

Bethe-Salpeter approach to three-body bound states with zero-range interaction

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Abstract. The Bethe-Salpeter equation for three scalar bosons, with zero-range interaction, is solved in Minkowski space by direct integration of the four-dimensional integral equation. The singularities occurring in the propagators are treated properly by standard analytical and numerical methods, without relying on any ansatz for the Bethe-Salpeter amplitude. The results for the binding energies and transverse amplitudes are compared with the results computed in Euclidean space. A fair agreement between the calculations is obtained.

1. Introduction

The Bethe-Salpeter (BS) equation [1, 2] comprises a reliable tool for the description of relativistic few-body systems in the non-perturbative regime. From the numerical point of view, the most straightforward way to solve this integral equation is to carry out its analytic continuation to the Euclidean space, through the Wick rotation [3]. Some physical quantities, e.g. binding energies, are unaltered under this transformation. However, as shown in [4], the Euclidean BS amplitude cannot be naively used to compute some dynamical observables. For such applications one needs the BS amplitude solution in Minkowski space. The two-scalar BS equation, with one-boson-exchange interaction, was solved successfully in Minkowski space by several research groups, see e.g. [5, 6, 7, 8]. The Nakanishi integral representation [9, 10] was then adopted to put the BS equation in a non-singular solvable form.

Unraveling the structure of relativistic three-body systems has important implications for applications in subatomic physics, but it is more difficult if compared to two-body systems. Most of the comprehensive studies in the three-body context have, so far, been carried out for the zero range interaction framework, which, despite of its simplicity, is quite useful. Investigations of the structure of three-body systems with short-range interactions are also important for Efimov physics which dominates the properties of the energy eigenstates with total vanishing angular momentum, composed by the maximally symmetric configuration. The low-energy properties of such systems are given by model independent correlations with few physical scales, namely the large scattering lengths and one characteristic three-body scale. Such universal correlations are limit cycles repeating themselves geometrically in the limit where the scattering length goes to infinity, as happens to the Efimov states with binding energies geometrically



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separated. For a more comprehensive discussion of the universality in few-body systems, see e.g. Ref. [14]. The mentioned properties are retained in a relativistic description of the three-boson equation considering a low momentum expansion with respect to the individual mass. The Efimov phenomena from the point of view of a relativistic model is associated with the infrared behavior of the associated momentum space integral equations. However, it is well-known that in non-relativistic three-body systems with zero-range interaction the binding energy is not bounded from below, i.e. they have a so-called Thomas collapse, which can be related to the Efimov effect due to the scale invariance of the ultraviolet form of the non-relativistic equations [11]. On the contrary, it was shown in Refs. [12, 13] that in a relativistic framework the Thomas collapse is not anymore equivalent to the Efimov effect, as the mass is a scale breaks the equivalence between the infrared region (non-relativistic) and the ultraviolet region. Furthermore, the Thomas collapse is eliminated by the appearance of an effective short-range repulsion due to the relativistic propagation of the constituents. Consequently, it is important to study the structure of such systems within fully relativistic frameworks.

The BS and Light-Front (LF) equations for the three-boson system with zero-range interaction were derived in [12]. The LF equation, which is obtained by performing the integration over k^- of the BS amplitude, only preserves the valence component of the BS amplitude, and was solved by Frederico in a limited range and the solution was subsequently generalized in Ref. [13]. Recently, in [15], we solved the BS equation, introduced in [12], in Euclidean space and it was then deduced that higher-Fock components beyond the valence have a great impact on the structure of the three-body system. As already pointed out, it is crucial to acquire the BS amplitude directly in Minkowski space. To this end, we solved in the recent work [17] the BS equation by direct integration in Minkowski space and some of the results are presented in this contribution. The results for the binding energies and transverse amplitudes are also compared with the ones obtained in Euclidean space.

