Extremal Black Holes in $\mathcal{N} = 2$ Supergravity

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Extremal Black Holes in $\mathcal{N} = 2$ Supergravity

Extremale Zwarte Gaten in $\mathcal{N} = 2$ Supergravitatie

(met een samenvatting in het Nederlands)

Proefschrift

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Chapter 1

Introduction

In recent years, interest in the study of black holes has been renewed due to a number of fascinating advances. Nowadays, it is widely accepted that most, if not all, centers of galaxies contain black holes that are responsible for some of the most exciting phenomena observed in these regions. On the other hand, stars of mass beyond a certain bound are known to result in a black hole state after gravitational collapse at the end of their life cycle. This ends a long period of speculation on the existence of such objects that started shortly after the discovery of Newton's law of gravitation and peaked with the advent of General Relativity.

In fact, both theories predict that if any object of given mass is small enough, one would need to travel at a speed greater than that of light to escape the gravitational attraction at its surface. To an outside observer, this surface would appear completely dark. These predictions were considered a mathematical oddity, until Einstein's theory of relativity made it clear that the speed of light is the highest possible speed in Nature, implying that if light cannot escape a black hole then nothing can. Such a unique property is much more interesting and far-reaching, so that the study of black holes, as these objects were later named, became one of the most active fields in Physics throughout the last century.

There are several aspects of black holes that are important from a theoretical point of view, as they are not only ubiquitous in all theories of gravity but, according to the so-called uniqueness theorems, they are generally described by a small number of parameters. This means that if one measures from a distance the global properties of a black hole, such as the mass, angular momentum and total charge, the full structure of the object is known uniquely. General Relativity predicts that beyond the point from which light cannot escape, called the event horizon, any observer will invariably reach the centre, a point of infinite density known as the singularity.

The presence of a singularity signals a breakdown of classical Einstein gravity, bringing up the need for a theory that can describe gravity at small scales. In this realm, quantum phenomena are unavoidable, so that one needs to introduce a quantum description of gravity. The quest for this theory is ongoing, as there are several candidates with varying merits and drawbacks but still no fully satisfactory theory reconciling General Relativity with Quantum Mechanics. One of the requirements on any quantum theory of gravity is that it must be able to reproduce the large scale features of a black hole and smooth out any singularities through quantum phenomena. At the same time, a proposed quantum description must be robust enough to comply with the uniqueness theorems, so that any degrees of freedom beyond the total mass and conserved charges do not affect the region far from the singularity. Using the properties mentioned above, one can argue that a black hole is the simplest nontrivial interacting system in any theory of quantum gravity, as it should be completely fixed by the global parameters.

A hint towards a microscopic theory is given by the so-called four laws of black hole mechanics, discovered in the 1970's. It turns out that under some general assumptions, it is possible to derive constraints on the variations of conserved quantities parametrising a continuous variety of black holes. The resulting equations bear a striking resemblance to the conventional laws of thermodynamics if one assigns a temperature and an entropy to the black hole, as we discuss in due course. Thus, a microscopic picture of a black hole must be in terms of a statistical system involving a large number of degrees of freedom, analogous to the atoms for a gas. An appropriate test on a candidate for a quantum gravity theory is that the proposed microscopic degrees of freedom can give rise to the macroscopic entropy when the laws of statistical mechanics are applied.

String theory has been successful in this respect, and this thesis is devoted to a comparison of macroscopic properties of black holes with string theoretic predictions. In this introductory chapter we first discuss in more detail some of the concepts mentioned above. Later, we turn to an exposition of the basic ideas behind the construction of black holes in theories of gravity that approximate string theory, through simple examples.

1.1 Black holes, statistical physics and string theory

The defining feature of a black hole is the presence of a so-called horizon, which can be thought of as the surface of the black hole, in the sense that any particle that crosses it will be trapped inside forever. Newton's laws imply that a particle can escape the attraction of a spherically symmetric object of mass M at distance r only if its velocity satisfies

$$\dot{r}^2 \geq 2\,\frac{G\,M}{r}\,, \tag{1.1.1}$$

where G is Newton's constant and $\dot{r}$ is the radial velocity of the particle. It has been observed already in the 18th century [1], that since there is no bound on the value of the potential as r decreases, the velocity required to escape the gravitational pull is greater as one approaches the centre. For example, at a distance $r_0 = 2M/c^2$ away from the centre, where c is the speed of light, only particles moving at the speed of light or faster would be able to escape.

Surprisingly, the above result is unchanged in the context of General Relativity. A similar analysis involves the study of test particles in the geometry produced by the presence of a single object of mass M, known as the Schwarzschild metric. The set of points with distance $r_0 = 2M/c^2$ from the centre defines the horizon of a Schwarzschild black hole. There are far more precise and formal definitions, but this is sufficient for most of our purposes.

In fact, the total mass is sufficient to describe not only the horizon, but the complete Schwarzschild geometry. This is a general feature of all stable black hole solutions and is sometimes called the no hair theorem, in the sense that only total conserved quantities are relevant. Except the mass, other such quantities are the angular momentum, J, and electric or magnetic charges q, p. Black holes carrying electromagnetic charges are usually called Reissner-Nordström black holes [2, 3], whereas rotating black holes are referred to as Kerr black holes [4]. Finally, charged rotating black holes are known as Kerr-Newman [5]. One can consider adding further charges, for example associated to non-abelian gauge fields, but we will not deal with such cases.

For the complete solution to make sense, there are usually constraints on the above parameters. The main source of such restrictions is the requirement of a finite horizon that hides the singularity at the centre of the black hole, also known as cosmic censorship. For a Reissner-Nordström black hole with electric charge Q, one finds that the mass must satisfy the relation

$$M \geq |Q| \,, \tag{1.1.2}$$

in geometrised units[1]. It follows that, unlike for Schwarzschild black holes, there is a nonzero minimum in the mass of charged black holes. Similar bounds exist in the presence of angular momentum and/or other charges.

Black holes that saturate the bound (1.1.2), i.e. have equal mass and charge are called extremal and are the subject of this thesis. These solutions are interesting from a theoretical perspective, as it should be impossible to extract mass/energy from them by any physical process, as long as the singularity at the centre remains hidden. In other words, they represent a kind of ground state for all black holes, that is expected to hold even quantum mechanically.

1.1.1 Black hole thermodynamics

In order to make the concept of a ground state more precise, one has to go beyond the classical regime, in which black holes only absorb matter and do not decay. Based on

[1] These units are defined as $G = c = k_B = 4\pi\,\epsilon_0 = 1$, where k_B is Boltzmann's constant and ϵ_0 is the vacuum permittivity.

this intuition, all black holes represent classical ground states of a gravitational system irrespective of extremality. The crucial property that distinguishes extremal solutions is a quantum effect, the so-called Hawking radiation.

As was first shown in [6] in the context of a semiclassical approximation, black holes are not completely black when quantum effects are taken into account, but emit thermal radiation just as any other object in nature. The associated temperature is not an independent parameter, but is also fixed in terms of the conserved quantities, e.g. for a Schwarzschild black hole it reads

$$T = \frac{\hbar\,\kappa}{2\pi} = \frac{\hbar}{8\pi}\frac{1}{M}\,. \tag{1.1.3}$$

Here, κ is known as the surface gravity of the black hole and is equal to the acceleration due to gravity experienced by a test particle at the horizon as measured at infinity. It is not surprising to note that the temperature is proportional to Planck's constant $\hbar$, as it is a purely quantum effect. However, it is also inversely proportional to the mass, which implies that the smaller a black hole is, the higher temperature it will have, radiating more energy and reducing its mass further. The result would presumably be a complete evaporation, and one concludes that the Schwarzschild black hole is not stable quantum mechanically.

On the other hand, this cannot happen when there is a minimum mass for the system, which is the case of a charged black hole. The temperature now takes the form

$$T = \frac{\hbar}{4\pi}\frac{\sqrt{M^2-Q^2}}{M(M+\sqrt{M^2-Q^2})}\,, \tag{1.1.4}$$

which has a zero when the bound in (1.1.2) is saturated. It is in the sense of zero temperature that extremal black holes are ground states of more general black holes and we use this fact to define extremality as a zero temperature condition. For Reissner-Nordström black holes all definitions are equivalent, but this is not the case for more general examples.

After introducing a fundamental thermodynamic quantity like the temperature for a black hole, one is faced with the obvious question of a possible thermodynamical structure behind black holes. If this is the case, the first thing to look for is a quantity that corresponds to the thermodynamic conjugate of temperature, the entropy, so that the first law of thermodynamics

$$\mathrm{d}M = T\,\mathrm{d}\mathcal{S}\,, \tag{1.1.5}$$

is valid. Here, we used that the mass M is the total energy of the system. For a Schwarzschild black hole with temperature given by (1.1.3), the entropy would be

$$\mathcal{S} = \frac{4\pi}{\hbar}M^2 = \frac{A}{4\,\hbar}\,, \tag{1.1.6}$$

where A is the area of the horizon at $r_0 = 2M$. This is known as the Bekenstein-Hawking entropy. It turns out that the entropy associated to all black holes in Einstein gravity (including charged and rotating black holes) is given by the area law, with the same normalisation as in (1.1.6). This result is inversely proportional to Planck's constant, making black holes the most efficient containers of entropy known. Even more surprisingly, the relation (1.1.5) generalises straightforwardly when more conserved charges are present, in the standard thermodynamical way

$$\mathrm{d}M = T\,\mathrm{d}\mathcal{S} + \Omega\,\mathrm{d}J + \Phi\mathrm{d}Q + \dots\,, \tag{1.1.7}$$

where Ω, Φ are the so-called angular velocity and the electrostatic potential respectively. The dots represent extra terms corresponding to possible further charges. For extremal black holes, the temperature is zero and the last relation is simply a statement about the fixed relation of the mass to the other conserved quantities.

The analogy can be extended to the other laws of thermodynamics [7–9]. The zeroth law is reflected on the constancy of the surface gravity over the horizon of all known black holes (cf. (1.1.4)), whereas the second law of black hole mechanics states that any classical process can never decrease the area of a black hole, and thus its associated entropy. At the semiclassi cal level, the sum of the entropy associated to the horizon and any emitted Hawking radiation is similarly constrained.

This surprisingly successful analogy with thermodynamics begs for an explanation in terms of an underlying statistical system. Consequently, the construction of a microscopic model that can reproduce the above relations has been a longstanding goal. Quite generally, any attempt to produce a microscopic description of something that could be interpreted as a black hole must include two basic ingredients: a structure that appears as a pointlike source[2] and the existence of appropriate internal degrees of freedom to account for the large entropy.

1.1.2 String theory and supergravity

String theory is a candidate for a quantum theory of gravity, and has provided us with theoretical models of the quantum structure of a class of extremal black holes. According to string theory, all matter can be described through different oscillating modes of spatially extended objects, called strings, that can only exist in a ten dimensional spacetime. To make contact with the four dimensions we observe, one has to assume that the extra six dimensions are curled up in very small sizes, an idea known as compactification.

In the last two decades, research has revealed that in order to make sense of the non-perturbative aspects of string theory, other extended objects must be included, the most important being the so-called D-branes. A string theoretic black hole involves a

[2] At least in the observed four dimensions.

collection of D-branes wrapped in the unseen compactified dimensions. Thus, from a four-dimensional point of view, they appear as heavy pointlike objects carrying charge and angular momentum, i.e. black hole-like objects. The entropy is then given by a standard statistical mechanical counting of the possible (microscopic) configurations of D-branes that result in the same (macroscopic) spacetime. Intuitively one can envisage this as counting the directions along which the D-branes can freely move in the compact space and treat them just like particles in a box. This was made precise for the first time in [10], followed by a large number of investigations of various aspects of black holes in string theory.

The main goal of the thesis at hand is to study various aspects of black holes in this context. As will be discussed in the following, one can make detailed comparisons between the D-brane (microscopic) and spacetime (macroscopic) pictures of a black hole within string theory. For example, the entropy of a given black hole as computed by the microstate counting of D-branes can be directly contrasted with the Bekenstein-Hawking entropy as computed from the area law (1.1.6). This brings up the issue of the gravity theories used to approximate string theory, which is especially relevant here in view of the fact that most of the following will be focused on macroscopic aspects of black holes. At low energies and large distances compared to their length, strings can be viewed as point particles, so that string theory can be described by a class of effective field theories of gravity coupled to scalars and gauge fields.

These theories are supersymmetric, like string theory, and are usually called supergravities. Supersymmetric field theories are an interesting subject on their own, as they place bosons and fermions in the same multiplet and allow for rotations between them. It follows that the parameters of supersymmetry transformations must themselves carry spin and are usually chosen in the smallest spinor representation available. The number of such independent parameters, $\mathcal{N}$, is used to characterise the amount of supersymmetry in a theory.

Demanding invariance of an action under supersymmetry implies strong restrictions on the possible interactions, which turn out to be expressed through a small number of functions. Although these couplings should be ultimately derived from a fundamental theory, it is possible to study the structure of supergravity theories independently, deferring the connection to a possible string theoretic origin until the end. This is the approach we will follow for black hole solutions in four- and five-dimensional supergravity in the following chapters. The results can be compared to the microscopic picture, with appropriate identifications.

1.2 Gauge equivalence and black holes

Throughout our study of extremal black hole solutions in supergravity we make use of a number of useful tools that require some introductory discussion. In this section we illustrate some of these techniques through a simple example, namely Einstein-Maxwell theory in four dimensions.

We initially present the theory, giving a qualitative discussion of its conformal formulation, which elucidates the overall structure of the theory and simplifies calculations. Subsequently, we turn to the corresponding equations of motion and consider the definition of conserved charges. In this context, the Noether procedure provides a versatile algorithm to construct conserved quantities associated to any kind of symmetry.

In later sections, we consider the addition of a scalar field to the theory, and display some of the known extremal black hole solutions. Finally, we discuss generic features of these solutions, emphasising the symmetry enhancement of the near-horizon region in the context of our explicit example.

1.2.1 Gauge equivalence

A central concept in this thesis is that of gauge equivalence between Einstein gravity and conformal gravity, extended by supersymmetry. A simple example of such an equivalence is given by a massive vector field, V_μ, described by the so-called Proca Lagrangian

$$\mathcal{L}_P = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}\,m^2 V^\mu V_\mu\,. \tag{1.2.1}$$

Here, $F_{\mu\nu} = 2\partial_{[\mu}V_{\nu]}$ and m denotes the mass of the vector field, which has four degrees of freedom. This is in contrast to a massless vector, which describes only three, due to the gauge invariance of (1.2.1) when the mass is set to zero. As is well known, this symmetry acts on a vector field, A_μ, as $\delta A_\mu = \partial_\mu\xi$, where ξ is an arbitrary function.

It is possible to reinstate this gauge invariance, by explicitly decomposing the massive vector into a massless one and a charged scalar, $\mathrm{e}^{-\mathrm{i}\phi}$, that carries the extra degree of freedom, as

$$V_\mu = A_\mu - \mathrm{i}\,m^{-1}\mathrm{e}^{\mathrm{i}\phi}\partial_\mu \mathrm{e}^{-\mathrm{i}\phi}\,. \tag{1.2.2}$$

The ambiguity in this definition is identical to the standard gauge symmetry variations for a photon and a charged scalar

$$\delta A_\mu = \partial_\mu\xi\,, \quad \delta\phi = m\,\xi\,. \tag{1.2.3}$$

The gauge covariant derivative of $\mathrm{e}^{-\mathrm{i}\phi}$ takes the form

$$D_\mu \mathrm{e}^{-\mathrm{i}\phi} = \partial_\mu \mathrm{e}^{-\mathrm{i}\phi} + \mathrm{i}\,m\,A_\mu\,\mathrm{e}^{-\mathrm{i}\phi}\,, \tag{1.2.4}$$

so that $V_\mu = -\mathrm{i}\, m^{-1} \mathrm{e}^{\mathrm{i}\phi} D_\mu \mathrm{e}^{-\mathrm{i}\phi}$ and the so-called Stueckelberg Lagrangian (1.2.1) can be rewritten in terms of A_μ and ϕ as

$$\mathcal{L}_P = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} - \tfrac{1}{2}\, D^\mu \mathrm{e}^{\mathrm{i}\phi}\, D_\mu \mathrm{e}^{-\mathrm{i}\phi}\,. \tag{1.2.5}$$

This Lagrangian, as the original one in (1.2.1), describes four physical degrees of freedom for any nonzero value of m, three for the gauge field and one for the scalar. The important difference is that in the second formulation one is dealing with two independent fields, A_μ and ϕ, i.e. we have reduced the massive vector field to a gauge vector and a scalar locally rather than solving for the longitudinal mode. Of course, the two theories are equivalent, as one can set the scalar to zero using (1.2.3), in which case the kinetic term of the scalar reduces to the mass term in (1.2.1).

This simple example exhibits the general idea that it is possible to reformulate a Lagrangian so that it is invariant under a larger set of symmetries, while decomposing the original independent fields. In a supersymmetric setting, the same procedure can be applied to reduce multiplets consisting of several bosons and fermions to simpler ones. We will make extensive use of this in the context of gravity, introducing its conformally invariant version.

Consider the following action, describing a scalar σ and an abelian gauge field A_μ coupled to Einstein gravity in four dimensions

$$S = \int d^4x\, \mathcal{L}_0 = -\frac{1}{16\pi} \int d^4x \sqrt{-g}\, \Big(-\partial^\mu \sigma\, \partial_\mu \sigma + \tfrac{1}{6}\, \sigma^2 \mathcal{R} + \tfrac{1}{2} F^{\mu\nu} F_{\mu\nu} \Big)\,, \tag{1.2.6}$$

which will be part of the supergravity theories that will appear in later applications. Here, $\mathcal{R}$ denotes the Ricci scalar defined by

$$\mathcal{R} = g^{\nu\sigma} \mathcal{R}_{\mu\nu}{}^{\mu}{}_{\sigma}\,, \qquad \mathcal{R}_{\mu\nu}{}^{\rho}{}_{\sigma} = 2\, \partial_{[\mu} \Gamma^{\rho}_{\nu]\sigma} - 2\, \Gamma^{\rho}_{\tau[\mu}\, \Gamma^{\tau}_{\nu]\sigma}\,, \tag{1.2.7}$$

$$\Gamma^{\rho}_{\mu\nu} = -\frac{1}{2} g^{\rho\sigma} \left(2\, \partial_{(\mu} g_{\nu)\sigma} - \partial_\sigma g_{\mu\nu} \right)\,, \tag{1.2.8}$$

and $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$ is the field strength of the gauge potential. From the point of view of Einstein gravity, this action describes a theory where the Newton constant is replaced by a dynamical scalar field, as in

$$G^{-1} = \tfrac{1}{6}\, \sigma^2\,. \tag{1.2.9}$$

The action is invariant under standard general coordinate and gauge transformations independently

$$\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu\,, \tag{1.2.10}$$

$$\delta A_\mu = \xi^\nu F_{\mu\nu} - \partial_\mu (\xi^\nu A_\nu) + \partial_\mu \xi\,, \tag{1.2.11}$$

where ξ^μ is the vector parameter of coordinate transformations and ξ is the parameter of gauge transformations. The covariant derivative ∇_μ contains the Christoffel symbol, as usual

$$\nabla_\mu \xi_\nu = \partial_\mu \xi_\nu + \Gamma^\rho_{\mu\nu} \xi_\rho \,, \tag{1.2.12}$$

and similarly for higher order tensors.

However, closer inspection shows that (1.2.6) is invariant under more local symmetries. The crucial one is the so-called scale invariance expressed through

$$\delta g_{\mu\nu} = \Lambda_D^{-2} g_{\mu\nu} \,, \quad \delta\sigma = \Lambda_D \sigma \,, \tag{1.2.13}$$

where the function Λ_D is the associated parameter. Note that the gauge field is inert under these transformation, consistent with the scale invariance of Maxwell theory at the classical level. On the other hand, there is a subtle interplay between the other two terms in (1.2.6). As one can verify, the Ricci scalar transforms as

$$\delta\mathcal{R} = \Lambda_D^2 \, \mathcal{R} + 3\, \partial^2 \Lambda_D \,, \tag{1.2.14}$$

so that it can be interpreted as a gauge connection associated to the transformation (1.2.13). Analogously to the mass term for the vector field in (1.2.5), the Ricci scalar appears as a covariantisation. Note that local scale symmetry only exists if the scalar kinetic term in has the sign appearing in (1.2.6), which is the opposite to that of a physical scalar. The theory is consistent only because the scale symmetry above implies that σ is not a gauge invariant degree of freedom.

In fact one can obtain an equivalent theory where the scalar does not appear, by gauge fixing the symmetry (1.2.6). To see this, note that the local scale invariance can be used to arrange that the transformed scalar $\sigma' = \Lambda_D \, \sigma$ is any nonzero constant, so that $\partial_\mu \sigma' = 0$. This transformation affects the metric as well, through (1.2.13). Since the first two terms in (1.2.6) are covariant under this transformation, they retain the same form. All in all, one can consistently drop the primes and choose the gauge $\sigma^2 = 6$ in (1.2.6), so that the Newton constant G in (1.2.9) is set to unity (in geometrised units) and the gauge-fixed action is simply

$$S = \int d^4x \, \mathcal{L}_0 = -\frac{1}{16\pi} \int d^4x \, \sqrt{-g} \left(\mathcal{R} + \tfrac{1}{2} F^{\mu\nu} F_{\mu\nu} \right) . \tag{1.2.15}$$

This is the standard Einstein-Maxwell theory in four dimensions, still invariant under (1.2.10)-(1.2.11), but not under scale transformations. Similar to the relation between (1.2.3) and (1.2.5), there is an extra symmetry and one more scalar. As in that example, one of the physical degrees of freedom contained in (1.2.15), namely the conformal scale factor of the metric, is described by an independent field, σ, in the conformally invariant Lagrangian (1.2.6).

Since the two theories are equivalent, one can freely choose which to use in any given application. Arguably, the conformal theory is more complicated as it contains more matter fields and symmetries. However, the decomposition in terms of conformal fields is instrumental in the context of supersymmetric theories, as the systematic construction of supersymmetric Lagrangians usually leads to such conformal theories.

1.2.2 Conserved charges

As is well known, it is generally possible to define global conserved charges corresponding to different symmetries of a theory, such as the total mass-energy, angular momentum and electric charge. One way to construct these quantities in an algorithmic way is through the so-called Noether potential associated to each of the symmetries of the action. In Appendix C.1 we provide a general discussion for various types of Lagrangians that are used throughout this thesis. Here, we exhibit the salient features of this construction in the context of the simple action (1.2.6)-(1.2.15).

Consider the effect of the variations in (1.2.10)-(1.2.11) on the theory in (1.2.6). Although the Lagrangian is strictly invariant under gauge transformations, it transforms as in

$$\delta_\xi \mathcal{L}_0 = \partial_\mu \left(\xi^\mu \mathcal{L}_0\right) , \tag{1.2.16}$$

under diffeomorphisms. On the other hand, one can also use Hamilton's principle of least action to obtain the equations of motion, again up to a total derivative

$$\delta \mathcal{L}_0 = E_\sigma \, \delta\sigma + E_{e}{}^{\mu}_{a} \, \delta e_\mu^a + E_A^\mu \, \delta A_\mu + \partial_\mu \theta^\mu(\delta\phi) \, , \tag{1.2.17}$$

where E_σ, $E_{e}{}^{\mu}_{a}$ and E_A^μ denote the equations of motion for the scalar, the metric and the gauge field respectively. The total derivative for the example in (1.2.6) is

$$\begin{aligned} 8\pi \, \theta^\mu(\delta\phi) =& \tfrac{1}{12}\sqrt{-g}\, g^{\mu[\rho} g^{\sigma]\nu} \Big(\mathcal{D}_\rho(\sigma^2)\delta g_{\sigma\nu} - \sigma^2 \mathcal{D}_\rho \delta g_{\sigma\nu}\Big) \\ & - \sqrt{-g}\, D^\mu\sigma \, \delta\sigma - \sqrt{-g}\, F^{\mu\nu}\delta A_\nu \, . \end{aligned} \tag{1.2.18}$$

Now, the crucial observation is that when the general variation in (1.2.17) is assumed to be a symmetry variation, the result must equal the one in (1.2.16). Let us consider gauge variations first, under which the Lagrangian is invariant. In this case, (1.2.17) implies that

$$\delta_\xi \mathcal{L}_0 = E_A^\mu \partial_\mu \xi - \tfrac{1}{8\pi}\partial_\mu \left(\sqrt{-g}\, F^{\mu\nu}\delta A_\mu\right) = 0 \, , \tag{1.2.19}$$

where we took into account that the only field subject to gauge transformations is the gauge potential. It then follows that one can define the so-called Noether current,

$$J_e^\mu = \tfrac{1}{8\pi}\sqrt{-g}\,F^{\mu\nu}\delta A_\mu = \tfrac{1}{8\pi}\sqrt{-g}\,F^{\mu\nu}\partial_\mu\xi\,, \qquad (1.2.20)$$

which is conserved when evaluated on any solution of the equations of motion, by virtue of (1.2.19). The usual procedure at this point would be to integrate the timelike component of the conserved current over space to obtain the associated charge. However, the equations of motion for the gauge field imply that the Noether current is itself a total derivative

$$\partial_\mu\left(\sqrt{-g}\,F^{\mu\nu}\right) = 0\,, \quad \Rightarrow \quad J_e^\mu = \tfrac{1}{8\pi}\partial_\mu\left(\sqrt{-g}\,F^{\mu\nu}\,\xi\right) \equiv \partial_\mu Q_e^{\mu\nu}\,, \qquad (1.2.21)$$

where we defined the Noether potential Q. This observation is in line with the general lore that the charge calculated from a conserved current associated to any local symmetry vanishes identically. Despite this, the existence of the Noether potential allows for a definition of the total electric charge of a solution by

$$q_e = \int_S Q_e^{\mu\nu}dS_{\mu\nu} = \tfrac{1}{8\pi}\int_S \sqrt{-g}\,F^{\mu\nu}\,\xi\,dS_{\mu\nu}\,. \qquad (1.2.22)$$

At this stage ξ is still arbitrary and must be chosen such that the gauge field is invariant under (1.2.11). Only then does the current (1.2.20) vanish and the charge (1.2.22) is conserved for any surface S (see the discussion in C.1). In the case at hand, this implies that ξ is a constant, so that the last equation is identical to the standard definition of electric charge in Maxwell theory.

The algorithm above can be applied to the more interesting example of diffeomorphism invariance. In this case, specialising the variation in (1.2.17) to diffeomorphisms and comparing to (1.2.16) leads to the conclusion that

$$\begin{aligned} 8\,\pi\,J^\mu = & -\tfrac{1}{12}\sqrt{-g}\sigma^2 g^{\mu[\rho}g^{\sigma]\nu}\left[\nabla_\nu\nabla_\rho\xi_\sigma + \mathcal{R}_{\lambda\nu\rho\sigma}\xi^\lambda\right] + \tfrac{1}{3}\,g^{\mu[\rho}g^{\sigma]\nu}\nabla_\rho\sigma^2\nabla_{(\nu}\xi_{\sigma)} \\ & - \sqrt{-g}\,D^\mu\sigma\,\xi^\nu\partial_\nu\sigma - \sqrt{-g}\,F^{\mu\nu}\left[\xi^\rho F_{\nu\rho} - \partial_\nu(\xi^\rho A_\rho)\right] - \xi^\mu\mathcal{L}_0\,, \end{aligned} \qquad (1.2.23)$$

is conserved for any solution of the equations of motion. As before, one can show that this current is again a total derivative for any solution of the equations of motion. The exact expression for the Noether potential reads

$$8\,\pi\,Q^{\mu\nu} = -\tfrac{1}{12}\sqrt{-g}\sigma^2\nabla^{[\mu}\xi^{\nu]} + \tfrac{1}{3}\,\nabla^{[\mu}\sigma^2\xi^{\nu]} - \sqrt{-g}\,\xi^\rho A_\rho\,F^{\mu\nu}\,. \qquad (1.2.24)$$

Once more, the parameter ξ^μ must be chosen appropriately. A vector for which the metric is invariant under (1.2.10), is known as a Killing vector. If we further demand that the gauge field is also invariant under this diffeomorphism according to (1.2.11), the integral of the Noether potential over a sphere can be used to define the conserved charge associated to the symmetry described by the Killing vector.

An important subtlety though is that unlike for gauge symmetry, there can be multiple Killing vectors, for example along the time or angular directions, leading to one or several conserved charges, such as mass and angular momentum. As explained in more detail in section C.2.2, in such a case the equality of the integral of Q over a sphere at infinity and the horizon leads to a nontrivial relation between the asymptotic mass, angular momentum and other global charges and the quantity

$$\mathcal{S} = 2\pi \int_{hor} Q_{\mu\nu}\, dS^{\mu\nu}\,. \tag{1.2.25}$$

In this integral, the relevant Killing vector is the one generating the null horizon (in the sense that it is both normal and tangent to it), rescaled so that $\nabla_\mu \xi_\nu = \epsilon_{\mu\nu}$, $\xi^\mu = 0$ and $\epsilon_{\mu\nu}$ is the binormal of the horizon.

For a generic horizon, in the example given by (1.2.24) this quantity takes the simple form

$$\mathcal{S} = \frac{1}{24}\sigma_h^2\, A = \frac{1}{4\,G} A\,, \tag{1.2.26}$$

where A is the area of the horizon and we used the definition (1.2.9). We further assumed the scalar to take a constant value σ_h at the horizon. The last equation is exactly the Bekenstein-Hawking area law in (1.1.6). However, the Wald entropy is more general, as it applies to any Lagrangian density (defined to transform as in (1.2.16)), even if it contains higher derivative terms. In C.2, we provide generic formulae for such Lagrangians and discuss ways to define the Wald entropy even for some noncovariant Lagrangians, which are of central importance in the following.

1.2.3 Extremal black holes and stabilisation equations

Given the above considerations, we now turn to the central subject of this thesis, namely black hole solutions in theories containing scalars. The generic features of extremal black holes can be exhibited in the context of the simple example in (1.2.6), to which we can add a single physical scalar through the replacement of σ^2 by the product $\sigma_-\sigma_+$. The Lagrangian is given by

$$16\pi\, S = -\int d^4x\, \sqrt{-g}\, \left(\tfrac{1}{6}\sigma_-\sigma_+\mathcal{R} - \partial_\mu\sigma_-\partial^\mu\sigma_+ + \tfrac{1}{2}\left(\tfrac{\sigma_+}{\sigma_-}\right)^\beta F^{\mu\nu}F_{\mu\nu} \right)\,, \tag{1.2.27}$$

which can be identified with (1.2.15) upon setting $\sigma_+ = \sigma_-$. Moreover, we have modified the gauge kinetic term by a coupling to the scalars, with coupling constant β. Note that since this coupling is expressed in terms of the ratio of two scalars transforming as in (1.2.13), it is scale invariant. The Poincaré version of this action is obtained by the gauge choice

$$\sigma_-\sigma_+ = 6\,, \qquad \sigma_\pm = \sqrt{6}\,\mathrm{e}^{\pm\phi/\sqrt{6}}\,, \tag{1.2.28}$$

where the second relation is a convenient parametrisation in terms of a physical scale invariant scalar ϕ. The resulting gauge fixed action is

$$16\pi\,S = -\int d^4x\,\sqrt{-g}\,\Big(\mathcal{R} + \partial_\mu\phi\,\partial^\mu\phi + \tfrac{1}{2}\,\mathrm{e}^{\sqrt{2}\beta\phi/\sqrt{3}}F^{\mu\nu}F_{\mu\nu}\Big)\,. \tag{1.2.29}$$

This model, known as the Einstein-Maxwell-dilaton model, has a long history [11–13] and several black hole solutions are known. The simplest one is for $\beta = 0$, in which case the scalar decouples from the gauge field and (1.2.27) reduces to Einstein-Maxwell theory. Another interesting value is $\beta = 3\sqrt{2}$, for which this theory can be viewed as the result of reducing five-dimensional Einstein gravity on a circle, the so-called Kaluza-Klein reduction. In all cases, the physical scalar ϕ is massless without a potential and is often called a modulus.

In this thesis, we will be interested in extremal black holes, which are restricted objects in four dimensions. Here we concentrate on static spherically symmetric solutions, for simplicity. Extremality implies that the spatial part of the metric is proportional to the flat three dimensional metric. Under these restrictions, there is only one degree of freedom describing the metric, which we parametrise through a function $U(r)$ as

$$\mathrm{d}s^2 = -\mathrm{e}^{2U}\mathrm{d}t^2 + \mathrm{e}^{-2U}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,, \tag{1.2.30}$$

where $i, j \dots = 1, 2, 3$. Imposing that the metric asymptotes to the Minkowski metric as $r \to \infty$, one finds that the boundary condition for e^U near infinity is

$$\mathrm{e}^{-2U} \simeq 1 + \frac{2\,M}{r} + \mathcal{O}(\frac{1}{r^2})\,. \tag{1.2.31}$$

The constant M is then identified with the ADM mass of the solution. All methods to construct solutions are based on the idea of viewing e^U on the same footing as the scalars and solve for all at the same time. Indeed, a generic scalar σ comes with a boundary condition similar to (1.2.31)

$$\sigma \simeq \sigma_0 + \frac{2\,\Sigma_\sigma}{r} + \mathcal{O}(\frac{1}{r^2})\,. \tag{1.2.32}$$

Here, the asymptotic value of the modulus, σ_0, and Σ_σ (also known as the scalar charge), are the two integration constants of the scalar equations of motion. Now, the crucial observation for extremal black hole solutions is that Σ_σ for any scalar is fixed in terms of the electric and magnetic charges and the asymptotic moduli. This follows from the requirements of having finite values for the scalars on the horizon and smoothness of the horizon itself. Generically, only one combination of σ_0 and Σ_σ is compatible with these assumptions and we conventionally choose σ_0 as independent.

The gauge field is similarly restricted for a static metric of the type in (1.2.30). In fact, one can solve for the field strength independently of the rest of the Lagrangian, since it can be shown that the Maxwell equations are equivalent to

$$\epsilon^{ijk}\partial_i F_{jk} = 0\,, \qquad \epsilon^{ijk}\partial_i G_{jk} = 0\,, \tag{1.2.33}$$

where $G_{\mu\nu}$ is the electric displacement tensor, given by

$$G_{\mu\nu} \equiv \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\,\frac{\partial\mathcal{L}}{\partial F_{\rho\sigma}} = \tfrac{1}{2}\sqrt{-g}\,\left(\tfrac{\sigma_+}{\sigma_-}\right)^{\beta}\varepsilon_{\mu\nu\rho\sigma}\,F^{\rho\sigma}\,. \tag{1.2.34}$$

The solution is

$$F_{jk} = \varepsilon_{ijk}\partial_k H^p\,, \qquad G_{jk} = \varepsilon_{ijk}\partial_k H_q\,, \tag{1.2.35}$$

where H^p, H_q are harmonic functions in three-dimensional flat space

$$\nabla^2 H^p = \nabla^2 H_q = 0\,. \tag{1.2.36}$$

Spherical symmetry only allows for point charges at the centre and both harmonic functions must only depend on the distance from that centre, which leaves us with

$$H^p = c_p + \frac{p}{r}\,, \qquad H_q = c_q + \frac{q}{r}\,. \tag{1.2.37}$$

Here, q, p are the electric and magnetic charges and $c_{p,q}$ are integration constants. By inverting (1.2.34) one can obtain F_{ti}, G_{ti}, which will be given in terms of H^p, H_q, the scalars and the metric.

Given this, the scalars, e^U, $\sigma_\pm$ can be found by solving using the Einstein and scalar equations of motion. Equivalently, one can consider the effective one-dimensional Lagrangian that results once the obtained by imposing the above results on (1.2.27)

$$S = \int dr\, r^2\Big(\tfrac{1}{6}\sigma_-\sigma_+\mathrm{e}^{-2U}\,\left(\partial_r \mathrm{e}^U\right)^2 - \partial_r\sigma_-\partial_r\sigma_+ + V_{\mathrm{bh}}\Big)\,, \tag{1.2.38}$$

where the so-called black hole potential reads

$$V_{\mathrm{bh}} = \mathrm{e}^{2U}\,\left(\left(\tfrac{\sigma_-}{\sigma_+}\right)^{\beta}q^2 + \left(\tfrac{\sigma_+}{\sigma_-}\right)^{\beta}p^2\right)\,. \tag{1.2.39}$$

We have therefore reduced the problem of finding extremal black hole solutions to (1.2.27) to the mechanical system in (1.2.38), with potential as in (1.2.39). The concept of a black hole potential is central in all investigations of non-supersymmetric stationary black hole solutions to theories that contain scalars [12]. In later chapters we will encounter other examples in supergravity, with similar structure. Note that the effective action is consistent with scale transformations, so that only e^U and the ratio σ_+/σ_-, describing the physical scalar are relevant.

Consider first $\beta = 0$, so that the scalars are decoupled from the gauge field. In this case, there are solutions for which $\sigma_\pm$ are constant throughout spacetime. The only unknown is e^U in the metric (1.2.30), which can be found from the Einstein equation and reads

$$\mathrm{e}^{-2U} = (H^p)^2 + H_q^2 \,. \tag{1.2.40}$$

This solution is the extremal limit of the familiar Reissner-Nordstöm black hole, carrying an electric and a magnetic charge. The horizon lies at $r = 0$, which is not a point but a sphere, as one can show using (1.2.30). It is straightforward to construct more general solutions, with non-constant scalar. In fact, the assumption of constant scalars was not used when solving for the gauge field, therefore one only needs to solve the Einstein equation for e^U, by the same method.

Rather than showing such a solution, we turn to the case $\beta = 3\sqrt{2}$, which is closer to the supersymmetric models[3] but still solvable analytically. The solution [14–16] can be written in the conformal formulation as

$$\mathrm{e}^{-2U}\begin{pmatrix}\sigma_-^2\\ \sigma_+^2\end{pmatrix} = \begin{pmatrix}\tilde{H}_q H_q\\ \tilde{H}^p H^p\end{pmatrix} \,, \tag{1.2.41}$$

where $\tilde{H}^p$, $\tilde{H}_q$ are harmonic functions as in (1.2.37) with the same poles but different constant parts, fixed in terms of c_p, c_q. We stress that at this stage all fields still transform under conformal scalings. As one can verify using the general transformation law (1.2.13) for the metric in (1.2.30), e^U scales with the same weight as $\sigma_\pm$, so that (1.2.41) is conformally invariant, as it should. Equivalently, one can consider the gauge fixed action. Using (1.2.28) one can solve for the physical fields as

$$\mathrm{e}^{-4U} = 36\, \tilde{H}_q H_q \tilde{H}^p H^p \,, \quad \mathrm{e}^{\sqrt{\frac{2}{3}}\beta\phi} = \frac{\tilde{H}^p H^p}{\tilde{H}_q H_q} \,. \tag{1.2.42}$$

In either case, it is not possible to solve for $\sigma_\pm$ individually, since they are not physical degrees of freedom. The solution is fully described by the two charges q, p and the two arbitrary constants in (1.2.37), which control the asymptotic values of e^U and the physical scalar.

Using this relatively simple example, it is possible to highlight some of the generic features of all extremal black hole solutions in theories with scalar-dependent kinetic terms for the gauge fields, as in (1.2.27). This includes the extended supergravities considered later in this thesis. As already mentioned, it is generally possible to solve for the gauge fields in terms of harmonic functions that define the electric and magnetic charges of the solution. Subsequently, one solves for the scalars, including the fundamental ones (such as $\sigma_\pm$ in our example), the scale factor e^U coming from the metric and possibly others resulting from the reduction of additional tensor fields in a given theory.

[3]In fact one can show that (1.2.27) for this value of β can be always embedded in four-dimensional $\mathcal{N} = 2$ supergravity.

The solution for the scalars is thus expressed in terms of a number of harmonic functions, related to the ones describing the gauge fields, as in the example (1.2.41). Such a set of equations are known as stabilisation equations. They appear in a large class of theories in which the numbers of gauge fields and physical scalars are balanced, so that the scalars can be uniquely fixed in terms of harmonic functions. In supersymmetric theories, as the ones we will be interested in, this is automatically true, among other useful properties, since all vectors are components of multiplets that also contain scalars. Much of this thesis deals with the derivation of stabilisation equations in various cases.

1.2.4 Black hole attractors

There is a further important aspect of extremal black holes, connected to the stabilisation equations. Consider the region near the horizon of the black hole described by (1.2.41), which we obtain by taking the limit $r \to 0$. All constant terms in the harmonic functions are suppressed and the result reads

$$r^2 \mathrm{e}^{-2U} \begin{pmatrix} \sigma_-^2 \\ \sigma_+^2 \end{pmatrix} = \begin{pmatrix} q^2 \\ p^2 \end{pmatrix} , \quad \Rightarrow \quad \mathrm{e}^{-2U} = 6\, p\, q\, r^{-2} , \quad \mathrm{e}^{\sqrt{\frac{2}{3}}\beta\phi} = p/q \, . \tag{1.2.43}$$

Observe that the radial dependence affects only the metric through e^U, whereas the scalar takes a fixed value that depends only on the charges.

This is a general feature of all extremal solutions that stems from two crucial properties. First, it is possible to show that the near-horizon region of an extremal black hole is itself a well defined solution to the equations of motion. This is by no way obvious in general, and it is not generally known whether non-extremal black holes share this feature. The second property is that the near-horizon region must be invariant under a $SO(1,2)$ isometry group [17, 18]. In other words, for a four dimensional static black hole with full metric like in (1.2.30), the metric near the horizon is not only spherically symmetric, but the time and radial parts must also come in a particular combination. The result is the so-called Bertotti-Robinson spacetime

$$\mathrm{d}s^2 = \left[-\frac{r^2}{v_1^2}\, \mathrm{d}t^2 + \frac{v_1^2}{r^2} \mathrm{d}r^2 \right] + v_2^2\, \left[\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2 \right] , \tag{1.2.44}$$

where v_1, v_2 are constants. For the example above, one can easily see that using the solution implied by (1.2.43), the general metric (1.2.30) reduces to the last equation with $v_1^2 = v_2^2 = p\, q$.

The terms in the first bracket in (1.2.44) constitute the metric of AdS_2, a maximally symmetric space similar to a two-sphere, S^2, in several ways. Just as the sphere is a space of constant positive curvature, AdS_2 has constant negative curvature. Both metrics are invariant under three Killing vectors, which for an S^2 combine into the familiar $SO(3)$ invariance, whereas for AdS_2 they reflect the $SO(1,2)$ invariance mentioned above.

The requirement of extremality thus puts stringent constraints on the geometry near the horizon, which must also be reflected in all fields. Starting from the gauge fields, there are only two components allowed for a field strength tensor, if it is assumed to be invariant under all symmetries of the spacetime

$$F_{rt} = e\,, \qquad F_{\theta\phi} = \frac{p}{4\pi}\sin\theta\,, \tag{1.2.45}$$

where p is again the magnetic charge and e is a constant. The second relation follows from spherical symmetry and is valid in the full solution due to the Bianchi identity (see (1.2.35)-(1.2.37)), whereas the first holds only in the near-horizon region and follows from the symmetry of AdS_2 in a similar fashion.

Turning to the scalar sector, spherical symmetry prohibits any dependence on the angular coordinates and AdS_2 symmetry implies the same for the r, t coordinates, so that all scalars must be constant. Generally, these constant values are of two very different kinds. First, some scalars may be fixed in terms of the electric and magnetic charges of the solution, as is the case for the scalar in the example solution (1.2.43). Such a scenario is automatically realised in black hole solutions that preserve some supersymmetry, for which all scalars are attracted to particular charge dependent values near the horizon, irrespectively of their values at infinity.

Another possibility, which arises in non-supersymmetric solutions, is that the constant values of some scalars in the near-horizon region are arbitrary. Such a behaviour is usually a result of a global symmetry of the Lagrangian and extends beyond the near-horizon region, throughout spacetime. In other words, one now deals with a continuous variety of black holes, labeled by the values of the scalars subject to this symmetry. We will see examples of such solutions in the following chapters.

In view of a potential microscopic interpretation of black hole entropy in terms of an underlying theory, it is crucial that the entropy depends only on conserved charges and not on arbitrary integration constants. In this respect, the attractor mechanism for supersymmetric solutions is a satisfactory result, since all possible quantities in the near horizon region, including the entropy, can be ultimately expressed in terms of conserved charges. On the other hand, the possibility of unfixed scalars in non-supersymmetric attractors does not spoil the above general expectation, due to the observation that any global symmetries of the scalars will also be symmetries of the Noether potential for diffeomorphisms (1.2.24), as it only involves derivatives with respect to the Riemann tensor. It follows that the result for the entropy must depend only on appropriate invariants.

There is a systematic way of studying the near-horizon values of scalars in generic theories, based on the so-called entropy function [19, 20]. In the following we will deal both with solutions preserving supersymmetry or not. For supersymmetric attractors, the values of the scalars follow directly from symmetry considerations. In the non-supersymmetric case, we will be constructing complete solutions, rather than the near-horizon geometry, so that the attractor values for the scalars can be found by simply taking a near-horizon limit. Therefore, we refrain from giving more details on the entropy function, referring the interested reader to [21] for a review.

1.3 This thesis

In this chapter, we gave a basic exposition of the properties of extremal black holes in theories containing scalars, based on a model theory. The focus of this thesis is the study of such black hole solutions for a particular class of supersymmetric gravity theories in four and five dimensions, which are known as $\mathcal{N} = 2$ supergravities and will be introduced in the next chapter.

The presence of supersymmetry in these theories facilitates the analysis in several ways. For example, one can restrict attention to the subset of all black hole solutions that preserve some supersymmetry–the so-called BPS solutions. This puts very strong constraints on the geometry and the matter fields, to the point that supersymmetric solutions can be completely characterised by first order differential equations without making use of the full equations of motion. In the next chapter, we give a concise summary of supersymmetric solutions both in four- and five-dimensional supergravity.

A further simplification implied by supersymmetry is that the theory is encoded in a small number of arbitrary functions. For example, the $\mathcal{N} = 2$ supersymmetric extension of Einstein-Maxwell theory is completely fixed by a single function of the scalars which controls all couplings, similar to the ratio σ_+/σ_- in the action (1.2.27). This property is crucial for the two main applications we consider in this thesis, namely the effects of higher derivative terms on BPS solutions and the construction of non-supersymmetric solutions. In chapters 3 and 4 we present a discussion of the four derivative terms allowed by $\mathcal{N} = 2$ supersymmetry in four and five dimensions respectively. Each of these invariants is parametrised by an arbitrary function, which can be fixed when one considers the embedding in a microscopic theory, such as string theory. Restricting to the class of theories obtained by string theoretic models, one can connect to microscopic constructions of black holes.

Constructing exact solutions of higher derivative theories is a complicated problem, which has nevertheless been considered with success in the past, at least for simple supersymmetric solutions. However, one can make general statements about BPS black holes based on the near-horizon region, which has been shown to preserve full $\mathcal{N} = 2$ supersymmetry. It follows that supersymmetric attractors for an arbitrary Lagrangian can be constructed directly, based on the enhanced symmetry. In chapters 3 and 4 we give a comprehensive treatment in both four and five dimensions, commenting on the relation between the two theories. We compare our results to the corresponding string theoretic predictions in chapter 2 and establish full agreement between macroscopic and microscopic predictions.

The highly restricted structure of extended supergravity theories reflects itself on the solutions, even if one abandons the requirement that they preserve some supersymmetry. In chapter 5 we discuss some recent results on the construction of non-BPS solutions in $\mathcal{N} = 2$ theories. As it turns out, many of the properties of BPS black holes, such as the description in terms of first order equations and the attractor mechanism, are a result of their extremality rather than the unbroken supercharges. In order to investigate these similarities, we first introduce a rather special class of five-dimensional solutions which are supersymmetric on local patches of spacetime, but nevertheless completely break supersymmetry globally. The new solutions obtained turn out to allow for a number of features that are present in generic extremal solutions, but forbidden in the supersymmetric case. Based on this fact, we then go on to propose a framework for constructing more general extremal solutions, including rotating black holes.

Chapter 2

Supergravity and Black holes

In this chapter we introduce supergravity theories invariant under eight supercharges in four and five spacetime dimensions. The construction of the relevant Lagrangians requires basic concepts of multiplet calculus, which we introduce along the way. We consider superconformal multiplets, which are off-shell (with the exception of the hypermultiplets), so that the superconformal symmetries close without the need for imposing field equations. The multiplet calculus consists of various multiplication and decomposition rules, as well as invariant density formulae. With these results one can construct rather general classes of invariant actions that go beyond the two derivative level, as will be seen in due course.

The structure of supergravity theories in four and five dimensions is very similar, as will become clear in the following. In appendix B we present the various multiplets of the superconformal algebra in both cases. The multiplet describing the gravitational background is the Weyl multiplet. Its bosonic part includes the vielbein e_μ^a, the spin connection ω_μ^{ab} and two more gauge fields of the conformal group, b_μ and f_μ^a, associated to local scale transformations and inversions. In addition, there are gauge fields associated to particular local symmetries in each case, $\mathrm{SU}(2) \times \mathrm{U}(1)$ in four and $\mathrm{USp}(2)$ in five dimensions respectively. Finally, one needs to introduce two auxiliary bosonic fields, a real scalar D and a tensor T_{ab}. The fermionic part of the Weyl multiplet contains two gauge fields, the gravitino ψ_μ^i and ϕ_μ^i, which arises when applying bosonic conformal transformations on the gravitino. There is also a fermionic auxiliary field, denoted by χ^i. Here, indices $i, j, \ldots$ denote the fundamental representation of SU(2) or USp(2), depending on the theory.

Similar patterns arise for matter multiplets, as one can see from tables B.3 and B.6. In the following, we briefly discuss the construction of supergravity theories in four and five dimensions, both in the full conformal setting and in the Poincaré frame. In addition, we present a concise review of BPS black hole solutions in these theories and their string theoretic interpretation, that will be used as reference in the following chapters. As there are many excellent sources on the subject we will be concise, emphasizing ideas and results rather than explicit derivations, referring to [22–26] for further details.

2.1 Five-dimensional supergravity

We start from the five-dimensional theory, which was introduced in the Poincaré frame in [27] Its natural interpretation in the context of M-theory compactified on Calabi-Yau manifolds was presented in [28]. The superconformal theory and further generalisations were considered in [29, 30]. We introduce the relevant superconformal multiplets, namely the Weyl, vector, linear and hypermultiplet, in Appendix B.1.

2.1.1 The Lagrangian and gauge fixing

In five space-time dimensions, composite linear multiplets play an important role in the construction of supersymmetric invariants involving vector multiplets [29–32]. For instance, at the linearized level in flat space, one can start with the field Y^{ij} belonging to a vector supermultiplet and generate a linear multiplet upon the following identification,

$$\begin{aligned} L^{ij} &\to 2\,Y^{ij}\,, \\ \varphi^i &\to \mathrm{i}\partial\!\!\!/\Omega^i\,, \\ E_\mu &\to \partial^\nu F_{\nu\mu}\,, \\ N &\to \Box\sigma\,, \end{aligned} \tag{2.1.1}$$

as the reader can easily verify by explicit calculation using (B.1.9). At this point one can generate a new vector multiplet, by starting with the field N and identifying it with a new field σ, etcetera, at the price of including higher and higher powers of derivatives. It is easy to see that the linear multiplet precisely corresponds to the field equations of the vector multiplet. Conversely, the vector multiplet will arise as the field equations of the linearized tensor multiplet action in flat space.

This relationship is clearly embodied in the invariant density formula for a product of a vector with a linear supermultiplet. In the following, we will take into account the presence of a general superconformal background, and the corresponding expression takes the following form

$$\begin{aligned} e^{-1}\mathcal{L}_{\rm vt} \;=\;& \big(Y_{ij} - \tfrac{1}{2}\bar\Omega_i\gamma^\mu\psi_\mu{}^k\,\varepsilon_{kj}\big)L^{ij} + \sigma\big(N - \tfrac{1}{2}\bar\varphi_i\gamma^\mu\psi_\mu{}^i\big) + \mathrm{i}\,\bar\Omega_i\varphi^i \\ & + \tfrac{1}{6}\mathrm{i}e^{-1}\varepsilon^{\mu\nu\rho\sigma\lambda}\,W_\mu\,\partial_\nu E_{\rho\sigma\lambda} + \tfrac{1}{4}\mathrm{i}\,\sigma\,L^{ij}\,\bar\psi_{\mu i}\gamma^{\mu\nu}\psi_\nu{}^k\varepsilon_{kj}\,. \end{aligned} \tag{2.1.2}$$

Note that the existence of this formula is tied to the gauge fields W_μ and $E_{\mu\nu\rho}$ of the two multiplets, which must be inert under scalings, fixing all other conformal weights in both multiplets.

By using composite linear multiplets, this density formula enables the construction of superconformally invariant actions. This represents a standard way of constructing actions that is also well-known in the context of four space-time dimensions where it has been exploited extensively to couple matter to $\mathcal{N}=1$ and $\mathcal{N}=2$ supergravity, but in five dimensions the restrictions are stronger.

