

A new class of (2 + 1)-dimensional topological superconductors with \mathbb{Z}_8 topological classification

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Abstract. The classification of topological states of matter depends on spatial dimension and symmetry class. For non-interacting topological insulators and superconductors, the topological classification is obtained systematically and non-trivial topological insulators are classified by either integer or \mathbb{Z}_2 . The classification of interacting topological states of matter is much more complicated and only special cases are understood. In this paper we study a new class of topological superconductors in (2 + 1) dimensions which has time-reversal symmetry and a \mathbb{Z}_2 spin conservation symmetry. We demonstrate that the superconductors in this class are classified by \mathbb{Z}_8 when electron interaction is considered, while the classification is \mathbb{Z} without interaction.



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1. Introduction

Topological states of matter (TSM) are ground states of gapped quantum systems which cannot be adiabatically tuned to a topologically trivial state without going across a phase transition. The first TSM discovered in nature are integer and fractional quantum Hall states [1, 2]. Many TSM have gapless surface states on the boundary which are topologically protected within the given symmetry class. For example, the integer quantum Hall states have chiral fermion edge states which propagate unidirectionally along the edge of the two-dimensional (2D) system, and are robust against arbitrary perturbation as long as the bulk remains gapped. The number of chiral channels is an integer which determines the Hall conductance and characterizes the corresponding quantum Hall state.

Since the recent discovery of time-reversal invariant topological insulators [3–5], TSM has been understood much more systematically in generic spatial dimensions and symmetries [6–8]. For gapped non-interacting fermion systems, i.e. band insulators without electron correlation and gapped superconductors in the mean-field theory sense, there are ten symmetry classes in each spatial dimension [9, 10] if we only consider symmetries that preserve the positions of electrons, such as time-reversal symmetry and particle–hole symmetry. In each spatial dimension, TSM exist in five of the ten symmetry classes, which always include three classes with integer classification and two classes with \mathbb{Z}_2 classification [7, 8]. The topological classification depends on the spatial dimension and symmetry class in a periodic way [8] which originates from the Bott periodicity [11]. When electron interaction is considered, the problem is much more complicated and the topological classification in generic dimensions and symmetry classes has not been completely understood. Topological invariants based on a single-particle Green’s function has been proposed for interacting TSM [12–15]. In one dimension (1D), the classification of gapped states has been studied systematically by making use of matrix product states and entanglement spectrum [16, 17]. In particular, Fidkowski and Kitaev [18] proposed an explicit example of 1D time-reversal invariant superconductors for which the topological classification is \mathbb{Z} for non-interacting states but is reduced to \mathbb{Z}_8 when interaction is considered. In dimensions higher than one, a systematical approach of constructing interacting symmetry-protected topological states (defined as the TSM in a certain symmetry class which becomes trivial when the symmetry is broken) in spin models has been proposed based on group

cohomology [19]. The relation between such states and the non-interacting TSMs in generic cases remains an open question¹.

In this paper, we propose a new class of TSM (labeled by D') which in 2Ds is classified by \mathbb{Z}_8 . This symmetry class is defined by two discrete symmetries, the time-reversal symmetry and an additional global \mathbb{Z}_2 symmetry. Physically, this symmetry class consists of time-reversal invariant superconductors of spinful fermions with the number of spin-up electrons (and also that of spin-down electrons) conserved modular 2. Without interaction, the topological classification is \mathbb{Z} , the same as time-reversal breaking superconductors (class D) in 2D [20]. With electron interactions, we show that the classification is reduced to \mathbb{Z}_8 , by analyzing the edge states and topological defects. This state can be considered as a higher dimensional generalization of the \mathbb{Z}_8 topological superconductor (TSC) proposed in [18].

2. Definition of the symmetry class and the classification of non-interaction systems

Consider a 2D electron system with the generic Hamiltonian $H = H(c_{i\sigma}, c_{i\sigma}^\dagger)$, with $\sigma = \uparrow, \downarrow$ two spin components, and i labels the lattice sites. The symmetries we consider are time-reversal symmetry defined by

$$T^{-1}c_{i\uparrow}T = c_{i\downarrow}, \quad T^{-1}c_{i\downarrow}T = -c_{i\uparrow} \quad (1)$$

and the \mathbb{Z}_2 spin conservation symmetry named by R :

$$R^{-1}c_{i\uparrow}R = c_{i\uparrow}, \quad R^{-1}c_{i\downarrow}R = -c_{i\downarrow}. \quad (2)$$

It should be noticed that T is an anti-unitary symmetry and R is a unitary symmetry. In an explicit form $R = (-1)^{N_\downarrow}$ with N_\downarrow the total number of spin-down electrons. An equivalent definition of T and R symmetries which is basis independent is

$$\begin{aligned} T^2 = F, \quad F^2 = 1, \quad R^2 = 1, \quad T^{-1}RT = FR, \\ T^{-1}iT = -i, \quad R^{-1}iR = i, \end{aligned} \quad (3)$$

in which $F = (-1)^{N_\uparrow + N_\downarrow}$ is the fermion number parity. The second line means that T is anti-unitary and R is unitary. We label the symmetry class defined by symmetries T and R as class D' due to its relation to the class D of time-reversal symmetry breaking superconductors, which will be made clear in the later part of the draft.