2. Three-body Bethe-Salpeter equation

We consider a system of three bosons, with constituent masses m , with zero-range interaction. The BS equation for the Faddeev component of the vertex function then reads [12]

$$v(p, q) = 2iF(M_{12}) \int \frac{d^4k}{(2\pi)^4} \frac{i}{[k^2 - m^2 + i\epsilon]} \frac{i}{[(p - q - k)^2 - m^2 + i\epsilon]}, \quad (1)$$

where p denotes the total four momentum of the three-body system and q is the four momentum of the spectator particle. Furthermore, $F(M_{12})$ is the two-body scattering amplitude and the squared mass of the two-body subsystem is given by $M_{12}^2 = (p - q)^2$.

The BS equation (1) comprises a highly-singular integral equation and is thus challenging to solve numerically. If the purpose simply is to compute well-defined quantities, such as binding energies, one can transform Eq. (1) to the complex plane through the Wick rotation. In the rest frame ($\vec{p} = \vec{0}$), the Euclidean BS equation is given by [15]

$$v_E(q'_4, q'_v) = 2F(-M_{12}'^2) \int_{-\infty}^{\infty} dk'_4 \int_0^{\infty} \frac{dk'_v}{(2\pi)^3} \frac{\Pi_E(q'_4, q'_v, k'_4, k'_v)}{(k'_4 - \frac{i}{3}M_3)^2 + k'_v^2 + m^2}, \quad (2)$$

with the kernel

$$\Pi_E(q'_4, q'_v, k'_4, k'_v) = \frac{k'_v}{2q'_v} \log \frac{(k'_4 + q'_4 + \frac{i}{3}M_3)^2 + (q'_v + k'_v)^2 + m^2}{(k'_4 + q'_4 + \frac{i}{3}M_3)^2 + (q'_v - k'_v)^2 + m^2}. \quad (3)$$

In the derivation of (2), we performed the change of variables $k = k' + \frac{p}{3}$ and $q = q' + \frac{p}{3}$, so that the Wick rotation could be accomplished without crossing any singularities of the integrand in Eq. (1).

The three-body LF equation, introduced in [12], is completely defined in Minkowski space, but it only gives access to the valence component. For realistic calculations of dynamical observables, it is thus essential to also study the full solution of Eq. (1) directly in Minkowski space. Following the technique introduced in [16], we write the propagator $[k^2 - m^2 + i\epsilon]^{-1}$ as

$$\frac{1}{k^2 - m^2 + i\epsilon} = PV \frac{1}{k_0^2 - \varepsilon_k^2} - \frac{i\pi}{2\varepsilon_k} [\delta(k_0 - \varepsilon_k) + \delta(k_0 + \varepsilon_k)], \quad (4)$$

with $\varepsilon_k = \sqrt{k_v^2 + m^2}$ and $k_v = |\vec{k}|$.

Eq. (1) can then be transformed to the partially non-singular form [17]

$$\begin{aligned} v(q_0, q_v) = & \frac{\mathcal{F}(M_{12})}{(2\pi)^4} \int_0^\infty k_v^2 dk_v \left\{ \frac{\pi i}{\varepsilon_k} [\Pi(q_0, q_v; \varepsilon_k, k_v) v(\varepsilon_k, k_v) + \Pi(q_0, q_v; -\varepsilon_k, k_v) v(-\varepsilon_k, k_v)] \right. \\ & - 2 \int_{-\infty}^0 dk_0 \left[\frac{\Pi(q_0, q_v; k_0, k_v) v(k_0, k_v) - \Pi(q_0, q_v; -\varepsilon_k, k_v) v(-\varepsilon_k, k_v)}{k_0^2 - \varepsilon_k^2} \right] \\ & \left. - 2 \int_0^\infty dk_0 \left[\frac{\Pi(q_0, q_v; k_0, k_v) v(k_0, k_v) - \Pi(q_0, q_v; \varepsilon_k, k_v) v(\varepsilon_k, k_v)}{k_0^2 - \varepsilon_k^2} \right] \right\}, \end{aligned} \quad (5)$$

where the propagator singularities have been eliminated by subtractions. The kernel Π has now only weak (logarithmic) singularities which will be treated numerically and is given explicitly in [17].

