actions $\mathcal{O}^{(q)} \cdot \mathcal{R}_{\mathfrak{g}}$ on various irreducible representations $\mathcal{R}_{\mathfrak{g}}$ of $\mathfrak{g}$. F denotes fundamental representation, Λ^n denotes the irreducible n-index antisymmetric representation, S^2 denotes the irreducible 2-index symmetric representation, and S and C denote irreducible spinor and cospinor representations. Bar on top of a representation denotes the complex conjugate of that representation. F of $\mathfrak{so}(2m)$ is left invariant by the action of $\mathcal{O}^{(2)}$.

As an example consider the $6d$ theory given by



(4.2)

The theory includes $2n$ hypers in F. The outer automorphism $\mathcal{O}^{(2)}$ of $\mathfrak{su}(n)$ descends to a discrete symmetry of the theory whose action on the hypermultiplets can be manifested as follows: we divide the $2n$ hypers into two ordered sets such that each set contains n hypers and then we exchange these two sets with each other.

4.2.3 Discrete symmetries from permutation of tensor multiplets

We can also consider discrete symmetries that arise from permutations of tensor multiplets $i \to S(i)$ such that

$$\mathfrak{g}_{S(i)} = \mathfrak{g}_i \tag{4.3}$$

$$\Omega^{S(i)S(j)} = \Omega^{ij} \tag{4.4}$$

for all i, j. This is a symmetry of $\mathfrak{T}$ if representations also preserve it:

$$\mathcal{R}_{S(i)} \simeq \mathcal{R}_i \tag{4.5}$$

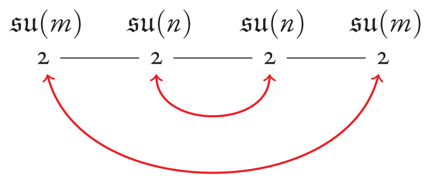
$$\mathcal{R}_{S(i)S(j)} \simeq \mathcal{R}_{ij} \tag{4.6}$$

for all i, j.

As another example, consider the $6d$ theory given by

$$\begin{array}{ccccccc} \mathfrak{su}(m) & & \mathfrak{su}(n) & & \mathfrak{su}(n) & & \mathfrak{su}(m) \\ 2 & \text{———} & 2 & \text{———} & 2 & \text{———} & 2 \end{array} \tag{4.7}$$

which has the following permutation as a symmetry of the theory



(4.8)

The untwisted matrix element $[\Omega^{ij}]$ is

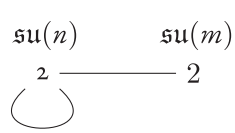
$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

and hence the twisted one is $[\Omega_S^{\alpha\beta}]$ is

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

.

which represents the following graph:



(4.9)

To understand the low energy $5d$ theory for (4.9), one needs to have a clear view of the action of this twist to the bifundamental of the central algebra. The two $\mathfrak{su}(m)$ get identified to a single $\mathfrak{su}(m)$ algebra. Similarly, the two $\mathfrak{su}(n)$ get identified to a single $\mathfrak{su}(n)$ algebra. Thus the $5d$ gauge algebra is $\mathfrak{h} = \mathfrak{su}(n) \oplus \mathfrak{su}(m)$. The bifundamentals of $\mathfrak{su}(m) \oplus \mathfrak{su}(n)$ descend to a single bifundamental of $\mathfrak{h}$. The bifundamental of $\mathfrak{su}(n) \oplus \mathfrak{su}(n)$ descends to S^2 of $\mathfrak{su}(n)$. Furthermore, we obtain $n - m$ extra fundamentals of $\mathfrak{su}(n)$ and $2m - n$ extra fundamentals of $\mathfrak{su}(m)$. Thus, the low energy $5d$ theory is an $\mathfrak{su}(n) \oplus \mathfrak{su}(m)$ gauge theory with a bifundamental plus $(2m - n)\mathsf{F}$ of $\mathfrak{su}(m)$ plus $(n - m)\mathsf{F} \oplus \mathsf{S}^2$ of $\mathfrak{su}(n)$.

4.2.4 General discrete symmetries

We can now combine the two types of twists discussed in Sections 4.2.2 and 4.2.3. That is, we consider actions of the form

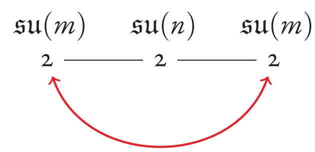
$$\left(\prod_i \mathcal{O}^{(q_i)}\right) S \tag{4.10}$$

where S is a permutation of the tensor multiplets and $\mathcal{O}^{(q_i)}$ is an outer automorphism of order q_i of gauge algebra $\mathfrak{g}_i$, where each $q_i \in \{1, 2, 3\}$ and $q_i = 1$ denotes the identity automorphism.

A nice demonstration of the combined twist is the following example, consider the $6d$ SCFT

$$\overset{\mathfrak{su}(m)}{2} \text{ --- } \overset{\mathfrak{su}(n)}{2} \text{ --- } \overset{\mathfrak{su}(m)}{2} \tag{4.11}$$

Suppose we want to perform the outer-automorphism $\mathcal{O}^{(2)}$ for the middle $\mathfrak{su}(n)$ node. Recall from the discussion around (4.2) that the outer automorphism of $\mathfrak{su}(n)$ exchanges the fundamental hypers in pairs. However, the graph in (4.11) indicates that the fundamental hypers of the middle $\mathfrak{su}(n)$ algebra are part of bifundamental representations formed by taking the tensor product with the fundamental representations of the neighboring $\mathfrak{su}(m)$ algebras. Therefore, if we want $\mathcal{O}^{(2)}$ to be a symmetry of the theory, we must permute the two neighboring $\mathfrak{su}(m)$ as well. Thus, $\mathcal{O}^{(2)}$ by itself is not a symmetry of the theory, but its combination with the permutation



(4.12)

is a symmetry of the theory. Thus, we see that in general it is not possible to decompose a general symmetry of the form (4.10) into more basic symmetries discussed earlier. However, we can perform the combined twist which gives rise to

$$\begin{array}{ccc} \mathfrak{su}(n)^{(2)} & & \mathfrak{su}(m)^{(1)} \\ 2 & \overset{}{-\!\!-\!\!-}\ {\scriptstyle 2}\ \longrightarrow & 2 \end{array} \tag{4.13}$$

4.3 Prepotential for 5d KK theories

The goal of this section is to propose a formula for the prepotential of a $5d$ KK theory $\mathfrak{T}^{KK}_{S,\{q_\alpha\}}$ starting from the tensor branch description of the corresponding $6d$ SCFT $\mathfrak{T}$. This includes incorporating the KK reduction of the theory together with any possible twists performed.

4.3.1 Prepotential

Compactify a $6d$ SCFT $\mathfrak{T}$ on a circle with a twist S, $\{q_\alpha\}$ around the circle. Let us analyze the low energy theory. Every node α in $\Sigma^{S,\{q_\alpha\}}_{\mathfrak{T}}$ gives rise to a low energy $5d$ gauge algebra $\mathfrak{h}_\alpha = \mathfrak{g}_\alpha/\mathcal{O}^{(q_\alpha)}$ which is the subalgebra of $\mathfrak{g}_\alpha$ left invariant by the action of outer automorphism $\mathcal{O}^{(q_\alpha)}$. In this work, our choice of outer automorphisms is such that the invariant subalgebras are those listed in Table 4.6. For each node α, we obtain an additional $\mathfrak{u}(1)_\alpha$ gauge algebra in the low energy $5d$ theory

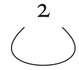
$\mathfrak{g}_\alpha$ $\Omega^{\alpha\alpha}$	Comments
$\mathfrak{su}(n)^{(2)}$ 1	$n = 3, 4$
$\mathfrak{su}(n)^{(2)}$ 2	$n \geq 3$
$\mathfrak{su}(3)^{(2)}$ 3	
$\mathfrak{so}(2n)^{(2)}$ 4	$n \geq 5$
$\mathfrak{so}(8)^{(q)}$ k	$1 \leq k \leq 4; q = 2, 3$
$\mathfrak{so}(10)^{(2)}$ 2	
$\mathfrak{so}(\widehat{12})^{(2)}$ 2	
$\mathfrak{e}_6^{(2)}$ k	$k = 2, 4, 6$
$\mathfrak{su}(n)^{(1)}$ 2 	$n \geq 1$; non-geometric

Table 4.5: List of all the new nodes that can appear in graphs associated to $5d$ KK theories. We also list all the possibilities where an edge starts and ends on the same node. The comment "non-geometric" for the last entry refers to the fact that there is no completely geometric description of this node. See also a node appearing in Table 4.1. If a KK theory involves either of these two kinds of nodes, then it does not admit a conventional geometric description.

$\mathfrak{g}$	$\mathcal{O}^{(q)}$	$\mathfrak{h}$	$\mathcal{R}_\mathfrak{g} \to \mathcal{R}_\mathfrak{h}$
$\mathfrak{su}(2m)$	$\mathcal{O}^{(2)}$	$\mathfrak{sp}(m)$	$\mathsf{F} \to \mathsf{F}, \bar{\mathsf{F}} \to \mathsf{F}, \Lambda^2 \to \Lambda^2 \oplus 1$
$\mathfrak{su}(2m+1)$	$\mathcal{O}^{(2)}$	$\mathfrak{sp}(m)$	$\mathsf{F} \to \mathsf{F} \oplus 1, \bar{\mathsf{F}} \to \mathsf{F} \oplus 1$
$\mathfrak{so}(2m)$	$\mathcal{O}^{(2)}$	$\mathfrak{so}(2m-1)$	$\mathsf{F} \to \mathsf{F} \oplus 1, \mathsf{S} \to \mathsf{S}, \mathsf{C} \to \mathsf{S}$
$\mathfrak{e}_6$	$\mathcal{O}^{(2)}$	$\mathfrak{f}_4$	$\mathsf{F} \to \mathsf{F} \oplus 1, \bar{\mathsf{F}} \to \mathsf{F} \oplus 1$
$\mathfrak{so}(8)$	$\mathcal{O}^{(3)}$	$\mathfrak{g}_2$	$\mathsf{F} \to \mathsf{F} \oplus 1, \mathsf{S} \to \mathsf{F} \oplus 1, \mathsf{C} \to \mathsf{F} \oplus 1$

Table 4.6: The table displays the invariant algebra $\mathfrak{h}$ when $\mathfrak{g}$ is quotiented by $\mathcal{O}^{(q)}$. An irrep $\mathcal{R}_\mathfrak{g}$ of $\mathfrak{g}$ decomposes to an irrep $\mathcal{R}_\mathfrak{h}$ of $\mathfrak{h}$ and this decomposition is displayed (for representations relevant in this section) in the column labeled $\mathcal{R}_\mathfrak{g} \to \mathcal{R}_\mathfrak{h}$. 1 denotes the singlet representation.

coming from the reduction of a tensor multiplet B_i on the circle where i lies in the orbit α.

Now we determine the spectrum of hypermultiplets charged under $\oplus_\alpha \mathfrak{h}_\alpha$ under the low energy $5d$ theory. First of all, for every node i in $6d$ theory, we define $\mathcal{T}_i = \oplus_j \mathcal{R}_{ij,i}^{\oplus \dim(\mathcal{R}_{ij,j})}$. Recall that $\mathcal{T}_i \subseteq \mathcal{R}_i$ and hence the $6d$ theory contains hypermultiplets charged under representation $\mathcal{S}_i$ of $\mathfrak{g}_i$ where $\mathcal{S}_i$ is defined such that $\mathcal{S}_i \oplus \mathcal{T}_i = \mathcal{R}_i$. $\mathcal{S}_i$ is the representation formed by those hypers that are only charged $\mathfrak{g}_i$ and not under any other gauge algebra $\mathfrak{g}_j$ with $j \neq i$.

As detailed in Table 4.6, irreducible representations $\mathcal{R}_{\mathfrak{g}_\alpha}$ of $\mathfrak{g}_\alpha$ can be viewed as irreducible representations of $\mathcal{R}_{\mathfrak{h}_\alpha}$. We can thus view hypers transforming in representation $\mathcal{S}_i$ of $\mathfrak{g}_i$ as transforming in a representation of $\mathfrak{h}_\alpha$. Let us denote this representation of $\mathfrak{h}_\alpha$ by $\tilde{\mathcal{S}}_\alpha$. The outer automorphism $\mathcal{O}^{(q_\alpha)}$ then permutes constituent irreps inside $\tilde{\mathcal{S}}_\alpha$ and thus acts on $\tilde{\mathcal{S}}_\alpha$ as an automorphism. The low energy $5d$ theory then contains hypers transforming in the representation

$$\mathcal{S}_\alpha := \tilde{\mathcal{S}}_\alpha / \mathcal{O}^{(q_\alpha)} \tag{4.14}$$

These hypers are only charged under $\mathfrak{h}_\alpha$ and not under any other gauge algebra $\mathfrak{h}_\beta$ with $\beta \neq \alpha$.

Now consider other hypermultiplets that are charged under multiple gauge algebras in the $6d$ theory. These descend to hypermultiplets charged under multiple gauge algebras in the low energy $5d$ theory plus some hypers only charged under the individual algebras. Consider the mixed representation $\mathcal{R}_{ij} = \mathcal{R}_{ij,i} \otimes \mathcal{R}_{ij,j}$ of $\mathfrak{g}_i \oplus \mathfrak{g}_j$ in the $6d$ theory. Let i and j lie in orbits α and β respectively. Let $\mathcal{R}_{ij,i}$ decompose as $\mathcal{R}_{\alpha\beta,\alpha} \oplus n_{\alpha\beta,\alpha}1$ when viewed as a representation of $\mathfrak{h}_\alpha$, where $\mathcal{R}_{\alpha\beta,\alpha}$ is the full subrepresentation that is charged non-trivially under $\mathfrak{h}_\alpha$. Similarly, let $\mathcal{R}_{ij,j}$ decompose as $\mathcal{R}_{\alpha\beta,\beta} \oplus n_{\alpha\beta,\beta}1$ when viewed as a representation of $\mathfrak{h}_\beta$, where $\mathcal{R}_{\alpha\beta,\beta}$ is the full subrepresentation that is charged non-trivially under $\mathfrak{h}_\beta$. Then, under the twist, $\mathcal{R}_{ij}$ descends to a mixed representation $\mathcal{R}_{\alpha\beta}$ of $\mathfrak{h}_\alpha \oplus \mathfrak{h}_\beta$ plus representations $\mathcal{S}_{\alpha\beta,\alpha}$ and $\mathcal{S}_{\alpha\beta,\beta}$ of $\mathfrak{h}_\alpha$ and $\mathfrak{h}_\beta$ respectively. Here $\mathcal{R}_{\alpha\beta} = \mathcal{R}_{\alpha\beta,\alpha} \otimes \mathcal{R}_{\alpha\beta,\beta}$, $\mathsf{S}_{\alpha\beta,\alpha} = n_{\alpha\beta,\beta}\mathcal{R}_{\alpha\beta,\alpha}$, and $\mathsf{S}_{\alpha\beta,\beta} = n_{\alpha\beta,\alpha}\mathcal{R}_{\alpha\beta,\beta}$.

In addition to the above, we also obtain hypers in the symmetric product $\text{Sym}^2(\mathcal{R}_{ij,i})$ for all $j \neq i$ such that both j and i are in the same orbit α. Thus, the full representation $\mathcal{R}_\alpha$ formed by hypers under $\mathfrak{h}_\alpha$ is

$$\mathcal{R}_\alpha = \oplus_{j\in\alpha}\text{Sym}^2(\mathcal{R}_{ij,i})|_{\mathfrak{h}_\alpha} \oplus \mathcal{S}_\alpha \oplus_\beta \left(\mathcal{R}_{\alpha\beta,\alpha}^{\oplus\dim(\mathcal{R}_{\alpha\beta,\beta})} \oplus \mathcal{S}_{\alpha\beta,\alpha}\right) \tag{4.15}$$

where $\text{Sym}^2(\mathcal{R}_{ij,i})|_{\mathfrak{h}_\alpha}$ means that we view $\text{Sym}^2(\mathcal{R}_{ij,i})$ as a representation of $\mathfrak{h}_\alpha$. Note that in the above expression, i is a fixed node in the orbit α, j cannot equal i, and β cannot equal α. There are no hypers charged under $\mathfrak{u}(1)_\alpha$. Just as the representations $\mathcal{R}_i$ and $\mathcal{R}_{ij}$ for all i and j determine the full matter content for $6d$ SCFTs, the representations $\mathcal{R}_\alpha$ and $\mathcal{R}_{\alpha\beta}$ for all α and β determine the full matter content for $5d$ KK theories. An example of this form was demonstrated in (4.9).

The low energy $5d$ gauge theory also contains tree-level Chern-Simons terms that arise from the reduction of (4.1) on the circle. These can be written as

$$\Omega_S^{\alpha\beta} A_{0,\alpha} \wedge \text{tr}(F_\beta^2) \tag{4.16}$$

where $A_{0,\alpha}$ is the gauge field corresponding to the $\mathfrak{u}(1)_\alpha$ obtained by reducing B_α on the circle and F_β is the gauge field strength for $\mathfrak{h}_\beta$. In writing (4.16), we have used the fact that the index of $\mathfrak{h}_\beta$ in $\mathfrak{g}_\beta$ is one which is true for our choice of $\mathfrak{h}$ listed in Table 4.4. (4.16) contributes the following tree-level term to the prepotential

$$6\mathcal{F}^{\text{tree}}_{S,\{q_\alpha\}} = 6\sum_{\alpha,\beta}\frac{1}{2}\Omega_S^{\alpha\beta}\varphi_{0,\alpha}\left(K_\beta^{ab}\varphi_{a,\beta}\varphi_{b,\beta}\right) \tag{4.17}$$

where $\varphi_{0,\alpha}$ is the scalar living in the vector multiplet corresponding to $\mathfrak{u}(1)_\alpha$ and $\varphi_{a,\beta}$ are scalars living in the vector multiplets corresponding to $\mathfrak{u}(1)_{a,\beta}$ which parametrize the Cartan of $\mathfrak{h}_\beta$. Here K_β^{ab} is the Killing form on $\mathfrak{h}_\beta$ normalized such that its diagonal entries are minimum positive integers while keeping all the other entries integer valued.

Let $\mathfrak{h} = \oplus_\alpha \mathfrak{h}_\alpha$ be the total gauge algebra visible at low energies. The low energy hypermultiplets form some representation $\mathcal{R}$ of $\mathfrak{h}$ which decomposes into irreducible representations of $\mathfrak{h}$ as $\mathcal{R} = \oplus_f \mathcal{R}_f$. Note that it is possible to have $f \neq f'$ such that $\mathcal{R}_f = \mathcal{R}_{f'}$. In other words, the index f distinguishes multiple copies of representation $\mathcal{R}_f$. Now we can add the one-loop contribution to the prepotential (4.17) to obtain

$$6\mathcal{F}_{S,\{q_\alpha\}} = \sum_{\alpha,\beta}3\Omega_S^{\alpha\beta}\varphi_{0,\alpha}\left(K_\beta^{ab}\varphi_{a,\beta}\varphi_{b,\beta}\right) + \frac{1}{2}\left(\sum_r |r\cdot\varphi|^3 - \sum_f\sum_{w(\mathcal{R}_f)}|w(\mathcal{R}_f)\cdot\varphi + m_f|^3\right) \tag{4.18}$$

where r are the roots of $\mathfrak{h} = \oplus_\alpha\mathfrak{h}_\alpha$, $w(\mathcal{R}_f)$ parametrize weights of $\mathcal{R}_f$ and $m_f \in \mathbb{R}$ is a mass term for each full[3] hypermultiplet f. The notation $w\cdot\varphi$ denotes the scalar product of the Dynkin coefficients of the weight w with Coulomb branch parameters. Note that similar approaches for computing prepotentials of 5d theories have appeared in the literature—see for example [33,34,82].

In (4.18) we must impose that mass terms for hypers belonging to $\mathcal{S}_{\alpha\beta,\alpha}$ and $\mathcal{S}_{\alpha\beta,\beta}$ equal the mass term for hypers belonging to $\mathcal{R}_{\alpha\beta}$. This is because $\mathcal{R}_{\alpha\beta}$, $\mathcal{S}_{\alpha\beta,\alpha}$ and $\mathcal{S}_{\alpha\beta,\beta}$ all descend from the same $6d$ representation $\mathcal{R}_{ij}$ which has only a single $\mathfrak{u}(1)$ symmetry rotating it. The Wilson lines for this $\mathfrak{u}(1)$ around the compactification circle gives rise to the mass terms for $\mathcal{R}_{\alpha\beta}$, $\mathcal{S}_{\alpha\beta,\alpha}$ and $\mathcal{S}_{\alpha\beta,\beta}$, and hence all

[3] Half-hypermultiplets do not admit mass parameters unless completed into a full hypermultiplet.

these mass terms must be equal.

We propose that (4.18) is the full exact prepotential for $\mathfrak{T}^{KK}_{S,\{q_\alpha\}}$ where we have ignored the terms involving the mass parameter $\frac{1}{R}$ where R is the radius of compactification. We are justified in doing so since these terms do not play any role in this section. Moreover, only the part of $6\mathcal{F}_{S,\{q_\alpha\}}$ that is cubic in Coulomb branch parameters $\varphi_{a,\alpha}$ is relevant to the discussion in this chapter; so, for convenience, we denote the part of the prepotential cubic in Coulomb branch parameters by $6\mathcal{F}^{\varphi}_{S,\{q_\alpha\}}$.

Notice that fixing the relative values of $\varphi_{a,\alpha}$ and m_f fixes the signs of the terms inside absolute values in (4.18). As the relative values of $\varphi_{a,\alpha}$ and m_f are changed, the sign of some of the terms in (4.18) changes. This leads to jumps in the coefficients of various terms in the resulting $6\mathcal{F}^{\varphi}_{S,\{q_\alpha\}}$. This means that different relative values of $\varphi_{a,\alpha}$ and m_f lead to different phases inside the Coulomb branch of the $5d$ KK theory.

4.3.2 Shifting the prepotential

Consider a $6d$ theory $\mathfrak{T}$ with gauge algebras $\mathfrak{g}_i$ on its tensor branch. Consider further compactifying $\mathfrak{T}$ on a circle of finite size without a twist. On a generic point of the resulting $5d$ Coulomb branch, the massive BPS spectrum includes W-bosons for the corresponding untwisted affine gauge algebras $\mathfrak{g}_i^{(1)}$. In other words, the abelian gauge algebra visible at low energies on the Coulomb branch is $\oplus_a \mathfrak{u}(1)_{a,i}$ parametrizing the Cartan of $\mathfrak{g}_i$ plus a $\mathfrak{u}(1)_{0,i}$ responsible for affinization. The $\mathfrak{u}(1)_i$ arising from the reduction of tensor multiplet B_i is central to $\oplus_a \mathfrak{u}(1)_{a,i} \oplus \mathfrak{u}(1)_{0,i}$. The untwisted Lie algebras are listed in Table B.2 along with their Coxeter and dual Coxeter labels.

We now generalize the above statements to the twisted case. Consider compactifying $\mathfrak{T}$ on a circle of finite size with a twist $S, \{q_\alpha\}$. On a generic point of the resulting $5d$ Coulomb branch, the massive BPS spectrum includes W-bosons for the corresponding twisted/untwisted affine gauge algebras $\mathfrak{g}_\alpha^{(q_\alpha)}$. In other words, the abelian gauge algebra visible at low energies on the Coulomb branch is $\oplus_a \mathfrak{u}(1)_{a,\alpha}$ parametrizing the Cartan of $\mathfrak{h}_\alpha$ plus a $\mathfrak{u}(1)_{0,\alpha}$ responsible for affinization. The $\mathfrak{u}(1)_\alpha$ arising from the reduction of tensor multiplet B_i (with i in orbit of α) is central to $\oplus_a \mathfrak{u}(1)_{a,\alpha} \oplus \mathfrak{u}(1)_{0,\alpha}$. The twisted Lie algebras are listed in Table B.3 along with their Coxeter and dual Coxeter labels.

The charge under $\mathfrak{u}(1)_{b,\alpha}$ (corresponding to a simple co-root $e_b^\vee$) of a W-boson W_a (corresponding to simple root e_a of $\mathfrak{g}_\alpha^{(q_\alpha)}$) is given by the element A_{ab} of the Cartan matrix. Now consider the $\mathfrak{u}(1)$ embedding into $\oplus_{b=0}^{r_\alpha}\mathfrak{u}(1)_{b,\alpha}$ by the map $e^{i\theta} \;\to\; \oplus_{b=0}^{r_\alpha}\left(e^{id_b^\vee\theta}\right)_b$ where $\left(e^{id_b^\vee\theta}\right)_b$ is the element $e^{id_b^\vee\theta}$ of $\mathfrak{u}(1)_{b,\alpha}$ and $d_b^\vee$ are dual Coxeter labels of $\mathfrak{g}_\alpha^{(q_\alpha)}$ listed in Tables B.2 and B.3. Since all the W-bosons W_a are uncharged under this $\mathfrak{u}(1)$, it follows that this $\mathfrak{u}(1)$ can be identified with the central $\mathfrak{u}(1)_\alpha$. The charge of a particle n_α under $\mathfrak{u}(1)_\alpha$ can be written as $\sum_{b=0}^{r_\alpha} d_b^\vee n_{b,\alpha}$ where $n_{b,\alpha}$ is the charge of the particle under $\mathfrak{u}(1)_{b,\alpha}$.

The truncated prepotential $6\mathcal{F}^{\varphi}_{S,\{q_\alpha\}}$ is written in terms of Coulomb branch parameters $\varphi_{b,\alpha}$ (with $1 \;\le\; b \;\le\; r_\alpha$) corresponding to $\mathfrak{u}(1)_{b,\alpha}$ and $\varphi_{0,\alpha}$ corresponding to $\mathfrak{u}(1)_\alpha$. To facilitate comparison with geometry, we wish to write the prepotential in terms of Coulomb branch parameters corresponding to $\mathfrak{u}(1)_{b,\alpha}$ for $0 \le b \le r_\alpha$. This is achieved by performing the following replacement in $6\mathcal{F}^{\varphi}_{S,\{q_\alpha\}}$

$$\varphi_{b,\alpha} \to \varphi_{b,\alpha} - d_b^\vee \varphi_{0,\alpha} \tag{4.19}$$

for all $1 \;\le\; b \;\le\; r_\alpha$ and for all α.[4] We will call the prepotential obtained after this shift as $\tilde{\mathcal{F}}_{S,\{q_\alpha\}}$. The Coulomb branch parameter $\varphi_{0,\alpha}$ in $\tilde{\mathcal{F}}_{S,\{q_\alpha\}}$ corresponds to $\mathfrak{u}(1)_{0,\alpha}$ rather than $\mathfrak{u}(1)_\alpha$.

A Mathematica notebook, using[72], accompanying the submission of the original paper[30] can be used to compute the contribution to $6\tilde{\mathcal{F}}$ (in any gauge-theoretic phase) from a single node or two nodes connected by an edge.

[4]Note that the shift (4.19) has been studied before the in the literature in relation to resolutions of elliptically fibered Calabi-Yau threefolds; in these examples, the effect of the shift is to expand the Kähler form J in basis of primitive divisors—see for example[81].

4.4 Geometries associated to 5d KK theories

In this section, we will show that we can associate (at least one) genus-one fibered Calabi-Yau threefold $X_{S,\{q_\alpha\}}$ to every $5d$ KK theory[5] $\mathfrak{T}^{KK}_{S,\{q_\alpha\}}$. Compactifying M-theory on $X_{S,\{q_\alpha\}}$ produces the Coulomb branch of $\mathfrak{T}^{KK}_{S,\{q_\alpha\}}$.

Some of the results appearing below also appeared in [64,67,66,63,65,70,68,69,55,29,28]

4.4.1 General features

In this subsection, we start with a description of general features of the geometric structure of $X_{S,\{q_\alpha\}}$ and the relationship between this geometry and the low energy effective theory governing the Coulomb branch of the KK theory $\mathfrak{T}^{KK}_{S,\{q_\alpha\}}$.

We will show that $X_{S,\{q_\alpha\}}$ can be realized as a local neighborhood of a collection of irreducible compact holomorphic surfaces intersecting with each other pairwise transversely. As we will see, the surfaces fall into families indexed by α. We denote the irreducible surfaces in each family α as $S_{a,\alpha}$ where $0 \leq a \leq r_\alpha$ (where r_α is the rank of $\mathfrak{h}_\alpha$). The Kähler parameters associated to $S_{a,\alpha}$ are identified as the Coulomb branch parameters $\varphi_{a,\alpha}$ of the corresponding $5d$ KK theory discussed in the previous section. Whenever $\mathfrak{h}_\alpha$ is trivial, the rank of $\mathfrak{h}_\alpha$ is zero and hence there is only a single surface $S_{0,\alpha}$ associated to the node α in that case.

Triple intersection numbers and the prepotential

A key role in the relationship between $X_{S,\{q_\alpha\}}$ and $\mathfrak{T}^{KK}_{S,\{q_\alpha\}}$ is played by the shifted prepotential $6\tilde{\mathcal{F}}_{S,\{q_\alpha\}}$. The coefficients $c_{a\alpha,b\beta,c\gamma}$ of $\varphi_{a,\alpha}\varphi_{b,\beta}\varphi_{c,\gamma}$ in $6\tilde{\mathcal{F}}_{S,\{q_\alpha\}}$ capture the triple intersection num-

[5] We remind the reader that this statement is not completely true for KK theories involving the last node in Table 4.5. For such KK theories, we only propose an algebraic description whose structure closely mimics the structure of genus-one fibered Calabi-Yau threefolds to be discussed in the next subsection 4.4.1.

bers of surfaces in $X_{S,\{q_\alpha\}}$ as follows:

$$c_{a\alpha,a\alpha,a\alpha} = S_{a,\alpha} \cdot S_{a,\alpha} \cdot S_{a,\alpha} \tag{4.20}$$

$$c_{a\alpha,a\alpha,b\beta} = 3S_{a,\alpha} \cdot S_{a,\alpha} \cdot S_{b,\beta} \tag{4.21}$$

$$c_{a\alpha,b\beta,c\gamma} = 6S_{a,\alpha} \cdot S_{b,\beta} \cdot S_{c,\gamma} \tag{4.22}$$

where $(a,\alpha),(b,\beta),(c,\gamma)$ denote distinct non-equal indices.

A triple intersection product of three surfaces can be computed via intersection numbers inside any one of the three surfaces. To explain it, let us first first define the notion of "gluing curves". Consider the intersection locus $\mathcal{L}_{a\alpha,b\beta}$ between two distinct surfaces $S_{a,\alpha}$ and $S_{b,\beta}$ in $X_{S,\{q_\alpha\}}$. $\mathcal{L}_{a\alpha,b\beta}$ splits into geometrically irreducible components as $\sum_i \mathcal{L}^i_{a\alpha,b\beta}$. Each $\mathcal{L}^i_{a\alpha,b\beta}$ appears as an irreducible curve $C^i_{a,\alpha;b,\beta}$ in $S_{a,\alpha}$ and an irreducible curve $C^i_{b,\beta;a,\alpha}$ in $S_{b,\beta}$. In other words, we can manufacture the intersection of $S_{a,\alpha}$ and $S_{b,\beta}$ by identifying the curves

$$C^i_{a,\alpha;b,\beta} \sim C^i_{b,\beta;a,\alpha} \tag{4.23}$$

with each other for all i. Identifying pairs of curves in the above fashion can be thought of as "gluing together" two surfaces along those curves[6]. The reducible curve $C_{a,\alpha;b,\beta} := \sum_i C^i_{a,\alpha;b,\beta}$ is called the "total gluing curve" in $S_{a,\alpha}$ for the intersection of $S_{a,\alpha}$ and $S_{b,\beta}$. Similarly, $C_{b,\beta;a,\alpha} := \sum_i C^i_{b,\beta;a,\alpha}$ is called the total gluing curve in $S_{b,\beta}$ for the intersection of $S_{a,\alpha}$ and $S_{b,\beta}$.

As two distinct surfaces $S_{a,\alpha}$ and $S_{b,\beta}$ can intersect each other, so can a single surface $S_{a,\alpha}$ intersect itself. Much as above for the intersection of two distinct surfaces, the self-intersection of $S_{a,\alpha}$ can be captured in terms of gluings

$$C^i_{a,\alpha} \sim D^i_{a,\alpha} \tag{4.24}$$

[6]On multiple occasions throughout this chapter, we abuse the language and denote the identification of two curves as "gluing" of the two curves.

where $C^i_{a,\alpha}$ and $D^i_{a,\alpha}$ are irreducible curves in $S_{a,\alpha}$.

Then the triple intersection numbers can be expressed as:

$$S_{a,\alpha} \cdot S_{a,\alpha} \cdot S_{a,\alpha} = K'_{a,\alpha} \cdot K'_{a,\alpha} \tag{4.25}$$

$$S_{a,\alpha} \cdot S_{a,\alpha} \cdot S_{b,\beta} = K'_{a,\alpha} \cdot C_{a,\alpha;b,\beta} = C^2_{b,\beta;a,\alpha} \tag{4.26}$$

$$S_{a,\alpha} \cdot S_{b,\beta} \cdot S_{c,\gamma} = C_{a,\alpha;b,\beta} \cdot C_{a,\alpha;c,\gamma} = C_{b,\beta;c,\gamma} \cdot C_{b,\beta;a,\alpha} = C_{c,\gamma;a,\alpha} \cdot C_{c,\gamma;b,\beta} \tag{4.27}$$

where

$$K'_{a,\alpha} := K_{a,\alpha} + \sum_i \left(C^i_{a,\alpha} + D^i_{a,\alpha} \right) \tag{4.28}$$

and $K_{a,\alpha}$ denotes the canonical class of $S_{a,\alpha}$.