Now we consider the simple situation when the Hamiltonian is quadratic in $c_{i\sigma}, c_{i\sigma}^\dagger$. Apparently, any term mixing spin-up and -down electrons such as $c_{i\uparrow}^\dagger c_{j\downarrow}$ and $c_{i\uparrow} c_{j\downarrow}$ is odd under the \mathbb{Z}_2 symmetry R , such that the quadratic Hamiltonian H_{quad} is decoupled into a direct sum of that of the spin-up and -down subsystems:

$$\begin{aligned} H_{\text{quad}} &= H_\uparrow + H_\downarrow, \\ H_\sigma &= \sum_{i,j} \begin{pmatrix} c_{i\sigma}^\dagger & c_{i\sigma} \end{pmatrix} h_{ij}^\sigma \begin{pmatrix} c_{i\sigma} \\ c_{i\sigma}^\dagger \end{pmatrix}, \quad \sigma = \uparrow, \downarrow. \end{aligned} \quad (4)$$

¹ After finishing this work, I notice a recent paper [30] which generalizes the group cohomology approach to fermion systems. Whether the result of the current paper can be understood by the approach of [30] is an interesting question.

The single-particle Hamiltonians h_{ij}^\uparrow and h_{ij}^\downarrow are related by the time-reversal invariance condition

$$T^{-1}H_\uparrow T = H_\downarrow \Rightarrow h_{ij}^{\uparrow*} = h_{ij}^\downarrow. \quad (5)$$

Thus all non-interacting states in this class are direct products of spin-up and -down superconductor ground states, and the spin-down state is determined by the spin-up state due to time-reversal symmetry. Consequently, the topological classification of H_{quad} reduces to that of the spin-up subsystem H_\uparrow , which is a Hamiltonian of a generic 2D superconductor, *without* time-reversal symmetry. (The time-reversal symmetry is recovered only when the spin-down state is also included.)

It is well known that the 2D time-reversal breaking superconductors are classified by integer, in the same way as integer quantum Hall states [20]. The topological invariant characterizing this class of TSC is the Chern number of the single-particle Hamiltonian. For a translation-invariant system, if we denote $|n, \mathbf{k}\rangle$ as the eigenstates of the single-particle Hamiltonian h^\uparrow , with correspondingly eigenvalues $E_n(\mathbf{k})$, the Chern number is defined as the total flux of the Berry phase gauge field in the Brillouin zone: $C_1 = \frac{1}{2\pi} \int d^2\mathbf{k} (\partial_x a_y - \partial_y a_x)$ with $a_i = -i \sum_{E_n < 0} \langle n, \mathbf{k} | \partial / \partial k_i | n, \mathbf{k} \rangle$. For a system with Chern number $C_1 = N$, on an open boundary there are N channels of chiral Majorana edge states, described by the following Hamiltonian:

$$H_{\text{edge}}^\uparrow = \int dk vk \sum_{a=1}^{|N|} \eta_{\uparrow ak}^\dagger \eta_{\uparrow ak} \quad (6)$$

with k the momentum along the edge and $\eta_{\uparrow ak}$ the edge state quasiparticle operator which is a superposition of electron and hole operators. $\eta_{\uparrow ak}$ satisfies the Majorana condition $\eta_{\uparrow ak}^\dagger = \eta_{\uparrow a, -k}$ so that the edge state fermion is its own anti-particle. In particular, $\eta_{\uparrow a, k=0} = \eta_{\uparrow a, k=0}^\dagger$ is a Majorana zero mode, carrying half of the degree of freedom of a complex fermion [20, 21]. For $N > 0$ ($N < 0$) v is positive (negative) respectively, corresponding to left and right moving edge states. For simplicity, we have set the velocity of the Majorana edge states to be the same. Generically the velocity can be different for different a but it does not change the topological properties. According to equation (5) determined by time-reversal symmetry, it can be shown straightforwardly that h^\downarrow and h^\uparrow have opposite Chern number. Correspondingly the edge states of them consist of the same number of Majorana edge states with opposite chirality. For $C_1^\uparrow = -C_1^\downarrow = N > 0$, the edge theory of the whole system is

$$H_{\text{edge}} = \int dk vk \sum_{a=1}^{|N|} \left[\eta_{\uparrow ak}^\dagger \eta_{\uparrow ak} - \eta_{\downarrow ak}^\dagger \eta_{\downarrow ak} \right]. \quad (7)$$

Since $\eta_{\uparrow(\downarrow)ak}$ is a superposition of $c_{\uparrow(\downarrow)}$, the action of R symmetry on $\eta_{\uparrow(\downarrow)ak}$ is $R^{-1} \eta_{\uparrow ak} R = \eta_{\uparrow ak}$, $R^{-1} \eta_{\downarrow ak} R = -\eta_{\downarrow ak}$. Consequently, no mixing term such as $\eta_{\uparrow ak}^\dagger \eta_{\downarrow ak}$ is allowed in the Hamiltonian H_{edge} if we consider generic perturbations preserving the R symmetry.

In summary, from the analysis above we conclude that the 2D superconductors with T , R symmetries are classified by integer, and the class labeled by integer N has an edge theory of $|N|$ flavors of non-chiral Majorana fermions, which are protected to be gapless by the two symmetries.

3. The reduction of \mathbb{Z} to \mathbb{Z}_8 with interaction

Now we consider the effect of electron interaction in this system. With interaction the terms mixing spin-up and -down electrons are allowed by the R symmetry, such as a pair hopping term $H_{\text{ph}} = \sum_{ijkl} (t_{ijkl} c_{i\uparrow}^\dagger c_{j\uparrow}^\dagger c_{k\downarrow} c_{l\downarrow} + \text{h.c.})$ with h.c. denotes the Hermitian conjugate of the first term. The time-reversal symmetry leads to some requirements on the matrix elements $t_{ijkl} = t_{klij}$. Other terms such as $c_{i\uparrow}^\dagger c_{j\uparrow}^\dagger c_{k\downarrow}^\dagger c_{l\downarrow}^\dagger$ can also be considered. To see the consequence of such interaction terms on the \mathbb{Z} topological invariant, it is most convenient to study the stability of the edge states described by the Hamiltonian (7) under interaction [22].² In the following we will show that the Hamiltonian (7) with $N = 8$ can be gapped by an interacting term without breaking the T, R symmetries, so that the classification of the bulk topological states can at most be \mathbb{Z}_8 .