It is not possible to directly compare the Minkowski vertex function $v(q_0, q_v)$ with the corresponding Euclidean one. However, one can use, in the BS amplitude, instead of $k = (q_0, q_v)$ the LF variables $q = (q_-, q_+, \vec{q}_\perp)$, with $q_\mp = q_0 \mp q_z$ and $\vec{q}_\perp = (q_x, q_y)$. The double integrals of the Minkowski BS amplitude over q_+ and q_- , and of the corresponding Euclidean ones over q_0 , q_z , are then the same.

The contribution $L_1(\vec{k}_{1\perp}, \vec{k}_{2\perp})$ to the transverse amplitude takes the form [17]

$$\begin{aligned} L_1(\vec{k}_{1\perp}, \vec{k}_{2\perp}) = & -i \int_{-\infty}^\infty dk_{1z} \left\{ \frac{i\pi}{2\tilde{k}_{10}} \left[\chi(\tilde{k}_{10}, k_{1z}; \vec{k}_{1\perp}, \vec{k}_{2\perp}) v_M(\tilde{k}_{10}, k_{1v}) + \chi(-\tilde{k}_{10}, k_{1z}; \vec{k}_{1\perp}, \vec{k}_{2\perp}) v_M(-\tilde{k}_{10}, k_{1v}) \right] \right. \\ & - \int_0^\infty dk_{10} \frac{\chi(-\tilde{k}_{10}, k_{1z}; \vec{k}_{1\perp}, \vec{k}_{2\perp}) v_M(-\tilde{k}_{10}, k_{1v}) - \chi(-\tilde{k}_{10}, k_{1z}; \vec{k}_{1\perp}, \vec{k}_{2\perp}) v_M(-\tilde{k}_{10}, k_{1v})}{k_{10}^2 - \tilde{k}_{10}^2} \\ & \left. - \int_0^\infty dk_{10} \frac{\chi(k_{10}, k_{1z}; \vec{k}_{1\perp}, \vec{k}_{2\perp}) v_M(k_{10}, k_{1v}) - \chi(\tilde{k}_{10}, k_{1z}; \vec{k}_{1\perp}, \vec{k}_{2\perp}) v_M(\tilde{k}_{10}, k_{1v})}{k_{10}^2 - \tilde{k}_{10}^2} \right\}, \end{aligned} \quad (6)$$

where

$$\tilde{k}_{10} = \sqrt{k_{1z}^2 + \vec{k}_{1\perp}^2 + m^2}. \quad (7)$$

Similarly to the treatment of the BS equation, we have here used subtractions to eliminate the propagator singularities at $k_0 = \pm\tilde{k}_{10}$.

3. Results and discussion

In Ref. [17], we solved Eq. (5) by adopting a bi-cubic spline decomposition of the vertex function $v(p, q)$. In Table 1 is shown the computed eigenvalue λ , which multiplies the right-hand side of the BS equation, for three different values of the two-body scattering length. In the calculations we used also as input the three-body binding energy obtained by solving the Euclidean BS equation derived in [15]. In the table, an eigenvalue of $\lambda = 1.0$ indicates that Minkowski and

Euclidean solutions are consistent. It is seen that for all three cases that the real part of λ is close to one. For reasons of numerical stability, we used in the Minkowski-space computations cut-offs on the variables q_0 and q_v . On the contrary, in the solution of the Euclidean BS equation the full integration domain was retained. This could explain the small imaginary parts and the error in the real parts.

am	B_3/m	λ
-1.280	0.006	$0.999 - 0.054i$
-1.500	0.395	$1.000 + 0.002i$
-1.705	1.001	$0.997 + 0.106i$

Table 1. Eigenvalues of the three-body ground state for three scattering lengths, a , computed by using the Euclidean three-body binding energies.

