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Open String Renormalization Group Flow as a Field Theory

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This article shows that the integral flow-lines of the RG-flow of open string theory can be interpreted as the solitons of a Hořova–Lifshitz sigma-model of open membranes. The authors argue that the effective background description of this model implies the g-theorem of open string theory. Its close connection to boundary string field theory is described. Additionally, the study endows the Hilbert space of the open membrane with a graded non-commutative, associative, cyclic algebra and construct an open membrane field theory, whose action measures the energy difference between different backgrounds in open string field theory. The authors use an identity-based membrane field to proof Sen's conjecture. Finally, the ideas are applied to the topological string and it is shown that the membrane action is quantized in equivariant K-theory of the moduli space of framed instantons.

1. Introduction

We set the stage by considering the basic construction of a background independent open string theory.[1–3] Let $\mathfrak{B}$ be the space of open/closed string theories. It is a supermanifold with $U(1)$-symmetry or ghost-number N_g. Every point in $\mathfrak{B}$ is represented by a boundary perturbation $\mathcal{O} = cV$ of a sigma model W_0 of open-closed strings on the disk D, where c is the ghost field, which we define later. The full theory is thus

$$W = W_0 + \frac{1}{2\pi}\int_{\partial D} ds\, V(s)\,, \tag{1}$$

with W_0 the Polyakov action plus a ghost action but we will only consider situations, where ghost and matter sectors are completely decoupled. Due to the possibility to parametrize the boundary operator $\mathcal{O}$ by couplings Φ^I in the form $\mathcal{O} = \Phi^I \mathcal{O}_I$, another term for $\mathfrak{B}$ is the space of couplings. There is a non-degenerate fermionic two-form ω on $\mathfrak{B}$ with $N_g = -1$. This odd symplectic structure defines a Poisson bracket $\{\cdot,\cdot\}$ on $\mathfrak{B}$. The symmetries of ω are generated by a cohomological vector field β with $\beta^2 = 0$ and $N_g = 1$. Thus, $d\omega = 0$ and $0 = L_\beta \omega = d(i_\beta \omega)$. The symplectic form ω defines a skew-symmetric pairing on the tangent space $T_p\mathfrak{B}$ at every point $p \in \mathfrak{B}$ and a metric G_{IJ}

$$G_{IJ} = \frac{1}{2}\oint dsds' \left\langle \mathcal{O}_I(s)\,\mathcal{O}_J(s')\right\rangle_{disk}, \tag{2}$$

where $\langle \dots \rangle_{disk}$ is the expectation value that corresponds to the BPZ inner product of operators $\mathcal{O}$ of the underlying conformal field theory. These are necessary ingredients to define a Batalin–Vilkovisky or BV-action S indirectly through $dS = i_\beta\, \omega$. In local coordinates

$$dS = d\Phi^I \frac{\partial S}{\partial \Phi^I} = d\Phi^I \frac{1}{2}\oint dsds' \left\langle \mathcal{O}_I(s)\,\beta_{\mathcal{O}}(s')\right\rangle_{disk}. \tag{3}$$

Furthermore, the two-dimensional sigma model (1) comes equipped with a BRST charge Q and $\{Q,\mathcal{O}\} = \beta_{\mathcal{O}}$. It is easy to see that $\{S,S\} = 0$. The perturbation in (1) induces a scale-dependence into the theory and triggers a worldsheet RG-flow

$$\frac{\partial \Phi^I(t)}{\partial t} = -\beta^I(\Phi) = -\left(1-\Delta_{\mathcal{O}}\right)\Phi^I + O\left(\Phi^2\right), \tag{4}$$

whereby $\Delta_{\mathcal{O}}$ is the conformal dimension of the perturbation and $t = \log|\zeta|$ is the logarithmic energy scale. The flow lines $\Phi^I(t)$ parametrize continuous curves C_Φ in $\mathfrak{B}$, which start at $\Phi^I(t_i)$ and end at $\Phi^I(t_f)$. There is a submanifold $\mathcal{N} \subset \mathfrak{B}$ with $N_g = 0$ delineated by W_0, which corresponds to the perturbation by the identity. A general perturbation defines a potential function S on $\mathfrak{B}$ to which a vector field, the so-called beta function $\beta_{\mathcal{O}}$, is associated. The zeros of β correspond to the conformally invariant backgrounds and to the equations of motion obtained from (1) by the variation principle. Now we will show that the flow lines of the RG-flow (4) can be realized as instantons $\Phi^I(t)$ in a Hořova–Lifshitz theory of membranes and we derive some well-known formulas of the BV-action (3). The special trait of sigma-models W_0 is that the couplings Φ^I are identical to background fields on the $D = 26$ dimensional target manifold X. Put differently, the partition function $\mathcal{Z}$ at leading order in the string coupling is equal to the effective background theory S_{eff}, from which the equations of motion

$$\delta S_{eff}[\Phi] = \beta^I(\Phi) = 0 \tag{5}$$

can be derived. This implies that $\mathcal{N}$ can be interpreted as the phase space of $S_{eff}[\Phi]$ associated to W_0 and $\mathfrak{B}$ is the extended phase space in the BV-quantization. For open string boundary

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conditions discussed in the next section and without any boundary perturbation turned on, S_{eff} is the Dirac-Nambu-Goto action. In our normalization this is the D25-brane tension

$$T_{25} = \mathcal{Z}_{disk} = -\frac{\mathcal{Z}_{CFT}}{vol\,(\mathrm{PSL}\,(2,\mathbb{R}))} = \frac{1}{2\pi^2}\mathcal{Z}_{CFT} = \frac{1}{2\pi^2 g_c}. \tag{6}$$

At the fixed points $\beta^I(\Phi) = 0$, where the background fields are constant, T_{25} and S coincide. Going into the next sections, a key point to understand is that the background fields of the Hořova–Lifshitz theory we are going to discuss, are exactly the t-dependent flow lines $\Phi^I(t)$ parametrizing the curves C_Φ.

2. Membranes as Gradient Flow-Lines

2.1. Lifshitz Field Theory

Constructing an action functional that produces equations of motion of similar type as (4) is not hard. Consider a Lifshitz field theory S_L with critical exponent $z = 2$

$$S_L\left[\Phi^I(\boldsymbol{x}, t)\right] = \int dt\, d^D\boldsymbol{x} \sum_I \dot{\Phi}^I G_{IJ}\dot{\Phi}^J - \frac{\delta S[\Phi]}{\delta \Phi^I} G^{IJ} \frac{\delta S[\Phi]}{\delta \Phi^J}, \tag{7}$$

where $S[\Phi^I(\boldsymbol{x}, t)]$ is an action functional in D dimensions, G^{IJ} is the metric on the space of fields and $\dot{\Phi} = \partial\Phi/\partial t$. The classical solutions, which are minimizers of the energy, are subject to the flow equation

$$\dot{\Phi}^I = \pm\frac{\delta S}{\delta \Phi^J} G^{IJ}. \tag{8}$$

The existence of non-trivial solutions is determined by the number of distinct vacua, which in this case coincide with the critical points of S_{eff} as is shown below. Simple examples are the one-dimensional scalar-field models, giving rise to a SUSY-chain of Schrödinger operators with reflectionless Pöschl–Teller potentials, the most prominent of which the sine-Gordon model is. If one chooses S to be the Chern–Simons action, then (8) would be the equivalent to the self-duality condition for Yang–Mills instantons. The SUSY-structure of Lifshitz action functionals (7) is reflected in the fact that the Hamiltonian can be written as

$$H = \int d^D x\, \bar{Q}_I G^{IJ} Q_J, \tag{9}$$

whereby

$$Q_I = i\hat{P}_I + \frac{\delta S}{\delta \Phi^I} \tag{10}$$

$$\hat{P}_I = -i\frac{\delta}{\delta \Phi^I}. \tag{11}$$

The eigenstate $\Upsilon_0[\Phi^I(x)]$ of (9) with vanishing energy satisfies

$$Q_I \Upsilon_0\left[\Phi^I\right] = \left(\frac{\delta}{\delta\Phi^I} + \frac{\delta S}{\delta \Phi^I}\right)\Upsilon_0\left[\Phi^I\right] = 0. \tag{12}$$

Therefore, the solution is simply

$$\Upsilon_0\left[\Phi^I\right] = e^{-S[\Phi^I]}. \tag{13}$$

An interesting property of the density $\Upsilon_0[\Phi^I]^2$ of the Schrödinger functional is that it is equal to the path integral density

$$\Upsilon_0\left[\Phi^I\right]^2 = \Upsilon_0^*\left[\Phi^I\right]\Upsilon_0\left[\Phi^I\right] = \mathcal{D}\Phi\, e^{-S[\Phi^I]}. \tag{14}$$

Thus, given an inner product $\langle -, - \rangle$ on the state space $\mathfrak{H}$, the path integral $\mathcal{Z}[\Phi^I]$ associated to the action W has the form

$$\mathcal{Z}\left[\Phi^I\right] = \left\langle \Upsilon_0\left[\Phi^I\right], \Upsilon_0\left[\Phi^I\right]\right\rangle = \int \mathcal{D}\Phi\, e^{-S[\Phi^I]}. \tag{15}$$

Equipped with this knowledge, we would like to determine the form of S for the case when the Lifshitz fields $\Phi^I(t, \boldsymbol{x})$ are the couplings parametrizing the space $\mathfrak{B}$ in the last section. Comparing (8) with the RG flow (4), we see that S must satisfy

$$\frac{\delta S}{\delta \Phi^J} G^{IJ} = \beta^I(\Phi). \tag{16}$$

Multiplying by G_{JI} we get

$$\frac{\delta S}{\delta \Phi^J} = \beta^I(\Phi)\, G_{JI}. \tag{17}$$

Now using (3), the RHS becomes

$$\frac{\delta S}{\delta \Phi^J} = \int ds ds' \left\langle \mathcal{O}_J(s)\ \beta_{\mathcal{O}}(s')\right\rangle_{disk}. \tag{18}$$

Therefore, S is the BV-action (3) and in (7), $D = 26$. As stated previously and by comparing (5) and (16),

$$\frac{\delta S}{\delta \Phi^J} G^{IJ} = \frac{\delta S_{eff}}{\delta \Phi_I}. \tag{19}$$

Plugging this into (8) we see that the equations of motion are the gradient flow equations of the effective background action of the open bosonic string

$$\dot{\Phi}^I = \frac{\delta S_{eff}}{\delta \Phi_I}. \tag{20}$$

At a fixed point $S_{eff} = T_{25}$ and $\dot{\Phi}^I(t_i) = \dot{\Phi}^I(t_f) = 0$ as required. Additionally, along the flow lines, the action S changes monotonically

$$\frac{\partial S}{\partial t} = -\frac{\partial \Phi^I}{\partial t}\frac{\delta S}{\delta \Phi^I} = -\beta^I\beta^J G_{IJ}. \tag{21}$$

Therefore, S provides an entropy functional for the corresponding evolution. Moreover, in the case of boundary conformal field theories (BCFTs) it is the boundary entropy, and, in this context, the expression (21) is nothing else than the g-theorem[4,5] for open string theory. Unitarity of the underlying theory implies that G_{IJ} is positive definite, which means that S decreases along flows. If the perturbation is marginal or the identity we can instead write $\Upsilon_0[\Phi^I] = e^{-S_{eff}[\Phi^I]}$.

2.2. Membrane Sigma-Model

At this point we could ask if the 27-dimensional Lifshitz field theory is itself the effective background theory of some sigma-model. The answer is yes, and this is relatively straight forward to see. To first order, the effective action is $\mathcal{Z}_{disk} = S_{eff}$, whereby

$$\mathcal{Z}_{disk} = \int \mathcal{D}\gamma_{ab}\mathcal{D}X_{\mu}\, e^{-W_0 - \frac{1}{2\pi}\int_{\partial D} ds\ V(s)} \tag{22}$$

is the partition function of (1). This partition function can be written in the spirit of (15) as

$$\left\langle \Upsilon_{\hat{I}}\left[\gamma_{ab}, X_{\mu}\right], \Upsilon_{\hat{I}}\left[\gamma_{ab}, X_{\mu}\right]\right\rangle_{BPZ} = \mathcal{Z}_{disk}. \tag{23}$$

Here the product on the space of states $\mathfrak{H}$ is the BPZ inner product and $\Upsilon_{\hat{I}}$ indicates that the boundary is perturbed by the unit operator $\hat{I}$. This allows us to reengineer the Lifshitz sigma-model from which the functional $\Upsilon_{\hat{I}}[\gamma_{ab}, X_\mu]$ originates as a zero mode of the Hamiltonian (9). It is helpful to first consider the situation, where there is no boundary perturbation present

$$\Upsilon_{\hat{I}}\left[\gamma_{ab}, X_{\mu}\right] = e^{-W_0\left[\gamma_{ab}, X_{\mu}\right]}. \tag{24}$$

In this case the action W_0 is the Polyakov action on the disk

$$W_0\left[\gamma_{ab}, X_{\mu}\right] = \frac{1}{4\pi\alpha'}\int_D d^2\sigma\ \sqrt{\gamma}\gamma^{ab}\partial_a X^{\mu}\partial_b X_{\mu}, \tag{25}$$

where $\mu = 1, \dots, 26$ and $a, b = 1, 2$ are spacetime and worldsheet indices, respectively. Additionally, the fields $X_\mu(\tau, \sigma)$ are subject to Neumann boundary conditions

$$\partial_\sigma X^{\mu}\ (\tau, 0) = \partial_\sigma X^{\mu}\ (\tau, \pi) = 0. \tag{26}$$

Using (12), the functional (24) must satisfy

$$\mathcal{Q}_I \Upsilon_{\hat{I}}\left[\phi(x)\right] = \left(\frac{\delta}{\delta\phi_I} + \frac{\delta W_0}{\delta\phi_I}\right)\Upsilon_{\hat{I}}\left[\phi(x)\right] = 0. \tag{27}$$

The variation of (25) yields

$$\frac{\delta W}{\delta X_{\mu}} = \frac{1}{\sqrt{\gamma}}\partial_a\left(\sqrt{\gamma}\gamma^{ab}\partial_b X^{\mu}\right) = \Delta X^{\mu} \tag{28}$$

$$4\pi\alpha'\frac{1}{\sqrt{\gamma}}\frac{\delta W}{\delta\gamma^{ab}} = \partial_a X^{\mu}\partial_b X_{\mu} - \gamma_{ab}\ \left(\gamma^{cd}\partial_c X^{\mu}\partial_d X_{\mu}\right) = T_{ab}. \tag{29}$$

The quantum operators associated to the canonical momenta of the dynamical fields are

$$\hat{P}_{\mu} = -i\frac{\delta}{\delta X^{\mu}} \tag{30}$$

$$\hat{\pi}_{ab} = -i\frac{\delta}{\delta\gamma^{ab}}. \tag{31}$$

Thus, the constraints read

$$\left(\frac{\delta}{\delta\gamma_{ab}} + \frac{\sqrt{\gamma}}{4\pi\alpha'}T^{ab}\right)\Upsilon_{\hat{I}}\left[\gamma_{ab}(x), X^{\mu}(x)\right] = 0 \tag{32}$$

$$\left(\frac{\delta}{\delta X_{\mu}} + \frac{\sqrt{\gamma}}{4\pi\alpha'}\Delta X^{\mu}\right)\Upsilon_{\hat{I}}\left[\gamma_{ab}(x), X^{\mu}(x)\right] = 0. \tag{33}$$

The $2+1$-dimensional Lifshitz sigma-model with such a ground state is

$$S_{HLM} = \int dt d^2\sigma\sqrt{\gamma}\left(\frac{1}{\kappa^2}\dot{\gamma}_{ab}G^{abcd}\dot{\gamma}_{ab} + \frac{1}{\alpha}(\dot{X})^2 - \frac{\alpha}{(4\pi\alpha')^2}(\Delta X)^2 - \frac{\kappa^2}{4(4\pi\alpha')^2}T^{ab}G_{abcd}T^{cd}\right), \tag{34}$$

whereby G^{abcd} is the metric on the space of metrics similar as the DeWitt metric in general relativity and G_{abcd} its inverse. This model was first considered by Hořova[6] for closed worldsheets. It arises from a suitable diffeomorphism-invariant worldvolume gravity on $\Sigma_3 = \Sigma_2\times\mathbb{R}$. Additionally, it is a membrane sigma-model with a ground state of prescribed characteristics, and the equations of motion describe the worldsheet gradient flow due to our choice of t as the logarithmic scale of the theory. In (34) γ_{ab} is the metric on the spatial slice of constant t in a codimension-one foliation with respect to the ADM-decomposition of the three-dimensional membrane metric γ_{ij}. The gauge symmetries of the general version of the membrane model enjoys invariance under the so-called foliation-preserving diffeomorphisms (FPD), which consist of spacetime-dependent spatial diffeomorphisms and time-dependent time reparametrizations. The associated gauge fields are the lapse $N(t)$ and shift functions $N^a(\sigma^a, t)$. However, since our t is not physical time but rather the flow-time of the RG-flow, the lapse and shift functions are fixed to be $N = 1$ and $N^a = 0$. Then the full theory of Hořova reduces to (34) and the FPD reduce in the non-relativistic limit to infinitesimal spatial diffeomorphisms and infinitesimal time reparametrizations

$$\delta\sigma^a = \xi^a\ (t, \sigma^a),\ \delta t = \rho(t). \tag{35}$$

The metric changes according to

$$\delta\gamma_{ab} = \partial_a\xi^c\gamma_{bc} + \partial_b\xi^c\gamma_{ac} + \partial_c\xi^c\gamma_{ab} + \rho\dot{\gamma}_{ab}. \tag{36}$$

It is necessary to extend the boundary conditions (26) onto the membrane worldvolume. The boundaries are the surfaces delineated by $\sigma = 0$ and $\sigma = \pi$

$$X^{\alpha} = X^{\alpha}\ (t, \tau, 0) \tag{37}$$

$$X^{\alpha} = X^{\alpha}\ (t, \tau, \pi). \tag{38}$$

Hence, the Neumann boundary conditions for the membrane are

$$\partial_\sigma X^{\alpha}\ (t, \tau, 0) = \partial_\sigma X^{\alpha}\ (t, \tau, \pi) = 0. \tag{39}$$

Finally, the minimizers of the energy satisfy the worldsheet flow equations

$$\dot{X} = -\Delta X^{\mu} \tag{40a}$$

$$\dot{\gamma}_{ab} = -G_{abcd}\ T^{cd} = -K\gamma. \tag{40b}$$

The former equation is the mean curvature flow on the disk, while the latter is the Ricci-flow on the disk with K being the Gauss-curvature. The uniqueness and existence of the flow depends on the initial condition $\gamma(t_i)$ and the time-dependent boundary conditions. In two-dimensions the flow deforms the metric conformally, which does not change the physics if only irrelevant boundary perturbations are present. For a locally defined function u on D we can replace the Ricci-flow locally by the logarithmic fast diffusion equation

$$\frac{\partial u}{\partial t} = e^{-2u}\,\Delta u = -K \tag{41}$$

It is important to keep in mind that t represents the scale of the disk. This could have the effect of regularizing the flow by counteracting the formation of singularities. The space of solutions of (40b) is the space of Ricci-solitons on the disk with vertex operator insertions at the boundary $\mathfrak{R}_D$ for given initial conditions $(D,\gamma(t_i))$ and possibly time-dependent boundary conditions up to diffeomorphisms. There is a local description of $\mathfrak{R}_D$ in terms of $u(t)$. In our previous example the vertex operator was the unit operator, which represents no deformation. The corresponding Ricci-flow does not represent RG-flows in the space $\mathfrak{B}$. To see this, we endow the disk with the hyperbolic metric $\gamma(0) = \gamma_h$

$$\gamma_h = \frac{4dzd\bar{z}}{(1-|z|)^2} \tag{42}$$

and consider the flow (40b). There exists a solution for $t \in [0,\infty)$ of the form

$$\gamma\,(t) = (1+2t)\,\gamma_h, \tag{43}$$

which expands the disk. The local function in (41) is simply $u(t) = t$ because $K = -1$. This is the Ricci soliton $D \times \mathbb{R}_+$ with metric $ds^2 = -dt^2 + e^{2u}\gamma_{ab}d\sigma^a d\sigma^b$, which is just a t-dependent conformal transformation that does not affect the unperturbed classical theory (25).