This conclusion is obtained based on the results of [18] in $(1+1)$ -dimensions. It was shown explicitly in [18] that the following interacting term induces a mass to eight channels of free Majorana fermions given by equation (7):

$$H_{\text{int}} = H_{1234} + H_{5678} + H_{1256} + H_{3478} + H_{3456} + H_{1278} - H_{2367} - H_{1458} - H_{2358} - H_{1467} + H_{1357} \\ + H_{2468} - H_{1368} - H_{2457},$$

$$H_{abcd} = \int dx M \sum_{e,f,g,h} \epsilon^{abcdefgh} \eta_{e\uparrow} \eta_{f\uparrow} \eta_{g\downarrow} \eta_{h\downarrow}, \quad (8)$$

in which the indices a, b, \dots, h run from 1 to 8, and $\epsilon^{abcdefgh}$ is the eight-dimensional totally antisymmetric tensor. For eight Majorana fermions with the same velocity, there is an $SO(8)$ symmetry between the Majorana fermions generated by the operators $J_{ab} = \frac{1}{4i}([\eta_{a\uparrow}, \eta_{b\uparrow}] + [\eta_{a\downarrow}, \eta_{b\downarrow}])$, and the Majorana fermion operators η_a form vector representation of this $SO(8)$. The interaction term given above breaks the $SO(8)$ symmetry to $SO(7)$, but the $SO(7)$ subgroup is not the one which preserves some vector n_a of $SO(8)$, but the one which preserves some spinor ψ_α of $SO(8)$. In the appendix A we provide an explicit explanation why the interaction term (8) leads to such a non-conventional symmetry breaking.

It is straightforward to verify that H_{int} defined in equation (8) preserves both T and R symmetries. The existence of such a mass term suggests that the quadratic Hamiltonian (4) with Chern number $N = 8$ for spin-up electrons is topologically trivial, since it can be adiabatically deform to a trivial state with $N = 0$. The adiabatic deformation path can be constructed by considering the $N = 8$ system on a closed manifold, and creating a trivial $N = 0$ region on the manifold. By moving the boundary between the two regions one can expand the trivial region until it covers the whole manifold. Since the edge states between the $N = 0$ and 8 regions can be completely gapped, the whole system remains gapped during this process, so that the $N = 8$ state is equivalent to $N = 0$.³ For $N > 8$ one can add the same mass term for any eight of the Majorana fermions, leading to the conclusion that the Hamiltonians with Chern number N and $N - 8$ have topologically equivalent edge states. If the edge states for $N < 8$ is stable, one is lead to the conclusion that the topological classification of T, R invariant TSC is \mathbb{Z}_8 rather than \mathbb{Z} .

² For similar discussions in the edge state of topological insulators, see [22].

³ I acknowledge a private communication with Alexei Kitaev who explained this proof of bulk–boundary relation for generic symmetry protected topological states.

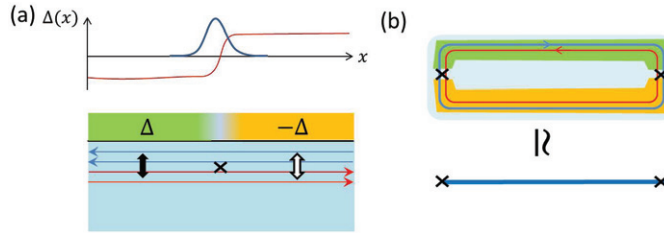


Figure 1. (a) Illustration of the T-breaking mass domain wall along the edge of the 2D TSC. The mass is induced by a proximity effect of the TSC with an s-wave superconductor, and the mass domain wall corresponds to a Josephson junction between two such s-wave superconductors (see text). (b) Illustration of a strip of 2D TSC with the edge states on the two edges gapped by opposite mass terms m and $-m$, which is topologically equivalent to a 1D TSC in the BDI class, with Majorana zero modes at the end.

4. Dimensional reduction and the stability of \mathbb{Z}_8 classification

Now we present further evidence that the systems with Chern number $N = 0, 1, \dots, 7$ are indeed topologically distinct from each other. Consider an open boundary system and add the following spatially inhomogeneous mass term to the Hamiltonian:

$$H_{\Delta} = \sum_i i\Delta_i (c_{i\uparrow}^{\dagger} c_{i\downarrow}^{\dagger} - \text{h.c.}). \quad (9)$$

Here $\Delta_i \in \mathbb{R}$ is finite in a neighborhood of the edge, and vanishes in the bulk. The Hamiltonian (9) is a special pairing term where each electron creation or annihilation operator is paired with its time reversal: $H_{\Delta} = \sum_{i\sigma} \frac{i\Delta_i}{2} [c_{i\sigma}^{\dagger} (T^{-1} c_{i\sigma}^{\dagger} T) + c_{i\sigma} (T^{-1} c_{i\sigma} T)]$. Such a generic definition of this pairing term is basis independent [23]. Therefore in the low-energy edge theory (7), the pairing term leads to a mass term $H_{\Delta} = \int dx \sum_{a=1}^{|N|} i\Delta(x) \eta_{\uparrow a}(x) \eta_{\downarrow a}(x)$.