In the superconformal setting (2.1.1) must, however, be modified in view of the additional restrictions posed by superconformal symmetries. For instance, the fields L^{ij} and Y^{ij} behave differently under scale transformations, as can be seen from table B.3 and L^{ij} is invariant under S-supersymmetry, whereas Y^{ij} is not. Nevertheless the relationship can still be established provided one gives up linearity, replacing the first component of (2.1.1) by (for a single multiplet),

$$L^{ij} \to 2\,\sigma\, Y^{ij} + \tfrac{1}{2}\mathrm{i}\,\varepsilon^{k(i}\bar{\Omega}_k\Omega^{j)}\,, \tag{2.1.3}$$

To establish the existence of this composite linear multiplet one verifies that the lowest component has the correct Weyl weight and is S-supersymmetric, and furthermore, that its supersymmetry variation is expressed in terms of a simple doublet spinor which can then act as the representative of the linear multiplet spinor φ^i. If these criteria are not met, the result will not be a superconformal linear multiplet consisting of $8+8$ degrees of freedom, but with a much larger multiplet. When dealing with several vector multiplets, labeled by indices $A, B, \dots = 1, 2, \dots, n_\mathrm{v}$, the expression (2.1.3) generalizes only slightly. It remains quadratic on the vector multiplets and depends on it in a symmetric fashion. Hence we start with

$$L^{ij(AB)} = 2\,\sigma^{(A}\, Y^{ijB)} + \tfrac{1}{2}\mathrm{i}\,\varepsilon^{k(i}\,\bar{\Omega}_k{}^{(A}\Omega^{j)B)}\,, \tag{2.1.4}$$

For clarity of the notation, we will henceforth suppress the explicit indices (A,B) on the right-hand side. In the presence of several vector multiplets, σ^2 generalizes to $\sigma^{(A}\sigma^{B)}$, $\sigma\Omega^i$ to $\sigma^{(A}\Omega^{iB)}$, etcetera.

The other components of the corresponding linear multiplet follow by applying successive supersymmetry variations and one finds the following expressions, all manifestly quadratic in the vector multiplet components,

$$\begin{aligned}
\varphi^{i(AB)} &= \mathrm{i}\,\sigma \not{D}\Omega^i + \tfrac{1}{2}\mathrm{i}\not{D}\sigma\,\Omega^i - 8\,\sigma^2\,\chi^i + Y^{ij}\varepsilon_{jk}\Omega^k - \tfrac{1}{4}(\hat{F}_{ab} - 6\,\sigma\,T_{ab})\gamma^{ab}\Omega^i\,,\\
E^{a(AB)} &= \tfrac{1}{8}\mathrm{i}\varepsilon^{abcde}\hat{F}_{bc}\hat{F}_{de} + D_b(\sigma\,\hat{F}^{ba} - 6\,\sigma^2\,T^{ba}) + \cdots\,,\\
N^{(AB)} &= \tfrac{1}{2}D^a D_a\sigma^2 - \tfrac{1}{2}(D_a\sigma)^2 + |Y^{ij}|^2\\
&\quad - \tfrac{1}{4}\hat{F}_{ab}\hat{F}^{ab} + 6\,\sigma\,\hat{F}_{ab}T^{ab} - \sigma^2\big(4\,D + \tfrac{39}{2}T^2\big) + \cdots\,,
\end{aligned} \tag{2.1.5}$$

where supercovariant terms of higher-order in the fermion fields have been suppressed. It is also possible to derive the expression for the three-rank tensor gauge field associated with this multiplet, by requiring (B.1.14),

$$\begin{aligned} E_{\mu\nu\rho}^{(AB)} &= \tfrac{1}{2}\mathrm{i}\, e\, \varepsilon_{\mu\nu\rho\sigma\lambda}\,(\sigma\,\hat{F}^{\sigma\lambda} - 6\,\sigma^2\,T^{\sigma\lambda}) + \tfrac{3}{2}W_{[\mu}F_{\nu\rho]} + \tfrac{1}{4}\bar{\Omega}_i\gamma_{\mu\nu\rho}\Omega^i \\ &\quad - \tfrac{3}{2}\mathrm{i}\sigma\,\bar{\Omega}_i\gamma_{[\mu\nu}\psi_{\rho]}{}^i - \tfrac{3}{4}\sigma^2\,\bar{\psi}_{[\mu i}\gamma_\nu\psi_{\rho]}{}^i\,. \end{aligned} \tag{2.1.6}$$

The above construction can be generalized to non-abelian vector multiplets as well [33]. More generally, a linear multiplet can also be constructed from the Weyl multiplet, as will be shown in 4.1, or hypermultiplets, but the resulting linear multiplet will not be fully realized off-shell.

Using these results, the construction of the two derivative Lagrangian for vector multiplets follows straightforwardly using (2.1.2). The composite linear multiplet constructed from vector multiplets will be written by means of a symmetric three-rank constant tensor C_{ABC}. The lowest component of the linear multiplet associated with the symmetrized product of two vector multiplets will thus be identified with $-\frac{1}{6}\,C_{ABC}\,L^{ij(BC)}$, where we make use of (2.1.4). Higher components are defined accordingly.

After these definitions we introduce the expression for the bosonic terms in the Lagrangian for vector multiplets, with convenient normalisation, to which we add the hypermultiplet Lagrangian.

$$\mathcal{L} = \mathcal{L}_{\text{vvv}} + \mathcal{L}_{\text{hyper}}\,. \tag{2.1.7}$$

Here the Lagrangian cubic in vector multiplet fields equals,

$$\begin{aligned} 8\pi\, e^{-1}\mathcal{L}_{\text{vvv}} &= \tfrac{1}{2}\,C_{ABC}\sigma^A\Big[\tfrac{1}{2}\mathcal{D}_\mu\sigma^B\,\mathcal{D}^\mu\sigma^C + \tfrac{1}{4}F_{\mu\nu}{}^B F^{\mu\nu C} - Y_{ij}{}^B Y^{ijC} \\ &\qquad -3\,\sigma^B F_{\mu\nu}{}^C T^{\mu\nu} - \tfrac{1}{24}\mathrm{i}\,e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}W_\mu{}^A F_{\nu\rho}{}^B F_{\sigma\tau}{}^C\Big] \\ &\quad -C(\sigma)\Big[\tfrac{1}{8}\mathcal{R} - 4\,D - \tfrac{39}{2}T^2\Big]\,, \end{aligned} \tag{2.1.8}$$

where we also use the notation $C(\sigma) = \frac{1}{6}\,C_{ABC}\,\sigma^A\sigma^B\sigma^C$. The Lagrangian for hypermultiplets (one of which plays the role of a compensating supermultiplet) reads,

$$8\pi\, e^{-1}\mathcal{L}_{\text{hyper}} = -\tfrac{1}{2}\Omega_{\alpha\beta}\,\varepsilon^{ij}\mathcal{D}_\mu A_i{}^\alpha\,\mathcal{D}^\mu A_j{}^\beta + \chi\Big[\tfrac{3}{16}\mathcal{R} + 2\,D + \tfrac{3}{4}T^2\Big]\,. \tag{2.1.9}$$

We remind the reader that $\mathcal{R}$ and $\mathcal{R}_{ab}$ refer to the Ricci scalar and tensor. The factor $8\pi^2$, which equals four times the volume of the unit sphere S^3, has been included to avoid explicit factors of π when defining electric charges.[1] Note the presence of the characteristic Chern-Simons term in (2.1.8), which implies that the corresponding action is only gauge invariant up to boundary terms. This term, and similar ones appearing

[1] In four space-time dimensions one extracts a factor equal to two times the volume of the unit sphere S^2. In this way the Coulomb potential has the same normalization in four and in five dimensions, without factors of π.

in the four derivative Lagrangian, may lead to difficulty in cases where the gauge fields are not globally defined, as we shall discuss in due course.

To appreciate the implications of the above results, let us first consider the Lagrangian (2.1.8) alone. When suppressing the coupling to the fields T_{ab}, D and to the metric, we are dealing with a Lagrangian based on scalar fields σ^A, gauge fields W_μ^A and auxiliary fields $Y^{ij\,A}$. Upon re-introducing the fermion fields, this Lagrangian is invariant under rigid superconformal transformations. Note that the overall sign of the Lagrangian is irrelevant, as it can be absorbed into an overall sign of the vector supermultiplet fields. To identify the kinetic terms one may expand about some constant values of the fields σ^A. The values of these constants are arbitrary and in fact they can be changed by a uniform scale transformation that acts on all the fields and belongs to the rigid superconformal symmetry group.

When coupled to the fields of the Weyl multiplet, this Lagrangian is invariant under local superconformal transformations. However, it is inconsistent because the field D acts as a Lagrange multiplier which requires $C(\sigma)$ to vanish. To avoid this difficulty one must also introduce the superconformally invariant Lagrangian of at least one hypermultiplet. Introducing the hypermultiplets, the field equation for the auxiliary D implies that

$$\chi = -2\, C(\sigma)\,. \tag{2.1.10}$$

In view of the local invariance under scale transformations, $C(\sigma)$ can be fixed to a constant. We choose the gauge condition

$$\tfrac{1}{6}\, C_{ABC}\, \sigma^A \sigma^B \sigma^C = 1\,, \tag{2.1.11}$$

so that the Ricci scalar will appear in the Lagrangian with a multiplicative factor $(16\pi^2)^{-1}$. This convention is related to the more conventional one, where one adopts a prefactor $(16\pi\, G_{\rm N})^{-1}$, just as in four space-time dimensions, by choosing Newton's constant[2] as $G_{\rm N} = \pi$. As a result of this convention, the Bekenstein-Hawking area law, leads to the area in Planck units, $A/G_{\rm N}$, with proportionality factor $(4\pi)^{-1}$.

Making use of the local SU(2) transformations, three phases contained in the hypermultiplet scalars can be fixed as well, due to the $SU(2)$ invariance of the hypermultiplet target space. Combing these two gauges, it follows that one of the hypermultiplets will not correspond to physical degrees of freedom. Finally, one can choose a gauge in which $b_\mu = 0$ and eliminate the auxiliary fields, Y^{ij} and T_{ab}, by their (algebraic) field equations, which yields

$$Y^{ij} = 0\,, \qquad T_{ab} = (24\, C(\sigma))^{-1} C_{ABC}\sigma^A\sigma^B F_{ab}^C\,. \tag{2.1.12}$$

The equation of motion for the $SU(2)$ gauge field is also algebraic, but this field will not be needed in what follows. Hence one is left with the bosonic part of Poincaré

[2] A generic value for the Newton constant can be introduced by rescaling the tensor C_{ABC} and the scalars accordingly, see [30]

five-dimensional supergravity coupled to vector multiplets and hypermultiplets

$$\begin{aligned} 8\pi\, e^{-1}\mathcal{L} &= -\tfrac{1}{2}\mathcal{R} - Q_{AB}\Big[\tfrac{1}{2}\mathcal{D}_\mu\sigma^A\,\mathcal{D}^\mu\sigma^B + \tfrac{1}{4}F_{\mu\nu}{}^A F^{\mu\nu B}\Big] \\ &\quad - \tfrac{1}{48}\mathrm{i}C_{ABC}\,e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}W_\mu{}^A F_{\nu\rho}{}^B F_{\sigma\tau}{}^C \\ &\quad - \tfrac{1}{2}\Omega_{\alpha\beta}\,\varepsilon^{ij}\mathcal{D}_\mu A_i{}^\alpha\,\mathcal{D}^\mu A_j{}^\beta\,, \end{aligned} \tag{2.1.13}$$

where the index α now runs over one less hypermultiplet[3] and

$$Q_{AB} = \tfrac{9}{2}\,\sigma_A\sigma_B - \tfrac{1}{2}\,C_{ABC}\sigma^C\,, \qquad \sigma_A \equiv \frac{1}{6}\,C_{ABC}\sigma^B\sigma^C\;. \tag{2.1.14}$$

Observe that all the original vector gauge fields are present, even though there is one less vector multiplet scalar due to the constraint (2.1.11). This is consistent, as one linear combination of gauge fields is reassigned to the Poincaré gravity multiplet through (2.1.12), and is usually called the graviphoton in the context of the two-derivative theory.

In what follows, we will sometimes restrict attention to the case where the scalars take values in a symmetric space, for the sake of simplicity. Such a requirement puts constraints on the tensor C_{ABC} [27], through the identities

$$\frac{4}{3}\delta_{A(D}C_{EFG)} = C_{ABC}C_{B'(DE}C_{FG)C'}\delta^{BB'}\delta^{CC'}\,, \tag{2.1.15}$$

$$\sigma^A = \frac{9}{2}C^{ABC}\sigma_B\sigma_C\,, \tag{2.1.16}$$

where $C^{ABC} \equiv \delta^{AA'}\delta^{BB'}\delta^{CC'}C_{A'B'C'}$ and the constraint in (2.1.11) was used in obtaining (2.1.16).

2.1.2 BPS black holes in five dimensions

The solutions of the Poincaré theory that preserve half of the supercharges possess a so-called (generalised) Killing spinor, ϵ, such that all supersymmetry variations of the fermions vanish. According to the general classification in [25, 35], there are two distinct classes, depending on whether the Killing vector, $\bar\epsilon\gamma^\mu\epsilon$, constructed from ϵ is time-like or null. We will briefly summarise the time-like case, for which the BPS conditions imply that the metric is a time-like fibration over a hyper-Kähler 4-manifold.

Denoting the hyper-Kähler base space metric by h_{mn}, we choose the associated complex structures, $X^{(i)}$, as anti-self-dual and assume for the moment that they are unique. Given these data, the metric and gauge fields of a supersymmetric solution can be

[3] This is only possible for a restricted class of target space geometries. As the hypermultiplets will not play any role in the following, we assume this is the case for simplicity, referring to [30, 34] for more details.

written locally as:

$$\begin{aligned} ds^2 &= -f^2(dt+\hat{\omega})^2 + f^{-1}h_{mn}dx^m dx^n \\ F^A &= 2\,\partial_{[\mu}(\sigma^A e^0_{\nu]})\,dx^\mu dx^\nu + \Lambda^A_{mn}\,dx^m dx^n . \end{aligned} \tag{2.1.17}$$

Here, $e^0 = f(dt+\hat{\omega})$, the indices, $m, n, \ldots = 1, \ldots, 4$, label coordinates on the base, and $f > 0$, $\hat{\omega}$ are a function and a one-form globally defined on the base that define the time fibration. The $\Lambda^A(x^m)$ are arbitrary closed self-dual forms on the base (they have no time component). A positive orientation is chosen using $e^0 \wedge \eta$ as the volume form, where η is a positive orientation on the base manifold.

Once the Λ^A are chosen, the function f and the σ^A are determined by solving the Maxwell equations, expressed as a Poisson equation on the base:

$$\Delta\left(f^{-1}\sigma_A\right) = \frac{1}{12} C_{ABC}\Lambda^B_{mn}\Lambda^{mn\,C}\,, \tag{2.1.18}$$

where Δ is the Laplacian on the hyper-Kähler manifold. Finally, the one-form, $\hat{\omega}$, is determined by solving:

$$2\,f\,\partial_{[m}\omega_{n]} = G_{mn} \equiv G^+_{mn} + G^-_{mn}, \qquad \sigma_A\Lambda^A = -\tfrac{2}{3}G^+\,, \tag{2.1.19}$$

where $G^\pm$ the are self- and anti-self-dual parts of the form $f\,\partial_{[m}\omega_{n]}$ on the hyper-Kähler base. These BPS solutions are invariant under an $SU(2)$ subgroup of the base space isometry group and, in an orthonormal basis, their respective Killing spinors satisfy $\gamma^0\epsilon = \mathrm{i}\epsilon$. Note that in case the complex structures are not unique, it may be possible to construct more BPS solutions than described by the generic equations above. This will be seen explicitly in the following, but for the moment we continue to assume uniqueness.

In this thesis we are interested in solutions with a four-dimensional interpretation, so we will demand that the base space has a compact isometry along which we can perform dimensional reduction. Under the assumption that the associated Killing vector is triholomorphic (i.e. leaves the complex structures invariant), and generates a symmetry of the complete solution, the base space can only be a Gibbons-Hawking space [36]. In this case the above equations can be solved explicitly [35]. For a Gibbons-Hawking space [37], which is itself a fibration over a flat Euclidean base, the metric, h_{mn}, can be written:

$$h_{mn}dx^m dx^n = H^{-1}\left(d\psi + \chi_i dx^i\right)^2 + H\delta_{ij}dx^i dx^j, \tag{2.1.20}$$

$$\epsilon^{ijk}\nabla_i \boldsymbol{\chi}_j = \nabla^k H. \tag{2.1.21}$$

Here, ∇ is the standard vector derivative on the Euclidean 3-space, $\mathbb{R}^3$, with coordinates x^i, $i, j = 1, 2, 3$, $H(x^i)$ is a harmonic function on $\mathbb{R}^3$ and $0 \leq \psi \leq 4\pi$. The isometry group is $SU(2) \times U(1)$, where the $U(1)$ is generated by the Killing vector $\partial/\partial\psi$. Two important examples we will use is flat space, corresponding to either $H = 1$ or $H = 1/|\mathbf{x}|$, and Taub-NUT space, for which $H = h^0 + p^0/|\mathbf{x}|$.

The complex structures associated with (2.1.20) are given by

$$X^{(i)} = \left(d\psi + \chi_j dx^j\right) \wedge dx^i - \tfrac{1}{2} H \epsilon_{ijk} dx^j \wedge dx^k. \tag{2.1.22}$$

Using (2.1.21), one can easily show that $dX^{(i)} = 0$. Imposing that they are anti-self-dual fixes the orientation of the base space, so that the volume form is

$$H d\psi \wedge dx^1 \wedge dx^2 \wedge dx^3. \tag{2.1.23}$$

The complex structures (2.1.22) are globally defined and, in all but one case, unique. The exception is flat space (in coordinates such that $H = 1$, $\chi = 0$), where one can also choose the opposite relative sign in (2.1.22) and the forms, $X^{(i)}$, remain closed. This is a consequence of the larger $SO(4) \cong SU(2)_L \times SU(2)_R$ isometry group of flat space compared to the generic $SU(2) \times U(1)$ isometry of Gibbons-Hawking spaces. It follows that for flat space there are two triplets of complex structures: one self-dual and one anti-self-dual, each invariant under a different $SU(2)_{L,R}$ isometry. This observation implies a slight enlargement of the set of BPS solutions in that case, to which we return in section 5.2.1.

The explicit BPS solution for a Gibbons-Hawking base can be described in terms of H and $2n_{\rm v} + 1$ additional harmonic functions [35]. The self-dual forms Λ^A, defined in (2.1.17), can be written as:

$$\Lambda^A = (d\psi + \chi_k dx^k) \wedge \partial_j \left(\frac{K^A}{H}\right) dx^j + \tfrac{1}{2} H \epsilon_{ijk} \partial_k \left(\frac{K^A}{H}\right) dx^i \wedge dx^j \,, \tag{2.1.24}$$

where the K^A are arbitrary harmonic functions related to the magnetic charges of the solution. Given these functions, the scalar equation (2.1.18) can be solved as

$$f^{-1}\sigma_A = \tfrac{1}{6} H^{-1} C_{ABC} K^B K^C + \tfrac{2}{3} L_A \,, \tag{2.1.25}$$

where L_A are arbitrary harmonic functions associated with the electric charges. Finally, decomposing the one-form ω as

$$\hat{\omega}_m dx^m = \omega_i dx^i + \omega_5 (d\psi + \chi_j dx^j) \,, \tag{2.1.26}$$

one finds that the solution of (2.1.19) reads

$$\omega_5 = \tfrac{1}{6} H^{-2} C_{ABC} K^A K^B K^C + H^{-1} L_A K^A + M \tag{2.1.27}$$

$$\nabla \times \omega = H \nabla M - M \nabla H + K^A \nabla L_A - L_A \nabla K^A \,. \tag{2.1.28}$$

Here, M is again a harmonic function, which controls the angular momentum along the ψ direction.

It is worth mentioning the so-called spectral flow transformations [38] acting on the BPS solutions above. It is easy to verify that (2.1.25)-(2.1.28) are invariant under the symmetry transformations

$$\begin{aligned} H &\rightarrow H\,, \\ K^A &\rightarrow K^A + k^A H\,, \\ M &\rightarrow M - k^A L_A + \tfrac{1}{4}\,C_{ABC}k^B k^C\,K^A + \tfrac{1}{12}C_{ABC}k^A k^B k^C\,H\,, \\ L_A &\rightarrow L_A - \tfrac{1}{2}\,C_{ABC}k^B\,K^C - \tfrac{1}{4}\,C_{ABC}k^B k^C\,H\,, \end{aligned} \tag{2.1.29}$$

for any real k^A. We stress that these are exact symmetries that act on the full solution, similar to the electric-magnetic dualities in four-dimensional theories (see section 2.2.2). A useful result is that this symmetry can be used to eliminate any poles of the K^A coinciding with poles of the Gibbons-Hawking harmonic function H.

2.1.3 Explicit solutions and supersymmetry enhancement

One can straightforwardly construct explicit solutions using the above relations, by choosing the harmonic functions appropriately. We will discuss two important examples, namely the BMPV (or spinning) black hole [39] and the supersymmetric black ring of [40]. The base is usually taken to be either flat or Taub-NUT, depending on the desired asymptotics. In what follows we will always choose

$$H = h + \frac{p^0}{r}\,, \tag{2.1.30}$$

which encompasses both cases, as it describes Taub-NUT for $h \neq 0$ and flat space[4] for $h = 0$.

BMPV black hole

If one assumes that all harmonic functions are centered at a single point, the observation under (2.1.29) shows that the K^A, and thus the magnetic parts of the field strengths in (2.1.17), can be taken to be trivial. The resulting solution carries only electric charges and is known as the BMPV black hole.

The poles of the nonvanishing harmonic functions are equal to the electric charges and the angular momentum

$$L_A = l_A + \frac{q_A}{r}\,, \qquad M = m + \frac{J_\psi}{r}\,, \tag{2.1.31}$$

[4] Since we fixed the periodicity of ψ in (2.1.20), this gives flat space with a $\mathbb{Z}_{p^0}$ identification along that direction.

where l_A, m are arbitrary constants. The full metric and the gauge fields follow from (2.1.17) and in these conventions the horizon is located at $r = 0$. The entropy, as defined using the area law, reads

$$\mathcal{S} = 2\pi \sqrt{Q^3 - \tfrac{1}{4}J_\psi^2}\,, \tag{2.1.32}$$

$$Q^{3/2} = \tfrac{1}{12}\,C_{ABC}q^A q^B q^C \qquad q_A = \tfrac{1}{4}\,C_{ABC}q^B q^C\,. \tag{2.1.33}$$

Multi-black hole solutions can be obtained by simply allowing for more poles in the harmonic functions in (2.1.31), centered at different positions.

It is worthwhile discussing the near horizon geometry. Taking the limit $r \to 0$ in the above equations, all constants in the harmonic functions can be neglected and the solution simplifies. As in the example in section 1.2.3, all scalars take constant values fixed in terms of the charges according to

$$f_0^{-1}\sigma_A = \tfrac{2}{3}q_A\,, \qquad f = f_0\, r\,, \tag{2.1.34}$$

where f_0 is a constant. The metric reads

$$ds^2 = -f_0^2\, r^2 (dt + \hat{\omega})^2 + \frac{p^0}{f_0}\left[\left(\frac{d\psi}{p^0} + \cos\theta d\phi\right)^2 + \frac{dr^2}{r^2} + \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right]\,,$$
$$\hat{\omega} = \frac{J_\psi}{r}(d\psi + p^0\, \cos\theta d\phi)\,, \tag{2.1.35}$$

and the gauge fields follow from (2.1.17). This solution is invariant under all eight supercharges, whereas the global solution preserves only four. Supersymmetry enhancement near the horizon is a general feature of all BPS black hole solutions [41–43], that extends the spacetime symmetry enhancement observed for extremal black holes in section 1.2.4. The reader can easily verify that (2.1.35) describes the local product of a circle with the $AdS_2 \times S^2$ geometry in (1.2.44).

Turning this observation around, one can determine the near horizon solution by imposing full supersymmetry. When higher derivative corrections are taken into account, it becomes increasingly cumbersome to find complete solutions, so the possibility of studying supersymmetric attractors is a welcome result. Chapter 4 is devoted to supersymmetric attractors in five-dimensional supergravity with four derivative corrections.

BPS black ring

A more involved solution is the BPS black ring, which is a magnetically charged object. In five dimensions, magnetic monopoles are necessarily string-like objects, as they couple naturally to the magnetic gauge fields, which are two-forms. It follows that magnetic charges can be carried only by black holes with ring topology. The harmonic functions describing the full solution are

$$K^A = \frac{p^A}{\Delta}\,, \qquad L_A = l_A + \frac{\tilde{q}_A}{\Delta}\,, \qquad M = m + \frac{J_\psi}{\Delta}\,, \tag{2.1.36}$$

where p^A are the five-dimensional magnetic dipole moments, $\tilde{q}_A$ are related to the electric charges q_A through

$$\tilde{q}_A = q_A - \tfrac{1}{2} C_{ABC} p^A p^B, \tag{2.1.37}$$

and l_A, m are constants. Finally, $\Delta = |\vec{r} - \vec{r}_0|$ denotes the distance from a point $\vec{r}_0$ in $\mathbb{R}^3$. The ring described by these harmonic functions is localised in the base, and wrapped around the ψ-circle at $\vec{r}_0$.

The metric and gauge fields follow as above, but obtaining the full physical solution is a bit more complicated. In particular, there are nontrivial constraints on the constants and charges appearing in (2.1.36) coming from requirements such as asymptotic flatness (or Taub-NUT) and the absence of closed timelike curves. For example, the position of the ring is not arbitrary, but is proportional to the angular momentum J_ψ, as is intuitively clear from the requirement that the spin of the ring balances the gravitational force to the centre. We will not be dealing with most of these subtleties, but we point out that the charges are not simply given by the poles of the harmonic functions, but there are shifts in the L_A's. The combination appearing in (2.1.37) has a physical meaning, as it is additive when one considers multi-ring configurations [35], as will be seen in section 4.5.

On the other hand, the near horizon solutions is oblivious to these complications, as the integration constants in (2.1.36) are dropped in the $\Delta \to 0$ limit. Once more, the scalars are constant, given by

$$f_0^{-1} \sigma_A = \tfrac{1}{6} C_{ABC} p^B p^C\,, \qquad f = \frac{p^0 f_0}{|\vec{r}_0|}\, \Delta^2\,, \tag{2.1.38}$$

where f_0 is again a constant. The metric reads [35]

$$ds^2 = -4\sqrt{\frac{\hat{q}_0}{P}}\, dt\, d\Delta + \frac{8}{P}\, \Delta\, dt\, d\psi + \frac{\hat{q}_0}{P}\, d\psi^2 + P^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\,, \tag{2.1.39}$$

where

$$P^3 = \frac{1}{6}\, C_{ABC}\, p^A p^B p^C\,, \qquad \hat{q}_0 = J_\psi - J_\varphi - \frac{1}{4} C^{AB} \tilde{q}_A \tilde{q}_B\,, \tag{2.1.40}$$

and $C^{AB} \equiv [C_{ABC} p^C]^{-1}$. This metric is locally isometric to $AdS_3 \times S^2$ and is fully supersymmetric, as expected [25]. Together with (2.1.35), these are the only two physically inequivalent fully supersymmetric geometries that are admissible as near horizon limits for a black hole solution. Note that upon dimensional reduction over the ψ direction both reduce to $AdS_2 \times S^2$, which is the only allowed near-horizon geometry in four dimensions.

The area law gives the following entropy

$$\mathcal{S} = 2\pi \sqrt{\frac{1}{6}\, \hat{q}_0\, C_{ABC}\, p^A p^B p^C}\,. \tag{2.1.41}$$

In view of the poles of the harmonic functions above, the result dependents on $\tilde{q}_A$ rather than q_A directly. Note that despite the presence of two angular momenta, it is the difference $J_\psi - J_\varphi$ that appears in the entropy [44], whereas the ϕ-component of the angular momentum is necessarily fixed by

$$J_\varphi = p^I (q_I - \tfrac{1}{6} C_{IJ} p^J)\,, \tag{2.1.42}$$

and is not an independent conserved charge.

2.2 Four-dimensional $\mathcal{N} = 2$ supergravity

We now turn to four-dimensional $\mathcal{N} = 2$ supergravity, for which the superconformal multiplet calculus was developed in [45–48]. The multiplets relevant to the subsequent discussion are introduced in Appendix B.2, which include the Weyl multiplet, the chiral multiplet and the hypermultiplet. Some of the theories in this class can be viewed as compactifications of five-dimensional theories or, upon gauge fixing to the Poincaré frame, as Calabi-Yau compactifications of Type II string theory and M-theory. Most techniques and results are similar to the corresponding five-dimensional ones, discussed in the previous section. An extra feature is the presence of electric-magnetic dualities that act on the equations of motion for vectors and scalars, which we briefly introduce.

2.2.1 The conformal and Poincaré Lagrangians

The construction of the Lagrangian for vector multiplets in four dimensions is conventionally based on a chiral multiplet rather than a tensor (linear) multiplet, even though that is also possible in principle. This is because the covariant fields in vector multiplets comprise chiral multiplets that satisfy a constraint, as explained in section B.2.3, so that it is natural to combine them into an unconstrained chiral multiplet, for which a density formula analogous to (2.1.2) exists.

This formula is based on a scalar chiral multiplet with $w = 2$, implying that its highest component, C, has Weyl weight 4, and chiral weight 0, as is appropriate for a conformally invariant Lagrangian in four dimensions. It follows that the expression

$$\begin{aligned} e^{-1}\mathcal{L} = {}& C - \varepsilon^{ij}\, \bar\psi_{\mu i}\gamma^\mu \Lambda_j - \tfrac{1}{8}\bar\psi_{\mu i} T_{ab\,jk}\gamma^{ab}\gamma^\mu \Psi_l\, \varepsilon^{ij}\varepsilon^{kl} - \tfrac{1}{16} A (T_{ab\,ij}\varepsilon^{ij})^2 \\ & - \tfrac{1}{2}\bar\psi_{\mu i}\gamma^{\mu\nu}\psi_{\nu j}\, B_{kl}\, \varepsilon^{ik}\varepsilon^{jl} + \varepsilon^{ij}\bar\psi_{\mu i}\psi_{\nu j}(G^{-\mu\nu} - \tfrac{1}{2} A\, T^{\mu\nu}{}_{kl}\, \varepsilon^{kl}) \\ & - \tfrac{1}{2}\varepsilon^{ij}\varepsilon^{kl} e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_{\mu i}\psi_{\nu j}(\bar\psi_{\rho k}\gamma_\sigma \Psi_l + \bar\psi_{\rho k}\psi_{\sigma j}\, A) + \text{h.c.} \end{aligned} \tag{2.2.1}$$

is invariant under local superconformal transformations, including a conformal supergravity background.

In order to find an action for vector multiplets, consider the case where the chiral multiplet in this formula is composite, expressed in terms of vector multiplets labeled by indices $I, J, \dots = 0, 1, \dots, n_{\mathrm{v}}$. It is possible to show (cf. (B.2.18)) that the relevant terms of such a composite multiplet are given by[5]

$$\begin{aligned} A &= -\tfrac{\mathrm{i}}{2} F(X)\,, \\ C &= -\tfrac{\mathrm{i}}{2} F(X)_I\, C^I + \tfrac{\mathrm{i}}{8} F(X)_{IJ} \big[B_{ij}{}^I B_{kl}{}^J\, \varepsilon^{ik} \varepsilon^{jl} - 2\, G_{ab}^{-}{}^I G^{-abJ} \big]\,, \end{aligned} \tag{2.2.2}$$

where F_I and F_{IJ} are the first and second derivative of the function F, known as the prepotential. The bottom composite component, A, must have $w = 2$, so that $F(X)$ must be homogeneous of degree two in the vector multiplet scalars X^I. Taking into account the constraints in (B.2.19), the bosonic terms of the Lagrangian following from (2.2.1) read

$$\begin{aligned} 8\pi\, e^{-1} \mathcal{L} \;=\;& \mathrm{i}\mathcal{D}^\mu F_I\, \mathcal{D}_\mu \bar{X}^I - \mathrm{i} F_I\, \bar{X}^I (\tfrac{1}{6}\mathcal{R} - D) - \tfrac{1}{8} \mathrm{i} F_{IJ}\, Y_{ij}^I Y^{Jij} \\ & + \tfrac{1}{4} \mathrm{i} F_{IJ} (F_{ab}^{-I} - \tfrac{1}{4} \bar{X}^I T_{ab}^{ij} \varepsilon_{ij})(F^{-Jab} - \tfrac{1}{4} \bar{X}^J T^{ijab} \varepsilon_{ij}) \\ & - \tfrac{1}{8} \mathrm{i} F_I (F_{ab}^{+I} - \tfrac{1}{4} X^I T_{abij} \varepsilon^{ij}) T_{ij}^{ab} \varepsilon^{ij} - \tfrac{1}{32} \mathrm{i} F (T_{abij} \varepsilon^{ij})^2 + \text{h.c.} \\ & - \tfrac{1}{2} \varepsilon^{ij}\, \Omega_{\alpha\beta}\, \mathcal{D}_\mu A_i{}^\alpha\, \mathcal{D}^\mu A_j{}^\beta + \chi (\tfrac{1}{6}\mathcal{R} + \tfrac{1}{2} D)\,, \end{aligned} \tag{2.2.3}$$

where in the last line we added the Lagrangian for the hypermultiplets. As in the five-dimensional theory, the vector multiplet Lagrangian alone is inconsistent and the presence of at least one hypermultiplet as a compensating multiplet is necessary.

With this addition, the equation of motion for the auxiliary scalar D is

$$\chi = -2\, \mathrm{i} (F_I\, \bar{X}^I - \bar{F}_I\, X^I)\,, \tag{2.2.4}$$

where the quantity in the right hand side coincides with the factor multiplying the Ricci scalar in the action and should be used in the gauge fixing of local scale transformations. We choose the gauge condition

$$\mathrm{i}(F_I\, \bar{X}^I - \bar{F}_I\, X^I) = 1\,, \tag{2.2.5}$$

to obtain the canonical normalisation of the Einstein term with Newton's constant set to unity. In addition, one can fix the local $U(1)$ symmetry by setting the phase of one of the X^I to a desired value, but we will refrain form doing that explicitly. The end result is that only $2n_{\mathrm{v}}$ of the original $2n_{\mathrm{v}}+2$ scalars are physical, due to the constraints. These are usually denoted as z^A, with $A = 1, \dots, n_{\mathrm{v}}$ and parametrise a so called special

[5] The function $\mathcal{G}$ in (B.2.18) is conventionally chosen as $\mathcal{G}(X^I) = -\frac{\mathrm{i}}{2} F(X)$ in this context.

Kähler target space manifold. A convenient choice that we will adopt is

$$z^A = \frac{X^A}{X^0}\,, \qquad z^0 = \frac{X^0}{X^0} = 1\,, \tag{2.2.6}$$

which are scale and chiral invariant. It can be shown that the associated Kähler metric and potential are given by

$$g_{A\bar{B}} = \partial_A \partial_{\bar{B}} K\,, \qquad K = -\log\Big(\mathrm{i}(F_I\,\bar{z}^I - \bar{F}_I\,z^I)\Big)\,, \tag{2.2.7}$$

where F_I are viewed as functions of the z^A. We will be generally using the variables $X^I(z)$ even in the Poincaré theory, with the understanding that they satisfy (2.2.5) and are subject to the local $U(1)$ invariance.

As in the five-dimensional theory, one can set $b_\mu = 0$ and use the $SU(2)$ gauge transformations to eliminate three phases from the hypermultiplet target space. The auxiliary fields Y^{ij} and T_{ab}^{ij} can be expressed in terms of dynamical fields using their equations of motion

$$Y^{ij} = 0\,, \qquad \varepsilon_{ij} T_{ab}^{ij} = 4\,(\bar{X}^K N_{KL} \bar{X}^L)^{-1} N_{IJ} \bar{X}^J F_{ab}^{-I}\,, \tag{2.2.8}$$

where $N_{IJ} = 2\,\mathrm{Im}\,F_{IJ}$. The $SU(2)$ and $U(1)$ gauge fields also have algebraic equations of motion, but we will only need the expression for the latter

$$A_\mu = \tfrac{\mathrm{i}}{2} N_{IJ}(X^I \partial_\mu \bar{X}^J - \bar{X}^I \partial_\mu X^J) = \mathrm{Im}(\partial_A K\,\partial_\mu z^A)\,. \tag{2.2.9}$$

The final form of the Poincaré supergravity Lagrangian is

$$\begin{aligned} 8\pi\, e^{-1}\,\mathcal{L} \;=\;& -\tfrac{1}{2}\mathcal{R} + \mathrm{i}\mathcal{D}^\mu F_I\,\mathcal{D}_\mu \bar{X}^I - \mathrm{i}\mathcal{D}^\mu \bar{F}_I\,\mathcal{D}_\mu X^I \\ & + \tfrac{\mathrm{i}}{4}\bar{\mathcal{N}}_{IJ} F_{ab}^{-I} F^{-Jab} - \tfrac{\mathrm{i}}{4}\mathcal{N}_{IJ} F_{ab}^{+I} F^{+Jab} \\ & - \tfrac{1}{2}\varepsilon^{ij}\,\Omega_{\alpha\beta}\,\mathcal{D}_\mu A_i{}^\alpha\,\mathcal{D}^\mu A_j{}^\beta\,, \end{aligned} \tag{2.2.10}$$

where the covariant derivatives on the scalars contain the composite gauge field (2.2.9) and

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + \mathrm{i}\frac{N_{IK} X^K N_{JL} X^L}{X^M N_{MN} X^N}\,. \tag{2.2.11}$$

Similar to the convention followed in five dimensions, we implicitly keep the same index naming for hypermultiplets, assuming the index α runs over one less value (see footnote 3).

By restricting the form of the prepotential, it is possible to interpret the four-dimensional theory as a Kaluza-Klein reduction of five-dimensional supergravity under some assumptions. We will return to this point in section 2.3.

2.2.2 Symplectic reparametrisations and Kähler geometry

The most characteristic feature of extended supergravity in four spacetime dimensions is the presence of continuous symmetries of the equations of motion that generalise the electric-magnetic duality transformations of Maxwell electrodynamics [49]. Generically, such transformations are not symmetries of the Lagrangian, but rather correspond to a reparametrisation of the same equations of motion in terms of a different Lagrangian.

In $\mathcal{N} = 2$ supergravity, scalars are in the same multiplet as gauge fields and control all couplings, so they necessarily transform under symplectic transformations [47, 50]. Here, we discuss the basic aspects of the structure appearing in this case, casting the variables of the on-shell action (2.2.10) in a suggestive form that will be useful in the following.

Define the dual field strength tensor, generalising the electric displacement tensor in (1.2.34) by:

$$G_I^{-\,\mu\nu} := -\frac{16\pi\,\mathrm{i}}{e}\frac{\partial\mathcal{L}}{\partial F^{I\,-}_{\mu\nu}} = \mathcal{N}_{IJ}F^{J\,-\,\mu\nu}\,, \tag{2.2.12}$$

in terms of which the Maxwell equations and Bianchi identities take the suggestive form:

$$\begin{aligned} \varepsilon^{\mu\nu\rho\sigma}\partial_\mu F^I_{\nu\rho} &= 0\,,\\ \varepsilon^{\mu\nu\rho\sigma}\partial_\mu G_{I\,\nu\rho} &= 0\,. \end{aligned} \tag{2.2.13}$$

These equations imply that the electric and magnetic charges conventionally defined as

$$\begin{aligned} p^I &= \frac{1}{4\pi}\int_{S^2} F^I\,,\\ q_I &= \frac{1}{4\pi}\int_{S^2} G_I\,, \end{aligned} \tag{2.2.14}$$

where S^2 is any closed surface near infinity, are conserved.

As is clear from (2.2.13), there is no essential difference between the field strength and its dual. In fact one can mix them by an arbitrary constant matrix and still satisfy the equations of motion and Bianchi identities. One then introduces a matrix $\mathcal{O}$ as

$$\begin{pmatrix}\check{F}^I\\ \check{G}_J\end{pmatrix} = \begin{pmatrix}U^I{}_K & Z^{IL}\\ W_{JK} & V_J{}^L\end{pmatrix}\begin{pmatrix}F^K\\ G_L\end{pmatrix} \quad\Leftrightarrow\quad \check{\mathcal{F}} = \mathcal{O}\mathcal{F}\,. \tag{2.2.15}$$

Here and in the following we will generally use calligraphic capital symbols to denote electric-magnetic pairs of objects, as for F^I and G_I in the last relation. Now, in order for the transformed field strengths to be interpreted as coming from a Lagrangian of the type (2.2.10), the matrix $\mathcal{O}$ must be symplectic, $\mathcal{O} \in Sp(2n_{\rm v}, \mathbb{R})$, implying that

$$\mathcal{O}^T \Omega \mathcal{O} = \Omega \,, \qquad \Omega = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \,, \tag{2.2.16}$$

or in terms of the block matrices:

$$U^T W - W^T U = 0 = Z^T V - V^T Z \,, \qquad U^T V - W^T Z = \mathbb{I} \,. \tag{2.2.17}$$

Such a transformation leaves the equations of motion of the gauge fields invariant, but since (2.2.12) involves the scalars, they must also transform nontrivially so that

$$\check{\mathcal{N}}_{IJ} = [V\mathcal{N} + W]_{IL}[(U + Z\mathcal{N})^{-1}]^L{}_J \,. \tag{2.2.18}$$

The correct rule can be found by pairing the scalars X^I with the derivatives of the prepotential F_I into a symplectic vector

$$\mathcal{V} = \begin{pmatrix} X^I \\ F_I \end{pmatrix} \,, \qquad \check{\mathcal{V}} = \mathcal{O}\mathcal{V} \,, \tag{2.2.19}$$

as they are the only quantities with appropriate indices that scale with unit weight. Then, one can show that this transformation induces (2.2.18), whereas the analog of (2.2.12) follows from (2.2.11) and the homogeneity of the prepotential

$$\mathcal{N}_{IJ} X^J = F_I \,. \tag{2.2.20}$$

As will be discussed in section 5.4, it is possible to write the full action (2.2.10) in a manifestly symplectically covariant way, at least for stationary backgrounds. For the moment, we exhibit the Kähler structure of the scalar manifold in terms of symplectically covariant quantities, referring to [51] for more details.

First, note that the gauge condition (2.2.5) is invariant under symplectic transformations, as it can be recast using the antisymmetric inner product of $Sp(2n_{\rm v}, \mathbb{R})$

$$\mathrm{i}\bar{\mathcal{V}}^T \Omega \mathcal{V} \equiv \mathrm{i}\langle \bar{\mathcal{V}}, \mathcal{V} \rangle = 1 \,, \tag{2.2.21}$$

where we introduced the bracket $\langle \, , \rangle$ as shorthand. The scale and chiral transformations of the scalars imply that $\mathcal{V}$ is a section of a complex line bundle with connection $Q_A = \partial_A K$. The corresponding covariant derivative is given by

$$\mathcal{D}_A \mathcal{V} \equiv \partial_A \mathcal{V} + \tfrac{1}{2}(\mathcal{D}_A K)\mathcal{V} \,, \qquad \mathcal{D}_{\bar{A}} \mathcal{V} \equiv \partial_{\bar{A}} \mathcal{V} - \tfrac{1}{2}(\mathcal{D}_{\bar{A}} K)\mathcal{V} = 0 \,. \tag{2.2.22}$$

These relations also imply the constraints

$$\langle \bar{\mathcal{V}}, \mathcal{D}_A \mathcal{V} \rangle = 0 \,, \qquad \langle \mathcal{D}_A \mathcal{V}, \mathcal{D}_{\bar{B}} \bar{\mathcal{V}} \rangle = \mathrm{i} g_{A\bar{B}} \,, \tag{2.2.23}$$

whereas all inner products of only holomorphic or anti-holomorphic objects vanish as well. It follows that it is possible to use the section $\mathcal{V}$, its derivatives $\mathcal{D}_A\mathcal{V}$ and their complex conjugates as an alternative base for symplectic vectors.

To this end, we introduce two operations realised as symplectic matrices through

$$\diamond \equiv \begin{pmatrix} -\,(\mathrm{Im}\,\mathcal{N})^{-1|IK}\,\mathrm{Re}\,\mathcal{N}_{KJ} & (\mathrm{Im}\,\mathcal{N})^{-1|IJ} \\ -\,\mathrm{Im}\,\mathcal{N}_{IJ} - \mathrm{Re}\,\mathcal{N}_{IK}\,(\mathrm{Im}\,\mathcal{N})^{-1|KL}\,\mathrm{Re}\,\mathcal{N}_{LJ} & \mathrm{Re}\,\mathcal{N}_{IK}\,(\mathrm{Im}\,\mathcal{N})^{-1|KJ} \end{pmatrix}\,, \qquad (2.2.24)$$

$$\mathcal{M} = \Omega\,\diamond\,, \qquad \mathcal{M}^T = \mathcal{M}\,. \qquad (2.2.25)$$

Using these definitions, one can show that

$$\diamond\,\mathcal{V} = -\mathrm{i}\mathcal{V}\,, \qquad \diamond\,\mathcal{D}_A\mathcal{V} = \mathrm{i}\mathcal{D}_A\mathcal{V}\,, \qquad (2.2.26)$$

and

$$\diamond \begin{pmatrix} \star F^I \\ \star G_J \end{pmatrix} = \begin{pmatrix} F^I \\ G_J \end{pmatrix}. \qquad (2.2.27)$$

These relations together with $\diamond^2 = -1$ allow for an interpretation of $\diamond$ as a kind of Hodge duality. If the prepotential is such that the theory can be viewed as a Calabi-Yau compactification of Type IIB supergravity, $\diamond$ is identified as the Hodge operator on the Calabi-Yau manifold.

In the basis defined by (2.2.26), any real symplectic vector can be written as

$$E = 2\,\mathrm{Im}[\,-Z(E)\,\bar{\mathcal{V}} + g^{A\bar{B}}\,\mathcal{D}_A Z(E)\,\bar{\mathcal{D}}_{\bar{B}}\bar{\mathcal{V}}\,]\,, \qquad (2.2.28)$$

where the central charge Z is given by

$$Z(E) = \langle E, \mathcal{V}\rangle = E^I F_I - E_I X^I\,. \qquad (2.2.29)$$

One can further show that the antisymmetric inner product can be alternatively written as

$$\langle E_1, E_2\rangle = 2\,\mathrm{Im}[\,Z(E_1)\,\bar{Z}(E_2) - g^{A\bar{B}}\,\mathcal{D}_A Z(E_1)\,\bar{\mathcal{D}}_{\bar{B}}\bar{Z}(E_2)\,]\,, \qquad (2.2.30)$$

where $g^{A\bar{B}}$ is the inverse matrix of $g_{A\bar{B}}$ and the Kähler covariant derivative on Z follows from (2.2.22).

Finally, we note that the symmetric negative definite matrix $\mathcal{M}$ allows for a symmetric product defined as

$$\|E_1,\, E_2\| \equiv -E_1^T\mathcal{M}E_2 = 2\,\mathrm{Re}[\,Z(E_1)\bar{Z}(E_2) + g^{A\bar{B}}\,\mathcal{D}_A Z(E_1)\,\bar{\mathcal{D}}_{\bar{B}}\bar{Z}(E_2)\,]\,. \qquad (2.2.31)$$

An application that will be of use later is provided by the black hole potential [52]

$$V_{\rm bh} = \tfrac{1}{2}\|\Gamma\|^2 = |Z(\Gamma)|^2 + g^{A\bar{B}}\,\mathcal{D}_A Z(\Gamma)\,\bar{\mathcal{D}}_{\bar{B}}\bar{Z}(\Gamma)\,, \tag{2.2.32}$$

for a given vector of electric and magnetic charges Γ.

2.2.3 BPS solutions in four dimensions

The supersymmetric black hole solutions of $\mathcal{N} = 2$ supergravity have been classified some time ago in [22, 24]. The existence of a Killing spinor implies that the spacetime has a timelike Killing vector and that the three-dimensional base space is flat, so that the metric takes the form

$$ds^2 = -\mathrm{e}^{2U}(dt+\omega)^2 + \mathrm{e}^{-2U}d\vec{x}^2\,, \tag{2.2.33}$$

where e^{2U} and ω are a function and a one-form in three dimensions. The electric and magnetic field strengths are given by

$$\mathcal{F} = 2\,\epsilon_{ijk}\partial_k\mathcal{H}\,dx^i dx^j - 2\,\partial_{[\mu}\left(e^0_{\nu]}\,\mathrm{Re}(\mathrm{e}^{2U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V})\right)\,dx^\mu dx^\nu\,, \tag{2.2.34}$$

where $e^0 = \mathrm{e}^U(dt+\omega_i\,dx^i)$ and $\mathcal{H}$ is a symplectic vector of harmonic functions in $\mathbb{R}^3$. The scale factor and scalar fields are found by solving the so called stabilisation equations

$$2\,\mathrm{Im}\big(\mathrm{e}^{-U-\mathrm{i}\alpha}\mathcal{V}\big) = \mathcal{H}\,, \tag{2.2.35}$$

where the arbitrary phase α is subject to $U(1)$ gauge transformations and is introduced to make the left hand side invariant under chiral $U(1)$ transformations. Finally, the one-form ω can be found by

$$\nabla\times\omega = \langle\nabla\mathcal{H},\mathcal{H}\rangle\,. \tag{2.2.36}$$

All the above results are manifestly covariant under scale and symplectic transformations. Their derivation will be briefly discussed in section 5.4, where the duality invariant formulation of the action will be introduced.

It is worthwhile to list a few useful relations implied by (2.2.35) for future reference [53]. Taking the intersection product with $\mathcal{V}$ and using (2.2.21), one finds

$$\mathrm{e}^{-U+\mathrm{i}\alpha} = \langle\mathcal{H},\mathcal{V}\rangle \equiv Z(\mathcal{H})\,, \tag{2.2.37}$$

so that e^{-U} and α are given by the absolute value and phase of the central charge, respectively. For any single centre solution, where the harmonic functions are defined by the charge vector Γ and some arbitrary constants $\mathcal{H}_\infty$ through

$$\mathcal{H} = \mathcal{H}_\infty + \frac{\Gamma}{r}\,, \tag{2.2.38}$$

the square of Z is naturally related to the entropy formula as follows. Using (2.2.35), one can show that $|Z(\mathcal{H})|^2$ is homogeneous of degree two under rescalings of $\mathcal{H}$, so that the area law gives

$$\mathcal{S} = \frac{1}{4}A = \pi \lim_{r\to 0} r^2 \left|Z(\mathcal{H}_\infty + \Gamma/r)\right|^2 = \pi \left|Z(\Gamma)\right|^2 . \tag{2.2.39}$$

On the other hand, in all known examples the entropy of extremal black holes in $\mathcal{N} = 2$ supergravity is given in terms of a quartic invariant, $I_4(\Gamma)$, which is a function of the charges as

$$\mathcal{S} = \pi \sqrt{I_4(\Gamma)} . \tag{2.2.40}$$

This implies that all scalars must take appropriate charge-dependent values at the horizon, so that $|Z(\Gamma)|^2 = \sqrt{I_4(\Gamma)}$. Extending to the full solution, the scale factor resulting from (2.2.35) is

$$\mathrm{e}^{-2U} = \sqrt{I_4(\mathcal{H})} , \tag{2.2.41}$$

a result that holds even for multi-centre solutions. In the next section, we will show an example of such an invariant (cf. (2.3.12)), in the context of a cubic prepotential.

As in five dimensions, taking the near-horizon limit of BPS solutions leads to a solution preserving full $\mathcal{N} = 2$ supersymmetry. Making use of the observation above (2.2.39), the metric in (2.2.33) reduces to

$$ds^2 = -\frac{r^2}{\sqrt{I_4(\Gamma)}}\,\mathrm{d}t^2 + \sqrt{I_4(\Gamma)}\,\frac{\mathrm{d}r^2}{r^2} + \sqrt{I_4(\Gamma)}\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) . \tag{2.2.42}$$

This is exactly of the form (1.2.44), with equal radii for AdS_2 and S^2, which is indeed fully supersymmetric [54, 55]. The scalars follow in a similar fashion, and are given by the so-called BPS attractor equations

$$2\,\mathrm{Im}\big(\bar{Z}\,\mathcal{V}\big) = \Gamma , \tag{2.2.43}$$

which is analogous to (1.2.43). Note that the scale invariance of $\mathcal{V}$ can be used to eliminate the overall factor $\bar{Z}$ in the left hand side of this equation. The gauge fields are given by (2.2.34) as always.