An important point is that on the disk the c-ghost has three zero-modes, and the b-anti-ghost has zero. The zero-modes correspond to the three conformal killing vector fields generating global isometries of the disk. A different perspective is that the ghost anomaly on the disk violates the ghost number by -3 units. On $D \times \mathbb{R}_+$ however, there is the field ρ in (35) generating infinitesimal translation in t-direction. The associated ghost has a zero-mode, which corresponds to global time-translations and therefore the CKV-group is $SL(2,\mathbb{R}) \times U(1)$. This implies that the ghost number of the membrane vacuum $D \times \mathbb{R}_+$ is -4. In this scenario, up to a normalization factor, the state $|\Upsilon_{\hat{I}}\rangle$ in (24), when evaluated on the disk, is the $SL(2,\mathbb{R})$-vacuum $|0\rangle$, defined by the state-operator mapping in CFT from the insertion of the unit operator 1 at the origin into the path integral

$$|0\,\rangle = \int^{X(|z|=1)} \mathcal{D}\gamma_{ab}\mathcal{D}X_\mu\; e^{-W_0}. \tag{44}$$

Every other state $|A\rangle \in \mathfrak{H}$ can be defined by acting on the vacuum with the corresponding vertex operator $V_A(0)|0\rangle$. Its dual state $\langle A| \in \mathfrak{H}^*$ corresponds to the action of the vertex operator inserted at infinity acting on the dual vacuum $\langle 0|\,V_A(\infty)$. The vacuum state of the full three-dimensional theory $|\Upsilon_{\hat{I}}\rangle_M$ describes the continuous scaling of the $SL(2,\mathbb{R})$-vacuum. Beginning at $D \times \{0\}$ with (44), a scaling due to the flow can be reabsorbed by a rescaling of the disk-metric. However, the boundary conditions in the path integral are also scaled and while the state remains the same, the parametrization changes. In the half string representation of the Schrödinger functional it is visible that the $SL(2,\mathbb{R})$-vacuum is not reparameterization-invariant. Yet under a change of parametrization the functional (24) changes only by a constant factor. Therefore, in the unperturbed case the effective background theory of the membrane theory is the D25-brane tension T_{25}. Hence, the potential S is constant and there is no flow inside $\mathfrak{B}$. The inner product $\langle -,-\rangle_M$ on the Hilbert space $\mathfrak{H}_M$ of (34) agrees with the BPZ inner product when restricted to D at the fixed-points. If the potential is constant, then

$$\langle \Upsilon_{\hat{I}}, \Upsilon_{\hat{I}} \rangle_M = \langle 0\,|0\,\rangle_{BPZ} = \mathcal{Z}_{disk} = T_{25}. \tag{45}$$

A general expression for the inner product will be given in a moment. Beforehand let us consider the insertion of a vertex operator into the boundary, which is topologically a puncture. These punctures act as a source of negative curvature. Hence, on one hand they influence the Ricci-flow by changing the initial condition $\gamma(t_f)$ and on the other hand they change the boundary conditions for the fields X^μ. The conformal transformation described by the flow will act differently around the punctures depending on their conformal dimensions. In principle one could choose local coordinates around the punctures, where the metric is flat. The infinite possibilities of a choice of transition functions then would be described by an infinite dimensional fiber above each point in the moduli space of the n-punctured disk. This would also induce an infinite-dimensional bundle over the space $\mathfrak{R}_D$. Then every point of the fibre over a point in $\mathfrak{R}_D$ corresponds to a flow with given initial condition plus a choice of transition functions. For the remainder of this paper, we will not be concerned with the solutions of (40b). Instead, we proceed by assuming that we perturb the boundary by the tachyon profile $V = T(X)$

$$W = W_0 + \frac{1}{4\pi}\int_{\partial D} ds\; T\,(X)\,. \tag{46}$$

This also modifies the Neumann boundary conditions for the fields to

$$\partial_\sigma X_\mu + \partial_\mu T = 0. \tag{47}$$

It spoils the conformal invariance on the boundary, which is equal to the statement that the fields pick up a t-dependence $\dot{X} \neq 0$ and $\dot{\gamma}_{ab} \neq 0$ on ∂D. By design, a suitable vacuum functional Υ_T satisfying the constraints (32, 33) and the modified boundary conditions (47) has the form

$$\Upsilon_T\left[\gamma_{ab}, X_\mu\right] = e^{-W_0 - \frac{1}{4\pi}\int_{\partial D} ds\; T(X(s))}. \tag{48}$$

Therefore,

$$\left\langle \Upsilon_T\left[\gamma_{ab}, X_\mu\right], \Upsilon_T\left[\gamma_{ab}, X_\mu\right]\right\rangle_{BPZ} = \int \mathcal{D}\gamma_{ab}\mathcal{D}X_\mu\; e^{-W_0 - \frac{1}{2\pi}\int_{\partial D} ds\; T(X(s))}. \tag{49}$$

The state $|\Upsilon_T\rangle_M \in \mathfrak{H}_M$ corresponds to the continuous scaling of the vacuum $|0\rangle$ with the insertion of $T(X)$ at the boundary. For a constant tachyon profile, it describes the flow from the UV-fixed point to the IR-fixed point, which is the tachyon vacuum. We will use a parametrization in which the beta functions are strictly linear, such that the IR-fixed points always occur at infinity. Using (2), the metric on $\mathfrak{B}$ is

$$ds^2 = e^{-T} dT^2 \tag{50}$$

and the beta function is

$$\beta(T) = \beta^T \frac{d}{dT} = -T\frac{d}{dT}. \tag{51}$$

Hence, the action S of boundary SFT is

$$S[T] = T_{25}(1+T)\, e^{-T}. \tag{52}$$

The unstable UV-fixed at $T=0$ represents the D25-brane and the IR-fixed point at $T=\infty$ is the tachyon vacuum. The membrane state $|\Upsilon_T\rangle_M$ is associated to the curve C_T we mentioned earlier, which is parametrized by the background field $T(t)$ satisfying (4)

$$\frac{\partial T(t)}{\partial t} = -\beta^I(T) = -T \tag{53}$$

Hence,

$$T(t) = tT. \tag{54}$$

In terms of background fields, we can also denote the background membrane functional as $\Upsilon_0[T] = e^{-S_{eff}[T]}$. It follows, that

$$-\log \Upsilon_0\left[\Phi^I\right] = S_{eff}\left[\Phi^I\right] \tag{55}$$

This implies that the membrane state can be written as the regularized infinite product

$$\Upsilon_0\left[\Phi^I\right] = \det(\mathcal{D})^{-1/2} = \left(\prod_n \lambda_n\right)^{-1/2}, \tag{56}$$

where λ_n are eigenvalues of the operator $\mathcal{D}$ with a complete set of orthonormal eigenfunctions $\psi = \sum_n \psi_n c_n$ such that the bosonic string action is $W = \frac{1}{2}\sum_n c_n^2 \lambda_n$ with respect to the appropriate Neumann boundary conditions. Additionally, $\Upsilon_0[\Phi^I]$ satisfies

$$\partial_t \Upsilon_0\left[\Phi^I\right] = -\partial_t S_{eff}\left[\Phi^I\right]\, e^{-S_{eff}\left[\Phi^I\right]} = \beta^I \beta_I \Upsilon_0\left[\Phi^I\right] \tag{57}$$

The flow (57) is equivalent to the constraint (12) and implies that at fixed-points

$$\partial_t \Upsilon_0\left[\Phi^I(t)\right]\Big|_{t_i,t_f} = 0. \tag{58}$$

Notice that S in (52) has the general form

$$S[\Phi] = \left(1 + \beta^I(\Phi)\frac{d}{d\Phi^I}\right)\mathcal{Z}(\Phi) \tag{59}$$

with $\mathcal{Z}(T) = \mathcal{Z}(0)e^{-T}$, where $\mathcal{Z}(0)$ is the unperturbed partition function on the disk $\mathcal{Z}_{disk} = T_{25}$. The decay of the D-brane is reflected by the fact that $\mathcal{Z}(T)$ is exponentially supressed for large T. Therefore, the tachyon vacuum contains no open string excitations.

2.3. Membrane Correlation Functions

Let's discuss some properties of the inner product $\langle -,-\rangle_M$. At fixed points $D\times\{t_i\}$ and $D\times\{t_f\}$, $\langle -,-\rangle_M$ is equivalent to the BPZ-inner product on the Hilbert space of the respective BCFTs. Take for example the ground state associated to the theory perturbed by V

$$\begin{aligned} \left\langle \Upsilon_V \middle| \Upsilon_V \right\rangle_M\Big|_{t_i} &= \left\langle \Upsilon_V \middle| \Upsilon_V \right\rangle_{BPZ}^{BCFT_0} = S\left[\Phi\left(t_i\right)\right] \\ \left\langle \Upsilon_V \middle| \Upsilon_V \right\rangle_M\Big|_{t_f} &= \left\langle \Upsilon_V \middle| \Upsilon_V \right\rangle_{BPZ}^{BCFT_*} = S\left[\Phi\left(t_f\right)\right] \end{aligned} \tag{60}$$

Moreover, for arbitrary membrane states $\mathcal{A},\mathcal{B} \in \mathfrak{H}_M$ representing flows of arbitrary string states A, B, the product is a function interpolating continuously between two BPZ-inner products

$$\begin{aligned} \langle \mathcal{A}|\mathcal{B}\rangle_M\big|_{t_i} &= \langle A|B\rangle_{BPZ}^{BCFT_0} = \int \mathcal{D}\gamma_{ab}\mathcal{D}X_\mu\, e^{-W_0}\, V_A(\infty)\, V_B(0) \\ \langle \mathcal{A}|\mathcal{B}\rangle_M\big|_{t_f} &= \langle A|B\rangle_{BPZ}^{BCFT_*} = \int \mathcal{D}\gamma_{ab}\mathcal{D}X_\mu\, e^{-W_0-\frac{1}{2\pi}\int_{\partial D} ds\, V(X(s))}\, V_A(\infty)\, V_B(0). \end{aligned} \tag{61}$$

That is, for every string state of the $BCFT_0$ (which corresponds to W_0) there is a membrane state representing its flow under the RG-flow when $BCFT_0$ is perturbed. The set of all membrane states represents the RG-flow of $BCFT_0$. This implies that they constitute a function that maps the reference Hilbert space onto another. In terms of D-brane categories this is a functor. In general terms, the function $\langle -,-\rangle_M$ is

$$\langle \mathcal{A}|\mathcal{B}\rangle_M = \left(1+\beta^I(\Phi)\frac{d}{d\Phi^I}\right)\langle A|B\rangle_{BPZ}^{BCFT_*} \tag{62}$$

The dependence of the two-point function on the couplings Φ^I is due to the dependence of the partition function on Φ^I. It is the evolution equation of the two-point functions in $BCFT_0$ under change of scale. The action (59) is the special case

$$\left\langle \Upsilon_V \middle| \Upsilon_V \right\rangle_M = \left(1+\beta^I(\Phi)\frac{d}{d\Phi^I}\right)\int \mathcal{D}\gamma_{ab}\mathcal{D}X_\mu\, e^{-W_0-\frac{1}{2\pi}\int_{\partial D} ds\, V(X(s))}. \tag{63}$$

This implies that the norm $\langle \Upsilon_V|\Upsilon_V\rangle_M$ defines a function on the space of string theories, which is equivalent to the g-function $S[\Phi]$. This extends to the n-point correlation functions of the membrane model

$$\begin{aligned} \left\langle \prod_{i=1}^n \tilde{\mathcal{O}}_i\left(z_i,t\right)\right\rangle_M &= \left(1+\beta^I(\Phi)\frac{d}{d\Phi^I}\right)\int \mathcal{D}\gamma_{ab}\mathcal{D}X_\mu\, e^{-W[\Phi]}\prod_{i=1}^n \mathcal{O}_i\left(z_i\right). \\ &= \left\langle e^{-\int_{\partial D}\Phi^I V_I}\prod_{i=1}^n \mathcal{O}_i\left(z_i\right)\right\rangle_0 + \beta^I(\Phi)\frac{d}{d\Phi^I}\left\langle e^{-\int_{\partial D}\Phi^I V_I}\prod_{i=1}^n \mathcal{O}_i\left(z_i\right)\right\rangle_0, \end{aligned} \tag{64}$$

where the expectation value is expressed with respect to the unperturbed theory $BCFT_0$. The existence of the membrane theory would imply background independence of the open string. However, one must be careful with the possible collision between the perturbation and the operators if the latter are boundary operators $\mathcal{O}_i(s_i)$. This can be avoided by cutting out disk D_i around each $\mathcal{O}_i$ and integrating over ∂D minus the disks. Assuming that the perturbation is small, we can expand it to first order in the path integral.

$$\left\langle e^{-\int_{\partial D}\Phi^I V_I}\prod_{i=1}^n \mathcal{O}_i(z_i)\right\rangle_0 \approx \left\langle \prod_{i=1}^n \mathcal{O}_i(z_i)\right\rangle_0 - \delta\left\langle \prod_{i=1}^n \mathcal{O}_i(z_i)\right\rangle_0$$
$$= \left\langle \prod_{i=1}^n \mathcal{O}_i(z_i)\right\rangle_0 - \Phi^I \int_{\partial D - \cup_i D_i}\left\langle V_I(s)\prod_{i=1}^n \mathcal{O}_i(z_i)\right\rangle_0. \quad (65)$$

Thus, in the vicinity of a zero of the beta-function (4) and at first order, we find

$$\left\langle \prod_{i=1}^n \mathcal{O}_i(z_i,t)\right\rangle_0 - \Phi^I \int_{\partial D - \cup_i D_i}\left\langle V_I(s)\prod_{i=1}^n \mathcal{O}_i(z_i)\right\rangle_0$$
$$+ (1-\Delta_{\mathcal{O}})\Phi^I \int_{\partial D - \cup_i D_i}\left\langle V_I(s)\prod_{i=1}^n \mathcal{O}_i(z_i)\right\rangle_0. \quad (66)$$

If we perturb by a marginal operator, then the membrane correlation functions are given in the vicinity of a fixed-point at first order by the correlation function evaluated with respect to the perturbed theory W

$$\left\langle \prod_{i=1}^n \tilde{\mathcal{O}}_i(z_i)\right\rangle_M = \left\langle \prod_{i=1}^n \mathcal{O}_i(z_i)\right\rangle_0$$
$$- \Phi^I \int_{\partial D - \cup_i D_i}\left\langle V(s)\prod_{i=1}^n \mathcal{O}_i(z_i)\right\rangle_0 = \left\langle \prod_{i=1}^n \mathcal{O}_i(z_i)\right\rangle_*. \quad (67)$$

This is expected since the beta functions annihilates the marginal perturbation and (67) is valid to all orders. For the case of a constant Tachyon profile discussed above all open string correlation function vanish, and the open string Hilbert space of the IR fixed-point is trivial. This means that the image of the flow, which represents Tachyon condensation is the kernel of the flow that creates a marginally deformed D-brane out of the Tachyon vacuum. In other words it is an exact sequence. The BRST-charge in the perturbed background is $Q + \delta Q$. Consequently, the t-dependent BRST-charge of the membrane theory Q_M satisfies.

$$Q_M\big|_{t_i} = Q = \oint_{C_\alpha} j_{BRST} \quad (68)$$

$$Q_M\big|_{t_f} = Q + \delta Q. \quad (69)$$

Since (64) is a t-dependent function, another quantity of interest is

$$-\int_{t_i}^{t_f} dt\, \partial_t \left\langle \prod_{i=1}^n \tilde{\mathcal{O}}_i(z_i,t)\right\rangle_M = \left\langle \prod_{i=1}^n \mathcal{O}_i(z_i)\right\rangle_0 - \left\langle \prod_{i=1}^n \mathcal{O}_i(z_i)\right\rangle_*. \quad (70)$$

The picture we have drawn indicatives that we can measure the difference in energy between different fixed-points by interpreting the perturbed BCFT as a particle in the potential S on the space $\mathfrak{B}$, evolving according to C_Φ

$$\Delta E = -\int_{C_\Phi} \beta^J(\Phi) G_{IJ} = -\int_{\Phi(t_i)}^{\Phi(t_f)} d\Phi^I \beta^J(\Phi) G_{IJ}$$
$$= -\int_{t_i}^{t_f} dt \frac{d\Phi^I}{dt}\beta^J(\Phi) G_{IJ} = \int_{t_i}^{t_f} dt\, \beta^I \beta^J G_{IJ}$$
$$= -\int_{t_i}^{t_f} dt\, \frac{dS(\Phi^I(t))}{dt} = S(\Phi^I(t_i)) - S(\Phi^I(t_f)). \quad (71)$$

In the case of the constant tachyon potential, we have $\Delta E = T_{25}$. Another hint that the D-brane has condensed completely. There is a large amount of evidence for the assumption that every BCFT corresponds to a string field $|\Psi\rangle$ that solves the equations of motion of cubic string field theory (see Appendix A). This indicates that for every path in $\mathfrak{B}$ there could exists a membrane field connecting different solutions of cubic string field theory and a membrane field theory. It is unlikely that the membranes exhibit dynamics different from those inherited by their boundary string theories. Therefore, we do not expect any equations of motion but rather a purely topological term, which, similar as (71), measures the change in background. We find that the construction of the MFT and its non-commutative, associative operations are easiest in the sliver coordinate frame (Appendix A).

