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Poisson Deformed Super-Algebras and Moyal Quantization

A. Ballesteros[1], **E. Celeghini**[2] **and M.A. del Olmo**[3]

[1] Departamento de Física, Universidad de Burgos, E-09006, Burgos, Spain
[2] Dipartimento di Fisica, Università di Firenze and INFN–Sezione di Firenze, I50019 Sesto Fiorentino, Firenze, Italy
[3] Departamento de Física Teórica, Universidad de Valladolid, E-47005, Valladolid, Spain.

E-mail: angelb@ubu.es, celeghini@fi.infn.it, olmo@fta.uva.es

Abstract.
The procedure to obtain the semi-classical limit of Lie-Hopf and quantum algebras introduced in preceding papers has been extended to Lie and quantum super-algebras. With a different definition of the limit the approach is the same: an $\hbar$-depending family of Hopf super-algebras is constructed starting from the super-bialgebra; then the singular limit $\hbar \to 0$ allows to move from quantum to semi-classical framework in the spirit of Moyal quantization. The Poisson undeformed $osp^P(1|2)$ and deformed $osp^P_q(1|2)$ super-algebras are exhibited, both as a paradigmatic example and because of their interest in applications.

1. Introduction

Group theory provides the natural mathematical language to formulate symmetry principles and to derive their consequences in Mathematics and Physics. Among groups, Lie ones are the most considered, because structures particularly manageable owing to their very high level of symmetry and of their holomorphic nature [1]. It is sufficient to consider the connection between quantum mechanics and the rotation group to see that quantum theories are almost inseparable from the underlying group structure.

This so advantageous approach has in itself a paradox: we assume that two objects (two elementary particles, two reference frames, etc..) are absolutely equal and, for this reason, indistinguishable. But this means that we are not able to identify them as separate objects and, thus, to state any relation between them. The solution of the paradox is, of course, the idea that a symmetry breaking always exists but that it is irrelevant for the problem we are considering: the electron at our left has the same properties of the electron at our right because the fact to be at our left is unimportant at the considered degree of accuracy. Also from this epistemological argument -that could find room in a presocratic fragment- an interest is born in describing a symmetry with a breaking that can be adiabatically reduced to zero.

For this reason, from the beginning of development of quantum groups we worked to find in terms of quantum groups a mathematical description of broken symmetries in quantum physics, the field where symmetry has played the largest role up to now. However, as symmetry is relevant also in classical physics, we have recently considered the idea of extending the quantum group approach to Hamiltonian mechanics [2, 3]. A broken symmetry has been thus described in a consistent way also at this level.

Technically the idea is to include the deformation in the Moyal quantization [4] to relate Poisson brackets and algebraic structure of quantum mechanics. Starting from a quantum algebra (where a deformation parameter z is already present) another parameter -the Planck constant $\hbar$- has been thus included in the scheme in such a way that it is possible to consider the limit where the Planck constant goes to zero in order to introduce in the semi-classical description an analytically broken symmetry.

However in the Hilbert space framework the full product of operators is defined. Thus, when we fix an algebra (and not also its representation) we have to choose between commutators and anticommutators. We are used to taking into account the commutators because a null commutator of two observables implies that the two observables can be simultaneously diagonalized but this argument does not play any role for not null cases. Indeed, super-algebras, where both commutators and anticommutators appear -i.e. where products are partially antisymmetrical and partially symmetrical- should be taken into account on the same footing. For this reason we focus here on the semi-classical limit of super-algebras and their deformation.

In sect. 2 a brief resumé of the results for the Moyal quantization in algebras is presented to introduce the notations and the basic ideas of the approach. For a better understanding, the method is illustrated in sect. 3 by a brief description of the Poisson-Hopf standard quantum deformations $su_q^P(2)$ and $su_q^P(3)$. Sect. 4 introduces the peculiarities of grading in connection with the relations between quantum and classical schemes. In sect. 5 a particular attention has been dedicated to the connection between the relevant cases of $osp(1|2)$ and $osp_q(1|2)$, and their semi-classical limits $osp^P(1|2)$ and $osp_q^P(1|2)$ respectively (that look interesting in describing the configuration and momentum spaces of quantum mechanics). Harmonic oscillator and its deformation are explicitly described from this new point of view. Finally sect. 6 illustrates the general philosophy of the approach.