Consistency of gluings: volume matching, the Calabi-Yau condition, and irreducibility

Not every pair of curves can be identified with one another to form a consistent gluing. First of all, the topology of the two curves must be identical. This implies that a geometrically irreducible curve in one surface can only be identified with a geometrically irreducible curve in another surface, and furthermore that the genera (as defined in Appendix B.3.1) of the two curves must be identical and non-negative. If $C \subset S$ is an irreducible curve, then a necessary condition that must be satisfied by C is that for any other irreducible curve $C' \subset S$ such that $C \neq C$, the intersection product must be non-negative:

$$C \cdot C' \geq 0. \tag{4.29}$$

In this chapter, some of the algebraic examples are non-geometric (i.e. do not admit a conventional geometric description satisfying these consistency conditions) because they involve gluings which identify a geometrically *reducible* curve in one surface with a geometrically irreducible curve in another surface. Despite this apparent pathology, these examples nevertheless satisfy the remaining conditions described below.

In addition to the above topological constraints, the volumes of a pair of gluing curves must be the same. The volume of a curve C is computed by intersecting the curve with the Kahler class J via

$$\mathrm{vol}(C) = -J \cdot C \tag{4.30}$$

where

$$J = \sum_{a,\alpha} \varphi_{a,\alpha} S_{a,\alpha} + \sum_f m_f N_f \tag{4.31}$$

where m_f are mass parameters and N_f are non-compact surfaces corresponding to those mass parameters. The contribution of mass parameters to the volume will not play a prominent role in this chapter, so we define a truncated Kahler class J^φ which only keep track of the contribution of Coulomb branch parameters to the volume

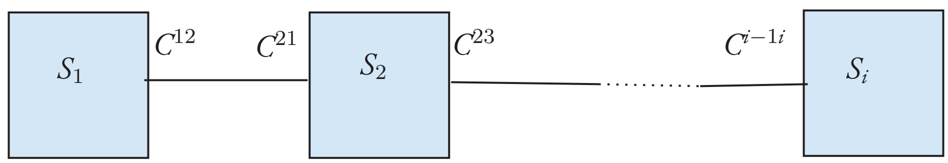
$$J^\varphi = \sum_{a,\alpha} \varphi_{a,\alpha} S_{a,\alpha} \tag{4.32}$$

The volume of C equals the mass of the BPS state obtained by wrapping an M2 brane on C because the intersection number

$$-S_{a,\alpha} \cdot C \tag{4.33}$$

captures the charge under $\mathfrak{u}(1)_{a,\alpha}$ of the BPS state arising from M2 brane wrapping C. If C lies in $S_{a,\alpha}$, then the intersection (4.33) is computed via

$$S_{a,\alpha} \cdot C = K'_{a,\alpha} \cdot C \tag{4.34}$$



$$\textit{Calabi} - \textit{Yau Condition}: \qquad \left(C^{ij}\right)^2 + \left(C^{ji}\right)^2 = 2g - 2$$

Figure 4.1: Gluing using the Calabi-Yau condition

If C lies in some other surface $S_{b,\beta}$, then (4.33) is computed via

$$S_{a,\alpha} \cdot C = C_{b,\beta;a,\alpha} \cdot C \tag{4.35}$$

Now, for (4.23) to be consistent we must have

$$J^{\varphi} \cdot C^{i}_{a,\alpha;b,\beta} = J^{\varphi} \cdot C^{i}_{b,\beta;a,\alpha} \tag{4.36}$$

which is an important consistency condition for constructing $X_{S,\{q_\alpha\}}$. We have checked that (4.36) is satisfied for all the geometries presented in this chapter.

Finally, the gluing curves also have to satisfy the *Calabi-Yau condition* which states that

$$\left(C^{i}_{a,\alpha;b,\beta}\right)^2 + \left(C^{i}_{b,\beta;a,\alpha}\right)^2 = 2g - 2 \tag{4.37}$$

where g is the genus of $C^{i}_{a,\alpha;b,\beta}$. See [111,28] for more details.

Weights, phase transitions and flops

A hypermultiplet transforming in a representation $\mathcal{R}_f$ of the $5d$ gauge algebra $\mathfrak{h} = \oplus_\alpha \mathfrak{h}_\alpha$ appears as a collection of curves inside $X_{S,\{q_\alpha\}}$. These curves are characterized as follows. Let m_f be the mass parameter corresponding to $\mathcal{R}_f$. For each weight $w(\mathcal{R}_f)$ of $\mathcal{R}_f$, define a quantity vol $\left(w(\mathcal{R}_f)\right)$, which we call the *virtual volume*, by shifting the quantity

$$w(\mathcal{R}_f) \cdot \varphi + m_f \tag{4.38}$$

by the shift (4.19) for all α. Then, one can find a holomorphic curve $C_{w(\mathcal{R}_f)}$ in $X_{S,\{q_\alpha\}}$ such that

$$\text{vol}\left(C_{w(\mathcal{R}_f)}\right) = |\text{vol}\left(w(\mathcal{R}_f)\right)| \tag{4.39}$$

In general, the curve $C_{w(\mathcal{R}_f)}$ can be a positive linear combination of curves living inside various irreducible surfaces. However, some of the curves $C_{w(\mathcal{R}_f)}$ turn out to be living purely inside a single irreducible surface $S_{a,\alpha}$. If such a curve C_w has genus zero and self-intersection -1 inside $S_{a,\alpha}$, then one can perform a *flop transition*[7] on $X_{S,\{q_\alpha\}}$ by flopping C, which corresponds to a phase transition in the Coulomb branch of the $5d$ gauge theory described in previous section. We refer to such a flop transition as a "gauge-theoretic flop transition" to distinguish it from the flop transitions associated to more general -1 curves not associated to any hypermultiplet.

Let the geometry obtained after the flop transition associated to C_w be $X'_{S,\{q_\alpha\}}$. As for $X_{S,\{q_\alpha\}}$, there exist curves $C'_{w(\mathcal{R}_f)}$ in $X'_{S,\{q_\alpha\}}$ associated to weights $w(\mathcal{R}_f)$ such that

$$\text{vol}\left(C'_{w(\mathcal{R}_f)}\right) = |\text{vol}'\left(w(\mathcal{R}_f)\right)| \tag{4.40}$$

where vol$'\left(w(\mathcal{R}_f)\right)$ is the shift of the quantity (4.38) computed in the new phase. The relationship

[7] This transition corresponds to blowing down C inside $S_{a,\alpha}$ and performing a blow-up in the neighboring surfaces intersecting C transversally. We will explain such transitions via various illustrations throughout this chapter. More detailed background can be found in Section 2 of [28].

between the two virtual volumes $\mathrm{vol}'\left(w(\mathcal{R}_f)\right)$ and $\mathrm{vol}\left(w(\mathcal{R}_f)\right)$ is

$$\mathrm{vol}'\left(w(\mathcal{R}_f)\right) = \mathrm{vol}\left(w(\mathcal{R}_f)\right) \tag{4.41}$$

for all $w(\mathcal{R}_f) \neq w$, and

$$\mathrm{vol}'(w) = -\mathrm{vol}(w) \tag{4.42}$$

with a minus sign.

Affine Cartan matrices and intersections of fibers

For each surface $S_{a,\alpha}$ in $X_{S,\{q_\alpha\}}$, we define a canonical fiber $f_{a,\alpha}$ inside it:

- If $\mathfrak{g}_\alpha$ is non-trivial, then $S_{a,\alpha}$ will always be a Hirzebruch surface[8] whose fiber class is the canonical fiber $f_{a,\alpha}$. An M2 brane wrapping this curve gives rise to the W-boson $W_{a,\alpha}$ discussed in last section.
- If the node α is

$$\underset{2}{\mathfrak{su}(1)^{(1)}} \tag{4.43}$$

  then it turns out that there is a single corresponding surface $S_{0,\alpha} = \mathbb{F}_0^2$ which is self-glued since $e-x$ and $e-y$ are identified with each other where x and y are the exceptional curves corresponding to the two blowups. Due to the self-gluing, the fiber class of $S_{0,\alpha}$ intersects itself inside the threefold $X_{S,\{q_\alpha\}}$ and appears as an elliptic curve with a nodal singularity. It is this fiber class that we refer to as the canonical fiber $f_{0,\alpha}$ in this case.

[8] In this chapter, by a "Hirzebruch surface", we refer to a Hirzebruch surface possibly with blowups at generic or non-generic locations. Some background on Hirzebruch surfaces can be found in Appendix B.1.3.

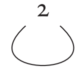
- If the node α is

$$\underset{1}{\mathfrak{sp}(0)^{(1)}_\theta} \tag{4.44}$$

then it turns out that there is a single corresponding surface $S_{0,\alpha} = dP_9$. The del Pezzo surface[9] dP_9 admits a unique elliptic fiber class $3l - \sum x_i$ which we refer to as the canonical fiber $f_{0,\alpha}$ in this case.

- If the node α is

$$\underset{2}{\mathfrak{su}(1)^{(1)}} \tag{4.45}$$



then it turns out that there is no completely geometric description. We provide an algebraic description in terms of algebraic properties of the curves inside the surface $S_{0,\alpha} = \mathbb{F}_1^2$ which is self-glued since x and y are identified with each other. The canonical fiber in this case is $f_{0,\alpha} = 2h + f - 2x - 2y$ which is a genus one curve of self-intersection zero.

For each α we find that

$$f_{a,\alpha} \cdot S_{b,\alpha} = -A_{ab} \tag{4.46}$$

where A_{ab} is the Cartan matrix of $\mathfrak{g}_\alpha^{(q_\alpha)}$ and $A_{ab} \equiv A_{00} = 0$ whenever $\mathfrak{g}_\alpha$ is trivial. This means that the fibers of Hirzebruch surfaces $S_{a,\alpha}$ for a fixed α intersect in the fashion of Dynkin diagram associated to affine Lie algebra $\mathfrak{g}_\alpha^{(q_\alpha)}$.

Intersection (4.46) is of the form $C \cdot S_{a,\alpha}$ where C is some curve in the threefold $X_{S,\{q_\alpha\}}$ and $S_{a,\alpha}$ is a surface inside the threefold. Like the triple intersection numbers of surfaces inside a threefold, such intersections can also be computed in terms of intersection numbers inside a surface. If C is a

[9] In this chapter, by a "del Pezzo surface dP_n", we refer to a surface which is an n point blowup of $\mathbb{P}^2$ but the blowups can be at non-generic locations. Some background on del Pezzo surfaces can be found in Appendix B.1.3.

curve inside $S_{a,\alpha}$, then

$$C \cdot S_{a,\alpha} = C \cdot K'_{a,\alpha} \tag{4.47}$$

and if C is a curve inside a surface $S_{b,\beta}$ that is distinct from $S_{a,\alpha}$, then

$$C \cdot S_{a,\alpha} = C \cdot C_{b,\beta;a,\alpha} \tag{4.48}$$

In fact one can show that the Calabi-Yau we are describing is genus-one fibered and we have summarized it in subsection B.4.1.

Let us now discuss the relationship between fibers f_α and the radius of compactification circle R. In general, we can find at least one node μ such that

$$n_\mu f_\mu \sim n_{\mu,\alpha} f_\alpha \tag{4.49}$$

with $n_{\mu,\alpha} \geq n_\mu \geq 1$ for all α . Then the curve

$$f := l_\mu n_\mu f_\mu \tag{4.50}$$

with l_μ defined in Section 4.2.3 can be identified with the KK mode of unit momentum in $\mathfrak{T}^{KK}_{S,\{q_\alpha\}}$ and has mass $\frac{1}{R}$ where R is the radius of the circle on which the $6d$ theory $\mathfrak{T}$ has been compactified. Thus, all the f_α can be identified as fractional KK modes with mass $\frac{1}{n_\alpha R}$ where $n_\alpha = l_\mu n_{\mu,\alpha}$. This generalizes the condition in the untwisted unfrozen case where the KK mode is identified with

$$f := f_i \tag{4.51}$$

for any i, which is consistent since $f_i \sim f_j$ for all i, j.

Untwisted Simple Node example

Let us illustrate through a simple example of the KK theory specified by the graph

$$\underset{2}{\mathfrak{su}(3)^{(1)}} \tag{4.52}$$

This theory has six hypers in fundamental of $\mathfrak{su}(3)$. The Dynkin coefficients of the positive roots of $\mathfrak{su}(3)$ are $(2,-1)$, $(1,1)$ and $(-1,2)$. The Dynkin coefficients for the weights of fundamental are $(1,0)$, $(-1,1)$ and $(0,-1)$. The Killing form is

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and $\Omega_S^{\alpha\beta}$ is a 1×1 matrix which equals 2. Without loss of generality, we can take $r \cdot \varphi$ for positive roots to be positive. This implies that $r \cdot \varphi$ for negative roots is negative.

Let us first fix all the mass terms to be zero. Then the first weight $(1,0)$ contributes with a positive sign since the positivity of $r \cdot \varphi$ for positive roots implies that φ_1 is positive. Similarly, the third weight $(0,-1)$ contributes with a negative sign to the prepotential. However, the sign of second weight $(-1,1)$ cannot be determined uniquely, and hence the theory has two phases when all mass parameters vanish. These two phases are distinguished by the sign s of the contribution due to the weight $(-1,1)$. The prepotential can be written as

$$\begin{aligned} 6\mathcal{F}^{\varphi} = 6\mathcal{F} =& 12\varphi_0\left(\varphi_1^2+\varphi_2^2-\varphi_1\varphi_2\right) + \left(\left(2\varphi_1-\varphi_2\right)^3 + \left(\varphi_1+\varphi_2\right)^3 + \left(2\varphi_2-\varphi_1\right)^3\right) \\ & - 3\left(s\left(\varphi_2-\varphi_1\right)^3 + \varphi_1^3 + \varphi_2^3\right) \end{aligned} \tag{4.53}$$

Here $12\varphi_0\left(\varphi_1^2+\varphi_2^2-\varphi_1\varphi_2\right)$ is the contribution coming from the Green-Schwarz term in $6d$,

$\left(2\varphi_1-\varphi_2\right)^3+\left(\varphi_1+\varphi_2\right)^3+\left(2\varphi_2-\varphi_1\right)^3$ is the contribution coming from the positive and negative roots, and $-3\left(s\left(\varphi_2-\varphi_1\right)^3+\varphi_1^3+\varphi_2^3\right)$ is the contribution coming from the weights of six hypers in fundamental.

When we turn on mass parameters, the sign of the weights corresponding to different hypers can be changed. For example, consider turning on a mass parameter for one of the fundamentals m_1 while keeping the mass parameters for the other five fundamentals zero. Now we obtain contributions from terms of the form $|m_1+\varphi_1|$, $|m_1-\varphi_1+\varphi_2|$ and $|m_1-\varphi_2|$. Depending on the value of m_1, we go through various new phases of the theory which are parametrized by choices of signs of these three terms. For example, suppose that m_1 is positive and very large, so that all the three terms are positive. Moreover, assume that $\varphi_2-\varphi_1$ is positive, so that $s=+1$. Then the resulting phase is governed by the following prepotential

$$
\begin{aligned}
6\mathcal{F}=&12\varphi_0\left(\varphi_1^2+\varphi_2^2-\varphi_1\varphi_2\right)+\left(\left(2\varphi_1-\varphi_2\right)^3+\left(\varphi_1+\varphi_2\right)^3+\left(2\varphi_2-\varphi_1\right)^3\right)\\
&-\frac{5}{2}\left(\left(\varphi_2-\varphi_1\right)^3+\varphi_1^3+\varphi_2^3\right)-\frac{1}{2}\left(\left(\varphi_2-\varphi_1+m_1\right)^3+(\varphi_1+m_1)^3+(-\varphi_2+m_1)^3\right)
\end{aligned}
\tag{4.54}
$$

which implies that the truncated prepotential is

$$
\begin{aligned}
6\mathcal{F}^{\varphi}=&12\varphi_0\left(\varphi_1^2+\varphi_2^2-\varphi_1\varphi_2\right)+\left(\left(2\varphi_1-\varphi_2\right)^3+\left(\varphi_1+\varphi_2\right)^3+\left(2\varphi_2-\varphi_1\right)^3\right)\\
&-3\left(\left(\varphi_2-\varphi_1\right)^3+\varphi_1^3\right)-2\varphi_2^3
\end{aligned}
\tag{4.55}
$$

Shifting the prepotential according to4.19

For illustrative purposes, we note that the shift for our example (4.52) is

$$\varphi_1 \to \varphi_1 - \varphi_0$$
$$\varphi_2 \to \varphi_2 - \varphi_0$$

which means that the shifted prepotential corresponding (4.53) is

$$6\tilde{\mathcal{F}} = 8\varphi_0^3 + 8\varphi_1^3 + 2\varphi_2^3 - 6\varphi_1\varphi_0^2 + 6\varphi_1\varphi_2^2 - 6\varphi_2\varphi_0^2 - 12\varphi_2\varphi_1^2 \tag{4.56}$$

where we have chosen the phase $s = +1$.

The shifted prepotential for (4.55) is

$$6\tilde{\mathcal{F}} = 7\varphi_0^3 + 8\varphi_1^3 + 3\varphi_2^3 - 6\varphi_1\varphi_0^2 + 6\varphi_1\varphi_2^2 - 3\varphi_2\varphi_0^2 - 3\varphi_0\varphi_2^2 - 12\varphi_2\varphi_1^2 \tag{4.57}$$

Reading off the Geometry according to 4.4.1

We propose that the associated geometry is as follows. Since there is a single node, we drop the index α and only display the index a. The surfaces are $S_0 = \mathbb{F}_0, S_1 = \mathbb{F}_2, S_2 = \mathbb{F}_4^6$. The gluing curves between S_0 and S_1 are $C_{0;1} = e, C_{1;0} = e$. The gluing curves between S_1 and S_2 are $C_{1;2} = h, C_{2;1} = e$. The gluing curves between S_2 and S_0 are $C_{2;0} = h - \sum x_i, C_{0;2} = e$.

Now we can check that the intersections of these curves indeed give rise to the various coefficients in (4.56):

- First of all, recall from (B.22) that $K^2 = 8 - b$ for $\mathbb{F}_n^b$. Indeed, the coefficients of φ_a^3 in (4.56) equal K_a^2.
- One third the coefficient of $\varphi_0\varphi_1^2$ is zero which matches $C_{0;1}^2 = (e^2)_{S_0}$ where $(e^2)_{S_0}$ denotes that the intersection number e^2 is computed inside S_0 and that in particular the curve e is

inside S_0. The coefficient also matches $K_1 \cdot C_{1;0} = (K \cdot e)_{S_1} = 0$. One third of the coefficient of $\varphi_2\varphi_0^2$ is -2 which indeed matches $C_{2;0}^2 = \left((h - \sum x_i)^2\right)_{S_2} = \left(h^2 - \sum x_i^2\right)_{S_2} = 4 - 6 = -2$ and $K_0 \cdot C_{0;2} = (K \cdot e)_{S_0} = -2$. Similarly, we can check the matching of such intersection numbers with one third the coefficients of other terms of the form $\varphi_a\varphi_b^2$.

- One sixth the coefficient of $\varphi_0\varphi_1\varphi_2$ is zero which matches $C_{0;1} \cdot C_{0;2} = (e^2)_{S_0} = 0$, $C_{1;2} \cdot C_{1;0} = (h \cdot e)_{S_1} = 0$, and $C_{2,0} \cdot C_{2;1} = \left((h - \sum x_i) \cdot e\right)_{S_2} = 0$.

On the other hand, the geometry associated to (4.57) has $S_0 = \mathbb{F}_0^1$, $S_1 = \mathbb{F}_2$ and $S_2 = \mathbb{F}_4^5$. The gluing curves between S_0 and S_1 are $C_{0;1} = e$, $C_{1;0} = e$. The gluing curves between S_1 and S_2 are $C_{1;2} = h$, $C_{2;1} = e$. The gluing curves between S_2 and S_0 are $C_{2;0} = h - \sum x_i$, $C_{0;2} = e - x$. Here x denotes the exceptional curve of the blowup of S_0 and x_i denote the exceptional curves of the blowups of S_2. One can check that the intersections of these curves indeed give rise to the various coefficients in (4.57). As an example, in what preceded above we discussed the geometry associated to (4.56). We can check that (4.36) is satisfied for all the gluing curves in the geometry. For instance,

$$J^{\varphi} \cdot C_{0;1} = \varphi_0 \left(K_0 \cdot C_{0;1}\right) + \varphi_1 C_{0;1}^2 + \varphi_2 \left(C_{0;2} \cdot C_{0;1}\right) \tag{4.58}$$

$$= \varphi_0 \left(K \cdot e\right)_{S_0} + \varphi_1 \left(e^2\right)_{S_0} + \varphi_2 \left(e^2\right)_{S_0} \tag{4.59}$$

$$= -2\varphi_0 \tag{4.60}$$

and comparing it with

$$J^{\varphi} \cdot C_{1;0} = \varphi_0 C_{1;0}^2 + \varphi_1 \left(K_1 \cdot C_{1;0}\right) + \varphi_2 \left(C_{1;2} \cdot C_{1;0}\right) \tag{4.61}$$

$$= \varphi_0 \left(e^2\right)_{S_1} + \varphi_1 \left(K \cdot e\right)_{S_1} + \varphi_2 \left(e \cdot h\right)_{S_1} \tag{4.62}$$

$$= -2\varphi_0 \tag{4.63}$$

we find that indeed the gluing $C_{0;1} \sim C_{1;0}$ is consistent. Similarly, it can be checked that all the other gluings are consistent as well. In a similar fashion, one can also check that all of the gluings in the geometry associated to (4.57) discussed above satisfy (4.36).

Flops as discussed in 4.4.1:

We know from the above analysis that the canonical $5d$ gauge theory associated to (4.52) is an $\mathfrak{su}(3)$ gauge theory with six fundamental hypers. The Dynkin coefficients of the weights of fundamental are $(1,0)$, $(-1,1)$ and $(0,-1)$. We call these weights w_1, w_2 and w_3 respectively. We can compute

$$\mathrm{vol}(w_1) = -\varphi_0 + \varphi_1 \tag{4.64}$$

$$\mathrm{vol}(w_2) = -\varphi_1 + \varphi_2 \tag{4.65}$$

$$\mathrm{vol}(w_3) = \varphi_0 - \varphi_2 \tag{4.66}$$

Recall that the phase (4.56) corresponds to $\mathrm{vol}(w_1)$ and $\mathrm{vol}(w_2)$ being positive and $\mathrm{vol}(w_3)$ being negative for all the six fundamentals. Now compute the volume of one of the blowups x_i living in the surface S_2 in the geometry corresponding to (4.56):

$$\mathrm{vol}(x_i) = -\varphi_0 + \varphi_2 \tag{4.67}$$

Thus we see that C_{w_3} for each fundamental is x_i. The reader can check that $C_{w_2} = f_2 + x_i$ and $C_{w_1} = f_1 + f_2 + x_i$ where f_a denotes the fiber of the Hirzebruch surface S_a.

In fact, the geometries corresponding to (4.56) and (4.57) are related by a flop transition. We first blow down one of the blowups, say x_6, inside S_2. Under this blowdown the identity of S_2 changes from $\mathbb{F}_4^6$ to $\mathbb{F}_4^5$. Since x_6 intersects the gluing curve $h - \sum_{i=1}^6 x_i$ at one point, the gluing curve after the blowdown becomes $h - \sum_{i=1}^6 x_i + x_6 = h - \sum_{i=1}^5 x_i$. The other gluing curve inside S_2 is unaffected since x_6 does not intersect with it. Correspondingly, since the gluing curve for S_1 in S_2 does not intersect x_6, the surface S_1 is unaffected by the flop transition. However, since the gluing curve for S_0 in S_2 intersects x_6, we have to blowup S_0 at a point lying on the gluing curve for S_2 inside S_0. Under the blowup the identity of S_0 changes from $\mathbb{F}_0$ to $\mathbb{F}_0^1$. The gluing curve for S_2 inside S_1 is changed to $e - x$.

Recall that the phase (4.57) corresponds to turning on a large mass m for one of the fundamentals such that

$$\mathrm{vol}(w_3) = \varphi_0 - \varphi_2 + m \tag{4.68}$$

for this fundamental is positive. Correspondingly, we can compute that

$$\mathrm{vol}(x) = \varphi_0 - \varphi_2 \tag{4.69}$$

which indeed matches (4.68) up to the contribution from mass parameter, thus verifying (4.42). We are not keeping track of non-compact surfaces in this work, so we are only able to verify (4.42) up to the contribution from m.

4.4.2 Geometry for each node

In this section we will describe the surfaces $S_{a,\alpha}$ along with their intersections associated to a single node α.

We will capture the data of the surfaces and their intersections by using a graphical notation that would be a simpler version of the graphical notation used in [28,29]. We will provide some examples to demonstrated the form of the geometries mainly associated to examples discussed in the previous subsections. An exhaustive presentation of our results can be found in [30].

Intersection Matrix as the Affine Cartan Matrix according to 4.4.1:

We can compute that

$$f_0 \cdot S_0 = (K \cdot f)_{S_0} = -2 \tag{4.70}$$

$$f_1 \cdot S_1 = (K \cdot f)_{S_1} = -2 \tag{4.71}$$

$$f_2 \cdot S_2 = (K \cdot f)_{S_2} = -2 \tag{4.72}$$

$$f_0 \cdot S_1 = C_{0;1} \cdot f_0 = (e \cdot f)_{S_0} = 1 \tag{4.73}$$

$$f_1 \cdot S_2 = C_{1;2} \cdot f_1 = (h \cdot f)_{S_0} = 1 \tag{4.74}$$

$$f_2 \cdot S_0 = C_{2;0} \cdot f_2 = \Big((h - \sum x_i) \cdot f\Big)_{S_0} = 1 \tag{4.75}$$

$$f_1 \cdot S_0 = C_{1;0} \cdot f_1 = (e \cdot f)_{S_1} = 1 \tag{4.76}$$

$$f_2 \cdot S_1 = C_{2;1} \cdot f_2 = (e \cdot f)_{S_2} = 1 \tag{4.77}$$

$$f_0 \cdot S_2 = C_{0;2} \cdot f_0 = (e \cdot f)_{S_0} = 1 \tag{4.78}$$

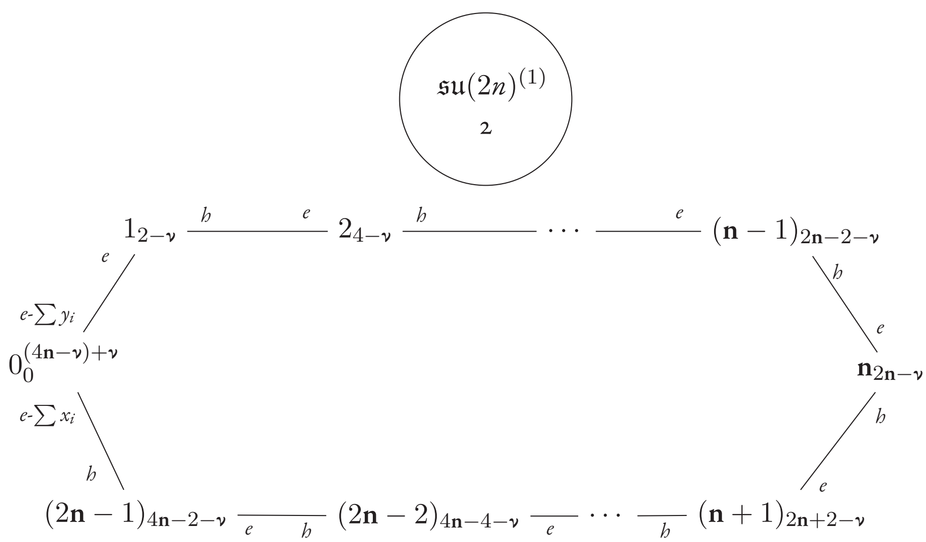
Thus we see that $f_a \cdot S_b$ indeed reproduces the negative of Cartan matrix of affine Lie algebra $\mathfrak{su}(3)^{(1)}$. We can similarly check that the geometry associated to (4.57) also leads to the Cartan matrix of $\mathfrak{su}(3)^{(1)}$.

Untwisted

In this subsection, we collect our results for nodes of the form

$$\begin{matrix} \mathfrak{g}^{(1)} \\ k \end{matrix} \tag{4.79}$$

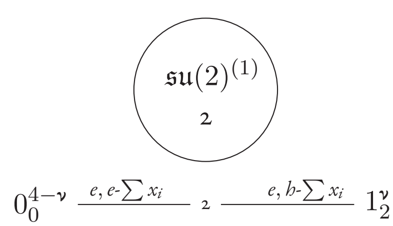
with 6d $\mathfrak{su}(n)$ symmetries. More general gauge symmetries were studied in the original work. That is, we restrict ourselves to the case where the associated affine Lie algebra is untwisted. All such nodes are displayed in Table 4.1 and Table 4.2. Most such cases were first studied in[28,29]. We will be able to recover their results. We will associate a collection of geometries parametrized by ν to each node of the form (4.79). Geometries for different values of ν are flop equivalent as long as there are no neighboring nodes, but might cease to be flop equivalent in the presence of neighboring nodes. The geometries associated to (4.79) in [28] are obtained as $\nu = 0, 1$ versions of the geometries associated in this chapter.



(4.80)

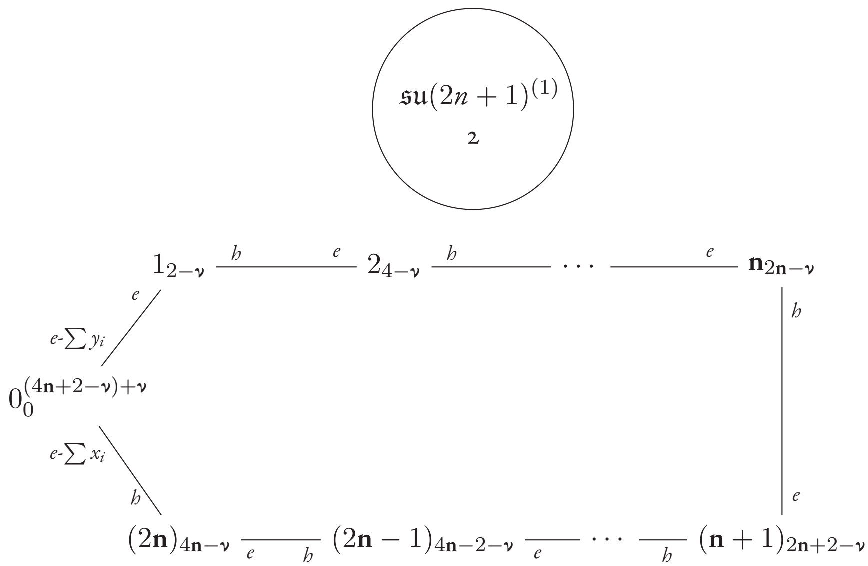
where $0 \leq \nu \leq 4n$ and $n \geq 2$.

For $n = 1$, we have



(4.81)

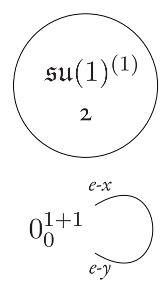
where $0 \leq \nu \leq 4$.



(4.82)

where $0 \leq \nu \leq 4n + 2$ and $n \geq 1$.

For $n = 0$, we claim that the geometry is



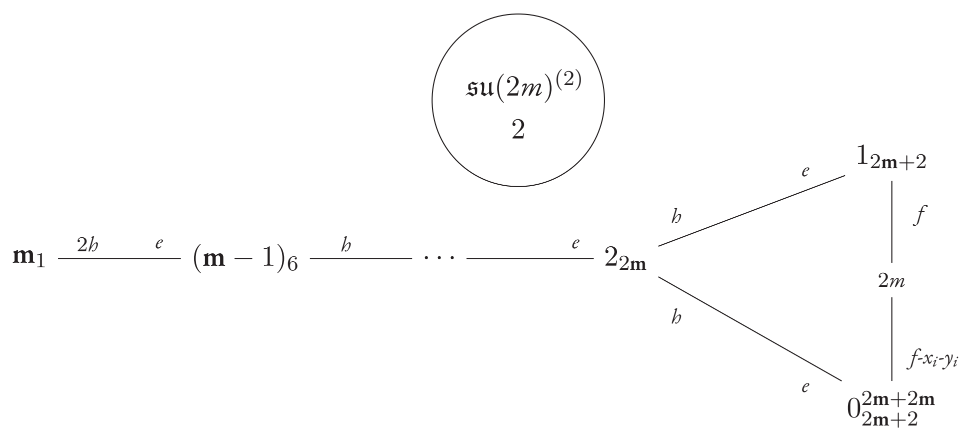
(4.83)

which can be recognized as a limit of $\nu = 1$ phase of (4.82). A detailed analysis of this was done in Appendix B of[30].

In the previous example we saw how one can start from a 6d SCFT, write down the 5d prepotential and successful read off the geometry. The example was done for the untwisted case in order to demonstrate the general methods. However, in the previous sections we demonstrated how automorphisms act on the prepotential and the theory in general and hence incorporating those too we end up with the 5d geometry of the twisted compactification.

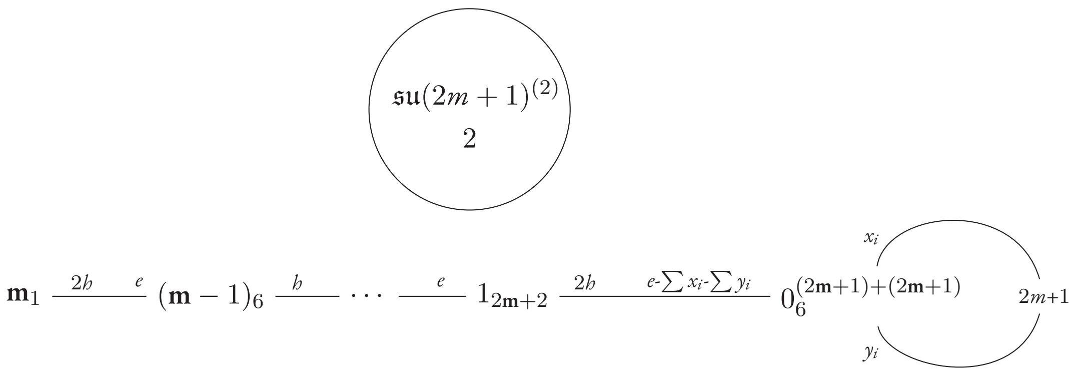
We would like to demonstrate some of the geometries associated to a single node which will be used for gluing, all such nodes are listed in Table 4.5. Here we will not try to be exhaustive as the results are in the original paper[30] but we will focus on single nodes associates with the 6d gauge symmetry $\mathfrak{su}(n)$ and consider geometric descendants.