In the case of supersymmetric attractors as in this section, there is an alternative way to view (2.2.43). Due to extremality, one can study the near-horizon in its own right, without any reference to the interpolating solution. The extra ingredient of full supersymmetry enhancement implies that all scalars are constant and imposes (2.2.34) independently of the values of $\mathcal{V}$. Then, the attractor equations simply express the definition of electric and magnetic charges in (2.2.14). It is important to stress that all the steps prior to the definition of charges follow only from symmetry arguments and a choice of Lagrangian is not necessary. In the next chapter we will show how to extent these results to more general Lagrangians using this method.

2.3 The 4D/5D connection

The discussion in the last two sections makes it clear that there is a close relation between five-dimensional supergravity and four-dimensional $\mathcal{N}=2$ supergravity. In fact, all supergravity theories in five dimensions are equivalent to a four-dimensional theory upon Kaluza-Klein reduction on a circle [56–58]. The opposite is not true, as there are no restrictions on the form of the prepotential F in (2.2.3) other than homogeneity, whereas a five-dimensional theory requires a cubic structure as in (2.1.8). The result for the prepotential in four dimensions is

$$F = -\frac{1}{12}\frac{C_{ABC}X^A X^B X^C}{X^0}\,. \tag{2.3.1}$$

Here and henceforth, we identify all but the first value of the index $I, J, K\ldots$ in four dimensions with its five-dimensional counterpart $A, B, C, \ldots$, as is customary, unless otherwise stated. In this section we comment on the precise dictionary between the two theories and the BPS solutions, both in the Poincaré frame and in the full conformal setting.

The most straightforward way to do this is to reduce the on-shell five-dimensional action (2.1.13) on a circle to arrive at the corresponding one (2.2.10) in four dimensions. We generally follow [57, 59], but use slightly different conventions.

The reduction is implemented through a standard Kaluza-Klein ansatz for the metric

$$ds^2_{(5)} \;=\; e^{2\phi}\, ds^2_{(4)} + e^{-4\phi}\,(d\psi - A^0)^2\,, \tag{2.3.2}$$

where $ds^2_{(5)}$, $ds^2_{(4)}$ are the five- and four-dimensional metrics, ϕ is a scalar and A^0 is the Kaluza-Klein gauge field in four dimensions. Here, the coordinate along the circle, ψ, runs over $0 \leq \psi \leq 4\pi$, fixing the four-dimensional Newton constant: $G_4 = G_5/4\pi$. Similarly, a suitable ansatz for the vectors and scalars is

$$\begin{aligned} W^A &= A^A + C^A\,(d\psi - A^0)\,,\\ \hat{\sigma}^A &= e^{-2\phi}\,\sigma^A\,, \end{aligned} \tag{2.3.3}$$

where the A^A are gauge fields and C^A are scalars in four dimensions.

Note that there is an extra gauge multiplet in four dimensions, represented by the zeroth index, originating from the off-diagonal part of the five-dimensional metric and gravitino. The scalars are complexified in the reduction, with the n_v pseudoscalars C^A coming from the gauge fields paired with the n_v physical scalars σ^A and the Kaluza-Klein scalar ϕ. All relative phases are fixed in this way, up to an unphysical overall phase[6]. A way out

[6] Remember that we refrained from fixing the $U(1)$ invariance of the conformal action in (2.2.10).

of this problem is to work in special coordinates

$$z^A = C^A + \mathrm{i}\hat{\sigma}^A\,, \tag{2.3.4}$$

which can be identified with the definition in (2.2.6).

For our purposes, we will only be interested in five-dimensional solutions that can be written as a time-like fibration over a Gibbons-Hawking base space as in (2.1.17) and (2.1.20). Under this restriction, there are significant simplifications. In particular, the Kaluza-Klein scalar, ϕ, and the four-dimensional metric, are written as:

$$ds^2_{(4)} = -\mathrm{e}^{2U}(dt+\omega)^2 + \mathrm{e}^{-2U}d\vec{x}^2\,, \tag{2.3.5}$$

$$\mathrm{e}^{-4U} = \frac{H^2}{f^2}e^{-4\phi} = H\,f^{-3} - (H\,\omega_5)^2\,. \tag{2.3.6}$$

Moreover, the Kaluza-Klein gauge potential seen in four dimensions is given by the expression:

$$A^0 = \omega_5\,H^2\,\mathrm{e}^{4U}\,(dt+\omega_i\,dx^i) - \chi_i\,dx^i\,, \tag{2.3.7}$$

where the notation in (2.1.26) is used for ω.

A useful application of the above formulae is to connect BPS solutions in five dimensions, given by (2.1.25), to the corresponding four-dimensional ones, given by (2.2.35). First, we define convenient rescaled variables in four dimensions by

$$\tilde{\mathcal{V}} \equiv \big(Y^I\,;\;F_I(Y)\big) = \mathrm{e}^{-U+\mathrm{i}\alpha}\mathcal{V}\,, \qquad \mathrm{e}^{-2U} = \mathrm{i}\langle\bar{\tilde{\mathcal{V}}},\tilde{\mathcal{V}}\rangle\,, \tag{2.3.8}$$

whose imaginary parts are directly given by the magnetic harmonic functions. Following [57, 60], we parametrise the real parts as

$$Y^0 = \tfrac{1}{2}(\phi^0 + i\,H^0) \quad , \qquad Y^A = -\frac{|Y^0|}{\sqrt{H^0}}\,x^A + \frac{Y^0}{H^0}\,H^A\,. \tag{2.3.9}$$

The real quantities x^A are determined by the other half of the stabilisation equations $F_I - \bar{F}_I = \mathrm{i}H_I$, which for the prepotential (2.3.1) read

$$\frac{1}{4}C_{ABC}x^Bx^C = H_A + \frac{1}{4\,H^0}\,C_{ABC}H^BH^C \equiv \Delta_A\,, \tag{2.3.10}$$

$$\mathrm{e}^{-2U}\,\phi^0 = H^0\,(H^IH_I) + \frac{1}{6}C_{ABC}H^AH^BH^C\,. \tag{2.3.11}$$

Here, we used the second relation in (2.3.8), which takes the form

$$\mathrm{e}^{-4U} = \tfrac{4}{9}H^0\,(x^A\Delta_A)^2 - (H^0)^{-2}\Big(H^0\,(H^IH_I) + \tfrac{1}{3}C_{ABC}H^AH^BH^C\Big)^2\,. \tag{2.3.12}$$

Using these formulae it is easy to find the connection with the corresponding five-dimensional equations (2.1.25)-(2.1.28). First, (2.3.10) is identical to (2.1.25), leading to

$$H^0 \equiv H\,, \quad H^A \equiv K^A\,, \quad H_A \equiv L_A\,, \tag{2.3.13}$$
$$x^A = f^{-1/2}\sigma^A\,, \qquad f^{-3/2} = \tfrac{1}{3}\,x^A\Delta_A\,. \tag{2.3.14}$$

Moreover, the five-dimensional rotation one-form is related to the electric charge associated to A^0 and the four-dimensional angular momentum, as expected. Its fifth component in (2.1.27) is part of the scalar in the Kaluza-Klein multiplet, as

$$\mathrm{e}^{-2U}\,\phi^0 = H^2\,\omega_5\,, \qquad M \equiv H_0\,. \tag{2.3.15}$$

whereas (2.2.36) is equal to the one in (2.1.28), using the identifications of the various harmonic functions given above. Finally, the physical scalars in four dimensions, as defined in (2.3.4), are given by

$$z^A = -\frac{\phi^0}{2|Y^0|}\frac{x^A}{\sqrt{H^0}} + \frac{H^A}{H^0} + \mathrm{i}\frac{\sqrt{H^0}}{2|Y^0|}\,x^A\,. \tag{2.3.16}$$
$$= -\frac{f}{H}\left(H\,\omega_5 - \mathrm{i}\,\mathrm{e}^{-2U}\right)\sigma^A + \frac{K^A}{H}\,, \tag{2.3.17}$$

where in the second line we display the expression in terms of five-dimensional quantities, for future reference. This can be shown to be consistent with (2.3.4) using (2.1.24) and (2.3.5).

In contrast to the above, finding the explicit relation between the charges of a solution to the five-dimensional theory and its four-dimensional reduction can be subtle, especially in the presence of magnetic charges. An example of this is given by the shifted poles in the electric harmonic functions in (2.1.36) and (2.1.37) compared to (2.2.38), despite the identification (2.3.13). For this reason, we refrain to give any explicit general formulae, using the appropriate definitions in each case.

It is worth mentioning that the reduction can be reformulated so as to keep four-dimensional conformal invariance manifest. In this setting, the metric ansatz can be taken to be

$$ds^2_{(5)} \;=\; ds^2_{(4)} + \phi^{-2}\,(d\psi - A^0)^2\,, \tag{2.3.18}$$

so that the four and five-dimensional metrics have the same Weyl weight, whereas the scalar ϕ has $w = 1$. Therefore, the scalars in four dimensions (2.3.4) can be viewed as special coordinates parametrising the scalars

$$X^A = \mathrm{e}^{\mathrm{i}\varphi}(\sigma^A + \mathrm{i}\,\phi\,C^A)\,, \qquad X^0 = \mathrm{e}^{\mathrm{i}\varphi}\phi\,, \tag{2.3.19}$$

where $e^{i\varphi}$ is an arbitrary phase subject to four-dimensional $U(1)$ transformations. The corresponding gauge field arises from the off-diagonal components of the tensor T_{ab} as

$$T_{a5} \propto e_a{}^{\mu}(A_\mu - \partial_\mu \varphi)\,. \tag{2.3.20}$$

We refrain from setting up such a correspondence in detail, pointing out that there is no difference with the on-shell procedure described above in the context of the two derivative action, but the conformal approach is the only practical way to dimensionally reduce a higher derivative action. The subtleties in the definition of conserved quantities also apply in this case, especially in the presence of nontrivial topology in five dimensions, similar to the reduction of magnetically changed solutions discussed above.

In the following, we will use the above formulae both explicitly and implicitly, to give a four-dimensional interpretation to five-dimensional results. More explicit details can be found in [59].

2.4 Microscopic considerations

A major motivation for the study of black hole solutions in $\mathcal{N} = 2$ supergravity is the possibility of identifying a precise microscopic model of these objects within string theory. For systems that preserve some supersymmetry, this has been worked out in detail in several cases, both in four and five dimensions. Partial results also exist for non-supersymmetric extremal and near extremal black holes in extended supergravity. In the context of string theory, black holes arise as a system of D-branes wrapped on internal manifolds. Here we give a general discussion of the ideas involved in the constructions describing BPS black holes, based on the by now classic reviews [61] and [62], as well on [63], to which we refer for more details and further references.

The starting point for modeling black holes is a system of intersecting D-branes wrapped on some compact internal manifold in flat transverse space, carrying a set of p-form charges. The reason that more than one kind of branes is needed is purely technical, imposed by the requirements of having everywhere regular scalars and a nonzero horizon area of the final black hole. The kinds of D-branes and intersections are not arbitrary either and are chosen in such a way that the resulting state preserves some of the initial supersymmetry, which means that it is a BPS state. Then, one considers a large number of individual branes, which nevertheless are noninteracting due to supersymmetry.

Just as one would do for a classical gas, a counting of the free modes is performed, which corresponds to a counting of the degeneracy of the system. In practise, the fluctuations of a D-brane correspond to a gas of open strings ending on it so that the low energy degrees of freedom are described by (classically) massless open strings stretching between the D-branes. In the presence of more than one D-brane or, even better, different kinds of D-branes , the number of possibilities for the positions of the open string end-points is dramatically increased. In the thermodynamic limit (equivalent to a large charge limit), the degeneracy is essentially the exponent of the entropy, which can indeed be arranged to be of the order of magnitude as found in classical General Relativity. Note that such a limit is necessary in any comparison, as the macroscopic supergravity theory is only valid in this limit and so is the entropy that is derived by it.

All the above correspond to a small string coupling description of the system and are far from the familiar spacetime picture of say, the Schwarzschild black hole. This description is valid when $g_s N \ll 1$, where g_s is the string coupling and N is the number of D-branes (or the typical scale of that number for the case of different kinds of branes).

The black hole description in terms of a curved spacetime is found by letting the coupling become strong: $1 \ll g_s N < N$, where the string coupling is still kept small ($g_s < 1$) to avoid large string loop corrections. At strong coupling the flat space description is no longer relevant. Instead, one is dealing with a curved gravitational background carrying the same set of charges as the higher dimensional D-branes. The above mentioned microscopic entropy is then associated to this black hole. The extrapolation of the degeneracy between the two regimes is possible because the number of microscopic states in a supersymmetric configuration is given by a coupling-independent object, known as an index. Therefore, calculating the index at one value of the string coupling is enough to know its value at any other coupling, as long as a number of conditions is met[7]. These ideas where pioneered in [10], followed by more general constructions in [39, 67, 68]. Here, we merely record the nature of the microscopic description of the D-brane systems in weak coupling for the cases considered in the next chapters without going in any detail. Instead, the focus will be on the supergravity description of the system in strong coupling.

2.4.1 Supergravity vs D-brane charges

The origin of the charges carried by a string theoretic black hole can be traced to the charges carried by the D-branes used to model it. Any supersymmetric Dp-brane is charged under a corresponding p-form gauge field present in the spectrum of perturbative string theory. Upon compactification on an internal manifold, the p-form fields reduce to a number of gauge fields, whereas the corresponding Dp-branes wrapped on cycles of the same manifold are charged under the resulting gauge fields.

One of the requirements on the final solution is that of independence on the internal coordinates. In order to satisfy it, the position of each brane has to be 'smeared', meaning that one should integrate over the possible position of every brane over the volume of the part of the internal manifold it is not wrapped on. This can be done in the higher

[7]The index can jump along surfaces in the moduli space, but we disregard this possibility in this discussion. See [64–66] for a recent treatment.

dimensional setting by constructing a periodic lattice of branes along the directions that are to be compactified and then take the lattice spacing to zero. For example, the electric field of a two-form gauge field created by a collection of Q_1 parallel $D1$-branes is described by a harmonic function in the transverse $\mathbb{R}^8$, given by $H_1 = 1 + Q_1/r^6$. When the same $D1$-branes are wrapped over a cycle of a five-dimensional torus, T^5, one has to consider a modified harmonic function

$$H_1 = 1 + \frac{1}{V_{T^4}} \int \frac{Q_1}{r^6} \quad \Rightarrow \quad H_1 = 1 + \frac{1}{32\pi^2} \frac{Q_1}{r^2}. \tag{2.4.1}$$

The integral is over all of T^4 and expresses the zero spacing limit of the lattice. The general rule is that the higher dimensional harmonic functions for the branes reduce to harmonic functions over the remaining noncompact transverse directions. This is explicit in the above example, since $1/r^2$ is a solution to the five-dimensional Laplace equation.

As is clear from the discussion in sections 2.1.2 and 2.2.3, the harmonic functions in the lower dimensional theory define the charges of a solution. Such a correspondence can be made more precise in the context of a specific theory, such as Type IIB strings compactified on Calabi-Yau manifolds. We choose this theory for convenience, but $\mathcal{N} = 2$ compactifications of other theories can be related to this by dualities and/or mirror symmetry. In this case, the lower dimensional gauge fields are viewed as reductions of the ten-dimensional five-form field strength, so that their charges arise from the RR charges of the original $D3$-branes. More concretely, dimensional reduction of any string theory on a compact n-dimensional manifold M_n involves an expansion of the p-form fields on the associated cohomology elements of that manifold. In our example, the five-form field strength $\mathcal{F}$ of the IIB theory is assumed to take values in $\Omega^2(M_4) \otimes H^3(M_{\mathrm{CY}}, \mathbb{Z})$, where $\Omega^2(M_4)$ represents the space of two-forms on four-dimensional spacetime. Therefore, it can be written as

$$\mathcal{F} = F^I \otimes \alpha_I - G_I \otimes \beta^I\,, \tag{2.4.2}$$

where $\{\alpha_I, \beta^J\}$ is the canonical symplectic basis for the third integral cohomology $H^3(M_{\mathrm{CY}}, \mathbb{Z})$. Here, F^I are the field strengths of the four-dimensional theory and the G_I can be shown to be equal to the dual field strengths in (2.2.12), so that the component expansion in (2.4.2) can be identified with $\mathcal{F}$ in (2.2.15). By integrating the five-form field strength over an appropriate two-sphere in space as in (2.2.14), we recover the charges as the coefficients of the three-form $\Gamma \in H^3(M_{\mathrm{CY}}, \mathbb{Z})$

$$\Gamma = \frac{1}{4\pi} \int_{S^2} \mathcal{F} = p^I \alpha_I - q_I \beta^I\,, \tag{2.4.3}$$

so that in terms of the Poincaré dual homology base of three-cycles $\{A^I, B_J\}$, the charges are

$$p^I = \int_{A^I} \Gamma \equiv \int_{M_{\mathrm{CY}}} \Gamma \wedge \beta^I\,, \qquad q_I = \int_{B_I} \Gamma \equiv \int_{M_{\mathrm{CY}}} \Gamma \wedge \alpha_I\,. \tag{2.4.4}$$

Intuitively, one can think of this correspondence as an extension of the S^2 in (2.4.3) to a surface that incorporates the basis elements $\{A^I, B_J\}$ as in

$$p^I = \int_{S^2 \cup A^I} \mathcal{F} \,, \qquad q_I = \int_{S^2 \cup B_I} \mathcal{F} \,. \tag{2.4.5}$$

Using Poincaré duality between cycles and cohomology classes one can relate this to brane sources, as there is a one to one correspondence between m-cycles and $n-m$-degree cohomology classes for a compact manifold. Thus, for any $\alpha_I\,, \beta^I$ above, there is a cycle that lies in directions transverse to that form's support and is contained in the interior of $S^2 \cup \{A^I, B_J\}$. By using this, the sum of the equations in (2.4.5) expresses the standard definition of charges for $\mathcal{F}$ in ten dimensions, with sources along the Poincaré dual cycles of the basis forms.

Similar reasoning applies in Type IIA and M-theory compactifications, with the further complication that the lower dimensional gauge fields do not arise from a unique RR form. It follows that the electric and magnetic charges arise from homology elements of different dimension in the compactification manifold, that must nevertheless add to its total dimension. Assuming that $\mathcal{F}$ is now a sum of forms of different order and that the corresponding integrals are performed over appropriate cycles, all the above formulae go through for the more general cases as well.

For example, the microscopics of four-dimensional black holes are more easily modeled in Type IIA string theory or M-theory. In Type IIA the only D-branes available are even dimensional. When wrapped on the cycles of a Calabi-Yau manifold, there are two possible pairs of electric and magnetic branes according to the rules sketched: a $D0$-$D6$ and a $D2$-$D4$ pair. In fact, both must be used in order to get a BPS black hole in four dimensions with a horizon area of macroscopic size. This can be seen heuristically from the fact that the $D0$ and $D2$ branes are worldvolume instantons of the $D4$ and $D6$ branes respectively, so that including one dual pair would generically induce (some combination of) charges of the other pair as well. For example, a system of $D2$-$D4$ branes will in general include $D0$ branes as worldvolume instantons on the $D4$ branes. This is a consistent reduction of the general $D6$-$D2$-$D4$-$D0$ system.

A more general and useful picture is the M-theory lift of this system. Then, the $D2$-$D4$ pair is viewed as a $M2$-$M5$ brane pair with the $M5$ brane wrapped on the M-circle, whereas the $D0$-$D6$ pair is now seen as electric and magnetic Kaluza Klein charges coming from the nondiagonal elements of the eleven-dimensional metric along the M-circle. This is the most general configuration used in the microscopic description of all kinds of 'black object' solutions for Calabi-Yau compactifications of M-theory, like black rings and supertubes [26]. The entropy for the four-dimensional black hole has not yet been accounted for microscopically in the general setting, but only for zero magnetic Kaluza Klein (or $D6$) charge [67] and this will be the case considered here as well.

2.4.2 Wrapped $M5$ branes and black holes in four dimensions

The microscopic description of four-dimensional black holes [67] is based on the assumption of a $M5$ brane wrapping the M-circle and the 4-cycles of the Calabi-Yau threefold and $M2$ branes wrapping orthogonal 2-cycles. Adding momentum on the effective string along the M-circle induces an extra Kaluza-Klein charge. This configuration preserves 1/8 of the original supersymmetries as is appropriate to describe a solitonic object in the resulting $\mathcal{N} = 2$ supergravity in the four noncompact dimensions. The charges arise in the standard way as wrapping numbers for $M5$ and $M2$ branes as already described. We therefore have as many electric charges q_I as 2-cycles on the CY and as many magnetic charges p^I as 4-cycles, which are both equal to the $(1,1)$ Hodge number of the Calabi-Yau due to Poincaré duality. To these, one must add the Kaluza-Klein electric charge q_0 from the compactification on the circle (the magnetic one is set to zero). The low energy modes are governed by a $(0,4)$ supersymmetric two-dimensional theory on the effective string worldvolume, which flows to a CFT in the low energy limit.

The microscopic entropy then arises as the logarithm of the number of states that preserve the right-moving supersymmetry and have arbitrary excitations on the left moving side. This constrains the momentum along the circle in eleven dimensions to be left-moving, or $q_0 > 0$. The final result of the degeneracy counting for this system is the entropy

$$\mathcal{S} = 2\pi\sqrt{\frac{\hat{q}_0}{6}(\mathcal{C}_{ABC}p^A p^B p^C + c_{2\,A}p^A)}, \qquad \hat{q}_0 = q_0 + \frac{1}{12}D^{AB}q_A q_B \tag{2.4.6}$$

where D^{AB} is the inverse of $\mathcal{C}_{ABC}p^K$ and $\mathcal{C}_{ABC}$ are the intersection numbers of the Calabi-Yau manifold, defined through the basis elements a_A of $H^2(CY)$ by

$$\mathcal{C}_{ABC} = \int_{CY} a_A \wedge a_B \wedge a_C. \tag{2.4.7}$$

Finally, $c_{2\,A}$ is the second Chern class of the tangent bundle on the Calabi Yau manifold, expanded on a basis of $H^4(CY)$ that is Poincaré dual to the basis of two cycles used above. Without these subleading corrections, results for other than Calabi-Yau compactifications have been obtained in [69].

On the macroscopic side, this system will correspond to a four-dimensional BPS black hole with all charges except p^0 turned on. The two derivative solution follows from the results shown in section 2.2.3, which correctly reproduces the leading microscopic entropy (i.e. for $c_{2\,A} = 0$). A match with the full microscopic result requires the inclusion of higher derivative corrections in the supergravity action. This was worked out in a series of papers [54, 70–72] from a purely four-dimensional supergravity point of view and will be reproduced in the next chapter based on the near horizon region.

2.4.3 Black rings in five dimensions

As the subject of black holes started in the physical case of four dimensions, extensive research has led to the uniqueness theorems for black holes in that number of dimensions [73]. Under some rather general physical assumptions, these theorems show that the only relevant topology of a black hole horizon in four dimensions is that of $R \times S^2$. This however is not the case in higher dimensions, where more 'exotic' objects can be constructed. In five dimensions, one encounters the black ring solutions [74], with horizon topology that of $R \times S^1 \times S^2$. Thus, when dealing with more than four dimensions, one does not have a unique black hole corresponding to a set of charges, but a finite number of black objects with different horizon topologies. From a microscopic standpoint one can think of these situations as different 'phases' of the same system.

Here, we will consider a version of the black ring embedded in five-dimensional supergravity [40, 75]. This solution can be viewed from an M-theory perspective as the result of a set of $M2$ branes wrapping 2-cycles and $M5$ branes wrapping 4-cycles of a Calabi-Yau manifold describing six of the eleven dimensions. Note that the $M5$ branes are not completely wrapped in the internal directions, giving rise to a remaining magnetically charged string-like object in five-dimensional spacetime. This is thought of as forming a closed loop that carries angular momentum, and supports a ring-like horizon. This description is exactly the same as for the four-dimensional black hole and, not surprisingly, the two solutions are related upon dimensional reduction along the ring.

However, an important subtlety should be taken into account, namely that in the present case the $M5$ branes are contractible in the five noncompact dimensions. This implies that there are really no conserved magnetic charges associated to them from a purely five-dimensional point of view. In fact, a macroscopic observer in five dimensions would see a closed electrically charged loop with a current flowing around it, giving rise to a magnetic dipole moment.

Despite this, the dipole moments appear in the entropy, which is very closely related to the four-dimensional one. The microscopic counting [26, 76–79] gives the result:

$$\mathcal{S} = 2\pi\sqrt{\frac{q_0'}{6}\,(\mathcal{C}_{ABC}p^A p^B p^C + c_{2\,A}p^A)}\,, \tag{2.4.8}$$

which is of the same form as (2.4.6), but the q_0' is now slightly different

$$q_0' = -J_\psi + \frac{1}{12}D^{AB}q_A q_B + \frac{1}{24}\,\mathcal{C}_{ABC}p^A p^B p^C\,. \tag{2.4.9}$$

Again, the five-dimensional supergravity solution corresponding to this system has all electric and magnetic charges turned on, as well as angular momentum. The explicit two derivative solution can be found using the results presented in section 2.1.2 with $H = 1/r$, so that the Taub-NUT charge is trivial. The resulting Bekenstein-Hawking entropy of the solution given in (2.1.41) agrees with (2.4.8) for $c_{2\,A} = 0$. The corresponding supergravity result with nonzero $c_{2\,A}$ that control the higher order corrections, will be considered in section 4.5.

2.4.4 Five-dimensional BPS black holes

A further restricted five-dimensional system is that of $M2$-branes wrapping 2-cycles in a Calabi-Yau manifold. Upon reduction to four dimensions, this is equivalent to a black hole arising from $D2$-branes wrapped on the same manifold and is U-dual to the system in section 2.4.2. However, in the five-dimensional theory, electric/magnetic duality does not exist and this is system is a priori distinct from the black ring of the previous section.

The presence of the $M2$-branes gives rise to electric charges as usual, whereas the space-time angular momentum is described by the genus of the 2-cycle they wrap in the Calabi-Yau[8]. The microscopic theory was developed in [10, 39, 68], resulting in the lowest order entropy formula

$$\mathcal{S} = 2\pi \sqrt{p^0 Q^3 - \frac{1}{4}(p^0 J_\psi)^2}\,, \tag{2.4.10}$$

where the quantities Q and q^A are defined in terms of the electric charges q_A by

$$Q^{3/2} = \tfrac{1}{6}\,\mathcal{C}_{ABC}q^A q^B q^C \qquad q_A = \tfrac{1}{2}\,\mathcal{C}_{ABC}q^B q^C \tag{2.4.11}$$

which reproduces the macroscopic prediction in (2.1.33).

The effect of higher order corrections was investigated in [68, 80], but a closed analytic formula for the entropy is only known for vanishing angular momentum. In that case, the correction to the two derivative result is effected by modifying the definition of the q^A in (2.4.11) to

$$q_A = \tfrac{1}{2}\,\mathcal{C}_{ABC}q^B q^C - \tfrac{1}{8}\,c_{2\,A}\,, \tag{2.4.12}$$

keeping all other relations the same. The comparison with the corresponding result in supergravity including higher derivative corrections is the subject of section 4.4.

[8]This is also true in the four-dimensional theory, with the difference that angular momentum is not an independent charge in that case, but is completely fixed in terms of the charges.

Chapter 3

BPS Attractors in four dimensional supergravity

The attractor phenomenon for BPS black branes [41–43] is caused by full supersymmetry enhancement at the horizon, which induces stringent restrictions on the values of the fields and the space-time geometry. When supersymmetry is realized off shell, the resulting attractor equations can be analyzed in a way that is independent of the action. In this way universal results can be obtained even when the action contains higher-derivative couplings, as was first demonstrated for $N = 2$ supergravity in four dimensions [71].

In the previous chapter we introduced the standard methods to construct supergravity Lagrangians in theories with eight supercharges, restricting ourselves to actions containing at most two derivatives of the various fields. Using the same techniques, one can go beyond the two derivative level to construct invariants containing an arbitrary number of derivatives. Effective theories describing the low energy limit of $\mathcal{N} = 2$ string theory compactifications naturally include such invariants as corrections to the lowest order Lagrangian, arising either as loop or worldsheet corrections. In this chapter, we will see how to set up a detailed comparison between the microscopic descriptions of black holes and supergravity solutions of these effective theories in four dimensions.

We start by introducing two independent higher derivative invariants, containing up to four derivatives, characterised by terms quadratic in the Riemann curvature and quartic in gauge field strengths respectively. In this context conformal methods are indispensable, as supersymmetry is realised without using equations of motion on the conformal multiplets, irrespective of a Lagrangian. Compared to the Poincaré theory, where one must construct the action and the transformation rules simultaneously order by order in a derivative expansion, the simplification is considerable.

When considering such higher derivative actions, it is cumbersome, if not impractical, to gauge fix to the Poincaré theory, since the action contains kinetic terms for nominally auxiliary fields, such as T_{ab}. The standard lore is to consider such an operation order by order in the parameters of the underlying fundamental theory that organise the corrections. When the higher invariants are viewed as string theory corrections, they are controlled by appropriate powers of the string tension α' or string coupling g_s, which are the appropriate expansion parameters. We will not implement such a procedure when dealing with higher derivative actions, but we will rather stay in the full superconformal setting.

The analysis of supersymmetric attractors is presented in section 3.2. Following [54] we derive general model-independent results for the entropy and scalars. In particular, we discuss the non-renormalisation theorem of [81], according to which the invariant quartic in the field strengths is irrelevant for BPS attractors. As the remaining higher order invariant contains interactions quadratic in the Riemann tensor, it is necessary to use the Wald entropy [82] rather than the area law. The interested reader can find a concise review of this formalism in Appendix C.

Finally, we specialise to models describing Type II string theory compactifications on Calabi-Yau manifolds, and compare with microscopic predictions. In view of the restriction on the charges in the result of section 2.4.2, this cannot be done in the most general case. Imposing the same restriction on the supergravity results, we find precise agreement with the microscopic counting.

3.1 Higher derivative invariants in four dimensions

In section 2.2.1 we saw how to construct the two derivative Lagrangian by combining a set of vector into a composite unconstrained chiral multiplet. A moment's thought shows that this is a rather restricted choice. In fact, any kind of chiral multiplets, even if they are composite (constrained or not), can be combined to give other chiral multiplets. Therefore, one could construct other couplings from any chiral multiplet derived from the fundamental multiplets of interest. In this section, we give a brief account of the ideas behind the construction of such composites and the resulting Lagrangians.

A first choice would be to use the chiral multiplet constructed from the Weyl multiplet in (B.2.22). As there are no other chiral anti-selfdual multiplets to couple it to, we consider its square (B.2.24), which can be straightforwardly included in the prepotential of section 2.2.1, as $F(X, \hat{A})$. Since $\hat{A}$ has $w = 2$ and F must still have weight two, the relation

$$X^I F_I + 2\hat{A}\, F_A = 2F\,, \tag{3.1.1}$$

is imposed, where F_A denotes the derivative of $F(X, \hat{A})$ with respect to $\hat{A}$. Once this modification is introduced, the extended action follows from the multiplication rule

(B.2.18). The result reads

$$
\begin{aligned}
8\pi\, e^{-1}\,\mathcal{L} \;=\;& \mathrm{i}\mathcal{D}^\mu F_I\,\mathcal{D}_\mu \bar X^I - \mathrm{i}F_I\,\bar X^I(\tfrac{1}{6}\mathcal{R}-D) - \tfrac{1}{8}\mathrm{i}F_{IJ}\,Y^I_{ij}Y^{Jij} - \tfrac{1}{4}\mathrm{i}\hat B_{ij}\,F_{AI}Y^{Iij}\\
&+\tfrac{1}{4}\mathrm{i}F_{IJ}(F^{-I}_{ab} - \tfrac{1}{4}\bar X^I T^{ij}_{ab}\varepsilon_{ij})(F^{-Jab} - \tfrac{1}{4}\bar X^J T^{ijab}\varepsilon_{ij})\\
&-\tfrac{1}{8}\mathrm{i}F_I(F^{+I}_{ab} - \tfrac{1}{4}X^I T_{abij}\varepsilon^{ij})T^{ab}_{ij}\varepsilon^{ij} + \tfrac{1}{2}\mathrm{i}\hat G^{-ab}\,F_{AI}(F^{-I}_{ab} - \tfrac{1}{4}\bar X^I T^{ij}_{ab}\varepsilon_{ij})\\
&+\tfrac{1}{2}\mathrm{i}F_A\hat C - \tfrac{1}{8}\mathrm{i}F_{AA}(\varepsilon^{ik}\varepsilon^{jl}\hat B_{ij}\hat B_{kl} - 2\hat G^-_{ab}\hat G^{-ab}) - \tfrac{1}{32}\mathrm{i}\,F(T_{abij}\varepsilon^{ij})^2 + \text{h.c.}\\
&-\tfrac{1}{2}\varepsilon^{ij}\,\bar\Omega_{\alpha\beta}\,\mathcal{D}_\mu A_i{}^\alpha\,\mathcal{D}^\mu A_j{}^\beta + \chi(\tfrac{1}{6}\mathcal{R} + \tfrac{1}{2}D)\,,
\end{aligned}
\tag{3.1.2}
$$

where the components of (B.2.24) are still to be substituted to find the fully explicit result.

Since we used a holomorphic prepotential in the chiral density formula (2.2.1), this action is similar to (2.2.3) in many ways, with the Riemann tensor, appearing through $\hat G^-_{ab}$, in the role of the gauge field strengths and couplings that depend on the vector multiplet scalars. Therefore it expresses the supersymmetric completion of an R^2-type interaction, in the same sense that the lowest order action is the $N = 2$ completion of the Maxwell term. It is easy to verify that (3.1.2) contains up to four derivatives on the fields.

There are a priori more four derivative interactions involving the Riemann tensor and the gauge fields, namely $R\,F^2$ and F^4-type interactions. A supersymmetric action containing an $R\,F^2$ term[1] would have to be constructed from the Weyl multiplet (B.2.22) directly and not its square. This seems highly unlikely to be possible in a Lorentz covariant theory. On the other hand, it was shown in [83] that a F^4-type supersymmetric Lagrangian can be constructed in the rigid limit, if one includes chiral and anti-chiral multiplets in a symmetric way.

Such invariants can be also constructed in the presence of a Weyl background, using yet another composite chiral multiplet, the so called kinetic chiral multiplet. The term 'kinetic' multiplet was first used in the context of the $N = 1$ tensor calculus [84], because this is the chiral multiplet that enables the construction of the kinetic terms, conventionally described by a real superspace integral, in terms of a chiral superspace integral. In flat $N = 1$ superspace, this construction is simply effected by the conversion,

$$
\int \mathrm{d}^2\theta\;\mathrm{d}^2\bar\theta\;\Phi\,\bar\Phi' \approx \int \mathrm{d}^2\theta\;\Phi\,\mathbb{T}(\bar\Phi')\,,
\tag{3.1.3}
$$

up to space-time boundary terms. Here Φ and Φ' are two chiral superfields and $\bar\Phi'$ is the anti-chiral field obtained from Φ' by complex conjugation. The kinetic multiplet equals $\mathbb{T}(\bar\Phi') = \bar D^2\bar\Phi'$, where $\bar D$ denotes the supercovariant $\bar\theta$-derivative. Obviously the kinetic multiplet contains terms linear and quadratic in space-time derivatives, so that, upon

[1] The invariants discussed in this chapter do include individual terms proportional to the Riemann tensor and quadratic in gauge field strengths. Here we are referring to the possibility of an invariant purely quadratic in the vector multiplets.

identifying Φ and Φ', the right-hand side of (3.1.3) does indeed give rise to the kinetic terms of an $N=1$ chiral multiplet.

In [46] a corresponding kinetic multiplet was identified for $N=2$ supersymmetry, which now involves four rather than two covariant $\bar\theta$-derivatives, i.e. $\mathbb{T}(\bar\Phi)\propto \bar D^4\bar\Phi$. As a result, $\mathbb{T}(\bar\Phi)$ contains up to four space-time derivatives, so that the expression

$$\int \mathrm{d}^4\theta\;\mathrm{d}^4\bar\theta\;\Phi\,\bar\Phi' \approx \int \mathrm{d}^4\theta\,\Phi\,\mathbb{T}(\bar\Phi')\,, \tag{3.1.4}$$

does not correspond to a kinetic term, but to a higher-order derivative coupling. Expressing the chiral multiplets in terms of (products of) reduced chiral multiplets, invariants such as (3.1.4) will correspond to higher-derivative couplings of vector multiplets. In the presence of a conformal supergravity background, the Weyl weight of the kinetic multiplets is relevant. Both in $N=1,2$ supergravity the kinetic multiplet carries Weyl weight $w=2$. The conversion starts from a $w=1$ chiral multiplet for $N=1$ and from a $w=0$ chiral multiplet for $N=2$ supersymmetry, respectively.

To demonstrate this in more detail, consider an anti-chiral $N=2$ supermultiplet in the presence of the superconformal background. Its supersymmetry transformations follow from taking the complex conjugate of (B.2.16). Precisely for $w=0$ we note that the field $\bar C$ is invariant under S-supersymmetry and transforms under Q-supersymmetry as the lowest component of a chiral supermultiplet with $w=2$. This observation proves that we are dealing with a $w=2$ chiral supermultiplet, as is also confirmed by the weight assignments specified in table B.5. What remains is to identify the various components of this multiplet in terms of the underlying $w=0$ multiplet. This can be done by applying successive Q-supersymmetry transformations on $\bar C$, something that requires rather tedious calculations in the presence of a superconformal background.

Denote the components of a $w=0$ chiral multiplet by (A,Ψ,B,G^-,Λ,C), out of which we construct the components of $\mathbb{T}(\bar\Phi_{w=0})$, denoted by $(A,\Psi,B,G^-,\Lambda,C)|_{\mathbb{T}(\bar\Phi)}$. In [81] the following relation was established,

$$\begin{aligned}
A|_{\mathbb{T}(\bar\Phi)} =&\,\bar C\,,\\
\Psi_i|_{\mathbb{T}(\bar\Phi)} =&\,-2\,\varepsilon_{ij}\not{D}\Lambda^j-6\;\varepsilon_{ik}\varepsilon_{jl}\chi^j B^{kl}-\tfrac14\varepsilon_{ij}\varepsilon_{kl}\,\gamma^{ab}T_{ab}{}^{jk}\overset{\leftrightarrow}{\not{D}}\,\Psi^l\,,\\
B_{ij}|_{\mathbb{T}(\bar\Phi)} =&\,-2\,\varepsilon_{ik}\varepsilon_{jl}\big(\Box_{\mathrm{c}}+3\,D\big)B^{kl}-2\,G^+_{ab}\,R(\mathcal{V})^{ab\,k}{}_i\,\varepsilon_{jk}\\
&-6\,\varepsilon_{k(i}\,\bar\chi_{j)}\Lambda^k+3\,\varepsilon_{ik}\varepsilon_{jl}\bar\Psi^{(k}\not{D}\chi^{l)}\,,\\
G^-_{ab}|_{\mathbb{T}(\bar\Phi)} =&\,-\big(\delta_a{}^{[c}\delta_b{}^{d]}-\tfrac12\varepsilon_{ab}{}^{cd}\big)\\
&\quad\times\big[4\,D_cD^eG^+_{ed}+(D^e\bar A\,D_cT_{de}{}^{ij}+D_c\bar A\,D^eT_{ed}{}^{ij})\varepsilon_{ij}\big]\\
&+\Box_{\mathrm{c}}\bar A\,T_{ab}{}^{ij}\varepsilon_{ij}-R(\mathcal{V})^-{}_{ab}{}^i{}_k\,B^{jk}\,\varepsilon_{ij}+\tfrac18T_{ab}{}^{ij}\,T_{cdij}G^{+cd}-\varepsilon_{kl}\,\bar\Psi^k\overset{\leftrightarrow}{\not{D}}\,R(Q)_{ab}{}^l\\
&-\tfrac94\varepsilon_{ij}\,\bar\Psi^i\gamma^c\gamma_{ab}D_c\chi^j+3\,\varepsilon_{ij}\bar\chi^i\gamma_{ab}\not{D}\Psi^j+\tfrac38T_{ab}{}^{ij}\varepsilon_{ij}\,\bar\chi_k\Psi^k\,,
\end{aligned}$$

$$
\begin{aligned}
\Lambda_i|_{\mathbb{T}(\bar\Phi)} = {} & 2\,\Box_{\rm c}\not{D}\Psi^j\varepsilon_{ij} + \tfrac14\gamma^c\gamma_{ab}(2\,D_cT^{ab}{}_{ij}\,\Lambda^j + T^{ab}{}_{ij}\,D_c\Lambda^j) \\
& - \tfrac12\varepsilon_{ij}\big(R(\mathcal{V})_{ab}{}^j{}_k + 2\mathrm{i}\,R(A)_{ab}\delta^j{}_k\big)\,\gamma^c\gamma^{ab}D_c\Psi^k \\
& + \tfrac12\,\varepsilon_{ij}\big(3\,D_bD - 4\mathrm{i}D^aR(A)_{ab} + \tfrac14 T_{bc}{}^{ij}\overset{\leftrightarrow}{D}_a T^{ac}{}_{ij}\big)\,\gamma^b\Psi^j \\
& - 2\,G^{+ab}\not{D}R(Q)_{abi} + 6\,\varepsilon_{ij}D\not{D}\Psi^j \\
& + 3\,\varepsilon_{ij}\,\big(\not{D}\chi_k\,B^{kj} + \not{D}\bar{A}\not{D}\chi^j\big) \\
& + \tfrac32\big(2\not{D}B^{kj}\varepsilon_{ij} + \not{D}G^+_{ab}\gamma^{ab}\,\delta^k_i + \tfrac14\varepsilon_{mn}T_{ab}{}^{mn}\,\gamma^{ab}\not{D}\bar{A}\,\delta_i{}^k\big)\chi_k \\
& + \tfrac94\,(\bar\chi^l\gamma_a\chi_l)\,\varepsilon_{ij}\gamma^a\Psi^j - \tfrac92\,(\bar\chi_i\gamma_a\chi^k)\,\varepsilon_{kl}\gamma^a\Psi^l\,, \\
C|_{\mathbb{T}(\bar\Phi)} = {} & 4(\Box_{\rm c} + 3\,D)\Box_{\rm c}\bar{A} - \tfrac12 D_a\big(T^{ab}{}_{ij}\,T_{cb}{}^{ij}\big)\,D^c\bar{A} + \tfrac{1}{16}(T_{abij}\varepsilon^{ij})^2\bar{C} \\
& + D_a\big(\varepsilon^{ij}D^aT_{bcij}\,G^{+bc} + 4\,\varepsilon^{ij}T^{ab}{}_{ij}\,D^cG^+_{cb} - T_{bc}{}^{ij}\,T^{ac}{}_{ij}\,D^b\bar{A}\big) \\
& + \big(6\,D_bD - 8\mathrm{i}D^aR(A)_{ab}\big)\,D^b\bar{A} + \cdots\,,
\end{aligned} \tag{3.1.5}
$$

where in the last expression we suppressed terms quadratic in the covariant fermion fields. Obviously terms involving the fermionic gauge fields, $\psi_\mu{}^i$ and $\phi_\mu{}^i$, are already contained in the superconformal derivatives. Observe that the right-hand side of these expressions is always linear in the conjugate components of the $w = 0$ chiral multiplet, i.e. in $(\bar{A}, \Psi^i, B^{ij}, G^+_{ab}, \Lambda^i, \bar{C})$.

Using the result (3.1.5) one can construct a large variety of superconformal invariants with higher-derivative couplings involving vector multiplets and the Weyl multiplet. The construction of the higher-order Lagrangians therefore proceeds in two steps. First one constructs the Lagrangian in terms of unrestricted chiral multiplets of appropriate Weyl weights, and subsequently one expresses the unrestricted supermultiplets in terms of the reduced supermultiplets in B.2.3. In these expressions it is natural to introduce a variety of arbitrary homogeneous functions.

Hence we start by writing down the bosonic terms of the Lagrangian (3.1.4), making use of the density formula (2.2.1) and of the product rule (B.2.17),

$$
\begin{aligned}
e^{-1}\mathcal{L} = {} & 4\,\mathcal{D}^2A\,\mathcal{D}^2\bar{A}' + 8\,\mathcal{D}^\mu A\,\big[\mathcal{R}_\mu{}^a - \tfrac13\mathcal{R}\,e_\mu{}^a\big]\mathcal{D}_a\bar{A}' + C\,\bar{C}' \\
& - \mathcal{D}^\mu B_{ij}\,\mathcal{D}_\mu B'^{ij} + (\tfrac16\mathcal{R} + 2\,D)\,B_{ij}B'^{ij} \\
& - \big[\varepsilon^{ik}\,B_{ij}\,F'^{+\mu\nu}\,R(\mathcal{V})_{\mu\nu}{}^j{}_k + \varepsilon_{ik}\,B'^{ij}\,F^{-\mu\nu}R(\mathcal{V})_{\mu\nu j}{}^k\big] \\
& - 8\,D\,\mathcal{D}^\mu A\,\mathcal{D}_\mu\bar{A}' + \big(8\,\mathrm{i}R(A)_{\mu\nu} + 2\,T_\mu{}^{cij}\,T_{\nu cij}\big)\mathcal{D}^\mu A\,\mathcal{D}^\nu\bar{A}' \\
& - \big[\varepsilon^{ij}\mathcal{D}^\mu T_{bcij}\mathcal{D}_\mu A\,F'^{+bc} + \varepsilon_{ij}\mathcal{D}^\mu T_{bc}{}^{ij}\mathcal{D}_\mu\bar{A}'\,F^{-bc}\big] \\
& - 4\big[\varepsilon^{ij}T^{\mu b}{}_{ij}\,\mathcal{D}_\mu A\,\mathcal{D}^cF'^+_{cb} + \varepsilon_{ij}T^{\mu bij}\,\mathcal{D}_\mu\bar{A}'\,\mathcal{D}^cF^-_{cb}\big] \\
& + 8\,\mathcal{D}_aF^{-ab}\,\mathcal{D}^cF'^+_{cb} + 4\,F^{-ac}\,F'^+_{bc}\,\mathcal{R}_a{}^b + \tfrac14 T_{ab}{}^{ij}\,T_{cdij}F^{-ab}F'^{+cd}\,.
\end{aligned} \tag{3.1.6}
$$

Note that the Lagrangian (3.1.6) vanishes whenever either one of the multiplets is equal to a constant, as the full superspace integral of a chiral or an anti-chiral field vanishes (up to total derivatives). This is reflected in the fact that the kinetic multiplet of a *constant* anti-chiral multiplet vanishes, as can be easily deduced from (3.1.5).

The above approach is a constructive one and in general it is hard to classify all these invariant couplings, say, in terms of a limited number of functions, as is often possible for supersymmetric theories. Here we are only interested in the construction of the invariant containing F^4-couplings, so we restrict attention to invariants proportional to a single kinetic multiplet, as given in (3.1.4) and correspondingly in (3.1.6). In that case, one obtains the supergravity-coupled invariants corresponding to the actions derived in [83, 85] in the abelian limit, expressing the composite chiral multiplets in terms of vector multiplets as

$$\Phi \to f(X^I)\,, \qquad \bar{\Phi}' \to \bar{g}(\bar{X}^I)\,, \tag{3.1.7}$$

where Φ^I denote the reduced chiral multiplets associated with vector multiplets. Upon expanding Φ and $\bar{\Phi}'$ in terms of the vector supermultiplets, making use of the material presented in appendices B.2.2 and B.2.3, one obtains powers of the vector multiplet components multiplied by derivatives of $f(X)$ and $\bar{g}(\bar{X})$, where as always, the X^I denote the complex scalars of the vector multiplets. Homogeneity implies that $X^I\, f_I(X) = 0 = \bar{X}^I\, \bar{g}_{\bar{I}}(\bar{X})$, where f_I and $\bar{g}_{\bar{I}}$ denote the first derivatives of the two functions with respect to X^I and $\bar{X}^I$, respectively.

As noted previously, the expression (3.1.6) vanishes whenever $f(X)$ or $\bar{g}(\bar{X})$ are constant and therefore the Lagrangian will depend exclusively on mixed holomorphic/anti-holomorphic derivatives of the product function $f(X)\,\bar{g}(\bar{X})$. By summing over an arbitrary set of pairs of functions $f^{(n)}(X)\,\bar{g}^{(n)}(\bar{X})$, we can further extend this function to a general function $\mathcal{H}(X,\bar{X})$ that is separately homogeneous of zeroth degree in X and $\bar{X}$. Because $\mathcal{H}(X,\bar{X})$ is only defined up to a purely holomorphic or anti-holomorphic function, it is thus subject to Kähler transformations

$$\mathcal{H}(X,\bar{X}) \to \mathcal{H}(X,\bar{X}) + \Lambda(X) + \bar{\Lambda}(\bar{X})\,, \tag{3.1.8}$$

just as for a rigid supersymmetry background. Hence $\mathcal{H}(X,\bar{X})$ can be regarded as a Kähler potential, which may be taken real (so that $\bar{\Lambda}(\bar{X}) = [\Lambda(X)]^*$).

Carrying out the various substitutions leads directly to the following bosonic contribution to the supersymmetric Lagrangian (for convenience, we assume $\mathcal{H}$ to be real),

$$\begin{aligned}
e^{-1}\mathcal{L} = {}& \mathcal{H}_{IJ\bar{K}\bar{L}}\Big[\tfrac{1}{4}\big(G_{ab}^{-\,I}\,G^{-ab\,J} - \tfrac{1}{2}Y_{ij}{}^{I}\,Y^{ijJ}\big)\big(G_{ab}^{+\,K}\,G^{+ab\,L} - \tfrac{1}{2}Y^{ijK}\,Y_{ij}{}^{L}\big) \\
&\qquad + 4\,\mathcal{D}_a X^I\,\mathcal{D}_b\bar{X}^K\big(\mathcal{D}^a X^J\,\mathcal{D}^b\bar{X}^L + 2\,G^{-\,ac\,J}\,G^{+\,b}{}_{c}{}^{L} - \tfrac{1}{4}\eta^{ab}\,Y_{ij}^{J}\,Y^{L\,ij}\big)\Big] \\
& + \Big\{\mathcal{H}_{IJ\bar{K}}\Big[4\,\mathcal{D}_a X^I\,\mathcal{D}^a X^J\,\mathcal{D}^2\bar{X}^K - \mathcal{D}_a X^I\,Y_{ij}^{J}\,\mathcal{D}^a Y^{K\,ij} \\
&\qquad - \big(G^{-ab\,I}\,G_{ab}^{-\,J} - \tfrac{1}{2}Y_{ij}^{I}\,Y^{Jij}\big)\big(\Box_{\rm c} X^K + \tfrac{1}{8}G_{ab}^{-\,K}\,T^{abij}\varepsilon_{ij}\big) \\
&\qquad + 8\,\mathcal{D}^a X^I G_{ab}^{-\,J}\big(\mathcal{D}_c G^{+\,cb\,K} - \tfrac{1}{2}\mathcal{D}_c\bar{X}^K T^{ij\,cb}\varepsilon_{ij}\big)\Big] + \text{h.c.}\Big\}
\end{aligned}$$

$$
\begin{aligned}
+\mathcal{H}_{I\bar{J}}\Big[&4\big(\Box_{\rm c}\bar{X}^I+\tfrac{1}{8}G^{+\,I}_{ab}\,T^{ab}{}_{ij}\varepsilon^{ij}\big)\big(\Box_{\rm c}X^J+\tfrac{1}{8}G^{-\,J}_{ab}\,T^{abij}\varepsilon_{ij}\big)+4\,\mathcal{D}^2X^I\,\mathcal{D}^2\bar{X}^J\\
&+8\,\mathcal{D}_aG^{-\,abI}\,\mathcal{D}_cG^{+c}{}_b{}^J-\mathcal{D}_aY_{ij}{}^I\,\mathcal{D}^aY^{ij\,J}+\tfrac{1}{4}T_{ab}{}^{ij}\,T_{cdij}\,G^{-ab\,I}G^{+cd\,J}\\
&+\big(\tfrac{1}{6}\mathcal{R}+2\,D\big)Y_{ij}{}^I\,Y^{ij\,J}+4\,G^{-ac\,I}\,G^{+}{}_{bc}{}^J\,\mathcal{R}_a{}^b\\
&+8\big(\mathcal{R}^{\mu\nu}-\tfrac{1}{3}g^{\mu\nu}\mathcal{R}+\tfrac{1}{4}T^{\mu}{}_b{}^{ij}\,T^{\nu b}{}_{ij}+\mathrm{i}R(A)^{\mu\nu}-g^{\mu\nu}D\big)\mathcal{D}_\mu X^I\,\mathcal{D}_\nu\bar{X}^J\\
&-\big[\mathcal{D}_c\bar{X}^J\big(\mathcal{D}^cT_{ab}{}^{ij}\,G^{-\,I\,ab}+4\,T^{ij\,cb}\,\mathcal{D}^aG^{-\,I}_{ab}\big)\varepsilon_{ij}+[\text{h.c.};I\leftrightarrow J]\big]\\
&-\big[\varepsilon^{ik}\,Y_{ij}{}^I\,G^{+ab\,J}\,R(\mathcal{V})_{ab}{}^j{}_k+[\text{h.c.};I\leftrightarrow J]\big]\Big]\,,
\end{aligned}
\tag{3.1.9}
$$

where (we suppress fermionic contributions),

$$
\begin{aligned}
G^{-\,I}_{ab}&=F^{-\,I}_{ab}-\tfrac{1}{4}\,\bar{X}^I\,T_{ab}{}^{ij}\varepsilon_{ij}\,,\\
\Box_{\rm c}X^I&=\mathcal{D}^2X^I+\big(\tfrac{1}{6}\mathcal{R}+D\big)\,X^I\,.
\end{aligned}
\tag{3.1.10}
$$

In view of the Kähler equivalence transformations (3.1.8), the mixed derivative $\mathcal{H}_{I\bar{J}}$ can be identified as a Kähler metric. The results for the metric, connection, and the curvature of the corresponding Kähler space are as follows

$$
\begin{aligned}
g_{I\bar{J}}&=\mathcal{H}_{I\bar{J}}\,,\\
\Gamma^I{}_{JK}&=g^{I\bar{L}}\,\mathcal{H}_{JK\bar{L}}\,,\\
R_{I\bar{J}K\bar{L}}&=\mathcal{H}_{IK\bar{J}\bar{L}}-g_{M\bar{N}}\,\Gamma^M{}_{IK}\,\Gamma^{\bar{N}}{}_{\bar{J}\bar{L}}\,.
\end{aligned}
\tag{3.1.11}
$$

The Lagrangian (3.1.9) can then be written in a Kähler covariant form, as shown in [81].