In particular, consider the configuration of $\Delta(x)$ with a domain wall where $\Delta(x)$ changes its sign, as shown in figure 1(a). This configuration corresponds to a Josephson junction between two s-wave superconductors in proximity with the TSC, as shown in figure 1(a). As is well known, such a mass term domain wall leads to Majorana zero modes localized on the domain wall [24]. For Chern number N there are $|N|$ zero modes on the domain wall. As is known for other symmetry classes, such as the quantum spin Hall state with T -breaking domain wall on the edge [25], such a domain wall of a symmetry breaking mass term can carry fractional quantum numbers (i.e., projective representation of remaining symmetries), in which case it can be used as a probe of the TSM even if electron interaction is considered. For example, in the case of quantum spin Hall insulator the T-breaking mass domain wall on the edge traps a fractional charge [25]. In the current system, to see if the zero modes carry fractional quantum numbers, one can first analyze the symmetry of the system with the domain wall. The mass term H_{Δ} in equation (9) breaks both T and R symmetries, but preserves the combined symmetry $\tilde{T} = T \circ R$. From the definition of T and R in equations (1) and (2) one can see that $\tilde{T}^{-1} c_{i\uparrow} \tilde{T} = c_{i\downarrow}$, $\tilde{T}^{-1} c_{i\downarrow} \tilde{T} = c_{i\uparrow}$, so that $\tilde{T}^2 = 1$. Thus \tilde{T} can be considered as the time-reversal symmetry for spinless fermions, which is the symmetry defining the BDI symmetry class [9, 10]. The domain wall zero modes are thus topologically equivalent to the edge zero modes

of a 1D TSC in the BDI class [18]. The relation between the domain wall zero modes and the edge state of 1D TSC can be seen most explicitly in the geometry shown in figure 1(b). A strip of 2D TSC with the upper and lower surface states gapped by opposite mass Δ and $-\Delta$ can be considered as a 1D TSC in the BDI class. With open boundary in the horizontal direction, the end of the 1D system becomes a mass domain wall between Δ and $-\Delta$ which has Majorana zero modes if the topological invariant is $N \neq 0 \pmod{8}$. In [16, 17, 26] it was shown that the edge zero modes of a 1D system are classified by projective representations of the symmetry group, or equivalently, the second cohomology of the symmetry group. Here the symmetries of the system are \tilde{T} and the fermion number parity F which is always a symmetry for fermion systems. \tilde{T} and F commutes with each other, and form the symmetry group of $\mathbb{Z}_2 \times \mathbb{Z}_2$. There are eight distinct projective representations of this group which corresponds to $N = 0, 1, \dots, 7$ number of Majorana zero modes. Therefore from the results on 1D BDI class, one can conclude that the domain wall zero modes for $N = 1, 2, \dots, 7$ are stable.

Since the \mathbb{Z}_8 classification of such Majorana zero modes is given by the topological classification of 1D BDI class, naively it seems possible to drive a boundary phase transition which changes the number of domain wall Majorana zero modes without affecting the bulk. However, the special mass term (9) guarantees that such boundary phase transition cannot occur, so that the classification of domain wall zero modes is indeed a bulk property. To prove this statement, it is sufficient to prove that a trivial bulk state coupled to the edge mass term (9) is *always* topologically trivial. Since the bulk is trivial, the system is equivalent to two purely 1D wires along the upper and lower half of the edge in the D' class, with the Hamiltonians $H_{u(l)} = H_{u(l)0} \pm H_\Delta$. Here $H_{u(l)0}$ preserves T and R symmetries, and H_Δ is the pairing term in equation (9) with a constant amplitude $\Delta(x) = \Delta$. This mass term has the following essential property:

- For an arbitrary quadratic $H_{u(l)0}$ in D' class, the Hamiltonian $H_{u(l)}$ always has a gap $E_g \geq |\Delta|$.

This property is a direct consequence of the time-reversal symmetry which determines the form of H_Δ . The detailed proof of this property is given in the appendix B. Since the open boundary system has a finite gap, there is no Majorana zero modes and the system must be topologically trivial. For $H_{u(l)0}$ with interaction, if the interaction strength V is much weaker than $|\Delta|$, the gap remains robust and thus the system remains trivial. Exactly parallel discussion can be made to a topologically non-trivial bulk state. For a bulk state with gap E_{bulk} and interaction strength $V \ll E_{\text{bulk}}$, one can always choose the boundary mass term with $V \ll |\Delta| \ll E_{\text{bulk}}$, in which case the argument above demonstrates that no boundary phase transition can occur, and thus the domain wall Majorana zero modes are completely determined by the bulk state. In summary, we reach the conclusion that the \mathbb{Z}_8 classification of 2D D' class is robust at least for weak interaction strength. In principle, it is possible to have new phases in strong interaction that is not adiabatically connected to any of the eight weak interacting states.

It is helpful to make more comments on the relation of the $N = 8$ edge state problem with the 1D BDI class. A most generic quadratic mass term allowed in the edge theory has the form of $H_m = i \int dx A_{ab} \eta_{a\uparrow}(x) \eta_{b\downarrow}(x)$. The Hermitivity of the Hamiltonian and \tilde{T} symmetry requires the matrix A to be real and symmetric. Such a massive Majorana fermion can be viewed as a 1D TSC in BDI class with a topological invariant $N = \text{Ind}(A)$ defined as the number of negative eigenvalues of mass matrix A . If only quadratic mass terms H_m are considered, the topological classification would be \mathbb{Z} since some eigenvalues of A_{ab} must vanish in order to

change the index N , leading to a topological phase transition. However, the quartic mass term H_{int} given in equation (8) is possible for $N = 8$, such that it is possible to have a cross over between two A_{ab} with index different by 8 [18]. For example, consider the mass term $H(\theta) = im \cos \theta \int dx \sum_{a=1}^8 \eta_{a\uparrow} \eta_{a\downarrow} + \sin \theta H_{\text{int}}$. For $m > 0$, the Hamiltonian at $\theta = 0, \pi$ has index 0 and 8, respectively. By adding the interaction term H_{int} , one obtains an interpolation between $H(0)$ and $H(\pi)$ without closing the gap. On the edge of 2D D' class TSC, the additional R symmetry requires the quadratic mass term A_{ab} to vanish, so that the edge theory can be viewed as the topological phase transition theory between different topological states in the (1+1)-D BDI class. At presence of the quartic mass term H_{int} , there are only eight distinct phases in (1+1)-D BDI class. Thus there are eight distinct phase transition theories at presence of the R symmetry, which becomes the robust topological edge states in the (2+1)-D case classified by \mathbb{Z}_8 .