3. Membrane Field Theory

3.1. Algebra and Action

If membrane fields represent paths in $\mathfrak{B}$, which are integral flow lines of the beta function, and those points are solutions to

$$\mathfrak{F}_\Psi = Q\Psi + \Psi \star \Psi = 0, \quad (72)$$

we may interpret (71) in terms of the action S_C of cubic SFT. In this picture membrane fields are paths of string fields starting and ending at critical points of S_C. Let Ψ_t with $t \in [0,1]$ be such a path with Ψ_0 and $\Psi_1 = \Psi'$ satisfying (72). Then

$$S_C[\Psi'] = S_C[\Psi_0] + \Delta E. \quad (73)$$

For $\Psi_0 = 0$ and $\Psi_t : t \to t\Psi$ the constant path, define

$$\mathfrak{F}_{\Psi_t} = tQ\Psi + t^2\Psi \star \Psi. \quad (74)$$

Then the action of the string field Ψ' is given by

$$\Delta E = S_C\left[\Psi'\right] = \int_0^1 dt \int \partial_t \Psi_t \star \mathfrak{F}_{\Psi_t}$$

$$= \int_0^1 dt \int \Psi \star \left(tQ\Psi + t^2 \Psi \star \Psi\right). \tag{75}$$

Integrating gives

$$S_C\left[\Psi'\right] = \int \Psi \star Q\Psi + \frac{2}{3}\Psi \star \Psi \star \Psi. \tag{76}$$

The path Ψ_t satisfies a string field analogue of the gradient flow Equation (4). A natural guess is that

$$\partial_t \Psi_t = \mathfrak{F}_{\Psi_t}, \tag{77}$$

but the ghost number is not conserved by the above equation. On the level of string field algebra (75) is non-trivial since the ghost number of the integrand is 3. We are integrating over a one-parameter family of inner products of string fields on the disk. Replacing $\partial_t \Psi_t$ according to (77) would cause the string field integral to vanish. Instead, we propose that the t-integration can be combined with the string field integration to an integration operation of membrane fields with ghost number −4, due to the additional ghost zero-mode corresponding to the infinitesimal translation in t. Now we are practically forced to replace terms according to (78) to raise the ghost number of the integrand by 1 and keep the expression non-trivial

$$\int_{t_i}^{t_f} dt \int \partial_t \Psi_t \star \mathfrak{F}_{\Psi_t} = \int \mathfrak{F}_{\Psi_t} \star \mathfrak{F}_{\Psi_t}. \tag{78}$$

Finally, we may replace the family of string fields by a single membrane field Υ and the star operation by a membrane field operation. This then implies that the energy difference ΔE has the general expression

$$S_{MFT} = \int \mathfrak{F}_\Upsilon \star \mathfrak{F}_\Upsilon, \tag{79}$$

whereby $\mathfrak{F}_\Upsilon = Q_M\,\Upsilon + \Upsilon \star \Upsilon$. For this to be true the membrane BRST-operator Q_M on $D \times \mathbb{R}$ must admit a decomposition into the standard BRST-operator Q on D and Q_t on $\mathbb{R}$. As usual, the charge Q is associated to the diffeomorphisms and Teichmüller deformations on D, whereas Q_t is the charge associated to the t-translations. It follows that the curvature $\mathfrak{F}_\Upsilon$ decomposes as

$$\mathfrak{F}_\Upsilon = Q_M\,\Upsilon + \Upsilon \star \Upsilon = Q_t \star \partial_t \Psi_t + \mathfrak{F}_{\Psi_t}. \tag{80}$$

Additionally, we define an operator $\mathfrak{o}_{\dim_{\mathbb{R}} \mathcal{K}}$ in the string and membrane star-algebra on the disk and on $D \times \mathbb{R}$, respectively, which maps elements with ghost number N_{gh} to elements with ghost number $\dim_{\mathbb{R}} \mathcal{K} - N_{gh}$. It defines an isomorphism on BRST-cohomology

$$\mathfrak{o}_{\dim_{\mathbb{R}} \mathcal{K}} : H^{N_{gh}}(Q) \to H^{\dim_{\mathbb{R}} \mathcal{K} - N_{gh}}(Q). \tag{81}$$

Here $\mathcal{K}$ is the conformal Killing group of $D \times \mathbb{R}$. Equivalently, it is the number of elements in the kernel of the projector $\mathcal{P}$

$$(\mathcal{P}c)_{ij} = \nabla_i\, c_j + \nabla_j c_i - g_{ij} \nabla_k c^k. \tag{82}$$

Hence, these elements correspond also to the ghost zero-modes on $D \times \mathbb{R}$. Their number is $\dim_{\mathbb{R}} \mathcal{K} = \dim_{\mathbb{R}} \ker \mathcal{P} = 4$. On D the same arguments lead to $\dim_{\mathbb{R}} \mathcal{K} = 3$. For a string field $\Psi = cV(0)$, with V matter primary of dimension 1, the operator $\mathfrak{o}_3$ acts by

$$\mathfrak{o}_3 cV\;(0) = c\partial c V(0) \tag{83}$$

and for the basis 1 of the zeroth BRST-cohomology

$$\mathfrak{o}_3 1 = c\partial c \partial^2 c\,(0). \tag{84}$$

Additionally,

$$\mathfrak{o}_{\dim_{\mathbb{R}} \mathcal{K}}\left(\mathfrak{o}_{\dim_{\mathbb{R}} \mathcal{K}} \Psi\right) = -(-1)^{N_{gh}\left(1+\dim_{\mathbb{R}} \mathcal{K}\right)} \Psi. \tag{85}$$

Again let $N_{gh}(\Psi) = 1$. It is

$$\mathfrak{o}_4\left(Q_t \star \Psi\right) = \mathfrak{o}_3\,\Psi. \tag{86}$$

The relations (75) are true for an arbitrary path and not just gradient flow-lines. Now the condition on $D \times \mathbb{R}$, which reads

$$\mathfrak{o}_4 \mathfrak{F}_\Upsilon = \mathfrak{F}_\Upsilon \tag{87}$$

implies that the membrane field Υ corresponds to a BCFT Ψ at t_f and t_i such that it satisfies the equation of motion $\mathfrak{F}_\Psi = 0$. Hence, it continuously connects fixed-points of the gradient-flow of the open bosonic string. It is

$$\mathfrak{o}_4 \mathfrak{F}_\Upsilon = \mathfrak{o}_3 \partial_t \Psi_t + Q_t \star \mathfrak{o}_3 \mathfrak{F}_{\Psi_t}. \tag{88}$$

Thus, the condition (87) can be expressed on D through

$$\partial_t \Psi_t = \mathfrak{o}_3 \mathfrak{F}_{\Psi_t}. \tag{89}$$

We will briefly discuss some of its solutions in Section 3.5 and proof that our consideration here reproduce the correct flows and energies. Furthermore, we may interpret the first term on the RHS in (71) as

$$S_{MFT} = \int Q_M L_C\left[\Psi_t\right], \tag{90}$$

where the integral is the membrane integral operation from (79). To prove (90), we will use the following membrane field algebra relations that follow from or previous discussion. That is, the algebra $\mathfrak{A}$ is non-commutative, cyclic, graded and differential associative. For $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathfrak{H}_M$ there is a star multiplication $\star : \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$ and a trace operation $\int : \mathfrak{A} \otimes \mathfrak{A} \to \mathbb{C}$ with

$$\mathcal{A}(\mathcal{B}\mathcal{C}) = (\mathcal{A}\mathcal{B})\,\mathcal{C} \tag{91}$$

$$N_{gh}(Q\mathcal{A}) = N_{gh}(\mathcal{A}) + 1 \tag{92}$$

$$N_{gh}(\mathcal{A}\mathcal{B}) = N_{gh}(\mathcal{A}) + N_{gh}(\mathcal{B}) \tag{93}$$

$$Tr(\mathcal{A}) = 0 \quad \text{if } N_{gh}(\mathcal{A}) \neq 4 \tag{94}$$

$$Tr\left(Q_M \mathcal{A}\right) = 0 \tag{95}$$

$$Tr\left(\mathcal{A}\mathcal{B}\right)=(-1)^{|\mathcal{A}||\mathcal{B}|}\,Tr\left(\mathcal{B}\mathcal{A}\right) \tag{96}$$

$$\mathcal{Q}_M\left(\mathcal{A}\mathcal{B}\right)=\mathcal{Q}_M\left(\mathcal{A}\right)\mathcal{B}+(-1)^{|\mathcal{A}|}\mathcal{A}\mathcal{Q}_M\left(\mathcal{B}\right). \tag{97}$$

$$Tr\left(\mathcal{A}^n\right)=0,\ n\geq 4 \tag{98}$$

The above structure is equivalent to the algebra of matrix-valued forms on a four-manifold. Using the nilpotency of $\mathcal{Q}_M$ and the cubic string field Lagrangian L_C

$$\begin{aligned}\int \mathcal{Q}_M L_C\left[\Psi_t\right] &= \int \mathcal{Q}_M\left(\Psi_t\star\mathcal{Q}_M\Psi_t+\frac{2}{3}\Psi_t\star\Psi_t\star\Psi_t\right)\\ &=\int \mathcal{Q}_M\Psi_t\star\mathcal{Q}_M\Psi_t+\frac{2}{3}\mathcal{Q}_M\Psi_t\star\Psi_t\star\Psi_t\\ &\quad -\frac{2}{3}\Psi_t\star\mathcal{Q}_M\Psi_t\star\Psi_t+\frac{2}{3}\Psi_t\star\Psi_t\star\mathcal{Q}_M\Psi_t\\ &=\int \mathcal{Q}_M\Psi_t\star\mathcal{Q}_M\Psi_t+\mathcal{Q}_M\Psi_t\star\Psi_t\star\Psi_t\\ &\quad +\Psi_t\star\Psi_t\star\mathcal{Q}_M\Psi_t\\ &=\int \mathfrak{F}_{\Psi_t}\star\mathfrak{F}_{\Psi_t},\end{aligned} \tag{99}$$

where we used the cyclicity in the form of (96) and (94)

$$\begin{aligned}\int \mathcal{Q}_M\Psi_t\star\Psi_t\star\Psi_t &= -\int \Psi_t\star\mathcal{Q}_M\Psi_t\star\Psi_t\\ &=\int \Psi_t\star\Psi_t\star\mathcal{Q}_M\Psi_t.\end{aligned} \tag{100}$$

In analogy to gauge theory, we can interpret $\mathcal{Q}_M L_C[\Psi_t]$ as a current on $D\times\mathbb{R}$ and S_{MFT} as the charge of Υ. On two-dimensional slices delineated by $t=0$ that represent the worldsheet, this reduces to the action of cubic OSFT. It leads to a string field version of Stokes theorem

$$\begin{aligned}&\left[\int \Psi\star\mathcal{Q}_M\Psi+\frac{2}{3}\Psi\star\Psi\star\Psi\right]_{\partial\Sigma_3}\\ &=\left[\int \mathcal{Q}_M\left(\Upsilon\star\mathcal{Q}_M\Upsilon+\frac{2}{3}\Upsilon\star\Upsilon\star\Upsilon\right)\right]_{\Sigma_3}=\left[\int \mathfrak{F}_\Upsilon\star\mathfrak{F}_\Upsilon\right]_{\Sigma_3}.\end{aligned} \tag{101}$$

The gauge invariance of the theory (79) is

$$\begin{aligned}\delta\Upsilon &= \mathcal{Q}_M\Lambda+\Upsilon\star\Lambda-\Lambda\star\Upsilon\\ \delta\mathfrak{F} &= \mathfrak{F}\star\Lambda-\Lambda\star\mathfrak{F},\end{aligned} \tag{102}$$

whereby Λ is an infinitesimal gauge parameter. The associativity of the flow is inherited by the associativity of the string star-algebra. The equations of motion are strictly zero. Finally, the second condition in (102) implies that $\mathfrak{F}$ behaves like a gauge field and needs to be gauge-fixed. An example of such a gauge-fixing condition is the self-dual gauge (87).

3.2. Membrane Field Operations in the Sliver Frame

Now we will give a precise definition of the membrane star algebra operations and illustrate them in the sliver frame. Mapping the disk to the semi-infinite strip $\mathcal{S}$ by the conformal transformation f_S described in the Appendix, we define a membrane vertex operator $\mathcal{V}_\mathcal{A}(0,t)$ to be the vertex operator corresponding to the flow of state of $BCFT_0$ under a boundary perturbation $\mathcal{O}$. On $\mathcal{S}\times\{t_i\}$ we have the state

$$|\mathcal{A}\rangle_M\big|_{t_i}=\mathcal{V}_\mathcal{A}\left(0,t_i\right)\left|\widetilde{0}\right\rangle_M=|A\rangle_0=V_A\left(0\right)|0\rangle_0. \tag{103}$$

Consequently, the membrane state $\mathcal{A}$ is defined as

$$\begin{aligned}|\mathcal{A}\rangle_M &= \mathcal{V}_\mathcal{A}\left(0,t\right)\left|\widetilde{0}\right\rangle_M=\left(1+\mathcal{Q}\mathcal{O}\right)V_A\left(0\right)\ |0\rangle_*\\ &=\left(1+\beta\right)V_A\left(0\right)|0\rangle_*,\end{aligned} \tag{104}$$

where $|\tilde{0}\rangle_M$ is represented by the semi-infinite cuboid $\mathcal{S}\times\mathbb{R}_+$ and $|0\rangle_*=|e^{-\int_{\partial D}b_{-1}\mathcal{O}}\rangle_0$ is the perturbed vacuum. Similarly, we define its dual by

$$\langle\mathcal{A}|_M=\left\langle\widetilde{0}\right|\mathcal{V}_\mathcal{A}\left(\infty,t\right). \tag{105}$$

Then the frame independent membrane star operation is defined as

$$\mathcal{A}\star\mathcal{B}=\left(T_1\circ F\left(\xi,t\right)\circ\mathcal{V}_\mathcal{A}\left(0,t\right)\right)\ \left(F\left(\xi,t\right)\circ\mathcal{V}_\mathcal{B}\left(0,t\right)\right). \tag{106}$$

Here $T_a(t)=z+a$ is a translation map such that the origin on $\mathcal{S}\times\{0\}$ coincides with V_B. In general, it can be t-dependent. $F(\xi,t)$ is a one-parameter family of frames $z(t)=f_t(\xi)$. Each regular linear frame delineates a vector field, which is analytic around $|z|=1$ on a slice of constant t

$$v\left(\xi\right)=\frac{f\left(\xi\right)}{f'\left(\xi\right)}. \tag{107}$$

This in return determines the quantity

$$\mathfrak{B}\left[v\right]=\oint\frac{d\xi}{2\pi i}v\left(\xi\right)b\left(\xi\right), \tag{108}$$

which is the anti-ghost operator in this frame. The gauge condition in perturbation theory is given by the annihilation of the classical string field by the anti-ghost zero mode

$$\mathfrak{B}_0\left[v\right]\left|\Psi_{cl}\right\rangle=0, \tag{109}$$

which are so-called regular linear b-gauges.[7,8] It means that we can interpolate continuously between different gauge-fixing conditions for the string fields at different fixed points. This is necessary because the MFT should be invariant under the choice of these gauge-fixings, and it is useful because one gauge might be more practical for a particular background than others. An example for such an interpolating family is

$$F\left(\xi,t\right)=\frac{1}{2}\frac{tan^{-1}\left(e^{-t}\xi\right)}{tan^{-1}\left(e^{-t}\right)}=\frac{1}{2}\frac{cot^{-1}\left(\mu/\xi\right)}{cot^{-1}\left(\mu\right)}. \tag{110}$$

It interpolates between the Siegel gauge and the Schnabl gauge, with the latter arising from the familiar sliver conformal frame. We may call the family (110) a scale-dependent frame. Then we can define

$$v(\xi,t)=\frac{F(\xi,t)}{F'(\xi,t)}=\frac{\left(\xi^2+e^{2t}\right)tan^{-1}\left(e^{-t}\xi\right)}{e^t} \tag{111}$$

with $F'=dF/d\xi$ and

$$\tilde{\mathfrak{B}}[v]=\int_{t_i}^{t_f}dt\oint\frac{d\xi}{2\pi i}v(\xi,t)\,b(\xi)\,. \tag{112}$$

It follows that the gauge fixing condition for the membrane fields is

$$\tilde{\mathfrak{B}}_0[v]\,|\Upsilon_{cl}\rangle=0. \tag{113}$$

For the sake of geometric visualization however, we will stay in the sliver coordinate frame for all t.

$$F(\xi)=\frac{2}{\pi}\arctan(\xi)\,. \tag{114}$$

Then (113) is automatically satisfied and the product of the membrane star algebra corresponds to the gluing of semi-infinite cuboids $\mathcal{S}\times\mathbb{R}$ with membrane vertex operator insertions. For example, if we have two cuboids (**Figure 1**) with width α and β and length ε, after gluing, we end up with a cuboid of width $\alpha+\beta$ and length ε. The length ε is the logarithm of the ratio of the initial and final length scale

$$\varepsilon(t)=\log\left|\frac{\zeta_f}{\zeta_i}\right|=t_f-t_i. \tag{115}$$

The integral operation glues the left boundary surface of the cuboid and the right one together.

$$\int\mathcal{A}\star\mathcal{B}=\left\langle\left(T_1\circ F(\xi,t)\circ\mathcal{V}_{\mathcal{A}}(0,t)\right)\left(F(\xi,t)\circ\mathcal{V}_{\mathcal{B}}(0,t)\right)\right\rangle_{\varepsilon}, \tag{116}$$

where the subscript ε indicates that the correlation function exists inside the volume between the two cylinders with correlation function of two different CFTs which are ε apart (**Figure 2**). The result is a correlation function on a space bounded by two concentric cylinders. The two correlation functions at t_i and t_f between which (116) interpolates, are

$$\langle A\,|\,B\rangle_{BPZ}^{BCFT_0}=\left\langle\left(\tau_1\circ f_{\mathcal{S}}\circ V_A(0)\right)\left(f_{\mathcal{S}}\circ V_B(0)\right)\right\rangle_C \tag{117}$$

and

$$\langle A\,|\,B\rangle_{BPZ}^{BCFT_*}=\left\langle\left(\tau_1\circ f_{\mathcal{S}}\circ\mathcal{V}_{\mathcal{A}}\left(0,t_f\right)\right)\left(f_{\mathcal{S}}\circ\mathcal{V}_{\mathcal{B}}\left(0,t_f\right)\right)\right\rangle_{C^+}. \tag{118}$$

Here the subscript C^+ denotes the cylinder with circumference scaled by the integral operation (116). For example, let the inner cylinder be of circumference $C=\alpha+1$ with correlation function (117), then the outer cylinder has circumference $C^+=\alpha+1+2\pi(t_f-t_i)$. Hence, the integral operation scales the outer strip units when forming the correlation function, which corresponds to a deformation of the CFT. We interpret the integral operation (116) as connecting the BPZ inner products of the respective BCFTs. Moreover, (116) is equivalent to the inner product (62) on the membrane Hilbert space of (34).