Fiber Twist:



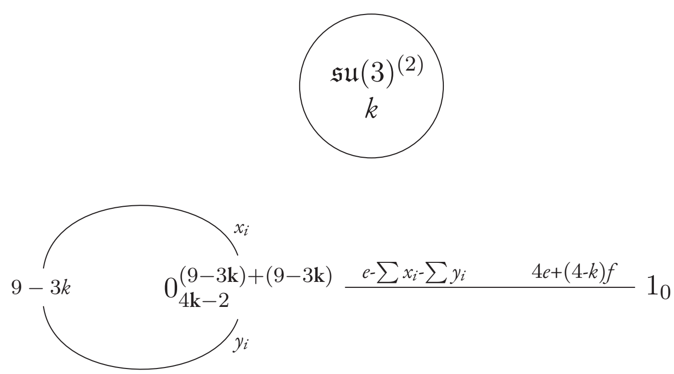
(4.84)

where $m \geq 3$. Notice that the Cartan matrix associated to this geometry is precisely that of $\mathfrak{su}(2m)^{(2)}$. Similar comments hold for all the geometries discussed below in this subsection. For each example below, one can check that $f_a \cdot S_b$ reproduces negative of Cartan matrix of the associated twisted affine algebra $\mathfrak{g}^{(q)}$.



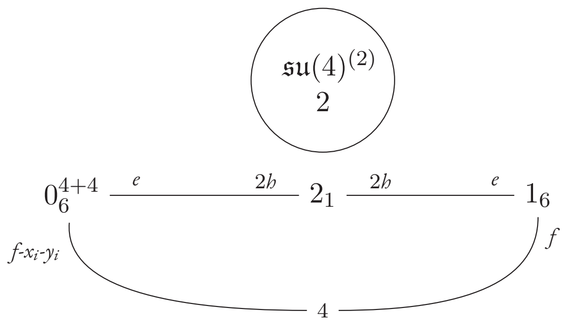
(4.85)

where $m \geq 2$.



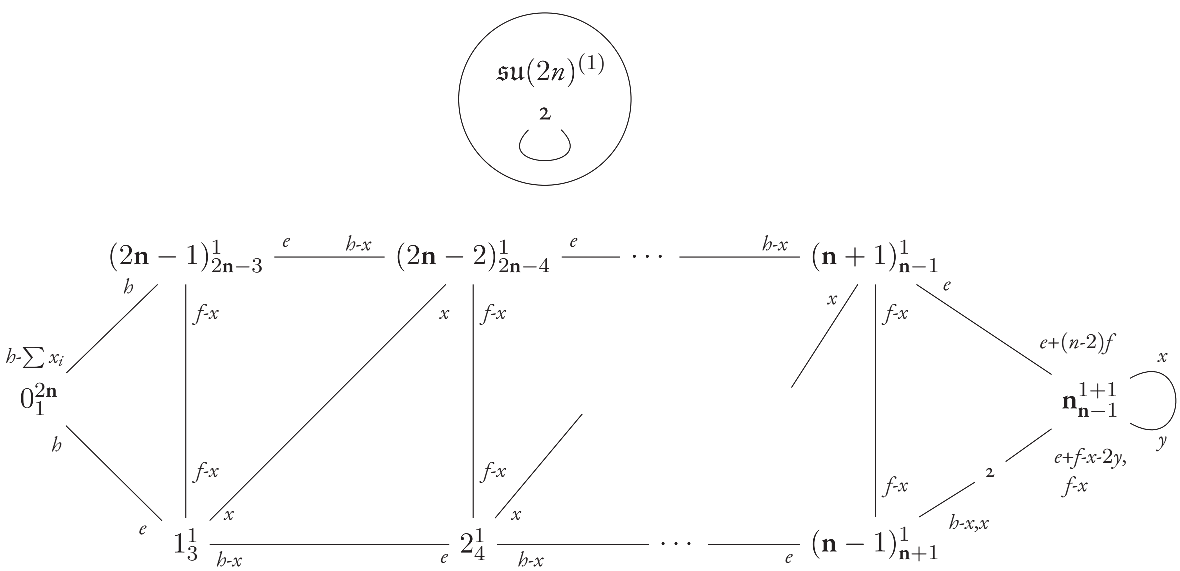
(4.86)

where $1 \leq k \leq 3$.



(4.87)

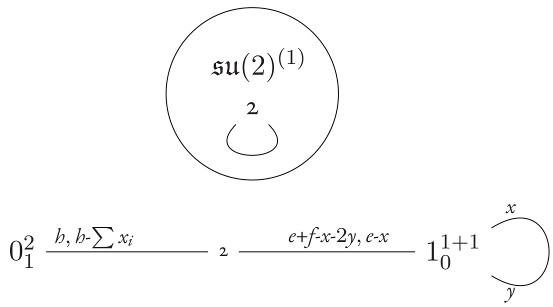
Base Twist:



(4.88)

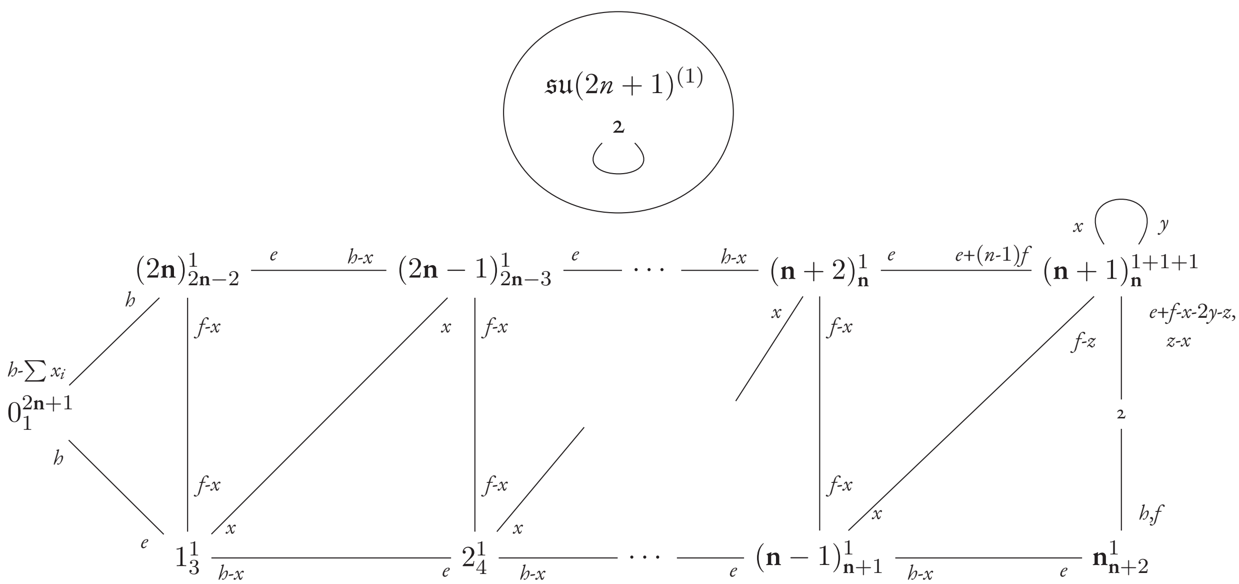
for $n \geq 2$.

For $n = 1$, we have



(4.89)

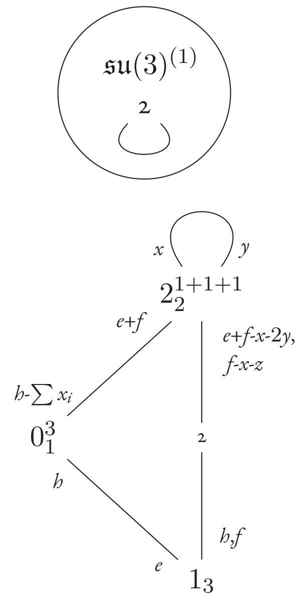
Now we discuss some examples which are not completely geometric:



(4.90)

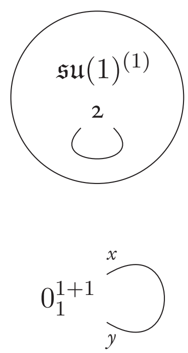
for $n \geq 2$.

For $n = 1$ we have



(4.91)

For $n = 0$ we have



(4.92)

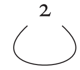
Let us now discuss the reasons why the above five examples are not completely geometric. Let us start with (4.92). The geometry for this example contains the -1 curve $h - x - y$ and hence an M2 brane wrapping this curve should give rise to a BPS particle. However, this BPS particle cannot appear in the associated $5d$ KK theory for the following reason. The existence of a particle associated to $h - x - y$ implies that the KK mode, which is associated to the elliptic curve $2h + f - 2x - 2y$, decomposes as a bound state of $h - x - y$ and $h + f - x - y$ but this is a contradiction since these two curves do not meet each other and hence there cannot be such a bound state.

Another reasoning is as follows. The volume of f is 2φ where φ is the Coulomb branch parameter associated to the above surface. On the other hand, the volume of $h - x - y$ is $-\varphi$. Requiring non-negative volumes for both curves implies that φ must be zero. In other words, there is no direction in the Coulomb branch where all BPS particles have non-negative mass. Thus, this geometry is not *marginal*, in the sense defined by [111], which is a condition that must be satisfied by geometries associated to KK theories.

The precise sense in which the above self-glued $\mathbb{F}_1$ surface is associated to the KK theory

$$\mathfrak{su}(1)^{(1)}$$

2



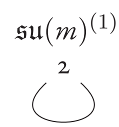
(4.93)

is as follows. The Mori cone of the surface is generated by $h - x - y, f - x, x, e$. However, since the curve $h - x - y$ does not correspond to a BPS particle, the generators of the Mori cone thus do not correspond to the fundamental BPS particles[10] in the associated KK theory (4.93). We propose that the fundamental BPS particles instead correspond to the curves $2h - x - 2y, f - x, x, e$. This set of curves satisfies all the properties that must be satisfied by the generators of the Mori cone of a surface. Thus, it is a complete set which can be consistently associated to fundamental BPS particles. The KK mode can be found as a bound state of $2h - x - 2y$ and $f - x$. One can check that this set of proposed BPS particles is marginal in the sense that it allows a direction in Coulomb branch with all BPS particles having non-negative volumes. See also Appendix B1 of [30] where we verify that this description of the KK theory allows the existence of an RG flow to an $\mathcal{N} = 2$ $5d$ SCFT, which is a fact well-known in the literature.

[10] We define a fundamental BPS particle to be a BPS particle that cannot arise as a bound state of other BPS particles.

There are two viewpoints one can take on the relationship between self-glued $\mathbb{F}_1$ and the KK theory (4.93). The first is that indeed compactifying M-theory on this surface leads to the KK theory (4.93), but the compactification has some extra ingredients which account for the mismatch between the set of Mori cone generators and the set of fundamental BPS particles[11]. The other viewpoint is that the relationship with self-glued $\mathbb{F}_1$ has no deep meaning and is probably a red herring. At the time of writing of this work, we do not know which of these two viewpoints, or if either of these two viewpoints, is the correct one. We leave this issue for future exploration, and only use the relationship between the two as an algebraic tool to build a formalism for KK theories from which one can explicitly perform RG flows to $5d$ SCFTs.

Now let us discuss the non-geometric nature of the KK theories



(4.94)

with $m > 1$. Consider as an example the case of $m = 3$. The surface S_2 contains a gluing curve $e + f - x - 2y$ and hence there must be a BPS particle associated to it. However, notice that it decomposes as $e + f - x - 2y = (e - x - y) + (f - y)$ such that the components $e - x - y$ and $f - y$ do not intersect each other. This leads to the same problem as discussed above, and we are forced to hypothesize that the fundamental BPS particles are distinct from the generators of Mori cone due to some non-geometric feature in the M-theory compactification. It is also evident that some of the components of the gluing curves in certain surfaces (which are identified with irreducible curves in adjacent surfaces as part of the gluing construction) fail to satisfy the necessary properties

[11] A similar situation occurs in the frozen phase of F-theory [31], where the set of generators of the Mori cone of the base of a threefold used for compactifying F-theory does not match the set of fundamental BPS strings arising in the associated $6d$ theory.

of irreducible curves that are described at the beginning of Section 4.4.1.[12] Similar comments apply to each of the $m > 1$ models presented above should be regarded as an algebraic proposal which retains many of the features of the local threefolds that seem to be necessary to compute RG flows to 5d SCFTs.

4.4.3 Gluing rules between two gauge theoretic nodes

In this section we will describe how to glue the surfaces $S_{a,\alpha}$ corresponding to a node α to the surfaces $S_{b,\beta}$ corresponding to another node β if there is an edge between α and β. The gluing rules are different for different kinds of edges between the two nodes but here we will give some simple examples based on theories with 6d $\mathfrak{su}(n)$ gauge symmetry. It turns out that the gluing rules between α and β are insensitive to the values of $\Omega^{\alpha\alpha}$ and $\Omega^{\beta\beta}$. This was also true for all of the cases studied in[28]. For this reason, we will often suppress the data of $\Omega^{\alpha\alpha}$ and $\Omega^{\beta\beta}$ in this subsection.

Undirected edges between untwisted algebras

The simplest untwisted example of such an edge was displayed in Table 4.3 and were first studied in[28]. We are able to reproduce their results using our methods.

Gluing rules for $\mathfrak{su}(n_\alpha)^{(1)}$ ——— $\mathfrak{su}(n_\beta)^{(1)}$: Here we allow $n_\alpha = \widehat{n}_\alpha$ and $n_\alpha = \tilde{6}$. We can take any geometry with $0 \leq \nu \leq 2n_\alpha - n_\beta$ for $\mathfrak{su}(n_\alpha)^{(1)}$, and any geometry with $0 \leq \nu \leq 2n_\beta - n_\alpha$ for $\mathfrak{su}(n_\beta)^{(1)}$. The gluing rules are:

- $f - x_1, x_{n_\beta}$ in $S_{0,\alpha}$ are glued to $f - x_1, x_{n_\alpha}$ in $S_{0,\beta}$.
- $x_i - x_{i+1}$ in $S_{0,\alpha}$ is glued to f in $S_{i,\beta}$ for $i = 1, \cdots, n_\beta - 1$.

[12] For example, in the case $m = 3$, one can see that the surface $\mathbf{2}_0^{1+1+1}$ contains a curve class $e + f - x - 2y$, which is identified with the curve class h in the surface $\mathbf{1}_3$. Since h is irreducible, this implies that $e + f - x - 2y$ must also be irreducible, but this leads to a contradiction (with smoothness) if the usual class $f - y$ remains among the generators of the Mori cone of $\mathbf{2}_0^{1+1+1}$.

- $x_i - x_{i+1}$ in $S_{0,\beta}$ is glued to f in $S_{i,\alpha}$ for $i = 1, \cdots, n_\alpha - 1$.

There is another possibility appearing in the twisted case that involves an undirected edge between two untwisted algebras. This was seen earlier and corresponds to:

$$\begin{array}{ccc} \mathfrak{su}(n_\alpha)^{(1)} & & \mathfrak{su}(n_\beta)^{(1)} \\ \underset{\circlearrowleft}{2} & \text{———} & 2 \end{array} \tag{4.95}$$

The gluing rules for this case are the same as the gluing rules for

$$\mathfrak{su}(n_\alpha)^{(1)} \text{ ——— } \mathfrak{su}(n_\beta)^{(1)} \tag{4.96}$$

presented above.

Directed edges

Now we move onto gluing rules for directed edges in the sense of example 4.13 with and without the fiber twist.

Gluing rules for $\mathfrak{su}(n_\alpha)^{(1)} \xrightarrow{\;2\;} \mathfrak{su}(n_\beta)^{(1)}$: We can take any geometry with $0 \le \nu \le 2n_\alpha - 2n_\beta$ for $\mathfrak{su}(n_\alpha)^{(1)}$, and any geometry with $0 \le \nu \le 2n_\beta - n_\alpha$ for $\mathfrak{su}(n_\beta)^{(1)}$. The gluing rules are:

- $f - x_1, x_{n_\beta} - x_{n_\beta+1}, x_{2n_\beta}$ in $S_{0,\alpha}$ are glued to $f - x_1, f, x_{n_\alpha}$ in $S_{0,\beta}$.

- $x_i - x_{i+1}, x_{n_\beta+i} - x_{n_\beta+i+1}$ in $S_{0,\alpha}$ are glued to f, f in $S_{i,\beta}$ for $i = 1, \cdots, n_\beta - 1$.

- $x_i - x_{i+1}$ in $S_{0,\beta}$ is glued to f in $S_{i,\alpha}$ for $i = 1, \cdots, n_\alpha - 1$.

Gluing rules for $\mathfrak{su}(n_\alpha)^{(1)} \xrightarrow{\;3\;} \mathfrak{su}(n_\beta)^{(1)}$: We can take any geometry with $0 \le \nu \le 2n_\alpha - 3n_\beta$ for $\mathfrak{su}(n_\alpha)^{(1)}$, and any geometry with $0 \le \nu \le 2n_\beta - n_\alpha$ for $\mathfrak{su}(n_\beta)^{(1)}$. The gluing rules are:

- $f - x_1, x_{n_\beta} - x_{n_\beta+1}, x_{2n_\beta} - x_{2n_\beta+1}, x_{3n_\beta}$ in $S_{0,\alpha}$ are glued to $f - x_1, f, f, x_{n_\alpha}$ in $S_{0,\beta}$.

- $x_i - x_{i+1}, x_{n_\beta+i} - x_{n_\beta+i+1}, x_{2n_\beta+i} - x_{2n_\beta+i+1}$ in $S_{0,\alpha}$ are glued to f, f, f in $S_{i,\beta}$ for $i = 1, \cdots, n_\beta - 1$.

- $x_i - x_{i+1}$ in $S_{0,\beta}$ is glued to f in $S_{i,\alpha}$ for $i = 1, \cdots, n_\alpha - 1$.

Gluing rules for $\mathfrak{su}(2n_\alpha)^{(2)} \overset{2}{\longrightarrow} \mathfrak{su}(n_\beta)^{(1)}$: We can take any geometry with $0 \leq \nu \leq 2n_\beta - 2n_\alpha$ for $\mathfrak{su}(n_\beta)^{(1)}$. The gluing rules are:

- $f - y_{n_\beta}, x_{n_\beta}, f - x_1, y_1$ in $S_{0,\alpha}$ are glued to $x_{2n_\alpha-1}, x_{2n_\alpha}, f - x_2, f - x_1$ in $S_{0,\beta}$.

- $x_i - x_{i+1}, y_{i+1} - y_i$ in $S_{0,\alpha}$ are glued to f, f in $S_{i,\beta}$ for $i = 1, \cdots, n_\beta - 1$.

- $x_i - x_{i+1}, x_{2n_\alpha-i} - x_{2n_\alpha-i+1}$ in $S_{0,\beta}$ are glued to f, f in $S_{i,\alpha}$ for $i = 1, \cdots, n_\alpha - 1$.

- $x_{n_\alpha} - x_{n_\alpha+1}$ in $S_{0,\beta}$ is glued to f in $S_{n_\alpha,\alpha}$.

Gluing rules for $\mathfrak{su}(2n_\alpha - 1)^{(2)} \overset{2}{\longrightarrow} \mathfrak{su}(n_\beta)^{(1)}$: We can take any geometry with $1 \leq \nu \leq 2n_\beta - 2n_\alpha + 1$ for $\mathfrak{su}(n_\beta)^{(1)}$. The (non-geometric) gluing rules are:

- $y_{n_\beta}, x_{n_\beta}, f - x_1, f - y_1, f, f$ in $S_{0,\alpha}$ are glued to $x_{2n_\alpha-1}, x_{2n_\alpha-1}, y_1, f - x_1, x_1 - x_2, f - x_2 - y_1$ in $S_{0,\beta}$.

- $x_i - x_{i+1}, y_i - y_{i+1}$ in $S_{0,\alpha}$ are glued to f, f in $S_{i,\beta}$ for $i = 1, \cdots, n_\beta - 1$.

- $x_{i+1} - x_{i+2}, x_{2n_\alpha-i-1} - x_{2n_\alpha-i}$ in $S_{0,\beta}$ are glued to f, f in $S_{i,\alpha}$ for $i = 1, \cdots, n_\alpha - 2$.

- $x_{n_\alpha} - x_{n_\alpha+1}$ in $S_{0,\beta}$ is glued to f in $S_{n_\alpha-1,\alpha}$.

4.4.4 Gluing rules involving non-gauge-theoretic nodes

There are only two such nodes which are listed below

$$\begin{array}{c}\mathfrak{su}(1)^{(1)}\\ 2\end{array} \qquad\qquad \begin{array}{c}\mathfrak{su}(1)^{(1)}\\ 2\\ \circlearrowleft\end{array} \tag{4.97}$$

We note that in fact there is one more class of non-gauge theoretic nodes[30] but we do not discuss it here. Similarly to the previous examples we can define the gluing rules as:

Gluing rules for $\begin{array}{ccc}\mathfrak{su}(1)^{(1)} & & \mathfrak{su}(2)^{(1)}\\ 2 & \text{———} & 2\end{array}$: We can choose any geometry with $1 \le \nu \le 4$ for $\mathfrak{su}(2)^{(1)}$. The (non-geometric) gluing rules are:

- $f - x - y$ in $S_{0,\alpha}$ is glued to f in $S_{0,\beta}$.
- x, y in $S_{0,\alpha}$ are glued to $f - x_1, x_1$ in $S_{1,\beta}$.

As in cases discussed in last subsection, the blowup x_1 in $S_{1,\beta}$ can be used for gluing o$\mathfrak{su}(2)^{(1)}$ with another neighbor such that the gluing rules for $\mathfrak{su}(2)^{(1)}$ with that neighbor allow a blowup on $S_{1,\beta}$ to be used for more than once.

Gluing rules for $\begin{array}{ccc}\mathfrak{su}(1)^{(1)} & & \mathfrak{su}(1)^{(1)}\\ 2 & \text{———} & 2\end{array}$:

- $f - x, x$ in $S_{0,\alpha}$ are glued to $f - x, x$ in $S_{0,\beta}$.

The blowups x in $S_{0,\alpha}$ and x in $S_{0,\beta}$ can be used for gluing to other $\mathfrak{su}(1)^{(1)}$ neighbors. See Appendix (B.3.3) for a derivation of the above gluing rules.

Gluing rules for $\begin{array}{ccc}\mathfrak{su}(2)^{(1)} & & \mathfrak{su}(1)^{(1)}\\ 2 & \text{———} & 2\\ \circlearrowleft & & \end{array}$:

- $f - x_1, x_1$ in $S_{0,\alpha}$ are glued to x, y in $S_{0,\beta}$.

- f in $S_{1,\alpha}$ is glued to $f - x - y$ in $S_{0,\beta}$.

Gluing rules for $\begin{array}{ccc} \mathfrak{su}(1)^{(1)} & & \mathfrak{su}(1)^{(1)} \\ 2 & \text{———} & 2 \\ \circlearrowleft & & \end{array}$:

- $2h - x - 2y, f - x$ in $S_{0,\alpha}$ are glued to $f - x, x$ in $S_{0,\beta}$.

The blowup x in $S_{0,\beta}$ can be used for gluing to other $\mathfrak{su}(1)^{(1)}$ neighbors.

Gluing rules for $\begin{array}{ccc} \mathfrak{su}(2)^{(1)} & & \mathfrak{su}(1)^{(1)} \\ 2 & \overset{}{\text{—}}\,{\scriptstyle 2}\,\longrightarrow & 2 \end{array}$: We can use any geometry with $1 \leq \nu \leq 3$ for $\mathfrak{su}(2)^{(1)}$. The gluing rules are:

- $f - x_1, x_1$ in $S_{0,\alpha}$ are glued to x, y in $S_{0,\beta}$.

- $f - x_1, x_1$ in $S_{1,\alpha}$ are glued to $f - x, f - y$ in $S_{0,\beta}$.

The blowups x_1 in $S_{0,\alpha}$ and x_1 in $S_{1,\alpha}$ can also be used for gluing to other neighboring nodes of $\mathfrak{su}(2)^{(1)}$ that carry some $\mathfrak{su}(n)^{(1)}$.

Gluing rules for $\begin{array}{ccc} \mathfrak{su}(2)^{(1)} & & \mathfrak{su}(1)^{(1)} \\ 2 & \text{—}\,{\scriptstyle 3}\,\longrightarrow & 2 \end{array}$: We can use any geometry with $1 \leq \nu \leq 3$ for $\mathfrak{su}(2)^{(1)}$. The gluing rules are:

- $f - x_1, x_1$ in $S_{0,\alpha}$ are glued to x, y in $S_{0,\beta}$.

- $f - x_1, x_1$ in $S_{1,\alpha}$ are glued to $2f - x, f - y$ in $S_{0,\beta}$.

The blowups x_1 in $S_{0,\alpha}$ and x_1 in $S_{1,\alpha}$ can also be used for gluing to other neighboring nodes of $\mathfrak{su}(2)^{(1)}$ that carry some $\mathfrak{su}(n)^{(1)}$.

Gluing rules for $\begin{array}{ccc} \mathfrak{su}(1)^{(1)} & & \mathfrak{su}(1)^{(1)} \\ 2 & \text{—}\,{\scriptstyle 2}\,\longrightarrow & 2 \end{array}$:

- $f-x, x$ in $S_{0,\alpha}$ are glued to $2f-x, x$ in $S_{0,\beta}$.

(Note that the gluing rules proposed above are non-geometric.) The blowups x in $S_{0,\alpha}$ and x in $S_{0,\beta}$ can be used to further glue to other neighboring $\mathfrak{su}(1)^{(1)}$.

Gluing rules for $\overset{\mathfrak{su}(1)^{(1)}}{2} \xrightarrow{\;3\;} \overset{\mathfrak{su}(1)^{(1)}}{2}$:

- $f-x, x$ in $S_{0,\alpha}$ are glued to $3f-x, x$ in $S_{0,\beta}$.

(Note that the gluing rules proposed above are non-geometric.) The blowups x in $S_{0,\alpha}$ and x in $S_{0,\beta}$ can be used to further glue to other neighboring $\mathfrak{su}(1)^{(1)}$.

4.5 Summary and Discussion

According to a conjecture reviewed in chapter 1 all the $5d$ SCFTs sit at the end points of RG flows emanating from $5d$ KK theories. Thus, this work can be viewed as providing a preliminary step towards an explicit classification of $5d$ SCFTs. In particular, in the original work associated to this chapter we provided a full list of all the possible vertices and edges that can appear in graphs associated to $5d$ KK theories. In principle, the Coulomb branch data of all $5d$ SCFTs is encoded in the properties of Calabi-Yau threefolds presented in this chapter (see Section 4.4). Explicitly, such RG flows can be performed through a sequence of flops and blowdowns on the Calabi-Yau threefolds associated to $5d$ KK theories. This method has been discussed in [111,29,28] and the explicit classification of $5d$ SCFTs up to rank three was done in [24] using the results of this work. Additionally, extensions of this work have been considered in [24,26,32,27]. The classification programs of SCFTs have been proven very fruitful and new constructions and new physics always seem to emerge through these programs. It would particularly interesting to study such classification programs for lower dimensional SCFTs. It would also be interesting to study in what extend similar ideas can be used in gravitational theories with the hope of classifying large classes of compact threefolds.

"String theory is a miracle through and through."

Edward Witten[45]

5

Conclusion

In this thesis various properties of UV completions in quantum gravity and quantum field theory were studied. Such questions are usually hard to study with only information from the low energy physics but string theory has shown to be a powerful guiding tool to uncover the secrets of UV physics. From the quantum field theory perspective it provides a way to construct explicit examples of UV CFTs which may contain tensionless strings and light solitonic objects. From the gravity side it provides a way to gain intuition about the potential basic properties of quantum gravity which are then supported by basic properties of gravity from the bottom up through he Swampland program. Strings theory has proven a useful tool to understand holography, dualities and microscopic information of black holes physics. In both cases it is important to attempt to understand similar characteristics without the help of string theory. This approach can help support the uniqueness of quantum gravity and its interplay with field theory or it can suggest new pathways forward and provide a deeper understanding of such UV questions.

A

Supersymmetry in diverse dimensions

In this appendix some important features of supersymmetries in diverse dimensions will be summarized. The following table summarizes the spinors in diverse dimensions.

s-t mod8	Spinors	SO(d)	SO(1,d-1)
0	MW= $\mathbb{R}(2^{d-2}/2) \oplus \mathbb{R}(2^{d-2}/2)$	SO(8)	SO(1,1/9)
1,7	M=$\mathbb{R}(2^{d-1}/2)$	SO(7)	SO(1,2/8/10)
2,6	$M^{\pm}$=$\mathbb{C}(2^{d-2}/2)$	SO(2)/SO(6)/SO(10)	SO(1,3/7)
3,5	SM= $\mathbb{H}(2^{d-3}/2)$	SO(3)/SO(5)/SO(11)	SO(1,4/6)
4	SMW= $\mathbb{H}(2^{(d-4)/2}) \oplus \mathbb{H}(2^{(d-4)/2})$	SO(4)	SO(1,5)

Table A.1: SO(t,s),MW=Majorana-Weyl,M=Majorana, $\mathcal{M}^{\pm}$ sign, SM=symplectic Majorana, SMW=Symplectic Majorana Weyl

The super-Poincare algebra has the following generators[178,160]:

- Translations: P_μ and Lorentz transformation: $\Lambda_{\mu\nu}$
- Supercharges: $Q^{\alpha i}$ and R-symmetry: T^A
- Central charge: $Z_{A_1 \cdots A_n}$

$SO(1,d-1)$	Spinors	Algebra	R-Symmetry
$SO(1,9)$	MW	$\{Q^i_+, Q^{jT}_-\} = P_+ C_- Z^{ij}$ $\{Q^i_\pm, Q^{jT}_\pm\} = P_\pm \gamma^\mu C_- P_\mu \delta^{ij}$	$SO(N_+) \times SO(N_-)$
$SO(1,8)_+/SO(1,10)_-$	M	$\{Q^i, Q^{jT}\} = \gamma^\mu C_\pm P_\mu \delta^{ij} + C_\pm Z^{ij}_\pm$	$SO(N)$
$SO(1,3)_-/SO(1,7)_+$	M $(Q^i_\pm)^c = Q^i_\mp$	$\{Q^i_+, Q^{jT}_+\} = P_+ \mathrm{C}_\pm Z^{ij}_\pm$ $\{Q^i_+, Q^{jT}_-\} = P_+ \gamma^\mu C_\pm P_\mu \delta^{ij}$	$U(N)$
$SO(1,4)_+/SO(1,6)_-$	SM, $\Omega^{ij}(Q^j)^c = Q^j$	$\{Q^i, Q^{jT}\} = \gamma^\mu C_\pm P_\mu \Omega^{ij} + C_\pm Z^{ij}_\mp$	$Usp(N)$
$SO(1,5)$	SMW $\Omega^{ij}_\pm (Q^j_\pm)^c = Q^j_\pm$	$\{Q^i_+, Q^{jT}_-\} = P_+ C_- Z^{ij}$ $\{Q^i_\pm, Q^{jT}_\pm\} = P_\pm \gamma^\mu C_- P_\mu \Omega^{ij}_\pm$	$Usp(N_+) \times Usp(N_-)$

Table A.2: Chirality Projection $P_\pm = \frac{1}{2}(1 \pm \gamma)$, Symplectic form $\Omega^{ij} = -\Omega^{ij}$, Central charge $Z^{ij}_\pm = \pm Z^{ij}_\pm$,Charge Conjugation $C_\pm$.

Massive representations with $Z^{ij} \neq 0$ satisfy

$$M \geq |Z| \tag{A.1}$$

called Bogomol'nyi-Prasad-Sommerfield (BPS) bound. States that saturate this bound are called BPS and are particularly important as they preserve some of the supersymmetry and hence provide stable solutions since they are the lightest charged states.

Adding extended objects that also preserve some of the supersymmetry will change the algebras by adding a corresponding central charge term. For example in string theory D-branes and NS branes have the following contributions:

$$\{Q^i_\pm, Q^{jT}_\pm\} = P_\pm \gamma^\mu C_- (P_\mu + T^{NS} Q^{NS}_M)\delta^{ij} \tag{A.2}$$

$$\{Q^i_+, Q^{jT}_-\} = \sum_p \frac{T^R_p}{p!} Q^R_{M_1 \cdots M_n}() \tag{A.3}$$

for the

$$Q^M = \int_{M^\perp_{d-p-1}} J \tag{A.4}$$

where the current $J = d * H_{p+1}$.

In fact the central charges for the BPS particles and strings are:

- 16 Supercharges: $m = Q_R$, $T = e^{-a\varphi}$ for some a that depends on the dimension.[1]
- 6d $\mathcal{N} = 1$: $T = \sum_i t_i J_i$ where t_i are the scalars in the tensor multiplets

[1] Note that for the $d \leq 5$ there are more string charges and hence more scalars similar to the $\mathcal{N} = 1$ cases.

- 5d $\mathcal{N} = 1$: $m = |Q_e\varphi|$ and $T = |Q_M\varphi^D|$ where $\varphi^D = \frac{\partial\mathcal{F}}{\partial\varphi}$ with φ the Coulomb branch parameters.

As stated above BPS states are states that their mass is equal to their charge. In a gravitational theory the mass of a particle can be thought of as a gravitational charge. In the absence of global symmetries in fact the central charge should depend on some combination of the gauge fields. But the BPS condition equates these two. In fact one would expect that the central charge would also have to do something with gravity since the mass does. Such a linear combination could be the graviphoton charges. For example, it explains why $M = Q_R$ for the theories with 16 supercharges.

A.1 Properties of BPS Strings

A.1.1 16 Supercharges

The supergravity multiplet for non-chiral theories with 16 supercharges includes an anti-symmetric 2-form tensor field B. By completeness of the spectrum in a gravitational theory we can consider a BPS string charged under this field and study the consequences of supersymmetry and unitary of the worldsheet CFT on this string.