2. Quantum (super)-algebras and Poisson-Hopf limit

In Moyal quantization [4] the non commutativity of the algebra of functions of the Poisson-Lie groups G is written as

$$[f_1, f_2] := f_1 \star f_2 - f_2 \star f_1 = \hbar \{f_1, f_2\} + o(\hbar^2),$$

where $\star$ denotes the non-commutative star product and $\{f_1, f_2\}$ is the Poisson bracket in $\mathrm{Fun}(G)$, related to the commutator by the Poisson limit

$$\{f_1, f_2\} = \lim_{\hbar \to 0} \frac{1}{\hbar} [f_1, f_2] .$$

Let us remember that quantization tries to associate to a commutative algebra of physical observables -$\mathrm{Fun}(M)$- of a classical mechanical system (with phase space M) a non-commutative algebra of linear operators -$Op(\mathcal{H})$- on the suitable complex Hilbert space, $\mathcal{H}$, of the quantum system under consideration. More precisely, the idea of quantization is to find a linear bijective map $\mathbb{Q} : \mathrm{Fun}(M) \mapsto Op(\mathcal{H})$ such that for all $f \in \mathrm{Fun}(M)$

$$\mathbb{Q}(\{f_1, f_2\}) = \frac{\mathbf{i}}{\hbar} [\mathbb{Q}(f_1), \mathbb{Q}(f_2)] . \qquad (1)$$

This condition comes from the fact that if f_2 is the classical Hamiltonian of the system and $\mathbb{Q}(f_2)$ its quantum counterpart then $\{f_1, f_2\}$ and $[\mathbb{Q}(f_1), \mathbb{Q}(f_2)]$ give the time evolution of the physical observable f_1 in the classical and in the quantum case, respectively. Unfortunately such a map $\mathbb{Q}$ is not known in general, even for very simple cases.

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When a Lie algebra g is involved at the level of quantum operators a solution of (1) can be found. Indeed when there is a set of quantum observables X_i closing a Lie algebra

$$[X_i, X_j] \;=\; f_{ij}^k\, X_k$$

in the (classical) Poisson-Hopf manifold a corresponding algebra can be defined by

$$\{\mathbb{Q}^{-1}(X_i), \mathbb{Q}^{-1}(X_j)\} \;:=\; \lim_{\hbar\to 0}\; \frac{\mathbf{i}}{\hbar}\; \mathbb{Q}^{-1}([X_i, X_j])\,,$$

$$\Delta(\mathbb{Q}^{-1}(X_i)) \;:=\; \lim_{\hbar\to 0}\; \mathbb{Q}^{-1}(\Delta(X_i))\,.$$

The dual space of g -g^*- can thus be equipped with a Poisson-Hopf structure whose fundamental brackets are

$$\{x^i, x^j\} \;=\; f_k^{ij}\, x^k, \tag{2}$$

where the x^i can be considered as local coordinates on g^*. So for any two functions on Fun(g^*) their Poisson bracket will be then given by

$$\{f_1, f_2\} \;=\; \frac{\partial f_1}{\partial x^i}\,\frac{\partial f_2}{\partial x^j}\; f_{ij}^k\; x^k\,.$$

Taking in mind all these ideas we see that we can quantize the Kostant-Kirillov-Souriau structure [5, 6] as follows

$$\{x^i, x^j\} = f_k^{ij} x^k \qquad \Longrightarrow \qquad [X_i, X_j] = \hbar\, f_{ij}^k\, X_k\,,$$

obtaining a "new" Lie algebra $g_\hbar$ isomorphic to g^* (and also to g as $f_{ij}^k = f_k^{ij}$) . Considering $\hbar$ as a free parameter we have thus a family of equivalent Lie algebras $g_\hbar$ determined by

$$[X_i, X_j] = \hbar\, f_{ij}^k\, X_k \tag{3}$$

that coincide with the original algebra when $\hbar = 1$.