We conclude that the most general four derivative Lagrangian for vector multiplets is the sum of (3.1.2) and (3.1.9). One can obtain more general couplings, for example R^2F^2- and R^4-type couplings by including the Weyl multiplet in the functions f, g in (3.1.7), as in the prepotential (3.1.1). The resulting invariants are similar to the ones in [86, 87]. We refer to [81] for a discussion of these possibilities.

3.2 BPS attractors

We now give a brief account of BPS attractors in $\mathcal{N}=2$ theories, following [54]. As mentioned in the introduction, the use of off-shell methods leads to results that are independent of a Lagrangian. In the next section we discuss the implementation of these results in the four derivative case.

In a fully supersymmetric background, all fermions and their supersymmetry variations must vanish for any choice of Q- and S-supersymmetry parameters ϵ^i, η^i. It is convenient to consider S-invariant combinations, so that all terms involve only the parameter ϵ. To this end, we introduce the spinors

$$
\zeta^{\rm H}_i=\chi^{-1}\bar{\Omega}_{\alpha\beta}A_i{}^\alpha\,\zeta^\beta\,,\qquad \zeta^{\rm V}_i=-i\,\mathrm{e}^K\Big[(\bar{F}_I-\bar{X}^JF_{IJ})\Omega^I_i-\bar{X}^IF_{IA}\,\hat{\Psi}_i\Big]\,,
\tag{3.2.1}
$$

both of which transform by constant shifts under S-supersymmetry

$$\delta\zeta_i^{\rm H} = \chi^{-1}\bar\Omega_{\alpha\beta}A_i{}^\alpha \not{D} A_j{}^\beta\,\epsilon^j + \varepsilon_{ij}\eta^j\,, \tag{3.2.2}$$

$$\begin{aligned}\delta\zeta_i^{\rm V} =& {\rm e}^K \not{\mathcal{D}} {\rm e}^{-K}\epsilon_i + i{\rm e}^K\left(\bar X^J \overset{\leftrightarrow}{\not{\mathcal{D}}}_\mu F_J - \bar F_J \overset{\leftrightarrow}{\not{\mathcal{D}}}_\mu X^J\right)\epsilon_i - \tfrac12 i\varepsilon_{ij}\,{\rm e}^{K/2}\langle \mathcal{F}^-_{ab}, \mathcal{V}\rangle\,\gamma^{ab}\epsilon^j \\ &+ {\rm e}^K\,N^{IJ}\Big[(\bar F_I - \bar F_{IK}\bar X^K)(F_{JA}\,\hat B_{ij} - \bar F_{JA}\,\varepsilon_{ik}\varepsilon_{jl}\hat B^{kl})\Big]\,\epsilon^j + 2\,\eta_i\,,\end{aligned} \tag{3.2.3}$$

where $\mathcal{F}^-$ is the anti-selfdual part of the electric and magnetic field strengths in (2.2.15). Adding the appropriate multiple of the above spinors to the supersymmetry variations of the fermionic fields, one can construct S-invariant combinations and infer the constraints on the bosons by demanding that all linearly independent terms of the resulting Q-variations vanish separately. First, the combination $2\,\zeta_i^{\rm H} - \varepsilon_{ij}\zeta^{\rm V\,j}$ is S-invariant, and its variation implies among other relations the expression (2.2.9) for the composite U(1) gauge field and that

$$\mathcal{D}_\mu(\chi^{-1}\,{\rm e}^K) = 0\,, \qquad {\rm e}^{-K} = {\rm i}(F_I\,\bar X^I - \bar F_I\,X^I)\,. \tag{3.2.4}$$

The scalar ${\rm e}^K$ is analogous to the one in (2.2.5), but here also depends on the Weyl multiplet through (3.1.1). Nevertheless, this is still the combination appearing as the coefficient of the Ricci scalar in (3.1.2) and is constrained to be proportional to the hyper-Kähler potential χ by the equation of motion for the scalar D, similar to (2.2.4).

Imposing that the S-invariant combination of the vector multiplet spinor in (B.2.19) vanishes one finds

$$\mathcal{F}^-_{ab} = \tfrac14\,\varepsilon_{ij}T^{ij}_{ab}\,\bar{\mathcal{V}}\,, \qquad \mathcal{D}_\mu({\rm e}^{K/2}\mathcal{V}) = 0\,, \tag{3.2.5}$$

The second relation implies that all vector multiplet scalars are constant, as expected. A similar analysis for a general chiral multiplet in (B.2.16) reveals that the bottom component A is constant, whereas all higher components vanish, consistent with the special case $w=1$ in (3.2.5) above[2]. Furthermore, the variations of $\hat\Psi_i$ in (B.2.24) and ζ^α in (B.2.25) imply that $\hat A$ and the hypermultiplet scalars $A_i{}^\alpha$ are also constant.

Turning to the Weyl multiplet, we impose the vanishing of the variations for χ^i in (B.2.1) and $R(Q)_{ab}{}^i$ in (B.2.23). Note that we do not demand that the variation of the gravitino vanishes, since that is too restrictive for a gauge field. We rather use its field strength $R(Q)_{ab}{}^i$. The results relevant to the discussion here are the following[3]

$$\begin{gathered}\mathcal{D}_c T^{ij}_{ab} = -\tfrac12\mathcal{D}_d K\left(\delta^d_c\,T^{ij}_{ab} - 2\delta^d_{[a}\,T^{ij}_{b]c} + 2\eta_{c[a}T^{ij\,d}_{b]}\right)\,,\\ R(V)_{ab}{}^i{}_j = 0\,, \qquad \hat R(M)_{ab}{}^{cd} = 0\,,\end{gathered} \tag{3.2.6}$$

where the last relation implies that the spacetime is conformally flat. Finally, it turns out that one needs to consider the variation of at least one fermionic derivative [54]. We

[2] Note that the tensor component of a reduced chiral multiplet is given by (B.2.19)

[3] See (B.2.9) for the definition of the modified curvature $\hat R(M)_{ab}{}^{cd}$.

choose to use the variation of $D_\mu \zeta_i^{\rm H}$, which results to

$$\mathcal{R}(\omega,e)_\mu{}^a - \tfrac{1}{6}\, e_\mu^a \mathcal{R}(\omega,e)_\mu{}^a = -\tfrac{1}{8} T_{\mu b}^{ij}\, T_{ij}^{ab} + \mathcal{D}_\mu \mathcal{D}^a K + \tfrac{1}{2}\mathcal{D}_\mu K\, \mathcal{D}^a K - \tfrac{1}{4}\, e_\mu^a\, (\mathcal{D}_c K)^2\,. \quad (3.2.7)$$

These results are sufficient to uniquely specify the spacetime background and the tensor T_{ab}^{ij}, once some gauge choices similar to those in section 2.2.1 are made. We fix $b_\mu = 0$, and e^K to a constant, which also implies that χ is a constant. In view of these choices, (3.2.6) and (3.2.7) imply that T_{ab} is covariantly constant and the metric is Ricci flat. As in (1.2.44), the metric is of the Bertotti-Robinson type

$$\mathrm{d}s^2 = -\frac{r^2}{|c|^2}\,\mathrm{d}t^2 + \frac{|c|^2}{r^2}\mathrm{d}r^2 + |c|^2\,\mathrm{d}\theta^2 + |c|^2\,\sin^2\theta\,\mathrm{d}\phi^2\,, \quad (3.2.8)$$

whereas the tensor is harmonic

$$\tfrac{1}{4}\,\varepsilon_{ij} T_{\hat a \hat b}^{ij} = \varepsilon_{\hat a \hat b \hat c}\,\partial_{\hat c}\frac{c}{r}\,, \qquad \tfrac{1}{4}\,\varepsilon_{ij} T_{t\hat a}^{ij} = i\partial_{\hat a}\frac{c}{r}\,. \quad (3.2.9)$$

In these relations, c is a complex constant and $\hat a, \hat b, \hat c$ denote spatial world indices.

Using the standard definition (2.2.14) for the charges, (3.2.5) becomes

$$\Gamma = 2\ \mathrm{Re}(\bar c \mathcal{V}) \quad \Rightarrow Z(\Gamma) = -\mathrm{i}\, c\,, \quad (3.2.10)$$

where the definition in (2.2.29) was used. The final form of the attractor equations is identical to (2.2.43)

$$2\,\mathrm{Im}\big(\bar Z\,\mathcal{V}\big) = \Gamma\,. \quad (3.2.11)$$

Note however that the section now implicitly involves the Weyl background through $F_I(X,\hat A)$, so that the attractor equations are significantly more complicated.

3.3 The attractor equations and the entropy

We now specialise the above results for the Lagrangian expressed as the sum of (3.1.2) and (3.1.9). Firstly, we point out the non-renormalisation theorem of [81], which follows from the remark on chiral multiplets of weight zero below (3.2.5). Inspection of the F^4 invariant in the form (3.1.6) shows that it is manifestly quadratic in components of $w = 0$ chiral multiplets that vanish in a fully supersymmetric background. Since the entropy and charges are expressed through first order variations of the action with respect to the Riemann tensor and the field strengths (see Appendix C), all contributions from the invariant (3.1.9) vanish identically in the BPS limit. Therefore, for the remainder of the section we concentrate on the Lagrangian in (3.1.2).

The entropy is given by the generic formula (C.2.16), which we repeat here with appropriate normalisation

$$\mathcal{S} = 2\pi \int_H \frac{\partial \mathcal{L}}{\partial \mathcal{R}_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma} \,, \tag{3.3.1}$$

where $\epsilon_{\mu\nu}$ denotes the binormal on the horizon. In order to compute the derivative of the Lagrangian with respect to the Riemann tensor, we again make use of the observation on the vanishing of all chiral multiplet components in the BPS limit except the bottom one. The only terms in (3.1.2) that are not quadratic in such higher components are the ones involving the Ricci scalar explicitly and $\hat{C}$, so that

$$\begin{aligned} 8\pi\, e^{-1} \frac{\partial \mathcal{L}}{\partial \mathcal{R}_{\mu\nu\rho\sigma}} &= -\tfrac{\mathrm{i}}{2} \left(F_I\, \bar{X}^I - \bar{F}_I\, X^I\right) g^{\mu\nu} g^{\rho\sigma} - \mathrm{Im}\left(F_A \frac{\partial \hat{C}}{\partial \mathcal{R}_{\mu\nu\rho\sigma}}\right) \\ &= -\tfrac{\mathrm{i}}{2} \left(F_I\, \bar{X}^I - \bar{F}_I\, X^I\right) g^{\mu\nu} g^{\rho\sigma} - 8\, g^{\nu\sigma}\ \mathrm{Im} F_A\, \varepsilon_{ij} T^{ij\,\mu a} \varepsilon^{kl} T_{kl}{}^{\rho}{}_a \,, \end{aligned} \tag{3.3.2}$$

where (3.2.6) was used to evaluate the last term. The entropy reads

$$\mathcal{S} = \pi \left(\mathrm{i} \left(F_I\, \bar{X}^I - \bar{F}_I\, X^I\right) |Z(\Gamma)|^2 - 256\ \mathrm{Im} F_A(X, \hat{A})\right) , \tag{3.3.3}$$

where the scalar $\hat{A} = -64\, \bar{Z}(\Gamma)^{-2}$ and we used the fact that F_A is a homogeneous function of weight zero. Once the attractor equations (3.2.11) are solved for the scalars, all quantities, including the entropy, are expressed only in terms of charges.

In string theory compactifications on Calabi-Yau manifolds, which is the prime example in this thesis, the leading correction to the prepotential in (2.3.1) takes the form

$$F(X, \hat{A}) = -\frac{1}{12} \frac{C_{ABC} X^A X^B X^C}{X^0} - \frac{1}{24} \frac{1}{64} \frac{c_A\, X^A}{X^0}\, \hat{A} \,. \tag{3.3.4}$$

We now exhibit the attractor equations above in more detail for this prepotential, both to compare with the microscopic prediction for the corrected entropy in section 2.4 and for future reference. We follow the same method as in section 2.3 for the two derivative action, using the rescaled variables introduced in (2.3.8), in the form

$$Y^I = \bar{Z}\, X^I \,, \qquad \Upsilon = \bar{Z}^2\, \hat{A} = -64 \,. \tag{3.3.5}$$

Parametrising the vector multiplet scalars by $Y^A = \frac{1}{2}(-\phi^A + \mathrm{i} p^A)$, the magnetic attractor equations of (3.2.11) are trivially satisfied. It remains to solve the electric equations, a task that is considerably simplified when some restrictions on the charges are imposed.

Consider first the case where all but one magnetic charges are zero, chosen as $p^A = 0$, $p^0 \neq 0$. We then obtain the following expressions for the entropy

$$\mathcal{S} = \frac{2\pi p^0}{(\phi^{0\,2} + p^{0\,2})^2} \left[\frac{1}{12} p^{0\,2} C_{ABC} \phi^A \phi^B \phi^C + \frac{1}{6}\, c_A \phi^A \phi^{0\,2}\right] , \tag{3.3.6}$$

and the remaining attractor equations

$$\begin{aligned} q_A &= \frac{p^0}{\phi^{0\,2}+p^{0\,2}}\Big[\frac{1}{4}\,C_{ABC}\phi^B\phi^C-\frac{1}{6}\,c_A\Big]\,, \\ q_0 &= -\frac{2\phi^0p^0}{(\phi^{0\,2}+p^{0\,2})^2}\Big[\frac{1}{12}\,C_{ABC}\phi^A\phi^B\phi^C-\frac{1}{6}\,c_A\phi^A\Big]\,. \end{aligned} \tag{3.3.7}$$

It is possible to find the corresponding expressions for non-zero p^A charges, using the symmetry of the theory based on (3.3.4) under the following transformations, which take the form of electric/magnetic dualities (see, e.g. [88]),

$$\begin{aligned} X^0 &\to X^0\,, \\ X^A &\to X^A+k^AX^0\,, \\ F_0 &\to F_0-k^AF_A+\tfrac{1}{4}\,C_{ABC}k^Bk^C\,X^A+\tfrac{1}{12}C_{ABC}k^Ak^Bk^C\,X^0\,, \\ F_A &\to F_A-\tfrac{1}{2}\,C_{ABC}k^B\,X^C-\tfrac{1}{4}\,C_{ABC}k^Bk^C\,X^0\,, \end{aligned} \tag{3.3.8}$$

where the parameters k^A are real. In principle there could be other dualities as well, depending on the specific form of the coefficients C_{ABC} and c_A. It follows that the electric and magnetic charges will exhibit similar transformations,

$$\begin{aligned} p^0 &\to p^0\,, \\ p^A &\to p^A+k^Ap^0\,, \\ q_0 &\to q_0-k^Aq_A+\tfrac{1}{4}\,C_{ABC}k^Bk^C\,p^A+\tfrac{1}{12}C_{ABC}k^Ak^Bk^C\,p^0\,, \\ q_A &\to q_A-\tfrac{1}{2}\,C_{ABC}k^B\,p^C-\tfrac{1}{4}\,C_{ABC}k^Bk^C\,p^0\,. \end{aligned} \tag{3.3.9}$$

Note that this is a symmetry of the equations of motion and holds irrespective of any supersymmetry preserved by a specific solution.

Let us now turn to the case of a black hole with $p^0=0$, but otherwise arbitrary charges. Then, (3.3.3) and (3.2.11) imply

$$\mathcal{S}=-\frac{2\pi}{\phi^0}\Big[\frac{1}{12}C_{ABC}\,p^Ap^Bp^C+\frac{1}{6}\,c_Ap^A\Big]\,, \tag{3.3.10}$$

with

$$\begin{aligned} q_A &= -\frac{1}{2\,\phi^0}C_{ABC}\,p^B\phi^C\,, \\ \hat{q}_0\equiv q_0-\tfrac{1}{6}D^{AB}q_Aq_B &= -\frac{1}{12}\frac{1}{\phi^{0\,2}}\Big[C_{ABC}\,p^Ap^Bp^C+2\,c_Ap^A\Big]\,, \end{aligned} \tag{3.3.11}$$

where D^{AB} is the inverse of $D_{ABC}p^C$. Just as before this gives rise to

$$\mathcal{S}=2\pi\,\sqrt{\tfrac{1}{12}|\hat{q}_0(C_{ABC}\,p^Ap^Bp^C+2\,c_Ap^A)|}\,. \tag{3.3.12}$$

Compared to the microscopic prediction in section 2.4.2, this expression agrees with the entropy in (2.4.6), upon identifying $C_{ABC} = 2\,\mathcal{C}_{ABC}$ and $c_A = c_{2\,A}$.

As the reader can easily verify, the expressions for $\hat{q}_0$ and for the entropy are invariant under the transformations (3.3.9) with $p^0 = 0$. Also the expression for the charges q_A is consistent with this symmetry as it acts on ϕ^A according to $\phi^A \to \phi^A + k^A\,\phi^0$, which follows straightforwardly from (3.3.8). When the R^2 correction is set to zero by $c_A = 0$, the above results agree with the ones presented in section 2.3. In particular, the reader can verify that (3.3.9) is a symmetry of (2.3.12), where all charges are present.

Chapter 4

BPS attractors in five dimensions

In this chapter we give a detailed discussion of supersymmetric black hole attractors in five spacetime dimensions, including four derivative interactions. There are a number of reasons to explore the five-dimensional setting. First, the space of solutions is richer compared to four dimensions, both due to the absence of electric/magnetic duality and because the BPS conditions allow for nontrivial angular momentum. It is very interesting to study the effects of this extra charge, especially for black rings, which have no static limit. A further motivation is that five-dimensional black holes are naturally related to four-dimensional ones by dimensional reduction, as in section 2.3. An investigation of this link can lead to a better understanding of the microscopic theories for both case, through a precise mapping of the relevant quantities.

The two varieties of five-dimensional BPS attractors, associated with the near-horizon geometry of the rotating black hole [39, 89], and of the black ring [40], have been studied in the context of the two-derivative effective action in [25, 35, 58, 89–91], using mostly on-shell methods. It is possible to include higher-derivative couplings into the conventional two-derivative supergravity action, such as the four-derivative supersymmetric action constructed in [32]. Both the two- and the four-derivative couplings involve a Chern-Simons term, which is a characteristic feature of five-dimensional supergravity. In the two-derivative case the Chern-Simons term is cubic in the gauge fields, whereas the higher-derivative mixed Chern-Simons term involves also the spin connection field. As a result the Lagrangian is only gauge invariant up to a total derivative, a feature that causes certain technical complications.

A study of BPS black holes and black rings that includes these higher-derivative interactions was initiated in [92–95]. In these works, a number of black hole solutions was constructed, and the corresponding attractors were studied by taking the near-horizon limit. In addition, the entropy function formalism [19] was used to determine the macroscopic entropy of these black holes, after reducing to four dimensions to restore gauge invariance of the action. A corresponding analysis for black rings was hampered by the difficulty in obtaining full asymptotically flat solutions.

Here, we present a comprehensive treatment of five-dimensional $N=2$ attractors in the presence of the same four-derivative couplings, using the tools provided by the off-shell calculus. This analysis of the near-horizon behaviour thus relies only on the full supersymmetry enhancement and does not take into account the more global aspects of possible solutions. In particular, no assumptions are made concerning the existence of interpolating solutions towards asymptotic infinity, and no use is made of any information from outside the near-horizon region. This is in line with the idea that the entropy of black branes should be determined fully by the horizon properties, in the spirit of the Bekenstein-Hawking area law.[1] As in the four-dimensional analysis, we find that the allowed space-time geometry is the same as for the two-derivative theory, which in the case at hand is described by the $AdS_2 \times S^2 \times S^1$ geometry of [97]. Because this geometry interpolates between the black hole and the black ring attractors, we can treat both types of five-dimensional attractors in a unified way for a large part of the analysis.

The higher-derivative corrections in the action enter into the expressions for both the entropy and the attractor equations pertaining to electric charges and angular momenta. For the Wald entropy [82, 98, 99] we obtain a universal formula expressed in terms of the horizon fields, which applies to both black holes and rings. This is an intriguing result, because the derivation in these two cases proceeds rather differently due to a number of subtleties associated with the mixed gauge-gravitational Chern-Simons term. Our treatment of this mixed Chern-Simons term is inspired by, but not completely identical to, the approach followed in [100]. The existence of a universal entropy formula is in line with previous results based on the entropy function upon reduction to four dimensions, and we confirm this by confronting the results with the four-dimensional near-horizon analysis.

The charges and the angular momenta can also be defined at the horizon. In view of the first law of black hole mechanics, this requires the use of the same Noether potential that enters into the determination of the Wald entropy. The evaluation of the full Noether potential is rather involved, and, as alluded to above, the relevant potentials do not take the same form for black rings and for black holes. The electric charges defined at the horizon are conserved by construction (although they are not invariant under large gauge transformations in the case of black rings). Subtleties arise with the proper definition of the gauge fields in the presence of the Chern-Simons terms, and those have important implications on the attractor equations for black hole and black ring charges.

As mentioned above, the BPS near-horizon geometries come in two varieties. In the case of a spacelike horizon cross section with spherical topology, we recover the $AdS_2 \times S^3$ near-horizon geometry of the rotating black hole [39, 89]. In the other case we

[1] See, however, [96], for a possibly different perspective.

find the $AdS_3 \times S^2$ near-horizon geometry of the supersymmetric black ring [40]. The latter constitutes a special limit of the generic BPS near-horizon geometry for which the spacelike cross section of the horizon has the topology of $S^2 \times S^1$, as is appropriate for a ring. Unlike the black hole, the black ring carries two independent angular momenta associated with rotations in two orthogonal planes. There are some other new features related to the non-contractible S^1. The first concerns the fact that this background allows for non-trivial magnetic charges on the circle, as magnetic charges are not pointlike in five dimensions, but stringlike objects. Hence the ring carries magnetic dipole charges. The second one concerns the non-trivial moduli associated with Wilson lines along the circle. We present a careful treatment of the gauge fields in this topology, recovering the correct electric charges and their associated attractor equations, following the strategy of [58].

Finally, we comment on the connection between corresponding black hole solutions and their associated entropy in four and in five dimensions. This connection is motivated by the fact that the four-dimensional theory can be obtained by dimensional reduction on a circle from the five-dimensional one [56–58], although there may be subtleties. Following this reasoning, our five-dimensional attractor equations are related to the four-dimensional attractors with a specific R^2-coupling. Comparing with the results in the previous chapter, we find agreement with four dimensions in the case of the black ring, except that the quantity $\hat{q}_0$ in four dimensions will only depend on the unmodified electric charges. For the case of the rotating black hole, we find a clear discrepancy in the contributions from the higher-derivative couplings to the electric charges. A similar, though somewhat different, deviation from the four-dimensional situation was observed in [93–95].

4.1 The R^2 invariant in five dimensions

As explained for the two derivative case in section 2.1.1, the construction of supersymmetric invariants in five dimensions depends on composite linear multiplets. There, we showed how a linear multiplet can be constructed out of a set of vector multiplets, but the same strategy can also be used to construct a linear multiplet from the square of the Weyl multiplet. In view of the fact that the transformations for the Weyl multiplet fields are nonlinear, this construction is considerably more complicated. The starting point, as before, is to define a composite field $L^{\mathrm{W}ij}$ in terms of the Weyl multiplet fields, which satisfies all the requirements for the lowest-dimensional component of a superconformal linear multiplet. This linear multiplet has originally been determined in [32]. In our conventions we find the following result,

$$L^{ij\mathrm{W}} = -\,\varepsilon^{k(i}\left[\tfrac{1}{32}\mathrm{i}\bar{R}_{abk}(Q)\,R^{j)\,ab}(Q) + \tfrac{32}{3}\mathrm{i}\bar{\chi}_k\chi^{j)} - \tfrac{1}{4}T^{ab}\,R_{abk}{}^{j)}(V)\right]\,, \tag{4.1.1}$$

which indeed is S-invariant and transforms under Q-supersymmetry into a spinor doublet. Furthermore it scales with Weyl weight 3, as is appropriate for a linear multiplet. By applying successive supersymmetry transformations, we identify the other components of this linear multiplet,

$$\begin{aligned}
\varphi^{i\mathrm{W}} =& \tfrac{1}{64} R_{ab}{}^{cd}(M)\, \gamma_{cd} R^{abi}(Q) + \tfrac{1}{32} R_{abj}{}^{i}(V) R^{abj}(Q) - \tfrac{3}{4} T^{ab} R_{ab}{}^{i}(S) \\
& - \tfrac{1}{6} R_{abj}{}^{i}(V) \gamma^{ab} \chi^{j} - \tfrac{3}{8} \mathrm{i}\, D_a T_{bc}\, \gamma^{c} R^{abi}(Q) + \tfrac{3}{16} T^{ab} T_{cd}\, \gamma^{cd} R_{ab}{}^{i}(Q) \\
& + 4\mathrm{i}\, T^{ab} \gamma_a D_b \chi^{i} - \tfrac{1}{2}\mathrm{i}\big(\gamma^{ab} \not{D} T_{ab} + 3\, \not{D} \gamma^{ab} T_{ab}\big)\chi^{i} + \tfrac{8}{3}\big(2\, D + 3\, T^{2}\big)\chi^{i} \,, \\
\hat{E}^{a\mathrm{W}} =& - \tfrac{1}{128} \mathrm{i}\, \varepsilon^{abcde} \Big[R_{bc}{}^{fg}(M)\, R_{defg}(M) + \tfrac{1}{3} R_{bcj}{}^{i}(V)\, R_{dei}{}^{j}(V) \Big] \\
& + \tfrac{3}{2} \mathrm{i}\, \varepsilon^{abcde}\, D_b \big[T_{cf} D^{f} T_{de} + \tfrac{3}{2} T_{cf} D_d T_{e}{}^{f} \big] \\
& - D_b \Big[\tfrac{3}{8} R(M)_{cd}{}^{ab}\, T^{cd} + 2\, T^{ab}\, D + \tfrac{3}{4} T^{ab}\, T^{2} - 9\, T^{ac} T_{cd} T^{db} \Big] + \cdots \,, \\
N^{\mathrm{W}} =& \tfrac{1}{64} R_{ab}{}^{cd}(M)\, R_{cd}{}^{ab}(M) + \tfrac{1}{96} R_{abj}{}^{i}(V)\, R^{ab}{}_{i}{}^{j}(V) + \tfrac{15}{8} T^{ab} T_{cd}\, R_{ab}{}^{cd}(M) \\
& + 3\, T^{ab} D^{c} D_a T_{bc} - \tfrac{3}{2} \big(D_a T_{bc}\big)^{2} + \tfrac{3}{2} D_c T_{ab}\, D^{a} T^{cb} \\
& - \tfrac{9}{4} \mathrm{i} \varepsilon_{abcde} T^{ab} T^{cd} D_f T^{fe} + \tfrac{8}{3} D^{2} + 8\, T^{2}\, D - \tfrac{33}{8} (T^{2})^{2} + \tfrac{81}{2} (T^{ac} T_{bc})^{2} \\
& + \cdots \,,
\end{aligned} \tag{4.1.2}$$

where the dots refer to fermionic terms, which we will not need for what follows.

In order to represent a linear multiplet, the vector $\hat{E}^{a\mathrm{W}}$ should satisfy the constraint $D_a \hat{E}^{a\mathrm{W}} = 0$, as a consequence of which this vector can be expressed in terms of a three-rank tensor field $E^{\mathrm{W}}_{\mu\nu\rho}$. In principle, we can determine the full expression of this composite tensor by verifying its supersymmetry transformation (B.1.14). This is how we originally obtained (2.1.6). For the Weyl multiplet, however, this calculation is considerably more involved, so that we restrict ourselves to the expression for the purely bosonic terms. The result reads as follows,

$$\begin{aligned}
E^{\mathrm{W}}_{\mu\nu\rho} \;=\;& -\tfrac{3}{16} \omega_{[\mu}{}^{ab} \left(\partial_\nu \omega_{\rho]\,ab} - \tfrac{2}{3} \omega_{\nu\, ac}\, \omega_{\rho]}{}^{c}{}_{b} \right) - \tfrac{1}{16} V_{[\mu i}{}^{j} \left(\partial_\nu V_{\rho j}{}^{i} - \tfrac{1}{3} V_{\nu j}{}^{k}\, V_{\rho]k}{}^{i} \right) \\
& - 9 \left(T_{\sigma[\mu}\, \mathcal{D}^{\sigma} T_{\nu\rho]} + \tfrac{3}{2} T_{\sigma[\mu}\, \mathcal{D}_\nu T_{\rho]}{}^{\sigma} \right) \\
& + \mathrm{i}\, e\, \varepsilon_{\mu\nu\rho\sigma\lambda} \left(\tfrac{3}{16} R(M)_{\kappa\tau}{}^{\sigma\lambda} T^{\kappa\tau} + T^{\sigma\lambda}\, D + \tfrac{3}{8} T^{\sigma\lambda} T^{2} - \tfrac{9}{2} T^{\sigma\kappa} T_{\kappa\tau} T^{\tau\lambda} \right) \\
& + \cdots \,,
\end{aligned} \tag{4.1.3}$$

where the dots represent the fermionic contributions. It is not difficult to verify that this expression is invariant under scale transformations and conformal boosts, up to tensor gauge transformations and up to terms proportional to fermions (we recall that the spin connection depends both on b_μ and $\psi_\mu{}^{i}$), and that the tensor field strength corresponding to it reproduces the bosonic terms in $\hat{E}^{a\mathrm{W}}$ shown in (4.1.2).

The construction of the relevant invariant follows from the results presented in section 2.1.1, by using (2.1.2) to couple the linear multiplet quadratic in the Weyl multiplet to a vector multiplet characterized by constants c_A, so that its scalar field equals $\tfrac{4}{3} c_A \sigma^{A}$, where the prefactor is chosen for later convenience. Employing the same normalizations as in section 2.1.1, the higher-derivative Lagrangian is given by

$$
\begin{aligned}
32\pi^2\, e^{-1}\mathcal{L}_{\rm vww} = &\tfrac{1}{3}\, c_A Y_{ij}{}^A\, T^{ab} R_{abk}{}^j(V)\, \varepsilon^{ki} \\
&+ c_A\sigma^A\Big[\tfrac{1}{48} R_{ab}{}^{cd}(M)\, R_{cd}{}^{ab}(M) + \tfrac{1}{72} R_{abj}{}^i(V)\, R^{ab}{}_i{}^j(V)\Big] \\
&- \tfrac{1}{96}\mathrm{i}e^{-1}\,\varepsilon^{\mu\nu\rho\sigma\tau}\, c_A W_\mu{}^A\,\Big[R_{\nu\rho}{}^{ab}(M)\, R_{\sigma\tau ab}(M) + \tfrac{1}{3} R_{\nu\rho j}{}^i(V)\, R_{\sigma\tau i}{}^j(V)\Big]\Big] \\
&+ \tfrac{1}{4} c_A\big(10\,\sigma^A\, T_{ab} - F_{ab}{}^A\big)\, R(M)_{cd}{}^{ab}\, T^{cd} \\
&+ 4\, c_A\sigma^A\Big[T^{ab}\mathcal{D}^c\mathcal{D}_a T_{bc} - \tfrac{1}{2}\big(\mathcal{D}_a T_{bc}\big)^2 + \tfrac{1}{2}\mathcal{D}_c T_{ab}\,\mathcal{D}^a T^{cb} \\
&+ \tfrac{8}{9}\mathcal{R}_{ab}(T^{ac}T^b{}_c - \tfrac{1}{2}\eta^{ab}T^2)D^2 + \tfrac{8}{3}\, T^2\, D - \tfrac{11}{8}(T^2)^2 + \tfrac{27}{2}(T^{ac}T_{bc})^2\Big] \\
&- \tfrac{4}{3}\, c_A F_{ab}{}^A\Big[T^{ab}\, D + \tfrac{3}{8} T^{ab}\, T^2 - \tfrac{9}{2}\, T^{ac} T_{cd} T^{db}\Big] \\
&+ \mathrm{i}\,\varepsilon^{abcde}\Big[c_A F_{ab}{}^A\big(T_{cf}\mathcal{D}^f T_{de} + \tfrac{3}{2} T_{cf}\mathcal{D}_d T_e{}^f\big) - 3\, c_A\sigma^A T_{ab} T_{cd}\,\mathcal{D}^f T_{fe}\Big]\,,
\end{aligned}
\tag{4.1.4}
$$

which should be added to the lowest order action (2.1.7). In the above result there are two terms which cannot be written in a manifestly gauge invariant form, related to the appearance of gravitational and SU(2) Chern-Simons terms. To avoid these Chern-Simons terms we have chosen to write their contribution in a form that is explicitly proportional to the gauge fields $W_\mu{}^A$. These non-covariant terms add to the two derivative Chern-Simons term and must also be handled carefully in the presence of magnetic charges.

The construction of this action parallels the corresponding four-dimensional one in (3.1.2). Both contain R^2-type interactions with scalar dependent couplings, very similar to the gauge theory action. In fact, upon dimensional reduction on a circle the five-dimensional R^2 invariant reduces to a sum of the Lagrangian in (3.1.2) with a prepotential as in (3.3.4), and the F^4 invariant (3.1.9), associated to the Kaluza-Klein gauge field.

4.2 BPS attractors

In this section we derive the conditions for full supersymmetry of the field configuration in five dimensions. Here we follow the systematic approach introduced for four spacetime dimensions in [54] and outlined in section 3.2. In this section the analysis is done entirely at the off-shell level, so that all results apply to any $\mathcal{N} = 2$ superconformal action, perhaps including higher derivative corrections. Our analysis differs from the one of [95], where on-shell information was already introduced at an earlier stage of the calculation. Only in the next section 4.3 we will make use of the supersymmetric action.

Although our analysis is different in spirit and covers a much larger class of supergravity theories, the results turn out to overlap substantially with those of [25].

4.2.1 Supersymmetry

To analyze supersymmetry one chooses a purely bosonic field configuration and requires that the supersymmetry variation of all fermion fields vanish up to a uniform S-supersymmetry transformation. In this context it is convenient to define two 'compensating' spinor fields, $\zeta^i_{\rm V}$ and $\zeta^i_{\rm H}$, belonging to the vector multiplet sector and the hypermultiplet sector, respectively, which transform linearly under S-supersymmetry,

$$\zeta^i_{\rm V} = \frac{1}{6\,C(\sigma)}\,C_{ABC}\sigma^A\sigma^B\,\Omega^{iC}\,, \qquad \zeta^i_{\rm H} = -\frac{2}{3\,\chi}\varepsilon^{ij}\,\Omega_{\alpha\beta}A_j{}^\alpha\zeta^\beta\,. \tag{4.2.1}$$

Here we have used a symmetric rank-three tensor C_{ABC} and the corresponding function $C(\sigma) = \frac{1}{6}\,C_{ABC}\sigma^I\sigma^J\sigma^K$ that may be identified with the ones used in the construction of the lowest order action in section 2.1.1. This is by no means necessary, as one could choose an arbitrary nonvanishing tensor, at the expense of making the following discussion somewhat more complicated.

It is straightforward to write down the supersymmetry variations of these two spinor fields (which both carry scaling weights equal to $\frac{1}{2}$),

$$\begin{aligned} \delta\zeta^i_{\rm V} &= \big(T_{ab} - \tfrac{1}{12}F_{ab}{}^A\partial_A\ln C(\sigma)\big)\gamma^{ab}\epsilon^i - \tfrac{1}{6}{\rm i}\not{\mathcal{D}}\ln C(\sigma)\,\epsilon^i - \tfrac{1}{3}\varepsilon_{jk}Y^{ijA}\partial_A\ln C(\sigma)\,\epsilon^k + \eta^i\,,\\ \delta\zeta^i_{\rm H} &= -\tfrac{1}{6}{\rm i}\not{\mathcal{D}}\ln\chi\,\epsilon^i + \tfrac{1}{3}{\rm i}\not{k}_j{}^i\,\epsilon^j + \eta^i\,, \end{aligned} \tag{4.2.2}$$

where here and henceforth we suppress terms proportional to the fermion fields. Furthermore we made use of the identity [34],

$$\chi^{-1}\Omega_{\alpha\beta}\,A_i{}^\alpha\mathcal{D}_\mu A_j{}^\beta = \tfrac{1}{2}\varepsilon_{ij}\,\mathcal{D}_\mu\ln\chi + k_{\mu i}{}^k\varepsilon_{kj}\,, \tag{4.2.3}$$

where $k_{\mu j}{}^i$ is proportional to the SU(2) Killing vectors of the underlying hyperkähler cone.

We now require that the S-supersymmetric linear combinations,

$$\begin{aligned} &\zeta^i_{\rm V} - \zeta^i_{\rm H}\,, \quad \zeta^\alpha - \tfrac{3}{2}A_i{}^\alpha\,\zeta^i_{\rm H}\,, \quad \Omega^{iA} - \sigma^A\zeta^i_{\rm V}\,,\\ &\varphi^i - 3\,\varepsilon_{jk}L^{ij}\zeta^k_{\rm V}\,, \quad \chi^i - \tfrac{3}{16}T_{ab}\gamma^{ab}\zeta^i_{\rm V}\,, \end{aligned} \tag{4.2.4}$$

do not transform under Q-supersymmetry. This leads to the following conditions,

$$\begin{aligned} \mathcal{D}_\mu(\chi^{-1/2}A_i{}^\alpha) &= 0\,, & k_{\mu j}{}^i &= 0\,,\\ \partial_\mu(C^{-1/3}(\sigma)\,\sigma^A) &= 0\,, & R_{\mu\nu i}{}^j(V) &= 0\,,\\ \mathcal{D}_\mu\big(C^{-1}(\sigma)\,L^{ij}\big) &= 0\,, & Y^{ijA} &= 0\,,\\ C(\sigma)\,\chi^{-1} &= \text{constant}\,, & N &= 0\,,\\ F_{ab}{}^A &= 4\,\sigma^A T_{ab}\,, & \hat{E}^a &= 0\,,\\ \mathcal{D}_{[a}\big(C^{1/3}(\sigma)\,T_{bc]}\big) &= 0\,, & D &= 0\,,\\ \mathcal{D}_b\big(C^{2/3}(\sigma)T^{ba}\big) &= \mathrm{i}\varepsilon^{abcde}T_{bc}T_{de}\,C^{2/3}(\sigma)\,, \end{aligned} \tag{4.2.5}$$

which were also given in [95] in the conventions of [31, 32]. However, there are further constraints in view of the fact that *all* fermionic quantities must vanish under supersymmetry. Experience from the corresponding analysis in four space-time dimensions [54] indicates that one must also consider the variations of

$$\begin{aligned} &R_{ab}{}^i(Q) - (T_{cd}\gamma_{cd}\gamma_{ab} - 4\,T_{ab})\zeta^i_{\rm V}\,,\\ &D_\mu\zeta^i_{\rm H} - \tfrac{1}{6}[\delta^i_j \not{\mathcal{D}}\ln\chi\gamma_\mu - 2\hat{\not{k}}_j{}^i\gamma_\mu - 6\mathrm{i}\,T_{\mu a}\gamma^a]\zeta^j_{\rm H}\,. \end{aligned} \tag{4.2.6}$$

Combining the result of the first variation with the previous results, one finds,

$$\begin{aligned} \mathcal{D}_c T_{ab} &= \tfrac{1}{2}\mathrm{i}\,\eta_{c[a}\varepsilon_{b]defg}\,T^{de}T^{fg}\\ &\quad - \tfrac{1}{3}\Big[2\,\mathcal{D}_{[a}\ln C(\sigma)\,T_{b]c} - \mathcal{D}_c\ln C(\sigma)\,T_{ab} - 2\,\mathcal{D}^d\ln C(\sigma)\,T_{d[a}\,\eta_{b]c}\Big]\,,\\ R_{ab}{}^{cd}(M) &= -2\Big[T^2\,\delta_{ab}{}^{cd} + 4\,T_{ab}T^{cd} + 4\,T_{[a}{}^cT_{b]}{}^d - 8\,T_{e[a}T^{e[c}\delta_{b]}{}^{d]}\Big]\,. \end{aligned} \tag{4.2.7}$$

In addition one considers the variation of the second combination in (4.2.6), subject to the conditions (4.2.5). This confirms the consistency of the previous results and, in addition, gives rise to one more condition,

$$\begin{aligned} f_\mu{}^a = &-\tfrac{1}{6}\mathcal{D}_\mu\mathcal{D}^a\ln\chi + \tfrac{1}{18}\mathcal{D}_\mu\ln\chi\mathcal{D}^a\ln\chi - 4\,T_{\mu b}T^{ab}\\ &+ \tfrac{1}{4}\Big[3\,T_{bc}T^{bc} - \tfrac{1}{9}(\mathcal{D}_b\ln\chi)^2\Big]\;e_\mu{}^a\,. \end{aligned} \tag{4.2.8}$$

Using the arguments presented in [54], we conclude that the above equations (4.2.5), (4.2.7) and (4.2.8) comprise all the conditions for a supersymmetric field configuration consisting of the Weyl multiplet, vector multiplets, linear multiplets and hypermultiplets, without imposing equations of motion. Because the fermionic equations of motion must be satisfied, simply because of supersymmetry, most of the bosonic equations of motion must be satisfied as well. There are, however, exceptions, such as the equation of motion associated with the scalar field D belonging to the Weyl multiplet, which does not appear as the supersymmetry variation of a fermionic expression.

Combining the second equation of (4.2.7) with (4.2.8), we derive the following equation for the Riemann tensor,

$$\begin{aligned} \mathcal{R}_{ab}{}^{cd}(\omega,e) &= 2\,e_a{}^{\mu}e_b{}^{\nu}\Big(\partial_{[\mu}\omega_{\nu]}{}^{cd} - \omega_{[\mu}{}^{ce}\omega_{\nu]e}{}^{d}\Big) \\ &= -\,8\big(T_{ab}T^{cd} + T_a{}^{[c}T_b{}^{d]}\big) + \delta^c_{[a}\delta^d_{b]}\big(4\,T_{ef}T^{ef} - \tfrac{2}{9}(\mathcal{D}_e\ln\chi)^2\big) \\ &\quad -\,\delta_{[a}{}^{[c}\Big(16\,T_{b]e}T^{d]e} + \tfrac{4}{3}\mathcal{D}_{b]}\mathcal{D}^{d]}\ln\chi - \tfrac{4}{9}\mathcal{D}_{b]}\ln\chi\,\mathcal{D}^{d]}\ln\chi\Big)\,. \end{aligned} \tag{4.2.9}$$

4.2.2 Space-time geometry

Before discussing the resulting space-time geometry we have to impose a number of gauge choices, as in section 3.2. We set the dilatational gauge field $b_\mu = 0$ (in fact, K-invariance implies that the equations found above are already independent of b_μ) and furthermore we set the function $C(\sigma)$ equal to a constant C. This implies that also χ becomes a constant. The ratio of the two constants C and χ will eventually be defined by the equation of motion for the field D, but at the moment we proceed without making reference to any particular Lagrangian. Note that the various fields will still be subject to constant scale transformations which are a remnant of the full space-time dependent dilatations. Physical results should, of course, be insensitive to these scale transformations. In addition we set the SU(2) gauge connections to zero, in view of the fact that their field strength is vanishing (c.f. (4.2.5)). In this situation the various scalar fields σ^I and L^{ij} are all constant.

The resulting geometry is now of a special type, as the tensor $T_{\mu\nu}$ is an example of a conformal Killing-Yano tensor [101]. Locally, in five space-time dimensions, this tensor generically induces a family of pairs of two-surfaces which together with the fifth orthogonal dimension foliate the space-time. It also leads to a Killing vector associated with this fifth dimension and a symmetric Killing tensor,

$$\xi^\mu = \mathrm{i}e^{-1}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,T_{\nu\rho}\,T_{\sigma\tau}\,, \qquad K_{\mu\nu} = T_{\mu\rho}\,T_\nu{}^{\rho}\,, \tag{4.2.10}$$

where $e = \det(e_\mu{}^a)$. Using the properties of the tensor $T_{\mu\nu}$ (in the gauge indicated above), we obtain the following results for the Riemann tensor and for the derivative of $T_{\mu\nu}$,

$$\begin{aligned} \mathcal{R}_{ab}{}^{cd} &= -8\big(T_{ab}T^{cd} + T_a{}^{[c}T_b{}^{d]}\big) - 16\,\delta_{[a}{}^{[c}\,T_{b]e}T^{d]e} + 4\,\delta^c_{[a}\delta^d_{b]}\,T_{ef}T^{ef}\,, \\ \mathcal{D}_\rho T_{\mu\nu} &= \tfrac{1}{2}g_{\rho[\mu}\,\xi_{\nu]}\,. \end{aligned} \tag{4.2.11}$$

Furthermore we note the results,

$$\begin{aligned} \mathcal{D}_\mu\xi_\nu &= -\mathrm{i}e\,\varepsilon_{\mu\nu\rho\sigma\tau}\xi^\rho T^{\sigma\tau}\,, &\qquad \xi^\mu T_{\mu\nu} &= 0\,, \\ \mathcal{D}_\rho K_{\mu\nu} &= -\tfrac{1}{2}\xi_{(\mu}T_{\nu)\rho}\,, & T^2 &\equiv (T_{ab})^2 = \text{constant}\,. \end{aligned} \tag{4.2.12}$$

From these equations it is clear that ξ^μ is indeed a Killing vector. Furthermore one may easily verify that the Riemann tensor satisfies the Bianchi identity.

If ξ^μ vanishes then the tensors $T_{\mu\nu}$ and $K_{\mu\nu}$ are covariantly constant and so is the Riemann tensor so that we are dealing with a locally symmetric space. In this particular case the space is in fact the product of a two- and a three-dimensional maximally symmetric space, as the Riemann tensor decomposes into two Riemann tensors corresponding to these subspaces satisfying $R_{\hat a\hat b}{}^{\hat c\hat d} \propto c\,T^2\,\delta_{[\hat a}{}^{\hat c}\delta_{\hat b]}{}^{\hat d}$, with proportionality $c=-16$ and $c=4$ for the two- and the three-dimensional subspace, respectively. Here the indices $\hat a, \hat b, \hat c, \hat d$ refer to the tangent-space projected onto the two- or three-dimensional subspaces.

Rather than considering this case any further, we concentrate on the more general case where $\xi^\mu \neq 0$ and return to the limit of vanishing ξ^μ at the end. Obviously the line element must reflect the isometry associated with the Killing vector ξ^μ. Choosing a coordinate ψ by $\xi^\mu\partial_\mu = \partial/\partial\psi$, we decompose the coordinates into ψ and four-dimensional coordinates x^m, where $m = 1,2,3,4$, without committing ourselves to a certain signature yet.[2] Correspondingly, the tangent-space indices $a = 1,2,\ldots,5$ are decomposed into $a = 5$ and indices $p,q,\ldots = 1,2,3,4$. Upon a suitable local Lorentz transformation, the fünfbein is brought into the form,

$$e_\mu{}^5\,\mathrm{d}x^\mu = \mathrm{e}^g\big[\mathrm{d}\psi + \sigma_m\,\mathrm{d}x^m\big]\,, \qquad e_\mu{}^p\,\mathrm{d}x^\mu = \mathrm{e}^{-g/2}\,\hat e_m{}^p\,\mathrm{d}x^m\,. \tag{4.2.13}$$

In view of the isometry corresponding to shifts of the coordinate ψ we may assume that g, σ_m and the vierbein field $\hat e_m{}^p$ do not depend on ψ. The corresponding inverse fünfbein components are given by,

$$e_5{}^\psi = \mathrm{e}^{-g}\,, \qquad e_5{}^m = 0\,, \qquad e_p{}^\psi = -\sigma_p\,\mathrm{e}^{g/2}\,, \qquad e_p{}^m = \mathrm{e}^{g/2}\,\hat e_p{}^m\,, \tag{4.2.14}$$

where, on the right-hand side, four-dimensional tangent-space and world indices are converted by the vierbein $\hat e_m{}^p$ and its inverse (so that, e.g. $\sigma_p = \hat e_p{}^m\,\sigma_m$, and the covariant derivative ∇_p contains the spin connection $\hat\omega_m{}^{pq}$, associated with the vierbein $\hat e_m{}^p$). This leads to the following expressions for the spin connection, $\omega_{abc} \equiv e_a{}^\mu\,\omega_{\mu cd}$,

$$\begin{aligned}
\omega_{pqr} &= \mathrm{e}^{g/2}\left[\hat\omega_{pqr} + \delta_{p[q}\nabla_{r]}g\right],\\
\omega_{5pq} &= \omega_{qp5} = \tfrac{1}{2}\mathrm{e}^{2g}\,\mathcal{Q}_{pq}\,,\\
\omega_{55p} &= -\mathrm{e}^{g/2}\,\nabla_p g\,,
\end{aligned} \tag{4.2.15}$$

where $\mathcal{Q}_{pg}$ equals,

$$\mathcal{Q}_{pq} = \hat e_p{}^m \hat e_q{}^n\,\mathcal{Q}_{mn}\,, \quad \mathcal{Q}_{mn} = \partial_m\sigma_n - \partial_n\sigma_m\,. \tag{4.2.16}$$

[2] At this point we are using Pauli-Källén metric conventions, where the signature is determined by making one of the coordinates purely imaginary. This enables us to consider all possible signatures at once, so that this analysis encompasses the solutions for minimal supergravity found in [25]. Momentarily we will assume that the Killing vector ξ^μ is spacelike.