At the end of this section, we would like to further clarify why we have not studied mass terms other than the s-wave pairing term (9). Other mass terms can certainly be written down which also gap the edge states. For example, for the edge theory we can write $H_{\Delta} = \int dx \sum_{a=1}^{|N|} i\Delta_a(x) \eta_{\uparrow a}(x) \eta_{\downarrow a}(x)$ with different pairing term Δ_a for different flavors a . However, it has been shown explicitly in the last section that the edge state of $N = 8$ system can be gapped without breaking T and R symmetries, so that the system can be adiabatically deform to the $N = 0$ trivial state. Consequently, the only two possibilities left are (i) the topological classification is \mathbb{Z}_8 ; and (ii) the topological classification is some quotient group of \mathbb{Z}_8 , if some of the $N = 1, 2, \dots, 7$ phases are actually trivial. Therefore as long as we can find a suitable mass term that proves the stability of the eight phases—which is what has been demonstrated above using the pairing term (9)—it is sufficient to conclude that the topological classification is \mathbb{Z}_8 . This is why we did not discuss other mass terms. If we choose other mass terms, we may get a weaker condition, which is not contradictory with the conclusion above, but does not provide the complete topological classification. For example if we consider the even topological classes $N = 2n$, and give the flavors $a = 1, 2, \dots, n$ pairing $\Delta(x)$ and the rest $a = n+1, n+2, \dots, 2n$ opposite pairing $-\Delta(x)$, and $\Delta(x)$ has the same configuration as is shown in figure 1, the corresponding 1D superconductor is in BDI class but with topological invariant 0 (due to cancelation of the two groups with opposite sign of pairing). Therefore we are not able to conclude whether the $N = 2n$ classes are trivial or non-trivial.

5. Conclusion and discussions

In summary we have defined a new class D' of TSC defined by two discrete symmetries T and R satisfying the conditions in equation (3). We demonstrate that TSC in D' class has a topological classification of \mathbb{Z}_8 . The (2+1)-D D' TSC is related to the (1+1)-D BDI class of TSC with symmetry \tilde{T} by adding one spatial direction and one additional discrete symmetry T . Such a relation between TSM in different dimensions applies to more generic cases. In non-interacting fermion systems, the single-particle Hamiltonian and anti-unitary symmetry generators anti-commute with each other and form a Clifford algebra, which determines the topological classification [8]. For eight of the ten symmetry classes that correspond to real Clifford algebras, the topological classification stays the same when the spatial dimension is increased by 1 and a new anti-unitary symmetry satisfying $T^2 = -1$ is added to the system, which anti-commutes with all existing generators of the Clifford algebra. With interaction the single-particle Hamiltonian and Clifford algebra approach do not apply, but our result shows that the relation between TSM in different dimensions with different symmetries can be generalized by

considering the domain wall geometry illustrated in figure 1(b). Consider a $(d + 1)$ -dimensional system with one of the dimensions (the vertical direction in figure 1(b)) small compared to other dimensions. When the added symmetry (T in the case of D' class) in the $d + 1$ dimensional system is broken on the boundary but preserved in the bulk, generically the surface states are gapped and such a $(d + 1)$ -dimensional system can be viewed as a d -dimensional TSM in the known symmetry class. However there are always two ways to gap the surface states, which corresponds to doing a T transformation to the boundary symmetry breaking terms. A domain wall forms between the two regions with opposite symmetry breaking mass terms, as is shown in figure 1(b). If the d -dimensional TSM obtained in this way is topological non-trivial, the domain wall must carry gapless modes which are topologically equivalent to the boundary states of the d -dimensional TSM. Following the same reasoning as we did for 2D D' class, for weak interacting states the domain wall gapless modes are completely determined by the bulk and can be used to characterize the bulk topological invariant.

We would like to emphasize an important difference between the cases of interacting and non-interacting systems. For non-interacting TSM, the D class with no symmetry and the D' class with T , R symmetries are completely equivalent, but they are distinct in interacting systems. This is consistent with the fact that the edge domain wall construction only works when a proper discrete symmetry is added, rather than removed. More systematical discussion on this dimensional reduction approach to TSM is reserved for future works.

A natural question is how to realize the D' class in real materials. One possible class of systems is inversion symmetric 2D films. The inversion symmetry suppresses the spin-flip terms such as the Rashba term, leading to an approximate realization of the D' class. More detailed study to this problem will be left for future work.

Acknowledgments

I acknowledge helpful discussions with Lukas Fidkowski and Alexei Kitaev. This work is supported by the Packard Foundation. Recently I became aware of several independent approaches to similar 2D TSM classified by \mathbb{Z}_8 [27–29].

Appendix A. Further analysis on the interaction Hamiltonian (8)

In this appendix we provide an explicit explanation on the symmetry property of the interaction term (8). The analysis is already presented in [18], but we would like to provide an alternative illustration that is hopefully simpler and more explicit.

