Quantum Information Processing (2024) 23:343
https://doi.org/10.1007/s11128-024-04545-0

Effect of noisy environment on secure quantum teleportation of unimodal Gaussian states

S. Mehrabankar[1] · P. Mahmoudi[2,3] · F. Abbasnezhad[4] · D. Afshar[4,5] · A. Isar[6]

Received: 17 October 2023 / Accepted: 23 September 2024

Abstract
Quantum networks rely on quantum teleportation, a process where an unknown quantum state is transmitted between sender and receiver via entangled states and classical communication. In our study, we utilize a continuous variable two-mode squeezed vacuum state as the primary resource for quantum teleportation, shared by Alice and Bob, while exposed to a squeezed thermal environment. Secure quantum teleportation necessitates a teleportation fidelity exceeding 2/3 and the establishment of two-way steering of the resource state. We investigate the temporal evolution of steering and teleportation fidelity to determine critical parameter values for secure quantum teleportation of a coherent Gaussian state. Our findings reveal constraints imposed by temperature, dissipation rate, and squeezing parameters of the squeezed thermal reservoir on the duration of secure quantum teleportation. Intriguingly, we demonstrate that increasing the squeezing parameter of the initial state effectively extends the temporal window for a successful secure quantum teleportation.

1 Introduction

Quantum teleportation has garnered significant attention in recent years, serving as both a fundamental quantum information protocol and an integral component of emerging quantum technologies [1–16]. This transformative process allows quantum

✉ S. Mehrabankar
mehso@uv.es; somayeh.mehrabankar@yahoo.com

1 Departamento de Física Teórica and IFIC, Universidad de Valencia-CSIC, 46100 Burjassot, Valencia, Spain

2 J. Heyrovský Institute of Physical Chemistry of the ASCR, v.v.i., , Dolejškova 2155/3, 182 23 Prague 8, Czech Republic

3 Department of Optics, Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic

4 Department of Physics, Faculty of Science, Shahid Chamran University of Ahvaz, Ahvaz, Iran

5 Center for Research on Laser and Plasma, Shahid Chamran University of Ahvaz, Ahvaz, Iran

6 Department of Theoretical Physics, National Institute of Physics and Nuclear Engineering, Bucharest-Magurele, Romania

Published online: 09 October 2024

information to be transmitted from one location to another by leveraging conventional communication methods and the utilization of entangled states as quantum resources [1]. The successful realization of quantum teleportation hinges upon the degree of entanglement shared within the system. A critical element of the burgeoning quantum internet is the achievement of high-fidelity quantum teleportation, enabling the transmission of information with fidelity surpassing classical limits [17–21]. In the context of a quantum communication network based on quantum teleportation [22–24], the secure transmission of quantum information over vast distances becomes paramount. Bennett [1] pioneered the first quantum teleportation protocol designed for discrete variable systems. This groundbreaking work later served as a foundation for the extension of quantum teleportation to continuous variable systems by Vaidman, Braunstein, and Kimble [4, 14, 21]. Specifically, the Braunstein–Kimble protocol [17, 18] laid the groundwork for traditional quantum teleportation in the realm of continuous variables. In the context of continuous variable quantum teleportation, legal entities can establish Gaussian entanglement even in the presence of lossy channels [11, 19, 21, 25]. Additionally, the utilization of a local oscillator as a filter for coherent detection has proven effective in reducing background noise [26, 27]. The transmission of quantum information through free space channels is currently gaining popularity, primarily due to its distinct advantages. Gaussian states hold significant importance in quantum optics and quantum information processing and transmission due to their ease of production and control. These states serve as vital resources in the field of quantum teleportation [5, 6, 28].

Dissipative effects stemming from interactions with the environment have substantial implications for real-world physical systems and significantly influence quantum information processing [29–32]. Consequently, recent research has prominently focused on quantum teleportation within open quantum systems [33–48]. In a prior study [13], we delved into an exploration and comparison of squeezed vacuum states and squeezed thermal states as initial resource states for secure quantum teleportation (SQT) within an open quantum system. This system consisted of two uncoupled harmonic oscillators interacting with a thermal environment. In the current paper, we employ the Gorini–Kossakowski–Lindblad–Sudarshan master equation, a foundational framework for the theory of open quantum systems based on completely positive dynamical semigroups, to characterize the time evolution of a system featuring two bosonic modes interacting with a squeezed thermal reservoir. This system state serves as the resource for teleporting a Gaussian coherent state. Successful SQT mandates both two-way steering of the resource state and teleportation fidelity exceeding 2/3. Our investigation reveals that quantum steering, teleportation fidelity, and thus SQT itself significantly depend on the parameters characterizing the resource state and the environment. Our primary objective is to delineate the dependencies of the allowable time intervals for SQT under these parameters, particularly within a shared squeezed thermal environment.

The structure of this paper is organized as follows. In Sect. 2, we introduce the concept of quantum teleportation, focusing specifically on continuous variable (CV) quantum teleportation. Moving to Sect. 3, we explore the realm of CV secure quantum teleportation, dissecting the quantum fidelity of teleportation and the phenomenon of two-way quantum steering. Section 4 examines the influence of a noisy environment

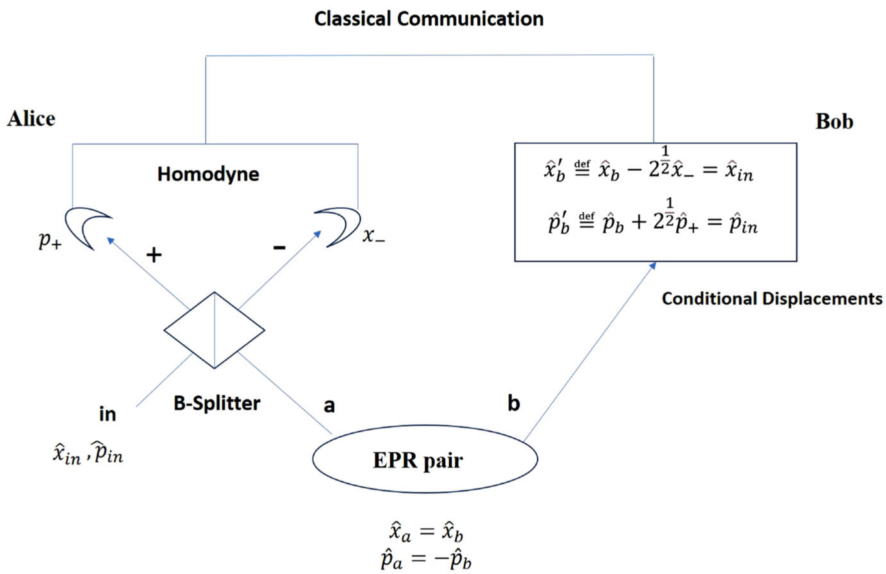


Fig. 1 Ideal teleportation with continuous variables [50]

on the CV secure quantum teleportation. In Sect. 5, we present a detailed analysis and discussion of our research findings. Finally, in Sect. 6, we offer concluding remarks, summarizing the key insights gleaned from our investigation.