This approach can be iterated for deformed algebras. Indeed, quantum groups were initially introduced as "quantizations" and the deformation parameter q was supposed to be related to the Planck constant $\hbar$ (for a detailed discussion, see [7, 8] and references therein). However, soon after, q became a parameter whose geometric and/or physical meaning has to be found for each particular case. In fact, quantum groups were soon seen as "abstract" Hopf algebras in order to explore whether they can be considered as symmetries of some physically relevant systems. The keystone of this idea was the discovery of the $su_q(2)$ invariance of the Heisenberg spin XXZ chain [9, 10], that was followed by a number of results exploiting quantum algebra symmetries in two-dimensional models [11]. Indeed, in the XXZ chain the "quantum" deformation parameter $z := \log(q)$ is identified with the anisotropy of the chain, which is completely independent of the (truly quantum) Planck constant. The same independence with respect to $\hbar$ can be traced in many other physical applications of quantum algebras and groups, like for instance lattice systems (where z is related to the lattice length) [12, 13], deformations of kinematical symmetries (in which the deformation parameter is a fundamental scale, see [14, 15] and references therein), effective nuclear models [16, 17], etc.

All these results suggest that $\hbar$ and q can be introduced separately in physics. It is exactly starting from this idea that a mathematical framework in which both the deformation parameter $z := \log q$ and the Planck constant $\hbar$ are independently and simultaneously considered has been introduced in quantum algebras [2, 3]. From an arbitrary quantum algebra $U_z(g)$ a one-parameter family of equivalent quantizations $U_{z,\hbar}(g)$, that depends explicitly on both $\hbar$ and z, have been constructed. Then, a q-Poisson-Hopf algebra was obtained as the $\hbar \to 0$ limit, getting -as in the undeformed case- a proper classical-mechanical limit of quantum-mechanical structure of symmetry.

From a technical point of view, to a one-parameter family of algebras $g_\hbar$, eq. (3), a one-parameter family of Lie bialgebras $(g_\hbar, \delta)$ is associated. By using the method of the analytical deformation, described in [18, 19], a two-parameter quantum algebra $(U_{z,\hbar}(g), \Delta_{z,\hbar})$, depending explicitly on $\hbar$ (as well as on z), is thus obtained. Poisson-Hopf super-algebra is then derived in the limit $\hbar \to 0$. Here we limit ourselves to summarize the procedure (see [2, 3] for details).

The starting point is the bialgebra (g, δ) [20], related to the quantum algebra, where g is a Lie algebra

$$[X_i, X_j] = f_{ij}^k \, X_k \tag{4}$$

and δ is the antisymmetric compatible cocommutator map $\delta : g \to g \otimes g$

$$\delta(X_i) \;:=\; z \, c_i^{jk} \, X_j \otimes X_k \,. \tag{5}$$

From the general properties of Hopf algebras and considering expression (4) and (5) as first orders in z of the quantum algebra, i.e.

$$[X_i, X_j] \;=\; f_{ij}^k \, X_k + o(z^2)$$

$$\Delta_z(X) = (X \otimes 1 + 1 \otimes X) + z \, \delta(X) + o(z^2),$$

the analytical form of the quantum algebra $(U_z(g), \Delta_z)$ was derived in [18, 19] as a first step.

Now, as a second step, eq.(3) has been substituted to (4) without varying (5), or -equivalently- the mapping

$$X_i \mapsto X_i/\hbar \,, \quad z \mapsto \hbar z \,. \tag{6}$$

has been realized inside (4) and (5). As the above described procedure does not change, a two-parameter quantum algebra $(U_{z,\hbar}(g), \Delta_{z,\hbar})$ was thus obtained.

As a third and last step, from $(U_{z,\hbar}(g), \Delta_{z,\hbar})$, the Poisson-Hopf algebra $(\mathrm{Fun}(g_z), \Delta_z^P)$ has been derived in the Poisson limit (i.e., $\hbar \to 0$). $(\mathrm{Fun}(g_z), \Delta_z^P)$ is just a Poisson-Lie structure on the quantum algebra $(U_z(g), \Delta_z)$ whose Lie bialgebra limit is characterized by the structure tensor f_i^{jk} of eq. (2). Concretely the Poisson bracket on $\mathrm{Fun}(g_z)$ is obtained by