To see the symmetry breaking induced by the interaction term (8), we introduce the Clifford algebra $\text{Cliff}(8, 0)$ with the generators Γ_a , $a = 1, 2, \dots, 8$ satisfying

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}. \quad (\text{A.1})$$

Γ_a can be represented by Hermitian matrices. If we do not require the representation of Clifford algebra to be real, the minimal dimension of faithful representation for the algebra $\text{Cliff}(2n, 0)$ is 2^n . If we require the representation to be real $\Gamma_a = \Gamma_a^*$, usually the minimal dimension is larger than 2^n . The special property of $\text{Cliff}(8, 0)$ is that the faithful representation with minimal dimension $2^4 = 16$ is also real. To obtain an explicit understanding, one can consider

the following matrix representation of Γ_a :

$$\begin{aligned}
 \Gamma_1 &= \nu_y \sigma_y \tau_0 \mu_0, \\
 \Gamma_2 &= -\nu_y \sigma_x \tau_0 \mu_y, \\
 \Gamma_3 &= \nu_y \sigma_x \tau_y \mu_z, \\
 \Gamma_4 &= \nu_y \sigma_x \tau_y \mu_x, \\
 \Gamma_5 &= -\nu_y \sigma_z \tau_x \mu_y, \\
 \Gamma_6 &= \nu_y \sigma_z \tau_y \mu_0, \\
 \Gamma_7 &= \nu_y \sigma_z \tau_z \mu_y, \\
 \Gamma_8 &= \nu_x \sigma_0 \tau_0 \mu_0.
 \end{aligned}
 \tag{A.2}$$

Here $\nu_i, \sigma_i, \tau_i, \mu_i$ are matrices in four independent 2×2 spaces, which are defined as the Pauli matrices for $i = x, y, z$, and 2×2 identity matrices for $i = 0$. The multiplication in the equation above shall be understood as direct product, so that Γ^a are 16 matrices. It can be directly verified that this representation of Γ^a is real, and satisfies the Clifford algebra defined in equation (A.1).

We define the generators T_{ab} of vector representation of $SO(8)$ as

$$[T_{ab}]^{cd} = -i (\delta_a^c \delta_b^d - \delta_a^d \delta_b^c). \tag{A.3}$$

A generic 8×8 orthogonal matrix is written as $O = \exp[i T_{ab} \theta^{ab}]$. The spinor representation of $SO(8)$ is generated by the commutators $\Gamma_{ab} = [\Gamma_a, \Gamma_b]/4i$. The elements of the spinor representation are $U = \exp[i \Gamma_{ab} \theta^{ab}]$. Γ^a carries the vector representation of $SO(8)$ under the similarity transformation defined by $\Gamma^a \rightarrow U^{-1} \Gamma^a U \equiv O_{ab} \Gamma_b$. In the choice of representation (A.2), all Γ_{ab} have the form of $\Gamma_{ab} = \nu_0 \dots$ or $\Gamma_{ab} = \nu_z \dots$. Consequently, all Γ_{ab} are block diagonal with two 8×8 blocks. In other words, the spinor representation is reducible. The two 8×8 blocks are the spinor representation and the conjugate spinor representation of $SO(8)$. Moreover, because Γ_a is real, Γ_{ab} is purely imaginary and the rotation in the spinor representation U is real. If we denote

$$\Gamma_{ab} \equiv \begin{pmatrix} \gamma_{ab}^+ & 0 \\ 0 & \gamma_{ab}^- \end{pmatrix}, \tag{A.4}$$

one can define the mappings from vector representation to the spinor representations

$$\begin{aligned}
 \varphi_+ : O = \exp [i T_{ab} \theta^{ab}] &\longrightarrow U_+ = \exp [i \gamma_{ab}^+ \theta^{ab}], \\
 \varphi_- : O = \exp [i T_{ab} \theta^{ab}] &\longrightarrow U_- = \exp [i \gamma_{ab}^- \theta^{ab}].
 \end{aligned}
 \tag{A.5}$$

Because U_{\pm} are also real orthogonal matrices, the two maps φ_{\pm} are *isomorphisms* between the vector representation and the spinor representations. This is the so-called triality property of $SO(8)$.

Now we consider eight Majorana fermion operators η_a . The Hilbert space of such a system is 16 dimensional, the same as that of four complex fermions. Since η_a satisfies the Clifford algebra in equation (A.1), we can take a basis in the Hilbert space so that the matrix representation of operators η_a is Γ_a . The time-reversal symmetry \tilde{T} acts on η_a as $\tilde{T}^{-1} \eta_a \tilde{T} = \eta_a$. Therefore in the representation given in equation (A.2), one can take the time-reversal symmetry to be $\tilde{T} = K$ which is the complex conjugation. Now we want to define a time-reversal invariant

Hamiltonian in this Hilbert space with a unique ground state. The simplest choice is a projection operator

$$H_{\text{proj}} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \dots & \\ & & & -1 \end{pmatrix}_{16 \times 16} \quad (\text{A.6})$$

which has all entries vanishing except the lower right corner. Such a projector Hamiltonian breaks the $SO(8)$ of the lower 8×8 block to $SO(7)$. However, this $SO(7)$ is the $SO(7)$ which rotates the first seven columns of the lower 8×8 block. This is not the usual $SO(7)$ subgroup of $SO(8)$ which preserves a given vector. Instead, it is the subgroup that preserves a given spinor $(0, 0, \dots, 1)$.