2 Continuous variable quantum teleportation

The concept of CV quantum teleportation was first introduced by Vaidman in the early 1990s [14] and later elaborated upon by Braunstein and Kimble [4]. Unlike discrete variable (DV) teleportation, where achieving precise Bell measurements poses significant challenges in experimental setups, CV teleportation utilizes simpler tools, which offer enhanced precision and ease of implementation [49].

In both CV and DV teleportation schemes, establishing entanglement between Alice and Bob remains crucial for accurate communication. However, CV teleportation demonstrates distinct advantages in terms of simplicity and accuracy. While DV teleportation relies heavily on the challenging task of performing Bell measurements to entangle the input qubit with the quantum channel, CV teleportation circumvents this complexity by employing basic tools like beam splitters and homodyne detectors [50].

These alternative techniques not only simplify the experimental setup but also enhance the fidelity and efficiency of the teleportation process. By leveraging these advantages, CV teleportation emerges as a promising approach for achieving high-quality quantum communication between distant parties.

Subsequent sections will delve into the details of the CV teleportation protocol and the underlying mechanisms, supported in Fig. 1, which illustrates the protocol execution [49, 50].

2.1 Bell measurement

Alice does a CV version of the Bell measurement by doing two things in a row with her modes:

A) Mixing with a beam splitter: Alice mixes the input mode with her own mode (part of the EPR pair) using a balanced (no loss) beam splitter; basically, she performs the following transformation:

$$\hat{x}_{\pm} = \frac{(\hat{x}_a \pm \hat{x}_{in})}{\sqrt{2}}, \quad \hat{p}_{\pm} = \frac{(\hat{p}_a \pm \hat{p}_{in})}{\sqrt{2}}, \tag{1}$$

where $\hat{x}_{in}$ and $\hat{p}_{in}$ are the input modes of an unknown state of Alice. Also, $\hat{x}_{\pm}$ and $\hat{p}_{\pm}$ are the quadratures of the output modes "positive" and "negative" of the beam splitter.

B) Homodyne Detection: Alice performs homodyne detection on the output modes $\pm$. Specifically, she measures the quadratures $\hat{x}_{-}$ and $\hat{p}_{+}$, applying the projectors $|x\rangle\langle x|$ to the "negative" mode and $|p\rangle\langle p|$ to the "positive" mode [51]. Denoted as $(\hat{x}_{-}, \hat{p}_{+})$, these outcomes cause the collapse in Eq. (1) as follows:

$$\begin{aligned} \hat{x}_a &= \hat{x}_{in} + \sqrt{2}\hat{x}_{-}, \\ \hat{p}_a &= -\hat{p}_{in} + \sqrt{2}\hat{p}_{+}. \end{aligned} \tag{2}$$

With the EPR property, we have:

$$\hat{x}_a - \hat{x}_b = \hat{p}_a + \hat{p}_a = 0. \tag{3}$$

Therefore, Bob quadratures x_b and p_b undergo immediate projection as dictated by Eq. (2), implying that:

$$\begin{aligned} \hat{x}_b &= \hat{x}_{in} + \sqrt{2}\hat{x}_{-}, \\ \hat{p}_b &= \hat{p}_{in} - \sqrt{2}\hat{p}_{+}. \end{aligned} \tag{4}$$

At this juncture, Alice transmits her measurement outcome (x_{-}, p_{+}) to Bob via a classical channel.

2.2 Conditional displacement

Bob employs the received classical information (x_{-}, p_{+}) to execute an appropriate conditional displacement on his mode b, facilitating the completion of the teleportation process:

$$\hat{x}_b \to \hat{x}'_b \equiv \hat{x}_b - \sqrt{2}x_- = \hat{x}_{in},$$
$$\hat{p}_b \to \hat{p}'_b \equiv \hat{p}_b + \sqrt{2}p_+ = \hat{p}_{in}. \tag{5}$$

In accordance with Eq. (5), mode b is ultimately described by a pair of complementary quadratures ($\hat{x}'_b$ and $\hat{p}'_b$), precisely matching those of the input ($\hat{x}_{\text{in}}$ and $\hat{p}_{\text{in}}$). In the Schrödinger representation, this process is tantamount to transferring the quantum state from the input mode in on Alice side to the output mode b on Bob side, achieving a fidelity of $F = 1$.

3 Secure CV quantum teleportation

In the context of quantum teleportation, the shared entangled state between the sender and receiver serves as a critical resource. Recent research has emphasized the stringent requirements for achieving secure quantum teleportation (SQT), which necessitates a fidelity of teleportation higher than the classical limit of 2/3 [52]. Additionally, the resource state must exhibit two-way steerability, a quantum correlation phenomenon that surpasses the strength of mere entanglement [13, 52–60]. These essential conditions for two-mode Gaussian states form the basis of our investigation into secure quantum teleportation.

Moreover, it is essential to distinguish the roles of secure teleportation, quantum key distribution (QKD) [61], and quantum secure direct communication (QSDC) [62]. While QKD enables two parties to produce a shared random secret key known only to them, providing security against eavesdropping on the key exchange process, quantum teleportation ensures security against eavesdropping on the transmitted quantum state. QKD is typically employed for encrypting and decrypting messages, ensuring secure communication channels. On the other hand, quantum teleportation serves as a means to transmit quantum information between distant nodes within a quantum network, preserving the integrity of the quantum state during transmission. Additionally, QSDC offers a mechanism to transmit secret messages directly over a quantum channel, without sharing a key, further diversifying the applications of quantum communication protocols [63, 64]. Sections 3.1 and 3.2 of this paper delve into the essential conditions for achieving secure quantum teleportation, providing a detailed analysis of the requirements and strategies involved.

3.1 Quantum fidelity of teleportation

The quantum fidelity between two states, expressed in terms of their density operators ρ_1 and ρ_2, is represented as:

$$F(\rho_1, \rho_2) = \left[\mathrm{Tr}\left(\sqrt{\sqrt{\rho_2}\rho_1\sqrt{\rho_2}}\right)\right]^2. \tag{6}$$

This expression quantifies the degree of similarity or overlap between two quantum states, with a fidelity value of 1 indicating a perfect match between the states [1].