The representation of the product given so far assumes the availability of a Schrödinger representation. This is not always the case. An alternative expression is given in terms of an arbitrary basis of states $|\Phi_i\rangle$ and the dual basis $|\Phi^i\rangle$ of $\mathfrak{H}_M$

$$\begin{aligned}\int\mathcal{A}\star\mathcal{B}&=\sum_i\left|\Phi_i\right\rangle\left\langle\Phi^i\right|^*\left|\mathcal{A}\mathcal{B}\right\rangle_M\\&=\sum_i|\Phi_i\rangle\left\langle\left(T_1\circ F(z,t)\circ\Phi^i(0)\right)\left(T_1\circ F(z,t)\circ\mathcal{V}_{\mathcal{A}}(0,t)\right)\right.\\&\quad\times\left.\left(F(z,t)\circ\mathcal{V}_{\mathcal{B}}(0,t)\right)\right\rangle_{\varepsilon}.\end{aligned} \tag{119}$$

Since the states in the Hilbert space of the three-dimensional theory interpolate between states of the worldsheet BCFTs, the basis $|\Phi_i\rangle$ also connects two bases in their respective BCFT Hilbert spaces. Hence, we can give the expression for the two string field products (119) connects, which are

$$\begin{aligned}\int A\star B&=\sum_i|\phi_i\rangle\langle\phi^i|^*|AB\rangle\\&=\sum_i|\phi_i\rangle\left\langle\left(\tau_2\circ f\circ\phi^i(0)\right)\left(\tau_1\circ f\circ V_A(0)\right)\left(f\circ V_B(0)\right)\right\rangle_C\end{aligned} \tag{120}$$

and

$$\begin{aligned}\int A\star B&=\sum_i|\phi_i\rangle|\phi^i|^*|AB\rangle\\&=\sum_i|\phi_i\rangle\left\langle\left(\tau_2\circ f\circ\phi^i(0)\right)\left(\tau_1\circ f\circ\mathcal{V}_{\mathcal{A}}\left(0,t_f\right)\right)\left(f\circ\mathcal{V}_{\mathcal{B}}\left(0,t_f\right)\right)\right\rangle_{C^+}\end{aligned} \tag{121}$$

The ϕ_i represent the basis of the space of states of the respective BCFTs. Now we are ready to define the membrane wedge state $\mathfrak{Q}^{\alpha,\varepsilon}$ and, most importantly, the membrane vacuum $\mathfrak{Q}^{1,\varepsilon}$

$$\mathfrak{Q}^{1,\varepsilon}=|\tilde{0}\rangle_M \tag{122}$$

in the sliver frame as a semi-infinite cuboid of length ε. There are two limits we should discuss, too. The first is $\varepsilon\to0$, which we call the identity membrane field $\mathfrak{Q}^{\alpha,\varepsilon\to0}=\mathfrak{I}^\alpha=Q^\alpha$. It is equal to the wedge state of the UV-fixed point. The second limit $\varepsilon\to\infty$ connects a wedge state of the inner CFT to the sliver Q^∞. We may write the state $|\mathcal{A}\rangle_M$ as

$$|\mathcal{A}\rangle_M=\mathcal{V}_{\mathcal{A}}(0,t)\left|\tilde{0}\right\rangle=\mathcal{A}\mathfrak{Q}^{1,\varepsilon(V)}, \tag{123}$$

whereby $\varepsilon(V)$ indicates that the length of the flow depends on the nature of the perturbation. Note that we can write $Q^\alpha=e^{-\alpha K}$, where K corresponds to a strip of infinitesimal width with a vertical line integral of the energy momentum tensor (**Figure 3**)

$$K=\oint_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\,T(z)\,. \tag{124}$$

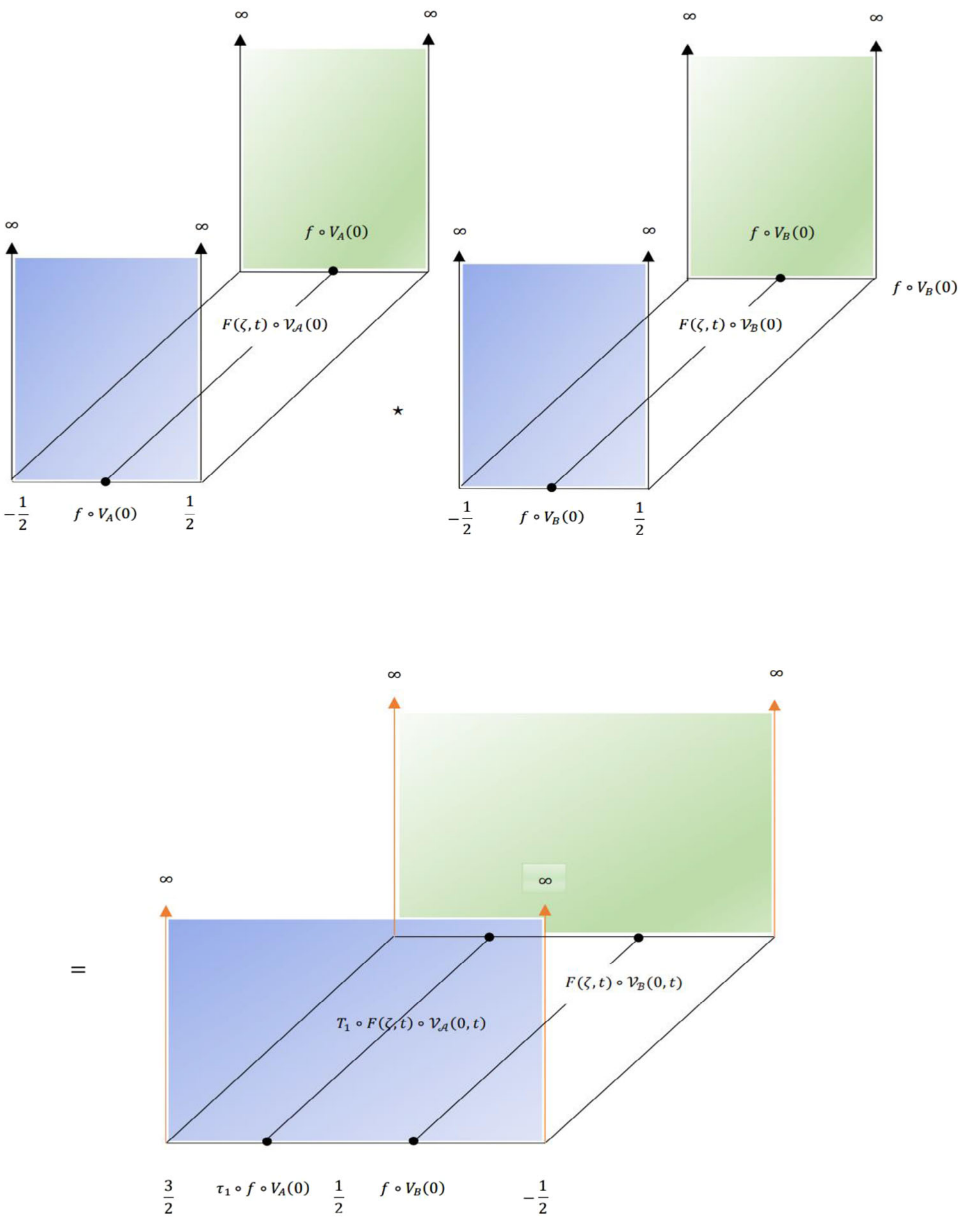

Figure 1. The membrane star product glues the right side of a semi-infinite cuboid to the left side of the other. Both cuboids are bounded by two string fields in the sliver coordinate frame, respectively. The membrane vertex operator bounded by two open string vertex operators.

Now consider a different representation of a wedge state

$$Q^{\alpha} = e^{-\alpha K} = \sqrt{Q}\, e^{-(\alpha-1)K}\sqrt{Q}. \tag{125}$$

Henceforth, the membrane field $\mathfrak{Q}^{\alpha,\varepsilon}$ connects a wedge state Q^{α} with

$$\begin{aligned} Q^{\alpha+2\pi\varepsilon} &= \sqrt{Q}e^{-(\alpha-1)K}Q^{2\pi\varepsilon}\sqrt{Q} = \sqrt{Q}e^{-(\alpha-1+2\pi\varepsilon)K}\sqrt{Q} \\ &= \sqrt{Q}e^{-(\lambda\alpha-1)K}\sqrt{Q} = Q^{\alpha\lambda} \end{aligned} \tag{126}$$

Consider as an example the path represented by the membrane field

$$\Upsilon = \mathfrak{I}^{1/2}\mathcal{c}\mathfrak{Q}^{0,\varepsilon}\mathcal{B}\mathfrak{Q}^{0,\varepsilon}\mathcal{c}\mathfrak{I}^{1/2} \tag{127}$$

shown in **Figure 6**. The field $\mathcal{B}$ and its correlation function are depicted in Figures 4 and 5. In the limit $\varepsilon \to 0$, the vacua reduce to wedge states and the field reduces to the zero-momentum tachyon,

$$\begin{aligned} \Upsilon_{\varepsilon\to 0} &= \sqrt{Q}cBc\sqrt{Q} = \sqrt{Q}c\,(1 - cB)\,\sqrt{Q} \\ &= \sqrt{Q}c\sqrt{Q} = c_1|0\rangle \end{aligned} \tag{128}$$

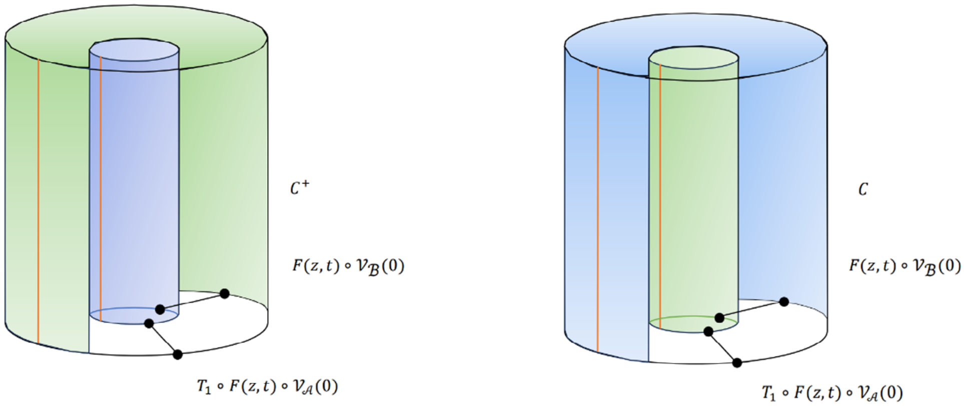

Figure 2. The membrane integral operation identifies the boundaries of the cuboid and glues them together. However, there are two ways of gluing them. (Left) The edges of the cuboid are glued together such that the edges of the blue strip are glued to form a cylinder without scaling it. This results in an up-scaling of the outer green cylinder, which must wind around the inner one at distance ε. The result is a correlation function inside a volume bound by two concentric cylinders. The circumference C^+ of the outer cylinder depends on their distance ε in the space of boundary CFT's. (Right) The second way to form the membrane integral operation glues the two-dimensional boundaries of the cuboid such that blue strip winds around the green strip without rescaling the blue strip. This results in a down-scaling of the inner correlation function. In this case, the distance ε is bounded from above because the scaled circumference C^- must remain finite. In both integral operations depicted, the blue strips have identical circumference. In a unitarian theory, only the left side is realized.

where we used $[B, c] = 1$. Forming the correlation function, in the limit $\varepsilon \to \infty$, Υ describes the phantom term of the tachyon vacuum

$$\Upsilon_{\varepsilon\to\infty} = \sqrt{Q}cBQ^{\infty}c\sqrt{Q}. \tag{129}$$

3.3. Multi-Boundary Strings and Membrane Fields

Ideas to identify different BCFT in open string field theory related by marginal deformations, involve insertions of exponentiated line integrals of boundary matter primaries V, which deform the open string boundary. Let $V = \kappa\hat{V}(0)|0\rangle$ be the string field corresponding to the boundary matter primary of weight 1 with κ the coupling constant of the boundary deformation. In this case we set the scaling factor to

$$\lambda = 1 + \kappa\hat{V}\int_0^{\infty} d\alpha\ Q^{\alpha}, \tag{130}$$

whereby $\hat{V}Q^{\alpha}$ tends to zero in the Fock space expansion for large α. It is reasonable to assume that K can be treated as real and positive variable and we absorb κ with $V = \kappa\hat{V}$ such that

$$V\int_0^{\infty} d\alpha\ Q^{\alpha} = \frac{V}{K} \tag{131}$$

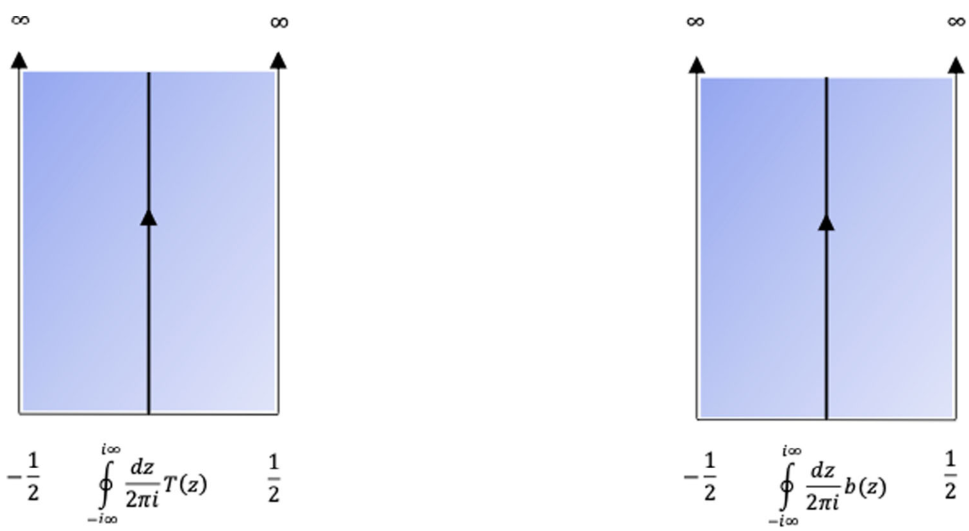

Figure 3. The string field K (left) defined by an infinitesimal strip with the insertion of a vertical line integral of the energy-momentum operator $T(z)$. (Right) The string field B defined through an infinitesimal strip with the vertical integration of the b-ghost. Here both string fields are depicted as insertions into the vacuum-strip.

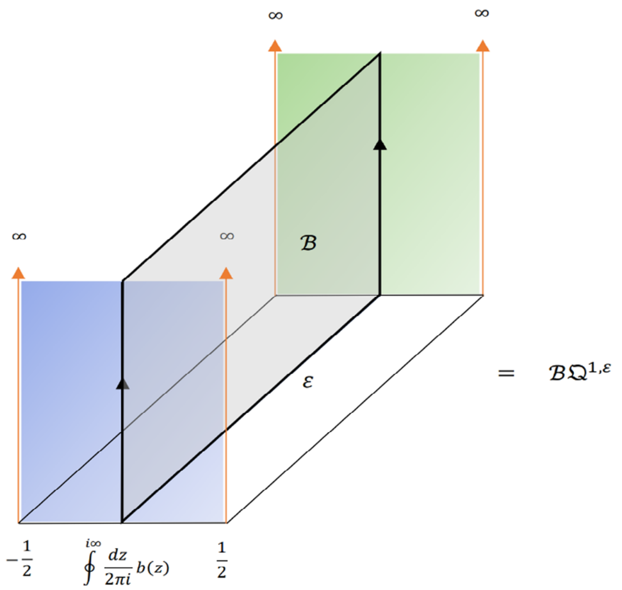

Figure 4. The membrane field $\mathcal{B}$ as a surface bounded by insertions of two vertical line integrals inserted into the membrane field vacuum.

is well-defined. In this way the marginally deformed background becomes

$$Q_V^\alpha = Q^{\lambda\alpha} = e^{-\lambda\alpha K} = e^{-\alpha(K+V)}. \tag{132}$$

Note that with (130) and (131) the relation for the ratio between the length of the marginal flow $\varepsilon = t_f - t_i$, the width of the wedge state α and the matter primary V deforming the boundary

$$\frac{V}{K} = \frac{2\pi\varepsilon}{\alpha}. \tag{133}$$

Hence, we get a family of flows for different regular marginal V, which annihilate the sliver, of length

$$\varepsilon\,(V;\alpha) = \frac{\alpha}{2\pi}\,\frac{V}{K}, \tag{134}$$

which implies that the ratio between the scales is

$$\frac{\zeta_f}{\zeta_i} = exp\left(\frac{\alpha}{2\pi}\,\frac{V}{K}\right). \tag{135}$$

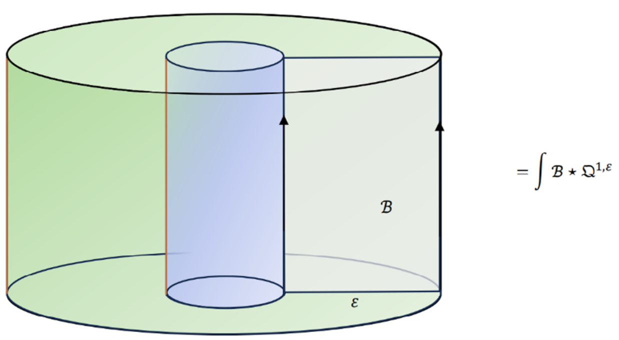

Figure 5. The integral operation of the membrane field $\mathcal{B}$ and the membrane vacuum.

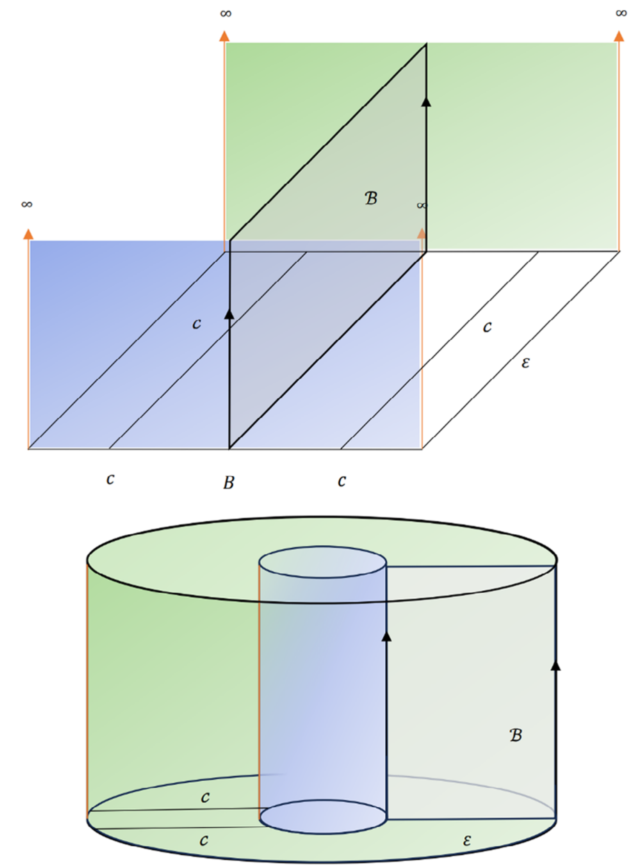

Figure 6. The membrane field Υ as the vacuum with insertions of the fields $\mathcal{B}$ and c and its correlation function.