Let us now return to (4.2.11) and consider the second equation, which we write in tangent-space indices as,

$$e_c{}^\mu\,\partial_\mu T_{ab} + 2\,\omega_{c[a}{}^d\,T_{b]d} = \tfrac{1}{2}\delta_{c[a}\delta_{b]5}\,\xi\,, \tag{4.2.17}$$

where we made use of the fact that $T_{5a} = 0$ and defined $\xi = \mathrm{i}\varepsilon^{pqrs}T_{pq}T_{rs}$, where $\varepsilon^{pqrs} = \varepsilon^{5pqrs}$ so that $\xi^\psi = \mathrm{e}^{-g}\,\xi$. Changing the overall sign of the epsilon tensor is irrelevant as it only corresponds to a sign change of the coordinate ψ.[3] Imposing the equations contained in (4.2.17) leads to the following results,

$$\partial_\psi T_{ab} = 0\,, \qquad \mathcal{Q}_{pq} = -2\mathrm{i}\mathrm{e}^{-2g}\,\varepsilon_{pqrs}T^{rs}\,, \qquad \nabla_p T_{qr} = 0\,, \qquad g = \text{constant}\,. \tag{4.2.18}$$

These results are consistent with what is found when considering the Riemann tensor from the connections (4.2.15) upon comparison with the first equation (4.2.11).[4] Here and henceforth we will be assuming that the four-dimensional subspace has signature $(-,+,+,+)$, so that the Killing vector ξ^μ is spacelike and ξ is real. The various curvature components read,

$$\begin{aligned}
\mathcal{R}_{pq5r} &= -\tfrac{1}{2}\mathrm{e}^{5g/2}\left[\nabla_r\mathcal{Q}_{pq} + \nabla_r g\,\mathcal{Q}_{pq} + \nabla_{[p}g\,\mathcal{Q}_{q]r} - \delta_{r[p}\,\mathcal{Q}_{q]s}\nabla^s g\right]\,,\\
\mathcal{R}_{5p5q} &= \mathrm{e}^g\left[\nabla_p\nabla_q g - \tfrac{1}{2}\delta_{pq}\,(\nabla_r g)^2 + 2\,\nabla_p g\,\nabla_q g\right] - \tfrac{1}{4}\mathrm{e}^{4g}\,\mathcal{Q}_{pr}\,\mathcal{Q}_q{}^r\,,\\
\mathcal{R}_{pqrs} &= \mathrm{e}^g\,\mathcal{R}_{pqrs}(\hat\omega) - 2\,\mathrm{e}^g\,\delta_{[p[r}\left[\nabla_{s]}\nabla_{q]}g + \tfrac{1}{2}\nabla_{s]}g\,\nabla_{q]}g - \tfrac{1}{4}\delta_{s]q]}\,(\nabla_u g)^2\right]\\
&\quad + \tfrac{1}{2}\,\mathrm{e}^{4g}\left[\mathcal{Q}_{pq}\mathcal{Q}_{rs} - \mathcal{Q}_{p[r}\mathcal{Q}_{s]q}\right]\,,
\end{aligned} \tag{4.2.19}$$

where the right-hand side is consistently written in four-dimensional notation. Obviously $\mathcal{R}_{pq5r}$ must vanish in order to be consistent with the first equation (4.2.11), and this is indeed what is implied by the earlier results (4.2.18). Likewise the expression for $\mathcal{R}_{5p5q}$ is consistent with the corresponding equation (4.2.11). Hence we are left to analyse the last equation of (4.2.19), which determines the four-dimensional Riemann tensor $\mathcal{R}(\hat\omega)$ according to

$$\mathcal{R}_{pqrs}(\hat\omega) = -16\,\mathrm{e}^{-g}\Big[4\,\delta_{[p[r}\,T_{s]t}T_{q]}{}^t - \delta_{p[r}\delta_{s]q}\,T^2\Big]\,. \tag{4.2.20}$$

The Ricci scalar, $\mathcal{R}_{ab}{}^{ab}(\hat\omega) = 0$. Further inspection shows that this Riemann tensor corresponds to a product of two two-dimensional spaces with equal radii, namely AdS_2 and S^2. The geometry thus takes the form of a circle (parametrized by the coordinate ψ) non-trivially fibered over an $AdS_2 \times S^2$ base space. We now adopt four-dimensional coordinates by writing the respective metrics in the standard form of a Bertotti-Robinson and a two-sphere metric, with coordinates t, r, and θ, φ, respectively, so that the five-dimensional line element takes the following form (r is non-negative and θ and φ have

[3] In Pauli-Källén notation we now fix convention such that $\varepsilon_{\mu\nu\rho\sigma\tau}x^\mu x^\nu x^\rho x^\sigma x^\tau = \mathrm{i}\,5!\,x^0x^1x^2x^3x^5$.

[4] We note that (4.2.18) has been derived from (4.2.17) assuming $\det[T] \neq 0$. For $\det[T] = 0$ one can arrive at the same result by also making use of (4.2.11) and (4.2.19).

periodicity π and 2π, respectively),

$$\begin{aligned} ds^2 &= \frac{1}{16\,v^2}\Big(-r^2\mathrm{d}t^2+\frac{\mathrm{d}r^2}{r^2}+\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2\Big)+\mathrm{e}^{2g}\big(\mathrm{d}\psi+\sigma\big)^2\,,\\ \sigma &= -\frac{1}{4\,v^2}\,\mathrm{e}^{-g}\left(T_{23}\,r\,\mathrm{d}t-T_{01}\,\cos\theta\,\mathrm{d}\varphi\right)\,, \end{aligned} \tag{4.2.21}$$

corresponding to,

$$\mathcal{Q}_{tr}=\frac{1}{4\,v^2}\,\mathrm{e}^{-g}\,T_{23}\,,\qquad \mathcal{Q}_{\theta\varphi}=-\frac{1}{4\,v^2}\,\mathrm{e}^{-g}\,T_{01}\,\sin\theta\,. \tag{4.2.22}$$

Here and henceforth we use the definition,

$$v=\sqrt{(T_{01})^2+(T_{23})^2}\,, \tag{4.2.23}$$

where T_{01} and T_{23} are the nonvanishing components of the tensor field T_{ab}, where the local Lorentz indices are $(0,1,2,3)$. Note that the vierbein fields can be chosen diagonally; their values can be read off from (4.2.21),

$$e_m{}^p\,\mathrm{d}x^m=\frac{1}{4\,v}\Big(r\,\mathrm{d}t,\;\frac{\mathrm{d}r}{r},\;\mathrm{d}\theta,\;\sin\theta\,\mathrm{d}\varphi\Big)\,,\qquad (p=0,1,2,3)\,. \tag{4.2.24}$$

In this Lorentz frame, the fields T_{ab} are constant. For future use we also list the nonvanishing spin-connection fields,

$$\begin{aligned} \omega_m{}^{pq} &= \overset{\circ}{\omega}_m{}^{pq}+\tfrac{1}{2}\sigma_m\,\mathrm{e}^{3g}\,\mathcal{Q}^{pq}\,,\\ \omega_m{}^{p5} &= \tfrac{1}{2}e_{mq}\,\mathrm{e}^{2g}\,\mathcal{Q}^{pq}\,,\\ \omega_\psi{}^{pq} &= \tfrac{1}{2}\mathrm{e}^{3g}\,\mathcal{Q}^{pq}\,, \end{aligned} \tag{4.2.25}$$

where $\overset{\circ}{\omega}_t{}^{01}=-r$ and $\overset{\circ}{\omega}_\varphi{}^{23}=\cos\theta$.

Observe that σ^I, T_{ab}, v and e^{-g} transform with weight $+1$ under the (constant) scale transformations inherited from the five-dimensional dilatations. As a result, the metric (4.2.21) scales uniformly with weight -2 and the one-form σ is inert under scale transformations. Note that σ is determined up to a four-dimensional gauge transformation associated with shifts of the coordinate ψ with a function depending on the four-dimensional coordinates. Such diffeomorphisms leave the form of the line element invariant.

Let us now further discuss the line element (4.2.21). Assuming that $T_{01}\neq 0$, we can rewrite the line element in the form,

$$\begin{aligned} ds^2 &= -\frac{\rho^4}{16\,v^2}\left(\frac{T_{01}}{v}\,\mathrm{d}t+\frac{T_{23}}{v\,\rho^2}\Big(\cos\theta\,\mathrm{d}\varphi+\frac{1}{p^0}\,\mathrm{d}\psi\Big)\right)^2\\ &\quad+\frac{1}{4\,v^2\rho^2}\left(\mathrm{d}\rho^2+\frac{\rho^2}{4}\Big(\mathrm{d}\theta^2+\mathrm{d}\varphi^2+\frac{1}{(p^0)^2}\,\mathrm{d}\psi^2+\frac{2}{p^0}\,\cos\theta\,\mathrm{d}\varphi\,\mathrm{d}\psi\Big)\right) \end{aligned} \tag{4.2.26}$$

where we used the definitions

$$\rho = \sqrt{r}\,, \qquad p^0 = \frac{\mathrm{e}^{-g}}{4\,v^2}\,T_{01}\,. \tag{4.2.27}$$

To make p^0 unambiguous we fix the periodicity interval for ψ to 4π. The second term of the line element then corresponds to a flat metric, up to an overall warp factor $(2v\rho)^{-2}$. To see this we combine the four Cartesian coordinates into two complex ones, which we parametrize as,

$$z_1 = \rho\,\cos\theta/2\,\exp\tfrac{1}{2}\mathrm{i}[\psi/p^0+\varphi]\,, \qquad z_2 = \rho\,\sin\theta/2\,\exp\tfrac{1}{2}\mathrm{i}[\psi/p^0-\varphi]\,. \tag{4.2.28}$$

Clearly for $|p^0| = 1$ we cover the whole four-dimensional space $\mathbb{R}^4$. For $|p^0| \neq 1$ we have a conical singularity at the origin. In all cases the three-dimensional horizon is located at $r = 0$ and its cross-sectional area is equal to

$$A_3 = \int_{\Sigma_{\rm hor}} = \pi^2 v^{-2}\,\mathrm{e}^g\,. \tag{4.2.29}$$

Observe that this result is not invariant under the scale transformations introduced earlier, which simply reflects the fact that the line element is not invariant either. Furthermore the bi-normal tensor at the horizon is the same in all cases when given with tangent space indices. Its only non-vanishing components are,

$$\varepsilon_{01} = \pm 1\,, \tag{4.2.30}$$

so that $\varepsilon_{\mu\nu}\varepsilon^{\mu\nu} = -2$. Both (4.2.29) and (4.2.30) can be derived by first determining the bi-normal tensor and the cross-sectional area in a coordinate frame that is non-singular at the horizon, and subsequently converting the results to the singular frame used in the text.

The line element (4.2.26) describes the near-horizon geometry of the spinning charged black hole [39] (see also, [89]), and we observe that the rotation is associated with a globally defined one-form on S^3, in view of

$$\mathrm{Im}\,[z_1\,\mathrm{d}z_1{}^* + z_2\,\mathrm{d}z_2{}^*] = \rho^2[(p^0)^{-1}\mathrm{d}\psi + \cos\theta\,\mathrm{d}\varphi]\,.$$

Clearly the angular momentum of the black hole is proportional to T_{23}. When $T_{23} = 0$ we are dealing with a static black hole and the near-horizon geometry is given by,

$$ds^2 = \frac{1}{16\,v^2}\left(-r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2}\right) + \frac{1}{4\,v^2}\,ds^2(S^3/\mathbb{Z}_{p^0})\,. \tag{4.2.31}$$

Finally we turn to the case $T_{01} = 0$ where we find,

$$ds^2 = \frac{1}{16\,T_{23}{}^2}\frac{\mathrm{d}r^2}{r^2} + \mathrm{e}^{2g}\,\mathrm{d}\psi^2 - \frac{\mathrm{e}^g}{2\,T_{23}}\,r\,\mathrm{d}\psi\,\mathrm{d}t + \frac{1}{16\,T_{23}{}^2}\,\mathrm{d}s^2(S^2)\,, \tag{4.2.32}$$

where $\mathrm{d}s^2(S^2)$ is the line element belonging to the unit two-sphere. The first three terms constitute a metric which is locally AdS_3 so that the near-horizon geometry is that of $AdS_3 \times S^2$. This is the near-horizon geometry of a supersymmetric black ring, or, when we drop the identification $\psi \cong \psi + 4\,\pi$, of an infinitely long black string.

4.2.3 Gauge fields and linear multiplets

We now turn to the gauge fields and the composite linear multiplets constructed in sections 2.1.1 and 4.1 from the product of two vector multiplets and from the square of the Weyl multiplet. Due to the intricate relationship between the vector and linear multiplets described in section 2.1.1, the equations of motion for the vector multiplets are expressed through the components of the two linear multiplets. As the reader can easily verify using the results in section 4.2.1, both vanish for BPS configurations, so that the equations of motion are satisfied.

In particular, the condition that the vector component, E_a, of any linear multiplet must vanish, imposes the Maxwell equations. The corresponding three-forms, denoted by $E_{\mu\nu\rho}$, are then required to be closed by (B.1.13), but do not necessarily vanish and provide the standard definition of the electric charges. Since these results are central to what follows, we will evaluate some of the corresponding expressions here.

According to (4.2.5), the field strengths $F_{\mu\nu}{}^A$ are determined in terms of the tensor field T_{ab},

$$F_{tr}{}^A = \frac{\sigma^A}{4\,v^2}\,T_{01}\,, \qquad F_{\theta\varphi}{}^A = \frac{\sigma^A}{4\,v^2}\,T_{23}\,\sin\theta\,. \tag{4.2.33}$$

At this point we can define magnetic charges associated with $Q_{\theta\varphi}$ and $F_{\theta\varphi}{}^A$. Employing the same conventions for these field strengths (apart from a relative sign between p^0 and p^A), we define

$$p^0 = \frac{\mathrm{e}^{-g}}{4\,v^2}\,T_{01}\,, \qquad p^A = \frac{\sigma^A}{4\,v^2}\,T_{23}\,, \tag{4.2.34}$$

with the same expression for p^0 as given in (4.2.27). In the five-dimensional context, the p^A will play the role of dipole magnetic charges. They are proportional to T_{23}, so they will vanish for a static black hole. The definition of the electric charges, which involves the equations of motion, will be discussed in section 4.3. From (4.2.33) we can determine the vector potentials,

$$W_\mu{}^A(x)\,\mathrm{d}x^\mu = -\,\frac{\sigma^A}{4\,v^2}\,(T_{01}\,r\,\mathrm{d}t + T_{23}\,\cos\theta\,\mathrm{d}\varphi) + \,\mathrm{d}\Lambda^A(x)\,, \tag{4.2.35}$$

up to an abelian gauge transformation, parametrized by $\Lambda^A(x)$.

Given an appropriate choice for this transformation, which should define the fifth component of the gauge field, $W_\psi{}^A$, we present some components of the tensor field $E^{(AB)}_{\mu\nu\rho}$, defined in (2.1.6). Subject to the BPS conditions, one obtains the following results,

$$\begin{aligned} E_{\psi\theta\varphi}^{(AB)} &= \frac{\sin\theta}{8\,v^2}\Big[-\mathrm{e}^g\,\sigma^A\sigma^B\,T_{01} + \sigma^{(A}W_\psi{}^{B)}\,T_{23}\Big]\,, \\ E_{\psi rt}^{(AB)} &= -\frac{1}{8\,v^2}\Big[\mathrm{e}^g\,\sigma^A\sigma^B\,T_{23} + \sigma^{(A}W_\psi{}^{B)}\,T_{01}\Big]\,, \\ E_{rt\varphi}^{(AB)} &= -\cos\theta\,\frac{T_{01}T_{23}\,\sigma^A\sigma^B}{32\,v^4}\,, \\ E_{\theta t\varphi}^{(AB)} &= r\sin\theta\,\frac{T_{01}T_{23}\,\sigma^A\sigma^B}{32\,v^4}\,. \end{aligned} \tag{4.2.36}$$

Note that these components are invariant under the scale transformations mentioned previously. The choice of the correct gauge transformation Λ^A depends on the presence of magnetic charges, which do not allow for a globally defined gauge field.

For the spinning black hole, where $T_{01} \neq 0$, the gauge potentials can be chosen to be globally defined on S^3. To see this one makes use of the observation preceding (4.2.31) in the previous subsection, which leads to,

$$W_\mu{}^A\mathrm{d}x^\mu = -\frac{\sigma^A}{4\,v^2}\left(T_{01}\,r\,\mathrm{d}t + T_{23}\left(\frac{\mathrm{d}\psi}{p^0} + \cos\theta\,\mathrm{d}\varphi\right)\right)\,. \tag{4.2.37}$$

In the case of the black ring, where $T_{01} = 0$, the gauge transformations in (4.2.35) introduce an uncontractible component corresponding to Wilson lines around the circle parametrized by ψ. The proper definition of the Wilson line moduli is subtle due to the presence of the charges p^A and the $S^1 \times S^2$ topology, as we shall discuss below. Generically, due to the presence of large gauge transformations (i.e. gauge transformations that cannot be connected continuously to the identity), these moduli a^A should be periodically identified and furthermore they should be defined such that they are not subject to small gauge transformations. At any rate, the gauge fields are expected to contain the following terms,

$$W_\mu{}^A\mathrm{d}x^\mu = -p^A\cos\theta\,\mathrm{d}\varphi + a^A\mathrm{d}\psi\,. \tag{4.2.38}$$

However, unlike in the case of the spinning black holes, the gauge fields are not globally defined, as is obvious from the fact that the monopole fields are sourced by Dirac strings. This phenomenon implies that the gauge fields should be defined in patches, connected by suitable gauge transformations. In the context of five space-time dimensions the Dirac strings are degenerate and one is actually dealing with Dirac membranes. Just as in the case of Dirac strings, the Dirac membranes are subject to constraints, some of them related to charge quantization (to appreciate this, the reader may consult [102, 103], where some of this is explained in the context of $2+1$ dimensions).

For a single black ring and for multiple concentric black rings, the appropriate sections have been considered in [58], guided by the explicit ring solutions [40] and [35]. Although these results were obtained without taking into account possible higher-derivative interactions, they should still apply to the general case, as the choice of the sections and the corresponding Dirac membranes is entirely based on the topology of the underlying charge configuration. With this in mind we replace (4.2.38) by the following sections (for a single ring),

$$W_\mu{}^A \mathrm{d}x^\mu = -p^A \left[\cos\theta\, \mathrm{d}\varphi \pm \mathrm{d}(\varphi + \tfrac{1}{2}\psi)\right] + a^A \mathrm{d}\psi\,, \tag{4.2.39}$$

where we note that $\cos\theta$ can be extended globally into the ring coordinate conventionally denoted by x [40, 104]. For $x = 1$ and $x = -1$ one is dealing with the inner and the outer part, respectively, of the two-dimensional plane that contains the ring. Hence the plus sign in (4.2.39) refers to the section that is singularity free in the outer part of plane, and the minus sign to the section that is singularity free in the inner part.

The nontrivial, and somewhat unexpected, feature of (4.2.39), is that the gauge transformation between the two patches involves a $\mathrm{d}\psi$ component, contrary to what one would expect based on intuition from four dimensions. Indeed, in the case of an infinite black string, this gauge transformation is just $\propto p^A \mathrm{d}\varphi$. However, the ring topology requires a more extended gauge transformation.

One way to understand this difference is to appreciate the fact that, in order that the Dirac membrane be unobservable, the gauge transformation between the patches must allow for general deformations of its worldvolume irrespective of its topology. Choosing a topologically trivial brane on each patch, say along the north and south pole of each sphere on the ring (see the two figures on the left-hand side of Fig. 4.1), leads to the gauge transformation $-2p^A \mathrm{d}\varphi$ between the patches. This is also the only possible choice for an infinite string. But in the case of a proper ring embedded in a four-dimensional space,[5] the topology of the spatial manifold $\mathcal{M}^4$ corresponding to the embedding space minus the ring is nontrivial. Possible Dirac branes are classified as the boundaries of three-dimensional spatial hypersurfaces. Thus it is important to know the third homology group $H_3(\mathcal{M}^4)$, since the Dirac brane can also be the boundary of a nontrivial hypersurface, as opposed to the trivial one discussed above.

In the case at hand it can be shown that $H_3(\mathcal{M}^4) = \mathbb{Z}$, so that the generator of the group is a hypersurface with no boundary that wraps the ring once. A corresponding Dirac brane is described as the boundary of the sum of the topologically trivial hypersurface and this generator. Such a brane starts at the north pole of the sphere at some point along the ring. When moving along the S^1 of the ring, this brane rotates to the south pole and subsequently it returns to the north pole when reaching the point of departure. A singular limit of this surface is shown on the right-hand side of Fig. 4.1. Using the construction based on de Rham currents in [102, 103, 105], this leads to a gauge transformation between the gauge field patches that is proportional to the Poincaré dual

[5] We assume a topologically trivial embedding space, like $\mathbb{R}^4$ or Taub-NUT, in the following discussion.

of the generator described above. A component along ψ is obviously necessary due to the plane in the centre. The relative coefficient in the gauge transformation $\mathrm{d}(\varphi + \frac{1}{2}\psi)$ has been fixed by demanding periodicity of this generator. Finally, note that higher wrappings would introduce integral multiples of $\mathrm{d}\psi$, and are therefore irrelevant in view of the integral shift symmetry of a^A.

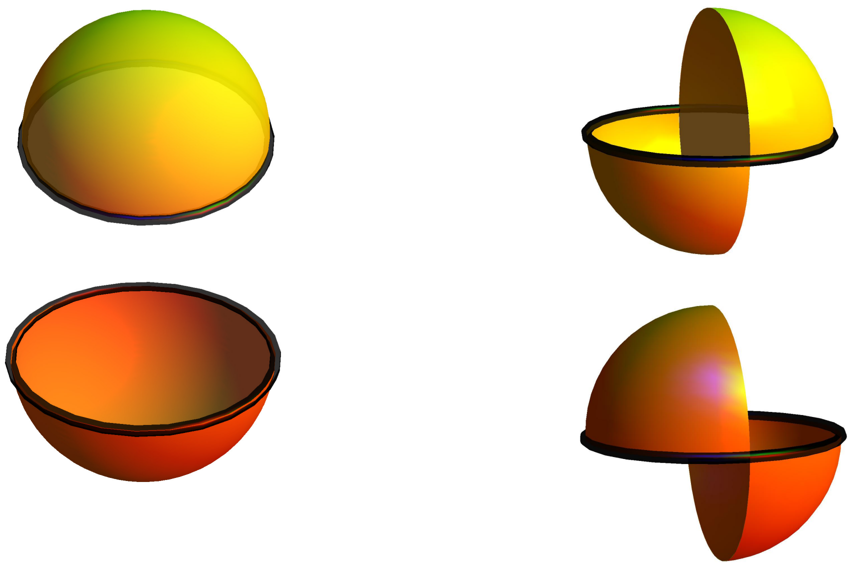

Figure 4.1: The two figures on the left-hand side correspond to the two different gauge field patches based on a topologically trivial choice for the two-dimensional Dirac brane. The three-dimensional hypersurface bounded by the two branes is a ball B^3. The gauge transformation associated with the transition between the two patches has only components along the angle φ not shown in the picture. On the right-hand side the two figures show a singular limit of the relevant but non-trivial choice for the Dirac brane. The three-dimensional hypersurface connecting the two branes is the sum of the B^3 above and the generator of $H_3(\mathcal{M}^4)$. The corresponding gauge transformation has an additional component along the ring circle, normal to the plane.

The way to measure the Wilson line moduli a^A now proceeds through the Chern-Simons charges of the ring, defined by the integral over the 3-cycle associated with the horizon Σ,

$$Q^{\rm CS}_A \propto \oint_\Sigma C_{ABC}\, W^B \wedge F^C \,, \tag{4.2.40}$$

as in (4.2.36). It was demonstrated in [58], by using the sections (4.2.39) and carefully evaluating the integral, that the Chern-Simons charges are linearly related to the moduli a^A, i.e., $Q^{\rm CS}_A \propto C_{ABC}\, a^B p^C$. The use of the sections (4.2.39) is essential for obtaining this relationship, so that the a^A, which are identified in this way, are no longer subject to small gauge transformations. Here it is relevant that the Chern-Simons charges are also invariant under small gauge transformations. This result is also consistent with large gauge transformations as both the a^A and the $Q^{\rm CS}_A$ change under a large gauge transformations by an integer (in proper units).

Although it is not the primary purpose of this discussion to consider multi-ring solutions, it is illuminating to briefly consider the situation of concentric rings [35]. Labeling the rings by an index i, one introduces the moduli ${a^A}_i$ and the charges ${p^A}_i$ of the i-th ring. Following the same logic as above, an extended set of sections generalizing (4.2.39) can be found that matches the one used in [58]. One can then derive the following relation,

$$Q_A^{\rm CS} \propto C_{ABC}\Big[\sum_i (2\,a^B+p^B)_i\, {p^C}_i - \Big(\sum_i {p^B}_i\Big)\Big(\sum_j {p^C}_j\Big)\Big]\,, \tag{4.2.41}$$

which, for a single ring, reduces to the previous result. The above relation indicates that the Chern-Simons charges are not additive, unlike the moduli $(a^A)_i$ and the charges $(p^A)_i$ associated with the various rings. In fact, as we will establish later in section 4.5, the best way to write this result is as follows,

$$Q_A^{\rm CS} - 6\,C_{ABC}P^BP^C = -12\,C_{ABC}\sum_i (a^B+\tfrac{1}{2}p^B)_i\,{p^C}_i\,, \tag{4.2.42}$$

where ${P^A}_i = \sum_i {p^A}_i$. This indicates that the expression on the left-hand side is in fact additive. We will return to this topic in section 4.5.

One encounters similar subtleties when considering the three–form $E^{\rm W}_{\mu\nu\rho}$, defined in (4.1.3). Using the BPS conditions, it is straightforward to derive the following expression

$$\begin{aligned} E^{\rm W}_{\mu\nu\rho} &= -\tfrac{3}{16}\omega_{[\mu}{}^{ab}\left(\partial_\nu\omega_{\rho]\,ab} - \tfrac{2}{3}\omega_{\nu\,ac}\,\omega_{\rho]}{}^c{}_b\right) \\ &\quad - \tfrac{3}{4}{\rm i e}\,\varepsilon_{\mu\nu\rho\sigma\lambda}\Big(T^2T^{\sigma\lambda} + 6\,T^{\sigma\kappa}T_{\kappa\tau}T^{\tau\lambda}\Big)\,, \end{aligned} \tag{4.2.43}$$

which is manifestly scale invariant. Adopting the gauge choice for the spin connection in (4.2.25), this leads to the following components

$$\begin{aligned} E^{\rm W}_{\psi\theta\varphi} &= -\tfrac{3}{8}\sin\theta\,{\rm e}^g\,T_{01}\,. \\ E^{\rm W}_{\psi rt} &= \tfrac{3}{8}\,{\rm e}^g\,T_{23}\,, \\ E^{\rm W}_{rt\varphi} &= \cos\theta\,\frac{T_{01}T_{23}}{16\,v^2}\,, \\ E^{\rm W}_{\theta t\varphi} &= -\,r\sin\theta\,\frac{T_{01}T_{23}}{16\,v^2}\,. \end{aligned} \tag{4.2.44}$$

However, just as for gauge fields, it is generally not possible to define the Chern–Simons term in (4.2.43) globally when the Euler density ${\rm Tr}[\mathcal{R}\wedge\mathcal{R}]$ has a nontrivial source and our choice above requires some explanation. For a black ring the Euler form is trivial, but for the spinning black hole its integral is related to the charge p^0, which is a topological property of the spatial base space. We will again employ two patches, similar to the magnetic monopole, connected by an appropriate closed but non-exact gauge transformation, β^0, defined as

$$\omega_{[\mu}{}^{ab}\left(\partial_\nu \omega_{\rho]\,ab} - \tfrac{2}{3}\omega_{\nu\,ac}\,\omega_{\rho]}{}^{c}{}_{b}\right)\Big|_1 = \omega_{[\mu}{}^{ab}\left(\partial_\nu \omega_{\rho]\,ab} - \tfrac{2}{3}\omega_{\nu\,ac}\,\omega_{\rho]}{}^{c}{}_{b}\right)|_2 + \beta^0_{\mu\nu\rho}\,. \tag{4.2.45}$$

Since the Chern-Simons term is a composite object constructed from gauge fields, rather a fundamental gauge field, the transformation between patches can be ultimately considered as a particular choice of gauge transformation on the spin connection.[6] Here, we concentrate on the construction of the patches, disregarding the fundamental gauge transformations. Any additional ambiguity on the Chern-Simons term is assumed to be of trivial cohomology and gauge fixed through (4.2.25).

Viewing the Chern-Simons term as a gauge potential coupled to a membrane in five spacetime dimensions, one concludes that the magnetic dual is a scalar field sourced at a single point (completely localised in spacetime). Such a magnetic monopole at the centre of the geometry gives rise to a Dirac string ending on it and reaching timelike infinity. Any given choice for the string defines a gauge patch and is connected to a different choice by a two-dimensional surface, which in turn defines the gauge transformation between the two patches. Now consider two Dirac strings, one moving off to infinite timelike future and one to infinite timelike past, so that they are connected by a surface spanning the time and radial coordinates. Similar to the gauge theory example in (4.2.39), the gauge transformation connecting the two patches is given by the normal form of this connecting surface. For the metric in (4.2.26) one finds

$$\beta^0 = \frac{T_{01}^2}{v^2 p^0}\ \sin\theta\ d\theta \wedge d\phi \wedge d\psi\,, \tag{4.2.46}$$

where the correct normalisation is fixed by demanding that in the static limit the flux of β^0 is equal to $2/p^0$, i.e. equal to the flux of $\mathrm{Tr}[\mathcal{R}\wedge\mathcal{R}]$ for a Gibbons–Hawking base space.

Assuming the Dirac strings are timelike, there will be exactly one intersection of one of the Dirac strings with a given time slice. The gauge transformation (4.2.46) on the Chern-Simons term changes the position of this intersection, shifting the integral of the above Chern-Simons term over a 3-surface, affecting the definition of the electric charges through the integral of (4.2.43). Note that while this is similar to the situation encountered above for the Wilson line moduli in (4.2.38), the ambiguity in this case is only related with the position of the Dirac brane singularity on each time slice. In this context, there are only two reasonable choices, namely spatial infinity and behind the horizon, contrary to the infinitely many equivalent choices for the α^A in (4.2.38).

In a purely five-dimensional setting one can push the Dirac brane to infinity and calculate all physical charges in the standard way. The connection in (4.2.44) is chosen according

[6] As the argument presented here is independent of the connection used, one can consider the Christoffel connection in computing the Chern–Simons term.

to this requirement, in line with the general view that a nontrivial Taub-NUT charge is not considered to be part of the black hole in the centre, so there should be no singularity coming from the $\mathcal{R} \wedge \mathcal{R}$ term there. This parallels the microscopic counting [10, 39, 68], where the large charge limit is taken for the electric charges but not for the Taub-NUT charge.

On the other hand, when considering the corresponding four-dimensional solution obtained by dimensional reduction, it is not acceptable to have a gauge dependent singularity present near spatial infinity. In this case, a physical solution can be obtained if the singularity is hidden behind the horizon, which amounts to a change of patch as in (4.2.46), which adds a delta source singularity to the Euler density. The charge computed using the patch appropriate from a four-dimensional point of view is then different from the one computed in the five-dimensional theory, due to the nonzero integral of (4.2.46) over the three-sphere.

The above difference between four and five-dimensional charges can be also seen by reducing the Chern–Simons term in (4.1.4) to four dimensions and computing the conserved charges of the resulting covariant theory [106]. This confirms the shift of the electric charge by the integral of (4.2.46), independent of supersymmetry.

4.3 The Lagrangian and conserved charges

The results of the previous section are generic for five-dimensional supergravity, in the sense that they follow from symmetry considerations without using a specific Lagrangian. One of the consequences is that, even though supersymmetry imposes enormous restrictions on all fields, these are of a kinematical nature. For example, all scalars must be constant but there are no restrictions on their values and only the magnetic parts of gauge fields could be explicitly related to conserved charges.

In order to proceed in the study of attractors, one needs to make use of the standard conserved charges present in any theory containing gauge fields and gravity, namely the electric charges and angular momentum[7]. Moreover, the entropy of a black hole is a further conserved quantity that has to be identified. The definition of such quantities in the context of a given theory requires the choice of a Lagrangian. Here, we consider extending the two derivative Lagrangian (2.1.7) by the addition of the R^2 invariant introduced in the previous chapter. The total Lagrangian reads

$$\mathcal{L} = \mathcal{L}_{\text{vvv}} + \mathcal{L}_{\text{hyper}} + \mathcal{L}_{\text{vww}} \,, \tag{4.3.1}$$

where the first two terms are given in (2.1.8) and (2.1.9), whereas the higher derivative correction was displayed in (4.1.4).

[7]The mass is also a conserved charge, but since we are considering only extremal solutions it is not independent.

We now proceed with the definition of the conserved charges following from this Lagrangian. For the evaluation of the entropy and the angular momentum, we make use of the Noether potential associated with space-time diffeomorphisms [82, 98, 99]. A concise introduction to the algorithmic construction of the Noether current is given in Appendix C.1, to which we refer for details.

4.3.1 The electric charges

We first discuss the electric charges, which follow from the three-forms of the composite linear multiplets used in the definition of the action in (2.1.8) and (4.1.4). The invariant density (2.1.2) implies that the gauge field equation of motion is equal to E_μ, so the relevant integral that defines the electric charge is

$$q_A = \frac{1}{16\pi^2} \int \mathrm{d}\theta\, \mathrm{d}\varphi\, \mathrm{d}\psi \left[-\tfrac{1}{2}\, C_{ABC}\, E^{BC}_{\psi\theta\varphi} + \tfrac{4}{3} c_A\, E^{\mathrm{W}}_{\psi\theta\varphi} \right] , \tag{4.3.2}$$

where one has to integrate over the 3-cycle that encloses the black hole or the black ring and the relative factor 3 results from the fact that the Lagrangian (2.1.8) is cubic in the vector multiplets, whereas the Lagrangian (4.1.4) is only linear. Making use of the results (4.2.36) and (4.2.44), one obtains the following result,

$$q_A = \frac{1}{16\, v^2} C_{ABC} \left(\sigma^B \sigma^C\, \mathrm{e}^g T_{01} - \frac{1}{4} \sigma^B [W_\psi{}^C]\, T_{23} \right) - \frac{1}{2} c_A\, \mathrm{e}^g T_{01} \,, \tag{4.3.3}$$

where we used the definition

$$[W_\psi{}^A] = \frac{1}{16\, \pi^2} \int \mathrm{d}\theta\, \mathrm{d}\varphi\, \mathrm{d}\psi\, \sin\theta\, W_\psi{}^A \,. \tag{4.3.4}$$

which is gauge invariant under periodic gauge transformations. For spinning black holes, where the gauge fields are globally defined, (4.3.3) takes the form

$$q_A = \frac{\mathrm{e}^g}{16\, T_{01}} \left[C_{ABC} \sigma^B \sigma^C - 8\, c_A {T_{01}}^2 \right] . \tag{4.3.5}$$

Observe that the above results are scale invariant.

To derive the corresponding result for the black ring is more subtle in view of the fact that the gauge fields are not globally defined, as was discussed in subsection 4.2.3. This will be discussed in subsection 4.5 and the resulting expression for the charges will be given in (4.5.7).

The charges can also be determined by making use of the Noether potential associated with abelian gauge transformations. Consider, for instance, a Lagrangian in five space-time dimensions consisting of an invariant Lagrangian depending on the abelian field strength $F_{\mu\nu}$, its space-time derivatives $\nabla_\rho F_{\mu\nu}$, and matter fields denoted by ψ and their derivatives $\nabla_\mu\psi$, plus an abelian Chern-Simons term,

$$\mathcal{L} = \mathcal{L}_0(F_{\mu\nu}, \nabla_\rho F_{\mu\nu}, \psi, \nabla_\mu\psi) + \varepsilon^{\mu\nu\rho\sigma\tau} A_\mu F_{\nu\rho} F_{\sigma\tau} \,. \tag{4.3.6}$$

As explained in C.2.1, for this Lagrangian, the Noether potential reads as follows,

$$Q^{\mu\nu}_{\rm gauge} = 2\,\mathcal{L}_F^{\mu\nu}\xi - 2\,\nabla_\rho\mathcal{L}_F^{\rho,\mu\nu}\xi + \mathcal{L}_F^{\rho,\mu\nu}\partial_\rho\xi + 6\,e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}\,\xi A_\rho F_{\sigma\tau}\,, \tag{4.3.7}$$

where ξ is the infinitesimal local parameter associated with the gauge transformations. Here we use the notation,

$$\delta\mathcal{L}_0 = \mathcal{L}_F^{\mu\nu}\,\delta F_{\mu\nu} + \mathcal{L}_F^{\rho,\mu\nu}\,\delta(\nabla_\rho F_{\mu\nu}) + \mathcal{L}_\psi\,\delta\psi + \mathcal{L}_\psi^\mu\,\delta(\nabla_\mu\psi) \tag{4.3.8}$$

It is straightforward to verify that $\partial_\nu Q^{\mu\nu}$ is equal to the field equation, up to terms proportional to $\partial_\nu\xi$. The electric charge defined in (4.3.2) can be now written as

$$q = \int_{\Sigma_{\rm hor}} \varepsilon_{\mu\nu}\, Q^{\mu\nu}_{\rm gauge} \,. \tag{4.3.9}$$

where $\varepsilon_{\mu\nu}$ is the binormal tensor associated with the horizon and the gauge parameter ξ must be taken constant so that the underlying field configuration is invariant and the corresponding Noether current vanishes on-shell.

4.3.2 Entropy and angular momentum

We now turn to the definition of the entropy and angular momentum, specialising some of the general expressions given in Appendix C.1 for the Lagrangian specified above. This Lagrangian contains two different Chern-Simons terms, one of the type $W\wedge F\wedge F$, which is cubic in the abelian gauge fields, and a mixed one of the type $W\wedge \mathrm{Tr}[\mathcal{R}\wedge\mathcal{R}]$, which is linear in the gauge fields and quadratic in the Riemann curvature. The derivation of the corresponding Noether potential is straightforward but subtle.

We first evaluate this potential for a Lagrangian that depends on the Riemann tensor, the field strengths of abelian gauge fields, and on an anti-symmetric tensor field $T_{\mu\nu}$ with at most first-order space-time derivatives $\nabla_\mu T_{\nu\rho}$. This Lagrangian does *not* contain the two Chern-Simons terms, which are considered separately. Its Noether potential associated with space-time diffeomorphisms is a slight extension of (C.2.10) (which does not include the tensor field), and reads

$$\begin{aligned} Q_0^{\mu\nu}(\xi^\rho) \;=\;& -2\,\mathcal{L}_{\rm R}^{\mu\nu\rho\sigma}\,\nabla_\rho\xi_\sigma + 4\,\nabla_\rho\mathcal{L}_{\rm R}^{\mu\nu\rho\sigma}\,\xi_\sigma \\ & - 2\,\mathcal{L}_{\rm F\,A}^{\mu\nu}(\xi^\sigma W_\sigma^A) + 2\,\nabla_\rho\mathcal{L}_{\rm F\,A}^{\rho,\mu\nu}(\xi^\sigma W_\sigma^A) - \mathcal{L}_{\rm F\,A}^{\rho,\mu\nu}\partial_\rho(\xi^\sigma W_\sigma^A) \\ & + \big[\mathcal{L}_{\rm T}^{\mu,\rho\sigma}\,T^\nu{}_\sigma + \mathcal{L}_{\rm T}^{\rho,\mu\sigma}\,T^\nu{}_\sigma + \mathcal{L}_{\rm T}^{\nu,\mu\sigma}\,T^\rho{}_\sigma - (\mu\leftrightarrow\nu)\big]\,\xi_\rho\,, \end{aligned} \tag{4.3.10}$$

where ξ^ρ parametrizes the diffeomorphisms, and $\mathcal{L}^{\mu\nu\rho\sigma}$ and $\mathcal{L}^{\mu,\nu\rho}$ denote partial derivatives of the Lagrangian according to

$$\delta\mathcal{L} = \mathcal{L}_{\rm R}^{\mu\nu\rho\sigma}\,\delta\mathcal{R}_{\mu\nu\rho\sigma} + \mathcal{L}_{\rm T}^{\mu,\nu\rho}\,\delta(\nabla_\mu T_{\nu\rho}) + \mathcal{L}_{\rm F\,A}^{\mu\nu}\,\delta(F_{\mu\nu}^A) + \mathcal{L}_{\rm F\,A}^{\mu,\nu\rho}\,\delta(\nabla_\mu F_{\nu\rho}^A)\,. \tag{4.3.11}$$

These derivatives are subject to the BPS attractor equations. As a result they take the following form on the horizon,

$$\begin{aligned} 32\pi^2\,\mathcal{L}_{\rm R}^{\mu\nu\rho\sigma} &= (-\tfrac{1}{2}C(\sigma) - c_A\sigma^A\,T^2)g^{\mu[\rho}g^{\sigma]\nu} + \tfrac{2}{3}c_A\sigma^A(T^{\mu\nu}T^{\rho\sigma} - T^{\mu[\rho}T^{\sigma]\nu})\,,\\ 32\pi^2\,\mathcal{L}_{\rm F\,A}^{\mu\nu} &= -\tfrac{1}{2}\,C_{ABC}\sigma^B\sigma^C\,T^{\mu\nu} + c_A\left(T^{\mu\nu}\,T^2 + 6\,T^{\mu\rho}T_{\rho\sigma}T^{\sigma\nu}\right)\,,\\ 32\pi^2\,\mathcal{L}_{\rm T}^{\rho,\mu\nu} &= -3{\rm i}\,c_A\sigma^A\,\varepsilon^{\mu\nu\sigma\lambda\tau}\,T_{\sigma\lambda}T_\tau{}^\rho\,, \end{aligned} \tag{4.3.12}$$

whereas $\mathcal{L}_{\rm F\,A}^{\mu,\nu\rho}$ vanishes. Obviously we also need the derivative $\nabla_\rho\mathcal{L}_{\rm R}^{\mu\nu\rho\sigma}$, which follows form (4.3.12) by means of the attractor equations. The result reads as follows,

$$32\pi^2\,\nabla_\rho\mathcal{L}_{\rm R}^{\mu\nu\rho\sigma} = \tfrac{5}{6}{\rm i}\,c_A\sigma^A(T^{\mu\nu}\varepsilon^{\sigma\rho\lambda\kappa\tau} - T^{\sigma[\mu}\varepsilon^{\nu]\rho\lambda\kappa\tau})T_{\rho\lambda}T_{\kappa\tau}\,. \tag{4.3.13}$$

Combining the above contributions we obtain an explicit expression for (4.3.10). In practice we need the contraction of the Noether potential with the bi-normal tensor (4.2.30) associated with the horizon. Therefore we evaluate the following expression for (4.3.10),

$$\begin{aligned} 8\pi^2\,\varepsilon_{\mu\nu}Q_0^{\mu\nu} = &-2\,\varepsilon_{01}\,C(\sigma)\,\nabla_{[0}\xi_{1]}\\ &-\tfrac{1}{3}\,\varepsilon_{01}\,c_A\sigma^A\left[3\,T_{23}{}^2\,\nabla_{[0}\xi_{1]} - 2\,T_{01}T_{23}\,\nabla_{[2}\xi_{3]} + 11\,T_{01}{}^2T_{23}\,\xi_5\right]\\ &+\varepsilon_{01}\,\xi^\rho W_\rho{}^A\,T_{01}\,\left[-2\,C_{ABC}\sigma^B\sigma^C + c_A(T_{23}{}^2 + 2\,T_{01}{}^2)\right]\,. \end{aligned} \tag{4.3.14}$$

These results will be used verbatim in subsequent sections both for spinning black holes and black rings, as they follow from the covariant terms in the Lagrangian. The issue of Chern-Simons terms will be dealt with in the next section.

By integrating the Noether potential over the horizon one obtains the entropy and the angular momentum from the Noether potential associated with the appropriate Killing vector. For the entropy the relevant Killing vector is the timelike[8] one,

$$\xi^\mu\partial_\mu = \partial/\partial t\,,\qquad \nabla_{[0}\xi_{1]} = \varepsilon_{01}\,.$$

According to the recipe in C.2.2, one furthermore drops all terms in the integrand except the ones proportional to $\nabla\xi$

$$\mathcal{S} = -\pi\int_{\Sigma_{\rm hor}}\varepsilon_{\mu\nu}Q^{\mu\nu}(\xi)\Big|_{\nabla_{[\mu}\xi_{\nu]}=\varepsilon_{\mu\nu};\,\xi^\mu=0}\,, \tag{4.3.15}$$

[8]This is the timelike vector that generates the horizon, which is a linear combination of the asymptotic timelike and rotational Killing vectors near the horizon.

where we included a conventional normalisation factor. For the angular momentum, the Killing vector is associated with the corresponding periodic isometry of the space-time, and we define

$$J(\xi) = \int_{\Sigma_{\rm hor}} \varepsilon_{\mu\nu} Q^{\mu\nu}(\xi)\ . \tag{4.3.16}$$

This agrees with the expression in (C.2.14), since C_{ξ_ϕ} vanishes by symmetry considerations for any angular Killing vector [58, 107].

4.3.3 Noether potential for Chern-Simons terms

The two Chern-Simons terms contained in (2.1.8) and (4.1.4),

$$32\pi^2\,\mathcal{L}_{\rm CS} = -\tfrac{1}{48}{\rm i}\,\varepsilon^{\mu\nu\rho\sigma\tau}\left[C_{ABC}\,W_\mu{}^A F_{\nu\rho}{}^B F_{\sigma\tau}{}^C + \tfrac{1}{2} c_A\,W_\mu{}^A \mathcal{R}_{\nu\rho}{}^{ab}\,\mathcal{R}_{\sigma\tau ab}\right]\ , \tag{4.3.17}$$

and their contribution to the Noether potential requires a more detailed discussion. Straightforward application of the standard formula (C.2.10) yields the result

$$\begin{aligned} 32\pi^2\,Q^{\mu\nu}_{\rm CS} \;\;=\;\; & \tfrac{1}{12}{\rm i}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,C_{ABC}\,\xi^\lambda W_\lambda{}^A W_\rho{}^B F_{\sigma\tau}{}^C \\ & + \tfrac{1}{24}{\rm i}\,\varepsilon^{\mu\nu\rho\sigma\tau} c_A W_\rho{}^A\,\mathcal{R}_{\sigma\tau}{}^{\kappa\lambda}\,\nabla_\kappa \xi_\lambda \\ & - \tfrac{1}{24}{\rm i}\,\varepsilon^{\rho\sigma\tau\lambda[\mu}\,c_A F_{\rho\sigma}{}^A\,\mathcal{R}_{\tau\lambda}{}^{\nu]\kappa}\,\xi_\kappa \\ & + \tfrac{1}{48}{\rm i}\ \varepsilon^{\rho\sigma\tau\lambda\kappa}\,c_A F_{\rho\sigma}{}^A\,\mathcal{R}_{\tau\lambda}{}^{\mu\nu}\,\xi_\kappa\ . \end{aligned} \tag{4.3.18}$$

A priori, there is no reason not to use the generic formula, as both terms are not gauge invariant, but are manifestly covariant under diffeomorphisms. As a result, the contribution in (4.3.18) is not gauge invariant. The same holds for the expression (4.3.10), containing terms linear in $\xi^\mu A_\mu$, which also seem troublesome at first sight. It was shown in [108] that the component of the gauge field along the Killing vector is in fact well defined under the integral. However, (4.3.18) is not of this type, especially the terms proportional to the square of the gauge potential, and is problematic when integrating to obtain the angular momentum.

Thus, (4.3.18) can be used as it stands only when the gauge potential is globally defined, or in other words in the absence of magnetic charges. This is the case with black holes, as explained in section 4.2.3. Evaluating the expression above at the horizon, using (4.2.9) and (4.2.37), one finds

$$\begin{aligned} 32\pi^2\,\varepsilon_{\mu\nu} Q^{\mu\nu}_{\rm CS} \;\;=\;\; & \tfrac{4}{3}\,\varepsilon_{01}\,T_{23}\,C_{ABC}\,\sigma^A W_5{}^B W_\lambda{}^C\,\xi^\lambda \\ & - \tfrac{4}{3}\varepsilon_{01}\,c_A W_5{}^A\left[-2\,T_{01}T_{23}\,\nabla_{[0}\xi_{1]} + (T_{01}{}^2 + 4\ T_{23}{}^2)\,\nabla_{[2}\xi_{3]}\right] \\ & - \tfrac{4}{3}\varepsilon_{01}\,T_{01}{}^2\,c_A\left[W_3{}^A\,\nabla_{[5}\xi_{2]} - W_2{}^A\,\nabla_{[5}\xi_{3]}\right] \\ & + \tfrac{8}{3}\,\varepsilon_{01}\,c_A\sigma^A\,T_{23}\left[6\,T_{01}{}^2 - T_{23}{}^2\right]\,\xi_5\ , \end{aligned} \tag{4.3.19}$$

which will be used for black holes in the following.

For black rings however, the situation is different and extra care is required, as the gauge fields are not globally defined. In such a situation there are two ways to proceed. A simple way out would be possible if a version of the action that is gauge invariant exists, perhaps sacrificing some other symmetry. Alternatively, one can try to regularise the integral of the Noether potential, using different gauge patches. As is clear by inspection, there is no way to rewrite the first Chern-Simons term of (4.3.17) in a gauge invariant form. Therefore, in section 4.5 we will present a recipe to modify the integral of the first term in (4.3.18) to a gauge covariant quantity.

Interestingly, we can employ an alternative treatment of the mixed Chern-Simons term which will lead to expressions that differ from (4.3.17)-(4.3.19). Consider a modification of the Lagrangian (4.1.4) proportional to $\varepsilon^{\mu\nu\rho\sigma\tau}W_\mu{}^A\mathcal{R}_{\nu\rho}{}^{ab}\mathcal{R}_{\sigma\tau ab}$, by adding a suitable total derivative. In this way the gauge field is converted to its field strength (which is globally defined), and the square of the curvature tensor $\mathcal{R}$ is converted to a corresponding Chern-Simons term. The alternative form of the mixed Chern-Simons term is thus,[9]

$$32\pi^2\mathcal{L}_{\rm CS} = -\tfrac{1}{48}\mathrm{i}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,c_A F_{\mu\nu}{}^A\omega_\rho{}^{ab}\left(\partial_\sigma\omega_{\tau ab} - \tfrac{2}{3}\omega_{\sigma ac}\,\omega_\tau{}^c{}_b\right)\,. \tag{4.3.20}$$

From the point of view of general coordinate invariance, this change does not seem crucial, as the Lagrangian (4.3.20) still transforms as a scalar. On the other hand, the spin-connection field $\omega_\mu{}^{ab}$ is a composite vector field associated with local Lorentz transformations. As a result of the explicit spin-connection, this form of the Lagrangian is no longer invariant under local Lorentz transformations, but transforms into a boundary term.

In this formulation diffeomorphism invariance of the relevant field configurations will be defined up to a local Lorentz transformation. Therefore Lorentz transformations have to be taken into account in the relevant Noether potential. In the previous form of the mixed Chern-Simons term given in (4.3.17), the local Lorentz transformations were avoided because that expression can be interpreted directly in the metric formulation without the need for including vielbein fields.

Under the combined variation of a diffeomorphism and a local Lorentz transformation with parameters ξ^μ and ε^{ab}, the Noether potential for the Lagrangian (4.3.20) corresponding to the mixed Chern-Simons term follows from (C.2.21), where some details of

[9] Note that for the remainder of this subsection we suppress the $W\wedge F\wedge F$ Chern-Simons term of (4.3.17), which is not affected by the conversion and whose effect has already been evaluated.

the derivation are given

$$
\begin{aligned}
32\pi^2 Q^{\mu\nu}_{\rm CS} \;=\; & -\tfrac{1}{24}\mathrm{i}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,c_A F_{\rho\sigma}{}^A\omega_\tau{}^{ab}\left[\varepsilon_{ab}-\tfrac{1}{2}\xi^\kappa\omega_{\kappa\,ab}\right]\\
& +\tfrac{1}{24}\mathrm{i}\,\varepsilon^{\mu\nu\rho\sigma\tau}\,c_A\,\xi^\lambda W_\lambda{}^A\omega_\rho{}^{ab}\left(\partial_\sigma\omega_{\tau ab}-\tfrac{2}{3}\omega_{\sigma ac}\,\omega_\tau{}^c{}_b\right)\\
& -\tfrac{1}{24}\mathrm{i}\,\varepsilon^{\rho\sigma\tau\lambda[\mu}\,c_A F_{\rho\sigma}{}^A\,\mathcal{R}_{\tau\lambda}{}^{\nu]\kappa}\,\xi_\kappa\\
& +\tfrac{1}{48}\mathrm{i}\;\varepsilon^{\rho\sigma\tau\lambda\kappa}\,c_A F_{\rho\sigma}{}^A\,\mathcal{R}_{\tau\lambda}{}^{\mu\nu}\,\xi_\kappa\,. \qquad (4.3.21)
\end{aligned}
$$

We note that the last two covariant terms proportional to $F\wedge R$ are identical to the corresponding terms given in (4.3.18). This expression should be evaluated for backgrounds that are invariant, which implies that the transformation parameter ε^{ab} should be chosen such that the vielbein is invariant under the diffeomorphisms. This implies that the diffeomorphism is again generated by a Killing vector ξ^μ, and

$$
\varepsilon^{ab} = -\nabla^{[a}\xi^{b]} + \xi^\lambda\omega_\lambda{}^{ab}\,. \qquad (4.3.22)
$$

This value for ε^{ab} should be substituted into the expression (4.3.21) for the Noether potential. The resulting expression is then expected to match the previous result (4.3.18) (without the contribution of the $W\wedge F\wedge F$ Chern-Simons term which has not been included above), when both the gauge fields and the spin connection field can be globally defined. This is not the case for the black hole and black ring solutions, so that only one of the two expressions will be applicable in either case. It should be of interest to compare the two formulae in more detail by making explicit use of coordinate patches.

4.4 Spinning BPS black holes

In this section we apply the material derived in the preceding sections to the case of spinning black holes. Subsequently we discuss various implications of our results and compare them to results that have been obtained elsewhere.