Using Eq. (6) in the context of the quantum teleportation protocol, we can determine the teleportation fidelity, which quantifies the similarity between the input state, denoted as $\rho_{in} = \rho_1$ and the output (teleported) state, $\rho_{out} = \rho_2$. The fidelity of teleportation, ranging from 0 to 1, serves as a measure of the efficiency of quantum teleportation. In practical experimental setups, it is common for the input and output states to not be identical, resulting in a fidelity of teleportation less than 1. Additionally, the fidelity of teleportation is influenced by dissipation and decoherence phenomena that manifest during interactions with the environment. For the teleportation of unimodal Gaussian states, the expression (6) for the teleportation fidelity is as follows [65, 66]:

$$F(\rho_{in}, \rho_{out}) = \frac{\exp\left[-\frac{1}{2}(\overline{X_{out}} - \overline{X_{in}})^{\mathrm{T}}(\sigma_{in} + \sigma_{out})^{-1}(\overline{X_{out}} - \overline{X_{in}})\right]}{\sqrt{\Delta + \Theta} - \sqrt{\Theta}}, \tag{7}$$

where σ_{in} and σ_{out} represent the covariance matrices of the input and output states, respectively. $\overline{X_{in}} = \mathrm{Tr}[\rho_{in}(X, P)^{\mathrm{T}}]$ and $\overline{X_{out}} = \mathrm{Tr}\left[\rho_{out}(X, P)^{\mathrm{T}}\right]$ denote the average values of the quadrature operators for the input and output states, respectively. Additionally, the parameters, Δ and Θ in Eq. (7), are defined as follows:

$$\Delta = \det(\sigma_{in} + \sigma_{out}) \geq 1$$

and

$$\Theta = 4\det\left(\sigma_{in} + \frac{\mathrm{i}}{2}J\right)\det\left(\sigma_{out} + \frac{\mathrm{i}}{2}J\right),$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

We employ an unimodal Gaussian coherent state as the input state for teleportation, characterized by the following covariance matrix:

$$\sigma_{in} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{8}$$

To derive the output teleported state, we adopt the characteristic function approach. Quantum teleportation is achieved through the utilization of both a classical communication channel and an entangled state shared by two parties, Alice (A) and Bob (B). At the outset of the teleportation process, parties A and B are presumed to be in an entangled squeezed vacuum state, characterized by the following characteristic function [53, 54, 56, 67]:

$$\chi_{AB}(\lambda_A, \lambda_B) = \exp\left\{-\frac{1}{2}\Lambda^{\mathrm{T}}\sigma(0)\Lambda - \mathrm{i}\Lambda^{\mathrm{T}}\overline{X_{AB}(0)}\right\}, \tag{9}$$

where $\sigma(0)$ represents the initial covariance matrix of the bipartite system AB, $\lambda_A = -\frac{\mathrm{i}}{\sqrt{2}}(x_A + \mathrm{i}p_A)$, $\lambda_B = -\frac{\mathrm{i}}{\sqrt{2}}(x_B + \mathrm{i}p_B)$ and $\Lambda = -\frac{\mathrm{i}}{\sqrt{2}}(x_A, x_B, p_A, p_B)^{\mathrm{T}}$. Additionally, $\overline{X_{AB}(0)}$ is defined as follows:

$$\overline{X_{AB}(0)} = \mathrm{Tr}\big[\rho(0)(X_A, P_A, X_B, P_B)^{\mathrm{T}}\big]. \tag{10}$$

These are the average values of the quadrature position and momentum operators $\Big(X_j = \frac{1}{\sqrt{2}}\big(a_j + a_j^\dagger\big)$ and $P_j = \frac{-\mathrm{i}}{\sqrt{2}}\big(a_j - a_j^\dagger\big),\ j = A, B\Big)$, with $a_j, a_j^\dagger$ representing the annihilation and creation operators, respectively. The symbol $\rho(0)$ denotes the initial density operator. If the time evolution of the bipartite system is Gaussian in nature, the shared entangled state retains its Gaussian characteristics throughout its interaction with the surrounding environment. At any given moment in time, t, the entangled state exhibits the following characteristic function:

$$\chi^t_{AB}(\lambda_A, \lambda_B) = \exp\left\{-\frac{1}{2}\Lambda^{\mathrm{T}}\sigma(t)\Lambda - \mathrm{i}\Lambda^{\mathrm{T}}\overline{X_{AB}(t)}\right\}, \tag{11}$$

where

$$\overline{X_{AB}(t)} = \mathrm{Tr}\big[\rho(t)(X_A, P_A, X_B, P_B)^{\mathrm{T}}\big] \tag{12}$$

represents the time-dependent averaged value of the quadrature operators. The covariance matrix of the shared state is denoted as follows [56, 67]:

$$\sigma(t) = \begin{pmatrix} A(t) & C(t) \\ C^{\mathrm{T}}(t) & B(t) \end{pmatrix}, \tag{13}$$

where $A(t) = \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix}$, $B(t) = \begin{pmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{pmatrix}$, and $C(t) = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$.

The expression of the covariance matrix of the output state can be derived by following the procedure outlined in Ref. [56], resulting in the following form:

$$\sigma_{out} = \begin{pmatrix} A_{11} + B_{11} - 2C_{11} + \frac{1}{2} & A_{12} - B_{12} + C_{12} - C_{21} \\ A_{12} - B_{12} + C_{12} - C_{21} & A_{22} + B_{22} + 2C_{22} + \frac{1}{2} \end{pmatrix}. \tag{14}$$

For the sake of simplicity, we assume that $\overline{X_{out}} = \overline{X_{in}}$. In this case, the teleportation fidelity (7) simplifies to:

$$F(\rho_{in}, \rho_{out}) = \frac{1}{\sqrt{\triangle}}, \tag{15}$$

with

$$\triangle = \det(\sigma_{in} + \sigma_{out}) = 1 + X + Y + XY - Z^2$$

and $\Theta = 0$, where X, Y, and Z are given by:

$$\begin{aligned} X &= A_{11} + B_{11} - 2C_{11}, \\ Y &= A_{22} + B_{22} + 2C_{22}, \\ Z &= A_{12} - B_{12} + C_{12} - C_{21}. \end{aligned} \tag{16}$$

By denoting

$$\Sigma = \begin{pmatrix} X & Z \\ Z & Y \end{pmatrix}, \tag{17}$$

the covariance matrix of the output state in Eq. (14) can be written as:

$$\sigma_{out} = \sigma_{in} + \Sigma \tag{18}$$

and $\triangle$ is as follows:

$$\triangle = 1 + \mathrm{Tr}\ \Sigma + \det \Sigma. \tag{19}$$

3.2 Two-way quantum steering

Steering is a form of quantum correlation initially introduced by Schrödinger in the context of the EPR paradox [57, 58]. Quantum steering refers to the capacity of one party in a bipartite system to influence the state of the other party through local measurements. This influence tends to be fundamentally asymmetric, as the effects of local measurements on the two subsystems often differ [60]. Due to this inherent property, steerable states find utility in various quantum processes where the outcome of measurements on one party cannot be fully trusted, such as quantum key distribution [68, 69]. Additionally, steerable states have proven valuable in tasks related to channel discrimination and secure quantum teleportation [70, 71].