The string with tensor charge Q couples to the bulk theory through the following term:

$$S^{\text{str}} = Q\int_{\mathcal{M}_d} B \wedge \prod_{a=1}^{d-2} \partial(x^a)dx^a = Q\int_{\mathcal{M}_2} B\,, \tag{A.5}$$

where we assume the string is located at the origin $x^a = 0$ of the transverse $\mathbb{R}^{d-2}$ directions.

The 1/2 BPS string will preserve $\mathcal{N} = (0,8)$ supersymmetry. This can be explained as follows. The supersymmetry algebra in the presence of the string should be comprised only of unbroken symmetry generators. In particular, the momentum generators along the transverse $\mathbb{R}^{d-2}$ cannot be parts of the algebra. First consider the 4d case. Suppose a string is stretched along x^0, x^3 directions in 4d supergravity. The 4d SUSY algebra involves $\{Q^I_\alpha, \bar{Q}_{\dot{\beta}J}\} \sim (\sigma^\mu P_\mu)_{\alpha\dot{\beta}}\delta^I_J$ with the Pauli matrices $\sigma^\mu = (-1, \sigma^i)$. Then for the absence of P_1, P_2 generators in this algebra the string will pick unbroken 8 supercharges $Q^I_+, \bar{Q}_{\dot{+}J}$ (or $Q^I_-, \bar{Q}_{\dot{-}J}$) whose currents turn out to be right-moving (or left-moving) in the 2d worldsheet. Also, half the supercharges in $d > 4$ supergravity preserved in the presence of 2d defects reduce to these chiral supercharges under toroidal compactification to 4d (with the string worldsheet transverse to the circles). This shows that the 8 supercharges have a definite chirality on the worldsheet for all d. They are right-moving in our convention.

We note that the B field transforms non-trivially under the local Lorentz and the gauge transformations with parameters Θ and Λ_i respectively:

$$\delta B = -\frac{1}{4}\mathrm{Tr}(\Lambda \cdot F) + \kappa\, \mathrm{tr}(\Theta R)\,, \tag{A.6}$$

with the gauge field strength F_i's and the curvature 2-form R in the bulk supergravity. We have used $a \cdot b = \Omega^{ij} a_i b_j$ for the dot product of two vectors in the charge lattice $\Gamma_{r_G,10-d}$ with respect to the $SO(r_G, 10-d)$-invariant metric Ω^{ij} with signture $(r_G, 10-d)$. Here, the gauge variations of B are fixed by invariance of the action under 16 supercharges[20,41,21]. On the other hand the variation under the local Lorentz transformation is from higher derivative corrections that cannot be fixed solely by supersymmetry. So the coefficient κ is yet to be determined by other means. The bulk action now includes the string action S^{str} that is not invariant under the symmetry transformations due to the variation rules (A.6) of B. Therefore, the presence of the string induces anomaly inflow toward the string worldsheet. This anomaly inflow must be cancelled by the anomaly coming from the worldsheet degrees of freedom.

The chiral degrees of freedom on the worldsheet CFT could have non-trivial anomaly and we expect that this worldsheet anomaly cancels the anomaly inflow from the bulk gravity theory discussed above. The cancellation of the anomaly inflow then restricts the anomaly polynomial of the worldsheet CFT on Q strings to the form,

$$\begin{aligned} I_4 &= Q\left[-\kappa\,\mathrm{tr}(R^2) + \frac{1}{4}\mathrm{Tr}(F\cdot F)\right] \\ &= Q\left[\frac{\kappa}{2}p_1(T_2) - \kappa\, c_2(SO(d-2)) + \frac{1}{4}\mathrm{Tr}(F\cdot F)\right] , \end{aligned} \tag{A.7}$$

where $p_1(T_2)$ is the first Pontryagin class of the tangent bundle T_2 on the 2d worldsheet and $c_2(SO(d-2))$ is the 2nd Chern-class of the $SO(d-2)$ normal bundle for the transverse $\mathbb{R}^{d-2}$ rotation. Here we used the decomposition $\mathrm{tr}R^2 = -\frac{1}{2}p_1(T_2) + c_2(SO(d-2))$.

When $d = 10$, for example, the anomaly polynomial with $\kappa = 1$ coincides with that of the 2d CFT on BPS strings in the 10d $\mathcal{N} = (1,0)$ supergravity with $E_8 \times E_8$ or $SO(32)$ gauge group computed in [119]. In this case the constant $\kappa = 1$ is fixed by the bulk anomaly cancellation by Green-Schwarz mechanism.

The worldsheet theory at low-energy reduces to a 2d conformal theory with at least $(0,8)$ supersymmetry. It is a priori conceivable that the supersymmetry gets enhanced in the IR to $(8,8)$. It is important to distinguish these two possibilities because the anomaly coefficients compute the left minus right contributions. Let us first discuss the case where there are no enhancements and the theory in the IR has $(0,8)$ supersymmetry only.

For the $(0,8)$ case we can easily compute the central charges of the 2d worldsheet theory from the anomaly polynomial. First, the coefficient of the gravitational anomaly $-\frac{1}{24}p_1(T_2)$ encodes the relative central charge $c_R - c_L$. In addition, we can obtain the right-moving central charge c_R using the $\mathcal{N} = (0,2)$ superconformal subalgebra in the $\mathcal{N} = (0,8)$ supersymmetric theory. The $U(1)_R$ R-symmetry group of the $\mathcal{N} = (0,2)$ superconformal algebra is chosen as an $SO(2)$ subgroup of the $SO(d-2)$ rotation group (which can be done since $d \geq 4$). Then the $(0,2)$ algebra relates the right-moving central charge with the 't Hooft anomaly k_R of the $U(1)_R$ symmetry such as $c_R = 3k_R = 12\kappa$ where κ is the 't Hooft anomaly coefficient for the $J_{SO(2)}$ current of $SO(2) \subset SO(d-2)$

bulk symmetry related to the $U(1)_R$ current by $J_R = 2J_{SO(2)}$. As a consequence, we find the central charges,

$$c_R = 12\kappa\,, \quad c_L = 24\kappa\,. \tag{A.8}$$

κ is quantized to be an integer because the $SO(2) \subset SO(d-2)$ is part of the Lorentz symmetry.

We note that this result involves the contributions from the center-of-mass degrees of freedom that come from the zero modes of broken symmetries in the presence of BPS strings. The center-of-mass modes form a free $(0,8)$ multiplet $(\sigma^a, Y^i, \lambda^+)$ with $a = 1, \cdots, 10-d$ and $i = 1, \cdots d-2$ where σ^a are right-moving compact scalars and Y^i are non-compact scalars (which realize the symmetry currents associated with the $10-d$ graviphotons), and λ^+ are 8 right-moving fermions. A simple counting yields their central charges $c_R^{\rm com} = 12, c_L^{\rm com} = d-2$. Thus the central charges of the interacting sector in the worldsheet theory are

$$\tilde{c}_R = c_R - c_R^{\rm com} = 12(\kappa-1)\,, \quad \tilde{c}_L = c_L - c_L^{\rm com} = 24\kappa + 2 - d\,. \tag{A.9}$$

The 't Hooft anomaly coefficients of flavor symmetry groups are identified with the levels of the Kac-Moody current algebra. For a string with $Q = 1$ the anomaly polynomial in (A.7) tells us that the 't Hooft anomalies k_i for flavor symmetry groups, which originate from the bulk gauge symmetries, are given by eigenvalues of the metric Ω^{ij}. In our convention, right-moving (or left-moving) current algebra provides negative (or positive) contribution to the associated 't Hooft anomaly. The k_i is a net contribution from both sectors. Therefore one can deduce that the 2d CFT on a single string must contain at least one current algebra for every symmetry group G_i realized in the right-moving sector if $k_i < 0$ or in the left-moving sector if $k_i > 0$.

A.1.2 Five Dimensional Anomaly Inflow

Monopole strings in five dimensions are two-dimensional magnetic sources for the low-energy Abelian gauge fields on the Coulomb branch. In particular, we shall consider 1/2 BPS monopole string configurations preserving 4 chiral supercharges in the 2d worldsheet.

The monopole string with magnetic charge q^I can be introduced by a delta function source in the Bianchi identity of the gauge field strength $F^I = dA^I$ as,

$$dF^I = q^I \prod_{\mu=2}^{4} \delta(x^\mu) dx^\mu \, . \tag{A.10}$$

We assume here that the monopole source is located at the origin $x^{2,3,4} = 0$ on the transverse $\mathbb{R}^3$. Completeness of charged string spectrum in a gravitational theory ensures existence of such monopole string states as long as Dirac quantization condition is obeyed[159,13] and the string tension is positive. In the following discussions, we shall focus on single BPS monopole string states for a given primitive magnetic charge q^I. The question of existence of such string states will be discussed later.

The string source supports a microscopic 2d theory that flows in the IR to a 2d $\mathcal{N} = (0,4)$ SCFT. The worldsheet SCFT involves chiral degrees of freedom coming from zero modes of the charged fields in the bulk gauge theory on the string background. The 2d chiral fields charged under the bulk symmetry develop non-trivial anomalies for the symmetries. The anomaly arising from the worldsheet degrees of freedom must be cancelled by an other source since otherwise, the monopole string configuration in the 5d supergravity will be inconsistent by the quantum anomaly along the string worldsheet.

The anomaly cancellation can be achieved by the anomaly inflow mechanism from the bulk gravity theory toward the string source. The anomaly inflow in the presence of BPS monopole strings in

5d supersymmetric theories was studied in[73,142,36] (See also[37,22,108,118,171,119] for anomaly inflow of BPS strings in other dimensions). We shall generalize these earlier studies and compute the anomaly inflow in the presence of the string sources in 5d supergravities. Using the result we will then compute gravitational and 't Hooft anomalies as well as central charges of the 2d SCFTs on monopole strings.

Let us first compute the anomaly inflow induced from the bulk Chern-Simons terms. The Chern-Simons terms in the bulk effective action is no longer invariant under the symmetry transformations when monopole strings are introduced.

When the string source with magnetic charge q^I in (A.10) is inserted, the cubic Chern-Simons term transforms under the local gauge transformation $\delta A^I = d\Lambda^I$ as[73],

$$\begin{aligned} \delta_\Lambda S_{\rm cs} &= \int_{M_5}\left(-\frac{1}{2}C_{IJK} d\Lambda^I \wedge F^J \wedge F^K\right) = C_{IJK}\int_{M_5} \Lambda^I F^J \wedge dF^K \\ &= C_{IJK} q^K \int_{M_2} \Lambda^I F^J \, . \end{aligned} \tag{A.11}$$

We here used the modified Bianchi identity in (A.10) for the second line. Thus the gauge variation does not vanish for general charge and Chern-Simons level. This non-vanishing gauge anomaly is the gauge anomaly inflow induced along the 2d string worldsheet.

The gravitational anomaly inflow computation for the local Lorentz transformation is more involved. In particular both the cubic and the linear Chern-Simons terms contribute to the gravitational anomaly inflow. To compute these contributions, we first solve the Bianchi identity in (A.10) of a string source by using the magnetic flux of the smoothed form[74,36]

$$F^I = -\frac{1}{2} q^I d\rho \wedge e_1^{(0)} \, , \tag{A.12}$$

where $\rho(r)$ is a smooth function of the radial direction r, with $\rho(0) = -1$ and $\rho(r) = 0$ for sufficiently large r, and $e_1^{(0)}$ is the 1-form in the descent relations $de_1^{(0)} = e_2$, $\delta e_1^{(0)} = de_0^{(1)}$ for the global angular form e_2 of the 2-sphere surrounding the monopole string. This smooths out the string source as

$$dF^I = q^I d(\rho e_2/2) \,. \tag{A.13}$$

In this case, the gauge field for the magnetic flux transforms under diffeomorphisms as $\delta A^I = -\frac{1}{2}q^I d(\rho e_0^{(1)})$. The following integrals for the 2-form e_2 on the 2-sphere bundle over the string worldvolume will prove to be useful for later discussion:

$$\int_{S^2} e_2 = 2 \,, \qquad \int_{S_\varepsilon(M_2)} e_0^{(1)} e_2 \wedge e_2 = 2\int_{M_2} p_1^{(1)}(N) \,, \tag{A.14}$$

where $p_1^{(1)}(N)$ is the 2-form in the descent relation, $dp_1^{(0)}(N) = p_1(N)$ and $\delta p_1^{(0)}(N) = dp_1^{(1)}(N)$, of the first Pontryagin class $p_1(N)$ of the $SU(2)_R$ normal bundle for the transverse $\mathbb{R}^3$ directions.

We now consider the local Lorentz transformation of the effective action on the background magnetic flux. One can compute the variation of both the cubic and the linear Chern-Simons terms under the local Lorentz transformation as[74,36]

$$\begin{aligned}
\delta_g S_{\rm cs} &= \frac{1}{48} C_{IJK} q^I q^J q^K \int_{M_5} d(\rho e_0^{(1)}) e_2^2 + \frac{1}{96} C_I q^I \int_{M_5} e_2 \wedge \delta p_1^{(0)}(T_5) \\
&= -\frac{1}{24} C_{IJK} q^I q^J q^K \int_{M_2} p_1^{(1)}(N) - \frac{1}{48} C_I q^I \int_{M_2} p_1^{(1)}(T_5) \,,
\end{aligned} \tag{A.15}$$

where $p_1(T_5)$ is the first Pontryagin class of the tangent bundle T_5 of the 5d spacetime. This non-vanishing variation of the bulk action is the gravitational anomaly inflow toward the monopole string.

The anomaly inflow for the gauge and the Lorentz transformations must be cancelled by the anomalies developed by the worldsheet degrees of freedom living on the monopole strings. This fact allows us to compute the quantum anomaly of the 2d CFT on the string worldsheet from the anomaly inflow that we just computed. Collecting the above results, we conclude that the 2d SCFT on the monopole string with magnetic charge q^I must have gauge and gravitational anomalies that are encoded in the 4-form anomaly polynomial of the form,

$$\begin{aligned} I_4 &= -I_4^{\text{inflow}} \\ &= -\frac{1}{2}C_{IJK}q^IF^JF^K + \frac{1}{24}C_{IJK}q^Iq^Jq^Kp_1(N) + \frac{1}{48}C_Iq^Ip_1(T_5) \\ &= -\frac{1}{2}C_{IJK}q^IF^JF^K - \frac{1}{6}\left(C_{IJK}q^Iq^Jq^K + \frac{1}{2}C_Iq^I\right)c_2(R) + \frac{1}{48}C_Iq^Ip_1(T_2)\,, \end{aligned} \tag{A.16}$$

where I_4^{inflow} is the anomaly inflow whose variation is related to the variation of the bulk action $I_2^{(1)} = \delta S_{5d}$ via the descent relations $I_4^{\text{inflow}} = dI_3,\ \delta I_3 = dI_2^{(1)}$. For the last line we used the relations of characteristic classes $p_1(T_5) = p_1(T_2) - 4c_2(R)$ and $p_1(N) = -4c_2(R)$, where $p_1(T_2)$ is the first Pontrygin class of the tangent bundle T_2 of the 2d worldsheet and $c_2(R)$ is the second Chern class of the $SU(2)_R$ Lorentz group transverse to the 2d worldsheet.

The anomaly polynomial I_4 of a 2d CFT encodes the left- and the right-moving central charges and the levels of the Kac-Moody current algebra coupled to the bulk gauge symmetry $\mathcal{G}$. The relative central charge $c_R - c_L$ can be read off from the coefficient of the gravitationaly anomaly term $-\frac{1}{24}p_1(T_2)$ in I_4. The right-moving central charge is $c_R = 6k_R$ where k_R is 't Hooft anomaly coefficient of the $SU(2)$ R-symmetry in the IR $(0,4)$ superconformal algebra. In order to compute the individual left- and right-moving central charges of the IR CFT, we thus need to know the exact value of k_R, which demands us to identify the correct $SU(2)$ R-symmetry in the IR CFT.

These central charges involve the contributions from the center-of-mass degrees of freedom. The center-of-mass modes consist of four bosons $(X^+_{\alpha\beta}, \varphi)$ and four fermions λ^α_+ in the right-moving sector and three bosons $X^-_{\alpha\beta}$ in the left-moving sector[142]. They form a free hypermultiplet of $\mathcal{N} = (0,4)$ supersymmetry. Their contribution to the central charges can be easily read off from the free field content as

$$c_L^{\rm com} = 3 \ , \quad c_R^{\rm com} = 6 \ . \tag{A.17}$$

The center-of-mass modes decouple from the interacting CFT in IR.

Therefore, the central charges $(\hat{c}_L, \hat{c}_R)$ of the interacting SCFT on a single string with magnetic charge q^I are given by

$$\begin{aligned}
\hat{c}_L &\equiv c_L - c_L^{\rm com} = C_{IJK}q^Iq^Jq^K + C_Iq^I - 3 \ , \\
\hat{c}_R &\equiv c_R - c_R^{\rm com} = C_{IJK}q^Iq^Jq^K + \frac{1}{2}C_Iq^I - 6 \ .
\end{aligned} \tag{A.18}$$

The worldsheet theory can carry the current algebras for the bulk gauge symmetry. The 't Hooft anomaly k_{IJ} of the current algebra can also be extracted from the anomaly polynomial as

$$k_{IJ} = C_{IJK}q^K \ , \tag{A.19}$$

for the mixed anomaly between two Abelian currents J^I and J^J. In our convention, the right- (or left-) moving charged fields add positive (or negative) contributions to the anomaly coefficient k_{IJ}.

As we discussed the bulk Abelian gauge symmetry can enhance to non-Abelian symmetry, say G_i, at some special points of the Coulomb branch. In this case, the string worldsheet theory can furnish a representation of the current algebra for the non-Abelian symmetry. The chiral fields realizing the current algebra yield 't Hooft anomaly, which can be read off from (A.16), of the form

$$-\frac{1}{4}k_i{\rm Tr}F_i^2 \quad \text{with} \quad k_i = -h_{i,\alpha}q^\alpha \ , \tag{A.20}$$

where F_i is the field strength of G_i and $h_{i,\alpha}$ is the coefficient in the gauge coupling h_i for G_i in the bulk effective action. The 't Hooft anomaly coefficient k_i for the non-Abelian symmetry is quantized as an integer number. The k_i is related to the level for the current algebra of the symmetry G_i. The level k current algebra of G_i realized by right-movers (or left-movers) provides $+k$ (or $-k$) contribution to the anomaly coefficient k_i.

A.1.3 Six Dimensional Anomaly Inflow

The anomaly inflow of 6d supergravity strings was studied in[119] and we review here. The 2d SCFT on strings with charge Q^I in the 6d supergravity theory has the anomaly polynomial of the form

$$I_4 = \Omega_{ij} Q^i (\frac{1}{2} a^i tr(R^2) + \frac{1}{4} b^i tr(F_i^2) + \frac{1}{2} Q^j \chi_4(N_4)) \tag{A.21}$$

$$= -\frac{Q \cdot a}{4} p_1(T_2) + \frac{1}{4} \sum_i Q \cdot b_i Tr(F_i^2) - \frac{Q \cdot Q - Q \cdot a}{2} c_2(R) + \frac{Q \cdot Q + Q \cdot a}{2} c_2(l) \tag{A.22}$$

where $\chi_4(N_4) = c_2(l) - c_2(R)$ is the Euler characteristic of $SO(4) = SU(2)_l \times SU(2)_R$ the normal bundle of the transverse to the string $\mathbb{R}^4$ and $tr(R^2) = -\frac{1}{2} p_1(T_2) + c_2(l) + c_2(R)$.

The center of mass modes consist of 4 bosons common to left and right movers and 4 right moving fermions and they form a free hypermultiplet $(X_{\alpha\dot{\alpha}}, \lambda_\alpha)$ where $\alpha\dot{\alpha}$ are indices for $SU(2)_l \times SU(2)_R$. The contribution to the anomalies is given by:

$$I_4^{com} = -\frac{1}{12} p_1(T_2) - c_2(l) \tag{A.23}$$

which needs to be subtracted from Equation A.21.

Therefore we can read off the central charges as

$$c_L = 3Q \cdot Q - 9Q \cdot a + 2 \quad c_R = 3Q \cdot Q - 3Q \cdot a \tag{A.24}$$

The levels for the G_i and $SU(2)_l$ current algebras are given by

$$k_i = b_i \cdot Q \quad k_l = \frac{Q \cdot Q + Q \cdot a + 2}{2} \tag{A.25}$$

B

5d Geometry: Compact and Non-Compact

B.1 Fun with Math

B.1.1 Mathematical facts and proofs

In this section, we describe the concepts and results which we need about divisors and linear systems on surfaces and threefolds. Our motivation is to understand the extent to which numerical conditions on divisor classes on compact Calabi-Yau threefolds do or do not guarantee the existence of a smooth surface. While no numerical criterion exists, there is a rich classical theory in algebraic geometry providing many results in that direction which will be useful for us.

A good general reference which touches on many of these issues is the book[129].

B.1.2 Definitions

We begin by stating the relevant definitions and notions from algebraic geometry. We consider divisors P on smooth projective varieties X of dimension n. A divisor can be expressed in terms of its irreducible components P_i as $P = \sum_i n_i P_i$ with $n_i \in \mathbb{Z}$. Our primary interest is $n = 2$ or 3.

Definition B.1.1 *The divisor P is* effective *if all $n_i \geq 0$.*

Definition B.1.2 *The divisor of a nonzero meromorphic function f on X is given by $(f) = (f)_0 - (f)_\infty$, where $(f)_0$ is the divisor of zeros of f including multiplicity and $(f)_\infty$ is the divisor of poles of f including multiplicity.*

To a divisor P, we associate the sheaf $\mathcal{O}_X(P)$ of meromorphic functions f on X with $(f) + P$ effective. By convention, the 0 function is also a section of $\mathcal{O}_X(P)$. Such an f can be viewed as a *holomorphic* section s_f[1] of $\mathcal{O}_X(P)$. For a general f (i.e. one not necessarily satisfying $(f) + P$ effective), the corresponding s_f might only be a meromorphic section of $\mathcal{O}_X(P)$. We can equivalently think of a meromorphic function f on X as either a function or a meromorphic section s_f of $\mathcal{O}_X(P)$. Conversely, identifying a nonzero meromorphic section s of $\mathcal{O}_X(P)$ with a meromorphic function f on X (so that $s = s_f$), we define the divisor (s) of s as $(f) + P$. In particular s is a holomorphic section of $\mathcal{O}_X(P)$ if and only if (s) is effective. This observation leads to conclude that we can always find an effective divisor associated to P if and only if $h^0(X, O_X(P)) \neq 0$ given by Proposition B.1.13.

[1] This is non-standard notation, introduced to expedite the discussion.

Definition B.1.3 *Two divisors P and P′ are* linearly equivalent*, denoted* $P \sim P'$*, if there exists a nonzero meromorphic function f on X with* $P' = (f) + P$*. The* complete linear system $|P|$ *of P is the set of all effective divisors linearly equivalent to P.*

Definition B.1.4 *A divisor P is* nef *if* $P \cdot C \geq 0$ *for all curves* $C \subset X$*.*

The term nef is in part intended as an acronym for "numerically eventually free". To say that a divisor is "eventually free" means that some positive multiple is base point free:

Definition B.1.5 *The linear system* $|P|$ *is called* base point free *when the intersection of all the divisors in* $|P|$ *is empty. A divisor P in X is called* semi-ample *if the linear system* $|mP|$ *is base point free for some* $m \in \mathbb{N}$*.*

A base point free linear system $|P|$ defines a mapping of X to projective space

$$\varphi_{|P|} : X \to \mathbb{P}^N, \qquad \varphi_{|P|}(x) = (s_0(x), \ldots, s_N(x)). \tag{B.1}$$

In (B.1), $\{s_0, \ldots, s_N\}$ is a basis for $H^0(X, \mathcal{O}_X(P))$. The map φ_P depends on the choice of basis, but is well-defined up to a linear change of homogeneous coordinates in $\mathbb{P}^N$.

If $|P|$ is not base point free, then $\varphi_{|P|}$ is not defined precisely at the base points. But we still get a rational map $\varphi_{|P|} : X - - \to \mathbb{P}^N$ whenever $|P|$ is not empty.

Definition B.1.6 *A divisor P is* very ample *if* $|P|$ *is base point free and the corresponding map* $\varphi_{|P|}$ *is an embedding. The divisor P is* ample *if mP is very ample for some* $m \in \mathbb{N}$*.*

We can always determine if a divisor is ample using the numerical criterion given by Theorem B.1.16. Moreover, we can always pass from an ample to a very ample divisor through Theorem B.1.24.

Definition B.1.7 *The* holomorphic Euler characteristic *of a divisor P in X is the alternating sum*

$$\chi(O_X(P)) = \sum_{i=1}^{3}(-1)^i h^i(O_X(P)), \tag{B.2}$$

where as usual $h^i(O_X(P)) = \dim H^i(O_X(P))$. *The holomorphic Euler characteristic of X is defined as* $\chi(\mathcal{O}_X) = 1 - h^{1,0}(X) + h^{2,0}(X) - \ldots + (-1)^n h^{n,0}(X)$.

Definition B.1.8 *A divisor P in X is* Big *if* $h^0(X, \mathcal{O}_X(mP)) \geq cm^n$ *for some* $c > 0$ *and all* $m \geq m_0$.

Remark. By Riemann-Roch we have $\chi(\mathcal{O}_X(mP)) \sim (P^n/n!)m^n$. If follows immediate that for divisors P satisfying the vanishing condition $H^i(X, \mathcal{O}_X(mP)) = 0$ for $i > 0$ and $m \geq m_0$, P is big if and only if $P^n > 0$. Vanishing theorems which imply such vanishing conditions will be discussed in Section B.1.5.

Let K_X be the canonical bundle of X and $P_m(X) = h^0(O_P(mK_X))$ be the m-th plurigenus[2] of X.

Definition B.1.9 *The* Kodaira dimension κ *of a smooth*[3] *surface X is defined as follows.*

$$\kappa(X) = min\{k| \ \frac{P_m(X)}{m^k} \ is\ bounded\ \} \tag{B.3}$$

When all plurigenera vanish we say $\kappa(X) = -\infty$.

We can similarly associate a Kodaira dimension to any line bundle L on X.

$$\kappa(X, L) = \begin{cases} -\infty & H^0(X, L^n) = 0 \text{ for all } n \geq 1 \\ \sup(\{\dim \varphi_{L^n}(X) \mid n \geq 1\}) & \text{otherwise} \end{cases} \tag{B.4}$$

[2]Note that $m = 1$ is the geometric genus $h^{2,0}(X)$.

[3]Singular surfaces do not have a canonical bundle in general. However, effective divisors in a smooth threefold are Gorenstein, hence their dualizing sheaves are again line bundles. We can generalize the notion of Kodaira dimensions to these surfaces if desired, but there is no known classification. We do not pursue this point further.

We can recover B.3 when $L = K_X$ i.e. $\kappa(X) := \kappa(X, K_X)$.

We now specialize to smooth projective surfaces, which we will denote by S instead of X. We use the standard notation and terminology of classical algebraic geometry.

Definition B.1.10 *The* geometric genus $p_g(S)$ *of S is the dimension of* $H^0(S, K_S) \simeq H^{2,0}(S)$. *The* irregularity $q(S)$ *of S is the dimension of* $H^{1,0}(S)$. *The* arithmetic genus $p_a(S)$ *is defined as* $p_g(S) - q(S)$.

In particular, $\chi(\mathcal{O}_S) = 1 + p_a(S)$.

When S is clear from context, we simply denote these by p_g, q, and p_a respectively.

Definition B.1.11 *A smooth projective surface S is* regular *if* $q = 0$. *The surface S is* irregular *if* $q > 0$.

Since $b_1(S) = 2q$, to say that S is regular is equivalent to the topological condition $b_1(S) = 0$.

Definition B.1.12 *A* (-1)-curve *is a curve $C \subset S$ with C isomorphic to $\mathbb{P}^1$ and $C^2 = -1$. A surface S is* minimal *if it has no (-1)-curves.*

By the adjunction formula, equivalently C is a -1-curve if and only if $C^2 = K_S \cdot C = -1$.

Any (-1)-curve $C \subset S$ can be blown down to a smooth surface. This means that we can find a smooth surface S_1 with a point $p \in S_1$ and a holomorphic mapping $f : S \to S_1$ such that $f(C) = p$ and f restricts to an isomorphism of $S - C$ to $S_1 - p$. If S_1 is not minimal, then it contains a -1 curve, which can be blown down to a surface S_2. It can be shown that this process terminates after finitely many steps and we wind up with a minimal surface S_n and a holomorphic birational map $S \to S_n$ which blows down n (-1)-curves in succession.

In this way, the classification of compact Kähler surfaces is reduced to the classification of minimal compact projective surfaces. In particular, in our case we are only interested in minimal surfaces because a surface with a (-1) curve can never be nef. We will futher restrict to the classification of compact algebraic surfaces, since all of our supergravity strings arise from wrapping surfaces which are algebraic instead of being merely Kähler, as will be explained in Appendix B.1.4. Kodaira's classification of minimal compact algebraic surfaces, organized by Kodaira dimension κ, is given in Table B.1.

κ	minimal projective algebraic surface S
$-\infty$	$\mathbb{P}^2$, ruled surface
0	$K3$, Enriques, hyperelliptic, abelian surface
1	minimal elliptic surface
2	minimal surface of general type

Table B.1: The first column is the Kodaira dimension κ of the surface S. The second column presents the Enriques-Kodaira classification of minimal projective algebraic surfaces.

The first class of surfaces with $\kappa = -\infty$ are ruled surfaces or $\mathbb{P}^2$, which are never nef divisors in a Calabi-Yau threefold. In particular, a ruled surface is a $\mathbb{P}^1$ bundle over a smooth curve C, which can have any genus g. If $g = 0$, the ruled surfaces are rational and are precisely the Hirzebruch surfaces $\mathbb{F}_n$. The ruled surfaces over curves of genus $g > 0$ have continuous complex structure moduli and there is no standard notation for them. We sometimes denote a ruled surface in one of these continuous families by $\mathbb{F}_n^g$.

The next three cases all represent semi-ample divisors. Enriques surfaces are regular algebraic surfaces which are $\mathbb{Z}_2$ quotients of K3. Hyperelliptic surfaces or bi-elliptic surfaces are finite abelian group quotients of a product of elliptic curves. Abelian surfaces are tori and all other elliptic surfaces other than those with $\kappa = -\infty$ or 0 have $\kappa = 1$.

B.1.3 Ruled and Rational Surfaces $\kappa = -\infty$

In this section, we will review some mathematical background useful for this thesis. We refer the reader to Section 2 of[28] for a more detailed background on various points discussed below.

Hirzebruch surfaces

A Hirzebruch surface is a $\mathbb{P}^1$ fibration over $\mathbb{P}^1$. We denote a Hirzebruch surface with a degree $-n$ fibration as $\mathbb{F}_n$. We refer to the fiber $\mathbb{P}^1$ as f and the base $\mathbb{P}^1$ as e. Their intersection numbers are

$$e^2 = -n \tag{B.5}$$

$$f^2 = 0 \tag{B.6}$$

$$e \cdot f = 1 \tag{B.7}$$

Another very important curve in $\mathbb{F}_n$ is

$$h := e + nf \tag{B.8}$$

whose genus is zero and intersection numbers are

$$h^2 = n \tag{B.9}$$

$$h \cdot e = 0 \tag{B.10}$$

$$h \cdot f = 1 \tag{B.11}$$

Note that $e = h$ for $\mathbb{F}_0$. The set of holomorphic curves, often referred to as *Mori cone*, for $\mathbb{F}_n$ with $n \geq 0$ is generated by e and f. For $\mathbb{F}_n$ with $n \leq 0$, the Mori cone is generated by h and f.

The canonical class K of $\mathbb{F}_n$ is an antiholomorphic curve which can be determined by the virtue of adjunction formula which states that for a surface S and a curve C inside S, the canonical class K_S

of S satisfies

$$(K_S + C) \cdot C = 2g(C) - 2 \tag{B.12}$$

where $g(C)$ is the genus of C. Demanding that K satisfies (B.12) for e, f determines it to be

$$K = -(e + h + 2f) \tag{B.13}$$

from which we can compute that

$$K^2 = 8 \tag{B.14}$$

Notice that $\mathbb{F}_n$ and $\mathbb{F}_{-n}$ are isomorphic to each other via the map

$$e \leftrightarrow h \tag{B.15}$$

$$f \leftrightarrow f \tag{B.16}$$

$$h \leftrightarrow e \tag{B.17}$$

Thus, we will restrict our attention to Hirzebruch surfaces with $n \geq 0$ in what follows. However, at various points in the main body of this work we find it useful to include Hirzebruch surfaces with negative degrees since they allow us to express answers in a more uniform way.

We also deal with surfaces which arise by performing b number of blowups on $\mathbb{F}_n$. The blowups will often be non-generic. We can obtain different surfaces by performing b blowups in different fashions on $\mathbb{F}_n$. In this work, we refer to all the different surfaces arising via b blowups of $\mathbb{F}_n$ as $\mathbb{F}_n^b$. The curves inside $\mathbb{F}_n^b$ can be described by adding the curves x_i with $i = 1, \cdots, b$ which are the exceptional divisors created by the blowups. We will use the convention that the total transforms[4] of the curves e, f and h are denoted by the same names e, f and h in $\mathbb{F}_n^b$. Thus, the intersection numbers

[4] If $B : \tilde{S} \to S$ is a blowup of a surface S, then the total transform of a curve C in S is the curve $B^{-1}(C)$ in $\tilde{S}$.

between e, f and h are those mentioned above, and their intersections with x_i are

$$x_i \cdot x_j = -\delta_{ij} \tag{B.18}$$

$$e \cdot x_i = 0 \tag{B.19}$$

$$f \cdot x_i = 0 \tag{B.20}$$

$$h \cdot x_i = 0 \tag{B.21}$$

The blowup procedure creates curves that can be written as

$$\alpha e + \beta f - \sum \gamma_i x_i \tag{B.22}$$

with $\alpha, \beta, \gamma_i \geq 0$. The important point is that the blowups x_i can appear with negative sign.