The membrane fields corresponding to these flows describe marginal deformations of the reference D-brane. Let the membrane field Φ correspond to the flow of a primary field and consider the inner product

$$\left\langle \Phi, \mathfrak{Q}^{\alpha,\varepsilon(V;\alpha)} \right\rangle_M \tag{136}$$

This interpolates between the BPZ inner product of

$$\langle \phi, Q^\alpha \rangle = \left\langle F\left(z, t_i\right) \circ \phi\,(0) \right\rangle_{C_{\alpha+1}} \tag{137}$$

and

$$\left\langle \phi, Q^{\alpha\left(1+\frac{V}{K}\right)} \right\rangle = \left\langle e^{-\int_{1/2}^{\alpha+1/2} V(x)dx} F\left(z, t_f\right) \circ\phi\,(0) \right\rangle_{C_{\alpha+1}}$$
$$= \left\langle F\left(z, t_f\right) \circ\phi\,(0) \right\rangle_{C_{\lambda\alpha+1}} \tag{138}$$

The above formula follows from (67), where it was shown that the membrane integral operation is equal to the marginal perturbed BPZ inner product. If the flow shrinks the boundary, then there is the condition $\lambda > 0$ and

$$V \int_0^\infty d\alpha\; Q^\alpha < 1, \tag{139}$$

although such flows would increase the boundary entropy and are therefore not realized by unitary theories. Next, consider a disk with two different boundary components C_0 and C_* representing different D-branes. Let us define two operators σ, $\bar{\sigma}$ of conformal weight zero, called boundary condition changing operators[9–12] with

$$V = \sigma \partial \bar{\sigma}. \tag{140}$$

Let again $BCFT_*$ and $BCFT_0$ be the target and reference BCFTs, respectively. The BCCOs satisfy

$$\sigma(u)\,\bar{\sigma}(v) = e^{-\int_{C_*} V(z)dz} \tag{141}$$

$$\lim_{\substack{v \to u \\ \text{clockwise}}} \sigma(v)\,\bar{\sigma}(u) = e^{-\int_{\partial D} V(z)dz} = 1_{BCFT_*} \tag{142}$$

$$\lim_{\substack{u \to v \\ \text{clockwise}}} \sigma(u)\,\bar{\sigma}(v) = 1_{BCFT_0}. \tag{143}$$

Hence, the length of the flow can be expressed in terms of σ and $\bar{\sigma}$ and the marginal deformed wedge state is given by

$$Q_V^{\alpha} = e^{-\alpha(K+V)} = e^{-\alpha(K+\sigma\partial\bar{\sigma})} = e^{-\alpha\sigma K\bar{\sigma}} = \sigma Q^{\alpha}\bar{\sigma}. \tag{144}$$

From the previous discussion we see that our three-dimensional considerations provide an equivalent description. In fact, for regular marginal boundary perturbations our picture is equivalent to a worldsheet of a string with two different boundaries. At t_i and t_f the boundary consists purely of a single boundary condition of either BCFT, while for intermediate values of t the real line between 1 and −1 can be occupied by two different boundary conditions. This represents a string stretched between two different Dp-branes. The upshot is that these strings and the membrane fields are equivalent. From the viewpoint of Dp-brane categories, this implies that its morphisms have two dual descriptions: either in terms of strings stretched between various Dp-branes and tachyonic backgrounds, or as the interpolating membrane fields. The condition changing operators or their generalizations, the intertwining fields, live in the state space of stretched strings. The connection between membrane fields and boundary condition changing operators is also visible by evaluating their matter two-point function on the disk

$$\begin{aligned} &\lim_{\theta\to 0^+} \left\langle \bar{\sigma}(1), \sigma\left(e^{i\theta}\right)\right\rangle_{disk} = g_* \\ &\lim_{\theta\to 2\pi^-} \left\langle \bar{\sigma}(1), \sigma\left(e^{i\theta}\right)\right\rangle_{disk} = g_0. \end{aligned} \tag{145}$$

On the other hand, we have

$$\left\langle \mathfrak{Q}^{1,\epsilon(V)} \middle| \mathfrak{Q}^{1,\epsilon(V)} \right\rangle_{disk}^{BCFT_*} = \left\langle Q^{\lambda}, Q^{\lambda}\right\rangle_{disk}^{BCFT_*} = \langle 1\rangle_{disk}^{BCFT_*} = g_* \tag{146}$$

$$\left\langle \mathfrak{Q}^{1,\epsilon(V)} \middle| \mathfrak{Q}^{1,\epsilon(V)} \right\rangle_{disk}^{BCFT_0} = \langle Q, Q\rangle_{disk}^{BCFT_0} = \langle 1\rangle_{disk}^{BCFT_0} = g_0. \tag{147}$$

The caveat is that for more singular BCCOs we cannot assume that $\sigma\bar{\sigma} = 1$ and should therefore consider more general intertwining fields. We will save this discussion for future papers. Expressions (146) and (147) appear to hold not only for V annihilating the sliver but also for more general operators. If some vertex operator V does not annihilate the sliver in (130), any wedge state will always flow to the sliver state Q^{∞}. The sliver, however, can be expressed as the limit $\lim_{\alpha\to\infty}(Q^{\alpha}cB)$, which squares to zero, which indicates that (146) vanishes. A vacuum described by the sliver would not contain open string excitations and would therefore be the tachyon vacuum. What we discussed here in the context of BCCO follows immediately from our definition of the membrane product (62) when a marginal deformation is present. Then the beta-function vanishes and

$$\left\langle \Upsilon_V \middle| \Upsilon_V \right\rangle_M = \left\langle e^{-\int_{\partial D} V(z)dz} \right\rangle_0, \tag{148}$$

which includes the BCCO. If we then insert a test membrane state Φ, we reproduce (138)

$$\langle \Upsilon_V | \Phi | \Upsilon_V \rangle_M = \left\langle e^{-\int_{\partial D} V(z)dz} \Phi \right\rangle_0 \tag{149}$$

According to our construction of the membrane integral operation, for perturbations $V = \Phi^I V_I$, the membrane inner product of $\mathfrak{Q}^{1,\epsilon(V)} := \Upsilon_V$ evaluates to the g-function

$$\left\langle \mathfrak{Q}^{1,\epsilon(V)} \middle| \mathfrak{Q}^{1,\epsilon(V)} \right\rangle_M = \left(1 + \beta^I(\Phi)\frac{d}{d\Phi^I}\right) \mathcal{Z}_{disk}[\Phi] = S[\Phi]. \tag{150}$$

3.4. The Phantom Term

Analytic solutions to the equations of motion can be constructed using a subalgebra of the string star algebra, which is called KBc-subalgebra

$$[B, c] = 1, [B, K] = 0,\ B^2 = c^2 = 0 \tag{151}$$

$$QB = K,\ Qc = cKc,\ QK = 0. \tag{152}$$

The perturbative vacuum can be described by the trivial string field $\Psi = 0$. In the KBc subalgebra there exists a one-parameter family of string fields representing the perturbative vacuum for $\mu < 1$

$$\Psi_{\mu} = \mu\sqrt{Q}c\frac{K}{1-\mu Q}Bc\sqrt{Q}. \tag{153}$$

It is convenient to write this as the sum

$$\Psi_{\mu} = \sum_{n=0}^{\infty} \mu^{n+1}\sqrt{Q}cKBQ^n c\sqrt{Q}. \tag{154}$$

This is a perturbative expansion about $\Psi = 0$, where the term of leading order is BRST exact

$$\sqrt{Q}cKBc\sqrt{Q} = \mathcal{Q}\left(\sqrt{Q}Bc\sqrt{Q}\right). \tag{155}$$

On the other hand, for $\mu \to 1^-$ there is a discontinuous change in background from the perturbative vacuum to the tachyon vacuum, where the D-brane has disappeared, and all open string amplitudes vanish. The distinction between both backgrounds is given by the phantom term. It is not part of the string star algebra. The phantom term represents strings stretched between the two backgrounds. The vanishing in the Fock space expansion is an indicator that there are no D-branes present, which would support open string states. Therefore, the tachyon vacuum is the sum of the perturbative vacuum and the phantom term

$$\Psi_{tv} = \Psi_{\mu} + \Psi_{ph} = \mu\sqrt{Q}c\frac{K}{1-\mu Q}Bc\sqrt{Q} + \sqrt{Q}c\frac{K}{1-Q}\frac{1-\mu}{1-\mu Q}Bc\sqrt{Q}. \tag{156}$$

Here we use a similar regularization as in Erler and Maccaferri.[13] Schnabl's solution is the limit $\mu \to 1^-$. Consequently

$$S_C\left[\Psi_{tv}\right] = S_C\left[\Psi_{\mu}\right] + S_C\left[\Psi_{ph}\right]. \tag{157}$$

If we compare this to (73) we see that

$$S_C\left[\Psi_{ph}\right] = \int \mathfrak{F}\star\mathfrak{F}, \tag{158}$$

which indicates that the action of the phantom term, associated to two gauge-inequivalent solutions, is equivalent to the action of the membrane field connecting them. This reflects the beforementioned duality between membrane fields and strings stretched between different backgrounds, because both terms in (158) "measure" the change in background. The phantom term is only non-zero if there is a singular left gauge transformation U between two solutions. Singular means that it is infinitesimally close to a gauge transformation or that $U+\epsilon$ for ϵ infinitesimal is invertible. In order to construct the phantom term, we have to find the gauge transformation. It is given by the shifted kinetic operator $Q_{\Psi_1\Psi_2}$, which represents the kinetic operator of a string stretched between two backgrounds Ψ_1 and Ψ_2, acting on a field b with ghost number -1.

$$U = Q_{\Psi_1\Psi_2}b = Qb + \Psi_1 b + b\Psi_2. \tag{159}$$

The phantom term is defined as

$$\Psi_{\Psi_1\Psi_2}(\epsilon) = \frac{\epsilon}{\epsilon+U}\left(\Psi_2-\Psi_1\right) \tag{160}$$

$$\Psi_{ph} = \lim_{\epsilon\to 0}\frac{\epsilon}{\epsilon+U}\left(\Psi_2-\Psi_1\right) \tag{161}$$

Only if U is singular (161) is non-trivial. It is easy to confirm that for the choices $\Psi_1 = 0$ and $\Psi_2 = \Psi_{tv}$,

$$U = Q_{\Psi_1\Psi_2}\left(B\frac{1-Q}{K}\right) = 1-\sqrt{Q}Bc\sqrt{Q} \tag{162}$$

and

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{\varepsilon+U}\left(\Psi_2-\Psi_1\right) = \sqrt{Q}cBQ^{\infty}c\sqrt{Q}. \tag{163}$$

In this case the change in background is equal to the tension of the reference D-brane

$$S_C\left[\Psi_{ph}\right] = \int \mathfrak{F}\star\mathfrak{F} = \frac{1}{2\pi^2}. \tag{164}$$

In general the dictionary that translates between phantom term and membrane field is unclear. This would require a better understanding of the underlying quantum symmetry algebra. For the ordinary bosonic membrane, it is well established that this algebra is a W_∞-algebra[14–16] and that the membrane action can be cast into the form of a (0 + 1)-dimensional Yang–Mills theory with gauge group $SU(N\to\infty)$ in Coulomb gauge.[17,18] Finally, the Ellwood invariant[19] is of similar structure as the MFT action in that it computes the shift of the closed string tadpole. This is a gauge invariant quantity defined by taking the trace with an additionally insertion of a closed string matter primary $\mathcal{V}_c = c\bar{c}V^m$ of weight (1, 1)

$$Tr_{\mathcal{V}}\left(\Psi_{\Psi_1\Psi_2}(\varepsilon)\right) = \left\langle \mathcal{V}_{cl}\left(i\infty,\overline{i\infty}\right)f\circ V_{\Psi}(0)\right\rangle_{C_1}, \tag{165}$$

where V_Ψ is the vertex operator associated to a string field. It determines the shift in the amplitude $\mathcal{A}(\mathcal{V}_{cl})$ of the emission and absorption of a single on-shell closed string off the D-brane between two backgrounds. It is determined precisely by the phantom term associated to these backgrounds

$$Tr_{\mathcal{V}}\left(\Psi_{\Psi_1\Psi_2}(\varepsilon)\right) = \mathcal{A}_2\left(\mathcal{V}_{cl}\right) - \mathcal{A}_1\left(\mathcal{V}_{cl}\right). \tag{166}$$

Like the membrane field, the phantom term contains all the information about the change of background and some of that information is extracted by the Elwood invariant. It would be interesting to study the connection of membrane fields to the invariant and its generalizations[20,21] more thoroughly. In conclusion, there is a duality between membrane fields and phantom terms, which supports the assumption that there is a duality between strings stretched between different backgrounds and membrane fields.

3.5. Explicit Solutions of the Flow

Finally, we want to describe some solutions to the gradient flow equations (89) and give a proof for (158) as well as for Sen's conjecture using our theory. Obviously, at the fixed points the path approaches a solution of cubic OSFT. As a warmup let Ψ be the identity-based $\Psi = -cK$, which is a solution to the equation of motions (72). Then the ansatz for a solution is

$$\Psi_t = f(t)\,\Psi = -f(t)\,cK. \tag{167}$$

Obviously, $f(t)$ must satisfy the logistic differential equation

$$\partial_t f(t) = -f(t)^2 + f(t) \tag{168}$$

together with appropriate initial conditions. For example, if $f(0) = \frac{1}{2}$, then

$$\Psi_t = -\left(\frac{e^t}{e^t+1}\right)cK. \tag{169}$$

To prove this, we plug it into the flow equation. The LHS gives

$$-\partial_t\left(\frac{e^t}{e^t+1}\right)cK = -\left(1-\frac{e^t}{e^t+1}\right)\left(\frac{e^t}{e^t+1}\right)cK \tag{170}$$

The RHS is

$$\mathfrak{o}_3\mathfrak{F}_{\Psi_t} = \mathfrak{o}_3\left(-\left(\frac{e^t}{e^t+1}\right)cKcK + \left(\frac{e^t}{e^t+1}\right)^2 cKcK\right)$$
$$= -\left(1-\left(\frac{e^t}{e^t+1}\right)\right)\left(\frac{e^t}{e^t+1}\right)\mathfrak{o}_3\,(cKcK)$$
$$= -\left(1-\left(\frac{e^t}{e^t+1}\right)\right)\left(\frac{e^t}{e^t+1}\right)cK, \tag{171}$$

which confirms the correctness of (169). This solution flows from the perturbative vacuum at $t=-\infty$ to the residual solution in the IR. In terms of the scale $\zeta = e^t$ the D-brane sits at $\zeta = 0$

$$\Psi_\zeta = \frac{|\zeta|\,cK}{|\zeta|+1}. \tag{172}$$

Doing the same calculation as above with the identity-based tachyon vacuum $\Psi = c(1-K)$, a more interesting solution, which describes tachyon condensation, is

$$\Psi_t = \frac{e^t c\,(1-K)}{e^t+1}. \tag{173}$$

We can consider also marginal deformations $\Psi = cV$. Since $Q(cV) = 0$ and $[c, V] = 0$, $f(t) = const.$ and cV is a static solution of the flow

$$\Psi_t = f(t)\,\Psi = cV = c\sigma\partial\bar{\sigma}. \tag{174}$$

Now, we want to give the promised proofs by first evaluating the membrane curvature of the residual solution.

$$\mathfrak{o}_4\mathfrak{F}_\Upsilon = \mathfrak{o}_3\partial_t\Psi_t + Q_t\star\mathfrak{o}_3\mathfrak{F}_{\Psi_t} = -\left(1-\frac{e^t}{e^t+1}\right)\left(\frac{e^t}{e^t+1}\right)cKcK$$
$$-\left(1-\left(\frac{e^t}{e^t+1}\right)\right)\left(\frac{e^t}{e^t+1}\right)Q_t cK. \tag{175}$$

Using the condition (87), the membrane curvature of this path is

$$\mathfrak{F}_\Upsilon = -\left(1-\frac{e^t}{e^t+1}\right)\left(\frac{e^t}{e^t+1}\right)\left(cKcK + Q_t cK\right)$$
$$= -\frac{1}{4}sech^2\left(\frac{t}{2}\right)\left(cKcK + Q_t cK\right). \tag{176}$$

The curvature is zero at the two BCFTs at $t=\pm\infty$ and has an extremum at $t=0$ as one would expect from a solitonic membrane. The maximum is located, where the "instantaneous" change in background happens. Similar calculations can be performed for the identity-based tachyon vacuum $\Psi = c(1-K)$.

$$\mathfrak{o}_4\mathfrak{F}_\Upsilon = \left(1-\frac{e^t}{e^t+1}\right)\left(\frac{e^t}{e^t+1}\right)\left(cKc - cKcK + Q_t\left(c\,(1-K)\right)\right) \tag{177}$$

The membrane curvature has exactly the same structure as (176), which is

$$\mathfrak{F}_\Upsilon = \frac{1}{4}sech^2\left(\frac{t}{2}\right)\left((cKc - cKcK) + Q_t\left(c\,(1-K)\right)\right)$$
$$= \frac{1}{4}sech^2\left(\frac{t}{2}\right)\left(Q + Q_t\right)c\,(1-K). \tag{178}$$

We want to calculate the difference in energy density between the perturbative vacuum and the tachyon vacuum with the flow (173). It should exactly match the D-brane tension T_{25} if our flow-equation is valid

$$\int \mathfrak{F}_\Upsilon\star\mathfrak{F}_\Upsilon = \int_{t_i}^{t_f} dt \int \partial_t\Psi_t\star\mathfrak{F}_{\Psi_t}. \tag{179}$$

Inserting the corresponding fields (173) and

$$\mathfrak{F}_{\Psi_t} = \frac{1}{4}sech^2\left(\frac{t}{2}\right)(cKc - cKcK) \tag{180}$$

into (179), we get

$$\int_{-\infty}^{\infty} dt \int \frac{1}{16}\text{sech}^4\left(\frac{t}{2}\right)c\,(1-K)\star(cKc - cKcK). \tag{181}$$

Performing the t-integration we arrive at

$$\frac{1}{6}\int c\,(1-K)\star\ \left((cKc - cKcK)\right) = \frac{1}{6}\int c\,(1-K)\star Qc\,(1-K). \tag{182}$$

Evaluating the integral naively would give zero, due to the identity-like form of the solution. However, as shown in ref. [22] we can use the gauge transformation $U = cBK + 1$ to map $\Psi = c(1-K)$ to the well-behaved Schnabl-Erler solution,[23] which is also known as the simple tachyon vacuum

$$\Psi_{simp} = (c + cKBc)\ \frac{1}{1+K}. \tag{183}$$

Note that $cKBc$ is a total derivative such that replacing $c(1-K)$ by Ψ_{simp} gives

$$\frac{1}{6}\int\frac{1}{1+K}c\frac{1}{1+K}cKc = \frac{1}{6}\int_0^\infty d\alpha\int_0^\infty d\beta e^{-\alpha-\beta}\int Q^\alpha cQ^\beta c\partial c$$
$$= \frac{1}{6}\int_0^\infty d\alpha\int_0^\infty d\beta e^{-\alpha-\beta}\left(\frac{\alpha+\beta}{\pi}\right)^2\sin^2\left(\frac{\pi\alpha}{\alpha+\beta}\right)$$
$$= \frac{1}{6}\,\frac{6}{\pi^3}\,\frac{\pi}{2} = \frac{1}{2\pi^2}. \tag{184}$$

Therefore, we managed to prove that the membrane field action is indeed equivalent to the action of the phantom term (158)

$$\frac{1}{6}\int\frac{1}{1+K}c\frac{1}{1+K}cKc = S_C\left[\Psi_{ph}\right] = \int\mathfrak{F}_\Upsilon\star\mathfrak{F}_\Upsilon\,, \tag{185}$$

where Ψ_{ph} is given in (156) in the limit $\mu\to 1^-$, which underlines the conjectured duality between strings with multiple boundary conditions and membrane fields. Additionally, we have given a new proof of Sen's conjecture. Usually, the kinetic term associated to the tachyon vacuum solution is normalized with the help of the equations of motion and Sen's conjecture

$$\frac{\pi^2}{3}\langle\Psi, \mathcal{Q}\Psi\rangle = -1, \tag{186}$$

whereas in our calculation the factor 1/6 arose from the t-integration in the membrane action (179). In the next and final section we want to apply the membrane model to the mathematically interesting area of topological string theory.