We assume arbitrary non-zero values of p^0. Using (4.2.29), we integrate the total Noether potential $Q = Q_0 + Q_{\rm CS}$ derived in (4.3.14) and (4.3.19) over the horizon. In this way we obtain the following expression for the entropy,

$$
\mathcal{S} = \frac{\pi\,\mathrm{e}^g}{16\,v^2}\left[C(\sigma) + \tfrac{16}{3}\,c_A\sigma^A\,T_{23}{}^2\right]. \qquad (4.4.1)
$$

The moduli are expressed in terms of the angular momentum J_ψ and the charges q_A and p^0 by the attractor equations. The black holes have only one component of angular momentum, associated with the Killing vector $\xi^\mu\partial_\mu = \partial/\partial\psi$. Here we refrain from introducing any additional normalization factor. This leads to $\xi_5 = \mathrm{e}^g$ and

$$
\nabla_{[0}\xi_{1]} = 2\,T_{23}\,\mathrm{e}^g\,, \qquad \nabla_{[2}\xi_{3]} = -2\,T_{01}\,\mathrm{e}^g\,. \qquad (4.4.2)
$$

Substituting these results into (4.3.14) and (4.3.19), and setting $\varepsilon_{01} = 1$, yields the following expression for J_ψ,

$$J_\psi = \frac{T_{23}\mathrm{e}^{2g}}{24\,T_{01}{}^2}\,\left[C_{ABC}\,\sigma^A\sigma^B\sigma^C - 32\,c_A\sigma^A\,T_{01}{}^2\right]\,. \tag{4.4.3}$$

Note that there is no other non-vanishing component of angular momentum in this case. The charges follow from (4.3.3) and (4.2.34),

$$q_A = \frac{\mathrm{e}^{g}}{16\,T_{01}}\,\left[C_{ABC}\sigma^B\sigma^C - 8\,c_A T_{01}{}^2\right]\,,\qquad p^0 = \frac{\mathrm{e}^{-g}}{4\,v^2}\,T_{01}\,. \tag{4.4.4}$$

It is convenient to express these results in terms of scale invariant variables defined by

$$\phi^A = \frac{\sigma^A}{4T_{01}}\,,\qquad \phi^0 = \frac{e^{-g}T_{23}}{4v^2} = \frac{p^0\,T_{23}}{T_{01}}\,. \tag{4.4.5}$$

In terms of these variables (4.4.1) reads

$$\mathcal{S} = \frac{\pi p^0}{(\phi^{0\,2}+p^{0\,2})^2}\left[\frac{1}{6}\,p^{0\,2}C_{ABC}\phi^A\phi^B\phi^C + \frac{1}{3}\,c_A\phi^A\,\phi^{0\,2}\right], \tag{4.4.6}$$

whereas the attractor equations for the electric charges q_A and the angular momentum J_ψ take the form,

$$\begin{aligned} q_A &= \frac{1}{4}\frac{p^0}{\phi^{0\,2}+p^{0\,2}}\left[C_{ABC}\phi^B\phi^C - \frac{1}{2}\,c_A\right], \\ J_\psi &= \frac{\phi^0 p^0}{(\phi^{0\,2}+p^{0\,2})^2}\left[\frac{1}{6}\,C_{ABC}\phi^A\phi^B\phi^C - \frac{1}{3}c_A\phi^A\right]. \end{aligned} \tag{4.4.7}$$

This result shows that ϕ^0 is proportional to the angular momentum, as is also obvious from (4.4.5). To understand the limit in which the charges become uniformly large, we consider uniform rescalings of the charges q_A and p^0 as well as of the moduli ϕ^A and ϕ^0. Obviously, the terms proportional to C_{ABC} in the attractor equations are consistent with this scaling, whereas the terms proportional to c_A are suppressed inversely proportional to the square of the charges and thus represent subleading contributions. The leading term of the entropy then scales as the square of the charges, while the correction terms proportional to c_A, which originate from the higher-order derivative couplings, represent the subleading contributions in the limit where all charges become large. These results are different from those obtained in [94], especially in the case of non-zero angular momentum. For details, we refer to the discussion at the end of this subsection.

The above attractor equations can be compared to the corresponding ones in four space–time dimensions discussed in section 3.2. It is possible to show that upon dimensional reduction, the Lagrangian in (4.3.1) leads to a four-dimensional theory including both the R^2 and F^4 invariants presented in chapter 3. Since the latter is irrelevant for supersymmetric attractors, we concentrate on the former, which turns out to be described by the prepotential (3.3.4), where the tensors C_{ABC} and c_A are those appearing in (2.1.8) and (4.1.4) respectively.

It is clear that the four- and five-dimensional expressions for the entropy, given by (3.3.6) and (4.4.6) respectively, are identical. The same holds for the angular momentum, assuming that the charge q_0 is identified with J_ψ when comparing (4.4.7) with (3.3.7). In order to compare the attractor equations for the electric charges q_A, one has to take into account the comments at the end of section 4.2.3 to give a four-dimensional interpretation to the electric charge. Using the gauge transformation (4.2.46) in (4.3.2) to move the Dirac brane singularity behind the horizon, the attractor equation (4.4.7) for q_A becomes

$$q_A|_2 \quad = \quad q_A - \frac{c_A}{24}\frac{{T_{01}}^2}{v^2} = \frac{1}{4}\frac{p^0}{{\phi^0}^2 + {p^0}^2}\Big[C_{ABC}\phi^B\phi^C - \frac{2}{3}\,c_A\Big]\,, \tag{4.4.8}$$

which is exactly the same as the four-dimensional one in (3.3.7). The difference between the four- and five-dimensional charges is therefore completely accounted for by the shift of the Chern–Simons terms and is of a purely topological nature. Somewhat related arguments that reconcile the five- and four-dimensional attractors have been presented in [94], but given that the attractor equations in that work differ from (4.4.7) for nonzero angular momentum, the relevant shift is also different in that case.

To investigate some of the consequences of this difference, we again consider the attractor equations (4.4.7), where we rescale the coefficients c_A in the attractor equations for q_A by $c_A \to \alpha\, c_A$ to account for the two expressions. Hence we set the parameter $\alpha = 1$ or $\frac{4}{3}$, depending on whether we consider $D = 5$ or 4 space-time dimensions, respectively.

Subsequently we solve the attractor equations for ϕ^A and ϕ^0 to first order in c_A, keeping the charges constant. To do this we first determine the solution for the case that $c_A = 0$,

$$\begin{aligned}\hat{\phi}^A \quad &\equiv \quad \frac{\phi^A}{\sqrt{{\phi^0}^2 + {p^0}^2}} \approx \frac{\hat{q}^A}{\sqrt{p^0}} + \mathcal{O}(c_A)\,, \\ \phi^0 \quad &\approx \quad \frac{J_\psi\, {p^0}^2}{\sqrt{4\,p^0 Q^3 - (p^0 J_\psi)^2}} + \mathcal{O}(c_A)\,,\end{aligned} \tag{4.4.9}$$

where the $\hat{q}^A$ are defined by the requirement that they satisfy the attractor equations in the limit of vanishing c_A. Therefore we have,

$$\begin{aligned} q_A \quad &= \quad \tfrac{1}{4}\,C_{ABC}\,\hat{q}^B\hat{q}^C\,, \\ Q^{3/2} \quad &= \quad \tfrac{1}{12}\,C_{AB}\,\hat{q}^A\hat{q}^B\,, \\ C_{AB} \quad &= \quad C_{ABC}\,\hat{q}^C\,. \end{aligned} \tag{4.4.10}$$

To first order in c_A this result changes into,

$$\begin{aligned} \hat{\phi}^A &\approx \frac{1}{\sqrt{p^0}}\left\{\hat{q}^A + \frac{\alpha\,(4\,p^0Q^3 - (p^0\,J_\psi)^2)}{16\,{p^0}^2\,Q^3}\,C^{AB}\,c_B\right\} + \mathcal{O}({c_A}^2)\,,\\ \phi^0 &\approx \frac{J_\psi\,{p^0}^2}{\sqrt{4\,p^0Q^3-(p^0J_\psi)^2}}\left\{1 - \frac{(3\,\alpha-8)\,c_A\,\hat{q}^A}{48\,p^0\,Q^{3/2}}\right\} + \mathcal{O}({c_A}^2)\,, \end{aligned} \tag{4.4.11}$$

where the matrix C^{AB} denotes the inverse of C_{AB}. Substituting these expressions into the entropy formula (4.4.6), one obtains,

$$\mathcal{S} \simeq 2\pi\,\sqrt{p^0Q^3 - \tfrac{1}{4}(p^0J_\psi)^2}\,\left(1 + \frac{\alpha}{16}\frac{c_A\,\hat{q}^A}{p^0Q^{3/2}}\right) + \mathcal{O}({c_A}^2)\,. \tag{4.4.12}$$

We note that the terms proportional to c_A are indeed subleading in the limit of large charges.

The expression (4.4.12) can be confronted with results from the literature. For the non-rotating case, where a direct comparison with microscopic counting is possible, the above result with $\alpha = 1$ agrees with the results of [68, 80] presented in section 2.4, provided the triple intersection tensor and the second Chern class are identified as $\mathcal{C}_{ABC} = \frac{1}{2}C_{ABC}$ and $c_2 = c_{2\,A}$. For the rotating black hole, no analytic microscopic results are available, but our results can be compared to the supergravity results of [94, 95]. Here there is a clear discrepancy originating from the different form of the attractor equations (4.4.3) for the electric charges and the angular momentum, which reflects itself in a different dependence on J in (4.4.12). This expression can also be compared to the results of [109], where the only higher-derivative coupling included into the action was the Euler density. For zero angular momentum one recovers the same relative factor for the subleading correction between the four- and five-dimensional entropies represented by the parameter α in (4.4.12). For finite angular momentum the subleading corrections determined by [94, 95] and [109] are mutually different and both fail to reproduce the expression (4.4.12).

In summary, it seems that, while in the static case the $\frac{4}{3}$ difference in the attractor equation for the charges between four and five dimensions is ubiquitous, discrepancies in the rotating case remain. On the other hand, there exist microscopic results for theories with 16 supercharges [110, 111], which could possibly be connected to the results above in certain asymptotic limits. It would be very interesting to investigate this further, in order to identify the correct dependence of the entropy on the angular momentum.

To further explore this difference between four and five space-time dimensions, let us also consider the case of small black holes, whose entropy depends sensitively on the higher-derivative couplings. We assume $C_{1ab} = \eta_{ab}$ and $c_a = 0$, which represents the typical situation for $\mathrm{K3}\times T^2$ heterotic string compactifications. From the attractor equations (including the parameter α as before) we obtain

$$
\begin{aligned}
q_1 &= \frac{1}{4}\frac{p^0}{\phi^{0\,2}+p^{0\,2}}\Big[\eta_{ab}\phi^a\phi^b - \frac{\alpha}{2}\,c_1\Big]\,, \\
q_a &= \frac{1}{2}\frac{p^0\,\phi^1}{\phi^{0\,2}+p^{0\,2}}\,\eta_{ab}\phi^b\,.
\end{aligned}
\tag{4.4.13}
$$

Using the above equations one easily derives,

$$
\mathcal{S} = \frac{\pi\,p^0\,\phi^1}{(\phi^{0\,2}+p^{0\,2})^2}\Big[2\,p^0 q_1(\phi^{0\,2}+p^{0\,2}) + \frac{1}{3}\Big(\frac{3\,\alpha}{4}p^{0\,2}+\phi^{0\,2}\Big)c_1\Big]\,, \tag{4.4.14}
$$

$$
\eta^{ab}q_aq_b = \frac{1}{4}\frac{p^0\,(\phi^1)^2}{\phi^{0\,2}+p^{0\,2}}\Big[4\,q_1 + \frac{\alpha\,c_1}{2}\,\frac{p^0}{p^{0\,2}+\phi^{0\,2}}\Big]\,. \tag{4.4.15}
$$

Let us now set $q_1 = 0$, so that we are describing small black holes. In that case one finds,

$$
\mathcal{S} = \pi\,\sqrt{|\tfrac{1}{2}\,\alpha\,c_1\,\eta^{ab}q_aq_b|}\,\left\{1 + \frac{4-3\,\alpha}{3\,\alpha}\,\frac{\phi^{0\,2}}{\phi^{0\,2}+p^{0\,2}}\right\}\,, \tag{4.4.16}
$$

where ϕ^0 is related to the angular momentum according to

$$
J_\psi = -\frac{\phi^0}{\phi^{0\,2}+p^{0\,2}}\,\frac{4-3\,\alpha}{3\,\alpha}\,\sqrt{|\tfrac{1}{2}\;\alpha\,c_1\,\eta^{ab}q_aq_b|}\,. \tag{4.4.17}
$$

This last relation only allows for a uniform rescaling of the charges in the way indicated before in the four-dimensional setting, for which $\alpha = 4/3$ and the angular momentum is necessarily zero. Indeed, a small black hole in four dimensions is characterised by vanishing q_0, p^1, p^a. In that case the entropy coincides with the four-dimensional result for small black holes.

The result (4.4.16) can be compared with the predictions of [80], where exact expressions for microscopic degeneracies of small static black holes in five space-time dimensions were derived. In this work it was found that the asymptotics of the entropy of the small black holes in five dimensions is the same as in four, with the same normalization. This disagrees with (4.4.16), which differs by an overall relative factor equal to $\sqrt{4/3}$ induced by the α-dependence. To resolve this puzzle it might perhaps be helpful to also have microscopic results for non-zero angular momentum, so that one has a more detailed test for (4.4.16). However, such results are quite difficult to obtain. As is well known, in four space-time dimensions the sub-leading contribution to the entropy of small black holes is problematic in the supergravity description, but the leading contribution is in perfect agreement with microstate counting arguments. The five-dimensional result thus poses a puzzle in this respect.

4.5 BPS black rings

In this final section we turn to the black rings, for which the relevant Noether potential has been derived in section 4.3.2. In particular we refer to the treatment of the mixed Chern-Simons term in subsection 4.3.3, which is crucial for the black ring. In this section we discuss the resulting expressions for the entropy, and for the charges and angular momenta, which are then confronted with results from the literature. As we shall see, the actual evaluation still involves a number of non-trivial issues related to the integration over the spacelike section Σ of the horizon.

The relevant Noether potential consists of (4.3.14) combined with the contributions from the Chern-Simons terms that can be extracted from (4.3.19) and (4.3.21). Using that $T_{01} = 0$ for the black ring, it is easy to see that (4.3.14) gives rise to the following contribution,

$$32\pi^2\, \varepsilon_{\mu\nu} Q_0^{\mu\nu} = -2\, \varepsilon_{01} \left[C(\sigma) + 4\, c_A \sigma^A\, {T_{23}}^2\right] \nabla_{[0}\xi_{1]}\,. \tag{4.5.1}$$

Subsequently we add the contributions from (4.3.21), together with the first term in (4.3.19) that originates from the $W \wedge F \wedge F$ Chern-Simons term,

$$\begin{aligned} 32\pi^2 Q^{\mu\nu}_{\rm CS} \;=\;& \tfrac{1}{12}\mathrm{i}\, \varepsilon^{\mu\nu\rho\sigma\tau}\, C_{ABC}\, \xi^\lambda {W_\lambda}^A {W_\rho}^B {F_{\sigma\tau}}^C \\ & - \tfrac{1}{24}\mathrm{i}\, \varepsilon^{\mu\nu\rho\sigma\tau}\, c_A {F_{\rho\sigma}}^A {\omega_\tau}^{ab} \left[\varepsilon_{ab} - \tfrac12 \xi^\kappa \omega_{\kappa\, ab}\right] \\ & + \tfrac{1}{24}\mathrm{i}\, \varepsilon^{\mu\nu\rho\sigma\tau}\, c_A\, \xi^\lambda {W_\lambda}^A {\omega_\rho}^{ab} \left(\partial_\sigma \omega_{\tau ab} - \tfrac23 \omega_{\sigma ac}\, {\omega_\tau}^c{}_b\right) \\ & - \tfrac{1}{24}\mathrm{i}\, \varepsilon^{\rho\sigma\tau\lambda[\mu}\, c_A {F_{\rho\sigma}}^A\, {\mathcal{R}_{\tau\lambda}}^{\nu]\kappa}\, \xi_\kappa \\ & + \tfrac{1}{48}\mathrm{i}\; \varepsilon^{\rho\sigma\tau\lambda\kappa}\, c_A {F_{\rho\sigma}}^A\, {\mathcal{R}_{\tau\lambda}}^{\mu\nu}\, \xi_\kappa\,. \end{aligned} \tag{4.5.2}$$

Observe that the last two terms in (4.5.2) have already been evaluated in (4.3.19). The third term of (4.5.2) vanishes as can be readily deduced from (4.2.44). Straightforwardly combining the various contributions gives rise to the following additional contribution to the Noether potential,

$$\begin{aligned} 32\pi^2\, \varepsilon_{\mu\nu} Q^{\mu\nu}_{\rm CS} \;=\;& \tfrac43\, \varepsilon_{01}\, T_{23} \Big[C_{ABC}\, \sigma^A {W_5}^B\, \xi^\lambda {W_\lambda}^C - 2 c_A \sigma^A\, {T_{23}}^2\, \xi_5\Big] \\ & - \tfrac83\, \varepsilon_{01}\, c_A \sigma^A\, {T_{23}}^2 \Big[\nabla_{[0}\xi_{1]} - \tfrac12 \xi^\lambda \omega_{\lambda 01}\Big]\,, \end{aligned} \tag{4.5.3}$$

where we have used that ${\omega_5}^{ab}$ vanishes with the exception of ${\omega_5}^{01} = -2\, T_{23}$.

From (4.5.3) we directly determine the expression for the entropy, which coincides with the corresponding expression (4.4.1) for the black hole,

$$\mathcal{S} = \frac{\pi\, \mathrm{e}^g}{16\, v^2} \left[C(\sigma) + \tfrac{16}{3}\, c_A \sigma^A\, {T_{23}}^2\right]. \tag{4.5.4}$$

Observe that, in order to obtain this result, it was crucial to use the alternative form of the Noether potential derived in subsection 4.3.3. Naive application of the Noether potential that was used earlier for the black hole, will yield a different result. In any case, we should stress that the mixed Chern-Simons term contributes to both the black hole and the black string entropy.

To obtain the expression (4.5.4) we had to integrate over the horizon, which, in the case at hand, was straightforward. However, to determine the electric charges and the angular momenta, one is confronted with an integration of terms that depend explicitly on gauge fields that are not globally defined. To perform the integral one therefore has to make use of patches, as was already explained in section 4.2.3, in such a way that the result will be invariant under 'small' gauge transformations continuously connected to the identity. The precise procedure for doing this has already been proposed in [58], and we will adopt it here.

We thus define two coordinate patches on the $S^1 \times S^2$ spacelike cross section Σ of the horizon. As we shall discuss in due time, these patches have to be also defined away from Σ, but for the moment we restrict our attention to Σ itself. One patch contains the north pole N of the S^2 factor. It is parametrized by $-1+\epsilon \leq \cos\theta \leq 1$, $0 \leq \varphi < 2\pi$ and $0 \leq \psi < 4\pi$. This patch has the topology of a solid two-torus. The second patch, which has the same topology, contains the south pole S of the S^2 factor, and is parametrized by $-1 \leq \cos\theta \leq -1+\epsilon$, $0 \leq \varphi < 2\pi$ and $0 \leq \psi < 4\pi$. The boundary of these two patches is a two-torus defined by $\cos\theta = -1+\epsilon$, where the parameter ϵ will be taken to zero at the end of the calculation. On these patches we define the gauge fields, $W_\mu^{\mathsf{N}\,A}$ and $W_\mu^{\mathsf{S}\,A}$, respectively, which are related by gauge transformations β^A. These gauge transformations move the Dirac brane singularities from the south to the north pole in a way that involves the ring coordinate ψ, as was already described in subsection 4.2.3 (in particular, see (4.2.39)). Hence,

$$
\begin{aligned}
W_\mu^{\mathsf{N}\,A}\,\mathrm{d}x^\mu &= -p^A\left[\cos\theta\,\mathrm{d}\varphi - \mathrm{d}(\varphi+\tfrac{1}{2}\psi)\right] + a^A\mathrm{d}\psi\,,\\
W_\mu^{\mathsf{S}\,A} &= W_\mu^{\mathsf{N}\,A} + \beta_\mu{}^A\,, \qquad \beta_\mu{}^A\mathrm{d}x^\mu = -2\,p^A\mathrm{d}(\varphi+\tfrac{1}{2}\psi)\,.
\end{aligned}
\tag{4.5.5}
$$

Integrals over the spacelike cross section Σ of the horizon, are now decomposed into integrals over the sections N and S and an additional integral over the boundary of the coordinate patches that involves the gauge transformations β^A. This last term must restore the gauge invariance of the integral under small gauge transformations [58]. The limit $\epsilon \downarrow 0$ is taken for convenience, so that the contribution from the section S will vanish, and the contribution from N will cover the whole horizon with the exception of the singular points related to the position of the Dirac brane.

Let us first consider the attractor equations for the electric charges q_A. From the evaluation of the charges for the black hole (c.f. (4.3.3)) it is clear that the only contribution originates from the $C_{ABC}\,W^A \wedge F^B \wedge F^C$ Chern-Simons term, since all other contributions vanish when $T_{01} = 0$. Therefore we focus directly on the Chern-Simons term, which requires to evaluate the integral of $C_{ABC}\,W^B \wedge F^C$ over the spacelike cross section Σ of the horizon. According to the prescription specified above, this integral is evaluated as follows,

$$\int_\Sigma C_{ABC}\, W^B \wedge F^C \;=\; \int_{\mathsf{N}} C_{ABC}\, W^{\mathsf{N}\,B} \wedge F^C + \int_{\mathsf{S}} C_{ABC}\, W^{\mathsf{S}\,B} \wedge F^C + 2\int_{\partial\mathsf{N}} C_{ABC}\, W^{\mathsf{N}\,B} \wedge \beta^C\,, \tag{4.5.6}$$

where the factor 2 arises because $F^A = 2\,\mathrm{d}W^A$. In the limit $\epsilon \downarrow 0$, the second integral vanishes. The third integral extends over the boundary, $\partial\mathsf{N} = -\partial\mathsf{S}$, of the two sections. Now, observe that $W^{\mathsf{N}\,J} \wedge F^K$ is proportional to $(a^J + \frac{1}{2}p^J)p^K\,\mathrm{d}\theta\wedge\mathrm{d}\varphi\wedge\mathrm{d}\psi$, while $W^{\mathsf{N}\,J}\wedge \beta^K$ is proportional to $(a^J - \frac{1}{2}(1-\epsilon)p^J)p^K\,\mathrm{d}\varphi\wedge\mathrm{d}\psi$. As it turns out, the contributions proportional to p^Jp^K from the first and the second integral cancel (in the limit $\epsilon \downarrow 0$), whereas the terms proportional to a^Jp^K add. This confirms the conclusion below (4.2.40) that the Chern-Simons charge should be proportional to $C_{ABC}\, a^B p^C$. From comparison with (4.3.3), one then easily determines the expression for the electric charges by substituting $[W_\psi{}^K] = 2\,a^K$. The attractor equations for the black ring charges are therefore summarized by

$$q_A = -\frac{1}{4\,T_{23}} C_{ABC}\,\sigma^B a^C\,, \qquad p^0 = 0\,, \qquad p^A = \frac{\sigma^A}{4\,T_{23}}\,. \tag{4.5.7}$$

It is important to realize that the prescription of [58] is based on the fact that $\mathrm{d}[W^B\wedge F^C] = \frac{1}{2}F^B\wedge F^C$ is gauge invariant. Upon extending the patches outside the horizon, we may calculate $F^B\wedge F^C$ over a four-dimensional manifold by extending the radial coordinate r, which can then be expressed as an integral over its three-dimensional boundary. This is the justification for the prescription (4.5.6), as Σ constitutes (part of) this boundary. However, we have simply ignored that the gauge fields must in principle be extendable outside the horizon in the two patches, and in the above calculation this feature does not seem to play a role as we obtain a result that is invariant under small gauge transformations. Indeed, one can repeat the calculation without any difficulty for a different choice of coordinate patches, such as, for instance, defined by $\cos\theta_0 \leq \cos\theta \leq 1$ for the N patch and $-1 \leq \cos\theta \leq \cos\theta_0$ for the S patch, so that the boundary is located at $\theta = \theta_0$. As it turns out the final result will not depend on θ_0 and simply remains the same.

However, the situation is different when considering the evaluation of the angular momenta and we shall see that the extension of the sections away from Σ will become an issue. The expression for the angular momenta follows from the Noether potential (4.5.3), which is again not gauge invariant so that the integral is again subtle. The troublesome term is the first one, depending on $W_5{}^J$, which originates form the $W\wedge F\wedge F$ Chern-Simons term shown in the first line of (4.5.2). This term leads to

$$8\pi^2\,\varepsilon_{\mu\nu}Q^{\mu\nu}_{\mathrm{CS}} = \tfrac{1}{6}\varepsilon_{01}\,\varepsilon^{\mu\nu\rho}\,C_{ABC}\,\xi^\lambda W_\lambda{}^A\,W_\mu{}^B F_{\nu\rho}{}^C + \cdots\,, \tag{4.5.8}$$

where the dots denote the remaining gauge invariant contributions in (4.5.3), which can be evaluated straightforwardly. Note that, unlike as on previous occasions, we converted the above expression to a density over Σ, so that its integration will require only the surface element $\mathrm{d}\psi\wedge\mathrm{d}\varphi\wedge\mathrm{d}\theta$.

In order that the integral over Σ of (4.5.8) is amenable to the same prescription as used above, it is important that Σ and the gauge potentials are invariant under the isometries associated with linear combinations, $\xi^\psi\partial_\psi+\xi^\varphi\partial_\varphi$, of the two Killing vectors associated with rotations over the angles ψ and φ. One then observes that $\mathrm{d}[(\xi\cdot W)\,W\wedge F]$ can be written as a linear combination of two terms. One is the contraction of the Killing vector with the five-form $W\wedge F\wedge F$ whose integral must vanish for symmetry reasons. The second term equals $(\xi\cdot W)\,F\wedge F$, which changes by a total derivative under gauge transformations, again because the gauge fields are invariant under the symmetry associated with the Killing vector. Hence the integral over the four-dimensional manifold is invariant under small gauge transformations, and, just as before, the integral of (4.5.8) over its boundary Σ can be decomposed into integrals over the patches N and S and an additional integral over the boundary $\partial\mathsf{N}$ of

$$\begin{aligned}\varepsilon^{\mu\nu\rho}\,C_{ABC}[\xi^\lambda W_\lambda{}^{\mathsf{N}\,A}\,W_\mu{}^{\mathsf{N}\,B}-\xi^\lambda W_\lambda{}^{\mathsf{S}A}\,W_\mu{}^{\mathsf{S}\,B}]F_{\nu\rho}{}^C =&\\ +\,\varepsilon^{\mu\nu\rho}\,C_{ABC}\,\partial_\mu\Big[\xi^\lambda W_\lambda{}^{\mathsf{S}\,A}\,\beta_\nu{}^B W_\rho{}^{\mathsf{S}\,C}-2\,\xi^\lambda\beta_\lambda{}^A\,\beta_\nu{}^B W_\rho{}^{\mathsf{S}\,C}\Big]&\\ -\,\tfrac{3}{2}\varepsilon^{\mu\nu\rho}\,C_{ABC}\,\xi^\lambda\beta_\lambda{}^A\,W_\mu{}^{\mathsf{S}\,B}F_{\nu\rho}{}^C\,.&\end{aligned}\qquad(4.5.9)$$

Here we insisted in writing the last two lines in terms of sections $W_\mu{}^{\mathsf{S}\,A}$, which are well defined at the south pole. Therefore, when writing the last term as a surface term over $\xi^\lambda\beta_\lambda{}^A\,W_\psi{}^{\mathsf{S}\,B}W_\varphi{}^{\mathsf{S}\,C}$, its contribution will vanish in the limit $\epsilon\downarrow 0$ because $W_\varphi{}^{\mathsf{S}\,C}$ vanishes at the south pole.

Combining the results above, the integral of (4.5.8) over Σ can therefore be written as

$$\begin{aligned}\int_\Sigma\varepsilon_{\mu\nu}Q^{\mu\nu}_{\mathrm{CS}} =& \frac{\varepsilon_{01}}{24\pi^2}\int\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}\psi\,C_{ABC}\,\xi^\lambda W_\lambda{}^{\mathsf{N}\,A}\,W_\psi{}^{\mathsf{N}\,B}F_{\theta\varphi}{}^C\\ &+\frac{\varepsilon_{01}}{12\pi^2}\int\mathrm{d}\varphi\,\mathrm{d}\psi\,C_{ABC}\,\beta^A_{[\varphi}W_{\psi]}{}^{\mathsf{S}\,B}\,(\tfrac12\xi^\lambda W_\lambda{}^{\mathsf{S}\,C}-\xi^\lambda\beta_\lambda{}^C)\Big|_{\theta=\pi}\,.\end{aligned}\qquad(4.5.10)$$

For both of these integrals the limit $\epsilon\downarrow 0$ can be taken without difficulty, so that the first one extends over the whole horizon section Σ and the second one over the boundary of the sections on the horizon. A straightforward calculation then leads to $2\,C_{ABC}p^Ap^B(a^C-\frac16 p^C)$ and $C_{ABC}p^A(a^Ba^C+a^Bp^C-\frac{1}{12}p^Bp^C)$, for J_φ and J_ψ, respectively.

The same calculation can be repeated for a different choice of the patches, namely such that, in the limit $\epsilon\downarrow 0$, the S patch will cover the whole horizon area Σ and the overlap of the N patch will shrink to the north pole. This requires to re-evaluate (4.5.9), but up to a few signs the calculations proceeds in the same way. However, now the result is *not* the same, and one finds instead, $-2\, C_{ABC}p^A p^B(a^C + \frac{1}{6}p^C)$ and $C_{ABC}p^A(a^B a^C - a^B p^C - \frac{1}{12}p^B p^C)$, for J_φ and J_ψ, respectively. The reason for this discrepancy resides in the last term in (4.5.9), which we dropped because it does not contribute at the south pole of the horizon.

However, one must verify whether there is no obstruction away from the horizon. If one assumes that the south poles are directed to the outward part of the ring, extending all the way to spatial infinity as in [104], one expects an obstruction which will result in an extra contribution from the integral at spatial infinity. On the other hand, for the inner region of the ring which contains the north poles, there is obviously no obstruction, so that the second result will be valid. In case the south poles are directed to the inward part of the ring, it is the first result that would be valid. In other words, a minimal understanding of the topological embedding of the near-horizon region in the global solution is essential in order to distinguish between the two prescriptions. It is possible that only one embedding leads to a solution that is globally BPS, in line with what was found in [112]. For a space that is asymptotically flat, both embeddings seem possible and lead to two inequivalent BPS solutions.

In light of the above we adopt the second result, which must be combined with the contributions from (4.5.3). Then we obtain the following result for the two independent angular momenta, associated with the two independent rotations of the ring in orthogonal planes,

$$\begin{aligned} J_\varphi &= -\tfrac{1}{2}\, C_{AB}p^A(a^B + \tfrac{1}{6}p^B) \\ J_\psi - J_\varphi &= -\frac{\mathrm{e}^{2g}}{8\,T_{23}}\left[C(\sigma) + \tfrac{16}{3}\, c_A\sigma^A\, T_{23}^2\right] + \tfrac{1}{4}\, C_{AB}(a^A + \tfrac{1}{2}p^A)(a^B + \tfrac{1}{2}p^B) \end{aligned} \tag{4.5.11}$$

where C^{AB} is the inverse of $C_{ABC}p^C$.

The above results are all invariant under scale transformations, as they should. Note that the Wilson line moduli a^A are scale invariant. As in the case of black holes, we introduce a scale invariant variable,

$$\phi^0 = \frac{\mathrm{e}^{-g}}{4\,T_{23}}\,, \tag{4.5.12}$$

so that the above expressions for the entropy and the electric charges take a manifestly scale invariant form,

$$\begin{aligned} \mathcal{S} &= \frac{\pi}{\phi^0}\left[\frac{1}{6}\, C_{ABC}\, p^A p^B p^C + \frac{1}{3} c_A p^A\right], \\ q_A &= -C_{ABC}\, p^B a^C\,. \end{aligned} \tag{4.5.13}$$

The angular momenta can be expressed as follows,

$$J_\psi - J_\varphi - \tfrac{1}{4}\, C^{AB}(q_A - \tfrac{1}{2}\, C_{AC}p^C)(q_B - \tfrac{1}{2}\, C_{BD}p^D) = -\frac{1}{2\,\phi^{0\,2}}\Big[C(p) + \frac{1}{3}\, c_A p^A\Big]\,,$$

$$J_\varphi = \tfrac{1}{2}\, p^A(q_A - \tfrac{1}{6} C_{AB}p^B)\,. \qquad (4.5.14)$$

The choice of the linear combination of the angular momenta in the first term is motivated by the explicit dimensional reduction of the known two-derivative solution [44], which showed that the rotation of the four-dimensional black hole cannot be identified with a rotation of the S^2 of the black ring but necessarily involves also a rotation along the ring. Likewise the dimensional reduction is over a circle generated by a simultaneous rotation around the ring and of the S^2. The corresponding generator equals the linear combination of two angular momenta, $J_\psi - J_\varphi$, which therefore corresponds to the charge associated with the Kaluza-Klein photon. Hence we introduce a modified charge $\hat{q}_0$ in the usual fashion,

$$\hat{q}_0 = J_\psi - J_\varphi - \tfrac{1}{4}\, C^{AB}(q_A - \tfrac{1}{2}\, C_{AC}p^C)(q_B - \tfrac{1}{2}\, C_{BD}p^D)\,. \qquad (4.5.15)$$

This expression coincides precisely with the one presented in [78].

With this definition the entropy takes its familiar form [67, 71],

$$\mathcal{S} = 2\pi\,\sqrt{|\tfrac{1}{12}\,\hat{q}_0(C_{ABC}\,p^Ap^Bp^C + 2\,c_Ap^A)|}\,, \qquad (4.5.16)$$

This result for the corrected entropy agrees with the microscopic counting of [76, 77], presented in section 2.4.3, using the same identifications for the intersection numbers and the second Chern class as above. Furthermore, the above results are generally in line with the AdS/CFT results for the black ring attractors [92, 113, 114]. As briefly reviewed in section 2.4, the entropy (4.5.16) is closely related to the microscopic entropy of a corresponding four-dimensional black hole.

Let us now confront the above expressions in more detail with the corresponding results in four space-time dimensions, again based on the function (3.3.4). Hence we are dealing with a black hole with $p^0 = 0$, which leads to (3.3.10)-(3.3.12). The symmetry transformations (3.3.9) can be considered in the five-dimensional case. In five dimensions there is no electric/magnetic duality but there is spectral flow [38], giving rise to the same transformations (cf. (2.1.29)). These transformations are precisely generated by integer shifts of the Wilson line moduli, $a^A \to a^A + k^A$. Observe that the angular momenta will also transform under these shifts, and we find the following results,

$$\begin{aligned} q_A &\to q_A - 12\, C_{ABC}p^Bk^C\,,\\ J_\varphi &\to J_\varphi - 12\, C_{ABC}p^Ap^Bk^C\,,\\ J_\psi &\to J_\psi - q_Ak^A - 6\, C_{ABC}p^Ap^Bk^C + 6\, C_{ABC}p^Ak^Bk^C\,, \end{aligned} \qquad (4.5.17)$$

which imply that $\hat{q}_0$ remains invariant.

The difference between (3.3.11) and (4.5.15) resides in the shifts of the electric charges proportional to $C_{ABC}p^Bp^C$. The presence of these shifts is consistent with many previous results, both from field theoretic solutions and from microstate counting [35, 38, 44, 58, 76–79], and is reflected in the two derivative solution through the harmonic functions (2.1.36) (see section 2.3). The modified charges $q_A - \frac{1}{2}\,C_{ABC}p^Bp^C$ in (4.5.14) are additive, as follows from a calculation similar to the one leading to the attractor equation for q_A, but now for a configuration of concentric rings. Such a calculation has been performed in [58] and resulted in the equations (4.2.41) and (4.2.42) that we discussed earlier. When combined with the attractor equation for q_A shown in (4.5.13), they establish the additivity of the shifted charges. The latter is manifest in the results of [35, 38]. The modified charges should therefore be used in the microscopic formula of [67] to match with the macroscopic result (4.5.14), as was already emphasized in [38, 77, 78]. Note, however, that in spite of the qualitative agreement of these conclusions, we should stress that we have adopted a different definition of the electric charges q_A, which is not based on the asymptotic fall-off of the electric fields at spatial infinity. Therefore the modified charges should be the same, but the electric charges may still be different.

The shifts in the electric charges cannot be removed in the four-dimensional results by a suitable duality transformation of the form (3.3.9), because that transformation induces shifts that are twice as large. The shifts are related to the terms $\pm\frac{1}{2}p^A\mathrm{d}\psi$ in the gauge field sections in (4.2.39). From the point of view of subsection 4.2.3, they arise due to the non-trivial topology of the full five-dimensional space-time. Therefore the four-dimensional black hole should be compared to the reduction of an infinite magnetic string in five dimensions, which is topologically trivial. In that case, both the terms $\pm\frac{1}{2}p^A\mathrm{d}\psi$ in (4.2.39) and the shifts in the electric charges in (4.5.11) will be absent, so that one obtains full agreement with the four-dimensional attractor results.

4.6 Concluding remarks

In this chapter, we gave a detailed discussion of four derivative corrections to five-dimensional supergravity and their effects on BPS attractors. Both for black holes and black rings we found full agreement with available microscopic results. In the case of black holes, this comparison is restricted to the static limit, due to the difficulty of obtaining microscopic results for rotating solutions. When including angular momentum, our results were compared to the supergravity predictions of [109] and [92–95]. Even though all supergravity predictions agree in the static limit, there are discrepancies related to attractor equations for the electric charges.

A somewhat related issue is the connection to the corresponding four-dimensional results presented in chapter 3. Upon dimensional reduction, the five-dimensional R^2 correction leads to a four-dimensional action including both the R^2 and F^4 invariants discussed in section 3.1, so that the results of section 3.3 apply. The comparison is however not straightforward, due to the presence of Chern–Simons terms in the five-dimensional action, which modify the relation between the four- and five-dimensional electric charges. After a careful analysis of the patch structure for the various gauge fields (section 4.2.3), we were able to identify the general form of the modifications, which depends only on the topology of spacetime. For a black ring, the electric charge is shifted due to the subtle gauge transformations between gauge patches in the presence of a spatially extended magnetic monopole and is related to the cubic Chern-Simons term. For the spinning black hole, a similar effect arises due to the nontrivial Euler density of spacetime, which does not allow for the gravitational Chern–Simon terms to be defined globally. Taking into account these results, we were able to give a precise four-dimensional interpretation to our results, which exactly match the corresponding expressions in chapter 3.

It would be very interesting to expand this understanding of the macroscopic properties of BPS black holes in connection with recent ideas on the supergravity interpretation of microstates [26, 115–118] and the associated partition function [119–121]. Parallel to that, the construction of higher derivative actions in supergravity is an interesting subject of its own, as knowledge of the possible invariants can elucidate the structure of the low energy effective description of string theory.

Chapter 5

Non-BPS black hole solutions

5.1 Introduction and Overview

In previous chapters we studied BPS black holes, including higher order corrections, and found very good agreement with the corresponding microscopic models. The presence of unbroken supercharges was instrumental, as they imply constraints that allow both for the explicit construction of the full solutions and for the matching to the microscopic theory.

In contrast, when the requirement that the solutions preserve some supersymmetry is abandoned, much less is known both about the general structure of the supergravity solutions and the microscopic theory behind them. The simplest generalisation of BPS black holes is to consider extremal black holes that do not preserve any supersymmetry. These are known to share some desirable features with the BPS branch, most importantly the attractor phenomenon [19, 52, 122].

For maximal and half maximal supergravity, where the constraints on the couplings are stronger, a number of general results have been obtained using symmetries of the action [123–126]. In theories with 8 supercharges coupled to vector multiplets in four and five dimensions,[1] the structure of general extremal black hole solutions is unclear, as only partial results are available. For instance, it is known that non-BPS attractors exhibit flat directions, in the sense that the scalars are not completely fixed at the horizon once the charges are chosen [127]. A few restricted examples can be obtained from BPS solutions simply by changing the sign of some charges [127, 128]. However, these examples are not generic enough – they contain (at least) one less than the minimum number of parameters required for the most general solution to be derived from them by dualities [126]. A solution that does contain enough parameters is called a seed solution in this context.

[1]Since the two are related by dimensional reduction, we do not make a distinction between them in this introduction.

For cubic prepotentials, an appropriate seed was found in [129, 130] and the full duality orbit for the *stu* model was subsequently derived in [131]. This full example clarifies how the non-supersymmetric solutions differ from their BPS counterparts in more than simply changing the signs of charges. In particular the scalar sector has flat directions, which are realised as symmetries acting along the full flow, including the horizon [132–134].

However, static solutions are not the end of the story, as non-supersymmetric extremal solutions can also rotate. If one allows for angular momentum, the space of solutions is extended into two types of single-centre extremal black holes [14–16], both displaying attractor behaviour [135]. The so-called over-rotating black holes feature an ergoregion and are continuously connected to the Kerr solution. Schematically, the entropy of these solutions is

$$\mathcal{S} \approx 2\pi \sqrt{J^2 - I_4(\Gamma)}\,, \tag{5.1.1}$$

where J is the angular momentum and $I_4(\Gamma)$ is the quartic invariant of the model, introduced in section 2.2.3. This relation shows that it may not be possible to set the angular momentum of some over-rotating black holes to zero. On the other hand, the under-rotating (or ergo-free) black holes always have a continuous limit to static charged black holes, as the entropy has the general form

$$\mathcal{S} \approx 2\pi \sqrt{I_4(\Gamma) - J^2}\,. \tag{5.1.2}$$

Such solutions correspond to the five-dimensional spinning black hole of section 2.1.3 and seem to be tractable using BPS-inspired techniques, as will be shown in this chapter.

Despite the existence of some known solutions, mentioned above, finding an organising principle for their general structure has proven challenging. The best developed approaches are based on four-dimensional supergravity, where electric-magnetic duality limits the possible structures. One such framework is provided by the timelike dimensional reduction of [12], which relates black holes, regardless of supersymmetry (or even extremality), to geodesics on an extended scalar manifold. Given sufficient symmetry on the scalar manifold, solutions may be generated with powerful group-theoretical methods, cf. [136–140], including multi-centre black holes. Recently, it has even been possible to show integrability of the scalar equations of motion in black hole backgrounds [141, 142], for specific models. Unfortunately, this comes at the expense of the results being expressed less explicitly.

A more direct perspective has been offered by the fake superpotential approach of [143], where it was pointed out that the rewriting of the effective Lagrangian as a sum of squares is not unique, leading to more than one type of first-order flow for the scalar fields. The flow, which in the supersymmetric case is governed by the absolute value of the central charge, may be more generally controlled by a different function, called the fake superpotential. The derivation of first-order equations based on a superpotential has been subsequently extended to static non-extremal black holes, and for a number of models the relevant superpotential has been identified explicitly [144–148] (see [149] for a synopsis of these developments and more references).

In this chapter, we summarise the investigations in [112, 150], where first order equations and stabilisation equations were introduced for some classes of non-BPS solutions. Along the way, we comment on some further generalisations considered in [151–154], as well as on the relation with the approaches mentioned above.

5.2 BPS versus almost-BPS solutions

A longstanding observation [127, 128, 155] in the context of four-dimensional $\mathcal{N} = 2$ supergravity is that one can generate a non-supersymmetric black hole solution from a supersymmetric one by changing the signs of charges. This operation is known to work for a very restricted set of solutions. The standard example, appearing in the following discussion, is the so called static electric solution, for which the harmonic functions in (2.2.34) are given by

$$\mathcal{H} = (H^0,\, 0\,;\;\; 0,\, H_A)\,. \tag{5.2.1}$$

Then, a BPS solution is given by solving (2.2.35) as usual, leading to the scale factor and scalars

$$e^{-4U} \propto H^0 C^{ABC} H_A H_B H_C\,, \qquad z^A \propto \mathrm{i} e^{2U} C^{ABC} H_B H_C \tag{5.2.2}$$

where we considered the case of a symmetric scalar manifold for simplicity. In order to have a well defined solution, the quartic product in (5.2.2) nuts be positive, imposing restrictions on the relative signs of the harmonic functions and thus the charges.

On the other hand, one can show that even if the harmonic functions are such that the product above is negative, a solution is also given by the above equations if one takes the absolute value. Equivalently, one can obtain this solution by solving the system (2.2.35) with $\mathcal{H}$ as in (5.2.1), whereas the gauge fields are instead controlled by the vector[2]

$$\mathcal{H}_c = (H^0,\, 0\,;\;\; 0,\, -H_A)\,. \tag{5.2.3}$$

The net effect is that the resulting solution does not preserve any supersymmetry, since the contributions from the geometry and the scalars in the Killing spinor equations are the same but the ones from the gauge fields are not. Based on this simple observation, there have been hopes of constructing the full class of non-BPS solutions generalising the relation between (5.2.1) and (5.2.3) to generic charges, as in [128].

However, a simple counting argument [130] shows that this is not possible, as the solution above is not generic enough to generate all other solutions in its class by electric-magnetic

[2] One can alternatively change the sign of H^0. In what follows we will consider both choices, depending on the context.

duality. Such a solution is called a seed solution and in four dimensions it must have at least one more independent parameter compared to the one in (5.2.2). For the BPS branch, a seed solution can be found by simply adding more harmonic functions, resulting in the supersymmetric solutions of (2.2.35). For the non-supersymmetric branch this is not true, as was shown in [129, 130] by explicit construction of the seed solution, which cannot be obtained by sign changes on a BPS solution. Therefore, it is necessary to develop new tools to construct non-BPS solutions in order to uncover their general structure, including the special case of sign flips.

5.2.1 Five-dimensional BPS solutions with flat base space

It turns out that an intuitive way of investigating the relation between the two solutions above is to consider the five-dimensional uplift of the supersymmetric solution, which is the static limit of the BMPV black hole in Taub-NUT (see section 2.1.3). In order to connect to the sign reversed solution, consider first the near-horizon region (2.1.35), which is described by a flat base space. In this context, it is possible to give a geometric interpretation to sign flip leading to the non-BPS solution described by (5.2.3), as we now show.

It was noted below (2.1.23) that there is some extra freedom in choosing the hyper-Kähler complex structures on flat space: they can be chosen to be either self- or anti-self-dual. This is a result of the larger group of rotations for $\mathbb{R}^4$ compared to a generic hyper-Kähler manifold, namely $SO(4) \equiv SU(2)_L \times SU(2)_R$ rather than $SU(2) \times U(1)$, so that the two triplets of complex structures rotate under different linear combinations of $SU(2)_R$ and $SU(2)_L$. In this special case, there is a second half-supersymmetric solution one can write down, in which all the fields are left invariant under the self-dual complex structures [25]. Explicitly, the relevant BPS conditions (2.1.17)-(2.1.19) for the flat case are modified to:

$$\begin{aligned} ds^2 &= -f^2(dt+\hat{\omega})^2 + f^{-1}h_{mn}dx^m dx^n\,, \\ F^A &= \pm 2\,\partial_{[\mu}(\sigma^A e^0_{\nu]})\,dx^\mu dx^\nu \pm \Lambda^A_{mn}\,dx^m dx^n\,, \\ \sigma_A\Lambda^A &= -\frac{2}{3}G^\pm\,, \\ \Delta\left(f^{-1}\sigma_A\right) &= \frac{1}{12}C_{ABC}\Lambda^B_{mn}\Lambda^{mn\,C}\,. \end{aligned} \tag{5.2.4}$$

Here, Λ^A are self-dual forms on the base for the upper sign and anti-self-dual for the lower sign. These pairs of BPS solutions are aligned with the two complex structures in the sense that they are invariant under the corresponding $SU(2)$ subgroups of the $SO(4)$ isometry group and, in an orthonormal basis, their respective Killing spinors satisfy $\gamma^0\epsilon = \pm i\epsilon$. Some examples of supersymmetric pairs were written down in [25], using right- and left- invariant one-forms on S^3.

An interesting property of these BPS pairs is that when reducing to four dimensions along a circle, it is not possible to retain both in the BPS spectrum. This is because the two sets of BPS solutions respect a different $SU(2)$ isometry in five dimensions, while any reduction ansatz can only respect one $SU(2) \equiv SO(3)$ isometry. Thus, choosing a particular $SU(2)$ to break in the reduction, all the BPS solutions aligned with it are lost as well. Equivalently, the Killing spinors of only one set of solutions are invariant under the $SU(2)$ that is being broken and will be trivially reduced to four dimensions. Those of the other set will be charged under the Kaluza-Klein gauge field, violating the natural assumption of invariance and leading to a non-BPS solution. A very similar situation was encountered in [156, 157], where it was shown that if one considers the full Kaluza-Klein tower, supersymmetry is recovered.

This "supersymmetry without supersymmetry" effect, was observed in [129, 158], in the context of lifting to five dimensions the near horizon geometry of the four-dimensional non-BPS black hole described by (5.2.3), which can be also written as a solution with flat base space. In the particular electric case discussed here, one has $\Lambda^A = 0$, $\omega = 0$ and $H = p^0/r$ for flat base space with a conical singularity (the coordinate ψ has a fixed range as in (2.1.20)). Then the two five-dimensional BPS solutions above are identical with respect to the metric and the scalars, but have opposite electric charges:

$$\begin{aligned} ds^2 &= -f^2 dt^2 + f^{-1}\left(\frac{p^0}{r} dx^i dx^i + \frac{r}{p^0}(d\psi + p^0 \cos\theta \phi)^2\right) \\ F^A &= \pm 2\, \partial_m (f \sigma^A)\, dx^m dt, \qquad f^{-1}\sigma_A = \frac{1}{3} L_A = \frac{1}{3}\frac{q_A}{r}. \end{aligned} \tag{5.2.5}$$

Upon reduction to four dimensions preserving the $SO(3)$ symmetry of the directions x^i, they give the BPS and the non-BPS attractor (5.2.1) and (5.2.3) for the plus and minus sign respectively[3].

It should be noted that this observation does not affect the 4D/5D connection for BPS solutions as described in [56, 159]. There, a Taub-NUT base that interpolates between a five-dimensional and a four-dimensional solution was used to argue that the BPS index is the same in four and five dimensions. As Taub-NUT has a unique triplet of complex structures, there is only one BPS solution for each choice of harmonic functions. Thus, all asymptotically Taub-NUT BPS solutions in five dimensions are mapped to asymptotically flat BPS solutions with appropriate charges, under dimensional reduction [57]. The same holds for any Gibbons-Hawking space except flat space.

This is explicitly seen from the asymptotically Taub-NUT extensions of the two attractor solutions. The solution with the plus sign in (5.2.5) is compatible with the anti-self-dual complex structures of $\mathbb{R}^4$ and its asymptotically Taub-NUT extension is the BMPV black hole [39] in the centre of Taub-NUT, described by the harmonic functions in (2.1.31). In view of the anti-self-duality of the complex structures of this base, it is a BPS solution

[3] If the other $SO(3)$ symmetry of $\mathbb{R}^4$ is chosen in the ansatz, one ends up with the same solutions in four dimensions, but their origins in five dimensions are interchanged.

[25]. The solution with the minus sign is instead compatible with the self-dual complex structures of $\mathbb{R}^4$. Its Taub-NUT extension was constructed in [129], and is the non-BPS static seed solution mentioned above. Interestingly, this solution can be viewed as a special case of a more general class of solutions, to which we now turn.

5.2.2 Almost-BPS solutions

One might wonder about the fate of the two different supersymmetric solutions in case the base space is not flat. This becomes especially interesting in view of the fact that both kinds of BPS solutions with a flat base (5.2.5) can be the near horizon region of (not necessarily BPS) black holes embedded in a more general space, as mentioned above. The existence of full interpolating Taub-NUT solutions for the two attractors in (5.2.5), only one of which is BPS, suggests that the two different solutions might survive as BPS/non-BPS pairs in this case.

Indeed, it is straightforward to show that as long as the base is Ricci flat, both expressions in (5.2.4)-(5.2.4) solve the equations of motion [112, 153, 154]. For a general base both solutions are non-supersymmetric, whereas for a hyper-Kähler base, anti-self-duality of the complex structures allows only for the one with the upper sign to be supersymmetric.

An intuitive picture of the relation between the two solutions can be given for a Gibbons-Hawking base space. Such a manifold is a $U(1)$ bundle over $\mathbb{R}^3$, so it can be trivialised into $\mathbb{R}^4$ by a suitable choice of coordinates on any local patch. One then has a choice between self-dual or anti-self-dual complex structures on every such patch as before, so that both expressions in (5.2.4)-(5.2.4) constitute BPS solutions.

By extending to the full base space, only the anti-self-dual structures on local patches can be integrated to the unique global complex structures (2.1.22). In contrast, the local self-dual structures can be integrated to the almost hyper-Kähler structures:

$$\tilde{X}^{(i)} = \left(d\psi + \chi_j dx^j\right) \wedge dx^i + \frac{1}{2} H \epsilon_{ijk} dx^j \wedge dx^k, \tag{5.2.6}$$

that are globally defined, but not integrable: $d\tilde{X}^{(i)} \neq 0$. The existence of the forms (5.2.6) allows one to construct globally defined fields by aligning local solutions on every patch with the appropriate restriction of these structures[4]. It is then clear why both signs in (5.2.4) provide a solution to the equations of motion, since they can be viewed as constructed locally from BPS solutions aligned with the forms in (5.2.6). The difference is that the one with the upper signs is compatible with the global complex structures and is a global BPS solution. The second solution fails to be supersymmetric only due to a global obstruction, providing an example of a non-supersymmetric solution with

[4]Note that the restriction of these forms on a patch is transformed to a constant by the coordinate transformation that trivialises the patch. It is the non-compatibility of these local coordinate transformations that makes the global forms non-integrable. This is also what prohibits the existence of a corresponding global Killing spinor.

the peculiar property of admitting four supercharges on local patches. In fact, it is expected to have all the local properties of a BPS solution, which are behind most of the computational simplifications in that case. This property, which is based on the existence of an almost hyper-Kähler structure, motivates the nickname, “almost-BPS”, for these solutions. By the same argument, almost-BPS solutions are supersymmetric on local patches for a base more general than Gibbons-Hawking if there exists a globally defined almost hyper-Kähler structure.