In this paper, we exclusively focus on two-mode squeezed vacuum states, which constitute a subset of continuous variable states characterized by a Gaussian Wigner function. These states are represented in phase space through a 4×4 covariance matrix, as illustrated in Eq. (13). Previous research has established that a bipartite Gaussian state is considered steerable if Alice can generate distinct states for the Bob subsystem by implementing different local measurements on her subsystem. Additionally, these states must be unattainable through a local hidden state model [70–73]. In accordance with this model, a bipartite Gaussian state with covariance matrix σ_{AB} is deemed $A \to B$ steerable with respect to Alice measurements, if and only if the following inequality is not satisfied [28]:

$$\sigma_{AB} + \frac{\mathrm{i}}{2}(0_A \oplus J) \geq 0. \tag{20}$$

This inequality is equivalent with the following conditions:

$$A > 0, \quad M_\sigma^B + \frac{\mathrm{i}}{2} J \geq 0, \tag{21}$$

where $M_\sigma^B = B - C^\mathrm{T} A^{-1} C$ represents the Schur complement of A in the covariance matrix σ_{AB} [70, 71]. It is important to note that the covariance matrix of each subsystem must adhere to physical constraints, which invariably ensures the satisfaction of the first condition in Eqs. (21). Consequently, σ_{AB} is considered $A \to B$ steerable only when the second condition is not met [57, 72]. This condition can be expressed as [73, 74]:

$$\nu^B \geq \frac{1}{2}, \tag{22}$$

where ν^B denotes the symplectic eigenvalue of M_σ^B. Moreover, the extent of $A \to B$ steerability can be gauged by quantifying the extent to which Eqs. (21) are violated, as follows:

$$S^{A \to B}(\sigma_{AB}) = \max\{0, -\ln(2\nu^B)\}. \tag{23}$$

This measure remains invariant under symplectic transformations and assumes a value of zero if and only if σ_{AB} is deemed non-steerable. Similarly, to ascertain $B \to A$ steerability, one can exchange the roles of A and B and substitute the symplectic eigenvalue of the Schur complement of B with that of A in Eq. (23). The quantification of steering, as described in Eq. (23), possesses a straightforward analytical expression in the case of two-mode Gaussian states [75]:

$$S^{A \to B}(\sigma_{AB}) = \max\left\{0, \frac{1}{2} \ln \frac{\det A}{4 \det \sigma_{AB}}\right\}. \tag{24}$$

Consequently, the prerequisite condition for the secure quantum teleportation (SQT) of coherent states can be expressed as follows [76]:

$$\mathcal{L} > 0,$$

where

$$\mathcal{L} = \min\left\{S^{A \to B}, S^{B \to A}, F - \frac{2}{3}\right\}. \tag{25}$$

4 The effect of noisy environment on SQT

We examine an open quantum system comprising two uncoupled bosonic modes immersed in a squeezed bosonic environment. The Hamiltonian governing this system is given by [49]:

$$H = \sum_{i=1}^{2} \hbar\omega a_i^\dagger a_i + \sum_{j=1}^{N} \hbar\omega b_j^\dagger b_j + \sum_{i=1}^{2} \hbar\big(G a_i^\dagger + G^\dagger a_i\big),$$
$$G = \sum_{j=1}^{2} g_j b_j, \tag{26}$$

where $a_i^\dagger$ and a_i, as well as $b_j^\dagger$ and b_j, represent the creation and annihilation operators for the system and the environment, respectively. The parameters g_j denote the strength of interaction between the system and the environment. The environment typically consists of radiation field modes that can exist in various states, including vacuum, squeezed, thermal, and squeezed thermal states. Consequently, the environment can be characterized by the correlation functions between environment operators at different times, as depicted below [49]:

$$\begin{aligned} &< b_j^\dagger(t) b_j(t') >= N_j \delta(t-t'), \quad < b_j(t) b_j^\dagger(t') >= (N_j+1)\delta(t-t'),\\ &< b_j(t) b_j(t') >= M_j \delta(t-t'), \quad < b_j^\dagger(t) b_j^\dagger(t') >= M_j^* \delta(t-t'), \end{aligned} \tag{27}$$

where N_j and M_j denote the mean photon numbers and the squeezing parameters of the environment, respectively [53]. When $M_j = N_j = 0$, the environment is in a vacuum state; for $N_j \neq 0$, $M_j = 0$, it adopts a thermal state, while for $M_j \neq 0$, $N_j \neq 0$, it corresponds to a squeezed thermal state. The Heisenberg uncertainty relation imposes the following constraint on N_j and M_j:

$$\mid M_j \mid^2 \leq N_j(N_j+1).$$

In the scenario of a shared squeezed thermal environment, which represents a correlated noisy channel, the following relationship holds:

$$N_1 = N_2 = N, \;\; M_1 = M_2 = M$$

and the irreversible time evolution of the analyzed open system is described, within the Markovian approximation, by the following Lindblad master equation in the interaction representation [77]:

$$\begin{aligned} \frac{\partial}{\partial t}\rho = &\sum_{i=1}^{2} \frac{\gamma}{2}\Big\{N\Big(2a_i^\dagger \rho a_i - a_i a_i^\dagger \rho - \rho a_i a_i^\dagger\Big) + (N+1)\Big(2a_i \rho a_i^\dagger - a_i^\dagger a_i \rho - \rho a_i^\dagger a_i\Big)\Big\}\\ &- \sum_{i\neq j=1}^{2} \frac{\gamma}{2}\Big\{M^*\Big(2a_i \rho a_j - a_i a_j \rho - \rho a_i a_j\Big) + M\Big(2a_i^\dagger \rho a_j^\dagger - a_i^\dagger a_j^\dagger \rho - \rho a_i^\dagger a_j^\dagger\Big)\Big\}, \end{aligned} \tag{28}$$

where γ represents the environment field decay rate.