We would like to express such a Hamiltonian in the Majorana fermion operators η_a , i.e. Γ_a in this representation. In general, a complete basis of 16×16 real matrices can be obtained by multiplying different Γ_a . The set of $\{1, \Gamma_a, \Gamma_a \Gamma_b, \Gamma_a \Gamma_b \Gamma_c, \dots, \Gamma_1 \Gamma_2 \dots \Gamma_8\}$ (with all $a, b, c \dots$ in each term different from each other) contains exactly

$$\sum_{n=0}^8 \binom{8}{n} = 2^8$$

independent real matrices. Because Γ^a are all off-diagonal, the multiplication of even Γ_a is non-vanishing in the diagonal 8×8 blocks and those of odd Γ_a is non-vanishing in the off-diagonal 8×8 blocks. In this set, the matrices with non-zero overlap with H_{proj} are those which are both symmetric and block-diagonal. Besides the identity 1, there are only two set matrices satisfying these two conditions: $\Gamma_a \Gamma_b \Gamma_c \Gamma_d$ and $F = \Gamma_1 \Gamma_2 \dots \Gamma_8$. In the representation (A.2) $F = -v_z$. Physically, F is the fermion number parity. Since the Hamiltonian (A.6) vanishes in the upper 8×8 block, we can focus on the basis of the lower 8×8 block obtained by the projection

$$\gamma_{abcd} = \frac{1}{2}(1 + F) \frac{1}{24} \epsilon^{abcdefgh} \Gamma_e \Gamma_f \Gamma_g \Gamma_h. \quad (\text{A.7})$$

The antisymmetric tensor $\epsilon^{abcdefgh}$ is introduced to extract only the components of $\Gamma_a \Gamma_b \Gamma_c \Gamma_d$ with a, b, c, d all different from each other. γ_{abcd} satisfies the orthogonality condition

$$\frac{1}{8} \text{Tr} [\gamma_{abcd} \gamma_{efgh}] = \frac{1+F}{2} \left[\frac{1}{24} \epsilon^{abcdijkl} \epsilon_{ijklfgh} + \epsilon_{abcdefgh} \right]. \quad (\text{A.8})$$

For example $\gamma_{1234} = \frac{1}{2}(1 + F) \Gamma_5 \Gamma_6 \Gamma_7 \Gamma_8 = \frac{1}{2}(\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 + \Gamma_5 \Gamma_6 \Gamma_7 \Gamma_8)$, which is orthogonal to other γ_{abcd} except $\gamma_{5678} = \gamma_{1234}$. The number of independent γ_{abcd} is therefore $\frac{8!}{2 \times 4!} = 35$ which is consistent with the number of symmetric traceless 8×8 matrices $35 = \frac{8 \times 9}{2} - 1$. Using the orthogonality condition one can expand the Hamiltonian to

$$H_{\text{proj}} = \frac{1+F}{16} \text{tr} \left[\frac{1+F}{2} H_{\text{proj}} \right] + \sum'_{abcd} \frac{1}{8} \text{tr} [\gamma_{abcd} H_{\text{proj}}] \gamma_{abcd} \quad (\text{A.9})$$

with the first term taking care of the trace part and the second term an expansion of the traceless part in γ_{abcd} . Here the sum \sum'_{abcd} stands for the sum over the 35 independent groups of $abcd$. For the particular Hamiltonian H_{proj} , we have

$$\text{tr} [\gamma_{abcd} H_{\text{proj}}] = - [\gamma_{abcd}]_{16,16} \quad (\text{A.10})$$

which is the last diagonal component of γ_{abcd} . In the representation given by equation (A.2), γ_{abcd} is also a direct product of Pauli matrices or identity. Consequently, the 16, 16 component is only non-zero if $\gamma_{abcd} = v_i \sigma_j \tau_k \mu_l$ with all $i, j, k, l = 0$ or z . Therefore, there are only $7 = 2^3 - 1$ terms with non-zero contributions. They can be explicitly found as

$$\begin{aligned} \gamma_{1234} &= -\frac{v_0 - v_z}{2} \sigma_z \tau_0 \mu_0, & \gamma_{1256} &= \frac{v_0 - v_z}{2} \sigma_z \tau_z \mu_0, & \gamma_{1278} &= -\frac{v_0 - v_z}{2} \sigma_0 \tau_z \mu_0, \\ \gamma_{1357} &= \frac{v_0 - v_z}{2} \sigma_z \tau_0 \mu_z, & \gamma_{1368} &= \frac{v_0 - v_z}{2} \sigma_0 \tau_0 \mu_z, & \gamma_{1458} &= -\frac{v_0 - v_z}{2} \sigma_0 \tau_z \mu_z, \\ \gamma_{1467} &= \frac{v_0 - v_z}{2} \sigma_z \tau_z \mu_z. \end{aligned} \quad (\text{A.11})$$

Therefore, we obtain

$$H_{\text{proj}} = -\frac{1+F}{16} - \frac{1}{8} [\gamma_{1234} + \gamma_{1256} + \gamma_{1278} + \gamma_{1357} - \gamma_{1368} - \gamma_{1458} - \gamma_{1467}]. \quad (\text{A.12})$$

γ_{abcd} can be written in the product of Γ_a , which in the operator form are η_a . For example $\gamma_{1234} = \frac{1}{2}(\eta_1 \eta_2 \eta_3 \eta_4 + \eta_5 \eta_6 \eta_7 \eta_8)$. Compare equation (A.12) with equation (8) one finds that except for a trivial constant term, they are identical if we only take the zero modes $\eta_{a,k=0\uparrow(\downarrow)}$ in equation (8). Consequently, we have demonstrated that the Hamiltonian (8) leads to a unique ground state in the Hilbert space of the eight Majorana zero modes.

Appendix B. More details on the gap induced by the mass term in equation (9)

In this section we will prove the essential property given in section 4:

- For an *arbitrary* quadratic H_{u0} in D' class, the Hamiltonian $H_u = H_{u0} + H_\Delta$ always has a gap $E_g \geq |\Delta|$.