4. Membrane Fields as Yang–Mills Instantons

4.1. Topological Flows

It was shown by Baulieu and Singer[24] that the topological field theories in four and two dimensions constructed by Witten, could be interpreted as BRST-quantized actions of purely topological terms. In two dimensions this is the instanton number associated to the maps of the open A-model (Appendix B)

$$W = \int dzd\bar{z}\ \omega_{ij}\partial\phi^i\bar{\partial}\phi^j, \tag{187}$$

whereby ω_{ij} is the symplectic two-form of the target X and $\phi : (\Sigma, J) \to X$ is a stable holomorphic map with Lagrangian boundary conditions. There is also the important coupling $e = 1/\hbar$. It is a well-established fact, that the theory is perturbatively exact such that we can evaluate the quantum theory in the weak-coupling limit $e \to \infty$. In this regime the path integral is dominated by holomorphic maps $\bar{\partial}_J\ \phi^i = 0$ classified by $H^2(X, \mathbb{Z})$. As prescribed, for $X = T^*\ Y$ these maps are necessarily constant and (187) vanishes. For general targets the topological Lifshitz theory is dominated by solutions of the equations of motion obtained by variation of the action

$$S = \int dt \int dzd\bar{z}\ \frac{1}{2}\left(\dot{\phi}^i\right)^2 - \left(\bar{\partial}_J\phi^i\right)^2. \tag{188}$$

Let $\mathcal{M}_J$ be the moduli space of pairs (J, ϕ). It is the base of the vector bundle $\mathcal{E} = \Omega^{0,1}(\Sigma_2) \times \phi^*(T^{1,0}X)$. The equations of motion $\bar{\partial}_J\phi^i$ are a section $s : \mathcal{M}_J \to \mathcal{E}$ defined by

$$(J, \phi) \mapsto \bar{\partial}_J\phi^i. \tag{189}$$

The holomorphic maps, which are instantons of the A-model, represent the zero-section $s^{-1}(0) \subset \mathcal{M}_J$ and satisfy

$$\bar{\partial}_J\phi^i = 0. \tag{190}$$

The Lifshitz fields define continuous flows on the space $\mathcal{M}_J$. We can think of them as one-parameter families of maps ϕ^i_t labeled by t, which interpolate between pairs (J, ϕ) in the zero-locus. In fact, $\mathcal{M}_J$ is non-compact and when there are only constant maps, the zero locus is at infinity and consists of virtual zeros. This is the degeneration-limit, in which the worldsheets of the A-model topological string degenerate into Feynman graphs in Y and strings into points. The membrane field is now a functional of maps $\phi_t : \Sigma_2 \times \mathbb{R}_t \to X \times T^*\mathbb{R}_t$ such that $\partial I \times \mathbb{R}$ is mapped into $Y \times \mathbb{R}$

$$\Upsilon\left[\phi_t, \ldots\right], \tag{191}$$

whereby the $(\ldots)$ represent fermi variables of full supersymmetric Lifshitz sigma model. It interpolates between two string field functionals $\Psi[\phi, \ldots]$ and $\Psi'[\phi, \ldots]$ of constant maps ϕ satisfying (190). We can extend the Chan–Paton bundle over $Y \times \mathbb{R}$. When the string fields reduce to flat $U(N)$ Lie algebra-valued connections on Y, the membrane fields reduce to $U(N)$ Yang–Mills instantons on $Y \times \mathbb{R}$. For clarity, let S_i be the boundary components of the topological Lifshitz membrane, which are oriented surfaces in $Y \times [t_i, t_f]$ with

$$\partial S_i = \left(-\{t_i\} \times K_0\right) \cup \left(\{t_f\} \times K_1\right). \tag{192}$$

That is, the S_i are knot cobordism between the knots K_0 and K_1. These knots represent the boundary components C_i of two topological open A-model strings ending on different D-branes. The boundary terms at these points t_i and t_f, which enter the path integral are given by the holonomy of the pullback of the $U(N)$ gauge connection on Y

$$\prod_i \mathrm{Tr} P\ exp \oint_{C_i} \phi^*\left(A\right). \tag{193}$$

The fermionic symmetry of the quantum theory is preserved if the variation of (193) vanishes. This is exactly the case if the field strength $F = dA + A \wedge A$ vanishes

$$\begin{aligned} &\delta \mathrm{Tr} P\ \exp \oint_C \phi^*\left(A\right) \\ &= \oint_C \delta\phi^i \frac{d\phi^j}{d\theta}\phi^*\left(F_{ij}\right) d\theta \ \cdot P\exp \int_C \phi^*\left(A\right) = 0\ \to F = 0. \end{aligned} \tag{194}$$

Now let A be the connection 1-form of the $U(N)$ bundle extended over $Y \times \mathbb{R}$ and $*_3$ the Hodge star on Y. In temporal gauge $A_0 = 0$, it can be interpreted as a one-parameter family of connections on Y. The space $\mathfrak{B}$ from the introduction becomes the space of connections on Y modulo automorphisms $\mathfrak{R}/\mathcal{G}$. First, we use (7) in order to determine the flow equation for the membrane fields.[1] Since the background theory is Chern–Simons theory, we have

$$S_L\left[A_t\right] = \int dtd^3x\ \frac{1}{2}\left(\partial_t A_t\right)^2 - \left(\frac{\delta S_{CS}}{\delta A_t}\right)^2, \tag{195}$$

with $S_{CS}[A]$ the Chern–Simons action. Therefore, the membranes couple to flow lines of the gradient flow

$$\partial_t A_t = *_3 F\left(A_t\right). \tag{196}$$

[1] The only degree of freedom of the membrane field is the self-dual background gauge field A it couples to. Hence, there is no distinction between the two.

The expression $*_3 F$ can be interpreted as a gradient vector field on $\mathfrak{R}/\mathcal{G}$. It is enlightening to use the standard manipulations in (71) to determine the form of the MFT action. The beta function is zero since the physics is scale independent. Hence, the action S of BSFT is simply the partition function $\mathcal{Z}_A$ of the A-model. Hence, we can replace S by the the Chern–Simons action

$$S = \mathcal{Z}_A = S_{CS}. \tag{197}$$

According to (71), the MFT action for a general path C_{A_t} in $\mathfrak{R}/\mathcal{G}$ is

$$\Delta E = -\int_{C_{A_t}} \frac{\delta}{\delta A_t} S_{CS}\left[A_t\right] = \int_{C_{A_t}} dAdvol_Y \left\langle -, *_3 F\left(A_t\right)\right\rangle$$
$$= \int_{t_i}^{t_f} dt \int dvol_Y \left\langle \frac{dA_t}{dt}, *_3 F\left(A_t\right)\right\rangle = \int_{t_i}^{t_f} dt\, \mathrm{Tr} \frac{dA_t}{dt} \wedge F\left(A_t\right). \tag{198}$$

We know that A_t must satisfy (196) and is self-dual and therefore

$$\int_{t_i}^{t_f} dt\, \mathrm{Tr} \frac{dA_t}{dt} \wedge F\,\left(A_t\right) = \frac{1}{2}\int_{Y\times[t_i,t_f]} \mathrm{Tr}\, F\wedge F = S_{CS}\left(t_f\right) - S_{CS}\left(t_i\right)$$
$$= \frac{1}{2}\left[8\pi^2 c_2\left(E\right)\left(Y\times\left[t_i, t_f\right]\right)\right] \tag{199}$$

It follows that

$$\int \mathfrak{F} \star \mathfrak{F} = \frac{1}{2}\int_{Y\times\mathbb{R}} \mathrm{Tr}\, F\wedge F. \tag{200}$$

Furthermore, the relation (73) of the last section turns into

$$S_{CS}\left[A\right] = S_{CS}\left[A'\right] + \frac{1}{2}\int_{Y\times\mathbb{R}} \mathrm{Tr}\, F\wedge F, \tag{201}$$

As before, the membrane action determines the change in background but now these string backgrounds are stacks of N topological A-model branes, between which the instantons interpolate, represented by topological sectors of Yang–Mills theory on $Y\times\mathbb{R}$. The change in background is simply the instanton charge k. For the remainder of this paper, we will be concerned with instantons on $\mathbb{R}^4 \simeq \mathbb{C}^2$. These correspond to open membranes, which are not necessarily of the tubular form. They are not solutions to the gradient flow equations (196) and the relation (195) does not hold for them either, however, their curvature vanishes at infinity, too. Their behavior is not so different from instantons on a tubular manifold, since by the homeomorphism $S^3\times\mathbb{R} \simeq \mathbb{R}^4\backslash\{0\}$ and the Uhlenbeck removing singularities theorem we can smooth out the instantons if the L^2-norm of the curvature is small enough. On the general four manifolds M, the gradient flow equations (196) resemble the ASD condition of the Yang–Mills curvature

$$F = *_4 F. \tag{202}$$

Here, we need to consider the more general relation for world volumes with boundary (90), which descends to

$$S_T\left(A\right) = \int_M \mathrm{Tr}\, F\wedge F = \int_M d\left(CS\left(A\right)\right) = \int_{\partial M} CS\left(A\right). \tag{203}$$

Remark: We might extend the effective background theory to the whole eight-dimensional manifold X with $Spin(7)$ or $SU(4)$ holonomy. Then the membrane field action reduces to an eight-dimensional topological Yang–Mills theory (see Appendix B) defined by the topological invariant

$$\int \mathfrak{F} \star \mathfrak{F} = \int_X \Omega_4 \wedge \mathrm{Tr}\,\left(F\wedge F\right), \tag{204}$$

where Ω_4 is the Cayley four-form. The CohFT[25,26] produced by BRST quantization of (204) appears to localize to the space of $Spin(7)$-instantons on X. From the membrane perspective it describes D7-branes wrapped on X. Upon dimensional reduction this theory should be equivalent to the seven-dimensional Chern–Simons theory, which is supposedly the background theory of the G_2-string[27–29]

4.2. Localization and Path Integrals

In this subsection we will describe that on M the path integral of (200) is given by the Nekrasov instanton partition function and that the quantum state $|\Upsilon\rangle$ is an element of $H^*_{\mathcal{T}}(\mathcal{M}_N)$. For a review on equivariant localization see Pestun.[30] It is known[31,32] that the BRST quantization of self-dual or topological Yang–Mills (200) produces a twisted $\mathcal{N}=2$ supersymmetric cohomological gauge theory, because $\mathcal{M}_{N,k}$ is the supersymmetric localization locus of Q-fixed points, which are BPS states representing cohomology classes in the chiral ring of twisted supersymmetric Yang–Mills. In more detail, let

$$\mathfrak{S}\left(G\right) = \mathfrak{G}\left(G\right)\times\Omega^0\left(X, \mathrm{ad}\; G\right)\times\Omega^*\;\left(\mathfrak{G}\left(G\right)\times\Omega^0\left(X, \mathrm{ad}G\right)\right)$$
$$= \mathfrak{V}\left(G\right)\times\Omega^*\left(\mathfrak{V}\left(G\right)\right), \tag{205}$$

whereby $\mathfrak{G}(G)$ is the space of G-connections which are flat at infinity. $\mathfrak{S}(G)$ is the infinite-dimensional space containing the fields of the untwisted $\mathcal{N}=2$ super Yang–Mills theory. We will take the fields $\varphi = 0 \in \Omega^0(X, \mathrm{ad}\; G)$. Consider the infinite-dimensional bundle $\mathfrak{W}(G)$ on $\mathfrak{S}(G)$, where the section represents the BPS equations

$$\mathfrak{W}\left(G\right) := \mathfrak{V}\left(G\right)\times_{\mathfrak{K}(G)}\Omega^{2,+}\left(X, \mathrm{ad}\; G\right), \tag{206}$$

whereby $\mathfrak{K}(G)$ is the group of automorphism of the G-bundle, trivial at infinity. Its section is

$$s\left(A,\varphi\right) = \left(F^+, \left(\partial + A\right)\varphi\right). \tag{207}$$

Via the techniques in equivariant localization with respect to $\mathfrak{K}(G)$, the partition function

$$\mathcal{Z} = \int_{\mathfrak{S}(G)} \alpha, \tag{208}$$

with $\alpha = e^{-S}\mathcal{O}$ a BRST-closed measure on $\mathfrak{S}(G)$, localizes to the zero section of $\mathfrak{W}(G)$. Let

$$\pi_* : H^*(\mathfrak{W}(G)) \to H^*(\mathfrak{B}(G)/\mathfrak{K}(G)) \tag{209}$$

be the Thom-isomorphism and $Th(\mathfrak{W}(G)) := \pi_*^{-1}(1)$ the Thom-class. The cohomology class $Th(\mathfrak{W}(G)) \in H^*(\mathfrak{W}(G))$ is the Poincaré-dual to the submanifold

$$\mathfrak{B}(G)/\mathfrak{K}(G) \subset \mathfrak{W}(G). \tag{210}$$

Let us pick $G = U(N)$. Then the zero locus is the instanton moduli space $\mathcal{M}_{N,k} := s^{-1}(0)$. As the observable $\mathcal{O}$ from above represents a form on $\mathfrak{B}(G)$, the localization property reads

$$\mathcal{Z} = \int_{\mathfrak{B}(U(N))/\mathfrak{K}(U(N))} s^*(Th(\mathfrak{W}(U(N))) \wedge \mathcal{O} = \int_{\mathcal{M}_{N,k}} f^*\mathcal{O}. \tag{211}$$

Here f is the inclusion map of $\mathcal{M}_{N,k}$ in the space of fields. In other words, α is the so-called Mathai–Quillen represantive of the Thom class of $\mathfrak{W}(G)$. It follows that the partition function localizes to a weighted sum over characteristic classes of vector bundles over $\mathcal{M}_{N,k}$. With a further so-called Ω-deformation introduced by Nekrasov, these integrals are interpreted as push forward maps in $\mathcal{T}$-equivariant cohomology, where $\mathcal{T} = (\mathbb{C}^*)^{N+2}$ is the maximal torus of the gauge group $U(N)$ combined with automorphisms of the four-dimensional manifold M acting on the moduli space of instantons. Considering the toroidal action of the group $\mathcal{T} = (\mathbb{C}^*)^{N+2}$ on the space $\mathcal{M}_{N,k}(M)$, the partition function of the membrane field theory descents to

$$\mathcal{Z}[\Upsilon] = \int \mathcal{D}\Upsilon e^{\int \mathfrak{F}\star\mathfrak{F}} = \sum_{k=0}^{\infty} \mathfrak{q}^k \int_{\mathcal{M}_{N,k}} e^{\omega + \mu_{\mathcal{T}}(\vec{\mathfrak{a}}, \epsilon_1, \epsilon_2)}, \tag{212}$$

where $\mu_{\mathcal{T}}$ is the moment map with respect to the action of $\mathcal{T}$, depending on the equivariant characters $\vec{\mathfrak{a}}$ and ϵ_1, ϵ_2. This is equivalent to the equivariant volumes of $\mathcal{M}_{N,k}$

$$\mathcal{Z}[\Upsilon] = \sum_{k=0}^{\infty} \mathfrak{q}^k \int_{\mathcal{M}_{N,k}} 1. \tag{213}$$

Notice that from our line of reasoning the Nekrasov instanton partition function[33] counts topological open membranes in $\phi : \Sigma_3 \to T^*M$ with $\partial(\Sigma_3) \subset M$. These membranes are simply points in M such that we can use a more direct method involving Hilbert schemes. The membrane map is completely characterized by the embedding of its boundaries S_i into the submanifold M. In the general case they wrap 2-cycles inside M. Let

$$\phi_*[S_i] = \lambda_i \gamma, i = 1, \dots, h \leq N, \tag{214}$$

where $\lambda_i \in \mathbb{Z}$ is the winding number and $\gamma \in H_2(M)$. We combine all winding numbers of the boundary components in non-increasing order into a partition $\lambda = (\lambda_1, \dots, \lambda_h)$ or Young-tableux Y of length $|\lambda| = k$. For multiple boundaries we can consider an N-tuple of partitions with $\vec{\lambda} = (\lambda_1, \dots, \lambda_N)$ with $|\vec{\lambda}| = |\lambda_1| + \dots + |\lambda_N| = k$ with most of the entries zero for larger N. In the case, where the target space is a cotangent bundle, there are no degenerate membrane instantons and the membrane shrink to points.