The asymptotic covariance matrix of the two bosonic modes is solely determined by the bath parameters and is expressed as follows [76]:

$$\sigma(\infty) = \begin{pmatrix} (N+1/2)I_2 & M\sigma_z \\ M\sigma_z & (N+1/2)I_2 \end{pmatrix}, \; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{29}$$

Here,

$$\begin{aligned} N &= n_{th}(\cosh^2 R + \sinh^2 R) + \sinh^2 R, \\ M &= -(2n_{th}+1)\cosh R \sinh R \end{aligned} \tag{30}$$

and R represents the squeezing parameter of the reservoir. The average thermal photon number of the bath can be computed as (assuming $\hbar = 1$, $\omega = 1$ and setting the Boltzmann constant $k_B = 1$):

$$n_{th} = \frac{1}{2}\left(\coth \frac{1}{2T} - 1\right), \tag{31}$$

where T is the temperature of the reservoir.

The solution of Eq. (28) in terms of the covariance matrix can be found in Refs. [13, 77], and it is given by

$$\sigma(t) = \Gamma\sigma(0) + (I_4 - \Gamma)\sigma(\infty), \tag{32}$$

where $\Gamma = e^{-\gamma t} I_4$; $\sigma(0)$ and $\sigma(\infty)$ represent the covariance matrices of the initial and final states, respectively. As initial state, we consider a squeezed vacuum state with the following covariance matrix [76]:

$$\sigma(0) = \frac{1}{2}\begin{pmatrix} \cosh 2r & 0 & \sinh 2r & 0 \\ 0 & \cosh 2r & 0 & -\sinh 2r \\ \sinh 2r & 0 & \cosh 2r & 0 \\ 0 & -\sinh 2r & 0 & \cosh 2r \end{pmatrix}, \tag{33}$$

where r is the squeezing parameter.

5 Results and discussion

This study is centered on the utilization of a two-mode squeezed vacuum state as a resource for quantum teleportation. This shared state is jointly held by Alice and Bob, and their system interacts with a shared squeezed thermal environment. Our primary goal is to explore the temporal evolution of both the fidelity of the teleportation process and quantum steering. This investigation aims to pint the essential parameters required for the successful SQT of an one-mode Gaussian coherent state. It is important to note

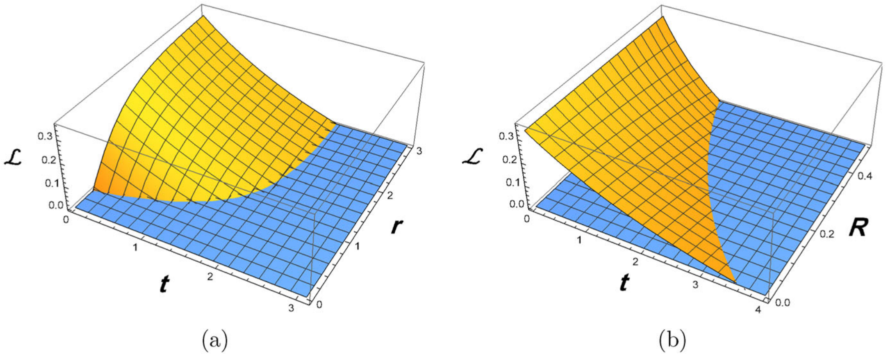


Fig. 2 The regions where $\mathcal{L} > 0$ versus time t and **a** squeezing r of the initial state for $R = 0.1$, $T = 1$; **b** squeezing R of the squeezed thermal environment for $r = 3$, $T = 0.7$. We set $\gamma = 0.1$.

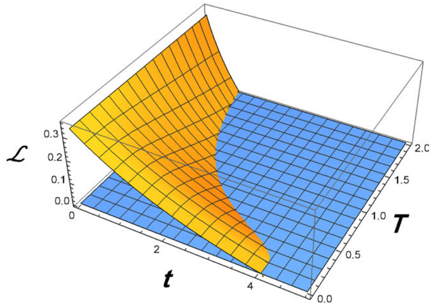
that the achievement of SQT hinges on two critical conditions: The teleportation fidelity must exceed 2/3, and the resource state must exhibit two-way steerability.

In Fig. 2a, we provide a plot that illustrates the correlation between the permissible time for SQT and the squeezing parameter of the initial resource state. It is evident that as the squeezing parameter increases, the feasible time window for SQT undergoes a substantial expansion. This observation implies that by enhancing the squeezing parameter, one can effectively prolong the duration within which successful SQT can be achieved. Conversely, we also note that for relatively small values of the squeezing parameter, the realization of SQT becomes unattainable.

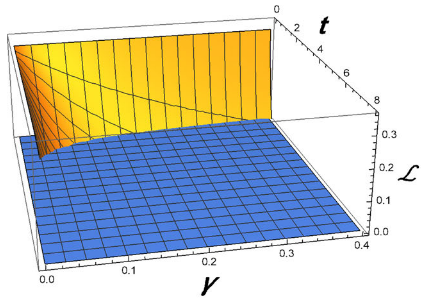
Figure 2b displays a plot that showcases the temporal and squeezing parameter dependencies of the conditions for SQT. In contrast to the previous scenario, it is evident that as the squeezing parameter of the environment increases, the available time frame for successful SQT diminishes. Furthermore, as we will demonstrate in the following, for relatively high values of the squeezing parameter of the thermal reservoir, the realization of SQT becomes unattainable.

In Fig. 3, we provide plots that delineate the regions in which the conditions for SQT are met, as functions of time and two crucial parameters: the temperature of the squeezed thermal reservoir (Fig. 3a) and the dissipation rate (Fig. 3b). As anticipated, our observations align with expectations. Specifically, we notice that if both the temperature of the reservoir and the dissipation rate increase, the temporal window during which SQT can be successfully realized shrinks.

Figure 4 showcases the region where the conditions for SQT are satisfied, with respect to the squeezing parameters of the initial state and the environment at a specific moment in time. Once again, we observe a clear trend: An increase in the squeezing parameter of the initial state contributes positively to the fulfillment of SQT conditions, while an increase in the squeezing parameter of the environment hampers these conditions. Furthermore, it is evident that for relatively small values of the squeezing parameter of the initial state and relatively large values of the squeezing parameter of the thermal reservoir, the realization of SQT remains unattainable.