To discuss the generic situation we write down the most generic lattice model form of H_{u0} . If the system is non-interacting, H_{u0} can be generically written in the Majorana basis as

$$H_{u0} = i \sum_{i,j} \eta_i A_{ij} \eta_j \quad (\text{B.1})$$

with i, j labeling all lattice sites and all internal band indices. The symmetry operations T and R act on the Majorana operators η_i as

$$T^{-1} \eta_i T = \mathcal{T}_{ij} \eta_j, \quad R^{-1} \eta_i R = \mathcal{R}_{ij} \eta_j. \quad (\text{B.2})$$

The matrices \mathcal{T} and \mathcal{R} satisfy

$$\mathcal{T}^2 = -1, \quad \mathcal{R}^2 = 1, \quad \{\mathcal{T}, \mathcal{R}\} = 0, \quad \{A, \mathcal{T}\} = 0, \quad [A, \mathcal{R}] = 0. \quad (\text{B.3})$$

The mass term H_Δ is defined by

$$H_\Delta = i\Delta \sum_i \eta_i (T^{-1} \eta_i T) = i\Delta \sum_i \eta_i \mathcal{T}_{ij} \eta_j. \quad (\text{B.4})$$

Therefore, the total Hamiltonian is

$$H_u = H_{u0} + H_\Delta = i \sum_{i,j} \eta_i (A_{ij} + \Delta \mathcal{T}_{ij}) \eta_j. \quad (\text{B.5})$$

Since $\{A, \mathcal{T}\} = 0$, the single-particle Hamiltonian $h_u = i(A + \Delta \mathcal{T})$ satisfies

$$h_u^2 = - (A^2 + \Delta^2 \mathcal{T}^2) = A^T A + \Delta^2. \quad (\text{B.6})$$

Therefore $h_u^2 - \Delta^2$ is semi-positive definite for any A , and thus all eigenvalues E_n of h_u satisfy $|E_n| \geq |\Delta|$. The derivation above proves that any quadratic Hamiltonian H_u which is a sum of an arbitrary T , R invariant Hamiltonian H_{u0} and the special mass term H_Δ is guaranteed to have a finite gap $|\Delta|$. Therefore H_u as a chain with open boundary can never have any topological edge states. In other words, we have proved that any H_u with the form of $H_u = H_{u0} + H_\Delta$ is topological trivial. The same proof applies to H_l . Therefore, we reach the conclusion that in non-interacting systems in \tilde{D} class with the boundary mass term H_Δ , there is no purely boundary contribution to the domain wall Majorana zero modes, and the number of Majorana zero mode is uniquely determined by the topological invariant N of the bulk.

References

- [1] von Klitzing K, Dorda G and Pepper M 1980 *Phys. Rev. Lett.* **45** 494
- [2] Tsui D C, Stormer H L and Gossard A C 1982 *Phys. Rev. Lett.* **48** 1559
- [3] Hasan M Z and Kane C L 2010 *Rev. Mod. Phys.* **82** 3045
- [4] Moore J E 2009 *Nature* **464** 194
- [5] Qi X-L and Zhang S-C 2011 *Rev. Mod. Phys.* **83** 1057–110
- [6] Qi X-L, Hughes T and Zhang S-C 2008 *Phys. Rev. B* **78** 195424
- [7] Schnyder A P, Ryu S, Furusaki A and Ludwig A W W 2008 *Phys. Rev. B* **78** 195125
- [8] Kitaev A 2009 *AIP Conf. Proc.* **1134** 22
- [9] Zirnbauer M R 1996 *J. Math. Phys.* **37** 4986
- [10] Altland A and Zirnbauer M R 1997 *Phys. Rev. B* **55** 1142
- [11] Bott R 1957 *Proc. Natl Acad. Sci. USA* **43** 933
- [12] Wang Z, Qi X-L and Zhang S-C 2010 *Phys. Rev. Lett.* **105** 256803
- [13] Wang Z, Qi X-L and Zhang S-C 2012 Topological invariants for interacting topological insulators with inversion symmetry arXiv:1201.6431
- [14] Gurarie V 2011 *Phys. Rev. B* **83** 085426
- [15] Essin A M and Gurarie V 2011 *Phys. Rev. B* **84** 125132
- [16] Fidkowski L and Kitaev A 2011 *Phys. Rev. B* **83** 075103
- [17] Turner A M, Pollmann F and Berg E 2011 *Phys. Rev. B* **83** 075102
- [18] Fidkowski L and Kitaev A 2010 *Phys. Rev. B* **81** 134509
- [19] Chen X, Gu Z-C, Liu Z-X and Wen X-G 2011 Symmetry protected topological orders and the group cohomology of their symmetry group arXiv:1106.4772
- [20] Read N and Green D 2000 *Phys. Rev. B* **61** 10267
- [21] Volovik G E 1999 *Pis'ma Zh. Eksp. Teor. Fiz.* **70** 601
Volovik G E 1999 *JETP Lett.* **70** 609 (Engl. transl.)
- [22] Xu C and Joel M 2006 *Phys. Rev. B* **73** 045322
Wu C, Bernevig A B and Zhang S-C 2006 *Phys. Rev. Lett.* **96** 106401
Zheng D, Wu C and Zhang G-M 2011 *Phys. Rev. B* **84** 205121
- [23] Wang Z, Qi X-L and Zhang S-C 2011 *Phys. Rev. B* **84** 014527
- [24] Jackiw R and Rebbi C 1976 *Phys. Rev. D* **13** 3398

- [25] Qi X-L, Hughes T and Zhang S-C 2008b *Nature Phys.* **4** 273
- [26] Chen X, Gu Z-C and Wen X-G 2011b *Phys. Rev. B* **83** 035107
- [27] Yao H and Ryu S 2012 Interaction effect on topological classification of superconductors in two dimensions arXiv:1202.5805
- [28] Ryu S and Zhang S-C 2012 arXiv:1202.4484
- [29] Gu Z-C and Levin M 2013 arXiv:1304.4569
- [30] Gu Z-C and Wen X-G 2012 Symmetry-protected topological orders for interacting fermions—fermionic topological nonlinear sigma-models and a group super-cohomology theory arXiv:1201.2648