$$\{\mathcal{X}_i, \mathcal{X}_j\} \;:=\; \lim_{\hbar \to 0} \; \frac{[X_i, X_j]}{\hbar}, \tag{7}$$

and the coproduct map by

$$\Delta_z^P(\mathcal{X}_i) \;:=\; \lim_{\hbar \to 0} \; \Delta_{z,\hbar}(X_i) \,, \tag{8}$$

where $\mathcal{X}_i (:= \mathbb{Q}^{-1}(X_i))$ can be seen as the local coordinates on the manifold ${g_z}^*$, so the Poisson brackets (7) are the so-called fundamental brackets and the coproduct (8) is a Poisson algebra homomorphism between $\mathrm{Fun}(g_z)$ and $\mathrm{Fun}(g_z) \otimes \mathrm{Fun}(g_z)$. Of course in the limit $z \to 0$ (i.e. starting from undeformed Lie algebra), coproducts are primitive and commutation rules linear: in this case, the Poisson limits (7) and (8) lead to the same formal structure where commutation rules have been just replaced by Poisson brackets. However, in quantum algebras nonlinear functions of the generators appear both in commutation rules and in coproducts: the q-Poisson structure given by the limit $\hbar \to 0$ can be formally different from the original quantum algebra structure and this is exactly what happens because, in the Poisson limit, contributions that arise from reordering disappear.

From a mathematical point of view, the dependence from $\hbar$ is regular for $\hbar \neq 0$ but the limit $\hbar \to 0$ is singular and does not commute with the procedure defined in eqs.(7,8), we can formally resume as

$$(\mathrm{Fun}(g_z), \Delta_z^P) := \lim_{\hbar\to 0} (U_{z,\hbar}, \Delta_{z,\hbar}),$$

that is just the Poisson-Lie structure on g_z whose algebra is determined by the dual δ^* of the Lie bialgebra map (5), (see [20, 8]).

The approach above described is extended in this paper to super-algebras introducing an adequate rescaling of odd generators (see eq.(13)). In conclusion, we have the following commutative diagram where g is an arbitrary algebra or super-algebra:

$$\begin{array}{ccccc} (U(g),\Delta_0) & \xleftarrow{\hbar=1} & (U_\hbar(g),\Delta_0) & \xrightarrow{\hbar\to 0} & (\mathrm{Fun}(g),\Delta_0^P) \\ \uparrow{\scriptstyle z\to 0} & & \uparrow{\scriptstyle z\to 0} & & \uparrow{\scriptstyle z\to 0} \\ (U_z(g),\Delta_z) & \xleftarrow{\hbar=1} & (U_{z,\hbar}(g),\Delta_{z,\hbar}) & \xrightarrow{\hbar\to 0} & (\mathrm{Fun}(g_z),\Delta_z^P)\,. \end{array}$$

Our interest is focused on the lower right corner, $(\mathrm{Fun}(g_z), \Delta_z^P)$, that defines in a constructive way from any (super)-bialgebra a related Poisson-Hopf deformed (super)-algebra.

3. Standard q-Poisson algebras $su_q^P(2)$ and $su_q^P(3)$

The general preceding discussion is now illustrated by the $su_q^P(2)$ and $su_q^P(3)$ examples. The $\hbar$-dependent family of Lie bialgebras $(su(2)_\hbar, \delta)$ related to the standard bialgebra is

$$[J_+, J_-] = 2\,\hbar\, J_3, \qquad [J_3, J_\pm] = \pm\,\hbar\, J_\pm\,, \tag{9}$$

$$\delta(J_3) = 0, \qquad \delta(J_\pm) = z\, J_3 \wedge J_\pm\,. \tag{10}$$

Since eqs.(10) are the same of the standard bialgebra and we have not reordering in the quantization, coproducts are not related to $\hbar$,

$$\Delta_z(J_3) = \Delta_0(J_3) = J_3 \otimes 1 + 1 \otimes J_3, \qquad \Delta_z(J_\pm) = e^{z\,J_3} \otimes J_\pm + J_\pm \otimes e^{-z\,J_3}. \tag{11}$$