Here, we restrict to Gibbons-Hawking base spaces for simplicity. In this case, the almost-BPS solutions can be specified through arbitrary harmonic functions as for the BPS case, following the algorithm in [35]. First write

$$\hat{\omega} = \omega_5 \left(d\psi + \chi_i dx^i\right) + \omega_i dx^i, \tag{5.2.7}$$

$$\Lambda^A = V_i^A \left(d\psi + \chi_j dx^j\right) \wedge dx^i - \frac{1}{2} H \epsilon_{ijk} V_k^A dx^i \wedge dx^j, \tag{5.2.8}$$

for the one-form ω and the anti-self-dual form in (5.2.4), where V^A are arbitrary one-forms to be determined.

Using (2.1.20), closure of Λ^A reduces to the relations:

$$\nabla \times \mathbf{V}^A = 0\,, \qquad \nabla \cdot \mathbf{V}^A = 0\,, \tag{5.2.9}$$

which in turn imply that locally:

$$\mathbf{V}^A = \nabla K^A\,, \tag{5.2.10}$$

for some harmonic functions K^A. The equation for $f^{-1}\sigma_A$ in (5.2.4) implies that

$$\nabla^2 \left(f^{-1}\sigma_A\right) = \frac{1}{3} H C_{ABC} \nabla K^B \nabla K^C = \frac{1}{6} H \nabla^2 \left(C_{ABC} K^B K^C\right)\,, \tag{5.2.11}$$

which can be solved up to a set of arbitrary harmonic functions L_A, given the K^A. Here, we will restrict to solutions of the slightly stronger relation:

$$\nabla \left(f^{-1}\sigma_A\right) = \frac{1}{6}\Big(H \nabla \left(C_{ABC} K^B K^C\right) - C_{ABC} K^B K^C \nabla H\Big) + \frac{2}{3} \nabla L_A\,, \tag{5.2.12}$$

even though there might be physically interesting solutions not captured by it. The advantage of this simplification is that the scalars are governed by a first order flow very similar to the BPS one.

Finally, we find the conditions on ω_5 and $\hat{\omega}_i$. Writing out the first equation in (5.2.4) using (2.1.19) and (5.2.10) one gets:

$$\nabla \times \omega + \nabla \left(H \omega_5\right) = 3\, f^{-1} H \sigma_A \nabla K^A \tag{5.2.13}$$

Taking the divergence of this gives the integrability condition

$$\nabla^2 (H\omega_5) = 3\,\nabla \left(f^{-1} H \sigma_A\right) \cdot \nabla K^A, \tag{5.2.14}$$

which can be solved up to an arbitrary harmonic function M, given the solution of (5.2.12). Substituting in (5.2.13), the one-form $\hat{\omega}$ can be determined up to a total derivative (removable by a change of coordinates).

Observe that, just like its supersymmetric partner, an almost-BPS solution is determined by $2n_v + 2$ harmonic functions H, K^A, L_A, M, which encode the charges as in section 2.1.2. In this respect, almost-BPS solutions are a five-dimensional analogue of the four-dimensional change of sign between (5.2.1) and (5.2.3). Comparing (5.2.11)-(5.2.14) with (2.1.25)-(2.1.28), shows that they are related in exactly that way if the K^A vanish, but the general case is more complicated than a change of relative signs, as it involves different powers of the harmonic function H.

When H is such that the base is flat, the BPS/almost-BPS pairs degenerate into the supersymmetric pairs of the previous section, so that there are no asymptotically flat five-dimensional non-supersymmetric solutions in the almost-BPS class. In this case, if one chooses $H = 1$, the supersymmetric pairs are always related by a sign change, but not if one takes $H = 1/r$. The latter choice typically leads to genuinely different BPS solutions that are not asymptotically flat, as in [25].

Choosing a Taub-NUT base instead, there are solutions that asymptote to $\mathbb{R}^3 \times S^1$, allowing for an interpretation as asymptotically flat solutions in four dimensions. It follows that reduction along the $\partial/\partial\psi$ direction results in non-supersymmetric solutions of four-dimensional $\mathcal{N} = 2$ supergravity that are more general in several ways than the ones produced by the four-dimensional sign change. In particular, almost-BPS solutions allow for some nontrivial moduli at infinity as in [129, 130] and multiple centres, unlike the four-dimensional flip of signs between (5.2.1) and (5.2.3), that only works for single-centre solutions with purely imaginary scalars.

However, almost-BPS solutions are far from being the most general non-BPS black holes, as they represent very special points in the duality orbits. For example, $D0$–$D6$ non-BPS black holes (see e.g [59, 130, 131]) are not captured by the almost-BPS equations, as they were derived from the BPS conditions, that do not allow for such objects. Similarly, it is known that non-BPS black holes generically exhibit flat directions all along the flow in both four and five dimensions [132–134, 160, 161]. This feature is also not captured by almost-BPS solutions, since the BPS solutions have no flat directions either. Therefore, in practice, the extra freedom in the almost-BPS equations has been used to produce seed solutions for non-BPS black holes, which can lead to more general solutions. In this respect, four-dimensional dualities are crucial, since the most general non-BPS solution in a specific theory can be explicitly generated from a restricted example, provided there are enough symmetries [130, 131]. In later sections we will discuss some the implications of symplectic reparametrisations, after reducing to four dimensions.

At the same time, the class of almost-BPS solutions can be enriched by relaxing the ansatz for the metric and gauge fields, as in [153, 154, 162–165]. In these works, new interesting classes of solutions were generated, mostly in the five-dimensional setting. One of the most exciting results is the construction of smooth, horizonless solutions with the same asymptotic charges as non-BPS black holes, which can be used as microstate geometries in the fuzzball approach [26, 115–118].

5.2.3 Non-BPS electric black hole

One of the remarkable properties of supersymmetric solutions is that they can usually be constructed analytically (see chapter 2 for a discussion in the two derivative case and [54, 93] for solutions to the four derivative theories). However, non-BPS solutions are significantly more complicated, as one can see by comparing the BPS equations with their almost-BPS counterparts, which can not be solved formally as in section 2.1.2 in the general case. Here we first discuss some of the general features of almost-BPS solutions, specialising to a single-centered example which is very interesting from a four-dimensional perspective. For further details, generalisations and more examples of explicit solutions we refer to [112, 151–154].

The most important feature of almost-BPS solutions is the special structure of their near horizon and asymptotic regions. Indeed, since the near horizon region of these solutions can be written as a timelike fibration over flat space, it preserves four supercharges by construction. In fact, it preserves all eight supercharges, since the near horizon region is maximally supersymmetric for BPS solutions. The same holds trivially for the asymptotic region, up to compactness of one coordinate in the case of asymptotically Taub-NUT solutions, as they asymptote to $\mathbb{R}^3 \times S^1$. It follows that the mass of any almost-BPS solution takes the same form as for its BPS partner, namely a simple sum of charges. This feature has been observed for non-supersymmetric solutions of the STU model in four dimensions [130], which can be lifted to almost-BPS solutions in five dimensions, as we show below. Note that the BPS nature of the attractors and the asymptotic region of our solutions is invisible from a four-dimensional perspective, as explained in section 5.2.1.

A second, more intricate, property that also follows from the almost-BPS conditions is the existence of a first order flow for the scalars as in [129, 143, 144]. For supersymmetric solutions, the function governing this flow is the central charge. The almost-BPS equations can then be used to obtain an analogous quantity for the non-supersymmetric case, both in the five and the four-dimensional theory, by dimensional reduction.

We now turn to an explicit single-centered example, which departs from the simple sign flip relation to it BPS partner and has two independent angular momenta [129, 130, 151]. Similar to the BMPV solution on the BPS side, we choose the harmonic functions as:

$$H = h^0 + \frac{p^0}{r}\,, \quad K^A = 0\,, \quad L_A = l_A + \frac{q_A}{r}\,, \quad M = b + \frac{J\,\cos\theta}{r^2}\,. \tag{5.2.15}$$

Here, $l_A\,, b$ are constants and $p^0\,, q_A\,, J$ are the Taub-NUT charge, the electric charges and the angular momentum of the solution. Considering the case of a symmetric scalar manifold for simplicity, one can solve the scalar equation explicitly as

$$\begin{aligned} f^{-1}\sigma_A &= \frac{2}{3}L_A\,, \quad \Rightarrow \\ f^{-3} &= \frac{4}{3}C^{ABC}L_AL_BL_C\,, \quad \sigma^A = 2\,f^2\,C^{ABC}L_BL_C\,, \end{aligned} \tag{5.2.16}$$

whereas the angular momentum one-form is given by

$$\omega_5 = \frac{M}{H}\,, \qquad \nabla\times\omega = -\nabla M\,, \qquad \omega = \frac{J\,\sin^2\theta}{r}\,\mathrm{d}\phi\,. \tag{5.2.17}$$

The metric and gauge fields follow straightforwardly from (5.2.4) with the lower sign.

Even though the five-dimensional geometry is interesting in its own right, we choose to emphasise the four-dimensional interpretation of this solution for future reference. Upon reduction along the ψ direction one obtains a single-centred under-rotating extremal solution that carries angular momentum along the ϕ direction, as in (5.2.17). Using (2.3.2), the four-dimensional metric is as in (2.3.5), with

$$e^{-4U} = \frac{4}{3}\,H\,C^{ABC}L_AL_BL_C - M^2\,, \qquad \lim_{r\to\infty} e^{-4U} = h^0l^3 - b^2 = 1, \tag{5.2.18}$$

$$l^3 = \frac{1}{3}\,l^Al_A, \qquad l^A = 4\,C^{ABC}l_Bl_C\,, \tag{5.2.19}$$

where we imposed four-dimensional asymptotic flatness and introduced some useful notation. The four-dimensional scalars (2.3.4) and gauge fields are given by:

$$z^A = \frac{f\,\sigma^A}{H}\left(-M + \mathrm{i}e^{-2U}\right)\,, \tag{5.2.20}$$

$$F^A_{(4)} = -d\left[\sigma^A f(1 + M^2\,e^{4U})(dt+\omega)\right]\,, \tag{5.2.21}$$

and $F^0 = dA^0$, with A^0 as in (2.3.7). Note that, unlike in (5.2.2), the scalars have a nontrivial (common) phase.

The ADM mass associated to this solution is simply found by expanding the metric to first order in $1/r$. The result is:

$$M_{ADM} = \frac{1}{4G_4}\left(p^0l^3 + h^0\,l^Aq_A\right)\,, \tag{5.2.22}$$

which is simply a sum of charges just like for its BPS partner, as anticipated above. Note the absence of angular momentum from this equation for the same reason, a feature that is also consistent with general expectations on four-dimensional under-rotating black holes.

Interestingly, this four-dimensional solution contains an extra parameter at infinity, b, which turns out to be exactly the missing degree of freedom for the construction of a static seed solution, mentioned in the beginning of the section. However, this constant is only the trivial part of a full harmonic function, M, which controls the angular momentum. Therefore, we find this an appropriate seed solution in four dimensions [151], and we expect that the most general under-rotating extremal black hole for a four-dimensional $\mathcal{N} = 2$ theory with a symmetric scalar manifold can be obtained by applying dualities on the above solution, along the lines of [131]. In the next section we discuss how to implement electric-magnetic duality for generic special Kähler manifolds.

Before doing that, we discuss two possible generalisations. The first is to the multi-centre case, by adding more centres in the harmonic functions $H\,, L_A$ in (5.2.15), generalising the base space to multi-Taub-NUT. Upon reduction to four dimensions, these solutions seem to agree qualitatively with the solutions of [137]. A second generalisation would be to turn on the magnetic harmonic functions. In the BPS case this is irrelevant due to spectral flow [38], as in (2.1.29). In contrast, equations (5.2.12)-(5.2.14) do not appear to have such an invariance. In this case, the functions $f\,, \omega_5$ diverge at the centre[5] as r^{-3} and r^{-4} respectively, making them unattractive at first sight. However, the near horizon geometry of these solutions is described by the BPS solutions considered in [25] (for the minimal theory). There, a number of curvature invariants were examined and were found to remain finite at $r = 0$, hinting at a regular solution.

5.3 Stabilisation equations and an ansatz

Having gained some insight for non-BPS solutions using the five-dimensional theory, we now return to its four-dimensional reduction. The reason is that, even though five-dimensional supergravity appears to be simpler, solution generating techniques are more powerful in the four-dimensional theory. The crucial difference is that the equations of motion of four-dimensional $N = 2$ supergravity are covariant under electric-magnetic duality, including the scalar sector, as discussed in section 2.2.2. One can use this covariance to generate new solutions from a given one by symplectic rotations. If the starting solution is generic enough, i.e. it is a proper seed, this process can lead to the most general solution of a given class. In this section, we consider the explicit electric solution of the last section and rewrite it in terms of symplectically covariant objects.

[5] One would naively expect to find a horizon at $r = 0$ in these coordinates, as in all other cases treated in this work.

Based on these variables, we can generate new solutions through symplectic rotations, aiming towards a generic ansatz that covers all single-centre solutions.

5.3.1 The four-dimensional solution and symplectic rotations

The importance of the rotating electric solution in the previous section stems from the fact that it can be used as a seed for four-dimensional under-rotating black holes [14–16] in theories with a cubic prepotential. Consequently, all under-rotating solutions, carrying arbitrary charges, can be generated from this by applying appropriate electric-magnetic rotations, as in (2.2.15).

This transformation does not act on the metric, but it does act on the symplectic section $\mathcal{V}$ according to (2.2.19), inducing projective linear transformations on the physical scalars, as can be seen using the special coordinates given by (2.2.6). As it is cumbersome to implement this in practice, especially in the case of non-symmetric scalar manifolds, we follow the opposite route, rewriting the known solution (5.2.20) for the physical scalars and the metric in terms of $\mathcal{V}$, so that symplectic rotations can be applied directly. It is worth noting that using a rotating black hole solution is very useful, as the presence of an extra harmonic function describing rotation provides guidance, minimising the ambiguity introduced in the process.

After reducing to four dimensions, it is useful to define a symplectic vector of harmonic functions describing the associated physical charges, as in[6]

$$\mathcal{H}_c = (-H^0,\, 0\,;\, 0,\, H_A)\,, \tag{5.3.1}$$

where the dictionary (2.3.13) was used to rename the harmonic functions. As the harmonic function M controls the angular momentum, it is invariant under symplectic transformations.

Using the 4D/5D dictionary of section 2.3 [56, 57, 159], we can rewrite the full solution given in five-dimensional notation above in terms of variables natural from the four-dimensional perspective. The metric is as in (2.2.33), which we repeat here

$$ds^2 = -\mathrm{e}^{2U}(\mathrm{d}t + \omega_i \mathrm{d}x^i)^2 + \mathrm{e}^{-2U} d\vec{x}^2\,, \tag{5.3.2}$$

while the resulting expressions for the gauge fields and scalars are[7]

$$\boldsymbol{\mathcal{F}} = \star_{\mathbf{0}}\mathbf{d}\mathcal{H}_c - 2\,\mathbf{d}(\mathrm{e}^{2U}\,\mathrm{Re}\,\tilde{\mathcal{V}}\omega)\,, \qquad 2\,\mathrm{Im}\,\tilde{\mathcal{V}} = \mathcal{J} \equiv \mathcal{H} + \mathcal{R} \tag{5.3.3}$$

[6]Here and henceforth we reverse the signs of all the charges of the non-BPS solution compared to the previous section for convenience, so that the BPS solution is found by changing the sign of p^0, rather than all the electric charges as is natural in five dimensions. This can be absorbed in an irrelevant overall sign change of $\mathcal{V}$.

[7] In the remainder of the chapter, we use form notation for quantities in the three spatial dimensions, which will be generally denoted by boldface symbols.

Here, $\star_0$ is the Hodge dual with respect to the flat three-dimensional metric δ_{ij} and we use again the shorthand $\tilde{\mathcal{V}} = \mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}$ as in section 2.3. The vector $\mathcal{J}$ is written in terms of a harmonic part, $\mathcal{H}$, and a part containing ratios of harmonic functions $\mathcal{R}$, given respectively by

$$\mathcal{H} = (H^0,\, 0\, ;\, 0,\, H_A)\,, \qquad \mathcal{R} = \Big(0,\, 0\, ;\, \frac{M}{H^0},\, 0\Big)\,. \tag{5.3.4}$$

Finally, the metric functions are re-expressed as

$$\star_0 \mathbf{d}\omega = -\mathbf{d}M = \mathbf{d}\langle\mathcal{H},\mathcal{R}\rangle\,, \qquad \mathrm{e}^{-2U} = \mathrm{i}\langle\tilde{\mathcal{V}},\bar{\tilde{\mathcal{V}}}\rangle = \sqrt{I_4(\mathcal{H}) - M^2}\,. \tag{5.3.5}$$

Here, I_4 is the quartic invariant that appears in the entropy formula for cubic prepotentials (see (2.3.12)), as usual.

The expression (5.3.3) for the scalars is in the form of stabilisation equations as they are known for BPS solutions [22]. In particular, the asymptotic values of the scalars are controlled by the constant parts of the harmonic functions $\mathcal{H}$ and M, whereas the attractor equations, obtained in the limit $r \to 0$, are controlled by the charges and the angular momentum [135].

The harmonic functions $\mathcal{H}$ are related to the physical ones (5.3.1) by a single sign flip, as in the example in the beginning of section 5.2, where the scalars were purely imaginary. The novel addition to $\mathcal{J}$ is a ratio of harmonic functions that was not present in previous attempts to write non-BPS stabilisation equations and allows for a nontrivial phase of the scalars. Note that the gauge field strengths are related to the corresponding expression in (2.2.34) by the sign flip of the harmonic functions only, whereas the term involving the angular momentum in (5.3.3) is exactly the same as for a BPS solution.

As this is the seed solution for under-rotating extremal black holes, the most general solution can be found by duality rotations on the stabilisation equations (5.3.3) and one concludes that a ratio of harmonic functions is generated in all other cases as well. For example, in the case of the *stu* model, one can explicitly dualise to the frame with only two charges present, corresponding to a D0-D6 brane system in Type IIA theory. For this model, the prepotential is as in (2.3.1) with $C_{ABC} = |\varepsilon_{ABC}|$ and the scalar sector is then described by the choice (no sum on $A = 1, 2, 3$):

$$\mathcal{H} = \big(H^0,\, \frac{1}{\lambda^A}H^A\, ;\, H_0,\, \lambda^A H_A\big)\,, \qquad \mathcal{R} = \frac{1}{8}\frac{M}{H_0^+}\Big(1,\, \frac{1}{\lambda^A}\, ; -1,\, -\lambda^A\Big)\,, \tag{5.3.6}$$

where

$$H_I = h_I + \frac{q_0}{r}\,, \qquad H_0^+ = \frac{1}{4}\Big(h_0 + \sum_A h_A\Big) + \frac{q_0}{r} \tag{5.3.7}$$

$$H^I = -\lambda^3 H_I\,, \qquad H^{0\,+} = -\lambda^3 H_0^+\,, \qquad C_{ABC}\lambda^A\lambda^B\lambda^C \equiv \lambda^3\,, \tag{5.3.8}$$

$$\mathrm{e}^{-4U} = I_4(\mathcal{H}) - M^2 = (H_0\, H^0)^2 - M^2\,, \tag{5.3.9}$$

and λ^3 must be a constant. Note that the individual constants λ^A appear only as multiplicative factors in $\mathcal{H}$ and $\mathcal{R}$, but not in e^{-U}, which depends only on the physical harmonic functions H_0 and H^0. It follows that the metric and gauge fields only depend on the combination λ^3, so that two of the λ^A correspond to flat directions. The structure in (5.3.6) seems to be generic for D0-D6 solutions for all cubic prepotentials.

It is interesting to note that for both solutions the angular momentum harmonic function can be invariantly characterised by $M = \langle \mathcal{H}, \mathcal{R} \rangle$ and that the flat directions described by the λ^A are zero modes of this equation as well. However, the harmonic part $\mathcal{H}$ is not related to the charges by sign flips, as one might expect on the basis of (5.3.3). In fact the electric solution is special, in the sense that the flat directions can be described through (5.3.3) by simply allowing for the missing harmonic functions to be constants, at the cost of making $\mathcal{R}$ more complicated, but still proportional to a single ratio and consistent with (5.3.5).

5.3.2 The ansatz in four dimensions

In view of the above observations, it is natural to propose an ansatz for the scalars in terms of the period vector that contains harmonic functions and ratios of harmonic functions. In fact, it is simple to see that imposing consistency of any generic ansatz $\mathrm{Im}\,\tilde{\mathcal{V}} \sim \mathcal{H} + \mathcal{R}$, leads to inverse harmonic functions. Since one can compute $\langle \mathcal{H}, \mathcal{R} \rangle$ in two ways:

$$\langle \mathcal{H}, \mathcal{R} \rangle = \mathrm{Im}\langle \tilde{\mathcal{V}}, \mathcal{R} \rangle = -\,\mathrm{Im}\langle \tilde{\mathcal{V}}, \mathcal{H} \rangle\,, \tag{5.3.10}$$

where $\mathcal{H}$ and $\mathcal{R}$ are a priori independent, it follows that $\langle \mathcal{H}, \mathcal{R} \rangle$ must be a scalar-independent quantity. The only other fields in the system are the scale factor and the rotation form ω in the metric, but in view of the scale invariance[8] of $\langle \mathcal{H}, \mathcal{R} \rangle$, it cannot depend on e^U, in accord with the explicit solution above, where $\langle \mathcal{H}, \mathcal{R} \rangle = M$. In the static limit M reduces to a constant, in which case the constraint can be solved even if $\mathcal{R}$ is harmonic, but in the rotating case one has to reproduce the full θ-dependent function M in (5.2.15). This implies a structure as in (5.3.3), which then must be present even when the angular momentum is turned off.

Based on the linearity of symplectic reparametrisations and that (5.3.4) and (5.3.6) are seed solutions, we expect the structure seen in the previous section to be universal for all under-rotating extremal black holes. In other words, we take the point of view that there is no essential difference between static non-supersymmetric and under-rotating black holes, since they are continuously connected by setting to zero the nonconstant part of a single harmonic function, as in (5.2.15). Therefore, we propose the following

[8]Here we refer to the invariance of the Poincaré Lagrangian under rigid scale transformations, inherited from the full conformal formulation of the theory and corresponds to $e^U \to e^D e^U$, $\mathcal{V} \to e^D \mathcal{V}$, for the scalars and $g_{ij} \to e^{2D} g_{ij}$ for the spacial metric, where D is constant [23].

form for the stabilisation equations for the scalars and the angular momentum:

$$2\,\mathrm{Im}\,\tilde{\mathcal{V}} = \mathcal{H} + \mathcal{R}\,, \tag{5.3.11}$$

$$\star_0 \mathbf{d}\omega = \langle \mathbf{d}\mathcal{H}, \mathcal{H}\rangle + \mathbf{d}\langle \mathcal{H}, \mathcal{R}\rangle\,, \tag{5.3.12}$$

where $\mathcal{H}$ is a vector of harmonic functions and $\mathcal{R}$ is a vector of ratios of harmonic functions. The integrability condition of the last equation implies that their symplectic inner product is a harmonic function, while the scale function of the metric is given by:

$$\mathrm{e}^{-2U} = \mathrm{i}\langle \tilde{\mathcal{V}}, \bar{\tilde{\mathcal{V}}}\rangle = \sqrt{I_4(\mathcal{H}+\mathcal{R})}\,. \tag{5.3.13}$$

The solution for the components of $\tilde{\mathcal{V}}$ can then be obtained in the same way as the solutions to supersymmetric stabilisation equations [57, 60]. Note that when $\mathcal{R} = 0$ and the charges carried by $\mathcal{H}$ are identified with the physical charges, one recovers the BPS stabilisation equations, as required. More generally, for a physically reasonable solution the harmonic and inverse harmonic functions in (5.3.11) are quite restricted due to various consistency constraints, both generic and based on known explicit solutions. The rest of this section is devoted to a discussion of these generic constraints and some of their implications.

A first requirement is that in the near-horizon limit the scale factor e^{-4U} of an under-rotating black hole must reduce to [135]:

$$\mathrm{e}^{-4U} \propto I_4(\Gamma) - J^2\cos^2\theta\,, \tag{5.3.14}$$

where $I_4(\Gamma)$ is the quartic invariant for the physical charges and J is the angular momentum. In the simple case of vanishing angular momentum, $\mathcal{R}$ is proportional to inverse harmonic functions and thus vanishes near the horizon. Therefore, a harmonic piece must always be present in the right hand side of (5.3.11), to make sense of the static solution in the near-horizon region. Similar comments apply to the full rotating case, hence it is impossible to have a physical solution for the scalars based purely on inverse harmonic functions. The poles of these harmonic functions must be such that $I_4(\mathcal{H}) > 0$ and that this quartic invariant should be related to the one of the physical charges by a sign flip in order to reproduce the first term in (5.3.14).

Going over to the constraints posed by the form of the full solution, observe that in the (necessarily static) BPS case the full scale function is simply $\mathrm{e}^{-4U} = I_4(\mathcal{H})$, where the charges are replaced by their corresponding harmonic functions. Similarly, for the *stu* model, where the most general non-BPS static black hole was explicitly constructed in [131] (using the seed solution of [129, 130]), it has been shown that the scale factor is shifted as $\mathrm{e}^{-4U} \sim I_4(\mathcal{H}) - b^2$, where b is a constant that does not depend on the charges.

Interestingly, for the known under-rotating seed solution the expression for e^{-U} in (5.3.5) can again be found from (5.3.14) by replacing the charges and angular momentum by harmonic functions. In this extended setting, the additional constant b of [131] is identified with the constant piece in the harmonic function for the angular momentum in (5.3.5). Therefore it is reasonable to expect that generically the scale factor is a function of the harmonic functions for the charges and angular momentum, thus allowing for the presence of a possible residual constant in the static solutions, when J is set to zero.

Now, for an ansatz of the type (5.3.11) to describe the known solutions, the vector $\mathcal{R}$ must be such that (5.3.13) is consistent with the above comments, in particular with (5.3.5), so that

$$\mathrm{e}^{-4U} = I_4(\mathcal{H}+\mathcal{R}) = I_4(\mathcal{H}) - \langle\mathcal{H},\mathcal{R}\rangle^2\,. \tag{5.3.15}$$

This equality poses very strong restrictions on $\mathcal{R}$, as it does not appear in linear, cubic or quartic terms. In particular, the components of $\mathcal{R}$ must be such that $I_4(\mathcal{R})$ and its first derivatives vanish, implying that it must have at most as many independent components as the charge vector of a two charge small black hole. Then, given $\mathcal{H}$ and a model in which I_4 is known, the linear term in $\mathcal{R}$ should vanish, further restricting its independent components. Indeed, $\mathcal{R}$ appears to completely fixed up to an overall nontrivial function in the explicit solutions, (5.3.4) and (5.3.6).

For symmetric cubic models this can be made more precise[9], by Taylor expanding the left hand side of (5.3.15) explicitly. For such models, it is possible to show that [166]

$$I_4(\mathcal{H}+\mathcal{R}) = I_4(\mathcal{H}) + 4\,I_{3,1}(\mathcal{H},\mathcal{R}) + 6\,I_{2,2}(\mathcal{H},\mathcal{R}) + 4\,I_{3,1}(\mathcal{R},\mathcal{H}) + I_4(\mathcal{R})\,, \tag{5.3.16}$$

where

$$\begin{aligned} I_4(\mathcal{H}) &= \tfrac{1}{2}\,K_{\hat{I}\hat{J}\hat{K}\hat{L}}\mathcal{H}^{\hat{I}}\mathcal{H}^{\hat{J}}\mathcal{H}^{\hat{K}}\mathcal{H}^{\hat{L}} \\ I_{3,1}(\mathcal{H},\mathcal{R}) &= \tfrac{1}{2}\,K_{\hat{I}\hat{J}\hat{K}\hat{L}}\mathcal{H}^{\hat{I}}\mathcal{H}^{\hat{J}}\mathcal{H}^{\hat{K}}\mathcal{R}^{\hat{L}} \\ I_{2,2}(\mathcal{H},\mathcal{R}) &= \tfrac{1}{2}\,K_{\hat{I}\hat{J}\hat{K}\hat{L}}\mathcal{H}^{\hat{I}}\mathcal{H}^{\hat{J}}\mathcal{R}^{\hat{K}}\mathcal{R}^{\hat{L}}\,. \end{aligned} \tag{5.3.17}$$

Here the hatted indices stand for pairs of electric-magnetic indices as $\mathcal{H}^{\hat{I}} \equiv \{\mathcal{H}^I,\,\mathcal{H}_I\}$ and $K_{\hat{I}\hat{J}\hat{K}\hat{L}}$ is a completely symmetric symplectic tensor. The previous discussion suggests that $\mathcal{R}$ should have only one independent component, so we assume it lies in a doubly critical orbit, characterised by [167]

$$3\,K_{\hat{I}\hat{J}\hat{K}\hat{L}}\mathcal{R}^{\hat{K}}\mathcal{R}^{\hat{L}} + \Omega_{\hat{I}\hat{K}}\mathcal{R}^{\hat{K}}\Omega_{\hat{J}\hat{L}}\mathcal{R}^{\hat{L}} = 0\,, \tag{5.3.18}$$

where Ω is the symplectic matrix (2.2.16). Contracting the last equation with $\mathcal{R}$ it is easy to see that $I_4(\mathcal{R}) = I_{3,1}(\mathcal{R},\mathcal{H}) = 0$ and that (5.3.16) reads

$$I_4(\mathcal{H}+\mathcal{R}) = I_4(\mathcal{H}) + I_{3,1}(\mathcal{H},\mathcal{R}) - \langle\mathcal{H},\mathcal{R}\rangle^2\,. \tag{5.3.19}$$

[9]The author wishes to thank Alessio Marrani for pointing out this version of the derivation.

Thus, the quadratic term reorganizes itself in the desired form without further assumptions. For a given model, $\mathcal{R}$ can then be determined by demanding that the linear term vanishes. Similar invariants constructed from two charge vectors have been considered recently in the context of multi-centre solutions [166, 168, 169].

Therefore, the only object missing for a complete characterisation of the ansatz for extremal solutions in principle, is an explicit form for $\mathcal{H}$ given a vector of physical charges. Unfortunately, solving this constraint is not a straightforward task. The only a priori requirement on $\mathcal{H}$ is that it must be “BPS” in the sense that $I_4(\mathcal{H}) > 0$ and that its quartic invariant should be related to the one of the physical charges by a sign flip. In fact the result should not be unique, as one can expect in view of the ambiguity is seen in (5.3.6), where the two extra unconstrained parameters in λ^A represent the flat directions of the scalar sector. On the other hand, the relation between $\mathcal{H}$ and the physical charges must be the same throughout the flow, as follows from (5.3.11), so that an attractor analysis would be sufficient for this purpose. In any case, one can always dualise the stabilisation equations for the seed solutions above to find any other solution and we comment on a possible way to construct $\mathcal{H}$ at the end of the next section.

5.4 Flow equations from the action

In the last section we argued in favour of a particular ansatz for under-rotating extremal black holes, motivated by symplectic covariance. In order to put it to use, one also needs to cast the equations of motion for $\mathcal{N} = 2$ supergravity in a symplectically covariant form, using the variables introduced in section 2.2.2. The standard action (2.2.10) is not invariant under electromagnetic duality rotations, and our first task is to rewrite it so as to make this invariance manifest, following [24]. Using the symplectic section in (2.2.19), it is straightforward to do this for the scalar kinetic term. However, it is not possible to obtain a duality invariant formalism for gauge fields in terms of Lorentz covariant quantities, so we are forced to use the alternative action of [170, 171].

As we are interested in asymptotically flat, stationary extremal black holes, we use (5.3.2) as an ansatz for the spacetime metric

$$ds^2 = -\mathrm{e}^{2U}(\mathrm{d}t + \omega_i \mathrm{d}x^i)^2 + \mathrm{e}^{-2U} d\vec{x}^2 \,, \tag{5.4.1}$$

with the condition U, $\omega_i \to 0$ as $r \to \infty$. We have restricted the ansatz to describe under-rotating black holes, by choosing a flat base space, so that the angular momentum enters only through the one-form ω.[10] It is convenient to introduce the following product for

[10] An appropriate ansatz for over-rotating solutions should have a more complicated base space [14–16, 135].

symplectic vectors of spatial 2-forms

$$(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}})=\frac{\mathrm{e}^{2U}}{1-w^2}\int_{M_{\rm CY}}\boldsymbol{\mathcal{X}}\wedge\big[\star_{\mathbf{0}}(\diamond\boldsymbol{\mathcal{Y}})-\star_{\mathbf{0}}(w\wedge\diamond\boldsymbol{\mathcal{Y}})\,w+\star_{\mathbf{0}}(w\wedge\star_{\mathbf{0}}\boldsymbol{\mathcal{Y}})\big], \tag{5.4.2}$$

where $\diamond$ is the scalar-dependent inner product defined in (2.2.24) and we defined $w=\mathrm{e}^{2U}\omega$.

With this notation, the two derivative action (2.2.10) can be recast in the form [170]

$$S=-\frac{1}{16\pi}\int dt\int_{\mathbb{R}^3}\big[2\,\mathbf{d}U\wedge\star_{\mathbf{0}}\mathbf{d}U-\tfrac{1}{2}\,\mathrm{e}^{4U}\,\mathbf{d}\omega\wedge\star_{\mathbf{0}}\mathbf{d}\omega$$
$$+2\,g_{a\bar{b}}\,\mathbf{d}z^a\wedge\star_{\mathbf{0}}\mathbf{d}\bar{z}^{\bar{b}}+(\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{F}})\Big]\,, \tag{5.4.3}$$

where we assume all time derivatives to vanish and $\boldsymbol{\mathcal{F}}=\mathbf{d}\mathcal{A}$ is the field strength associated to a symplectic vector of spatial gauge fields $\mathcal{A}$. In this formulation, $\boldsymbol{\mathcal{F}}$ is identified with the spatial components of $\mathcal{F}$ in (2.2.15), whereas the electric components of the field strength F and its dual G are non-dynamical fields, determined by the self-duality constraint (2.2.27). The corresponding charge vector is given by the integral of $\mathcal{F}$ through (2.2.14), as usual.

The scalars z^a, the metric factor e^U and the angular momentum one-form can be also repackaged in a similar form using the two-form

$$\boldsymbol{\mathcal{W}}=2\;\mathrm{Im}\,\star_{\mathbf{0}}\mathbf{D}(\tilde{\mathcal{V}})-2\;\mathrm{Re}\,\mathbf{D}(\tilde{\mathcal{V}}\,w), \tag{5.4.4}$$

where

$$\mathbf{D}=\mathbf{d}+\mathrm{i}(\mathbf{Q}+\mathbf{d}\alpha+\tfrac{1}{2}\mathrm{e}^{2U}\star_{\mathbf{0}}\mathbf{d}\omega)\,,\qquad \mathbf{Q}=\mathrm{Im}\,(\partial_aK\,\mathbf{d}z^a)\,, \tag{5.4.5}$$

and $\alpha(\mathbf{x})$ is an arbitrary function. Similar to (2.2.35), the combination $\tilde{\mathcal{V}}\equiv\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}$ is invariant under rigid scale and $U(1)$ transformations. The action is now simply

$$S=-\frac{1}{16\pi}\int dt\int_{\mathbb{R}^3}\Big[(\boldsymbol{\mathcal{W}},\boldsymbol{\mathcal{W}})+(\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{F}})-2\,(\mathbf{Q}+\mathbf{d}\alpha+\tfrac{1}{2}\mathrm{e}^{2U}\star_{\mathbf{0}}\mathbf{d}\omega)\wedge\mathbf{d}w\Big]\,, \tag{5.4.6}$$

which is manifestly a quadratic form of functionally independent terms.

A standard trick to obtain solutions to the equations of motion arising from a Lagrangian of this type, due to Bogomol'nyi, is to set to zero each of these terms. Direct application to (5.4.6) leads to $\boldsymbol{\mathcal{W}}=\boldsymbol{\mathcal{F}}=\mathbf{Q}=\omega=0$, which can be shown to describe a flat spacetime with constant scalars. In order to obtain more interesting solutions, one needs to find an equivalent rewriting of the Lagrangian by mixing the scalars with the gauge fields.

This was done in [24] to obtain the symplectically covariant form of the BPS solutions, making use of the combination $\boldsymbol{\mathcal{G}}=\boldsymbol{\mathcal{F}}-\boldsymbol{\mathcal{W}}$, so that the action takes the form

$$S=-\frac{1}{16\pi}\int dt\int_{\mathbb{R}^3}\Big[(\boldsymbol{\mathcal{G}},\boldsymbol{\mathcal{G}})-4\,(\mathbf{Q}+\mathbf{d}\alpha+\tfrac{1}{2}\mathrm{e}^{2U}\star_{\mathbf{0}}\mathbf{d}\omega)\wedge\mathrm{Im}\langle\boldsymbol{\mathcal{G}},\mathrm{e}^U\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}\rangle\Big]\,, \tag{5.4.7}$$

In this form, the Bogomol'nyi trick leads to

$$\boldsymbol{\mathcal{G}} = 0\,, \tag{5.4.8}$$

$$\mathbf{Q} + \mathbf{d}\alpha + \tfrac{1}{2}\mathrm{e}^{2U}\star_{\mathbf{0}}\mathbf{d}\omega = 0\,, \tag{5.4.9}$$

which are easily shown to be identical to (2.2.34)-(2.2.35). Note that the supersymmetry variations were not used in this approach, and one still has to check that the above relations solve the Killing spinor equations to establish supersymmetry.

Inspired by the relation between (5.2.1) and (5.2.3), it is possible to use different ways to rewrite the action to obtain non-supersymmetric equations as well, employing a specific rigid transformation (rotation) of the charge vector with a corresponding modification of the field strengths [143, 172]. A further generalisation was presented in [150], where no assumptions about this transformation or about the resulting modified (or 'fake') field strength were made. Consider a new two-form valued symplectic vector $\tilde{\boldsymbol{\mathcal{F}}}$, constrained to satisfy

$$\big(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{F}}\big) = \big(\tilde{\boldsymbol{\mathcal{F}}}, \tilde{\boldsymbol{\mathcal{F}}}\big) - \boldsymbol{\Xi}\,, \tag{5.4.10}$$

where we allowed for a possible extra term described by the three-form $\boldsymbol{\Xi}$, which will be determined in due course by consistency arguments. Note that, unlike the physical field strength, we do not require $\tilde{\boldsymbol{\mathcal{F}}}$ to be closed.

As in [24] the crucial step in deriving the manifestly duality-invariant equations of motion is to appropriately pair the derivatives of the scalars with the gauge fields and use the invariant product (5.4.2) to re-express the Lagrangian. As before, the combination

$$\tilde{\boldsymbol{\mathcal{G}}} = \tilde{\boldsymbol{\mathcal{F}}} - 2\,\mathrm{Im}\,\star_{\mathbf{0}}\mathbf{D}(\tilde{\mathcal{V}}) + 2\,\mathrm{Re}\,\mathbf{D}(\tilde{\mathcal{V}}\,w)\,, \tag{5.4.11}$$

leads to an action of the form

$$S = -\frac{1}{16\pi}\int d^4x \Big[(\tilde{\boldsymbol{\mathcal{G}}}, \tilde{\boldsymbol{\mathcal{G}}}) - 4\,(\mathbf{Q} + \mathbf{d}\alpha + \boldsymbol{\eta} + \tfrac{1}{2}\mathrm{e}^{2U}\star_{\mathbf{0}}\mathbf{d}\omega) \wedge \mathrm{Im}\langle\tilde{\boldsymbol{\mathcal{G}}}, \mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}\rangle\Big]\,, \tag{5.4.12}$$

where we introduced an (otherwise arbitrary) one-form $\boldsymbol{\eta}$, which we require to satisfy

$$\boldsymbol{\eta} \wedge \mathrm{Im}\langle\tilde{\boldsymbol{\mathcal{G}}}, \mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}\rangle = \langle\mathbf{d}\tilde{\boldsymbol{\mathcal{F}}}, \mathrm{Re}(\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V})\rangle\, + \tfrac{1}{4}\boldsymbol{\Xi}\,, \tag{5.4.13}$$

up to a total derivative.

In the form (5.4.12), it is again possible to find a stationary point of the action by requiring that the variations of the two terms vanish separately

$$\tilde{\boldsymbol{\mathcal{G}}} = 0\,, \tag{5.4.14}$$

$$\mathbf{Q} + \mathbf{d}\alpha + \boldsymbol{\eta} + \tfrac{1}{2}\mathrm{e}^{2U}\star_{\mathbf{0}}\mathbf{d}\omega = 0\,. \tag{5.4.15}$$

The second relation implies that $\mathbf{D} = \mathbf{d} - \mathrm{i}\boldsymbol{\eta}$, so that the first becomes

$$\tilde{\boldsymbol{\mathcal{F}}} - 2\,\mathrm{Im}\big[\star_{\mathbf{0}}(\mathbf{d} - \mathrm{i}\boldsymbol{\eta})(\tilde{\mathcal{V}})\big] + 2\,\mathrm{Re}\big[(\mathbf{d} - \mathrm{i}\boldsymbol{\eta})(\tilde{\mathcal{V}}\,w)\big] = 0\,. \tag{5.4.16}$$

Differentiating one finds:

$$\mathbf{d}\star_{\mathbf{0}}\mathbf{d}\,\mathrm{Im}\,\tilde{\mathcal{V}} - \mathbf{d}\left(\star_{\mathbf{0}}\boldsymbol{\eta}\,\mathrm{Re}\,\tilde{\mathcal{V}}\right) - \mathbf{d}\left(\boldsymbol{\eta}\wedge w\,\mathrm{Im}\,\tilde{\mathcal{V}}\right) = \tfrac{1}{2}\mathbf{d}\tilde{\boldsymbol{\mathcal{F}}}\,, \tag{5.4.17}$$

where we note that the modified field strength is not necessarily closed, as mentioned earlier.

It is now possible to derive $\boldsymbol{\Xi}$ in terms of the other quantities, using the observation that in the dynamical system we are considering, the electromagnetic part of the Lagrangian acts as a potential for the remaining fields. Therefore, when evaluated for solutions of the equations of motion with vanishing action, this potential is expected to equal the kinetic energy. Using (5.4.16) and (5.4.15), this requirement is imposed by comparing

$$\begin{aligned}\big(\tilde{\boldsymbol{\mathcal{F}}},\tilde{\boldsymbol{\mathcal{F}}}\big) &= 2\mathbf{d}U\wedge\star_{\mathbf{0}}\mathbf{d}U - \tfrac{1}{2}\mathrm{e}^{4U}\mathbf{d}\omega\wedge\star_{\mathbf{0}}\mathbf{d}\omega + 2g_{a\bar{b}}\mathbf{d}z^{a}\wedge\star_{\mathbf{0}}\mathbf{d}z^{\bar{b}}\\ &\quad + \mathrm{e}^{2U}\mathbf{d}w\wedge\star_{\mathbf{0}}\mathbf{d}\omega + 2\mathbf{d}\mathbf{Q}\wedge w\\ &= \big(\boldsymbol{\mathcal{F}},\boldsymbol{\mathcal{F}}\big) + \mathrm{e}^{2U}\mathbf{d}w\wedge\star_{\mathbf{0}}\mathbf{d}\omega + 2\mathbf{d}\mathbf{Q}\wedge w\,,\end{aligned} \tag{5.4.18}$$

with the original assumption (5.4.10) on $\boldsymbol{\mathcal{F}}$. Up to boundary terms, one obtains:

$$\boldsymbol{\Xi} = -2\boldsymbol{\eta}\wedge\mathbf{d}w\,, \tag{5.4.19}$$

which we assume to hold henceforth.

In summary, we have obtained first-order equations expressed in terms of a non-closed two-form valued symplectic vector $\tilde{\boldsymbol{\mathcal{F}}}$, implicitly related to the field strength $\boldsymbol{\mathcal{F}}$ through (5.4.10). This formulation comes at the price of introducing an auxiliary object, $\boldsymbol{\eta}$, related to the non-closure of $\boldsymbol{\mathcal{F}}$ through (5.4.13). It follows that in any given model for which nontrivial $\tilde{\boldsymbol{\mathcal{F}}}$, $\boldsymbol{\eta}$ can be found such that (5.4.10) and (5.4.13) are satisfied with $\boldsymbol{\Xi}$ as in (5.4.19), there exist solutions described by the first order flow in (5.4.16).

5.4.1 Solving for the flow

The simplest solutions to the flow equations above can be found in the special case that $\tilde{\boldsymbol{\mathcal{F}}}$ is closed. Then, $\boldsymbol{\Xi}$ and $\boldsymbol{\eta}$ must vanish by eq. (5.4.13) and equation (5.4.17) reduces to the Laplace equation, which can be solved through a vector of (possibly multi-centred) harmonic functions as

$$2\,\mathbf{d}\star_{\mathbf{0}}\mathbf{d}\,\mathrm{Im}\,\tilde{\mathcal{V}} = 0\,, \quad\Rightarrow\quad 2\,\mathrm{Im}\,\tilde{\mathcal{V}} = \mathcal{H}\,. \tag{5.4.20}$$

If one further assumes that $\tilde{\boldsymbol{\mathcal{F}}} = \boldsymbol{\mathcal{F}}$, the supersymmetric solutions of [24] are recovered, leading to (2.2.34)-(2.2.35). Dropping this identification, one finds a rather restricted[11] set of non-BPS solutions with charges as in (5.2.3) and vanishing axions [137, 172, 174]. As discussed above, $\tilde{\boldsymbol{\mathcal{F}}}$ is related to $\boldsymbol{\mathcal{F}}$ by sign reversals for these special solutions, and the same holds for the poles of the harmonic functions $\mathcal{H}$ in (5.4.20) compared to the physical charges.

In its general form, however, equation (5.4.16) cannot be solved directly, since the period vector $\mathcal{V}$, $\boldsymbol{\eta}$, ω and $\tilde{\boldsymbol{\mathcal{F}}}$ are all unknown and constrained by (5.4.10) and (5.4.13). One is then forced to introduce an appropriate ansatz either for $\tilde{\boldsymbol{\mathcal{F}}}$ or $\tilde{\mathcal{V}}$. In view of the results in the previous section, we adopt the ansatz in (5.3.11)-(5.3.12) for the scalars.

Compared to the first order equations (5.4.16), the non-closure of $\tilde{\boldsymbol{\mathcal{F}}}$ is linked to the presence of the non-harmonic part in the ansatz for the scalars. Differentiating both sides of (5.3.11) and taking the intersection product with the imaginary part of $\mathcal{V}$, one obtains the following expression for the Kähler connection

$$\mathbf{d}\alpha + \mathbf{Q} = \mathrm{e}^{U}\,\mathrm{Im}(\mathrm{e}^{-\mathrm{i}\alpha}\Psi) = -\tfrac{1}{2}\mathrm{e}^{2U}\langle \mathbf{d}\mathcal{J}, \mathcal{J}\rangle\,. \tag{5.4.21}$$

When combined with (5.4.15), it implies that $\boldsymbol{\eta}$ is given by

$$\boldsymbol{\eta} = \tfrac{1}{2}\mathrm{e}^{2U}\Big(\langle \mathbf{d}\mathcal{J}, \mathcal{J}\rangle - \star_{\mathbf{0}}\mathbf{d}\omega\Big) = \mathrm{e}^{2U}\langle \mathbf{d}\mathcal{R}, \mathcal{H}\rangle\,. \tag{5.4.22}$$

where in the second step we used the integrability conditions $\langle \mathbf{d}\mathcal{H}, \mathcal{H}\rangle = \langle \mathbf{d}\mathcal{R}, \mathcal{R}\rangle = 0$ implied by (5.3.12).

Using this information, we can show that the ansatz (5.3.11) automatically solves the constraint (5.4.13), which can be expressed purely in terms of $\mathcal{J}$ using (5.4.16). The left-hand side is clearly zero while on the right-hand side we have an intersection product that we know how to compute. Neglecting the total derivative, one finds

$$\mathrm{e}^{2U}\langle \mathbf{d}\star_{\mathbf{0}}\mathbf{d}\mathcal{J}, \mathrm{Re}\,\tilde{\mathcal{V}}\rangle = \boldsymbol{\eta}\wedge\left(\star_{\mathbf{0}}\boldsymbol{\eta} + \mathrm{e}^{2U}\,\mathbf{d}\omega\right)\,. \tag{5.4.23}$$

Using (5.3.13) and (5.3.15), one can show that

$$\langle \mathbf{d}\star_{\mathbf{0}}\mathbf{d}\mathcal{J}, \mathrm{Re}\,\tilde{\mathcal{V}}\rangle = -\tfrac{1}{2}\,\mathrm{e}^{2U}\langle\mathcal{R}, \mathcal{H}\rangle\,\langle \mathbf{d}\star_{\mathbf{0}}\mathbf{d}\mathcal{R}, \mathcal{H}\rangle = \mathrm{e}^{2U}\langle\mathcal{R}, \mathcal{H}\rangle\,\langle \star_{\mathbf{0}}\mathbf{d}\mathcal{R}, \mathbf{d}\mathcal{H}\rangle\,, \tag{5.4.24}$$

where the last step follows from the fact that $\langle\mathcal{R}, \mathcal{H}\rangle$ is a harmonic function. Finally, since $\mathcal{R}$ depends only on a single ratio of the form $\langle\mathcal{H}, \mathcal{R}\rangle/\bar{H}$ (with $\bar{H}$ a harmonic function, cf. (5.3.6)), it is possible to show that

$$\langle\mathcal{H}, \mathcal{R}\rangle\,\mathbf{d}\mathcal{R} = -\langle \mathbf{d}\mathcal{R}, \mathcal{H}\rangle\,\mathcal{R}\,. \tag{5.4.25}$$

[11] By charge redefinitions one can generate physically equivalent solutions also for other charge configurations [173].

Combining the last relation with (5.4.24) and (5.4.22) shows that the constraint (5.4.23) is identically satisfied. Therefore we have conclude that the choice of ansatz in (5.3.11)-(5.3.12), together with the additional requirements of section 5.3.2, is a solution of the constraint (5.4.13). The seed solution (5.3.3) also satisfies these relations, as expected.

This is a rather nontrivial result, as (5.4.23) is a quartic equation for $\mathcal{J}$ in view of (5.4.22). Assuming this to be the general solution, the only constraint remaining at this stage is (5.4.10), which generalises the definition of the fake superpotential for static black holes [158] to the case of under-rotating black holes. However, it is difficult to verify (5.4.10) and (5.4.16) explicitly for the seed solution (5.3.3), or find the general solution. In the next section, we give a more detailed comparison to the ansatz (5.3.11)–(5.3.12) in the static limit.

5.4.2 The static flow equations

The static limit of the results in section 5.4 leads to several simplifications, as all solutions are necessarily spherically symmetric. This implies that $\omega = 0$ and all quantities depend only on the radial coordinate. Similar to the actual field strength, spherical symmetry implies that the modified field strength $\tilde{\boldsymbol{\mathcal{F}}}$ is of the form:

$$\tilde{\boldsymbol{\mathcal{F}}} = \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\varphi \otimes \tilde{\Gamma}\,, \tag{5.4.26}$$

where now $\tilde{\Gamma}$ is an symplectic vector that dependents on r. By (5.4.10), it must reproduce the same black hole potential V_{bh} as the physical charge Γ

$$\tfrac{1}{2}\|\tilde{\Gamma}\|^2 = V_{\mathrm{bh}} = \tfrac{1}{2}\|\Gamma\|^2\,, \tag{5.4.27}$$

where we used (2.2.32). This relation was introduced in [143] in the context of finding a moduli dependent function $W(\Gamma)$, called the fake superpotential, such that

$$V_{\mathrm{bh}} = |Z(\Gamma)|^2 + g^{A\bar{B}}\,\mathcal{D}_A Z(\Gamma)\,\bar{\mathcal{D}}_{\bar{B}}\bar{Z}(\Gamma) = |W(\Gamma)|^2 + g^{A\bar{B}}\,\mathcal{D}_A W(\Gamma)\,\bar{\mathcal{D}}_{\bar{B}}\bar{W}(\Gamma)\,. \tag{5.4.28}$$

In what follows we indicate how to connect the flow equations above to this formalism.