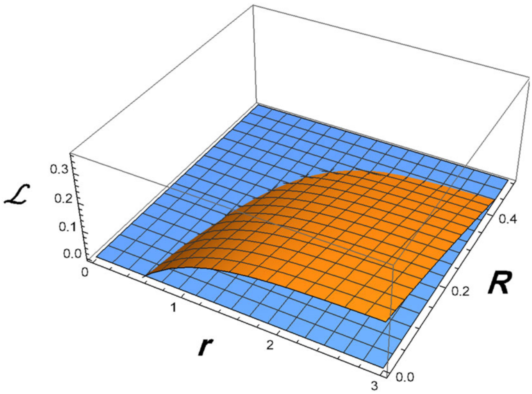
(a)



(b)

Fig. 3 The regions where $\mathcal{L} > 0$ versus time t and **a** temperature T of the squeezed thermal environment for $R = 0.2$, $\gamma = 0.1$; **b** dissipation rate γ for $T = 1.1$, $R = 0.09$. We set $r = 3$.

Fig. 4 The regions where $\mathcal{L} > 0$ versus squeezing r of the initial state and squeezing R of the squeezed thermal environment at the moment of time $t = 1$ for $T = 1$, $\gamma = 0.1$.



6 Conclusion

This study has offered valuable insights into the temporal dynamics and the impact of critical parameters on secure quantum teleportation (SQT). Our investigations have revealed noteworthy correlations and trends, providing valuable information regarding the feasibility and constraints of this quantum communication protocol.

One of the principal findings of our study pertains to the profound influence of environmental factors on the permissible time frame for SQT. Specifically, our investigation has unveiled the pivotal role played by the nature of the squeezed thermal environment in determining the temporal window within which successful and secure teleportation can be achieved. We meticulously examined the influence of the temperature, dissipation rate, and squeezing parameter of the squeezed thermal reservoir on the allowed duration for SQT. Our findings indicate that each of these environmental parameters imposes limitations on the feasible duration for SQT. Furthermore, we delved into the impact of the squeezing parameter of the initial state on the allowed time for SQT. Our results highlight that a significant trend and increase in the squeezing of the initial state has the capacity to notably extend the temporal range over which successful SQT can be realized.

These findings emphasize the critical importance of meticulous consideration of environmental conditions and associated parameters when conceiving and deploying SQT protocols. Through the optimization of squeezing parameters and the mitigation of diffusion and dissipation effects, researchers can substantially bolster the reliability and resilience of quantum teleportation. This, in turn, paves the way for more prolonged periods of secure and trustworthy information transfer in quantum communication networks.

To sum up, our study has advanced the comprehension of temporal limitations and parameter dependencies inherent to secure quantum teleportation. These insights hold potential implications for the advancement of practical quantum communication systems and offer valuable guidance for forthcoming research endeavors aimed at enhancing the efficiency and security of quantum teleportation protocols.

Acknowledgements The author S. Mehrabankar would like to acknowledge the support by the Spanish MCIN/AEI/10.13039/501100011033 grant PID2020-113334GB-I00, Generalitat Valenciana grant CIPROM/2022/66, the Ministry of Economic Affairs and Digital Transformation of the Spanish Government through the QUANTUM ENIA project call - QUANTUM SPAIN project, by the European Union through the Recovery, Transformation and Resilience Plan - NextGenerationEU within the framework of the Digital Spain 2026 Agenda, and by the CSIC Interdisciplinary Thematic Platform (PTI+) on Quantum Technologies (PTI-QTEP+). This project has also received funding from the European Union's Horizon 2020 research and innovation program under grant agreement 101086123- CaLIGOLA. A. Isar acknowledges the support by the Romanian Ministry of Research, Innovation and Digitization, through the Project PN 23 21 01 01/2023. D. Afshar would like to thank Shahid Chamran University of Ahvaz for the grant no. SCU.SP1402.812. This work was supported by the project PID2023-152724NA-I00, with funding from MCIU/AEI/10.13039/501100011033 and FSE+.

Author Contributions This statement outlines the specific contributions of each author to the manuscript. All the authors designed and coordinated the research. S. M , A. I, and D. A wrote the main manuscript text. F. A prepared figures 1-3. P. M conducted data analysis and statistical interpretation. All authors reviewed and provided critical feedback on the manuscript.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Declarations

Conflict of interest The authors declare no competing interests.

References

1. Bennett, C.H., Brassard, G., Crépeau, C., Jozsa, R., Peres, A., Wootters, W.K.: Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. Phys. Rev. Lett. **70**(13), 1895 (1993)