Again, using the adjunction formula (B.12) we can find the canonical class K for $\mathbb{F}_n^b$ to be

$$K = -(e + h + 2f) + \sum x_i \tag{B.23}$$

from which we compute

$$K^2 = 8 - b \tag{B.24}$$

An important isomorphism exists between $\mathbb{F}_0^1$ and $\mathbb{F}_1^1$ with the blowup on both surfaces being performed at a generic point. In fact, a single blowup of $\mathbb{F}_0$ is always generic. The map from $\mathbb{F}_1^1$ to $\mathbb{F}_0^1$ is

$$e \to e - x \tag{B.25}$$

$$f - x \to x \tag{B.26}$$

$$x \to f - x \tag{B.27}$$

It is easy to see that the above isomorphism only works when the blowups are generic. For, the non-generic one point blowup of $\mathbb{F}_1$ contains the curve $e-x$, which would be sent to $e-f$ inside $\mathbb{F}_0^1$. But $e-f$ is not a holomorphic curve in $\mathbb{F}_0^1$.

To differentiate between the different surfaces $\mathbb{F}_n^b$ for fixed n and b, we have to track the data of their Mori cone. One important point is that the gluing curves inside the surfaces must be the generators of Mori cone. The final point we want to address is that $\mathbb{F}_2$ and $\mathbb{F}_0$ are same up to decoupled states. This can be seen by noticing that the Mori cone of latter embeds into the Mori cone of former. This embedding $\mathbb{F}_0 \to \mathbb{F}_2$ is

$$e \to e+f \tag{B.28}$$

$$f \to f \tag{B.29}$$

This means that $\mathbb{F}_2$ equals $\mathbb{F}_0$ plus some decoupled states. Decoupling these states corresponds to performing a complex structure deformation $\mathbb{F}_2 \;\to\; \mathbb{F}_0$. When $\mathbb{F}_0$ and $\mathbb{F}_2$ carry blowups, this conclusion might be changed or unchanged depending on how the blowups are done.

del Pezzo surfaces

The discussion of del Pezzo surfaces starts with the discussion of complex projective plane $\mathbb{P}^2$ which contains a single curve l whose genus is zero and intersection number is

$$l^2 = 1 \tag{B.30}$$

(B.12) determines the canonical class to be

$$K = -3l \tag{B.31}$$

from which we compute

$$K^2 = 9 \tag{B.32}$$

Performing n blowups on $\mathbb{P}^2$ at generic locations leads to the del Pezzo surface dP_n. It can be described in terms of curve l and x_i with intersection numbers

$$x_i \cdot x_j = -\delta_{ij} \tag{B.33}$$

$$l \cdot x_i = 0 \tag{B.34}$$

Again, the blowups create new holomorphic curves which can be written as

$$\alpha l - \sum \gamma_i x_i \tag{B.35}$$

with $\alpha, \gamma_i \geq 0$. In the work, we abuse the notation and call a non-generic n point blowup of $\mathbb{P}^2$ as dP_n too. The canonical class for dP_n is

$$K = -3l + \sum x_i \tag{B.36}$$

with

$$K^2 = 9 - n \tag{B.37}$$

del Pezzo surfaces and Hirzebruch surfaces are related to each other by virtue of an isomorphism

$dP_1 \to \mathbb{F}_1$ which acts as

$$x \to e \tag{B.38}$$

$$l - x \to f \tag{B.39}$$

$$l \to h \tag{B.40}$$

A one point blowup of $\mathbb{P}^2$ is always generic and thus there is a unique dP_1 which appears in the above isomorphism.

The curve

$$F = 3l - \sum x_i \tag{B.41}$$

has the properties that

$$F^2 = 0 \tag{B.42}$$

and

$$K \cdot F = 0 \tag{B.43}$$

Thus, F is a fiber of genus one, or in other words a torus fiber inside dP_9.

dP_n for $n \geq 3$ admits the following basic automorphism. We first choose three distinct blowups x_i, x_j and x_k, and then implement

$$x_i \to l - x_j - x_k \tag{B.44}$$

$$x_j \to l - x_i - x_k \tag{B.45}$$

$$x_k \to l - x_i - x_j \tag{B.46}$$

$$l \to 2l - x_i - x_j - x_k \tag{B.47}$$

Combining this automorphism with permutations of blowups, we can obtain more general automorphisms of dP_n (with $n \geq 3$) which can be decomposed as a sequence comprising of above mentioned basic automorphisms and permutations of blowups. Notice that for dP_9, any such automorphism leaves the torus fiber (B.41) invariant.

Ruled and Rational Surfaces in Compact threefolds

Ruled and rational surfaces with $\kappa = -\infty$ and their blow ups are never semi-ample divisors in a compact Calabi-Yau threefold. Obviously, if we consider any surface with blowups it will automatically not be semi-ample because it will always contain a rational (-1) curve with $K \cdot C = -1$ coming from the blow up as mentioned above. Therefore, it is enough to consider minimal such surfaces and hence $\mathbb{P}^2$ or $\mathbb{F}_n^g$.

Hirzebruch surfaces P over a curve of genus $g > 1$ have $P^3 = 8(1-g) < 0$ therefore they are not ample. As for $g = 0$, the cohomology of the ordinary Hirzebruch surface $\mathbb{F}_n$ is generated by the section e with $e^2 = -n$ and f a fiber. We can inspect the intersection of the canonical divisor $K = -2e - (n+2)f$ with any section h satisfying $h^2 = n$, which gives us $K \cdot h = -(n+2) < 0$. For $g = 1$ we have $K = -2e - nf$, and now $K \cdot h = -n < 0$ for $n > 0$. Finally, if $n = 0$ and $g = 1$ we have $K = -2e$ and hence $K \cdot f = -2 < 0$. Therefore, any ruled surface cannot be semi-ample. Lastly, $\mathbb{P}^2$ has a canonical bundle which satisfies $K = -3\ell$ for the class $\ell^2 = 1$ with $K \cdot \ell = -3 < 0$. We conclude that smooth projective surfaces with $\kappa = -\infty$ are never semi-ample.

B.1.4 Theorems for compact Calabi-Yau threefolds

In this section, we let X be a smooth projective variety of dimension n. In several situations, we will specialize to the case where X is a Calabi-Yau threefold and add something more precise. We will always assume that such a Calabi-Yau has $SU(3)$ holonomy, so that $H^{1,0}(X) = H^{2,0}(X) = 0$.

Proposition B.1.13 *Given a divisor P, there is a 1-1 correspondence between $|P|$ and elements of the projective space $\mathbb{P}(H^0(X, \mathcal{O}_X(P))$. In particular, in which case $|P|$ is a projective space of dimension $h^0(X, \mathcal{O}_X(P)) - 1$.*

This is a well-known foundational result (e.g.[80]) but we provide a proof to fix ideas.

Proof. A holomorphic section $s \in H^0(X, \mathcal{O}_X(P))$ can be written as $s = s_f$ for $(f) + P$ effective. Thus $(s) = (f) + P \in |P|$. For any nonzero constant c we have $(cs) = (s)$. Thus the assignment $s_f \mapsto (f) + P$ induces a map $\mathbb{P}(H^0(X, \mathcal{O}_X(P)) \to |P|$.

In the other direction, let $P' \in |P|$. Let f be such that $P' = (f) + P$. For this f we have $(s_f) = P'$, which was assumed effective. Thus s_f is holomorphic, i.e. gives a section of $H^0(X, \mathcal{O}_X(P))$. If instead we choose a different f' with $P' = (f') + P$, then $(f'/f) = (P' - P) - (P' - P) = 0$. It follows that f'/f' is a holomorphic nonvanishing function on X, which must be constant since X is compact. So f, hence s_f, is unique up to scalar and we have defined an inverse map $|P| \to \mathbb{P}(H^0(X, \mathcal{O}_X(P))$.

Theorem B.1.14 ***(Bertini's Theorem)*** *The general member of a base point free linear system is smooth.*

In particular, if X is a Calabi-Yau threefold and P is semi-ample, then we can find a smooth surface in the linear system $|mP|$ for some $m \geq 1$.

Proposition B.1.15 *Semi-ample divisors are nef.*

Thus the nef condition is a purely numerical condition on a divisor which is automatically satisfied if it semi-ample, i.e. "eventually free".

Proof: Suppose $|mP|$ is base point free. Let C be an irreducible curve, and $p \in C$. Since p is not a base point of $|mP|$, there is a divisor D in $|mP|$ not containing p. This implies that D does not contain C, hence $D \cdot C = (mP) \cdot C \geq 0$ and finally $P \cdot C \geq 0$.

Recall the notion of an ample divisor from Appendix B.1.2.

Theorem B.1.16 ***(Nakai–Moishezon criterion for ampleness)*** *A divisor D in X is ample iff $D^k \cdot V > 0$ for all irreducible subvarieties $V \subset X$, where k is the dimension of V. In particular, if X is a (Calabi-Yau) threefold, D is ample iff the following three conditions hold: $D^3 > 0$, $D^2 \cdot S > 0$ for all irreducible surfaces $S \subset X$, and $D \cdot C > 0$ for all irreducible curves $C \subset X$.*

The Nakai-Moishezon criterion implies that the ampleness of D only depends on the class $[D] \in H^2(X, \mathbb{R})$. The cohomology classes of ample divisors span a cone in $H^2(X, \mathbb{R})$, the *ample cone* or *Kähler cone* $\mathcal{K}(X) \subset H^2(X, \mathbb{R})$. The reason for the interchangeable terminology is that the line bundles associated to ample divisors are precisely the line bundles which admit Kähler metrics. We will describe the cone spanned by the ample divisor classes as the Kähler cone to match usage in physics.

While the Nakai-Moishezon condition is a purely numerical condition, it is not completely satisfactory for our purposes since we have to know *all* surfaces $S \subset X$ in order to implement the criterion. As we will see presently, it is easier to work with the nef cone, which gives us almost as much information anyway.

The following theorem shows that the nef cone Nef(X) generated by nef divisors is the closure $\overline{\mathcal{K}(X)}$ of the ample cone, as we just replace "$>$" in the Nakai-Moishezon criterion with "$\geq$".

Theorem B.1.17 ***(Kleiman's Theorem*** [122]***)*** *Let D be a nef divisor on X. Then for any subvariety V of X we have $D^k \cdot V \geq 0$ where k is the dimension of V. In particular, if X is any threefold, such as a Calabi-Yau threefold, we see that $D^3 \geq 0$.*

Said differently, Kleiman's Theorem says that $\text{Nef}(X) = \overline{\mathcal{K}(X)}$ is dual to the Mori cone $M(X) \subset H_2(X, \mathbb{R})$, the cone generated by the classes of all irreducible curves $C \subset X$. The conclusion of Kleiman's theorem holds for semi-ample divisors, since semi-ample divisors are nef by Proposition B.1.15.

In general, the determination of the Kähler cone $\mathcal{K}(X) \subset \mathrm{Nef}(X)$ is more subtle. But it can be shown that the ample cone is the interior of the nef cone[122]. In particular, if the nef cone is known to be a polyhedral cone generated by finitely many nef divisors, then this fact determines the ample cone.

Remark. We have assumed that X is projective throughout this section, so these results do not apply to local Calabi-Yau threefolds. To see the issue, suppose that P is a smooth surface in a compact Calabi-Yau X. To say that P is nef means that $P \cdot C \geq 0$ for all curves $C \subset X$, *not just those contained in* P. In the local case, all compact curves are either contained in P or a deformation of curves contained in P, hence homologous to curves in P. So the condition becomes $P \cdot C \geq 0$ for all curves $C \subset P$ (as was studied in the context of 5D SCFT[111]), which is a substantially weaker condition than requiring that $P \cdot C \geq 0$ for all curves $C \subset X$ for any given compact Calabi-Yau X containing P.

For simplicity, we only state the Hirzebruch-Riemann-Roch theorem for Calabi-Yau threefolds.

Theorem B.1.18 ***(Hirzebruch-Riemann-Roch theorem)*** *The holomorphic Euler characteristic of a divisor P in a Calabi-Yau threefold X is given by*

$$\chi(O_X(P)) = \frac{1}{6}P^3 + \frac{1}{12}P \cdot c_2(X) \tag{B.48}$$

If $H^k(X, \mathcal{O}_X(P)) = 0$ for all $k > 0$, then $\dim |P| = \dim H^0(X, \mathcal{O}_X(P)) - 1 = \chi(\mathcal{O}_X(P)) - 1$, and we can compute the dimension of our moduli space of surfaces $|P|$ very simply by Hirzebruch-Riemann-Roch. We now give a few theorems which guarantee these vanishings of cohomology.

Theorem B.1.19 ***(Kodaira Vanishing theorem[123])*** *Let P be an ample divisor on a smooth projective variety X. Then $H^i(X, K_X(P)) = 0$ for any $i > 0$. In particular, if X is Calabi-Yau we have $H^i(X, \mathcal{O}_X(P)) = 0$ for any $i > 0$.*

Since $\mathrm{Nef}(X)$ is the closure of $\mathrm{Amp}(X)$, one might hope that the desired vanishing holds for nef divisors, but that is not true in general. A slight strenghtening of the nef hypothesis works which is more general than ample.

Theorem B.1.20 ***(Kawamata-Viehweg vanishing[123])*** *Let P be a nef and big divisor on X. Then* $H^i(X, K_X(P)) = 0$ *for any* $i > 0$*. In particular, if X is Calabi-Yau we have* $H^i(X, O_X(P)) = 0$ *for any* $i > 0$*.*

Specializing to a Calabi-Yau threefold for definiteness, we see that $\chi(\mathcal{O}_X(mP))$ grows like $(P^3/6)m^3$. If $P^3 > 0$, this is close to the condition for being big, but is not the same since $\chi(\mathcal{O}_X(mP))$ is not the same as $\dim H^0(X, \mathcal{O}_X(mP))$ in general. To conclude the required growth of $\dim H^0(X, \mathcal{O}_X(mP))$, the growth of $\dim H^2(X, \mathcal{O}_X(mP))$ must be controlled for $i > 0$. This can be done:

Proposition B.1.21 *Suppose X is a Calabi-Yau threefold, and P is nef and satisfies* $P^3 > 0$*. Then P is big.*

Since semi-ample divisors are nef, it follows immediately from the Kawamata-Viehweg vanishing theorem that we get the desired vanishings $H^i(X, \mathcal{O}_X(P)) = 0$ for $i > 0$ if P is nef and $P^3 > 0$.

Proof. Follows immediately from [129] Cor. 1.4.41. Indeed, the proof shows that $\dim H^i(X, \mathcal{O}_X(mP)) = O(m^{3-i})$.

If P is an effective divisor, nef already implies semi-ample. Hence nef and semi-ample are equivalent conditions on effective divisors:

Theorem B.1.22 [152] *If P is effective (or more generally if* $\kappa(X, P) \geq 0$*) and nef, then P is semi-ample.*

Corollary B.1.23 *If P is nef and* $|nP|$ *contains an effective surface for any* $n \geq 1$*, then* $|nmP|$ *contains a smooth surface for some* $m \geq 1$*.*

We can be more precise for ample divisors.

Theorem B.1.24 *(Oguiso-Peternell Theorem*[154] *)*

Let P be an ample divisor in a Calabi-Yau threefold. Then

1. $|mP|$ *is base point free for* $m \geq 5$

2. mP *is very ample for* $m \geq 10$

By Bertini's Theorem, we see that we can always find a smooth surface in $|5P|$.

We are primarily interested in smooth irreducible surfaces $P \subset X$. Note that since X is assumed projective, we have that S is automatically projective, stronger than merely Kähler. We will see presently that if P is a regular surface, we get the desired vanishings $H^i(X, \mathcal{O}_X(P)) = 0$ for $i > 0$ without any additional hypotheses on the linear system $|P|$. The reason is that the numerical invariants of the surface P are related to the properties of P as a divisor in X.

Proposition B.1.25 *For X and P as above, we have*

1. $\dim H^0(X, \mathcal{O}(P)) = p_g + 1$

2. $\dim H^1(X, \mathcal{O}(P)) = q$

3. $H^k(X, \mathcal{O}(P)) = 0$ *for* $k \geq 2$.

Corollary B.1.26 *If the surface P is regular, then* $H^i(X, \mathcal{O}_X(P)) = 0$ *for* $i > 0$ *and* $\dim |P| = p_g$.

Proof. We consider the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(P) \to \mathcal{O}_X(P)|_P \to 0. \tag{B.49}$$

By the adjunction formula and the Calabi-Yau condition, we have $\mathcal{O}_X(P)|_P \simeq K_P$. Using $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ which is part of the Calabi-Yau condition, the associated long exact sequence of cohomology splits up into a short exact sequence,

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X(P)) \to H^0(P, K_P) \to 0, \tag{B.50}$$

an isomorphism $H^1(X, \mathcal{O}(P)) \simeq H^1(P, K_P)$, and an exact sequence

$$0 \to H^2(X, \mathcal{O}_X(P)) \to H^2(P, K_P) \to H^3(X, \mathcal{O}_X) \to H^3(X, \mathcal{O}_X(P)) \to 0. \tag{B.51}$$

Taking dimensions in (B.50) gives 1. By Serre duality on P, we have $H^1(P, K_P) \simeq H^1(P, \mathcal{O}_P)^*$, which has dimension $h^{0,1}(P) = h^{1,0}(P) = q$. So 2 follows immediately from the isomorphism between (B.50) and (B.51). By Serre duality on X we get $H^3(X, \mathcal{O}_X(P)) \simeq H^0(X, \mathcal{O}_X(-P))^* = 0$, using the Calabi-Yau condition $K_X \simeq \mathcal{O}_X$. Since $H^2(P, K_P) \simeq H^{2,2}(P)$ and $H^3(X, \mathcal{O}_X)$ are each 1-dimensional, (B.51) implies that $H^2(X, \mathcal{O}_X(P)) = 0$. We trivially have $H^k(X, \mathcal{O}(P)) = 0$ for $k > 3$ for dimension reasons. This completes the proof of 3 and of the proposition.

Continuing to assume that P is a smooth surface, note that if in addition P is either ample, or more generally nef with $P^3 > 0$, then Kodaira vanishing or Kawamata-Viehweg vanishing implies that $H^1(X, \mathcal{O}(P)) = 0$, so that P is regular by Proposition B.1.25.

If P is merely semi-ample, then $|mP|$ is base point free for $m \gg 0$, hence its restriction to P is still basepoint free. Since P restricts to K_P on P, we see that $|mK_P|$ is base point free and hence $\kappa(P) \geq 0$. Furthermore P cannot contain any (-1) curve C, since $K_P \cdot C = -1$ on P is equivalent to $P \cdot C = -1$ on X, contradicting the fact that P is nef.

Furthermore, if P is ample, then the restriction K_P of $\mathcal{O}_X(P)$ to P is still ample, hence mK_P is very ample for $m >> 0$ and $\kappa(P) = 2$. If P nef with $P^3 > 0$, then $K_P^2 = P^3 > 0$ and P is a minimal surface with $\kappa \geq 0$ as we just saw. As a consequence of the Kodaira classification of minimal surfaces, we see that $K_P^2 = 0$ for minimal surfaces P with $\kappa = 0$ or 1. It follows that $\kappa = 2$ in this case as well.

Summarizing, we have proven

Proposition B.1.27 *Suppose that P is a smooth surface which is also semi-ample as a divisor in X. Then P is a minimal surface with $\kappa \geq 0$. If in addition P is ample, or more generally nef with $P^3 > 0$, then P is a regular surface of general type.*

Of course, if $|P|$ is very ample, then a general surface in $|P|$ is automatically smooth by Bertini's theorem.

Regarding c_2 we have

Theorem B.1.28 *If P is nef, then $c_2 \cdot P \geq 0$.*

This follows from [140] Theorem 1.1.

Corollary B.1.29 *If $[P]$ is an ample class, or more generally if $[P]$ is nef with $P^3 > 0$, then it has an effective representative P.*

Proof. By Kodaira vanishing in the ample case, or Proposition B.1.21 and Kawamata-Viehweg in the more general case, we get $H^i(X, \mathcal{O}(P)) = 0$ for $i > 0$. Then we have $h^0(\mathcal{O}(P)) = \chi(\mathcal{O}(P)) = P^3/6 + c_2.P/12 > 0$, the first equality coming from the vanishing of higher cohomology.

Theorem B.1.30 [130] *If P is nef and $c_2(X) \cdot P > 0$, then P is semi-ample.*

Theorem B.1.31 ***(Lefschetz hyperplane theorem)***

Let P be an effective ample divisor on a smooth projective variety X of dimension n. Then the restriction map $r_i : H^i(X, \mathbb{Z}) \to H^i(P, \mathbb{Z})$ is an isomorphism for $i \leq n-2$ and injective for $i = n-1$.

In particular, if X is a Calabi-Yau threefold, then $\dim H^1(P,\mathbb{Z}) = 0$ so that P is regular, and $\dim H^2(X,\mathbb{Z}) \leq \dim(H^2(P,\mathbb{Z}))$.

Theorem B.1.32 ***(Hodge index theorem)***

Assume P is a compact surface then the cupproduct form on $H^2(P,\mathbb{R})$, restricted to $H^{1,1}_{\mathbb{R}}(P)$, is non-degenerate and of signature $(1, h^{1,1}-1)$

Theorem B.1.33 ***(Noether bound)***

Let P be a smooth minimal surface of general type ($\kappa = 2$). Then

$$\frac{1}{2}K_P^2 \geq p_g(P) - 2 \tag{B.52}$$

In the case that P is a smooth surface with $\kappa = 2$, then $p_g = \frac{1}{6}P^3 + \frac{1}{12}P.c_2(X) - 1$ by Hirzebruch-Riemann-Roch and Proposition B.1.25. Since $K_P^2 = P^3$, we conclude that $4P^3 \geq P \cdot c_2(X) - 36$ if P^3 is even or $4P^3 \geq P \cdot c_2(X) - 30$ if P^3 is odd. In addition, all minimal smooth surfaces with $\kappa = 0$ have $c_2(P) \leq 36$, which can be found in [14] and hence satisfy the same inequality. This implies that smooth ample or semi-ample divisors that correspond to smooth surfaces with $\kappa = 0, 2$ satisfy $4P^3 \geq P \cdot c_2(X) - C$, where $C = 36$ when P^3 is even and $C = 30$ when P^3 is odd.

B.1.5 Proofs of Inequalities

Theorem B.1.34 ***(Inequality 4, Table 3.6)***

Let P be a smooth ample divisor inside the Calabi-Yau threefold X. Then

$$h^{1,1}(X) \leq P^3 + P \cdot c_2(X) - 2 \tag{B.53}$$

Proof. Since P is smooth and ample in X, the Lefschetz hyperplane theorem applies. Therefore the restriction map $r : H^2(X, \mathbb{Z}) \to H^2(P, \mathbb{Z})$ is an injection. Hence,

$$dim(H^2(X, \mathbb{C})) \leq dim(H^2(P, \mathbb{C})). \tag{B.54}$$

By the Hodge Decomposition we know that $dim(H^2(X, \mathbb{C})) = h^{1,1}(X)$ since $h^{2,0}(X) = 0$ and $dim(H^2(P, \mathbb{Z})) = h^{1,1}(P) + 2h^{2,0}(P) = h^{1,1}(P) + 2p_g$. In addition, since P is a regular surface, the topological Euler characteristic of P is given by $\chi(P) = 2 + b_2 = 2 + 2p_g + h^{1,1}(P)$, while from (3.98) we also know that $\chi(P) = P^3 + P \cdot c_2(X)$. Hence, $h^{1,1}(P) = P^3 + P \cdot c_2(X) - 2p_g - 2$ which implies that (B.54) becomes $h^{1,1}(X) \leq h^{1,1}(P) + 2p_g = P^3 + P \cdot c_2(X) - 2$.

Theorem B.1.35 ***(Inequality 5, Table 3.6)***

Let P be a smooth, big and nef divisor inside the Calabi-Yau threefold X. Then the number of rational (-2) curves on P is bounded by

$$N_{-2} \leq \frac{1}{6}(4P^3 + 5P \cdot c_2(X)) - 1. \tag{B.55}$$

Proof A smooth nef and big divisor inside the Calabi-Yau threefold is a minimal surface of general type. A consequence of the Hodge Index Theorem is that the number N_{-2} of rational -2 curves in a surface of general type P is bounded by $N_{-2} \leq \rho(P) - 1$,where $\rho(P)$ is the Picard number of P, the rank of the group of divisor classes. This claim can be found in [14,Prop.VII(2.5)]

In addition, the Picard number $\rho(P)$ is clearly bounded above by $h^{1,1}(P)$, as the Picard lattice of cohomology classes of divisors is a sublattice of $H^{1,1}(P, \mathbb{C}) \cap H^2(P, \mathbb{Z})$. Therefore $N_{-2} \leq h^{1,1}(P) - 1$. In the proof of (B.53), we saw that $h^{1,1}(P) = P^3 + P \cdot c_2(X) - 2p_g - 2$. But $\chi(\mathcal{O}(P)) = p_g + 1$ by Proposition B.1.25 and $\chi(\mathcal{O}(P)) = (2P^3 + c_2 \cdot P)/12$ by Hirzebruch-Riemann-Roch. Combining these formulas, we conclude that $N_{-2} \leq \frac{1}{6}(4P^3 + 5P \cdot c_2(X)) - 1$.

B.2 5D Compact Threefolds

In this section of the appendix, we collect examples supporting the discussion in the main text. We begin with an example of an ample divisor class with no smooth representative. We then follow with examples of $SU(2)$ and $SU(3)$ gauge theories which Higgs to the quintic.

B.2.1 An ample divisor class with no smooth representative

Referring to [38], we let X be a smooth Weierstrass elliptic fibration over $\mathbb{P}^2$, equivalently the blowup of a weighted hypersurface $\hat{X}$ of degree 18 in $\mathbb{P}(1,1,1,6,9)$. The closure $\overline{\mathcal{K}(X)}$ of the Kähler cone is generated by two classes, denoted by H and L. The dual Mori cone generators are denoted by h and ℓ. Each of the classes H and L are nef but not ample (we have $H \cdot \ell = 0$ and $L \cdot h = 0$). But $H + L$ is in the interior of the nef cone hence is ample (cf. the discussion following Theorem B.1.17). We study the surfaces in $|H + L|$ and show that all are singular.

The blowup of $\hat{X}$ is performed along the singular locus $x_1 = x_2 = x_3 = 0$ (a single point in $\hat{X}$ due to the imposition of the defining weight 18 equation), with exceptional divisor $E \simeq \mathbb{P}^2$. The blowup guarantees that the projection to the first three coordinates gives a well-defined map $X \to \mathbb{P}^2$ with elliptic fibers, the base being embedded in X as the section E. The Mori generator ℓ is a line in $E \simeq \mathbb{P}^2$ and the Mori generator h is the class of the elliptic fiber. The divisor class L is the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ to X, and in particular is represented by the proper transforms of any of the surfaces defined by $x_i = 0, i = 1,2,3$. The class H is defined as $3L + E$. In particular, H projects to a class of weight 3 in $\mathbb{P}(1,1,1,6,9)$. Thus $2H$ projects to a class of weight 6 and $3H$ projects to a class of weight 9. Furthermore, it can be checked that the proper transform of $x_4 = 0$ is in the class $2H$ and the proper transform of $x_5 = 0$ is in the class $3H$.

We now examine the class $H+L = 4L+E$, which has weight 4 after projection to $\mathbb{P}(1,1,1,6,9)$. But the only weight 4 polynomials in $\mathbb{P}(1,1,1,6,9)$ are just the degree 4 homogeneous polynomials $f(x_1, x_2, x_3)$ in the homogeneous coordinates of the base $\mathbb{P}^2$. The proper transform of $f = 0$ is in the class $4L$. Thus any effective divisor D in $|H + L|$ contains as a component a surface in $|4L|$, which is simply the restriction S of the elliptic fibration to a plane curve C in the base of degree 4. We conclude that $D = S \cup E$, which is singular along $S \cap E$. This last is just the curve C identified as a curve in the section E.

B.2.2 $SU(2)$

In our first example, an $SU(2)$ gauge theory, the geometry is a singular quintic with an A_1 singularity along a line L, and smooth otherwise. For definiteness, we choose homogeneous coordinates $(x_0, \ldots, x_4)$ on $\mathbb{P}^4$ so that L is defined by $x_0 = x_1 = x_2 = 0$. Then the equation of the quintic has the form

$$\sum_{i,j=0}^{2} x_i x_j f_{ij}(x_3, x_4) = 0, \tag{B.56}$$

where the f_{ij} are homogeneous polynomials of degree 3.

More generally, we can find a quintic with an $SU(2)$ on any curve C which can be defined by the simultaneous vanishing of a collection of homogeneous polynomials $q_i(x) = q_i(x_0, \ldots, x_4)$ of degrees $d_i \leq 2$. In addition to the case of the line above $\{d_i\} = \{1, 1, 1\}$, we will also consider the cases where C is a plane conic $\{d_i\} = \{1, 1, 2\}$ or a twisted cubic $\{d_i\} = \{1, 2, 2, 2\}$. In general, letting $f_{ij}(x) = f_{ij}(x_0, \ldots x_4)$ denote generic homogeneous polynomials of degrees $5 - d_i - d_j$, the quintic defined by the equation

$$\sum_{i,j} q_i(x) q_j(x) f_{ij}(x) = 0 \tag{B.57}$$

has an A_1 singularity at the generic point of C. The assumption $d_i \leq 2$ is needed to ensure that $5 - d_i - d_j > 0$ and so nonvanishing $f_{ij}(x)$ exist.

Note that we are not assuming that the q_i are independent (as in the case of the line above), so there could be more than one way to choose the $f_{ij}(x)$ to get a fixed quintic. The twisted cubic is an example where such an ambiguity arises, with a linear syzygy relating the three quadratic terms.

Returning to the case of an $SU(2)$ on a line L, we now identify the matter. At a point $(0, 0, 0, x_3, x_4) \in L$ (which hereafter we simply write as $(x_3, x_4) \in L$), the type of the transverse singularity can be identified by the matrix

$$A(x_3, x_4) = \begin{pmatrix} f_{00}(x_3, x_4) & f_{01}(x_3, x_4) & f_{02}(x_3, x_4) \\ f_{10}(x_3, x_4) & f_{11}(x_3, x_4) & f_{12}(x_3, x_4) \\ f_{20}(x_3, x_4) & f_{21}(x_3, x_4) & f_{22}(x_3, x_4) \end{pmatrix}. \quad \text{(B.58)}$$

We have a transverse A_1 singularity at (x_3, x_4) when $\det A(x_3, x_4) \neq 0$. We assume that the f_{ij} are chosen generically, so that $\det A(x_3, x_4)$ is a degree 9 homogeneous polynomial vanishing at 9 distinct points, which are generically A_2 singularities. The $SU(2)$ gauge theory therefore has $N_f = 9$, with the matter localized at the zeros of $\det A$. Similar methods can be used to locate the matter starting from equations of the form (B.57). However, in this work we primarily concerned with the value of N_f rather than the more precise information of the location of the matter. Later in this example, we will compute $N_f = 9$ by a different method which will generalize in a straightforward manner.

Blowing up the singular quintic gives a Calabi-Yau X with $h^{1,1}(X) = 2$. The cohomology generators are H, the proper transform of the hyperplane class of the quintic, and E, the exceptional divisor of the blowup. We need to compute the triple intersection numbers of H and E, their intersections with $c_2 = c_2(X)$, and the generators of the Kähler cone.

We proceed by first blowing up L inside $\mathbb{P}^4$ to obtain the blown-up fourfold $\widetilde{\mathbb{P}^4}$. Then $H^{1,1}(\widetilde{\mathbb{P}^4})$ is generated by $\mathbf{H}$, the proper transform of the hyperplane class of $\mathbb{P}^4$, and $\mathbf{E}$, the exceptional divisor.

We have

$$H = \mathbf{H}|_X, \qquad E = \mathbf{E}|_X. \tag{B.59}$$

Since X is obtained by blowing up a quintic (degree 5) with a multiplicity 2 singularity along L, we get for the class $[X] \in H^{1,1}(\widetilde{\mathbb{P}^4})$ of X

$$[X] = 5\mathbf{H} - 2\mathbf{E}. \tag{B.60}$$

To compute the triple intersections on X, we lift to classes to $\widetilde{\mathbb{P}^4}$ using (B.59) and then restrict the corresponding triple intersection on $\widetilde{\mathbb{P}^4}$ to X. Using (B.60) we get

$$\begin{aligned} H^3 &= \mathbf{H}^3\,(5\mathbf{H} - 2\mathbf{E})\,,\ H^2E = \mathbf{H}^2\mathbf{E}\,(5\mathbf{H} - 2\mathbf{E})\,, \\ HE^2 &= \mathbf{H}\mathbf{E}^2\,(5\mathbf{H} - 2\mathbf{E})\,,\ E^3 = \mathbf{E}^3\,(5\mathbf{H} - 2\mathbf{E})\,. \end{aligned} \tag{B.61}$$

To finish the calculation, we just need the four-fold intersection products $\mathbf{H}^i\mathbf{E}^{4-i}$ on $\widetilde{\mathbb{P}^4}$. This is a standard calculation in algebraic geometry, using Segre classes[77]. The Segre class $s(S,M)$ of a submanifold $S \subset M$ is the inverse of the total chern class of the normal bundle $N_{S,M}$ of S in M:

$$s(S,M) = c(N_{S,M})^{-1}, \tag{B.62}$$

a cohomology class on S. Now suppose that we have a birational mapping of manifolds $f : M \to N$ with $T = f(S)$ also a manifold. Then we have

$$f_*\left(s(S,M)\right) = s(T,N), \tag{B.63}$$

i.e. Segre classes are invariant under birational pushforward[77] P. 76.