Furthermore, in Fig. 1 it is shown as an example the calculated vertex function, $v(q_0, q_v = 0.5m)$, for the three-body binding energy $B_3/m = 0.395$. As discussed in more detail in [17], the vertex function has peaks at the values of q_v and q_0 , corresponding to that $M_{12}^2 = 0$ or $M_{12}^2 = 4m^2$. In the figure these positions are indicated with dashed lines.

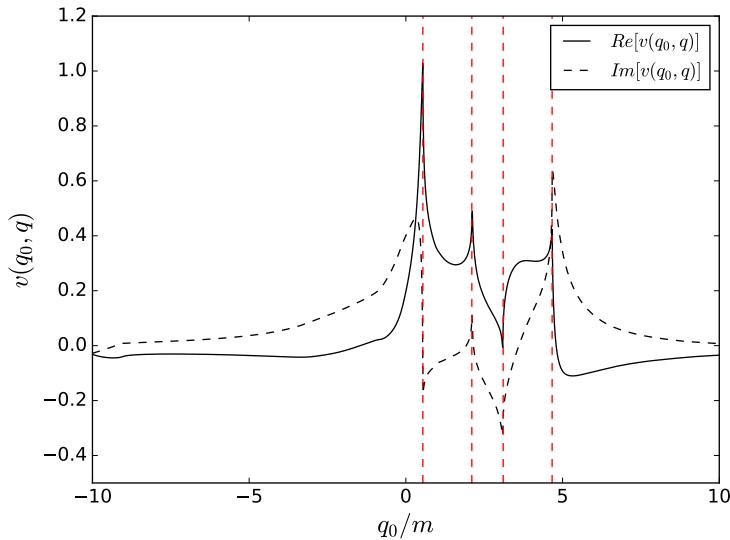


Figure 1. The vertex function, $v(q_0, q_v = 0.5m)$ with respect to q_0 for the parameters $am = -1.5$ and $B_3/m = 0.395$.

As already mentioned, we use for the three-body vertex function an expansion in terms of a finite number of spline functions. It is then important to make sure that the adopted number of basis functions is enough. To this end, we display in Fig. 2 the real and imaginary parts of $v(q_0, q_v = 0.5m)$, calculated by using different number of subintervals N_{qv} and N_{q0} , corresponding to the variables q_v and q_0 . The results in the figure corresponds to the parameters $am = 1.5$ and $B_3/m = 0.395$. It is clearly visible in the figure that for $N_{qv} \geq 40$ and $N_{q0} \geq 80$, the solution is well-converged.

Moreover, in Fig. 3 we show the modulus of the contribution $L_1(k_{1\perp}, k_{2\perp} = 0)$ to the transverse amplitude for $B_3/m = 0.395$, calculated by using Eq. (6). The results are also compared with the corresponding Euclidean ones. It is visible in the figure that the results are