2. Bouwmeester, D., Pan, J.-W., Mattle, K., Eibl, M., Weinfurter, H., Zeilinger, A.: Experimental quantum teleportation. Nature **390**(6660), 575–579 (1997)
3. Boschi, D., Branca, S., De Martini, F., Hardy, L., Popescu, S.: Experimental realization of teleporting an unknown pure quantum state via dual classical and Einstein-Podolsky-Rosen channels. Phys. Rev. Lett. **80**(6), 1121 (1998)
4. Braunstein, S.L., Kimble, H.J.: Teleportation of continuous quantum variables. Phys. Rev. Lett. **80**(4), 869 (1998)
5. Furusawa, A., Sørensen, J.L., Braunstein, S.L., Fuchs, C.A., Kimble, H.J., Polzik, E.S., Furusawa, A.: Unconditional quantum teleportation. Science **282**(5389), 706–709 (1998)
6. Bowen, W.P., Treps, N., Buchler, B.C., Schnabel, R., Ralph, T.C., Bachor, H.-A., Symul, T., Lam, P.K.: Experimental investigation of continuous-variable quantum teleportation. Phys. Rev. A **67**(3), 032302 (2003)
7. Zhang, T.C., Goh, K.W., Chou, C.W., Lodahl, P., Kimble, H.J.: Quantum teleportation of light beams. Phys. Rev. A **67**(3), 033802 (2003)
8. Riebe, M., Häffner, H., Roos, C.F., Hänsel, W., Benhelm, J., Lancaster, G.P., Körber, T.W., Becher, C., Schmidt-Kaler, F., James, D.F., Blatt, R.: Deterministic quantum teleportation with atoms. Nature **429**(6993), 734–737 (2004)
9. Yonezawa, H., Aoki, T., Furusawa, A.: Demonstration of a quantum teleportation network for continuous variables. Nature **431**(7007), 430–433 (2004)
10. Sherson, J.F., Krauter, H., Olsson, R.K., Julsgaard, B., Hammerer, K., Cirac, I., Polzik, E.S.: Quantum teleportation between light and matter. Nature **443**(7111), 557–560 (2006)
11. Yukawa, M., Benichi, H., Furusawa, A.: High-fidelity continuous-variable quantum teleportation toward multistep quantum operations. Phys. Rev. A **77**(2), 022314 (2008)
12. Yin, J., Ren, J.-G., He, L., Cao, Y., Yong, H.-L., Yu-Ping, W., Liu, C., Liao, S.-K., Zhou, F., Jiang, Y., et al.: Quantum teleportation and entanglement distribution over 100-kilometre free-space channels. Nature **488**(7410), 185–188 (2012)
13. Afshar, D., Abbasnezhad, F., Mehrabankar, S., Isar, A.: Two-mode Gaussian states as resource of secure quantum teleportation in open systems. Chin. J. Phys. **68**, 419–425 (2020)
14. Vaidman, L.: Teleportation of quantum states. Phys. Rev. A **49**(2), 1473 (1994)
15. Sheng, Y.-B., Deng, F.-G., Long, G.L.: Complete hyperentangled-Bell-state analysis for quantum communication. Phys. Rev. A **82**(3), 032318 (2010)
16. Kim, M.S., Hwang, M.-R., Jung, E., Park, D.K.: Scrambling and quantum teleportation. Quantum Inf. Process. **22**(4), 176 (2023)
17. Massar, S., Popescu, S.: Optimal extraction of information from finite quantum ensembles. Phys. Rev. Lett. **74**(8), 1259 (1995)
18. Horodecki, M., Horodecki, P., Horodecki, R.: General teleportation channel, singlet fraction, and quasidistillation. Phys. Rev. A **60**(3), 1888 (1999)
19. Banaszek, K.: Optimal quantum teleportation with an arbitrary pure state. Phys. Rev. A **62**(2), 024301 (2000)
20. Albeverio, S., Fei, S.-M., Yang, W.-L.: Optimal teleportation based on Bell measurements. Phys. Rev. A **66**(1), 012301 (2002)
21. Kimble, H.J.: The quantum internet. Nature **453**(7198), 1023–1030 (2008)
22. Furusawa, A., Takei, N.: Quantum teleportation for continuous variables and related quantum information processing. Phys. Rep. **443**(3), 97–119 (2007)
23. Pirandola, S., Eisert, J., Weedbrook, C., Furusawa, A., Braunstein, S.L.: Advances in quantum teleportation. Nat. Photonics **9**(10), 641–652 (2015)
24. Zhihua, Z., Yuyang, S.: Bidirectional quantum teleportation in multi-hop communication network. Quantum Inf. Process. **22**(5), 201 (2023)
25. Harraz, S., Cong, S., Nieto, J.J.: Optimal tripartite quantum teleportation protocol through noisy channels. Quantum Inf. Process. **22**(1), 83 (2023)
26. Xiaodong, W., Wang, Y., Guo, Y., Zhong, H., Huang, D.: Passive continuous-variable quantum key distribution using a locally generated local oscillator. Phys. Rev. A **103**(3), 032604 (2021)
27. Wang, S., Huang, P., Wang, T., Zeng, G.: Feasibility of all-day quantum communication with coherent detection. Phys. Rev. Appl. **12**(2), 024041 (2019)
28. Adesso, G., Ragy, S., Lee, A.R.: Continuous variable quantum information: Gaussian states and beyond. Open Syst. Inf. Dyn. **21**, 1440001 (2014)

29. Abbasnezhad, F., Mehrabankar, S., Afshar, D., Jafarpour, M.: Evolution of quantum correlations in the open quantum systems consisting of two coupled oscillators. Quantum Inf. Process. **16**, 1–17 (2017)
30. Abbasnezhad, F., Mehrabankar, S., Afshar, D., Jafarpour, M.: Markovian thermal evolution of entanglement and decoherence of GHZ state. Eur. Phys. J. Plus **133**, 1–11 (2018)
31. Afshar, D., Mehrabankar, S., Abbasnezhad, F.: Entanglement evolution in the open quantum systems consisting of asymmetric oscillators. Eur. Phys. J. D **70**, 1–8 (2016)
32. Mehrabankar, S., Afshar, D., Jafarpour, M.: Quantum fidelity evolution of Penning trap coherent states in an asymmetric open quantum system. Quantum Inf. Comput. **19**(5–6), 413–423 (2019)
33. Badziag, P., Horodecki, M., Horodecki, P., Horodecki, R.: Local environment can enhance fidelity of quantum teleportation. Phys. Rev. A **62**(1), 012311 (2000)
34. Verstraete, F., Verschelde, H.: Fidelity of mixed states of two qubits. Phys. Rev. A **66**(2), 022307 (2002)
35. Bandyopadhyay, S.: Origin of noisy states whose teleportation fidelity can be enhanced through dissipation. Phys. Rev. A **65**(2), 022302 (2002)
36. Kumar, D., Pandey, P.N.: Effect of noise on quantum teleportation. Phys. Rev. A **68**(1), 012317 (2003)
37. Jung, E., Hwang, M.R., Ju, Y.H., Kim, M.S., Yoo, S.K., Kim, H., Park, D., Son, J.W., Tamaryan, S., Cha, S.K.: Greenberger-Horne-Zeilinger versus W states: quantum teleportation through noisy channels. Phys. Rev. A **78**(1), 012312 (2008)
38. Bhaktavatsala Rao, D.D., Panigrahi, P.K., Mitra, C.: Teleportation in the presence of common bath decoherence at the transmitting station. Phys. Rev. A **78**(2), 022336 (2008)
39. Yeo, Y., Kho, Z.-W., Wang, L.: Effects of Pauli channels and noisy quantum operations on standard teleportation. Europhys. Lett. **86**(4), 40009 (2009)
40. Xueyuan, H., Ying, G., Gong, Q., Guo, G.: Noise effect on fidelity of two-qubit teleportation. Phys. Rev. A **81**(5), 054302 (2010)
41. Olivares, S., Paris, M.G.A., Rossi, A.R.: Optimized teleportation in Gaussian noisy channels. Phys. Lett. A **319**(1–2), 32–43 (2003)
42. Olivares, S., Paris, M.G.A., Bonifacio, R.: Teleportation improvement by inconclusive photon subtraction. Phys. Rev. A **67**(3), 032314 (2003)
43. Pirandola, S., Laurenza, R., Braunstein, S.L.: Teleportation simulation of bosonic Gaussian channels: strong and uniform convergence. Eur. Phys. J. D **72**, 1–20 (2018)
44. Tserkis, S., Dias, J., Ralph, T.C.: Simulation of Gaussian channels via teleportation and error correction of Gaussian states. Phys. Rev. A **98**(5), 052335 (2018)
45. Zhou, L., Sheng, Y.-B.: Complete logic Bell-state analysis assisted with photonic Faraday rotation. Phys. Rev. A **92**(4), 042314 (2015)
46. Xiao-Min, H., Zhang, C., Bi-Heng Liu, Yu., Cai, X.-J.Y., Guo, Yu., Xing, W.-B., Huang, C.-X., Huang, Y.-F., Li, C.-F., Guo, G.-C.: Experimental high-dimensional quantum teleportation. Phys. Rev. Lett. **23**, 230501 (2020)
47. Shen, S., Yuan, C., Zhang, Z., Yu, H., Zhang, R., Yang, C., Li, H., Wang, Z., Wang, Y., Deng, G., et al.: Hertz-rate metropolitan quantum teleportation. Light Sci. Appl. **12**(1), 115 (2023)
48. Hu, X.M., Guo, Y., Liu, B.H., Li, C.F., Guo, G.C.: Progress in quantum teleportation. Nat. Rev. Phys. **5**(6), 339–353 (2023)
49. Walls, D.F., Milburn, G.J.: Quantum Optics. Springer-Verlag, Berlin (1994)
50. Pirandola, S., Mancini, S.: Quantum teleportation with continuous variables: a survey. Laser Phys. **16**, 1418–1438 (2006)
51. D'Ariano, G.M., Presti, P.L., Sacchi, M.F.: Bell measurements and observables. Phys. Lett. A **272**, 32–38 (2000)
52. Grosshans, F., Grangier, P.: Quantum cloning and teleportation criteria for continuous quantum variables. Phys. Rev. A **64**(1), 010301 (2001)
53. He, Q., Rosales-Zárate, L., Adesso, G., Reid, M.D.: Secure continuous variable teleportation and Einstein-Podolsky-Rosen steering. Phys. Rev. Lett. **115**(18), 180502 (2015)
54. He, G., Zhang, J., Zhu, J., Zeng, G.: Continuous-variable quantum teleportation in bosonic structured environments. Phys. Rev. A **84**(3), 034305 (2011)
55. Cuzminschi, M., Zubarev, A., Isar, A.: Extractable quantum work from a two-mode Gaussian state in a noisy channel. Sci. Rep. **11**(1), 24286 (2021)
56. Zubarev, A., Cuzminschi, M., Isar, A.: Optimal fidelity of teleportation using two-mode Gaussian states in a thermal bath as a resource. Rom. J. Phys. **64**, 108 (2019)
57. Marian, P., Marian, T.A.: Continuous-variable teleportation in the characteristic-function description. Phys. Rev. A **74**(4), 042306 (2006)