In the special case where S is a divisor, we have $N_{S,M}$ is a line bundle, and $c_1(N_{S,M})$ is the restriction of the cohomology class of S itself to S. So $c(N_{S,M})$ is the restriction of $1 + S$ to S. Specialing $S \subset M$ to $\mathbf{E} \subset \widetilde{\mathbb{P}^4}$ and inverting, we get

$$s(\mathbf{E}, \widetilde{\mathbb{P}^4}) = \mathbf{E} - \mathbf{E}^2 + \mathbf{E}^3 - \mathbf{E}^4. \tag{B.64}$$

For the projection $\pi : \widetilde{\mathbb{P}^4} \to \mathbb{P}^4$ we have $\pi(E) = L$. Since $c_1(N_{L,\mathbb{P}^4}) = 3p$ (p being the class of a point), we invert $c(N_{L,\mathbb{P}^4}) = L + 3p$ on L and get

$$s(L, \mathbb{P}^4) = L - 3p. \tag{B.65}$$

Then $\pi_*(\mathbf{E} - \mathbf{E}^2 + \mathbf{E}^3 - \mathbf{E}^4) = L - 3p$ gives

$$\pi_*(\mathbf{E}) = 0,\ \pi_*(\mathbf{E}^2) = 0, \pi_*(\mathbf{E}^3) = L, \pi_*(\mathbf{E}^4) = 3p. \tag{B.66}$$

Letting h be the hyperplane class of $\mathbb{P}^4$ with $h^4 = 1$, and $H = \pi^* h$, we can now compute the four-fold intersections on $\widetilde{\mathbb{P}^4}$ by

$$\mathbf{H}^i\mathbf{E}^{4-i} = \pi_*(\mathbf{H}^i\mathbf{E}^{4-i}) = \pi_*((\pi^* h^i)\mathbf{E}^{4-i}) = h^i \pi_*(\mathbf{E}^{4-i}). \tag{B.67}$$

Combining with (B.66) we get

$$\begin{aligned} \mathbf{H}^4 = h^4 = 1,\ \mathbf{H}^3\mathbf{E} = h^3\pi_*(\mathbf{E}) = 0,\ \mathbf{H}^2\mathbf{E}^2 = h^2\pi_*(\mathbf{E}^2) = 0, \\ \mathbf{H}\mathbf{E}^3 = h\pi_*(\mathbf{E}^3) = hL = 1,\ \mathbf{E}^4 = 3. \end{aligned} \tag{B.68}$$

Plugging these into (B.61) we get

$$H^3 = 5,\ H^2E = 0,\ HE^2 = -2,\ E^3 = -1. \tag{B.69}$$

Since E is a ruled surface over $L \simeq \mathbb{P}^1$, it is the blowup of a Hirzebruch surface at N_f points. However, E^3 is the self-intersection of the canonical bundle of E, which is $8 - N_f$. So $E^3 = -1$ is equivalent to $N_f = 9$.

For a smooth surface S on any Calabi-Yau we have $S^3 + S \cdot c_2 = c_2(S)$. Applying this to H, a quintic surface in $\mathbb{P}^3$ with $c_2 = 55$ we get $H \cdot c_2 = 50$. From the description of E as the blowup of a Hirzebruch surface at 9 points we get $c_2(E) = 13$, as a Hirzebruch surface has $c_2 = 4$ and each blowup adds 9. Combining with $E^3 = -1$ we get $E \cdot c_2 = 14$. Summarizing:

$$H \cdot c_2 = 50, \qquad E \cdot c_2 = 14. \tag{B.70}$$

Finally, we turn to the Kähler cone, which is most easily computed from the dual Mori cone. The calculation is elementary albeit a bit lengthy. We provide all of the details in this case to illustrate the ideas. In the other examples in this and the following section, we omit details in the calculation of the Mori cone. In some cases, we do not have a mathematical proof that we have found all of the Mori generators, but we provide justification by checking consistency with physics.

We coordinatize the Mori cone by identifying the class $[D]$ of a curve $D \subset X$ with the ordered pair

$$(D \cdot H, D \cdot E) \in \mathbb{Z}^2. \tag{B.71}$$

Alternatively, if desired we could identify of pair of curve classes which generate $H_2(X, \mathbb{Z})$ and express all curve classes in terms of the two chosen generators. While that approach might clarify the geometry, using our coordinates is simpler.

We identify irreducible curves $D \subset X$ with help of the blowdown map $\pi : X \to Y$ which contacts E to the line L in the singular quintic $Y \subset \mathbb{P}^4$. The restriction of π to E exhibits E as a ruled surface over L. Let r be the class of the generic fiber. Since a general hyperplane in Y intersects L at one point, its proper transform H in X is disjoint from the fiber r over any other point of L. Thus

$H \cdot r = 0$. Furthermore, $E \cdot r = -2$ because the curve r can be viewed as the exceptional curve of a transverse A_1 singularity. Thus r has coordinates $(0, -2)$.

There are $N_f = 9$ special fibers which split into a pair of $\mathbb{P}^1$'s. Since each $\mathbb{P}^1$ in this pair is orthogonal to H, the two classes lie in the same 1-dimensional subspace of the two-dimensional $H_2(X, \mathbb{Z})$ and are therefore proportional. We conclude that each of these $\mathbb{P}^1$'s has class $r/2$ and coordinates $(0, -1)$.

If D is not contained in a fiber of π, then $\pi(D)$ is a curve in $Y \subset \mathbb{P}^4$ of some degree $d > 0$. For example, $\pi(D)$ can be a line, $d = 1$. There are two cases to consider separately: $\pi(D) = L$ or $\pi(D) \neq L$.

We consider the latter case first. Let d be the degree of $\pi(D)$ as a curve in $X \subset \mathbb{P}^4$, so that $H \cdot D = d$. Since $\pi(D) \neq L$, the curves $\pi(D)$ and L meet at finitely many points (possibly none). Equivalently, D and E meet at finitely many points. Putting $k = E \cdot D \geq 0$, we conclude that the coordinates of D are (d, k).

We now show that $k \leq d$. Choose a hyperplane $P \subset \mathbb{P}^4$ containing L but not containing $\pi(D)$. Then P meets $\pi(D)$ in d points (including multiplicity) by the definition of degree. On the other hand, P meets $\pi(D)$ in at least k points (including multiplicity), namely those contained in L. Thus $k \leq d$, as claimed.

We now exhibit a curve D with $d = k = 3$. Choose a two-plane $Q \subset \mathbb{P}^4$ containing L. Since $Y \subset \mathbb{P}^4$ is a quintic, we have that $Q \cap Y$ is a degree 5 plane curve, including multiplicities. However $L \subset Q \cap Y$, and L occurs with multiplicity 2 in $Q \cap Y$ due to the A_1 singularity. It follows that

$$Q \cap Y = 2L + D \tag{B.72}$$

for some degree 3 curve D, i.e. $d = 3$ for the curve D. Since D and L are contained in the same plane Q, they meet in exactly 3 points, and $k = 3$ as claimed. Thus D has coordinates $(3, 3)$.

Thus the Mori cone is spanned (over $\mathbb{Q}$) by the curve classes with coordinates $(0,-1)$, $(1,1)$, and the curves D with $\pi(D)=L$.

We are now ready to turn to the case $\pi(D)=L$, i.e. curves $D\subset E$, and show that these classes are already in the span of the curve classes found above.

Since the half-fibers have self-intersection -1, we can blow down either of the $\mathbb{P}^1$s in the 9 singular fibers and get a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Thus E is the blowup of a Hirzebruch surface. Note that the Mori cone of a blown up Hirzebruch surface is generated by the exceptional curves and some sections, as discussed for example in [111]. The exceptional curves which are not sections are among the half-fibers $r/2$ which we have already accounted for in the Mori cone.

To determine the possible coordinates of the sections, we describe E as a hypersurface inside $\mathbb{P}^1\times\mathbb{P}^2$ by viewing (x_0,x_1,x_2) as homogeneous coordinates for $\mathbb{P}^2$ and (x_3,x_4) as homogeneous coordinates for $\mathbb{P}^1$ in (B.56). Thus E is a hypersurface of bidegree $(3,2)$.

Now a section of E can be thought of as the image of a map $\mathbb{P}^1\to E\subset\mathbb{P}^1\times\mathbb{P}^2$. Thus any section D of E is the graph of a map $\mathbb{P}^1\to\mathbb{P}^2$. Let s be the degree of this map. We now compute the coordinates of D in terms of s.

First, we have $H\cdot D=1$, since H meets E in a fiber of E, which in turn meets D in exactly one point since D is a section.

Next, $E\cdot D$ is equal to the degree of the restriction $(K_E)|_D$ to D of the canonical bundle of E. Since the canonical bundle of $\mathbb{P}^1\times\mathbb{P}^2$ is $\mathcal{O}(-2,-3)$ and E is a section of $\mathcal{O}(3,2)$, the adjunction formula tells us that K_E is $\mathcal{O}(1,-1)$, which has degree $1-s$ after restricting to D. Thus the section D has coordinates $(1,1-s)$. Since these classes are all in the cone spanned by the curve classes with coordinates $(0,-1)$ and $(1,1)$, we see that the Mori cone is spanned by the curve classes with coordinates

$$(0,-1),\ (1,1). \tag{B.73}$$

Finally, the Kähler cone is generated by

$$H,\ H-E, \tag{B.74}$$

the dual basis to (B.73).

Before turning to other examples, we first make some general observations. Suppose we have a Calabi-Yau threefold Y with a generic A_1 singularity along a smooth curve C of genus g, enhancing to A_2 at N_f distinct points. Let $\pi : X \to Y$ be the blowup of C, with exceptional divisor E. Then E is a ruled surface over C with generic fiber r, and N_f special fibers consisting of pairs of $\mathbb{P}^1$'s, each of class $r/2$. Let $\{D_i\}$ be any collection of divisors in $H^2(Y,\mathbb{Z})$, and we continue to denote their pullbacks to X by the same symbols. Then by similar methods to the above example, we compute

$$D_iD_jE = 0,\ D_iE^2 = -2D_iC, E^3 = 8-8g-N_f, \tag{B.75}$$

while the triple intersections of the D_i are identical when computed on either X or Y. The intersection D_iC is computed on Y, while the triple intersections in (B.75) are computed on X.

We have done computations for quintics with $SU(2)$ on various curves C. Suppose that $C \subset Y \subset \mathbb{P}^4$ has degree c and genus g. Since $c_1(N_{C,\mathbb{P}^4}) = 5d+2g-2$, we compute $c(N_{C,\mathbb{P}^4}) = [C]+(5d+2g-2)p$ so that

$$s(C,\mathbb{P}^4) = C+(5d+2g-2)p. \tag{B.76}$$

Birational invariance of Segre classes then reads $\pi_*(\mathbf{E}-\mathbf{E}^2+\mathbf{E}^3-\mathbf{E}^4) = C-(5d+2g-2)p$, giving

$$\pi_*(\mathbf{E}) = 0,\ \pi_*(\mathbf{E}^2) = 0, \pi_*(\mathbf{E}^3) = C, \pi_*(\mathbf{E}^4) = (5d+2g-2)p, \tag{B.77}$$

hence

$$\mathbf{H}^4 = 1,\ \mathbf{H}^3\mathbf{E} = 0,\ \mathbf{H}^2\mathbf{E}^2 = 0,\ \mathbf{H}\mathbf{E}^3 = hC = d,\ \mathbf{E}^4 = 5d + 2g - 2. \tag{B.78}$$

It follows that

$$H^3 = 5,\ H^2E = 0,\ HE^2 = -2d,\ E^3 = 4 - 4g - 5d. \tag{B.79}$$

Since a $\mathbb{P}^1$-bundle over C has $K^2 = 8 - 8g$, the surface E must be a $\mathbb{P}^1$-bundle over C blown up at $N_f = (8 - 8g) - (4 - 4g - 5d) = 5d + 4 - 4g$ points.

Combining $c_2(E) = 4 - 4g + N_f = 5d + 8 - 8g$ (since a $\mathbb{P}^1$-bundle over a curve of genus g has $c_2 = 2(2-2g) = 4-4g$ and each blowup adds 1) with $c_2(E) = E^3 + Ec_2$, we get $Ec_2 = 10d+4-4g$.

We collect the results in the following table. Here d' is the degree of $C \subset \mathbb{P}^4$.

d'	g	Mori gens	Kahler gens	N_f	H^3	H^2E	HE^2	E^3	c_2H	c_2E
1	0	$(1,1),\ (0,-1)$	$H,\ H-E$	9	5	0	-2	-1	50	14
2	0	$(1,2),\ (0,-1)$	$H, 2H-E$	14	5	0	-4	-6	50	24
3	0	$(1,2),\ (0,-1)$	$H,\ 2H-E$	19	5	0	-6	-11	50	34

(B.80)

The only information in (B.80) which does not follow immediately from (B.79) and the following paragraphs are the Mori generators and the Kähler generators. We have continued to coordinatize the Mori cone by $[D] \mapsto (H \cdot D, E \cdot D)$. The Kähler generators are immediately deduced from the Mori generators by duality, so we need only describe the Mori generators.

The method is a straightforward adaptation of the case of a line. We consider the blowdown $\pi : X \to Y$ to a quintic Y with an A_1 singularity along C, and separately consider the cases $\pi(D) = C$ and $\pi(D) \neq C$. We have half-fibers $r/2$ with coordinates $(0, -1)$. If $\pi(D) \neq C$, then $[D]$ has coordinates (d, k), with $d > 0$ and $k \geq 0$ exactly as in the case $C = L$. We find a curve D which maximizes the slope of the ray from the origin through (d, k) and as before, we can show that the case $\pi(D) = C$ does not produce any new classes. Then the Mori cone is generated by (d, k) and $(0, -1)$.

In the case $d = 2, g = 0$, such curves are well-known to be contained in a unique two-plane Q[98] Example 6.4.2, which intersects Y in a degree 5 curve $Y \cap Q$. This intersection contains C with multiplicity 2. Considering degrees, we see that we must have

$$Y \cap Q = 2C + D \tag{B.81}$$

for some line D, $d = 1$. Since D meets C in 2 points by plane geometry, we see that $k = 2$ and D has coordinates $(1, 2)$ and slope 2.

We now show that any other irreducible curve D with coordinates (d, k) has slope $k/d < 2$. It follows that Mori generators are those appearing in the second line of (B.80).

To see this, our previous argument shows that the line D above is the *only* curve in the quintic Y other than C which is contained in Q. Since any other curve D' is not contained in Q, we can find a hyperplane P containing Q (hence containing C) which does not contain D'. The same argument as in the case of an $SU(2)$ on a line shows that $k \leq d$, and we are done since the slope of the rays associated to these curves are at most 1.

In the case where C is a twisted cubic $d = 3, g = 0$, we found the curve D whose coordinate ray has maximal slope experimentally by a computer search. We simply describe this curve.

First, we note that the curve C is defined by the vanishing of homogeneous polynomials ℓ_1, q_2, q_3, q_4 of degrees 1,2,2,2. The hypersurface $\ell_1 = 0$ intersects the singular quintic threefold Y in a quintic surface S which is also singular along the curve C. We let $\tilde{S}$ be the proper transform of S inside the Calabi-Yau X. The surface $\tilde{S}$ is the blowup of S along C, and is a smooth surface assuming that we have chosen the singular quintic Y containing C generically.

We now calculate intersections on $\tilde{S}$ by blowing up $S \subset \mathbb{P}^3$ along C using exactly the same method we previously used to find intersections on X by blowing up $Y \subset \mathbb{P}^4$ along C. We state results without providing all of the supporting calculations.

We denote the exceptional divisor of the blowup $\widetilde{\mathbb{P}^3}$ of $\mathbb{P}^3$ by $\mathbf{F}$, the proper transform of a hyperplane by $\mathbf{H}$, the projection $\widetilde{\mathbb{P}^3} \to \mathbb{P}^3$ by π, the restriction of $\mathbf{F}$ to $\tilde{S}$ by F and the restriction of $\mathbf{H}$ to $\tilde{S}$ by H. Calculating Segre classes as in the $SU(2)$ cases, we get $\pi_*(\mathbf{F}) = 0,\ \pi_*(\mathbf{F}^2) = -C,\ \mathbf{F}^3 = -10$, which yields

$$\mathbf{H}^3 = 1,\ \mathbf{H}^2\mathbf{F} = 0,\ \mathbf{H}\mathbf{F}^2 = -3,\ \mathbf{F}^3 = -10. \tag{B.82}$$

Since $\tilde{S}$ has class $5\mathbf{H} - 2\mathbf{F}$ in $\widetilde{\mathbb{P}^3}$, we calculate products of H and F in $\tilde{S}$ by replacing H and F by $\mathbf{H}$ and $\mathbf{F}$ respectively, multiplying by $5\mathbf{H} - 2\mathbf{F}$, then calculating the resulting intersection on $\widetilde{\mathbb{P}^3}$ using (B.82). We obtain after calculation

$$F^2 = 5,\ HF = 6,\ H^2 = 5. \tag{B.83}$$

The desired curve D has class $7H - 4F$. For this class, we compute $H \cdot D = 7H^2 - 4HF = 11$ and $F \cdot D = 7HF - 4F^2 = 22$, So D has coordinates $(11, 22)$, slope 2. This ray is indicated (B.80). To show that D is in the Mori cone, we just have to show that there is an effective curve in this class.

By the adjunction formula for $\tilde{S} \subset \widetilde{\mathbb{P}^3}$, the canonical class $K_{\tilde{S}}$ of $\tilde{S}$ is $((-4\mathbf{H} + \mathbf{F}) + (5\mathbf{H} - 2\mathbf{F}))|_{\tilde{S}} = H - F$. Since $K_{\tilde{S}}$ has degree $H(H - F) = -1 < 0$, the class $K_{\tilde{S}}$ is not effective, i.e. $p_g(\tilde{S}) = 0$ and hence $\chi(\mathcal{O}_{\tilde{S}}) = 1$. Riemann-Roch then gives $\chi(\mathcal{O}(D)) = (1/2)D(D - K_{\tilde{S}}) + 1 = (1/2)(7H - 4F)(6H - 3F) + 1 = 1 > 0$. Also $H^2(\mathcal{O}(D))$ is Serre dual to $H^0(K_{\tilde{S}} - D) = H^0(\mathcal{O}(-6H + 3F))$, which is zero since $-6H + 3F$ has negative degree $H(-6H + 3F) = -12$. Thus $1 = \dim H^0(\mathcal{O}(D)) - \dim H^1(\mathcal{O}(D))$ and so $H^0(\mathcal{O}(D))$ is nonzero. Hence D is effective, as claimed.

B.2.3 $SU(3)$

To achieve an $SU(3)$ geometry on a line L, we inspect the $SU(2)$ geometry (B.56) and see that by reinterpreting (x_0, x_1, x_2) as homogeneous coordinates on $\mathbb{P}^2$, the same equation describes E as a ruled surface over the $\mathbb{P}^1$ with homogeneous coordinates (x_3, x_4). The fibers are degree 2 curves in $\mathbb{P}^2$ which are generically isomorphic to $\mathbb{P}^1$, except over the $N_f = 9$ points where $\det A(x_3, x_4) = 0$. For those points, the degree 2 curve factors into a product of linear terms and the fiber is a pair of lines, corresponding to the geometry of an $SU(2)$ enhancement.

This description immediately suggests a way to achieve an $SU(3)$: we require the degree 2 curve in every fiber to factor. This can be achieved if the equation of E factors as

$$\left(x_0 g_0(x_3, x_4) + x_1 g_1(x_3, x_4) + x_2 g_2(x_3, x_4)\right) \left(x_0 h_0(x_3, x_4) + x_1 h_1(x_3, x_4) + x_2 h_2(x_3, x_4)\right) . \tag{B.84}$$

In (B.84), the degrees of the polynomials g_j and h_j are fixed by an integer $0 \leq n \leq 3$: the g_j all have degree n and the h_j all have degree $3 - n$. By construction, each term in (B.84) has degree 5 in the full set of variables $(x_0, \ldots, x_4)$ so is the equation of a quintic. But this is not a good quintic when viewed as a hypersurface in $\mathbb{P}^4$, since it visibly has two components, one of degree $n + 1$ and the other of degree $4 - n$. This is easily fixed by adding terms of order greater than two in (x_0, x_1, x_2)

$$\left(x_0 g_0 + x_1 g_1 + x_2 g_2\right) \left(x_0 h_0 + x_1 h_1 + x_2 h_2\right) + \ldots . \tag{B.85}$$

For generic g_j, h_j, and higher order terms, the quintic (B.85) has an $SU(3)$ geometry along a line, and no other singularities.

We will blow up this geometry twice to a smooth Calabi-Yau threefold which Higgses to the quintic. The choice of blowup depends on the ordering of the factors in (B.84). For this reason, the construction is not symmetric in g and h. In particular, switching g and h, and replacing n with

$3 - n$ gives the same geometry (B.85) but a different smooth Calabi-Yau. These distinct Calabi-Yaus are related by a flop.

We start by blowing up $\mathbb{P}^4$ along the line L, just as we did in the $SU(2)$ case, and now consider the proper transform Z of the quintic Y defined by (B.85). The exceptional divisor E is still fibered over L, and by construction it splits into two components, each component being a $\mathbb{P}^1$-bundle over L, i.e. a Hirzebruch surface. The two components intersect in a section F of E over L. While it is clear that Z is smooth away from F and at the generic point of F, there is nothing to prevent Z from having conifolds at finitely many points of F. We will perform a blowup of Z along one of the Hirzebruch surfaces which will both detect the conifolds and resolve them by small resolutions. A choice of small resolution will be made in the process.

As a preliminary, we show how blowing up a surface in a threefold can detect a singularity in the threefold. First, consider a smooth surface in a smooth threefold. We can choose local analytic coordinates (x, y, z) in the threefold so that the surface is defined by $z = 0$. Since there is only one equation, blowing up $z = 0$ does nothing, and the proper transform of the surface is isomorphic to the surface being blown up.

By contrast, suppose a smooth surface passes through a conifold point. We can choose local analytic coordinates (w, x, y, z) so that the conifold is defined by $wx = yz$ and the surface is defined by $w = y = 0$. Now if we blow up $w = y = 0$, we get two coordinate patches. In the first patch we have a new coordinate $u = w/y$, leaving coordinates (u, x, y, z) after eliminating w via $w = uy$. Making this substitution into the equation of the conifold and factoring out y, we get $ux = z$, i.e. z can be eliminated as well, leaving independent coordinates (u, x, y), i.e. this patch of the blowup is a smooth threefold. The blowdown map is seen to be

$$(u, x, y) \mapsto (w, x, y, z) = (uy, x, y, ux). \qquad \text{(B.86)}$$

The inverse image of the conifold point in this patch is $\{(u, 0, 0)\}$, a copy of $\mathbb{C}$.

We have a second coordinate patch described in terms of a new coordinate $v = y/w$. A similar calculation gives coordinates (v, w, z) and blowdown map

$$(v, w, z) \mapsto (w, x, y, z) = (w, vz, vw, z). \tag{B.87}$$

The inverse image of the conifold point in this patch is $\{(v, 0, 0)\}$, another copy of $\mathbb{C}$. Since the first coordinates in these two patches are related by $v = u^{-1}$, we see that the conifold gets blown up to $\mathbb{P}^1$, and we have a small resolution. Furthermore, the local forms (B.86), (B.87) of the blowdown map show that the exceptional $\mathbb{P}^1$ is identified with the exceptional $\mathbb{P}^1$ of the blowup of surface $w = z = 0$ with coordinates (x, y).

Now the divisor $w = 0$ in the singular threefold has two component divisors: $w = y = 0$ and $w = z = 0$. Blowing up the first introduces an exceptional $\mathbb{P}^1$ in its proper transform. We now show that the blowup does not change the other divisor. If we consider the divisor $w = z = 0$ and make the coordinate change $w = uy$, recalling that the exceptional divisor is y, we get the proper transform of this divisor is $u = z = 0$, or just $u = 0$ since $z = uy$ as discussed above. The inverse image via (B.86) of the conifold point inside $u = 0$ is just $(0, 0, 0)$, so the proper transform of the divisor $w = z = 0$ is isomorphic to the original divisor inside this coordinate patch. A similar calculation in the other coordinate patch completes the verification of our assertion.

This gives us our strategy for identifying and resolving the conifolds: by blowing up one Hirzebruch surface, we introduce exceptional $\mathbb{P}^1$'s in its proper transform without changing the other Hirzebruch surface. We will see this explicitly in our $SU(3)$ model after further calculation. This process involves a choice and is asymmetric, related by flops. Furthermore, since we are using algebraic blowups, the resulting smooth threefold is guaranteed to be Kähler.

We now implement this strategy by blowing up the Hirzebruch surface S corresponding to the first factor of (B.84). It can be shown that

$$S \simeq \begin{cases} \mathbb{F}_0 & n \text{ even} \\ \mathbb{F}_1 & n \text{ odd} \end{cases} \tag{B.88}$$

but we do not need this, as K_S (needed for Segre classes) can be computed by other techniques. Instead, we note that the exceptional divisor $\mathbb{E}$ of $\widetilde{\mathbb{P}^4}$ is a trivial $\mathbb{P}^2$-bundle over L, i.e. is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$, essentially because the homogeneous coordinates (x_0, x_1, x_2) on the fiber are independent of the coordinates on L. The equation of the surface S has degree n in the $\mathbb{P}^1$ variables and degree 1 in the $\mathbb{P}^2$ variables. If we let $h_1 \in H^2(\mathbb{P}^1, \mathbb{Z})$ and $h_2 \in H^2(\mathbb{P}^2, \mathbb{Z})$ be the respective generators, we therefore have for the class of S

$$[S] = nh_1 + h_2. \tag{B.89}$$

Then we can generate $H^2(S, \mathbb{Q})$ by $f = h_1|_S$ and $h = h_2|_S$, where f is a fiber of the Hirzebruch surface and h is a section. We compute

$$h^2 = h_2^2\,(nh_1 + h_2) = n. \tag{B.90}$$

We also have

$$f^2 = 0,\ hf = 1, \tag{B.91}$$

either by lifting to $\mathbb{P}^1 \times \mathbb{P}^2$ and intersection with the class $nh_1 + h_2$ of S, or more simply by noting that f is a fiber and h is a section.

For K_S we write $K_S = \alpha h + \beta f$ and solve for α and β using the adjunction formula

$$f(f + K_S) = -2, \qquad h(h + K_S) = -2. \tag{B.92}$$

We get

$$K_S = -2h + (2n-2)f \tag{B.93}$$

In preparation for blowing up S, we need to compute its Segre class. We have

$$s(S, \widetilde{\mathbb{P}^4}) = c(N_{S/\widetilde{\mathbb{P}^4}})^{-1} = c(T_S)c(T_{\widetilde{\mathbb{P}^4}}|_S)^{-1}. \tag{B.94}$$

We have

$$c(T_S) = [S] + \big(2h - (2n-2)f\big) + 4p. \tag{B.95}$$

To compute $c(T_{\widetilde{\mathbb{P}^4}})$, it is most convenient to use that $\widetilde{\mathbb{P}^4}$ is a toric variety, whose six torus-invariant divisors have classes $\mathbf{H}, \mathbf{H}, \mathbf{H}-\mathbf{E}, \mathbf{H}-\mathbf{E}, \mathbf{H}-\mathbf{E}, \mathbf{E}$. We get

$$c(T_{\widetilde{\mathbb{P}^4}}) = (1+\mathbf{H})^2\,(1+\mathbf{H}-\mathbf{E})\,(1+\mathbf{E})\,. \tag{B.96}$$

We now blow up S and let $\mathbf{E}_1$ denote the exceptional divisor of this second blowup. We then identify the class of the proper transform of $\mathbf{E}$ and denote it as

$$\mathbf{E}_2 = \mathbf{E} - \mathbf{E}_1. \tag{B.97}$$

Using invariance of the Segre class as we did in the $SU(2)$ case, we can compute all of the 4-fold intersections involving $\mathbf{H}, \mathbf{E}_1, \mathbf{E}_2$.

We now choose our singular quintic threefold Y to have multiplicity 2 along L, and furthermore, after blowing up as we did in the $SU(2)$ case, contains S. Our Calabi-Yau X is the proper transform of Y after our two blowups. We see that X has class

$$[X] = 5\mathbf{H} - 2\mathbf{E} - \mathbf{E}_1. \tag{B.98}$$

We then put

$$H = \mathbf{H}|_X,\ E_1 = \mathbf{E}_1|_X,\ E_2 = \mathbf{E}_2|_X \tag{B.99}$$

By construction, X has a resolved A_2 configuration over L. The surface E_1 is a Hirzebruch surface and E_2 is a blown up Hirzebruch surface. We can then compute all of the triple products of H, E_1, E_2 by lifting to the blowup and multiplying by $5\mathbf{H} - 2\mathbf{E} - \mathbf{E}_1$. In particular, N_f is deduced from E_2^3. We omit the calculations and state the results.

For all values of $0 \leq i \leq 3$, we get

$$\begin{aligned} H^3 = 5,\ H^2E_1 = H^2E_2 = 0,\ HE_1^2 = HE_2^2 = -2,\ HE_1E_2 = 1, \\ E_1^3 = -3,\ E_2^3 = 8, \end{aligned} \tag{B.100}$$

Since E_1 is a blown-up Hirzebruch surface by our general discussion and $K^2 = 8 - N_f$ for a Hirzebruch surface blown up N_f times, we conclude that $N_f = 11$. Note that E_2 is a Hirzebruch surface which has not been blown up, again consistent with our general discussion.

The other intersection numbers depend on n:

$$E_1^2E_2 = n + 3\, E_1E_2^2 = -n - 5. \tag{B.101}$$

From the description of E_1 as a Hirzebruch surface and of E_2 as a blown-up Hirzebruch surface, we get $c_2(E_1) = 4$ and $c_2(E_2) = 15$. From $c_2(E_i) = E_i^3 + c_2 \cdot E_i$ and (B.100) we get

$$c_2 \cdot E_1 = 18,\ c_2 \cdot E_2 = -4. \tag{B.102}$$

It remains only to describe the Mori cone and Kähler cone. As in the $SU(2)$ case, we choose a two-plane $Q \subset \mathbb{P}^4$ containing L and we again find a degree 3 curve $D \subset P \cap Y$ by (B.72). This curve again meets L in 3 points. The factorization in (B.85) tells us that after blowing up, n of these points meet E_1 and $3 - n$ meet E_2.

The curves D and the fiber r_1 of E_1 are again in the Mori cone. However, unlike the $SU(2)$ case, the two components of the reducible fibers of E_1 are asymmetric: since a fiber r_1 satisfies $r_1 \cdot E_2 = 1$, one of these two components must meet E_2 and the other one does not. We let r_1' be the component which intersects E_2. The other component is then $r_1 - r_1'$ and is disjoint from E_2. In principle, we might need both r_1' and $r_1 - r_1'$ to generate the Mori cone.

The intersection numbers of each of $D' \in \{D, r_1, r_2', r_2 - r_2\}$ are listed as an ordered triple $[D'] = (D' \cdot H, D' \cdot E_1, D' \cdot E_2)$.

$$[D] = (3, n, 3 - n),\ [r_2] = (0, 1, -2),\ [r_1'] = (0, -1, 1),\ [r_1 - r_1'] = (0, -1, 0). \qquad \text{(B.103)}$$

The coordinates of D can be found by an explicit geometric computation using the equation (B.85) of the singular quintic Y. Alternatively, since D is the complete intersection of the proper transform of two hyperplanes containing L, we get $D = (H - E_1 - E_2) \cdot (H - E_1 - E_2)$, and then the intersection numbers of D with H, E_1, and E_2 follow readily from (B.100) and (B.101).

The coordinates (B.103) make plain the relation $[r_1 - r_1'] = [r_2] + 2[r_1']$. So $[r_1 - r_1']$ is not needed to span the Mori cone.

We need another curve to generate the Mori cone. But it will be instructive to explain how we can find new curves iterative. We content ourselves with working out the case $n = 0$.

So let's assume that the Mori cone is actually spanned by D, r_2, r_1'. Then dually, the Kähler cone would be generated by

$$H,\ H - E_1 - E_2,\ H - 2E_1 - E_2. \qquad \text{(B.104)}$$

In particular $H - 2E_1 - E_2$. But we compute $(H - 2E_1 - E_2)^3 = -3 < 0$, contradicting Theorem B.1.17. Thus $H - 2E_1 - E_2$ is not nef.

But this class does have an effective representative. Interpreting (B.84) as an equation inside the exceptional divisor $\mathbb{P}^1 \times \mathbb{P}^2 \subset \tilde{\mathbb{P}^4})$ as before, without loss of generality we can change coordinates in $\mathbb{P}^2$ so that the first factor is just $x_0 = 0$. Then the proper transform $H - E = H - E_1 - E_2$ of $x_0 = 0$ contains the first Hirzebruch surface E_1. So when we blow up a second time, we have to subtract E_1 again. So the class of the proper transform S of $x_0 = 0$ after both blowups is $H - 2E_1 - E_2$, hence that class is effective.

Since S is not nef and S is a surface, a curve C satisfying $S \cdot C < 0$ is necessarily contained in S. This gives us a strategy for finding missed curves: look for curves in S. Since $|H - E_1 - E_2|$ is base point free[5], we know that we can find a representative where the intersection $S \cdot (H - E_1 - E_2)$ is an effective curve $K \subset S \subset X$, hence is in the Mori cone. Using $[S] = H - 2E_1 - E_2$, (B.100) and (B.101), we get for the coordinates of K

$$[K] = (2, 2, 0). \tag{B.105}$$

Comparing with (B.103) and recalling that $n = 0$, we see that $(1/3)[D] = (1/2)[K] + [r'_1]$. The Mori cone is generated by K, r_2, and r'_1.

Dually, the Kähler cone is generated by

$$H,\ H - E_1 - E_2,\ 2H - 2E_1 - E_2. \tag{B.106}$$

As a check, we compute $(2H - 2E_1 - E_2^3) = 14 > 0$. We have also checked in Section 3.3.1 that this example satisfies the physical requirements of supergravity strings.

[5] This linear system corresponds to hyperplanes in Y containing L. That linear system has L as a base locus, but it is immediately checked that the base locus is removed by the blowups.

B.3 5d Non-Compact Threefolds

B.3.1 Arithmetic genus for curves in a self-glued surface

When a surface has no self-gluings, then the arithmetic genus[6] of curves living inside the surface can be computed using the adjunction formula (B.12).