In order to make some concrete statements let $M = \mathbb{C}^2$. In the absence of membrane instantons, the membrane is just a set of partitioned points. It follows, that for our case the phase space of a membrane is the Hilbert scheme $Hilb_k(\mathbb{C}^2) \simeq \mathcal{M}_{1,k}$ of k points in the plane.[34,35] This is a special case of isomorphisms between closed and open AKSZ sigma-models.

$$\mathrm{Map}_{\mathfrak{g}}(T[1]\partial\Sigma_3, M) \simeq \mathrm{Map}(T[1]M, T^*[3]\mathfrak{g}[1])$$

for $\mathfrak{g} = \mathfrak{u}(1)$ (see Appendix B). In particular, a map ϕ corresponds to a monomial m_λ that generates an ideal $\mathcal{I} \in \mathbb{C}[p_1, p_2]$ in two variables with codimension k. Moreover, these subsets in return represent cycles in the cohomology of the composition space $\coprod_{k=1}^{\infty} Hilb_k$

$$H^*\left(\coprod_{k=1}^{\infty} Hilb_k\right) \simeq \mathbb{C}[p_1, p_2, \dots] \tag{215}$$

This space is a boson Fock space $\mathcal{F}$, which we call $H^*(\mathcal{M}_{1,k})$. The ideals $\mathcal{I}$ generated by monomials are fixed points of the torus action of $T = \mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C}[p_1, p_2]$. If we define ϵ_1, ϵ_2 to be equivariant parameters corresponding to the characters of T, which acts naturally on $Hilb_k(\mathbb{C}^2)$, we can define the equivariant cohomology

$$H^*_{\mathcal{T}}(\mathcal{M}_{1,k}) \simeq \mathbb{C}[p_1, p_2, \dots] \otimes \mathbb{C}[\epsilon_1, \epsilon_2]. \tag{216}$$

The tuple $\vec{\lambda} = (\lambda_1, \dots, \lambda_N)$ defines a direct sum of sheaves $\mathcal{S}_{\lambda_1,\dots,\lambda_N} = \mathcal{J}_{\lambda_1} \oplus \dots \oplus \mathcal{J}_{\lambda_N}$ in

$$\mathcal{M}_N = \coprod_{k_1+\dots+k_N} Hilb_{k_1} \times \dots \times Hilb_{k_N}, \tag{217}$$

corresponding to cycles in the cohomology of $\mathcal{M}_N$, on which there is an action of the group $\mathcal{T} = (\mathbb{C}^*)^{N+2}$. Denoting the additional equivariant parameters corresponding to $(\mathbb{C}^*)^N$ by $\mathfrak{a}_1, \dots, \mathfrak{a}_N$, the $\vec{\lambda}$ define elements in

$$H^*_{\mathcal{T}}(\mathcal{M}_N) \simeq \mathcal{F}^{\otimes N} \otimes \mathbb{C}[\epsilon_1, \epsilon_2, \mathfrak{a}_1, \dots, \mathfrak{a}_N]. \tag{218}$$

In other words, $\mathcal{S}_{\lambda_1,\dots,\lambda_N}$ are fixed points of the $\mathcal{T}$-action. The membrane partition function simply counts membrane maps. Hence, it can be written as the generating function of the $\mathcal{T}$-equivariant Euler characteristic classes $\mathfrak{E}_k^{\mathcal{T}} \in H^*_{\mathcal{T}}(\mathcal{M}_N)$ of the instanton moduli space. We can write down their generating function

$$\mathcal{Z}(\varepsilon_1, \varepsilon_2, \mathfrak{q}, \vec{\mathfrak{a}}) = \sum_{k=0}^{\infty} \mathfrak{q}^k \int_{\mathcal{M}_{N,k}} \mathfrak{E}_k^{\mathcal{T}}, \tag{219}$$

where $\mathfrak{q} = e^{2\pi i\tau}$ is a formal variable weighting the equivariant volumes of $\mathcal{M}_{N,k}$ with τ the complexified gauge coupling parameter. Via the localization with respect to $\mathcal{T}$, we can further express (219) as a sum over the $\mathcal{T}$-fixed-points ℓ

$$\mathcal{Z}(\mathfrak{q}) = \sum_{k=0}^{\infty} \mathfrak{q}^k \sum_{\ell \in \mathcal{M}_k^{\mathcal{T}}} \frac{\mathfrak{E}_k^{T}(\ell)}{\prod_i \mathfrak{w}_i(\ell)}, \tag{220}$$

whereby $\mathfrak{w}_i$ are weights. Reminding ourselves of the picture of the MFT action as membrane field charge, the K-theoretic Nekrasov partition function (220) could be interpreted as the quantization of this charge. This is consistent with the assumption among string theorists[36,37] that, under certain circumstances, D-brane charge is quantized in terms of K-theory classes, since in the case we discussed, the membrane fields represent D-branes wrapping M. In other words, the membrane states are elements of the equivariant K-theory of the instanton moduli space

$$|\Upsilon\rangle \in K^*_{\mathcal{T}}\mathcal{M}_{N,|\vec{Y}|}\left(\mathbb{C}^2\right). \tag{221}$$

4.3. Jack Polynomials

Theorems in equivariant localization imply that the fixed points must correspond to an orthogonal basis. This basis is given by generalized Jack polynomials.[38,39] There is a vacuum vector

$$|0\rangle \in \mathcal{F} \to 1 \in \mathbb{C}\left[p_1, p_2, \dots\right] \tag{222}$$

and the creation and annihilation operators

$$\alpha_{-k} = p_k \tag{223}$$

$$\alpha_k = k\frac{\partial}{\partial p_k} \tag{224}$$

acting on the vacuum

$$\alpha_{-k}|0\rangle = p_k \tag{225}$$

$$\alpha_k|0\rangle = 0. \tag{226}$$

It appears that the quantum topological open membrane defines a quantum integrable system whose classical phase space coincides with the moduli space of instantons. The commuting Hamiltonians are realized as cup-multiplication by the Chern classes of the tautological bundle $\mathfrak{T}$ over the moduli space $\mathcal{M}_{N,k}(\mathbb{C}^2)$. In the equivariant case they fail to be locally nilpotent. The origin of this integrable system might be a two-dimensional setting, where classical membrane fields form the moduli space of polystable Higgs-bundles. Nevertheless, the integrable system is given by the differential operator

$$c_1(\mathfrak{T}) = \sum_{l=1}^{N} H_l + \sum_{l_1<l_2} H_{l_1,l_2}. \tag{227}$$

As commuting integrals of motions, the H_l are realized[40–42] by the coefficients of the Taylor sum at $\mathfrak{a} = \infty$ of the matrix elements $\Gamma(\mathfrak{a})$ of $\mathfrak{R}(\mathfrak{a})$ acting on the tensor powers of $\mathcal{F}$

$$\mathfrak{R}\left(\mathfrak{a}_1, \dots, \mathfrak{a}_N\right) : \mathcal{F}^{\otimes N} \to \mathcal{F}^{\otimes N} \tag{228}$$

Define $|0\rangle$ as the vacuum corresponding to $1 \in \mathcal{F} \otimes \mathbb{C}[\epsilon_1, \epsilon_2, \mathfrak{a}_1, \dots, \mathfrak{a}_N]$ then $|0\rangle \otimes \mathcal{F}$ is a subspace in $\mathcal{F}^{\otimes 2}$ and

$$\Gamma(\mathfrak{a}) : (|0\rangle \otimes \mathcal{F})^{\otimes 2} \to (|0\rangle \otimes \mathcal{F})^{\otimes 2} \tag{229}$$

$$[\Gamma(\mathfrak{a}), \Gamma(\mathfrak{b})] = 0. \tag{230}$$

In terms of the generalized Jack polynomials or Uglov-functions[43] the matrix elements are diagonal. For $-\varepsilon_1/\varepsilon_2 = \beta$

$$H_l = \frac{1}{2}\sum_{k,m=1}^{\infty} \beta\alpha^{(l)}_{-k}\alpha^{(l)}_{-m}\alpha^{(l)}_{m+k} + \alpha^{(l)}_{-k-m}\alpha^{(l)}_{k}\alpha^{(l)}_{m} + \frac{1}{2}\sum_{k=1}^{\infty}\left(2\mathfrak{a}_l + (\beta-1)(n-1)\right)\alpha^{(l)}_{-k}\alpha^{(l)}_{k} \tag{231}$$

$$H_{l_1,l_2} = (1-\beta)\sum_{k=1}^{\infty} k\alpha^{(l_1)}_{-k}\alpha^{(l_2)}_{k} \tag{232}$$

The associated quantum integrable system is a generalization of the trigonometric Calogero–Moser–Sutherland model[44,45] with arbitrary many time-variables $p^{(l)}_k$. The eigenfunctions of (231) are called generalized Jack functions

$$J^{\beta}_{\vec{Y}}\left(p^{(l)}_k, \mathfrak{a}_l\right) \in \mathcal{F}^{\otimes N} \otimes \mathbb{C}\left[\epsilon_1, \epsilon_2, \mathfrak{a}_1, \dots, \mathfrak{a}_N\right] \tag{233}$$

$$\hat{c}_1 J^{\beta}_{\vec{Y}} = \left|\Theta_{\vec{Y}}\right| J^{\beta}_{\vec{Y}}, \tag{234}$$

where

$$\left|\Theta_{\vec{Y}}\right| = \Theta_{Y_1}\left(\mathfrak{a}_1\right) + \dots + \Theta_{Y_N}\left(\mathfrak{a}_N\right) \tag{235}$$

$$\Theta_Y(\mathfrak{a}) = \sum_{(s,t)\in Y}\left(\mathfrak{a} + (s-1) + (t-1)\beta\right), \tag{236}$$

where s and t are arm-length and leg-length of a box in the Young-tableaux. In the limit $\beta \to 1$ the system describes a set of N noninteracting CMS-Hamiltonians and the Jacks factorize to a product of N Schur functions s_Y. This is of course tied to the fact that one can interpret the equivariant parameters $\varepsilon_1 = -\varepsilon_2 = \hbar$. Furthermore, the second term contains level operators

$$\hat{N}_l = \sum_{k>0} kN^{(l)}_k = \sum_{k>0}\alpha^{(l)}_{-k}\alpha^{(l)}_{k} = \sum_{k>0} kp^{(l)}_k\frac{\partial}{\partial p^{(l)}_k}, \tag{237}$$

which correspond to the total energy of the oscillations since they weight the number of oscillations by the level k. They act on the generalized Jacks as

$$\hat{N}_l J_{\vec{Y}} = \left|\vec{Y}\right| J_{\vec{Y}}. \tag{238}$$

Note that this is also the rank of the tautological bundle

$$\hat{N}_i = rk\ (\mathfrak{T}) = k. \tag{239}$$

In conclusion, the partition function of open topological membranes is effectively given by the Nekrasov instanton partition function. The Hilbert space is identified with the equivariant cohomology of the instanton moduli space, which admits a description in terms of the tensor product of Fock modules. The basis for this space is given by the generalized Jack polynomials. Notice that a single module is the highest weight representation of the conformal Heisenberg group $\mathfrak{h}$ with generators $[\alpha_k, \alpha_m] = k\delta_{k+m}$

as shown by Nakajima, which is an important aspect in understanding the AGT-duality. What we outlined about the embeddings holds up to conjugacy. Thus, one should consider the conjugacy classes of N-tuples of Young tableaux $[\vec{Y}]$. The corresponding cycles are in bijective correspondence with the generalized Jack polynomials $[\vec{Y}] \to J_{\vec{Y}}$.

Appendix A: Open Bosonic String Field Theory in the Sliver Frame

Open string field theory (OSFT) is a framework for describing the dynamics of open strings, which are fundamental objects in string theory. It was first proposed by Witten[46] in the 1980s, a good review can be found in Erler[47] and references therein. The basic idea behind OSFT is to construct a field theory that describes the evolution of a collection of open strings. The field in this theory $\Psi[T, A, \ldots]$ is a function on the space of all possible open string configurations, and it encodes information about the shape, position, and orientation of each string in the collection. The open string field is a functional of the open string excitation modes, under which the massless ones are the tachyon T, a $U(1)$ gauge potential A as well as scalar fluctuation modes. Roughly, the string field can be expanded in terms of a basis of the Fock space. In the absence of null-states, the Fock modules are isomorphic to the Verma modules associated to the primary fields of the worldsheet CFT. In the oscillator basis the explicate expansion for a state is

$$|\Psi\rangle = \frac{1}{(2\pi)^{26}} \int d^{26}p \left[T(k)\, c_1 + A_\mu(k)\, \alpha^\mu_{-1} c_1 + \varphi_a(k)\, \alpha^a_{-1} c_1 + \frac{i}{\sqrt{2}} \eta(k)\, c_0 + \cdots \right] \left| k_\mu \right\rangle . \tag{A1}$$

Every term is an eigenstate of the Virasoro generators. The open string field takes values in an A_∞-algebra, which is a graded associative algebra $\mathfrak{A}$. There are non-commutative string field operations corresponding to the folding and gluing of strings. The one we need first is the multiplication $\star\ : \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$. Additionally, we have a BRST-operator Q acting on the string fields by increasing their ghost number N_{gh} or grade by 1. The other operation is the integral operation, sometimes also called the trace operation or vertex $\int \Psi \star \Psi$. Often the world sheet is parametrized by the unit disk in the upper half plane. However, for our purposes it is much more convenient to use a different parametrization. In the so-called sliver coordinate frame $f(\xi) = 2/\pi \arctan(\xi)$, where ξ is a local coordinate on the disk with $Im(\xi) \geq 0$ and $|\xi| \leq 1$ the world sheet is mapped to a semi-infinite strip.[48] The star (**Figure A1**) product corresponds to the gluing of these strips

$$A \star B = \left(\tau_1 \circ f \circ V_A(0)\right) \left(f \circ V_B(0)\right) . \tag{A2}$$

Here $\tau_a = z + a$ are translation maps, $f(\xi)$ is a conformal transformation from the unit half disk to the vertical strip and $V(0)$ is a vertex operator insertion at the boundary corresponding to the real line.

In the sliver frame, the trace (**Figure A2**) glues the edges of the unit strips to form a correlation function on a cylinder of circumference 2

$$\int A \star B = \left\langle \left(\tau_1 \circ f \circ V_A(0)\right) \left(f \circ V_B(0)\right) \right\rangle_{C_2} . \tag{A3}$$

In this way the trace is equal to the BPZ inner product of the underlying CFT. With these operations, Witten's action is

$$S[\Psi] = \int \Psi \star Q\Psi + \frac{2}{3} \Psi \star \Psi \star \Psi , \tag{A4}$$

where Q is the BRST operator. The string field enjoys the non-linear gauge invariance

$$\Psi' = \Psi + Q\Lambda + [\Psi, \Lambda] \tag{A5}$$

with Λ an infinitesimal gauge parameter. As an element of a A_∞-algebra, which is cyclic, graded and differential associative, string fields A, B, C have the following properties:

$$A(BC) = (AB)C \tag{A6}$$

$$N_{gh}(QA) = N_{gh}(A) + 1 \tag{A7}$$

$$N_{gh}(AB) = N_{gh}(A) + N_{gh}(B) \tag{A8}$$

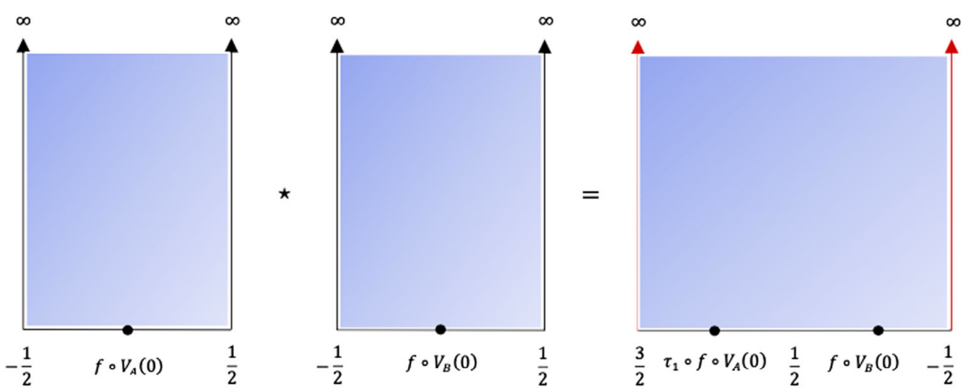

Figure A1. The star product of two string fields with vertex operators inserted at their origins, glues the right half of the first to the left half of other. The red edges ae identified with each other.

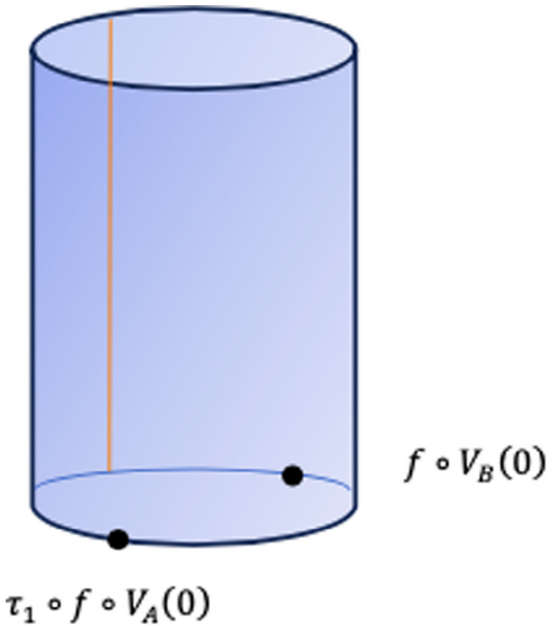

Figure A2. The integral operation identifies the right and the left side of the semi-infinite strip and glues them together. The result is a correlation function on a semi-infinite cylinder.

$$Tr(A) = 0 \text{ if } N_{gh}(A) \neq 3 \tag{A9}$$

$$Tr(QA) = 0 \tag{A10}$$

$$Tr(AB) = (-1)^{|A||B|}\, Tr(BA) \tag{A11}$$

$$Q(AB) = Q(A)B + (-1)^{|A|} AQ(B). \tag{A12}$$

The equations of motions are

$$Q\Psi + \Psi \star \Psi = 0. \tag{A13}$$

and solutions describe BCFTs, which are conformally invariant. Some examples are marginal solutions,[49,50] D-branes,[51] tachyon lumps, and various expressions for the tachyon vacuum.