However, starting from (9,11) we get

$$[J_+, J_-] = \hbar\, \frac{\sinh(2\,z\,J_3)}{z}, \qquad [J_3, J_\pm] = \pm\,\hbar\, J_\pm$$

and the $(\hbar, z)$-deformed Casimir operator

$$C_{z,\hbar} = \frac{\sinh^2(zJ_3)}{z^2}\cosh(z\hbar) + \frac{1}{2}\,[J_+\,, J_-]_+\,. \tag{12}$$

As the limit $\hbar \to 0$ is singular, the parameter $\hbar$ becomes essential to obtain the Poisson algebras. Introducing $\mathcal{J}_i := \mathbb{Q}^{-}1(J_i)$, the limit $\hbar \to 0$ gives us the q-Poisson-Hopf algebra $su_q^P(2) \equiv (\mathrm{Fun}(g_z), \Delta_z^P)$ as the commutative algebra of functions on g_z endowed with the Poisson bracket and coproduct map obtained by performing, respectively, the limits (7) and (8):

$$\{\mathcal{J}_+, \mathcal{J}_-\} = \frac{\sinh(2\,z\,\mathcal{J}_3)}{z}\,, \qquad \{\mathcal{J}_3, \mathcal{J}_\pm\} = \pm\mathcal{J}_\pm\;;$$

$$\Delta_z^P(\mathcal{J}_3) = \Delta_0^P(\mathcal{J}_3) = \mathcal{J}_3 \otimes 1 + 1 \otimes \mathcal{J}_3\,, \qquad \Delta_z^P(\mathcal{J}_\pm) = e^{z\,\mathcal{J}_3} \otimes \mathcal{J}_\pm + \mathcal{J}_\pm \otimes e^{-z\,\mathcal{J}_3}$$

and the q-deformed Casimir function obtained from (12) is:

$$C_z^P = \lim_{\hbar\to 0} C_{q,\hbar} = \frac{\sinh^2(z\,\mathcal{J}_3)}{z^2} + \mathcal{J}_+\,\mathcal{J}_- \ .$$

For the deformed $su_q(3)$ the procedure is the same. We start from the standard $\hbar$-depending bialgebra for $su(3)$ $(i,j,k=1,2,3)$:

$$\begin{aligned}
&[H_i,H_j]=0, \qquad\qquad [H_i,F_{jk}]=\hbar\,(\delta_{ij}-\delta_{ik})F_{jk},\\
&[F_{ij},F_{kl}]=\hbar\,(\delta_{jk}F_{il}-\delta_{il}F_{kj})+\hbar\,\delta_{jk}\delta_{il}(H_i-H_j)
\end{aligned}$$

$$\begin{aligned}
&\delta(H_i)=0,\\
&\delta(F_{ij})=\tfrac{1}{2}\,(H_i-H_j)\wedge F_{ij}+\textstyle\sum_{k=i+1}^{j-1}F_{ik}\wedge F_{kj} \qquad (i<j),\\
&\delta(F_{ij})=\tfrac{1}{2}\,(H_j-H_i)\wedge F_{ij}-\textstyle\sum_{k=j+1}^{i-1}F_{ik}\wedge F_{kj} \qquad (i>j).
\end{aligned}$$

The analytical approach of [18, 19] allows to obtain the quantum algebra $su_{z,\hbar}(3)$ on which the related $\hbar\to 0$ limit is defined. All the details can be found in [2, 3]. As an example of the fact that commutators and not only the Universal Enveloping Algebra are, also at formal level, different in semi-classical limit we exhibit the quantum commutator

$$[F_{12},F_{13}] = \frac{4}{\hbar}\,(\sinh\frac{z\hbar}{2})^2\,F_{12}\,F_{23}\,F_{12}$$

that, because the factor $\frac{4}{\hbar}\,(\sinh\frac{z\hbar}{2})^2$ is related to the reordering, reduces at semi-classical level to the Poisson bracket

$$\{\mathcal{F}_{12},\mathcal{F}_{13}\} = z^2\,\mathcal{F}_{12}^2\,\mathcal{F}_{23}\,.$$

4. (Quantum) Super-algebras and (Quantum) Super-Poisson

In quantum mechanics what is defined is the product of operators i.e. both commutator and anticommutator, but the commutator has been considered the relevant one because if two operators commute they are simultaneously diagonalizable, disregarding the fact that if they do not commute the commutator is an operator like all the others including anticommutator. Other reasons of our interest in commutators are that the elements of a Lie group are related to the commutators of the generators and that we are used to consider as the most relevant super-algebra in physics the super-algebra of the fermions $h(1|1)$.