58. Schrödinger, E.: Discussion of probability relations between separated systems. In: Mathematical Proceedings of the Cambridge Philosophical Society, vol. 31, pp. 555–563. Cambridge University Press (1935)
59. Liuzzo-Scorpo, P., Adesso, G.: Optimal secure quantum teleportation of coherent states of light. Quantum Photonic Devices **10358**, 71–81 (2017)
60. Einstein, A., Podolsky, B., Rosen, N.: Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. **47**(10), 777 (1935)
61. Bera, S., Gupta, S., Majumdar, A.S.: Device-independent quantum key distribution using random quantum states. Quantum Inf. Process. **22**(2), 109 (2023)
62. Roy, P., Bera, S., Gupta, S., Majumdar, A.S.: Device-independent quantum secure direct communication under non-Markovian quantum channels. Quantum Inf. Process. **23**(5), 170 (2024)
63. Zhang, W., Ding, D.-S., Sheng, Y.-B., Zhou, L., Shi, B.-S., Guo, G.C.: Quantum secure direct communication with quantum memory. Phys. Rev. Lett. **118**(22), 220501 (2017)
64. Sheng, Y.-B., Zhou, L., Long, G.-L.: One-step quantum secure direct communication. Sci. Bull. **67**(4), 367–374 (2022)
65. Scutaru, H.: Fidelity for displaced squeezed thermal states and the oscillator semigroup. J. Phys. A Math. Gen. **31**, 3659–3663 (1998)
66. Marian, P., Marian, T.A.: Uhlmann fidelity between two-mode Gaussian states. Phys. Rev. A **86**, 022340 (2012)
67. Cuzminschi, M., Isar, A.: Quantum steering of two bosonic modes in the two-reservoir model. Roman. Rep. Phys. **73**(2) (2021)
68. Wiseman, H.M., Jones, S.J., Doherty, A.C.: Steering, entanglement, nonlocality, and the Einstein-Podolsky-Rosen paradox. Phys. Rev. Lett. **98**(14), 140402 (2007)
69. Branciard, C., Cavalcanti, E.G., Walborn, S.P., Scarani, V., Wiseman, H.M.: One-sided device-independent quantum key distribution: security, feasibility, and the connection with steering. Phys. Rev. A **85**(1), 010301 (2012)
70. Quintino, M.T., Vértesi, T., Cavalcanti, D., Augusiak, R., Demianowicz, M., Acín, A., Brunner, N.: Inequivalence of entanglement, steering, and Bell nonlocality for general measurements. Phys. Rev. A **92**(3), 032107 (2015)
71. Jones, S.J., Wiseman, H.M., Doherty, A.C.: Entanglement, Einstein-Podolsky-Rosen correlations, Bell nonlocality, and steering. Phys. Rev. A **76**(5), 052116 (2007)
72. Piani, M., Watrous, J.: Necessary and sufficient quantum information characterization of Einstein-Podolsky-Rosen steering. Phys. Rev. Lett. **114**(6), 060404 (2015)
73. Giedke, G., Ignacio Cirac, J.: Characterization of Gaussian operations and distillation of Gaussian states. Phys. Rev. A **66**(3), 032316 (2002)
74. Fiurášek, J., Mišta, L., Jr.: Gaussian localizable entanglement. Phys. Rev. A **75**(6), 060302 (2007)
75. Williamson, J.: On the algebraic problem concerning the normal forms of linear dynamical systems. Am. J. Math. **58**(1), 141–163 (1936)
76. Kogias, I., Lee, A.R., Ragy, S., Adesso, G.: Quantification of Gaussian quantum steering. Phys. Rev. Lett. **114**(6), 060403 (2015)
77. Xiang, S.-H., Zhao, Y.-J., Zhu, X.-X., Song, K.-H.: Nonclassical correlation dynamics in a system of mesoscopic Josephson junction coupled to single-mode optical cavity. Int. J. Theor. Phys. **54**, 2881–2892 (2015)