Appendix B: AKSZ Sigma-Models

Named after its four authors,[52] the AKSZ formalism is a framework to construct action functionals of topological sigma models reproducing the BV-BRST formalism,[53,54] where the target space is a symplectic differential graded (dg-)manifold $\mathcal{X}$ of degree n and the worldvolume is a graded $n+1$-dimensional manifold $\mathcal{V}$. Let $(\mathcal{X}, \mathcal{Q}_{\mathcal{X}}, \omega)$ be a dg-manifold, or QP-manifold of degree $n = d-1$ with differential $\mathcal{Q}_{\mathcal{X}}$ of degree $+1$ with $\mathcal{Q}_{\mathcal{X}}^2 = 0$ and a graded symplectic structure ω of degree n. The symplectic structure ω defines a Poisson structure on $C^\infty(\mathcal{X})$. The cohomological vector field $\mathcal{Q}$ is generated by an element $\varrho \in C^\infty(\mathcal{X})$ of degree $n+1$ with respect to the graded Poisson bracket $\mathcal{Q} = \{\varrho, -\}$ called the cohomological function satisfying $\{\varrho, \varrho\} = 0$. For the second ingredient, let $(\mathcal{V}, \mathcal{Q}_{\mathcal{V}}, \mu)$ be a dg-manifold called the source, with a cohomological vector field $\mathcal{Q}_{\mathcal{V}}$ and a $\mathcal{Q}_{\mathcal{V}}$-invariant measure $\mu = d^d\,\tilde{z} := d^d\sigma d^d\theta$ of degree $-n-1$ with supercoordinates σ and θ. Now consider the space of smooth maps

$$\mathcal{M} = \text{Map}(\mathcal{V}, \mathcal{X}) \tag{B1}$$

from $(\mathcal{V}, \mathcal{Q}_{\mathcal{V}}, \mu)$ to $(\mathcal{X}, \mathcal{Q}_{\mathcal{X}}, \omega)$. For our purposes $\mathcal{V} = T[1]\Sigma_d$ is the graded tangent space of the d-dimensional worldvolume Σ_d. Obviously, $Diff(\mathcal{X}) \times Diff(\mathcal{V})$ acts on the mapping space (B1); therefore, $\mathcal{Q}_{\mathcal{V}}$ and $\mathcal{Q}$ induce a cohomological vector field $\mathcal{Q}$ on $\mathcal{M}$. Let $z^\mu \in \mathcal{V}$, $X^i \in \mathcal{X}$, $\boldsymbol{f} \in \mathcal{M}$ and define

$$\boldsymbol{D}(z, \boldsymbol{f}) = \mathcal{Q}_{\mathcal{V}}(z)\, d\boldsymbol{f}(z) \tag{B2}$$

$$\mathcal{Q}_{\mathcal{X}}(z, \boldsymbol{f}) = \mathcal{Q}_{\mathcal{X}}(z) \boldsymbol{f}(z) \tag{B3}$$

$$\mathcal{Q} = \boldsymbol{D} + \mathcal{Q}_{\mathcal{X}} \tag{B4}$$

as well as the evaluation map $ev : \mathcal{V} \times \mathcal{M} \to \mathcal{X}$ by $ev : (z, f) \to f(z)$ such that $\mu_* ev^* : \Omega^\bullet(\mathcal{X}) \to \Omega^\bullet(\mathcal{M})$. With it we can lift a n-form $a \in \Omega^n(\mathcal{X})$ to $\boldsymbol{a} := \mu_* ev^* a \in \Omega^n(\mathcal{M})$. In this way, the symplectic form ω on $\mathcal{X}$ induces a symplectic form $\boldsymbol{\omega}$ on $\mathcal{M}$. Additionally, we can lift the Q-structure from $\mathcal{X}$ to $\mathcal{M}$ by setting $\omega = d\rho$, for a Liouville potential ρ by defining $S_0 := \iota_D \mu_* ev^* \rho$ and $S_1 = \mu_* ev^* \varrho$. The form $\boldsymbol{\omega}$ induces a Poisson structure where the Poisson bracket coincides with the BV-antibracket $\{-,-\}_{BV}$ and $S = S_0 + S_1$ coincides with the BV-action and describes a Q-structure on $\mathcal{M}$. It is straight forward to see that $\{S, S\}_{BV} = 0$ and $\mathcal{Q} = \{S, -\}_{BV}$ with $\mathcal{Q}^\sharp = 0$. Thus, the QP-structure on $\mathcal{X}$ induces a QP-structure on the dg-manifold $\mathcal{M}$ with degree -1 and defines a topological field theory. In the course of this paper, we will only be interested in a submanifold $\mathcal{M}_{\mathcal{L}'} \subset \mathcal{M}$ consisting of maps from $\partial\Sigma_d$ into $\mathcal{L}'$, which we take to be a submanifold $\mathcal{L}' \subseteq \mathcal{L}$, where $\mathcal{L}$ is a Lagrangian submanifold of $\mathcal{X}$ such that $\partial\mathcal{V} = \partial T[1]\Sigma_d$ is mapped smoothly into $\mathcal{L}$. Thus, with the help of canonical transformations, which represent symplectomorphisms, one can show that the AKSZ-action on $\mathcal{M}_{\mathcal{L}'}$ is extended by a boundary term

$$S = S_0 + S_1 - \oint_{\partial T[1]\Sigma_d} d^{d-1}\tilde{z} ev^*(\beta) \tag{B5}$$

where β is a degree $d-1$ function with $\{\beta, \beta\} = 0$. Action (B5) describes open topological n-branes.

The A-Model

For $n = 1$ the QP-structure defines a Poisson structure and the AKSZ sigma-model is a Poisson sigma model. We can also obtain a complex structure for $n = 1$. In physics terms these correspond to the open A-model and B-model topological string.[55] In the A-model $\mathcal{X}$ is a Calabi–Yau three-fold and $\mathcal{L}' = \mathcal{L}$. We will take the target to be the kinematic phase space $\mathcal{X} = T^*[1]M$. Let x^i be degree 0 coordinates on the base M and ξ_i coordinates on the base the fibre of degree 1. The canonical symplectic form on $\mathcal{X}$ is given by

$$\omega = dx^i \wedge d\xi_i. \tag{B6}$$

Define the Liouville potential $\rho = \xi_i \, dx^i$. The Lagrangian submanifold is delineated by $\mathcal{L} = \{p_i = 0\}$, which is simply the base M. The most general Hamiltonian function is

$$\varrho = \frac{1}{2}\, \pi^{ij}(x)\, \xi_i \xi_j \tag{B7}$$

with $\pi^{ij}(x)$ a $(0,\ 2)$-tensor on M and the condition $\{\varrho, \varrho\} = 0$ implies $[\pi, \pi]_S = 0$, where $[-,-]_S$ is the Schouten bracket. Thus, π is the Poisson bivector and the upshot is that a QP-manifold with $n = 1$ defines a Poisson structure on M and a Lie algebroid on T^*M. The BV-action is given by

$$S = \int_{T[1]\Sigma_2} d^2\tilde{z}\, \boldsymbol{\pi}^{ij} \boldsymbol{\xi}_i \boldsymbol{\xi}_j. \tag{B8}$$

For M a Kähler manifold, π^{ij} is the inverse of the Kähler form. For open-strings, Σ_2 has boundary components and the boundary conditions must preserve the QP-structure. In the physical language this requires BRST-invariance at the boundary as well as the Grassman fields of the theory to take values in $\phi^*(T\mathcal{L})$, where $\phi : \Sigma_2 \to T^*M$. In other words, the conditions are fulfilled if $\phi(\partial\Sigma_2) \subset \mathcal{L}$. In string theory, open strings are charged under the $U(N)$ gauge field on the N D-branes they end on. Therefore, coupling the theory to Chan–Paton factors introduces an $U(N)$ gauge freedom. It can be shown that allowed connections must be flat to preserve the fermionic symmetry. The $U(N)$ bundle $E \to M$ corresponds to the physical setting where N topological branes wrap a special Lagrangian submanifold. Both topological string theories admit a description in terms of Witten's open bosonic string field theory. We are interested in the case where the target of the A-model is the cotangent space T^*Y of a closed three-dimensional manifold Y. This is a non-compact complex Calabi–Yau three-fold. The Lagrangian submanifold is the zero locus of the Liouville potential ρ, which is simply Y. Let ϕ^i and $\psi_{\mu i}$ be a scalar and a vector of degree 0 and c^i a scalar of degree 1. The fields in (B8) are superfields $(\boldsymbol{x}^i,\ \boldsymbol{\xi}_i)$ and can be expanded in the supercoordinate θ^μ as follows.

$$\boldsymbol{x}^i(\sigma, \theta) = \phi^i + \psi^{+i}_{\mu}\theta^\mu + \frac{1}{2} c^{+i}_{\mu\nu}\theta^\mu\theta^\nu \tag{B9}$$

$$\boldsymbol{\xi}_i(\sigma, \theta) = -c_i + \psi_{\mu i}\theta^\mu + \frac{1}{2}\phi^i_{i\mu\nu}\theta^\mu\theta^\nu \tag{B10}$$

The plus sign inside the superscript indicates that $c^{+i} =* \tilde{c}^i$, whereby $\tilde{c}^i$ is the corresponding anti-field and $*$ is the hodge star on Σ_2. The same holds for ϕ^{+i} and $\psi^{+i} = d\sigma^\mu\, \epsilon_{\mu\nu}\psi^{*\nu i}$. We choose $(z, \bar{z})$ as local coordinates on the world sheet and $(\dot{a}, a)$ as (anti-)holomorphic indices on the target space. Since the target is a complex Kähler manifold, we have a Kähler metric g_{ij} and a complex structure $J^i{}_j$. Hence, $\pi^{ij}(x)$ can be written as $\pi^{ij}(x) = -J^i{}_k g^{jk}$. Decomposing the fields into holomorphic parts ψ^{+i}_z, ψ_{zi}, anti-holomorphic parts $\psi^{+i}_{\bar{z}}, \psi_{\bar{z}i}$, as well as $\phi_{z\bar{z}i} = 2i\tilde{\phi}_i$, $c_{z\bar{z}}{}^{+i} = 2i\tilde{c}^i$ and introducing a gauge fixing term

$$\Theta = \int_{T[1]\Sigma_2} d^2z g_{a\dot{a}}(\phi)\left(\psi^{+\dot{a}}_z \partial_{\bar{z}}\phi^a - \psi^{+a}_{\bar{z}}\partial_z\phi^{\dot{a}}\right), \tag{B11}$$

the gauge fixing conditions are

$$c^{+i} = 0 \tag{B12}$$

$$\psi^{z}{}_{\dot{a}} = ig_{a\dot{a}}(\phi)\,\partial_{\bar{z}}\phi^a \tag{B13}$$

$$\psi^{\bar{z}}{}_a = -ig_{a\dot{a}}(\phi)\,\partial_z\phi^{\dot{a}} \tag{B14}$$

$$\phi^+_{z\bar{z}a} = -i\partial_{\bar{z}}\left(g_{a\dot{a}}(\phi)\,\psi^{+\dot{a}}_z\right) \tag{B15}$$

$$\phi^+_{z\bar{z}\dot{a}} = i\partial_z\left(g_{a\dot{a}}(\phi)\,\psi^{+a}_{\bar{z}}\right). \tag{B16}$$

After integrating out ψ_{za} and $\psi_{\bar{z}\dot{a}}$ we obtain

$$S_1 = \int_{\Sigma_2} dz \wedge d\bar{z}\, g_{a\dot{a}}(\phi)\,\partial_{\bar{z}}\phi^a\partial_{\bar{z}}\phi^a - \frac{1}{2}\left(\psi^{+a}_{\bar{z}} D_z c_a - \psi^{+\dot{a}}_z D_{\bar{z}} c_{\dot{a}} + \frac{1}{2} R_{a\dot{a}}{}^{b\dot{b}}\psi^{+a}_{\bar{z}}\psi^{+\dot{a}}_z c_b c_{\dot{b}}\right), \tag{B17}$$

where D_z is the covariant derivative. We define

$$\frac{1}{2i}c_a \equiv \chi_a \in \phi^*(TX), \tag{B18}$$

$$\psi^{+\dot{a}}_z \equiv \psi^{\dot{a}}_z \in \phi^*\left(T^{(0,1)}T^*Y\right) \tag{B19}$$

$$\psi^{+a}_{\bar{z}} \equiv \psi^a_{\bar{z}} \in \phi^*\left(T^{(1,0)}T^*Y\right). \tag{B20}$$

In order to delineate the field theoretic description of the A-model we have to evaluate the content of the string field $\Psi[\phi, \ldots]$. This was done by Witten. Since the theory is semiclassical exact, we can evaluate it in a regime where higher modes decouple and the maps ϕ are constant. Then the string field Ψ is a functional of the bose and fermi zero-modes q^a, χ^a

$$\Psi = A^{(0)}(q) + \sum_{p=1}^{3} \chi^{a_1}\cdots\chi^{a_p} A^{(p)}_{a_p\cdots a_p}. \tag{B21}$$

It follows that each of the terms can be interpreted as p-forms on M, which take values in $End(E)$, where E is a flat vector bundle over M. The BRST-operator Q can be identified with the exterior derivate and its cohomology with the deRham cohomology on Y. Classically, the string field has ghost number 1. Therefore, $\Psi = \chi^a A_a(q)$ is a flat $U(N)$ Lie algebra-valued connection one-form on E and the string field action reduces to Chern–Simons theory on Y. This is the background theory of the open A-model on T^*Y, which is exact. Roughly speaking the open A-model is a theory of world sheet instantons with Lagrangian boundary conditions. On general Calabi–Yau three-folds there also exist non-degenerate worldsheet instantons, which intersect the submanifolds at one-dimensional curves or knots. These honest instantons would contribute additional non-perturbative terms to the Chern–Simons action. Thus, the equivalence (B21) holds only perturbatively on a Lagrangian submanifold of the QP-target. In general, Lagrangian submanifolds $\mathcal{L}' \subset \mathcal{X}$ are defined by

$$\pi^{ab}q_b = \frac{\partial\Theta}{\partial\phi^a}. \tag{B22}$$

In a complex Calabi–Yau m-fold this is equivalent to the vanishing of the Kähler class

$$J|_{\mathcal{L}'} = d\rho|_{\mathcal{L}'} = g_{a\dot{a}} d\phi^a \wedge d\phi^{\dot{a}}|_{\mathcal{L}'} = 0, \tag{B23}$$

which in return implies that the gauge fermion (B11) vanishes. It follows immediately that the maps ϕ must be constant because their action $\int_{\Sigma_2} \phi^*(J) = \int_{\partial\Sigma_2} \phi^*(\rho)$ vanishes due to $\phi(\partial\Sigma_2) \subset \mathcal{L}' \cong Y$. In this case, the only degrees of freedom of the A-model are the flat gauge connections $A \in \Omega^1(Y, \mathfrak{u}\mathfrak{n}(N))$ it couples to, which are critical points of the Chern–Simons action functional. This is a direct consequence of the requirement that the boundary conditions must be consistent with the QP-structure. In particular, consider the inclusion map $i : T[1]\partial\Sigma_2 \to T[1]\Sigma_2$, which induces the measure $\mu_{\partial\Sigma_2}$ and the map $(i \times id)^* : \Omega^\bullet(\mathcal{U} \times \mathcal{X}) \to \Omega^\bullet(\partial\mathcal{U} \times \mathcal{X})$ restricting the bulk-graded differential forms on the mapping space to the boundary. Then from $S = \iota_D\ \mu_* ev^* \rho + \mu_* ev^* \varrho$ follows,

$$\{S, S\} = \iota_D\ \mu_{\partial\Sigma_{2*}} (i \times id)^* ev^* \rho + \mu_{\partial\Sigma_{2*}} (i \times id)^* ev^* \varrho. \tag{B24}$$

The first term vanishes if $\rho|_{\mathcal{L}'} = 0$ and the second vanishes if $\varrho|_{\mathcal{L}} = 0$. If these conditions are met, there is an equivalence between the following two structures, where $d - 1 = n$:

1) An AKSZ sigma-model with Lagrangian boundary conditions as above, with source $T[1]\Sigma_d$, where Σ_d is an open manifold with boundary and with target $\mathcal{X}$ a Lie n-algebroid with isotropic integrable subspace $\mathcal{L}'$ of dimension m delineated by (B23), coupled to target space fields on a vector bundle $E \to \mathcal{L}'$ with values in the Lie algebra $\mathfrak{g}$.
2) An AKSZ sigma-model with source $T[1]\mathcal{L}'$ and target, a Lie $(m-1)$-algebroid over a point, which is the QP-manifold $T^*[m-2]\mathfrak{g}[1]$. This statement implies that their mapping spaces are isomorphic:

$$\mathcal{M}_{\mathcal{L}'} = \mathrm{Map}_{\mathfrak{g}} \left(T[1]\,\partial\Sigma_d, \mathcal{L}' \right) \simeq \mathrm{Map} \left(T[1]\,\mathcal{L}', T^*[m-2]\,\mathfrak{g}[1] \right). \tag{B25}$$

Notice that $T^*[m-2]\mathfrak{g}[1] \simeq \mathfrak{g}[1] \oplus \mathfrak{g}^*[m-3]$. For $n = 1$ this implies the equivalence of the open A-model on T^*Y and Chern–Simons theory but also the equivalence between the open B-model and holomorphic Chern–Simons theory. However, open B-strings couple to holomorphic connections $\mathcal{E}^{(0,1)}$ on a holomorphic vector bundle. Here the subspace $\mathcal{L}'$ can be a space-filling complex cycle inside a complex Calabi–Yau three-fold CY_3. Thus, the real dimension of $\mathcal{L}'$ can be six and it can be identified with the Calabi–Yau manifold itself. Let $CY_3 = \mathcal{O}(-1)^{\oplus 2} \to \mathbb{CP}^1$. By the above proposal this would imply that the mapping space of the AKZS sigma-model for open B-strings is isomorphic to

$$\mathcal{M}_{\mathcal{O}(-\mathfrak{f})^{\oplus\mathfrak{f}} \to \mathbb{CP}^{\mathfrak{f}}} \simeq \mathrm{Map} \left(T^{(0,1)}[1]\,\mathcal{O}(-1)^{\oplus 2} \to \mathbb{CP}^1, T^*[4]\,\mathfrak{g}[1] \right) \tag{B26}$$

and that the six-dimensional AKSZ-action is holomorphic Chern–Simons theory, which is indeed the case. Since boundary terms can be obtained by twisting,[56] the duality implies the equivalence of Lie $(m-1)$-algebroid structures and twisted Lie n-algebroid structures, which, from a physical viewpoint, could be interpreted as equivalence between open topological strings and closed topological membranes on a Lie $(m-1)$-algebroid over a point if $n = 1$. In general, the Lie $(m-1)$-algebroids coincide with m dimensional non-abelian BF-theories.

Eight-Dimensional Topological Yang–Mills

The effective background theory of the topological membrane of Hořova-Lifshitz type can be generalized to the whole eight-dimensional target X. Here, it can be cast into the form of an 8d-topological Yang–Mills theory

$$S[A] = \frac{1}{2} \int_X \Omega \wedge Tr\ F \wedge F = \left[8\pi^2 \left(2c_2(E) - c_1(E)^2 \right) \right] \cup [\Omega], \tag{B27}$$

whereby Ω is a closed, covariantly constant self-dual four-form, which is called a calibration[57] of X and $\cup$ denotes the cup-product. Additionally, c_i are the i-th Chern-classes of the bundle E over X. Calibration means that for a point $x \in X$ and for every 4-plane $\zeta \in T_x X$ it holds that $\Omega|_\zeta \le vol_\zeta$ with respect to the metric g on X. Moreover, if we restrict the calibration Ω to a submanifold M it becomes its volume form

$$\int_M \Omega|_M = vol_M. \tag{B28}$$

In this context, M is called a calibrated submanifold or Cayley 4-plane and Ω is the Cayley four-form. If X is a 4-dimensional complex Calabi–Yau orbifold with Kähler form ω and holomorphic volume-form, $\Omega^{(m,0)}$ we can express the calibration as

$$\Omega = \frac{1}{2}\,\omega \wedge \omega + Re\left(\Omega^{(m,0)}\right). \tag{B29}$$

The $SU(4)$ holonomy of the Calabi–Yau orbifold induces a $Spin(7)$-structure (Ω, g) on X.[58–62] In this paper we will only consider complex Calabi–Yau four-folds with $Spin(7) \subset SO(8)$ holonomy.[63] The self-duality conditions in eight dimensions were analysed in ref. [64] and involve an antisymmetric tensor $T^{\mu\nu\rho\sigma}$, which is invariant under a total subgroup of $SO(8)$

$$\xi F^{\mu\nu} = \frac{1}{2}\,T^{\mu\nu\rho\sigma} F_{\mu\nu}, \tag{B30}$$

where ξ is an eigenvalue. If we take our prime example $X = T^*M$, the calibrated submanifold is the Lagrangian submanifold M and the action reduces on M to the usual topological Yang–Mills theory

$$S_T[A] = \frac{1}{2} \int_M \mathrm{Tr}\ F \wedge F. \tag{B31}$$

The self-duality conditions are unambiguous in four-dimensions.

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Conflict of Interest

The authors declare no conflict of interest.

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