However super-algebras are objects that can play a large role in the description of symmetry as it is well known and also shown in the following section. The preceding approach has thus been extended to super-algebras, both Lie and quantum, introducing a generalization of eq. (6) to describe the relation between Poisson anti-brackets and anticommutators. The $\hbar$ dependence is indeed different for even (we denote by X_i) and odd (we denote by v_l) generators. The super-algebra $\{X_i; v_l\}$ is defined by

$$[X_i,X_j] = f_{ij}^k\,X_k \qquad [X_i,v_l] = g_{il}^m\,v_m \qquad [v_l,v_m]_+ = h_{lm}^k\,X_k$$

where $[\bullet,\bullet]_+$ is the anticommutator, $f_{ij}^k = -f_{ji}^k$ and $h_{lm}^k = +h_{ml}^k$ but in the classical limit, while commutators go to zero, anticommutators reduce to the simple product of non ordered objects. I.e.

$$[\bullet,\bullet] \;\propto\; \hbar^1 + o(\hbar^2) \qquad\qquad [\bullet,\bullet]_+ \;\propto\; \hbar^0 + o(\hbar^1).$$

So eq. (6) is still valid for the even elements of the bialgebra but, to obtain the correct $\hbar \to 0$ limit, it must be improved by an appropriate relation for the odd part:

$$X_i \;\mapsto\; X_i/\hbar\,, \qquad v_l \;\mapsto\; v_l/\sqrt{\hbar}, \qquad z \;\mapsto\; \hbar\, z. \tag{13}$$

As shown in the following examples, this recipe is all we need to extend the limit to Lie and quantum super-algebras.

5. The Poisson-Hopf super-algebras $osp^P(1|2)$ and its quantization $osp^P_q(1|2)$

The smallest super-algebra is $h(1|1)$, the algebra of fermions. The transition to semi-classical world is trivial and the only possible deformation has not found application up to now. We thus consider $osp(1|2)$, the second super-algebra in order of "complication". It has three even generators $\{H, X_\pm\}$ and two odd ones $\{v_\pm\}$. $X_\pm$ can be considered as elements of the Universal Enveloping Algebra as, in the appropriate real form,

$$X_\pm \;=\; \frac{1}{2}[v_\pm, v_\pm]_+ \;=\; {v_\pm}^2.$$

The full algebra can be thus described in terms of $\{H, v_\pm\}$ by

$$[H, v_\pm] \;=\; \pm\, v_\pm \qquad [v_+, v_-]_+ \;=\; 2\,H.$$

The best known $osp(1|2)$ representation is $\{|n\rangle\,;\; n \in I_+\}$ where

$$H|n\rangle = |n\rangle(n+1/2) \qquad v_+|n\rangle = |n+1\rangle\sqrt{n+1} \qquad v_-|n\rangle = |n-1\rangle\sqrt{n}\,. \tag{14}$$

We find thus an alternative description of the harmonic oscillator in this representation of $osp(1|2)$. All other representations -characterized by a positive highest weight $k \neq 1/2$- have not, up to now, found applications in physics.

Using the recipe (13), the semi-classical form $osp^P(1|2)$ is easily obtained and it is formally equal to $osp(1|2)$. For this reason, we describe here only its quantum deformation $osp_q(1|2)$ and its semi-classical limit. $osp_q(1|2)$ is well known [21] and we simply apply the preceding procedure. The $\hbar$-depending super-bialgebra is

$$[H, v_\pm] \;=\; \pm\,\hbar\, v_\pm, \qquad [v_+, v_-]_+ \;=\; 2\,H$$
$$\delta(H) = 0\,, \qquad \delta(v_\pm) \;=\; z\; H \wedge v_\pm\;.$$