However, when the surface has self-gluings, the genus of the curve is modified. For example, consider gluing the exceptional curves x and y in a generic two point blowup of $\mathbb{F}_1$. The curve $h - x - y$ (which is a rational curve before gluing) looks like an elliptic fiber with nodal singularity after the gluing, so its arithmetic genus should be one instead of zero, which is what would be suggested by (B.12). This example suggests that the intersection numbers of a curve C with the curves C_1 and C_2 participating in a self-gluing should be used to modify (B.12) in order to obtain the correct arithmetic genus. However, not all such intersection numbers participate in such a modification. To see this, consider the curve $f - x$ in the above example. This curve remains rational even after gluing. Thus, even though it intersects x, its genus is correctly captured by (B.12).

The examples of $h - x - y$ and $f - x$ above suggest that the genus of a curve C should only be modified whenever an intersection with C_1 has a partner intersection with C_2. Thus our proposal for the computation of genus of an arbitrary curve C is as follows: Let n_1 and n_2 be the intersections of C with C_1 and C_2 respectively, and let $n = \min(n_1, n_2)$. Then, our proposal for computation of genus is

$$2g(C) - 2 = (K_S + C) \cdot C + 2n \tag{B.107}$$

(B.107) allows certain curves to have a non-negative genus even though they did not have a non-negative genus before self-gluing. For example, consider

[6] Throughout this work, we never use the geometric genus. Whenever the word "genus" appears in this work, it always refers to arithmetic genus.

- A surface $\mathbb{F}_m^2$ with x glued to y. The curve $e-x-2y$ has $g=0$ according to (B.107) while it has $g=-1$ according to (B.12) which is the formula we would use in the absence of self-gluing. $e-x-2y$ appears as a gluing curve in some of our geometries.

- A surface $\mathbb{F}_0^2$ with $e-x$ glued to $e-y$. The curve $2f-x$ has $g=0$ according to (B.107) while it has $g=-1$ according to (B.12). $2f-x$ appears as a gluing curve in the gluing rules for

$$\begin{array}{ccc}\mathfrak{su}(1)^{(1)} & & \mathfrak{su}(1)^{(1)} \\ 2 & \overset{}{\text{---}_{2}\longrightarrow} & 2\end{array} \tag{B.108}$$

B.3.2 Exceptional cases

In this Appendix we study some of the exceptional cases where the methods used in chapter 4 are not applicable in a straightforward manner.

B.3.3 Gluing rules between non-gauge theoretic nodes

As we combine non-gauge theoretic nodes via edges, the prepotential $6\tilde{\mathcal{F}}$ still remains zero. Thus, another method to compute the gluing rules presented in the main body of chapter 4 is desirable. The goal of this section is to provide this alternative derivation.

Gluing rules for $\begin{array}{ccc}\mathfrak{su}(1)^{(1)} & & \mathfrak{su}(1)^{(1)} \\ 2 & \text{———} & 2\end{array}$: It is known that this KK theory is equivalent to a $5d$ $\mathfrak{su}(3)$ gauge theory with an adjoint and Chern-Simons level zero. The geometry for $\mathfrak{su}(3)$ with CS level zero is

$$1_1 \;\overset{e\qquad\quad e}{\text{————}}\; 2_1 \tag{B.109}$$

The weight system for adjoint in this phase is

$$
\begin{gathered}
(1,1)^+ \\
(-1,2)^+ \; (2,-1)^+ \\
(0,0)^+ \; (0,0)^+ \\
(1,-2)^+ \; (-2,1)^+ \\
(-1,-1)^+
\end{gathered}
$$

The weight $(-1,-1)$ can be identified with a -1 curve living in a non-compact surface and intersecting both S_1 and S_2 at one point each. Flipping the sign of this weight leads to the appearance of a blowup on both S_1 and S_2

$$1_1^1 \overset{e,x}{\text{———}} {}_2 \overset{e,x}{\text{———}} 2_1^1 \tag{B.110}$$

Notice that both the blowups are glued to each other. This can be understood as a consequence of the fact that they both correspond to the same weight i.e. $(-1,-1)^-$, but since there is a single such weight, these two curves must be identified with each other. In this flop frame, the weight system is

$$
\begin{gathered}
(1,1)^+ \\
(-1,2)^+ \; (2,-1)^+ \\
(0,0)^+ \; (0,0)^+ \\
(1,-2)^+ \; (-2,1)^+ \\
(-1,-1)^-
\end{gathered}
$$

and the curves corresponding $(-1,2)^+$ and $(-2,1)^+$ can be identified as $(f-x)_{S_1}$ and $(f-x)_{S_2}$ respectively. Flopping both of these, flips the sign of both the weights $(-1,2)$ and $(-2,1)$ and leads to the geometry

$$\overset{x}{\underset{y}{\frown}}\; 1_0^{1+1} \xrightarrow{\;e\text{-}y,\, f\text{-}x\;} {}_2 \xleftarrow{\;e\text{-}y,\, f\text{-}x\;} 2_0^{1+1} \;\overset{x}{\underset{y}{\frown}} \tag{B.111}$$

which after performing an isomorphism of both the surfaces can be written as

$$\overset{e\text{-}x}{\underset{e\text{-}y}{\frown}}\; 1_0^{1+1} \xrightarrow{\;f\text{-}x,x\;} {}_2 \xleftarrow{\;f\text{-}x,x\;} 2_0^{1+1} \;\overset{e\text{-}x}{\underset{e\text{-}y}{\frown}} \tag{B.112}$$

leading to the same gluing rules as those presented in the main text.

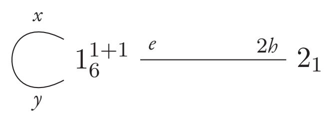
Gluing rules for $\overset{\mathfrak{su}(1)^{(1)}}{2} \longrightarrow {}_2 \longrightarrow \overset{\mathfrak{su}(1)^{(1)}}{2}$: It is known that this KK theory is equivalent to a $5d$ $\mathfrak{sp}(2)$ gauge theory with an adjoint and theta angle zero. The geometry for pure $\mathfrak{sp}(2)$ with zero theta angle is known to be

$$1_6 \overset{e \qquad\quad 2h}{\longrightarrow} 2_1 \tag{B.113}$$

The weight system for adjoint in this phase is

$$(2,0)^+$$
$$(0,1)^+$$
$$(-2,2)^+\ (2,-1)^+$$
$$(0,0)^+\ (0,0)^+$$
$$(2,-2)^+\ (-2,1)^+$$
$$(0,-1)^+$$
$$(-2,0)^+$$

Flipping the sign for $(-2,0)$ leads to the geometry



(B.114)

In this phase, the weight $(0,-1)^+$ can be identified with curves $f_1 - x$ and $f_1 - y$, along with a -1 curve z living in a non-compact surface and intersecting S_2 at one point. z is glued to $f_1 - x$ but not to $f_1 - y$. Since if it glues also to $f_1 - y$, then it would mean that $f_1 - x$ is glued to $f_1 - y$ resulting in another self-gluing of S_1, namely $f_1 - x \sim f_1 - y$. After this self-gluing, the volumes of $f_1 - x$ and $f_1 - y$ will be $\varphi_1 - \varphi_2$ leading to a contradiction with our starting step that their volume is $-\varphi_2$.

Now, to flip the sign of the weight $(0,-1)$, we have to flop $f_1 - x \sim z$ which automatically flops $f_1 - y$ since its volume is same. The flop of $f_1 - x$ creates a new blowup on S_1 that we call x'. Similarly, the flop of $f_1 - y$ creates a new blowup on S_1 that we call y'. Moreover the flop of z creates a blowup on S_2 that we call z'.

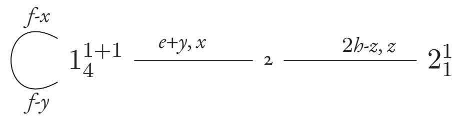
After the flop $S_1 = \mathbb{F}_4^2$ with $f_1 - x'$ glued to $f_1 - y'$ and $S_2 = \mathbb{F}_2^1$. The total gluing curve for S_2 in S_1 is $e_1 + x' + y'$, and the total gluing curve for S_1 in S_2 is $2h$. The gluing $f_1 - x \sim z$ transforms into the gluing $x' \sim z'$ in the new frame. Thus, the total gluing curve splits into two gluing curves:

$$e_1 + y' \sim 2h - z' \tag{B.115}$$

$$x' \sim z' \tag{B.116}$$

The reader can check that the curves involved on both sides in both of these gluings have same genus, and moreover (4.36) and (4.37) are satisfied for both gluings. Notice that if we would have tried to split the total gluing curve into three gluing curves e_1, x', y' glued respectively to $2h - 2z', z', z'$, we would have run into two problems. First is the same problem that we noted before the flop was performed, that this would imply a second self gluing $x' \sim y'$ of S_1 and the weight system won't match with the system of curves in the geometry anymore. Second, the genus of $2h_2 - 2z'$ is -1 and the genus of e_1 is $+1$, so the first gluing curve wouldn't make sense.

Thus at this step of the integration process, the geometry is

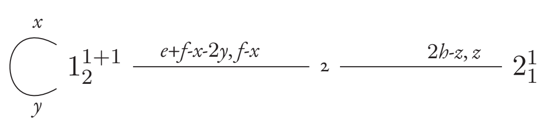


(B.117)

where we have dropped the primes on the blowups. The corresponding weight system is

$$
\begin{gathered}
(2,0)^+ \\
(0,1)^+ \\
(-2,2)^+ \ (2,-1)^+ \\
(0,0)^+ \ (0,0)^+ \\
(2,-2)^+ \ (-2,1)^+ \\
(0,-1)^- \\
(-2,0)^-
\end{gathered}
$$

By performing an isomorphism, we can write the geometry as



(B.118)

The weight $(2,-2)^+$ corresponds to the curve $x \sim y$, and the weight $(-2,1)^+$ corresponds to the curve $f_2 - z$. Upon flopping them, we obtain the geometry with adjoint matter completely integrated in

$$\underset{y}{\overset{x}{\big(}}\; 1_2^{1+1} \;\overset{e+f\text{-}y,\, f\text{-}x}{\rule{3em}{0.4pt}}\; {}_{2} \;\overset{2e+f\text{-}x\text{-}2y,\, f\text{-}x}{\rule{3em}{0.4pt}}\; 2_0^{1+1} \;\underset{y}{\overset{x}{\big)}} \tag{B.119}$$

After an isomorphism, we obtain

$$\underset{e\text{-}y}{\overset{e\text{-}x}{\big(}}\; 1_0^{1+1} \;\overset{f\text{-}x,\, x}{\rule{3em}{0.4pt}}\; {}_{2} \;\overset{2f\text{-}x,\, x}{\rule{3em}{0.4pt}}\; 2_0^{1+1} \;\underset{e\text{-}y}{\overset{e\text{-}x}{\big)}} \tag{B.120}$$

which shows that gluing rules are precisely those quoted in the main text.

Gluing rules for $\overset{\mathfrak{su}(1)^{(1)}}{2} \rule{3em}{0.4pt} \overset{\mathfrak{su}(1)^{(1)}}{\underset{\bigcirc}{2}}$: It is known that this KK theory is equivalent to a $5d$ $\mathfrak{sp}(2)$ gauge theory with an adjoint and theta angle π. Thus, the analysis for this case is similar to that of the last case which was

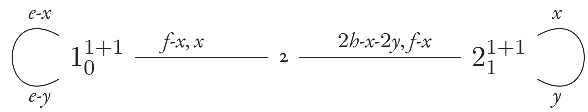
$$\overset{\mathfrak{su}(1)^{(1)}}{2} \;\text{—}\; {}_{2} \longrightarrow \overset{\mathfrak{su}(1)^{(1)}}{2} \tag{B.121}$$

since only the theta angle is different for these two cases. Following similar steps as above, the final "geometry"[7] analogous to (B.119) is found to be

$$\underset{y}{\overset{x}{\big(}}\; 1_2^{1+1} \;\overset{e+f\text{-}y,\, f\text{-}x}{\rule{3em}{0.4pt}}\; {}_{2} \;\overset{2h\text{-}x\text{-}2y,\, f\text{-}x}{\rule{3em}{0.4pt}}\; 2_1^{1+1} \;\underset{y}{\overset{x}{\big)}} \tag{B.122}$$

[7] We remind the reader that it should only be viewed as an algebraic description since the KK theory involves the non-geometric node.

which after an isomorphism becomes



(B.123)

which matches the gluing rules claimed in the text.

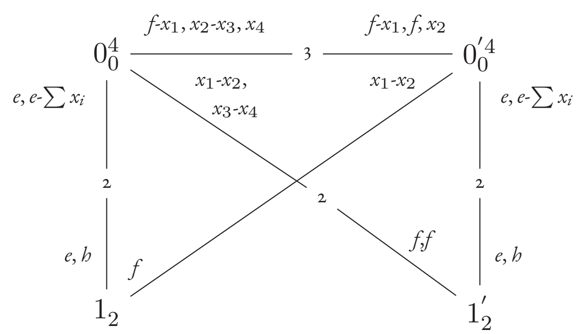
B.3.4 A concrete non-trivial check of our proposal

We devote this section to a concrete and non-trivial check of our proposal. It is known that[101] the KK theory

$$\begin{array}{ccc} \mathfrak{su}(2)^{(1)} & & \mathfrak{su}(2)^{(1)} \\ 2 & \overset{}{—\!\!—\; {\scriptstyle 2} \longrightarrow} & 2 \end{array} \tag{B.124}$$

is equivalent to the $5d$ gauge theory with gauge algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(4)$ with a hyper transforming in $\mathsf{F} \otimes \Lambda^2$. More precisely, the gauge-theoretic phase diagram for the $\mathfrak{su}(2) \oplus \mathfrak{su}(4)$ embeds into the phase diagram for the KK theory (B.124). In this section we will show this explicitly.

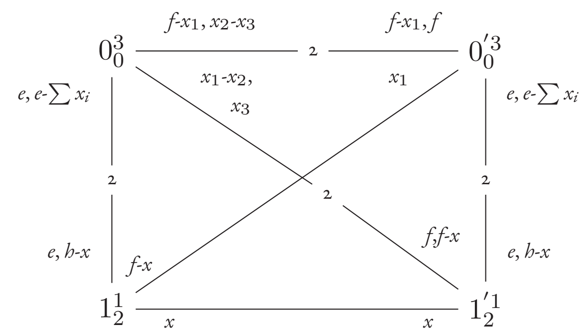
Let us start with the geometry assigned to (B.124) with ν chosen to be zero for both $\mathfrak{su}(2)^{(1)}$:



(B.125)

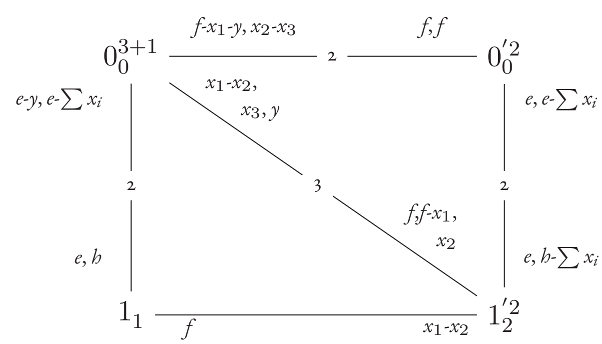
where the surfaces S_0 and S_1 correspond to the left $\mathfrak{su}(2)^{(1)}$ in (B.124), and the surfaces S'_0 and S'_1

correspond to the right $\mathfrak{su}(2)^{(1)}$ in (B.124). As visible in the above diagram, x_4 in S_0 is glued to x_2 in S_0'. Flopping this curve, we obtain



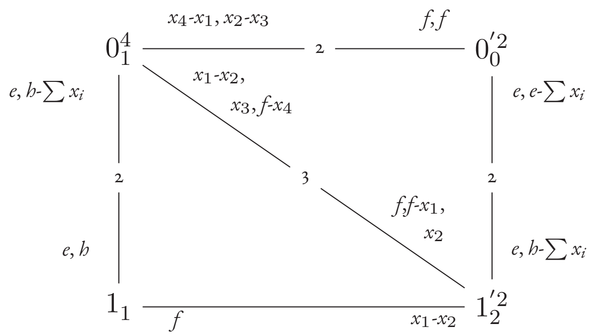
(B.126)

Now flopping $f-x$ in S_1 which is glued to x_1 in S_0', we obtain



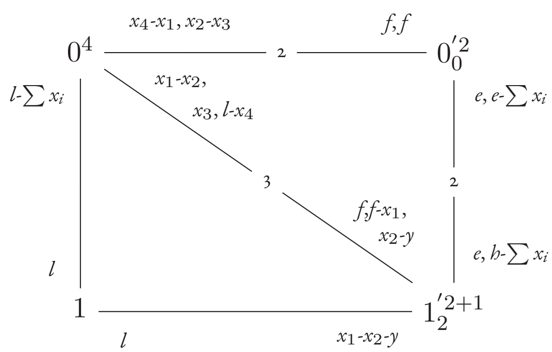
(B.127)

which after performing an isomorphism on S_0 can be written as



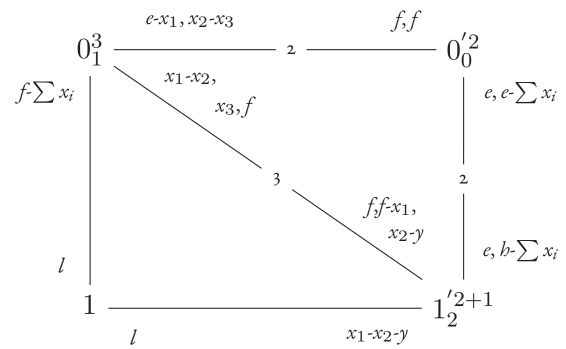
(B.128)

Now, flopping the e curves inside S_0 and S_1 (which are glued to each other), we obtain



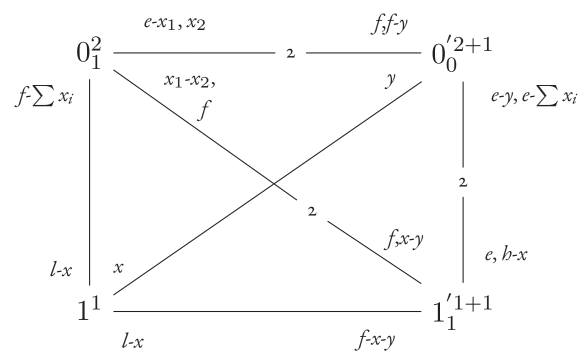
(B.129)

where a surface without a subscript denotes that the surface is a del Pezzo surface rather than a Hirzebruch surface. That is, $S_0 = dP_4$ and $S_1 = \mathbb{P}^2 = dP_0$. Let us use the blowup x_4 on S_0 to write S_0 in terms of the Hirzebruch surface $\mathbb{F}_1$



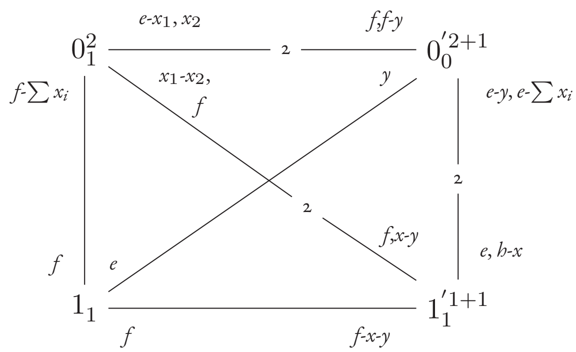
(B.130)

Flopping x_3 in S_0 glued to $f - x_1$ in S_1' gives rise to



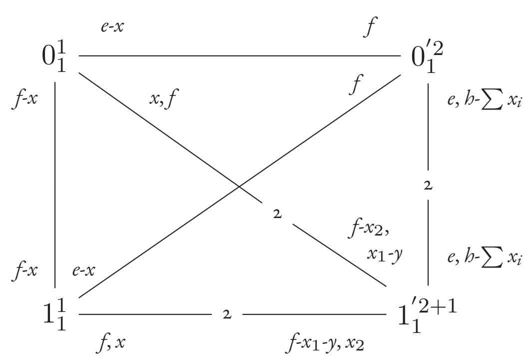
(B.131)

We use x in S_1 to write S_1 in terms of Hirzebruch surface $\mathbb{F}_1$



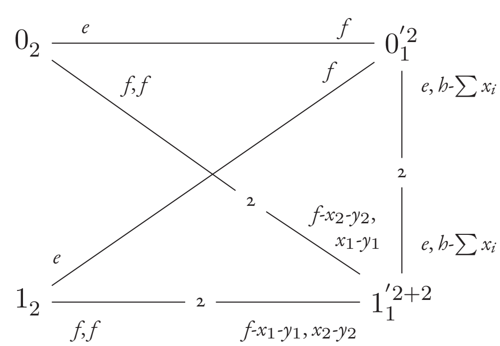
(B.132)

Flop x_2 in S_0 glued to $f-y$ in S_0' to obtain



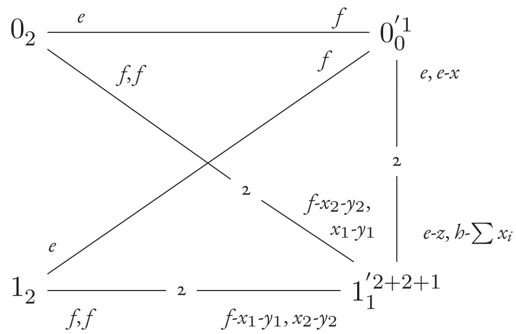
(B.133)

Now flopping $f-x$ in S_0 glued to $f-x$ in S_1, we obtain



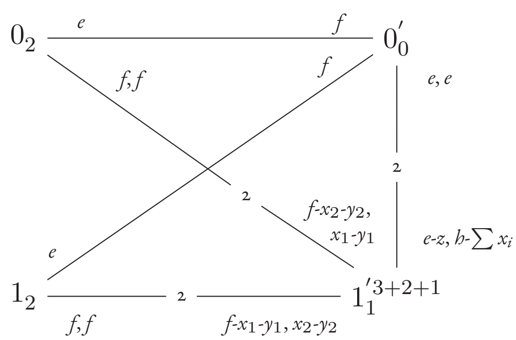
(B.134)

Flopping $f - x_2$ in S_0' we obtain



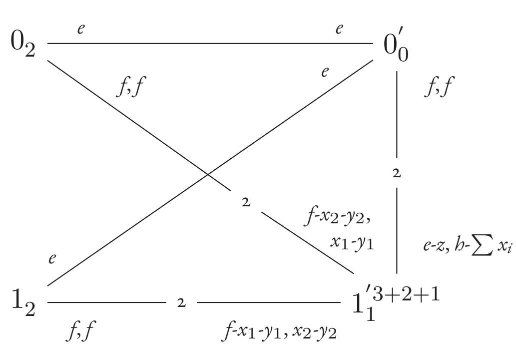
(B.135)

Now flopping x in S_0' we get



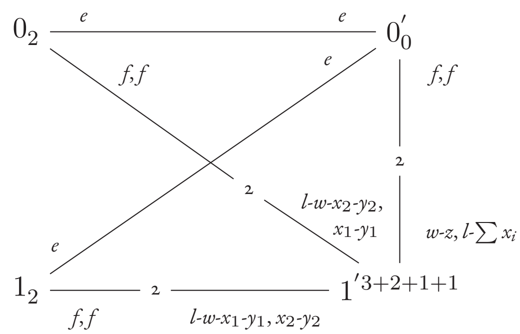
(B.136)

Performing the automorphism on S_0' that exchanges e and f, we obtain



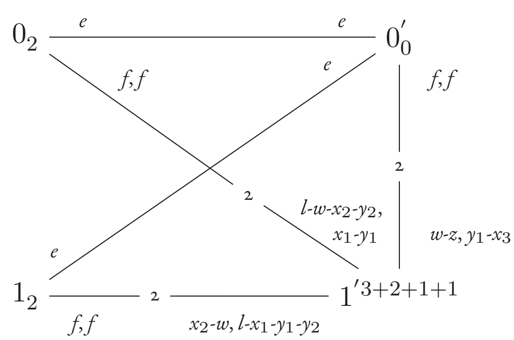
(B.137)

Now let us write S_1' as a del Pezzo surface. This rewrites the e curve as a blowup which we denote by w



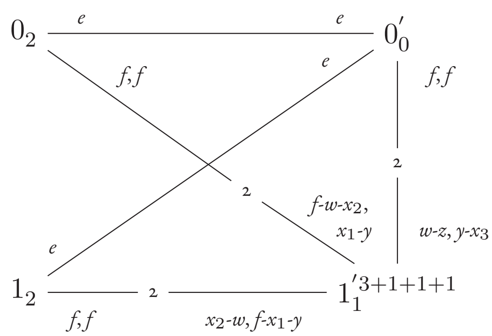
(B.138)

We can now perform a basic automorphism (of del Pezzo surfaces) on S_1' involving the three blowups x_1, x_2 and y_1 to obtain



(B.139)

Converting S_1' back into $\mathbb{F}_1$ using the blowup y_2, we obtain



(B.140)

This is the final form of the geometry that we wanted to obtain.

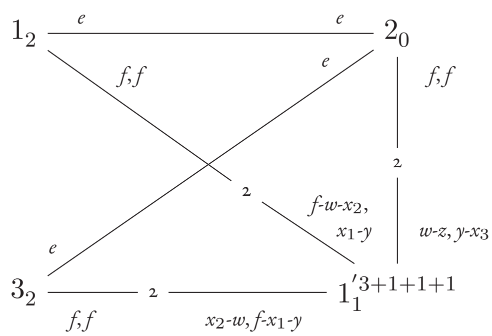
It is clear that S_0, S_0' and S_1 describe an $\mathfrak{su}(4)$ and S_1' describes an $\mathfrak{su}(2)$ in (B.140). This can be checked by intersecting the fibers of the corresponding Hirzebruch surfaces with these surfaces. The intersection matrix yields the Cartan matrix for $\mathfrak{su}(4) \oplus \mathfrak{su}(2)$. Now, let us show that the configuration of blowups indeed describes $\Lambda^2 \otimes \mathsf{F}$ of $\mathfrak{su}(4) \oplus \mathfrak{su}(2)$. For this we relabel the surfaces as

$$S_0 \to S_1 \tag{B.141}$$

$$S_0' \to S_2 \tag{B.142}$$

$$S_1 \to S_3 \tag{B.143}$$

thus rewriting the geometry as



(B.144)

The weight system for $\Lambda^2 \otimes \mathsf{F}$ can be written as

$$\begin{gathered}(0,1,0|1)\\(1,-1,1|1)\;(0,1,0|-1)\\(-1,0,1|1)\;(1,0,-1|1)\;(1,-1,1|-1)\\(-1,1,-1|1)\;(-1,0,1|-1)\;(1,0,-1|-1)\\(0,-1,0|1)\;(-1,1,-1|-1)\\(0,-1,0|-1)\end{gathered}$$

where the three entries on the left hand side of slash denote the weights with respect to $\mathfrak{su}(4)$ comprised by S_1, S_2 and S_3, and the entry on the right hand side of slash denotes the weight with respect to $\mathfrak{su}(2)$ comprised by S_1'.

From the geometry (B.144) we see that the holomorphic curves

$$\mathrm{vol}(x_1) = (1, 0, -1|1) \tag{B.145}$$

$$\mathrm{vol}(x_2) = (-1, 0, 1|1) \tag{B.146}$$

$$\mathrm{vol}(x_3) = (0, -1, 0|1) \tag{B.147}$$

$$\mathrm{vol}(y) = (-1, 1, -1|1) \tag{B.148}$$

$$\mathrm{vol}(f - z) = (0, 1, 0|1) \tag{B.149}$$

$$\mathrm{vol}(f - w) = (1, -1, 1|1) \tag{B.150}$$

match weights of the form $(x, y, z|1)$, and the antiholomorphic curves $x_1 - f, x_2 - f, x_3 - f, y - f, -z, -w$ match weights of the form $(x, y, z| - 1)$, where f denotes the fiber of Hirzebruch surface $S_1' = \mathbb{F}_1^6$. Thus we have reproduced the full weight system for $\Lambda^2 \otimes \mathsf{F}$, justifying our claim. More precisely, the geometry (B.144) describes the $\mathfrak{su}(4) \oplus \mathfrak{su}(2)$ gauge theory in the gauge-theoretic phase given by the following virtual volumes

$$\begin{gathered}
(0, 1, 0|1)^+ \\
(1, -1, 1|1)^+ \; (0, 1, 0| - 1)^- \\
(-1, 0, 1|1)^+ \; (1, 0, -1|1)^+ \; (1, -1, 1| - 1)^- \\
(-1, 1, -1|1)^+ \; (-1, 0, 1| - 1)^- \; (1, 0, -1| - 1)^- \\
(0, -1, 0|1)^+ \; (-1, 1, -1| - 1)^- \\
(0, -1, 0| - 1)^-
\end{gathered}$$

B.3.5 Comparisons with known cases in the literature

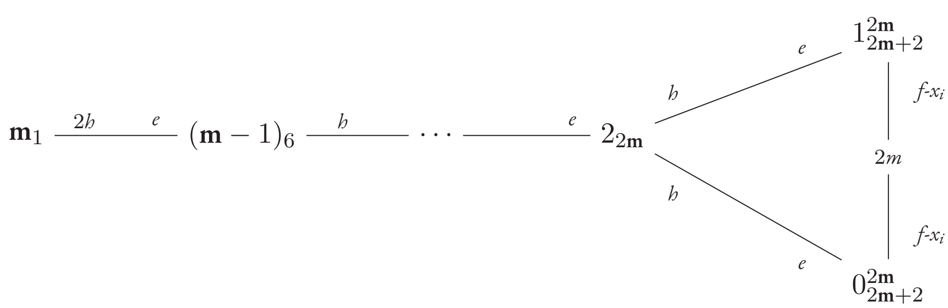
In this section we provide a comparison with some $5d$ KK theories known in the literature via other methods. In particular, we show that the geometries we obtain for these $5d$ KK theories allow us to see the $5d$ gauge theory descriptions of these $5d$ KK theories that have been proposed in the literature.

B.3.6 Twisted

Now, let us consider an example when we twist by an outer automorphism. It has been proposed in [101] that

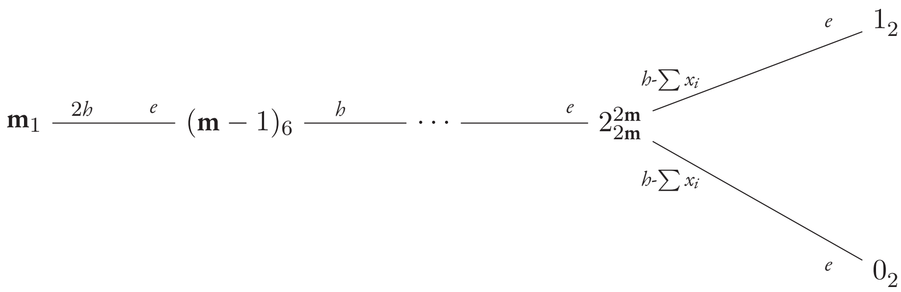
$$\underset{2}{\mathfrak{su}(n)^{(2)}} \tag{B.151}$$

can be described by $5d$ gauge theory with gauge algebra $\mathfrak{so}(n+2)$ and n fundamental hypers. First let us consider the case when $n = 2m$. In this case the geometry is displayed in (4.84). Flopping all the y_i, we obtain



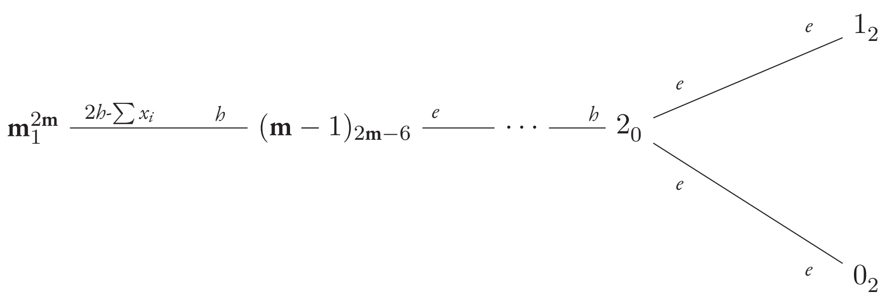
(B.152)

Now flopping all the $f - x_i$, we obtain



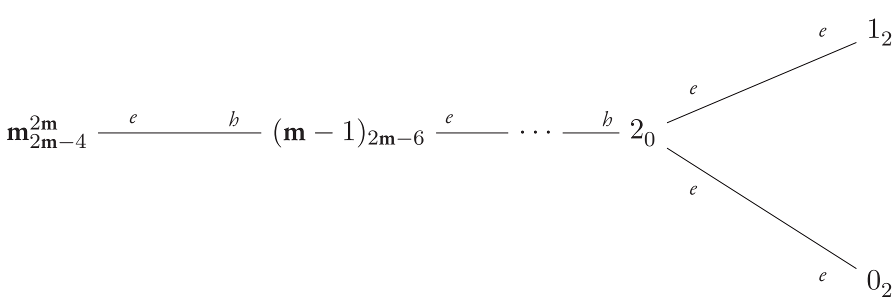
(B.153)

Now we can carry the $2m$ blowups onto S_m to obtain the geometry



(B.154)

which after an isomorphism on S_m can be rewritten as



(B.155)

The Cartan matrix associated to this geometry is indeed that for $\mathfrak{so}(2m+2)$ and the $2m$ blowups can be identified as $2m$ hypers in fundamental of $\mathfrak{so}(2m+2)$.