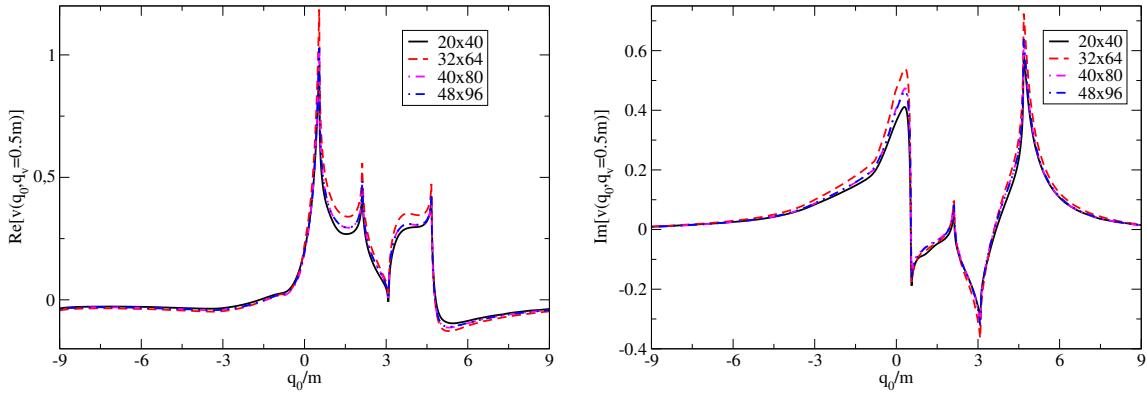


Figure 2. Convergence of the real (left panel) and imaginary (right panel) parts of the vertex function $v(q_0, q_v = 0.5m)$ with respect to the size of the basis, $N_{q_v} \times N_{q_0}$. The results correspond to the case $B_3/m = 0.395$.

in fair agreement with each other. As is clearly seen in Fig. 1, the vertex $v(q_0, q_v)$ with respect to q_0 is a non-smooth function. Despite of this, the computed transverse amplitude versus k_{\perp} is smooth, which makes the coincidence even more remarkable.

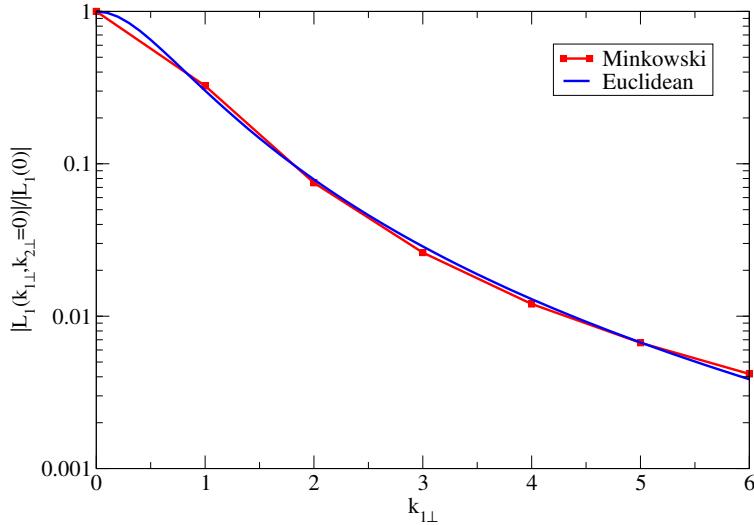


Figure 3. Transverse contribution, $L_1(k_{1\perp}, k_{2\perp} = 0)$, obtained in Minkowski space compared with the one computed in Euclidean space, for the parameters $am = -1.5$ and $B_3/m = 0.395$.

4. Conclusions

We have in this work, solved, directly in Minkowski space, the three-body BS equation derived in [12] for scalar bosons interacting by the two-body zero-range interaction. Our results show that both the three-body binding energies and transverse amplitudes, derived by direct integration of the Minkowskian BSE, agree with the Euclidean ones. However, the method is rather demanding from the numerical perspective, because of the appearance of many singularities which have to be treated properly. One possible way to the improve the numerical accuracy and also to be able to treat more realistic kernels is to transform the BS equation into a non-singular form by

using the Nakanishi integral representation [9, 10]. This is a work in progress and calculations based on this method will be undertaken soon.

Acknowledgments

We are grateful to Jaume Carbonell for stimulating discussions. This study was financed in part by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance code 001. J.H.A.N. acknowledges the support of the grants #2014/19094-8 and #2017/14695-1 and V.A.K. of the grant #2015/22701-6 from Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP). E.Y. thanks for the financial support of the grant #2016/25143-7 from FAPESP. We thank the FAPESP Thematic Projects grants #13/26258-4 and #17/05660-0. V.A.K. is also sincerely grateful to the group of theoretical nuclear physics of ITA, São José dos Campos, Brazil, for kind hospitality during his visit.

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