The two-parameters family of analytical quantum super-deformation is described by

$$\begin{aligned} &[H, v_\pm] \;=\; \pm\,\hbar\, v_\pm, && [v_+, v_-]_+ \;=\; \tfrac{1}{z}\sinh(2\,z\,H) \\ &\Delta_z(H) = \Delta_0(H) = H\otimes 1 + 1\otimes H, && \Delta_z(v_\pm) = v_\pm\otimes e^{-zH} + e^{zH}\otimes v_\pm, \end{aligned} \tag{15}$$

and the deformed representation with $k = 1/2$ is $\{|n\rangle\,;\; n \in I_+\}$

$$\frac{H}{\hbar}|n\rangle = |n\rangle(n{+}1/2) \quad \frac{v_+}{\sqrt{\hbar}}|n\rangle = |n{+}1\rangle\; f(n{+}1) \quad \frac{v_-}{\sqrt{\hbar}}|n\rangle = |n{-}1\rangle\; f(n) \qquad f(n) = \sqrt{\frac{\sinh(2zn)}{2\,z\cosh(z)}}$$

Now in the limit $\hbar \to 0$ we get the q-Poisson-Hopf super-algebra $osp^P_q(1|2)$

$$\begin{aligned} &\{\mathcal{H}, \mathbf{v}_\pm\} \;=\; \pm\mathbf{v}_\pm && \{\mathbf{v}_+, \mathbf{v}_-\}_+ = \tfrac{1}{z}\,\sinh(2\,z\,\mathcal{H}), \\ &\Delta^P_z(\mathcal{H}) = \Delta^P_0(\mathcal{H}) = \mathcal{H}\otimes 1 + 1\otimes\mathcal{H}\,, && \Delta^P_z(\mathbf{v}_\pm) = \mathbf{v}_\pm\otimes e^{-z\mathcal{H}} + e^{+z\mathcal{H}}\otimes\mathbf{v}_\pm \end{aligned}$$

formally identical to (15).

6. General Philosophy

The algebraic structures here presented illustrate the full "hierarchy of complexity" that Hopf (super)-algebras provide. Quantum (super)-algebras $U_{z,\hbar}(g)$ (as well as $U_z(g) \equiv U_{z,\hbar=1}(g)$) would be the richest and most complex structures (both non commutative since $\hbar \neq 0$ and non cocommutative since $z \neq 0$), the q-Poisson (super)-algebras would be somehow intermediate (being commutative and non cocommutative) and the Lie (super)-algebras would be the "simplest" ones (non-commutative and cocommutative).

In this context, the procedure to obtain a q-Poisson analogue of a given quantum (super) algebra is canonically defined. Firstly, one goes from $U_z(g)$ to $U_{z,\hbar}(g)$ taking into account explicitly from the beginning both parameters $(\hbar, z)$ in the analytical construction of the deformation. At first sight, this two-parameter deformation seems to be irrelevant since for any finite value of $\hbar$, by using the Lie algebra automorphism $\{ X \mapsto \hbar X \ , \ v \mapsto \sqrt{\hbar}\, v \}$ and by transforming the deformation parameter as $z \mapsto z/\hbar$, (inverse of (13)), the quantum algebra $(U_{z,\hbar}(g), \Delta_{z,\hbar})$ can be converted into $(U_z(g), \Delta_z)$. However, in order to obtain the Poisson analogue, one has to perform the $\hbar \to 0$ limit that *does not commute* with this automorphism and gives rise to the right result. The extension to counity and antipode has not been discussed because it is trivial. Note that a similar singular limit, that time in the deformation parameter z, allows to construct the contraction of quantum algebras [22].

With respect to the Poisson limit (7) and (8) we would like finally to stress that, because of the $\hbar$-dependence, it is essential to write the quantum algebra in terms of products with well defined symmetry. The so called q-commutators, without precise symmetry, cannot be used since their Poisson limit is not well-defined. This fact did not allow in the past the construction of q-Poisson analogues for algebras of rank greater than one. However, the analytical bases approach presented in [18, 19] provides a quantization framework based on pure commutators and anticommutators and lead thus to a well-defined Poisson limit for any quantum (super)-algebras. This is another argument to consider analytical bases as the right generalization for quantum algebras of Cartan bases.

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