Similarly, the geometry for $n = 2m+1$ is given in (4.85). Flopping $x_i \sim y_i$ living on S_0, we obtain

$$\mathbf{m}_1 \overset{2h \qquad e}{\text{———}} (\mathbf{m}-1)_6 \overset{h}{\text{——}} \cdots \overset{e}{\text{——}} 1^{\mathbf{2m+1}}_{\mathbf{2m+2}} \overset{2h\text{-}2\sum x_i \qquad\qquad e}{\text{——————}} 0_6 \tag{B.156}$$

After performing an isomorphism we can write the above geometry as

$$\mathbf{m}_1 \overset{2h \qquad\qquad e}{\text{————}} (\mathbf{m}-1)_6 \overset{h}{\text{——}} \cdots \overset{e\text{-}\sum x_i}{\text{———}} 1^{\mathbf{2m+1}}_1 \overset{2h \qquad\qquad e}{\text{————}} 0_6 \tag{B.157}$$

Now moving the blowups onto S_m we obtain

$$\mathbf{m}^{\mathbf{2m+1}}_1 \overset{2h\text{-}\sum x_i \qquad e}{\text{————}} (\mathbf{m}-1)_{\mathbf{2m-5}} \overset{h}{\text{——}} \cdots \overset{e}{\text{——}} 1_1 \overset{2h \qquad\qquad e}{\text{————}} 0_6 \tag{B.158}$$

which can be rewritten as

$$\mathbf{m}^{\mathbf{2m+1}}_{\mathbf{2m-3}} \overset{e \qquad\quad h}{\text{————}} (\mathbf{m}-1)_{\mathbf{2m-5}} \overset{e}{\text{——}} \cdots \overset{e}{\text{——}} 1_1 \overset{2h \qquad\qquad e}{\text{————}} 0_6 \tag{B.159}$$

which precisely describes $\mathfrak{so}(2m+3)$ with $2m+1$ hypers in fundamental of $\mathfrak{so}(2m+3)$.

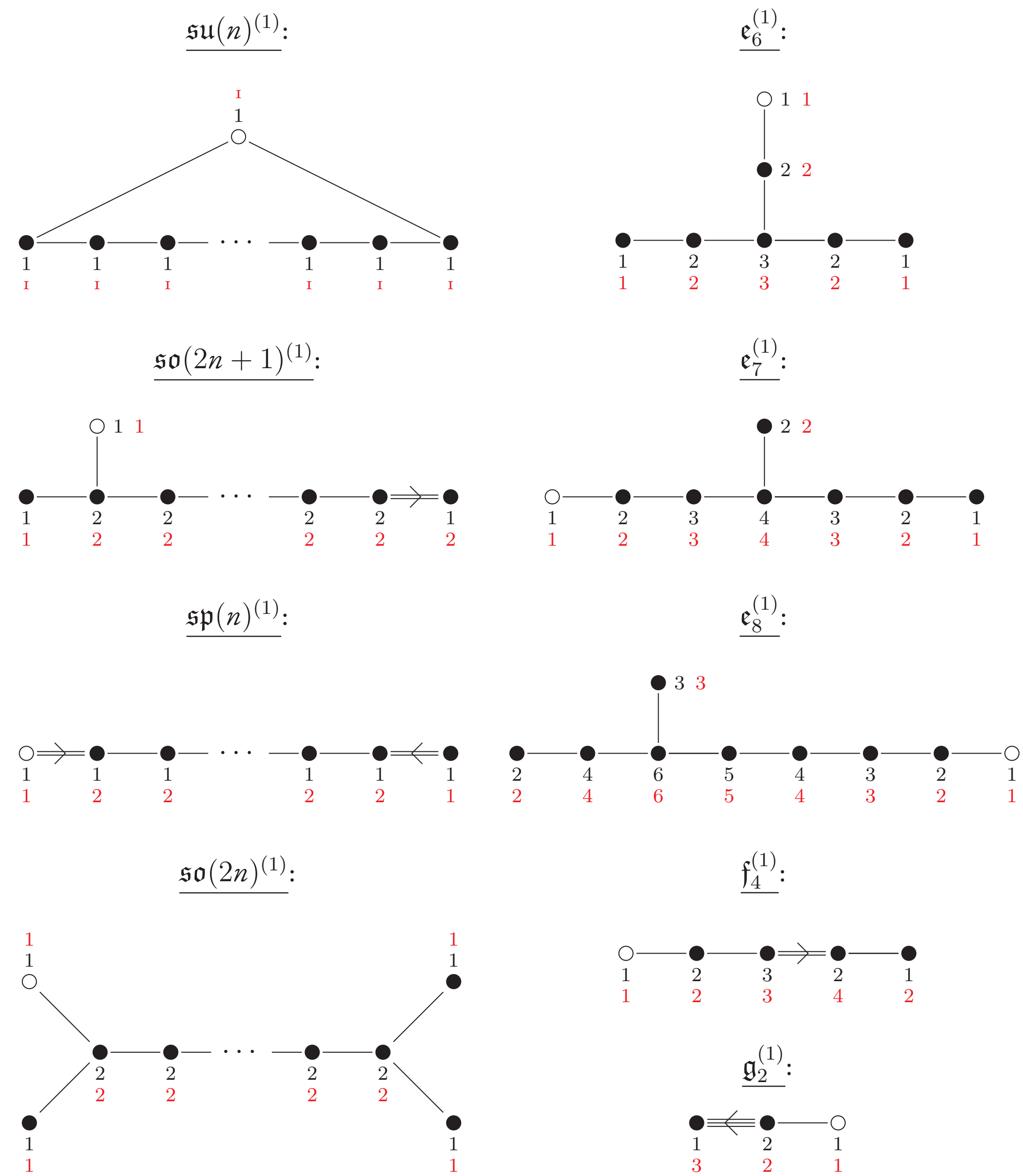


Table B.2: Untwisted affine Lie algebras. The affine node is shown as a hollow circle. The numbers in black $d_a^\vee$ denote the column null vector for the Cartan matrix, popularly known as dual Coxeter labels. The numbers in red d_a denote the row null vector for the Cartan matrix, popularly known as Coxeter labels.

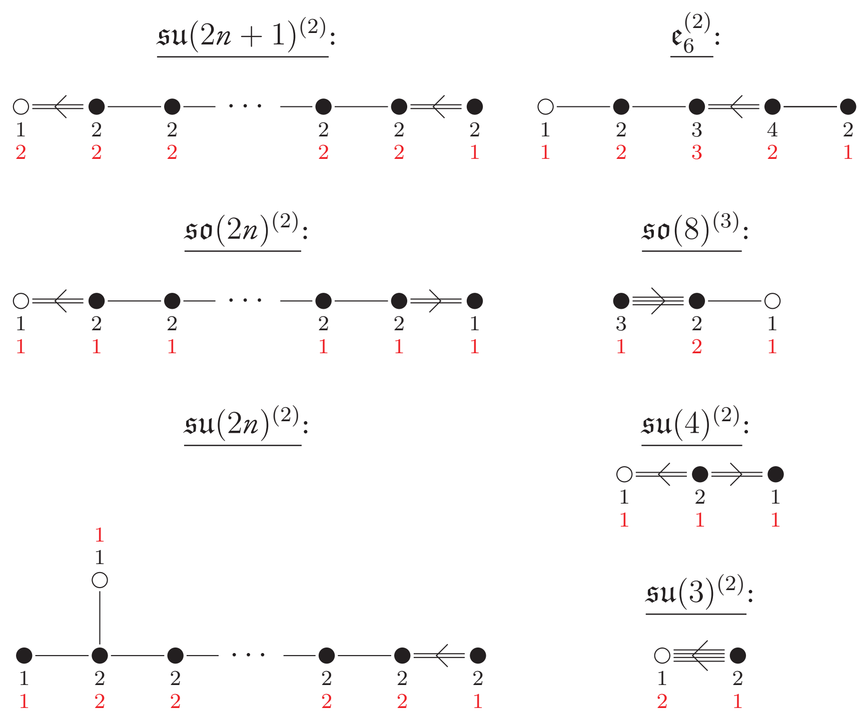


Table B.3: Twisted affine Lie algebras. The affine node is shown as a hollow circle. The numbers in black $d_a^\vee$ denote the column null vector for the Cartan matrix, popularly known as dual Coxeter labels. The numbers in red d_a denote the row null vector for the Cartan matrix, popularly known as Coxeter labels. The total number of nodes for $\mathfrak{su}(2n+1)^{(2)}$ is $n+1$, for $\mathfrak{so}(2n)^{(2)}$ is n, and for $\mathfrak{su}(2n)^{(2)}$ is $n+1$.

B.4 Lie Algebra

B.4.1 The genus one fibration

For each α, combining the fibers $f_{a,\alpha}$, let us define a fiber f_α via

$$f_\alpha = d_a f_{a,\alpha} \tag{B.160}$$

where d_a are Coxeter labels for $\mathfrak{g}_\alpha^{(q_\alpha)}$ listed (in red color) in Tables B.2 and B.3. If $\mathfrak{g}_\alpha$ is trivial, then $d_0 := 1$.

We claim that f_α is a genus one fiber. This means that f_α can be obtained by a degeneration of a torus. It is well-known that torus fibers can degenerate into Kodaira fibers, which are collections of rational curves[8] intersecting in the pattern of untwisted affine Dynkin diagrams of type $\mathfrak{su}(n)^{(1)}$, $\mathfrak{so}(2n)^{(1)}$ and $\mathfrak{e}_n^{(1)}$. The multiplicity of each rational component curve is given by the Coxeter label for the corresponding node in the affine Dynkin diagram. The fiber f_α, on the other hand, is composed of rational curves $f_{a,\alpha}$ with their multiplicity given by the Coxeter labels for affine Dynkin diagram $\mathfrak{g}_\alpha^{(q_\alpha)}$. Now, one can notice that every affine Dynkin diagram can be obtained by folding

[8] This means they have genus zero.

affine Dynkin diagrams of type $\mathfrak{su}(n)^{(1)}$, $\mathfrak{so}(2n)^{(1)}$ and $\mathfrak{e}_n^{(1)}$ as follows:

$$\mathfrak{so}(2n)^{(1)} \to \mathfrak{so}(2n-1)^{(1)} \to \mathfrak{so}(2n-2)^{(2)} \tag{B.161}$$

$$\mathfrak{e}_6^{(1)} \to \mathfrak{f}_4^{(1)} \to \mathfrak{so}(8)^{(3)} \tag{B.162}$$

$$\mathfrak{so}(8)^{(1)} \to \mathfrak{so}(7)^{(1)} \to \mathfrak{g}_2^{(1)} \tag{B.163}$$

$$\mathfrak{so}(4n)^{(1)} \to \mathfrak{su}(2n)^{(2)} \to \mathfrak{su}(2n-1)^{(2)} \tag{B.164}$$

$$\mathfrak{so}(8)^{(1)} \to \mathfrak{so}(7)^{(1)} \to \mathfrak{su}(4)^{(2)} \to \mathfrak{su}(3)^{(2)} \tag{B.165}$$

$$\mathfrak{e}_7^{(1)} \to \mathfrak{e}_6^{(2)} \tag{B.166}$$

Moreover, observe that the Coxeter numbers of two nodes are added if they are identified under gluing. This means that f_α can be obtained by identifying the rational components of the Kodaira fibers according to the above folding rules. This explicitly shows that f_α is a genus one fiber.

Moreover, we find that due to the virtue of gluing rules, f_α is glued to f_β as

$$q_\alpha(-\Omega_{\beta\alpha})f_\alpha \sim q_\beta(-\Omega_{\alpha\beta})f_\beta \tag{B.167}$$

This generalizes the condition in the untwisted unfrozen case[28] where $f_i \sim f_j$ whenever there is an edge between i and j in $\Sigma_{\mathfrak{T}}$. This shows that certain multiples of genus one fibers are identified with each other as one passes over from one collection of surfaces to another, allowing us to extend the fibration structure consistently throughout the threefold.

More formally, according to a theorem due to Oguiso and Wilson[153,188], a threefold X admits an genus one fibration structure if and only if there exists an effective divisor S_{T^2} satisfying

$$S_{T^2} \cdot S_{T^2} \cdot S_{T^2} = 0, \;\; S_{T^2} \cdot S_{T^2} \neq 0 \tag{B.168}$$

where S_{T^2} lives in the extended Kähler cone, possibly on the boundary. The extended Kähler cone is parameterized by all the Coulomb branch and mass parameters satisfying

$$J \cdot C \geq 0 \tag{B.169}$$

for all holomorphic curves C in X. Physically, the extended Kähler cone corresponds to the Coulomb branch of the (possibly mass deformed) $5d$ theory corresponding to X.

In all of geometries associated to $5d$ KK theories, we can find an S_{T^2} which lies in the extended Kähler cone satisfies (B.168). Pick any node α and define

$$S_{T^2} := \sum_{a=0}^{r_\alpha} d_a^\vee S_{a,\alpha} \tag{B.170}$$

where $d_a^\vee$ are dual Coxeter labels for the associated affine algebra $\mathfrak{g}_\alpha^{(q_\alpha)}$ (see Tables B.2 and B.3) and r_α is the rank of invariant subalgebra $\mathfrak{h}_\alpha$. If the node α carries a trivial gauge algebra, then we define $d_0^\vee = 1$ and take (B.170) to be the definition of S_{T^2}.

In the gauge theoretic case, the direction parametrized by (B.170) is special since all the fibers $f_{a,\alpha}$ have zero volume along this direction[9]

$$-S_{T^2} \cdot f_{a,\alpha} = \sum_b A_{ab} d_b^\vee = 0 \tag{B.171}$$

Similarly, in the non-gauge theoretic case

$$-S_{T^2} \cdot f_{0,\alpha} = -K'_{0,\alpha} \cdot f_{0,\alpha} = 0 \tag{B.172}$$

where the last equality can be checked to be true for every non-gauge theoretic case. Moreover, the

[9] In fact, non-negativity of the volumes of fibers implies that the only directions in the Coulomb branch when mass parameters are turned off are given by $\sum_a d_a^\vee S_{a,\alpha}$ for various α.

reader can check using the explicit description of geometries presented in this chapter 4 that

$$S_{T^2} \cdot C \geq 0 \tag{B.173}$$

for all other holomorphic C in the threefold $X_{S,\{q_\alpha\}}$. So, S_{T^2} as defiend in (B.170) lies in the extended Kähler cone of $X_{S,\{q_\alpha\}}$.

Now it can be easily checked for all the geometries presented in this work that

$$S_{T^2} \cdot S_{T^2} = -q_\alpha \Omega^{\alpha\alpha} \sum_{a=0}^{r_\alpha} (d_a f_{a,\alpha}) \neq 0 \tag{B.174}$$

where d_a are the Coxeter labels for $\mathfrak{g}_\alpha^{(q_\alpha)}$ with $d_0 := 1$ if α is a non-gauge theoretic node. We can now compute

$$S_{T^2} \cdot S_{T^2} \cdot S_{T^2} \propto \sum_{a=0}^{r_\alpha} (d_a f_{a,\alpha}) \cdot (\sum_{b=0}^{r_\alpha} d_b^\vee S_{b,\alpha}) = -\sum_{a,b=0}^{r_\alpha} d_a A_{ab} d_b^\vee = 0 \tag{B.175}$$

thus verifying both the conditions in (B.168) and establishing the presence of a genus one fibration in $X_{S,\{q_\alpha\}}$.

C

6d Supergravity

C.1 Infinite Families

The objective of section 3.1 is to show that there are only finitely many massless modes for a 6d $\mathcal{N} = 1$ theory and hence it is important to understand the possible infinite families that could occur. As was discussed in Section 3.1, the only theories with one simple gauge factor and unbounded size are presented in the first two rows of Table 3.1. However, they were both shown to be finite because of the existence of non-unitary BPS strings for arbitrarily large size. Therefore, of particular interest will be theories with multiple simple gauge factors drawn from Table 3.1 for which one can reduce $H - V$ by gauging matter.

Specifically, we will argue that the only matter that can be gauged is fundamental matter. This fact was shown in [124] to be true for $T < 9$ and can be generalized for any T. By considering the group theory coefficients A_R presented in [127] one can note that all representations except the fundamental contribute to $b_i \cdot b_j$ at least linear in N. However, as was discussed extensively in Section 3.1 b_i vectors belong to the string charge lattice and consequently are independent of N. Alternatively, one can also note from Table 3.1 that this would not be possible for the specific theories. For example, consider the gauge group $G_N \times G_M$ with gauge factors picked from Table 3.1 with matter of the form (R_N, R_M) charged under both gauge groups in representations R_N, R_M respectively. Assume now that at least one of representations is not fundamental and hence the only choices are symmetric /antisymmetric/adjoint. We may assume that R_M is such a representation and hence $(R_N, R_M) = (R_N, \frac{M(M\pm1)}{2}$ or $\mathbf{Adj})$. In order to form such a representation $SU(M)$ needs to have at least $\dim(R_N)$ representations in $M(M\pm1)/2$ or $\mathbf{Adj}$ but that is not possible since there is only a finite number of those for each theory. Therefore, we conclude that any matter charged under more than one gauge groups necessarily includes gauging of fundamental matter.

Furthermore, one should note from Table 3.1 that only matter charged under at most two groups can appear. For example, let us consider a theory of the form $SU(N) \times SU(M) \times SU(K)$, even though trifundamental matter is possible for finite N, M, K as shown in [125], it is not possible to construct it for unbounded size. This is because there needs to exist sufficiently many bifundamentals between every pair of simple factors and the theories we are considering for arbitrary size do not. Alternatively, one could note that for example a trifundamental of $SU(N) \times SU(M) \times SU(K)$ would require at least K bifundamentals (N, M) but in this case $b_i \cdot b_j$ would also grow with K which as discussed earlier is not consistent.

We also note that $SU(N)/Sp(N)$ with $16\,\square + 2/1\,\begin{smallmatrix}\square\\\square\end{smallmatrix}$ or any other group of finite size can not be a factor because gauging bifundamental matter between unbounded size gauge groups requires also an unbounded number of hypermultiplets in the fundamental representation. Therefore, we proceed by considering the possible ways three gauge groups from Table 3.1 can combine and produce a theory with unbounded size not restricted by anomaly cancellation.

- Firstly, consider matter charged as $(N, M, 1) + (1, M, K) + (N, 1, K)$.

  This type of matter is only possible if the inequalities $F_N \geq M + K, F_M \geq N + K, F_K \geq N + M$ are satisfied, where F_N, F_M, F_K represent the number of hypermultiplets in the fundamental representation. From Table 3.1 we know that there are three choices of each F_N, F_M, F_K e.g. $F_N = N - 8, N + 8, 2N$. Starting from the first possibility the above inequalities become:

$$F_N = N - 8 \geq M + K \tag{C.1}$$

  and

$$M + 2K + 8 \leq N + K \leq F_M = 2M \tag{C.2}$$

  with $F_M = 2M$ being the only consistent choice for $K > 0$. The third inequality becomes:

$$2M + K + 8 \leq N + M \leq F_K = 2K \tag{C.3}$$

  again with $F_K = 2K$ being the only consistent choice.

  Note that these inequalities can not be satisfied simultaneously for any combination of

N, M, K for large values because the combination of the last two implies:

$$N \leq \frac{K+M}{2} \tag{C.4}$$

While the first

$$N \geq M + K + 8 \tag{C.5}$$

One could also try $F_N = N + 8$ which leads to $F_K = 2K, F_M = 2M$ for large N, M, K and hence one would need to satisfy $N \leq \frac{K+M}{2}, N \geq M + K - 8$ for unbounded N, M, K which is impossible.

Lastly, one can choose $F_N = 2N$ which forces $F_K = 2K, F_M = 2M$ and substituting this in the above inequalities imply that $N = M = K$. In particular, this theory has gravitational anomaly $H - V = 3N^2 - 3(N^2 - 1) = 3$ and hence $T \leq 9$. More generally, we could consider such a loop for arbitrary number of factors $SU(N)^k$ which has $H - V = kN^2 - kN^2 + k = k$ and hence $k \leq 12$, $T \leq 9$.

- We now move on to charged matter of the form: $(N, M, 1) + (1, M, K)$

  Lets us consider the three simple gauge factors $G_N \times G_M \times G_K$ with each component drawn from Table 3.1. Then assuming that no matter is charged under more than one gauge group we have that the leading contribution to $H - V$ is given by $c_1 N^2 + c_2 M^2 + c_3 K^2$ with $c_i = \frac{1}{2}$ or 1 depending on the matter. Therefore, in order to make $H - V$ finite we can gauge matter in bifundamental representations which may eliminate the quadratic leading behavior of the contribution to the gravitational anomaly. As we did before the number of fundamental hypermultiplets F_N, F_M, F_K needs to satisfy $F_N \geq M, F_M \geq N + K, F_K \geq M$ with the leading behaviour $F_N \sim 2c_1 N, F_M \sim 2c_2 M, F_K \sim 2c_3 K$ and hence $2c_1 N \pm 2(1 -$

$c_1)8 \geq M, \quad 2c_2M \pm 2(1-c_2)8 \geq N+K, \quad 2c_3K \pm 2(1-c_3)8 \geq M$. Therefore, for large values of M, N, K we have the leading terms $M \sim 2c_1\alpha N, \quad K \sim 4c_1c_2\alpha\beta N$ with $\alpha \leq 1, \beta < 1$ and such that they satisfy the above inequalities. This implies that the contribution for each bifundamental is $2c_1\alpha N^2 + 8c_1^2c_2\alpha^2\beta N^2$ and hence this is the amount subtracted from $H-V$ when gauging F_N, F_M, F_K. We thus require that $c_1N^2 + c_2M^2 + c_3K^2 = c_1N^2 + c_2(2c_1a)^2N^2 + 16c_3(c_1c_2\alpha\beta)^2N^2 = 2c_1\alpha N^2 + 8\alpha^2c_1^2c_2\beta N^2$ in order to cancel the leading N behavior of the gravitational anomaly. The equation $N^2 + 4c_2c_1\alpha^2N^2 + 16c_3c_1c_2^2\alpha^2\beta^2N^2 = 2\alpha N^2 + 8\alpha^2c_1c_2\beta N^2$ has the following solutions that eliminate the quadratic behavior from $H-V$: $(\alpha, \beta, c_1, c_2, c_3) = (1, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}), (1, \frac{1}{2}, 1, 1, \frac{1}{2})$.

Starting from the first solutions and enforcing $F_N \geq M, F_M \geq N+K, F_K \geq M$ we can find the consistent solutions. The possibilities as indicated from $(\alpha, \beta, c_1, c_2, c_3) = (1, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2})$ are : $F_N = N-8, N+8, \quad F_M = 2M = 2N, \quad F_K = N-8, \quad N+8$. The first case requires $F_N = N-8 \geq M, M+K+8 \leq N+K \leq 2M$ and $K+8 \leq M \leq F_K$ we have $F_K = K+8$.

The second choice of F_N is similar to above with $N+8 \geq M, M+K-8 \leq N+K \leq 2M$ and $K-8 \leq M \leq F_K$. Consistency then requires $F_K = K-8$ (which is identical to the previous case)or $F_K = K+8$.

The second category with $(\alpha, \beta, c_1, c_2, c_3) = (1, \frac{1}{2}, 1, 1, \frac{1}{2})$ requires $2N \geq M, M \leq 2K$, $M \leq N+K \leq F_M$ with $F_M = 2M$.

All the theories that satisfy those conditions are summarized in Table C.1. The last column indicates the contribution to $H-V$ of each theory, one can see that although the quadratic terms are eliminated there can still be linear terms. This could have also been deduced by looking at each individual contribution of $H-V$ from Table 3.1 and noticing that the theories with $c_i = 1/2$ need to come in pairs such that the linear contributions cancel because gauging bifundamental matter does not affect the linear terms as long as the size is not restricted to be even. Therefore, only four theories do not have their dimensions restricted by anomalies.

$SU(N+8) \times SU(N) \times SU(N-8)$	(□,□,1) +(1,□,□)+(□□,1,1) +(1,1,⊟)	-53
$SU(N-8) \times SU(N) \times SU(N-8)$	(□,□,1) +(1,□,□)+(⊟,1,1) +(1,1,⊟)+16(1,□,1)	$15N-53$
$SU(N-8) \times SU(N) \times SU(N+8)$	(□,□,1) +(1,□,□)+(⊟,1,1) +(1,1,⊟)+16(1,1,□)	$15N+67$
$SU(N) \times SU(2N) \times SU(N)$	(□,□,1) +(1,□,□) +8 (1,□,1)+(1,⊟,1)	$15N-3$
$SU(N) \times SU(2N) \times SU(N+8)$	(□, □, 1) +(1, □, □) +(1,⊟,1) +16(1, 1, □)	$15N+61$
$SO(M+8) \times SU(M) \times SU(M-8)$	(□, □, 1) +(1, □, □) +(1,1,⊟)	-58
$SU(M+8) \times SU(M) \times Sp((M-8)/2)$	(□, □, 1) +(1, □, □) +(□□,1,1)	-58
$SO(M+8) \times SU(M) \times Sp((M-8)/2)$	(□, □, 1) +(1, □, □)	-57
$Sp((M-8)/2) \times SU(M) \times Sp((M-8)/2)$	(□, □, 1) +(1, □, □) +16 (1, □, 1)	$15N-57$
$SU(N) \times Sp(M) \times SU(N)$	(□, □, 1) +(1, □, □) +8 (1, □, 1)	$15N-2$

Table C.1: Theories with three simple gauge factors and $H-V$ at most linear in N.

In a similar way we can extend this analysis to more than 3 gauge factors. We may start by considering a linear chain of gauge groups of the form $G_1 \times G_2 \times \cdots \times G_k$ with bifundamental matter charged under every adjacent pair of groups G_i each of size $N_i = a_i N + c_i$. Let us start by considering G_i with matter in $F_{N_i} = N_i \pm 8$. Then the size of each adjacent gauge group is bounded as: $F_{N_i} \geq N_{i+1} + N_{i-1}$ which implies that $a_i N + c_i \pm 8 \geq a_{i-1}N + a_{i+1}N + c_{i+1} + c_{i-1}$. For large N this inequality can be translated to keeping only the linear terms in N given by:

$$a_i N \geq a_{i-1} N + a_{i+1} N \tag{C.6}$$

Assuming the same type of matter for the adjacent groups we have

$$a_{i+1}N \geq a_iN + a_{i+2}N,\ a_{i-1}N \geq a_iN + a_{i-2}N \tag{C.7}$$

Adding the first inequality and the last two gives:

$$0 \geq a_i + a_{i+2} + a_{i-2} \tag{C.8}$$

which can not be satisfied for positive a_i's. We can instead consider a different type of matter for one of the gauge groups($F_{i+1} = 2N_{i+1}$) satisfying:

$$2a_{i+1}N \geq a_iN + a_{i+2}N,\ a_{i-1}N \geq a_iN + a_{i-2}N \tag{C.9}$$

Combining these equations gives:

$$0 \geq \frac{1}{2}a_i + \frac{1}{2}a_{i+2} + a_{i-2} \tag{C.10}$$

which is also not satisfied for positive a_i's.

Next we can consider also $F_{i-1} = 2N_{i-1}$ which needs the following inequalities to be satisfied:

$$2a_{i+1}N \geq a_iN + a_{i+2}N, 2a_{i-1}N \geq a_iN + a_{i-2}N \tag{C.11}$$

Which can combine to:

$$0 \geq \frac{1}{2}a_{i+2} + \frac{1}{2}a_{i-2} \tag{C.12}$$

Similarly, if instead one had: $F_{i-1} = N_{i-1}, F_{i+1} = N_{i+1}, F_i = 2N_i$ then the following inequalities need to be satisfied:

$$2a_iN \geq a_{i-1}N + a_{i+1}N,\ a_{i+1}N \geq a_iN + a_{i+2}N,\ a_{i-1}N \geq a_iN + a_{i-2}N \qquad \text{(C.13)}$$

And hence

$$0 \geq a_{i+2} + a_{i-2} \qquad \text{(C.14)}$$

as above, which can not be satisfied for theories with more than three gauge groups. However, if the chain has only three gauge groups $a_{i+2}, a_{i-2} = 0$ and hence the C.14, C.12 inequalities are satisfied. In particular, both theories were found earlier in Table C.1 but the latter had $H - V$ linear in N while the former had $H - V$ constant as desired. Therefore, any infinite family should only have gauge groups away from the edges of type $SU(N) + 2N\Box$. However, even though the other simple gauge factors do not appear away from the edges of the chain, there is nothing wrong with them being the first and last factors. For example, if G_i was the first gauge group then $a_{i-1} = a_{i-2} = 0$ and hence the inequalities become: $a_i \geq a_{i+1}, 2a_{i+1} \geq a_i + a_{i+2}$ which can be solved.

Lastly, let us look at theories with more than three gauge groups starting with $SU(N) + 2N\Box$. In this case we require the following inequalities to be satisfied for the $SU(N) \times SU(a_1N + c_1) \times \cdots SU(a_kN + c_k)$ gauge group:

$$2 \geq a_1, 2a_1 \geq 1 + a_2,\ 2a_2 \geq a_1 + a_3, \cdots, 2a_{k-1} \geq a_k + a_{k-2}, F_K \geq a_{k-1}N \qquad \text{(C.15)}$$

We can start by investigating the different possible solutions for these inequalities. Let $a_1 = 1$ then there is a unique choice for all $a_i = 1$ but there are N fundamental hypermultiplets for the first and last gauge groups which have not been gauged and consequently give quadratic contributions to $H - V$. However, if $a_1 = 2$ then $a_2 = 2, 3$, for $a_2 = 2$ then all $a_i = 2$ but if $a_2 = 3$ then trying to saturate all inequalities we get increasing a_i: $a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 4, \cdots a_{k-1} = k, a_k = k + 1$. All these theories have a large number $\sim N$ of ungauged matter in the fundamental representation which lead to quadratic diverge in $H - V$ for large N irrespectively of the choice of F_K. Therefore, the only theories that may give potential infinite families are those with gauge groups of the form:

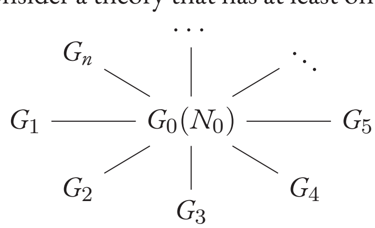
$$G_1 \times SU(N_2) \times \cdots \times SU(N_{k-1}) \times G_k \tag{C.16}$$

with G_k any group with matter that contains $(N_i \pm 8)\square$. All these cases are studied extensively in Section 3.1.

Here we have considered only linear chains of groups that could be potential infinite families.

C.2 Non-linear chains

Consider a theory that has at least one of the groups connected to n other groups.



We know from Table 3.1 each group $G_i(N_i)$ depends on a parameter N_i that control its size and the matter associated to that group can be either $N_i \pm 8$ or $2N_i$ fundamental hypermultiplets, which we will label as $k_i N_i \pm (2 - k_i)8$ with $k_i = 1$ or 2. Therefore, assuming that there is matter charged under each adjacent group the following inequalities for the leading contributions should hold:

$$\sum_{i=1}^{k} N_i \leq F_{N_0} = k_0 N_0, N_0 \leq F_{N_i} = k_i N_i \tag{C.17}$$

We can combine them to get

$$\sum_i \frac{1}{k_i} \leq k_0 \tag{C.18}$$

The largest number of gauge groups i is reached when $k_i = 2$ and $k_0 = 2$ for which $i = 4$. Therefore, $n = 4$ is the largest number of adjacent groups one can have. We have already studied the cases with $n = 0, 1, 2$ which corresponds to the linear chains. For $n \geq 3$ one would require $k_0 = 2$ and hence the only possibilities are $(k_1, k_2, k_3) = (1, 2, 2), (2, 2, 2)$ or $(k_1, k_2, k_3, k_3) = (2, 2, 2, 2)$.

For $n = 4$:

$$N_1 + N_2 + N_3 + N_4 = 2N_0 \tag{C.19}$$

and $N_i \geq N_0/2$ then the unique solution is:

$$\begin{array}{ccccc} & & SU(N) & & \\ & & | & & \\ SU(N) & \text{———} & SU(2N) & \text{———} & SU(N) \\ & & | & & \\ & & SU(N) & & \end{array}$$

with $H - V = 5$ and hence $T \leq 9$.

For $n = 3$:

$$\frac{3}{2} N_0 \leq N_1 + N_2 + N_3 = 2N_0 \tag{C.20}$$

and $N_i \geq N_0/2$ from which one could construct the following infinite families of theories :

$$\begin{array}{ccccccccc} & & SU(N) & & & & SU(N) & & \\ & & | & & & & | & & \\ SU(N) & \text{———} & SU(2N) & \text{- - - - - -} & SU(2N)^m & \text{- - - - - -} & SU(2N) & \text{———} & SU(N) \end{array}$$

with $H - V = 6 + m$ which implies that $T \leq 9$.

$$\begin{array}{ccccccccc} & & & & SU(N) & & & & \\ & & & & | & & & & \\ & & & & SU(2N) & & & & \\ & & & & | & & & & \\ SU(N) & \text{———} & SU(2N) & \text{———} & SU(3N) & \text{———} & SU(2N) & \text{———} & SU(N) \end{array}$$

with $H - V = 7$ which implies that $T \leq 9$.

$$\begin{array}{ccccccccccccc} SU(N) & \text{—} & SU(2N) & \text{—} & SU(3N) & \text{—} & SU(4N) & \text{—} & SU(3N) & \text{—} & SU(2N) & \text{—} & SU(N) \\ & & & & & & | & & & & & & \\ & & & & & & SU(2N) & & & & & & \end{array}$$

with $H - V = 8$ which implies that $T \leq 9$.

$$\begin{array}{ccccccccccccccc} SU(2N) & \text{—} & SU(4N) & \text{—} & SU(6N) & \text{—} & SU(5N) & \text{—} & SU(4N) & \text{—} & SU(3N) & \text{—} & SU(2N) & \text{—} & SU(N) \\ & & & & | & & & & & & & & & & \\ & & & & SU(3N) & & & & & & & & & & \end{array}$$

with $H - V = 9$ which implies that $T \leq 9$.

All these theories have $a^2 = 9 - T$, $b_i^2 = -2$, $b_i \cdot b_j = 1$ or 0 for $i \neq j$ depending on the inner product pattern presented on the diagrams above. These theories together with the cyclic $SU(N)$ we found earlier C.1 have anomaly lattices equal to the negative of the extended Cartan matrices from affine ADE.

The other case with

$$N_1 + N_2 + N_3 = 2N_0 \tag{C.21}$$

and $N_1 \geq N_0, N_{i\neq 1} \geq N_0/2$ does not seem to give infinite families.

We note that the list of theories might not be exhaustive since we have not considered any more exotic configurations where for example loops could appear.

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