In the static limit, it is convenient to choose the arbitrary function $\mathrm{e}^{\mathrm{i}\alpha}$ to be the phase of $\langle\tilde{\Gamma}, \mathcal{V}\rangle$, and eqs. (5.4.16)-(5.4.15) reduce to

$$2\,\partial_\tau\big(\mathrm{Im}\,\tilde{\mathcal{V}}\big) - 2\,\eta\,\mathrm{Re}\,\tilde{\mathcal{V}} = -\tilde{\Gamma}\,, \tag{5.4.29}$$

$$\eta = -\dot{\alpha} - Q_\tau = -\,\mathrm{Im}\,\frac{\langle\dot{\tilde{\Gamma}}, \mathcal{V}\rangle}{Z(\tilde{\Gamma})}\,, \tag{5.4.30}$$

where $Q_\tau = \mathrm{Im}(\partial_a K\,\dot{z}^a)$ and the dot stands for the derivative with respect to the inverse radial coordinate $\tau = 1/r$. Note that (5.4.30) is equivalent to (5.4.13), which has not been used. As before, the ansatz in (5.3.11) identically solves this constraint.

Imposing this solution for (5.4.30), the equations of motion are solved if one can construct a $\mathcal{J}$, along the lines of section 5.3.2, such that the other constraint is satisfied using (5.4.29). In this case, (5.4.27) and the definition of α are equivalent to

$$\tfrac{1}{2}\|\partial_\tau \mathcal{J}\|^2 = \tfrac{1}{2}\|\Gamma\|^2 + \mathrm{e}^{2U_{\mathcal{J}}}\langle \mathcal{H}, \partial_\tau \mathcal{R}\rangle^2 \,. \tag{5.4.31}$$

Once (5.3.11) is used for the scalars, this is a quadratic equation for the physical charges in terms of the harmonic functions $\mathcal{H}$. Therefore, finding a $\partial_\tau \mathcal{J}$ that satisfies (5.4.31) is equivalent to integrating the static equations of motion. Note that the presence of a nontrivial η is essential in the treatment of section 5.4 and is easiest to understand if one allows for angular momentum to be a priori present, as we have done here.

We have checked that $\mathcal{J}$ for the known explicit static solutions of section 5.3 is such that they satisfy (5.4.31) and hence are described by the flow equation (5.4.29) with η as in (5.4.22). As all static non-BPS solutions are related by symplectic rotations to these seed solutions, it follows that they satisfy the same duality covariant equations. The nontrivial η is reflected in the anharmonic part of (5.3.3), controlled by the constant b that remains after setting the angular momentum to zero in (5.2.15). This observation is in line with [130] where it was stressed that the crucial departure of the static non-BPS seed solution from a BPS-like ansatz is the presence of a parameter related to the asymptotic scalars, identified with this residual constant.

The approach above is similar in spirit, but different than the one of [146–148], were one seeks to rewrite the black hole potential in (5.4.27) through the fake superpotential in (5.4.28), which is a function of the physical charges and moduli z^a directly. In contrast, (5.4.31) is an equation relating the harmonic functions controlling the physical charges to the ones controlling the scalars through the period vector Ω.

In order to compare, we make use of the fact that the fake superpotential is defined through the flow equations for the metric function and the moduli. The corresponding equations can be found by taking the inner product of (5.4.29) with the basis elements, leading to

$$\dot{U} = -\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha}\langle\tilde{\Gamma}, \mathcal{V}\rangle \equiv -\mathrm{e}^{U} W \,, \tag{5.4.32}$$

$$\dot{z}^a = -\mathrm{e}^{U}\mathrm{e}^{-\mathrm{i}\alpha} g^{a\bar{b}}\langle\tilde{\Gamma}, \bar{\mathcal{D}}_{\bar{b}}\bar{\mathcal{V}}\rangle \equiv -2\mathrm{e}^{U} g^{a\bar{b}}\bar{\partial}_{\bar{b}} W \,, \tag{5.4.33}$$

where in the second step we identified the fake superpotential W in (5.4.28). When $W = Z(\Gamma)$, the flow is supersymmetric, in the same way as the BPS solutions arise in the framework of the last two sections when $\tilde{\boldsymbol{\mathcal{F}}} = \boldsymbol{\mathcal{F}}$.

Whenever W is explicitly known for a given model and charge configuration, a practical way to connect it with our approach may be to first look for a moduli-independent matrix S that rotates the physical charge vector Γ so that:

$$Z(\tilde{\Gamma}) := \langle S\Gamma, \mathcal{V}\rangle\,, \tag{5.4.34}$$

where S is a complex nonconstant symplectic matrix. The relation of Γ with $\tilde{\Gamma}$ is expressed indirectly through

$$\tilde{\Gamma} := \mathrm{i}\overline{Z(\tilde{\Gamma})}\,\mathcal{V} - \mathrm{i}g^{\bar{a}b}\overline{\mathcal{D}_a Z(\tilde{\Gamma})}\,\mathcal{D}_b\mathcal{V} + \mathrm{i}g^{a\bar{b}}\mathcal{D}_a Z(\tilde{\Gamma})\,\bar{\mathcal{D}}_{\bar{b}}\bar{\mathcal{V}} - \mathrm{i}Z(\tilde{\Gamma})\,\bar{\mathcal{V}}\,. \tag{5.4.35}$$

In the standard fake superpotential approach one assumes S to be a constant real symplectic matrix and identifies $\tilde{\Gamma} = S\Gamma$. However, in view of the discussion in section 5.4 this is not the generic situation and we find it simpler to keep S complex.

One can find the matrix S explicitly for the electric configuration, as in (5.3.3), assuming all physical scalars to have the same phase, say $\mathrm{e}^{\mathrm{i}f}$. The relevant superpotential was given in [59, 131] and reads

$$W = p^0\,F_0\,\mathrm{e}^{\mathrm{i}f} - q_A\,X^A = \langle S\Gamma, \mathcal{V}\rangle\,, \tag{5.4.36}$$

with S given by

$$S = \mathrm{diag}\big(\mathrm{e}^{-2\mathrm{i}f}, 1, 1, 1, \mathrm{e}^{2\mathrm{i}f}, 1, 1, 1\big)\,. \tag{5.4.37}$$

In terms of the parameters appearing in the solution of section 5.3 one can identify $\cot f = \mathrm{e}^{2U}M$ and check that the equations of motion (5.4.29) are satisfied. The one-form $\boldsymbol{\eta}$ is given by (5.4.22). If the scalars are assumed purely imaginary, M vanishes and $f = \pi/2$ so that S is constant (but not identity), while allowing for a τ-dependent f leads to more general non-supersymmetric solutions. It is worth noting that $\eta = 0$ whenever S is constant (cf. eq. (5.4.30)) and that if $S = \mathbb{I}$ we recover the supersymmetric solution.

Alternatively, one can rewrite (5.4.29) and (5.4.31) in terms of a real matrix T such that:

$$T\Gamma := \Gamma_T = \tilde{\Gamma} - 2\eta\,\mathrm{Re}\big(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}\big)\,, \tag{5.4.38}$$

$$\tfrac{1}{2}\langle\Gamma_T, \diamond\Gamma_T\rangle = V_{\mathrm{bh}} + \mathrm{e}^{-2U}\eta^2\,, \tag{5.4.39}$$

$$\mathrm{e}^{-\mathrm{i}\alpha}\langle\Gamma_T, \mathcal{V}\rangle = W - \mathrm{i}\mathrm{e}^{-U}\eta\,. \tag{5.4.40}$$

If T is known, it leads to simpler equations of motion for the scalars, that is

$$2\partial_\tau\,\mathrm{Im}\big(\mathrm{e}^{-U}\mathrm{e}^{-\mathrm{i}\alpha}\mathcal{V}\big) = -\Gamma_T\,, \tag{5.4.41}$$

which have the advantage of being directly integrable. For the electric example above, T takes the form:

$$T = \begin{pmatrix} \mathbb{I} & 0 \\ \mathcal{W} & \mathbb{I} \end{pmatrix}, \qquad \mathcal{W} = \begin{pmatrix} -\frac{\mathrm{e}^{-2U}\cot f}{(H^0)^2} & \frac{2q_a}{p^0} \\ \frac{2q_a}{p^0} & 0 \end{pmatrix}, \tag{5.4.42}$$

so that (5.3.4) can be written as $\mathcal{J} = -T\mathcal{H}_c$, if the constants in (5.3.1) are appropriately chosen. Similar to its complex counterpart S, it reduces to a constant matrix if the phase of the scalars is $f = \pi/2$.

It is interesting to point out that the matrix (5.4.42) is a (position dependent) element of the Peccei–Quinn group of transformations, defined as the largest subgroup of the symplectic group leaving the X^I's and the Kähler potential invariant. As was shown recently [175], applying such a transformation on the charges indeed shifts the black hole potential as in (5.4.39). For generic charges and phases of the scalars, the corresponding T can be found from the one in (5.4.42) by conjugation with the appropriate element of the symplectic group. Such a matrix would leave a certain combination of X^I's and F_I's unchanged, e.g. for the magnetic dual of the electric solution in (5.3.1) it would leave the F_I's invariant. Identifying the combinations that must be invariant for a given set of charges could be a way to determine T from first principles.

5.5 Concluding remarks

In this chapter, we presented various aspects of recent work on non-BPS black hole solutions of $\mathcal{N} = 2$ supergravity in both four and five dimensions, emphasising techniques based on similarities to the supersymmetric solutions in order to reduce the non-linear second order equations of motion to simpler ones.

Motivated by the observation that one can obtain simple non-BPS solutions by simply changing the sign of harmonic functions describing supersymmetric solutions, we were led to introduce the almost-BPS class of five-dimensional solutions. These are constructed so that they preserve four supercharges on local patches of spacetime but nevertheless break supersymmetry globally. Recently, this class has been explored in detail and further generalisations have appeared in the literature [151–154, 162–165]. Some of the results include a novel black ring solution and non-extremal configurations without a horizon, which are interesting in the context of the fuzzball proposal.

In order to gain a better understanding of the space of solutions, it is helpful to consider the reduction to four dimensions, where one can make use of the extended electric-magnetic duality. A particularly interesting almost-BPS solution is the rotating electric black hole of [151], discussed in section 5.2.3. Upon Kaluza-Klein reduction on a circle, this can be used as a seed solution to obtain all under-rotating black hole solutions in four-dimensional $\mathcal{N} = 2$ supergravity. Based on the form of the seed solution we proposed an ansatz that encompasses all known single centred extremal solutions. Subsequently, we presented a new derivation of first order equations for extremal black hole solutions that are compatible with the explicit seed solution and the generic ansatz, extending [24, 172]. As the proposed ansatz is highly restricted, the flow equations are essentially reduced to algebraic equations, which nevertheless are still not directly solvable. Hence the situation remains unsatisfactory.

It is important to note that our flow equations are by construction fully covariant with respect to electric-magnetic duality and are compatible with the general single centered seed solutions in four dimensions. It then follows that they capture the full orbit of non-BPS extremal solutions, regardless of the existence of other stationary points of the action, which should not be part of the standard non-BPS orbit of extremal black holes. We expect a similar pattern to exist beyond the single centre case, especially in view of the results of [151, 152, 154]. As we have not made any assumptions on the number of centres in deriving the flow equations in section 5.4, it would be interesting to investigate the possibility of constructing multi-centre solutions by allowing the harmonic functions in the ansatz of section 5.3.2 to have multiple centres. Such constructions would provide a test on the robustness of the assumption on the existence of stabilisation equations for generic extremal backgrounds.

On the microscopic side, it would be very interesting to reproduce the stabilisation equations (5.3.11). In the rotating case the ratio of harmonic functions survives the near-horizon limit and modifies the attractor equations [135], so one generally expects this structure to be accessible from microscopics. Given the model of [176], where the constant part of M in (5.3.4) is interpreted as the angle between wrapped D3 branes, one expects that the full angular momentum harmonic function might have a similar microscopic analogue. Similar considerations for over-rotating black holes have appeared recently in [177–179].

There is considerable ongoing research on the structure of non-BPS solutions, utilising both the powerful constraints following from the symplectic structure of the four-dimensional theory and the relative simplicity of the five-dimensional theory. Finding the appropriate set of variables to express all extremal black hole solutions is a very interesting problem, both from a mathematical point of view and because it can shed light on several related open questions about non-supersymmetric backgrounds in string theory.

Appendix A

Conventions

In this short appendix we specify some of the details of our conventions used in four and five dimensions. Throughout this thesis, especially when dealing with spinors, we use Pauli-Källén conventions. Unless otherwise indicated, space-time and Lorentz indices are denoted by $\mu, \nu, \ldots$, and $a, b, \ldots$, respectively; SU(2)-indices are denoted by $i, j, \ldots$. All (anti-)symmetrisations are defined with unit strength.

Our conventions for four-dimensional supergravity follow the notation used *e.g.* in [54]. In five dimensions we use closely related conventions as follows. We employ hermitean 4-by-4 gamma matrices γ_a, which satisfy

$$\begin{aligned} C\gamma_a C^{-1} &= {\gamma_a}^{\mathrm{T}}\,, \qquad C^{\mathrm{T}} = -C\,, \qquad C^{\dagger} = C^{-1}\,, \\ \gamma_{abcde} &= \mathbf{1}\,\varepsilon_{abcde}\,. \end{aligned} \tag{A.0.1}$$

Here C denotes the charge-conjugation matrix and gamma matrices with k multiple indices denote the fully antisymmetrised product of k gamma matrices in the usual fashion, so that we have, for instance, $\gamma_a\,\gamma_b = \mathbf{1}\,\delta_{ab} + \gamma_{ab}$. In view of the last equation of (A.0.1), gamma matrices with more than two multiple indices are not independent, and can be decomposed into the unit matrix, γ_a and γ_{ab}. Note that C, $C\gamma_a$ and $C\gamma_{ab}$ constitute a complete basis of 6 antisymmetric and 10 symmetric (unitary) matrices in spinor space. The gamma matrices commute with the automorphism group of the Clifford algebra, USp($2N$), where N denotes the number of independent spinors. Spinors can be described either as Dirac spinors, or as symplectic Majorana spinors. The latter description has the advantage that it makes the action of the USp($2N$) R-symmetry group manifest. We thus employ symplectic Majorana spinors ψ^i with $i = 1, 2, \ldots, 2N$, subject to the reality constraint,

$$C^{-1}\,{\bar{\chi}_i}^{\mathrm{T}} = \Omega_{ij}\,\chi^j\,, \tag{A.0.2}$$

where Ω is the symplectic USp($2N$) invariant tensor. The Dirac conjugate is defined by $\bar{\psi} = \psi^\dagger\gamma_5$. Observe that we adhere to the convention according to which raising or lowering of USp($2N$) indices is effected by complex conjugation.

The gravitini $\psi_\mu{}^i$ and associated supersymmetry parameters ϵ^i transform in the **2N** representation of USp($2N$). In principle we may also consider spinors transforming under more complicated representations of USp($2N$). For example, we note the following result for fermionic bilinears, with spinor fields ψ^i and φ^i and a spinor matrix Γ constructed from products of gamma matrices,

$$\bar{\psi}_i\Gamma\varphi^j = -\Omega_{ik}\,\Omega^{jl}\,\bar{\varphi}_l\,C^{-1}\,\Gamma^{\mathrm{T}}\,C\,\psi^k = (\bar{\varphi}_j\,\gamma_5\,\Gamma^\dagger\,\gamma_5\,\psi^i)^\dagger\,. \tag{A.0.3}$$

Hence $\mathrm{i}\,\bar{\psi}_i\,\varphi^j$, $\bar{\psi}_i\gamma_a\varphi^j$ and $\mathrm{i}\,\bar{\psi}_i\gamma_{ab}\varphi^j$ are pseudo-hermitean (provided $a, b, \ldots = 1, \ldots, 4$; in Pauli-Källén convention the time component associated with $a = 5$ acquires an extra minus sign) . Generalization of this result to spinors transforming according to more complicated USp($2N$) representations is straightforward.

Multiplication of symplectic Majorana spinors with spinor matrices Γ consisting of products of gamma matrices are not automatically symplectic Majorana spinors. This follows from

$$\overline{\Gamma\chi^i}^{\,\mathrm{T}} = \Omega_{ij}\,C\,\gamma_5(C^{-1}\Gamma^{\mathrm{T}}C)^\dagger\gamma_5\,\chi^j\,. \tag{A.0.4}$$

This means that $\mathrm{i}\gamma_a\chi^i$, $\gamma_{ab}\chi^i$, $\mathrm{i}\gamma_{abc}\chi^i$, $\gamma_{abcd}\chi^i$ are also symplectic Majorana spinors with the same reality phase as (A.0.2).

Appendix B

Conformal Supergravity

A convenient method for dealing with off-shell formulations of supergravity theories is provided by the superconformal multiplet calculus. This calculus was originally set up for $N=2$ supergravity in $d=4$ dimensions [45–48], following early work for $N=1, d=4$ supergravity [180, 181]. The $N=1$ case was worked out more fully in [182], and shortly thereafter the formalism was also applied to $N=1, d=6$ supergravity in [183]. For $d=5$ dimensions superconformal methods were developed relatively recently by several groups [29–32].

The field content of the various multiplets in four-dimensional $N=2$ and five-dimensional $N=1$ supergravity is rather similar, in view of the facts that spinors carry four components in both cases and that the R-symmetry groups are similar, equal to $\mathrm{SU}(2) \times \mathrm{U}(1)$ and $\mathrm{USp}(2)$, respectively. The only exception to this rule are the Weyl multiplets, which have a different number of degrees of freedom, as is shown in table B.1. The reason can be understood from the fact that the Weyl multiplet is conjugate to the smallest massive supersymmetry representation containing spin-2 and spin-3/2 as the highest spin states. For comparison we also display the situation for the $N=4$ Weyl multiplet in four dimensions, and the $N=2$ Weyl multiplet in five dimensions, with corresponding R-symmetry groups $\mathrm{U}(4)$ and $\mathrm{USp}(4)$, respectively. These two multiplets comprise the same number of degrees of freedom.

In this section we give a self-contained summary of the transformation rules of superconformal multiplets in four and five space-time dimensions, namely the Weyl multiplet, the chiral multiplet, the vector multiplet, the linear multiplet and the hypermultiplet for supergravity with eight supercharges. With the exception of the hypermultiplet, these multiplets define off-shell representations of the algebra of superconformal transformations. We refer to appendix A for spinor and space-time conventions.

	8 supercharges		16 supercharges	
field	d=4	d=5	d=4	d=5
$e_\mu{}^a$	5	9	5	9
$V_{\mu i}{}^j$	9	12	45	40
A_μ	3	-	-	4
$T_{ab}{}^{[ij]}$	6	10	36	50
$D_{[kl]}{}^{[ij]}$	1	1	20	14
$E_{(ij)}$	-	-	20	10
ϕ	-	-	2	1
$\psi_\mu{}^i$	16	24	32	48
$\chi^i{}_{[kl]}$	8	8	80	64
Λ_i	-	-	16	16
bosons+fermions	24+24	32+32	128+128	128+128

Table B.1: Bosonic and fermionic degrees of freedom of the Weyl multiplets in four and five dimensions for the case of four and sixteen supercharges. All degrees of freedom can be assigned to a product representation of the group of spatial rotations and the R-symmetry group. Decomposing the states in the second column under the group of 3-dimensional rotations yields a reducible multiplet comprising the states of the four-dimensional Weyl multiplet (given in the first column) and of an extra vector (or tensor) multiplet.

B.1 Superconformal multiplets in five dimensions

The bosonic gauge transformations are those of the conformal group, diffeomorphisms, local Lorentz transformations with generators M_{ab}, scale transformations with generator D and special conformal transformations (also called conformal boosts) with generators K_a. Furthermore there are local R-symmetry transformations. In five space-time dimensions, the R-symmetry group equals USp($2N$) so that for simple supergravity we have USp(2) $\cong$ SU(2). The fermionic gauge transformations are the conventional Q- and the special conformal S-supersymmetry transformations.

B.1.1 The Weyl multiplet

The Weyl multiplet of five-dimensional simple conformal supergravity is shown in table B.2. The independent fields consist of the fünfbein $e_\mu{}^a$, the gravitino field $\psi_\mu{}^i$, the dilatational gauge field b_μ, the R-symmetry gauge fields $V_{\mu i}{}^j$ (which is an anti-hermitean, traceless matrix in the SU(2) indices i, j) and a tensor field T_{ab}, a scalar field D and a spinor field χ^i. The three gauge fields $\omega_\mu{}^{ab}$, $f_\mu{}^a$ and ϕ_μ, associated with local Lorentz transformations, conformal boosts and S-supersymmetry, respectively, are not independent and will be discussed later. The infinitesimal Q, S and K transformations of the independent fields, parametrised by spinors ϵ^i and η^i and a vector $\Lambda_{\mathrm{K}}{}^a$, respectively, are

	Weyl multiplet										parameters	
field	$e_\mu{}^a$	$\psi_\mu{}^i$	b_μ	$\mathcal{V}_\mu{}^i{}_j$	T_{ab}	χ^i	D	$\omega_\mu{}^{ab}$	$f_\mu{}^a$	$\phi_\mu{}^i$	ϵ^i	η^i
w	-1	$-\frac{1}{2}$	0	0	1	$\frac{3}{2}$	2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Table B.2: Weyl weights w of the Weyl multiplet component fields and the supersymmetry transformation parameters.

as follows,

$$\begin{aligned}
\delta e_\mu{}^a &= \tfrac{1}{2}\bar\epsilon_i\gamma^a\psi_\mu{}^i\,, \\
\delta\psi_\mu^i &= \mathcal{D}_\mu\epsilon^i + \tfrac{1}{4}\mathrm{i}\,T_{ab}(3\,\gamma^{ab}\gamma_\mu - \gamma_\mu\gamma^{ab})\epsilon^i - \mathrm{i}\gamma_\mu\eta^i\,, \\
\delta V_{\mu i}{}^j &= 3\mathrm{i}\bar\epsilon_i\phi_\mu{}^j - 8\bar\epsilon_i\gamma_\mu\chi^j - 3\mathrm{i}\bar\eta_i\psi_\mu{}^j + \delta^i{}_j\,[-\tfrac{3}{2}\mathrm{i}\bar\epsilon_k\phi_\mu{}^k + 4\bar\epsilon_k\gamma_\mu\chi^k + \tfrac{3}{2}\mathrm{i}\bar\eta_k\psi_\mu^k]\,, \\
\delta b_\mu &= \tfrac{1}{2}\mathrm{i}\bar\epsilon_i\phi_\mu^i - 2\bar\epsilon_i\gamma_\mu\chi^i + \tfrac{1}{2}\mathrm{i}\bar\eta_i\psi_\mu^i + 2\Lambda_K{}^a e_{\mu a}\,, \\
\delta T_{ab} &= \tfrac{2}{3}\mathrm{i}\bar\epsilon_i\gamma_{ab}\chi^i - \tfrac{1}{8}\mathrm{i}\bar\epsilon_i R_{ab}{}^i(Q)\,, \\
\delta\chi^i &= \tfrac{1}{4}\epsilon^i D + \tfrac{1}{128}R_{\mu\nu j}{}^i(V)\gamma^{\mu\nu}\epsilon^j + \tfrac{3}{128}\mathrm{i}(3\,\gamma^{ab}\not{D} + \not{D}\gamma^{ab})T_{ab}\,\epsilon^i \\
&\quad - \tfrac{3}{32}T_{ab}T_{cd}\gamma^{abcd}\epsilon^i + \tfrac{3}{16}T_{ab}\gamma^{ab}\eta^i\,, \\
\delta D &= \bar\epsilon_i\not{D}\chi^i - \mathrm{i}\bar\epsilon_i T_{ab}\gamma^{ab}\chi^i - \mathrm{i}\bar\eta_i\chi^i\,.
\end{aligned} \tag{B.1.1}$$

Under local scale transformations the various fields and transformation parameters transform as indicated in table B.2. The derivatives $\mathcal{D}_\mu$ are covariant with respect to all the bosonic gauge symmetries with the exception of the conformal boosts. In particular we note

$$\mathcal{D}_\mu\epsilon^i = \big(\partial_\mu - \tfrac{1}{4}\omega_\mu{}^{cd}\,\gamma_{cd} + \tfrac{1}{2}\,b_\mu\big)\epsilon^i + \tfrac{1}{2}\,V_{\mu j}{}^i\,\epsilon^j\,, \tag{B.1.2}$$

where the gauge fields transform under their respective gauge transformations according to $\delta\omega_\mu{}^{ab} = \mathcal{D}_\mu\lambda^{ab}$, $\delta b_\mu = \mathcal{D}_\mu\Lambda_D$ and $\delta V_{\mu i}{}^j = \mathcal{D}_\mu\Lambda_i{}^j$, with $(\Lambda_i{}^j)^* \equiv \Lambda^i{}_j = -\Lambda_j{}^i$. The derivatives D_μ are covariant with respect to all the superconformal symmetries.

In order to discuss the dependent gauge fields, we first introduce the following curvature tensors,

$$\begin{aligned}
R_{\mu\nu}{}^a(P) &= 2\,\mathcal{D}_{[\mu}e_{\nu]}{}^a - \tfrac{1}{2}\bar\psi_{[\mu i}\gamma^a\psi_{\nu]}{}^i\,, \\
R_{\mu\nu}{}^{ab}(M) &= 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\,\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b - 8\,e_{[\mu}{}^{[a}f_{\nu]}{}^{b]} + \mathrm{i}\bar\psi_{[\mu i}\gamma^{ab}\phi_{\nu]}{}^i \\
&\quad -\tfrac{1}{4}\mathrm{i}T^{cd}\,\bar\psi_{[\mu i}(6\gamma^{[a}\gamma_{cd}\gamma^{b]} - \gamma^{ab}\gamma_{cd} - \gamma_{cd}\gamma^{ab})\psi_{\nu]}{}^i \\
&\quad -\tfrac{1}{2}\bar\psi_{[\mu i}(\gamma_{\nu]}R^{abi}(Q) + 2\,\gamma^{[a}R_{\nu]}{}^{b]i}(Q)) + 8\,e_{[\mu}{}^{[a}\,\bar\psi_{\nu]i}\gamma^{b]}\chi^i\,, \\
R_{\mu\nu}(D) &= 2\,\partial_{[\mu}b_{\nu]} - 4\,f_{[\mu}{}^a e_{\nu]a} - \mathrm{i}\bar\psi_{[\mu i}\phi_{\nu]}{}^i + 4\,\bar\psi_{[\mu i}\gamma_{\nu]}\chi^i\,. \\
R_{\mu\nu i}{}^j(V) &= 2\,\partial_{[\mu}V_{\nu]i}{}^j - V_{[\mu i}{}^k V_{\nu]k}{}^j \\
&\quad -6\mathrm{i}\,\bar\psi_{[\mu i}\phi_{\nu]}{}^j + 16\bar\psi_{[\mu i}\gamma_{\nu]}\chi^j + \delta_i{}^j\big[3\mathrm{i}\,\bar\psi_{[\mu k}\phi_{\nu]}{}^k - 8\bar\psi_{[\mu k}\gamma_{\nu]}\chi^k\big]\,, \\
R_{\mu\nu}{}^i(Q) &= 2\,\mathcal{D}_{[\mu}\psi_{\nu]}{}^i - 2\mathrm{i}\,\gamma_{[\mu}\phi_{\nu]}{}^i + \tfrac{1}{2}\mathrm{i}\,T_{ab}(3\,\gamma^{ab}\gamma_{[\mu} - \gamma_{[\mu}\gamma^{ab})\psi_{\nu]}{}^i\,.
\end{aligned} \tag{B.1.3}$$

The conventional constraints (which are not invariant under Q- and S-supersymmetry) are as follows,

$$\begin{aligned} R_{\mu\nu}{}^{a}(P) &= 0\,, \\ \gamma^{\mu} R_{\mu\nu}{}^{i}(Q) &= 0\,, \\ e_{a}{}^{\mu}\, R_{\mu\nu}^{ab}(M) &= 0\,. \end{aligned} \tag{B.1.4}$$

These conditions determine the gauge fields $\omega_{\mu}{}^{ab}$, $f_{\mu}{}^{a}$ and $\phi_{\mu}{}^{i}$. We only display the bosonic parts here

$$\begin{aligned} \omega_{\mu}^{ab} &= -\,2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_{\sigma}e_{\nu}{}^{c} - 2e_{\mu}{}^{[a}e^{b]\nu}b_{\nu}\,, \\ f_{\mu}{}^{a} &= \tfrac{1}{6}\mathcal{R}(\omega,e)_{\mu}{}^{a} - \tfrac{1}{48}\mathcal{R}(\omega,e)\big)e_{\mu}{}^{a}\,. \end{aligned} \tag{B.1.5}$$

The conventional constraints lead to additional constraints on the curvatures when combined with the Bianchi identities. In this way one derives $R_{[abc]d}(M) = 0 = R_{ab}(D)$ and the pair-exchange property $R_{abcd} = R_{cdab}$ from the first and the third constraint. The second constraint, which implies also that $\gamma_{[\mu\nu}R_{\rho\sigma]}{}^{i}(Q) = 0$, determines the curvature $R_{\mu\nu}{}^{i}(S)$, which we refrained from defining previously. It turns out to be proportional to $R_{\mu\nu}{}^{i}(Q)$ and derivatives thereof,

$$\begin{aligned} R_{\mu\nu}{}^{i}(S) \;=\;& -\mathrm{i}\not{D}R_{\mu\nu}{}^{i}(Q) - \mathrm{i}\gamma_{[\mu}D^{\rho}R_{\nu]\rho}{}^{i}(Q) - 4\,\gamma_{\mu\nu}T^{\rho\sigma}R_{\rho\sigma}{}^{i}(Q) \\ & +18\,\gamma_{\sigma}T^{\rho\sigma}\gamma_{[\mu}R_{\nu]\rho}{}^{i}(Q) - 5\,T^{\rho\sigma}\gamma_{\rho\sigma}R_{\mu\nu}{}^{i}(Q) - 12\,T^{\rho}{}_{[\mu}R_{\nu]\rho}{}^{i}(Q) \end{aligned} \tag{B.1.6}$$

The remaining curvature $R_{\mu\nu}{}^{a}(K)$ does not play a role in the applications considered here.

Whereas the first constraint is invariant under S- but not under Q-supersymmetry, the other two constraints are invariant under neither supersymmetry. This implies that the dependent gauge fields will acquire terms in their transformation rules proportional to the constrained curvature tensors,

$$\begin{aligned} \delta\omega_{\mu}{}^{ab} \;=\;& \mathcal{D}_{\mu}\lambda^{ab} + 4\Lambda_{K}{}^{[a}e_{\mu}{}^{b]} - \tfrac{1}{2}\mathrm{i}\bar{\epsilon}_{i}\gamma^{ab}\phi_{\mu}{}^{i} + \tfrac{1}{2}\mathrm{i}\bar{\eta}_{i}\gamma^{ab}\psi_{\mu}{}^{i} \\ & + \tfrac{1}{8}\mathrm{i}T^{cd}\,\bar{\epsilon}_{i}(6\gamma^{[a}\gamma_{cd}\gamma^{b]} - \gamma^{ab}\gamma_{cd} - \gamma_{cd}\gamma^{ab})\psi_{\mu}{}^{i} \\ & + \tfrac{1}{4}\bar{\epsilon}_{i}(\gamma_{\mu}R^{abi}(Q) + 2\,\gamma^{[a}R_{\mu}{}^{b]i}(Q)) + 4\,e_{\mu}{}^{[a}\,\bar{\epsilon}_{i}\gamma^{b]}\chi^{i}\,, \\ \delta\phi_{\mu}{}^{i} \;=\;& \mathcal{D}_{\mu}\eta^{i} + \tfrac{1}{4}\mathrm{i}\,T_{ab}(\gamma_{\mu}\gamma^{ab} - \gamma^{ab}\gamma_{\mu})\eta^{i} + \mathrm{i}f_{\mu}{}^{a}\gamma_{a}\epsilon^{i} - \mathrm{i}\Lambda_{K}{}^{a}\gamma_{a}\psi_{\mu}{}^{i} \\ & -\tfrac{1}{48}\mathrm{i}(2\,\gamma^{ab}\gamma_{\mu} - \gamma_{\mu}\gamma^{ab})R_{abj}{}^{i}(V)\epsilon^{j} + \tfrac{1}{4}\big(\not{D}T^{ab}\gamma_{ab}\gamma_{\mu} + D_{a}T^{ab}\gamma_{\mu}\gamma_{b}\big)\epsilon^{i} \\ & +\mathrm{i}\big(-\tfrac{3}{4}T^{ab}T^{cd}\gamma_{\mu abcd} + T_{\mu a}T_{bc}\gamma^{abc} - 4\,T_{\mu a}T^{ab}\gamma_{b} - \tfrac{3}{4}\gamma_{\mu}T^{2}\big)\epsilon^{i} \\ & -\tfrac{9}{4}\mathrm{i}\,\bar{\epsilon}_{j}\psi_{\mu}{}^{j}\,\chi^{i} + \tfrac{7}{4}\mathrm{i}\,\bar{\epsilon}_{j}\gamma_{a}\psi_{\mu}{}^{j}\,\gamma^{a}\chi^{i} - \tfrac{1}{8}\mathrm{i}\bar{\epsilon}_{k}\gamma^{ab}\psi_{\mu}{}^{k}\big(\gamma_{ab}\chi^{i} + \tfrac{1}{4}R_{ab}{}^{i}(Q)\big) \\ & +\tfrac{1}{4}\mathrm{i}\bar{\epsilon}_{k}\gamma^{ab}\psi_{\mu}{}^{i}\big(\gamma_{ab}\chi^{k} + \tfrac{1}{4}R_{ab}{}^{k}(Q)\big)\,, \\ \delta f_{\mu}{}^{a} \;=\;& \mathcal{D}_{\mu}\Lambda_{K}{}^{a} + \tfrac{1}{2}\eta_{i}\gamma^{a}\phi_{\mu}{}^{i} + \cdots\,, \end{aligned} \tag{B.1.7}$$

where here and henceforth $T^2 \equiv (T_{ab})^2$. With these results we obtain the following Q- and S-variations for ${R_{ab}}^i(Q)$ and ${R_{abi}}^j(V)$

$$
\begin{aligned}
\delta {R_{ab}}^i(Q) \;=\;& -\tfrac{1}{24}(\gamma^{cd}\gamma_{ab} - 4\,\delta_a{}^{[c}\delta_b{}^{d]})\, {R_{cdi}}^i(V)\epsilon^j - \tfrac{1}{4}{R_{ab}}^{cd}(M)\gamma_{cd}\epsilon^i \\
& + \tfrac{1}{2}\mathrm{i}\big(3\,D_{[a}T^{cd}\gamma_{cd}\gamma_{b]} - D_{[a}T^{cd}\gamma_{b]}\gamma_{cd} - \gamma_{[a}\not{D}T^{cd}\gamma_{cd}\gamma_{b]} - D_cT^{cd}\gamma_{ab}\gamma_d\big)\epsilon^i \\
& - 2\big(T_{ab}T_{cd}\gamma^{cd} + T_{ac}T_{bd}\gamma^{cd} + 2\,T_{c[a}T^{cd}\gamma_{b]d} + \tfrac{1}{4}T^2\gamma_{ab}\big)\epsilon^i \\
& + \big(\gamma^{cd}\gamma_{ab} - 4\,\delta_a{}^{[c}\delta_b{}^{d]}\big)\eta^i\,T_{cd}\,, \\
\delta {R_{abi}}^j(V) \;=\;& 3\mathrm{i}\,\bar\epsilon_i{R_{ab}}^j(S) + 16\,\bar\epsilon_i\gamma_{[a}D_{b]}\chi^j - 4\mathrm{i}\,\bar\epsilon_i\big(3\,\gamma_{[a}\gamma^{cd}\gamma_{b]} - \gamma^{cd}\gamma_{ab}\big)\chi^j\,T_{cd} \\
& - 3\mathrm{i}\,\bar\eta_i{R_{ab}}^j(Q) - 16\mathrm{i}\,\bar\eta_i\gamma_{ab}\chi^j - \text{trace}\,. \qquad \text{(B.1.8)}
\end{aligned}
$$

The above transformations agree with those of [29, 30], upon including a T-dependent S-supersymmetry transformation into the Q-supersymmetry variations and rescaling the tensor field by a factor 4/3. The difference with the conventions of [31, 32] are a bit more involved.

B.1.2 The vector supermultiplet

The vector supermultiplet consists of a real scalar σ, a gauge field W_μ, a triplet of (auxiliary) fields Y^{ij}, and a fermion field Ω^i. Under superconformal transformations these fields transform as follows,

$$
\begin{aligned}
\delta\sigma \;=\;& \tfrac{1}{2}\mathrm{i}\bar\epsilon_i\Omega^i\,, \\
\delta\Omega^i \;=\;& -\tfrac{1}{4}(\hat F_{ab} - 4\,\sigma T_{ab})\gamma^{ab}\epsilon^i - \tfrac{1}{2}\mathrm{i}\not{D}\sigma\epsilon^i - \varepsilon_{jk}\,Y^{ij}\epsilon^k + \sigma\,\eta^i\,, \\
\delta W_\mu \;=\;& \tfrac{1}{2}\bar\epsilon_i\gamma_\mu\Omega^i - \tfrac{1}{2}\mathrm{i}\sigma\,\bar\epsilon_i\psi_\mu^i\,, \\
\delta Y^{ij} \;=\;& \tfrac{1}{2}\varepsilon^{k(i}\,\bar\epsilon_k\not{D}\Omega^{j)} + \mathrm{i}\varepsilon^{k(i}\,\bar\epsilon_k(-\tfrac{1}{4}T_{ab}\gamma^{ab}\Omega^{j)} + 4\sigma\chi^{j)}) - \tfrac{1}{2}\mathrm{i}\varepsilon^{k(i}\,\bar\eta_k\Omega^{j)}\,. \qquad \text{(B.1.9)}
\end{aligned}
$$

where $(Y^{ij})^* \equiv Y_{ij} = \varepsilon_{ik}\varepsilon_{jl}Y^{kl}$, and the supercovariant field strength is defined as,

$$
\hat F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - \bar\Omega_i\gamma_{[\mu}{\psi_{\nu]}}^i + \tfrac{1}{2}\mathrm{i}\sigma\,\bar\psi_{[\mu i}{\psi_{\nu]}}^i\,. \qquad \text{(B.1.10)}
$$

We also note the transformation rule,

$$
\begin{aligned}
\delta(\hat F_{ab} - 4\,\sigma T_{ab}) \;=\;& -\bar\epsilon_i\gamma_{[a}D_{b]}\Omega^i - \tfrac{8}{3}\,\sigma\,\bar\epsilon_i\gamma_{ab}\chi^i \\
& + \tfrac{1}{4}\mathrm{i}\bar\epsilon_i(3\,\gamma_{[a}\gamma^{cd}\gamma_{b]} - \gamma^{cd}\gamma_{ab} - 8\,\delta_a^c\delta_b^d)\Omega^i\,T_{cd} + \mathrm{i}\bar\eta_i\gamma_{ab}\Omega^i \qquad \text{(B.1.11)}
\end{aligned}
$$

The fields behave under local scale transformations according to the weights shown in table B.3.

	vector multiplet			
field	σ	W_μ	Ω_i	Y_{ij}
w	1	0	$\frac{3}{2}$	2
	linear multiplet			
field	L^{ij}	E_a	φ_i	N
w	3	4	$\frac{7}{2}$	4
	hypermultiplet			
field	$A_i{}^\alpha$		ζ^α	
w	$\frac{3}{2}$		2	

Table B.3: Weyl weights w of the vector multiplet, the tensor (linear) multiplet, and the hypermultiplet component fields.

B.1.3 The linear supermultiplet

The linear multiplet consists of a triplet of scalars L^{ij}, a divergence-free vector $\hat{E}_a$, an (auxiliary) scalar N, and a fermion field φ^i. The superconformal transformation rules for these fields are as follows,

$$\begin{aligned} \delta L^{ij} &= -\,\mathrm{i}\,\varepsilon^{k(i}\,\bar\epsilon_k\varphi^{j)}\,, \\ \delta\varphi^i &= -\tfrac{1}{2}\mathrm{i}\,\varepsilon_{jk}\not{D}L^{ij}\epsilon^k + \tfrac{1}{2}(N-\mathrm{i}\hat{\not{E}})\epsilon^i + 3\varepsilon_{jk}L^{ij}\eta^k\,, \\ \delta\hat{E}_a &= -\tfrac{1}{2}\mathrm{i}\,\bar\epsilon_i\gamma_{ab}D^b\varphi^i + \tfrac{1}{8}\bar\epsilon_i(3\gamma_a\gamma^{bc}+\gamma^{bc}\gamma_a)\varphi^i T_{bc} - 2\bar\eta_i\gamma_a\varphi^i\,, \\ \delta N &= \tfrac{1}{2}\bar\epsilon_i\not{D}\varphi^i + \tfrac{3}{4}\mathrm{i}\bar\epsilon_i\gamma^{ab}\varphi^i T_{ab} - 4\mathrm{i}\,\varepsilon_{jk}\,\bar\epsilon_i\chi^k L^{ij} + \tfrac{3}{2}\mathrm{i}\bar\eta_i\varphi^i\,. \end{aligned} \quad \text{(B.1.12)}$$

The constraint on $\hat{E}^a$,

$$D_a\hat{E}^a = 0\,, \quad \text{(B.1.13)}$$

can be solved in terms of a three-rank anti-symmetric tensor gauge field $E_{\mu\nu\rho}$, which transforms as follows under the superconformal transformations,

$$\delta E_{\mu\nu\rho} = \tfrac{1}{2}\bar\epsilon_i\gamma_{\mu\nu\rho}\varphi^i - \tfrac{3}{2}\mathrm{i}\,\bar\epsilon_i\gamma_{[\mu\nu}\psi_{\rho]}{}^k\,\varepsilon_{jk}L^{ij}\,. \quad \text{(B.1.14)}$$

The corresponding supercovariant field strength associated with $E_{\mu\nu\rho}$ equals

$$\hat{E}^\mu = \tfrac{1}{6}\mathrm{i}\,e^{-1}\varepsilon^{\mu\nu\rho\sigma\lambda}\Big[\partial_\nu E_{\rho\sigma\lambda} - \tfrac{1}{2}\bar\psi_{\nu i}\gamma_{\rho\sigma\lambda}\varphi^i + \tfrac{3}{4}\mathrm{i}\,\bar\psi_{\nu i}\gamma_{\rho\sigma}\psi_\lambda{}^k\,\varepsilon_{jk}L^{ij}\Big]\,. \quad \text{(B.1.15)}$$

The behaviour under local scale transformations follow from the weights shown in table B.3. The tensor field $E_{\mu\nu\rho}$ is inert under scale transformations and thus carries zero weight.

B.1.4 Hypermultiplets

Hypermultiplets are necessarily associated with target spaces of dimension $4r$ that are hyperkähler cones [34, 184]. The supersymmetry transformations are most conveniently written in terms of the sections $A_i{}^\alpha(\phi)$, where $\alpha = 1, 2, \ldots, 2r$,

$$\begin{aligned} \delta A_i{}^\alpha &= \mathrm{i}\,\bar{\epsilon}_i \zeta^\alpha\,, \\ \delta\zeta^\alpha &= -\tfrac{1}{2}\mathrm{i}\not{D} A_i{}^\alpha \epsilon^i + \tfrac{3}{2} A_i{}^\alpha \eta^i\,. \end{aligned} \tag{B.1.16}$$

The $A_i{}^\alpha$ are local sections of an $\mathrm{Sp}(r) \times \mathrm{Sp}(1)$ bundle. The existence of such an associated bundle is known from general arguments [185]. We also note the existence of a covariantly constant skew-symmetric tensor $\Omega_{\alpha\beta}$ (and its complex conjugate $\Omega^{\alpha\beta}$ satisfying $\Omega_{\alpha\gamma}\Omega^{\beta\gamma} = -\delta_\alpha{}^\beta$), and the symplectic Majorana condition for the spinors reads as $C^{-1}\bar{\zeta}_\alpha{}^{\mathrm{T}} = \Omega_{\alpha\beta}\,\zeta^\beta$. Covariant derivatives contain the $\mathrm{Sp}(r)$ connection $\Gamma_A{}^\alpha{}_\beta$, associated with rotations of the fermions. The sections $A_i{}^\alpha$ are pseudo-real, i.e. they are subject to the constraint, $A_i{}^\alpha \varepsilon^{ij} \Omega_{\alpha\beta} = A^j{}_\beta \equiv (A_j{}^\beta)^*$. The information on the target-space metric is contained in the so-called hyperkähler potential,

$$\varepsilon_{ij}\,\chi = \Omega_{\alpha\beta}\, A_i{}^\alpha A_j{}^\beta\,. \tag{B.1.17}$$

For the local scale transformations we refer again to the weights shown in table B.3. The hypermultiplet does not exist as an off-shell supermultiplet. Closure of the superconformal transformations is only realized upon using fermionic field equations, but this fact does not represent a serious problem in our considerations.

B.2 Superconformal multiplets in four dimensions

We now turn to the transformation rules of the superconformal multiplets in four space-time dimensions. The superconformal algebra comprises the generators of the general-coordinate, local Lorentz, dilatation, special conformal, chiral SU(2) and U(1), supersymmetry (Q) and special supersymmetry (S) transformations.

B.2.1 The Weyl multiplet

The Weyl multiplet contains the gauge fields associated with general-coordinate transformations ($e_\mu{}^a$), dilatations (b_μ), chiral symmetry ($\mathcal{V}_\mu{}^i{}_j$ and A_μ) and Q-supersymmetry ($\psi_\mu{}^i$) are independent fields. The remaining gauge fields associated with the Lorentz ($\omega_\mu{}^{ab}$), special conformal ($f_\mu{}^a$) and S-supersymmetry transformations ($\phi_\mu{}^i$) are dependent fields. They are composite objects, which depend on the independent fields of the multiplet [46–48]. The corresponding supercovariant curvatures and covariant fields are contained in a tensor chiral multiplet, which comprises $24+24$ off-shell degrees of freedom. In addition to the independent superconformal gauge fields, it contains three other fields: a Majorana spinor doublet χ^i, a scalar D, and a selfdual Lorentz tensor T_{abij}, which is anti-symmetric in $[ab]$ and $[ij]$. The Weyl and chiral weights have been collected in table B.4.

Under Q-supersymmetry, S-supersymmetry and special conformal transformations the independent fields of the Weyl multiplet transform as follows,

$$\begin{aligned}
\delta e_\mu{}^a &= \bar\epsilon^i\,\gamma^a\psi_{\mu i} + \bar\epsilon_i\,\gamma^a\psi_\mu{}^i\,,\\
\delta\psi_\mu{}^i &= 2\,\mathcal{D}_\mu\epsilon^i - \tfrac18 T_{ab}{}^{ij}\gamma^{ab}\gamma_\mu\epsilon_j - \gamma_\mu\eta^i\\
\delta b_\mu &= \tfrac12\bar\epsilon^i\phi_{\mu i} - \tfrac34\bar\epsilon^i\gamma_\mu\chi_i - \tfrac12\bar\eta^i\psi_{\mu i} + \text{h.c.} + \Lambda_K^a e_{\mu a}\,,\\
\delta A_\mu &= \tfrac12\mathrm{i}\bar\epsilon^i\phi_{\mu i} + \tfrac34\mathrm{i}\bar\epsilon^i\gamma_\mu\,\chi_i + \tfrac12\mathrm{i}\bar\eta^i\psi_{\mu i} + \text{h.c.}\,,\\
\delta\mathcal{V}_\mu{}^i{}_j &= 2\,\bar\epsilon_j\phi_\mu{}^i - 3\bar\epsilon_j\gamma_\mu\,\chi^i + 2\bar\eta_j\,\psi_\mu{}^i - (\text{h.c. ; traceless})\,,\\
\delta T_{ab}{}^{ij} &= 8\,\bar\epsilon^{[i}R(Q)_{ab}{}^{j]}\,,\\
\delta\chi^i &= -\tfrac{1}{12}\gamma^{ab}\,\not{D}T_{ab}{}^{ij}\,\epsilon_j + \tfrac16 R(\mathcal{V})_{\mu\nu}{}^i{}_j\gamma^{\mu\nu}\epsilon^j - \tfrac13\mathrm{i}R_{\mu\nu}(A)\gamma^{\mu\nu}\epsilon^i + D\epsilon^i + \tfrac{1}{12}\gamma_{ab}T^{abij}\eta_j\,,\\
\delta D &= \bar\epsilon^i\,\not{D}\chi_i + \bar\epsilon_i\,\not{D}\chi^i\,.
\end{aligned} \tag{B.2.1}$$

Here ϵ^i and ϵ_i denote the spinorial parameters of Q-supersymmetry, η^i and η_i those of S-supersymmetry, and $\Lambda_K{}^a$ is the transformation parameter for special conformal boosts. The full superconformally covariant derivative is denoted by D_μ, while $\mathcal{D}_\mu$ denotes a covariant derivative with respect to Lorentz, dilatation, chiral U(1), and SU(2) transformations,

$$\mathcal{D}_\mu\epsilon^i = \big(\partial_\mu - \tfrac14\omega_\mu{}^{cd}\,\gamma_{cd} + \tfrac12\,b_\mu + \tfrac12\mathrm{i}\,A_\mu\big)\epsilon^i + \tfrac12\,\mathcal{V}_\mu{}^i{}_j\,\epsilon^j\,. \tag{B.2.2}$$

	Weyl multiplet											parameters	
field	$e_\mu{}^a$	$\psi_\mu{}^i$	b_μ	A_μ	$\mathcal{V}_\mu{}^i{}_j$	$T_{ab}{}^{ij}$	χ^i	D	ω_μ^{ab}	$f_\mu{}^a$	$\phi_\mu{}^i$	ϵ^i	η^i
w	-1	$-\frac12$	0	0	0	1	$\frac32$	2	0	1	$\frac12$	$-\frac12$	$\frac12$
c	0	$-\frac12$	0	0	0	-1	$-\frac12$	0	0	0	$-\frac12$	$-\frac12$	$-\frac12$
γ_5		+					+				$-$	+	$-$

Table B.4: Weyl and chiral weights (w and c) and fermion chirality (γ_5) of the Weyl multiplet component fields and the supersymmetry transformation parameters.

The covariant curvatures of the various gauge symmetries take the following form,

$$
\begin{aligned}
R(P)_{\mu\nu}{}^a &= 2\,\partial_{[\mu}\, e_{\nu]}{}^a + 2\, b_{[\mu}\, e_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}\, e_{\nu]b} - \tfrac{1}{2}(\bar\psi_{[\mu}{}^i\gamma^a\psi_{\nu]i} + \text{h.c.})\,,\\
R(Q)_{\mu\nu}{}^i &= 2\,\mathcal{D}_{[\mu}\psi_{\nu]}{}^i - \gamma_{[\mu}\phi_{\nu]}{}^i - \tfrac{1}{8}\,T^{abij}\,\gamma_{ab}\,\gamma_{[\mu}\psi_{\nu]j}\,,\\
R(A)_{\mu\nu} &= 2\,\partial_{[\mu}A_{\nu]} - \mathrm{i}\left(\tfrac{1}{2}\bar\psi_{[\mu}{}^i\phi_{\nu]i} + \tfrac{3}{4}\bar\psi_{[\mu}{}^i\gamma_{\nu]}\chi_i - \text{h.c.}\right)\,,\\
R(\mathcal{V})_{\mu\nu}{}^i{}_j &= 2\,\partial_{[\mu}\mathcal{V}_{\nu]}{}^i{}_j + \mathcal{V}_{[\mu}{}^i{}_k\,\mathcal{V}_{\nu]}{}^k{}_j + 2(\bar\psi_{[\mu}{}^i\,\phi_{\nu]j} - \bar\psi_{[\mu j}\,\phi_{\nu]}{}^i) - 3(\bar\psi_{[\mu}{}^i\gamma_{\nu]}\chi_j - \bar\psi_{[\mu j}\gamma_{\nu]}\chi^i)\\
&\quad - \delta_j{}^i(\bar\psi_{[\mu}{}^k\,\phi_{\nu]k} - \bar\psi_{[\mu k}\,\phi_{\nu]}{}^k) + \tfrac{3}{2}\delta_j{}^i(\bar\psi_{[\mu}{}^k\gamma_{\nu]}\chi_k - \bar\psi_{[\mu k}\gamma_{\nu]}\chi^k)\,,\\
R(M)_{\mu\nu}{}^{ab} &= 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\,\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b - 4f_{[\mu}{}^{[a}e_{\nu]}{}^{b]} + \tfrac{1}{2}(\bar\psi_{[\mu}{}^i\,\gamma^{ab}\,\phi_{\nu]i} + \text{h.c.})\\
&\quad + (\tfrac{1}{4}\bar\psi_\mu{}^i\,\psi_\nu{}^j\,T^{ab}{}_{ij} - \tfrac{3}{4}\bar\psi_{[\mu}{}^i\,\gamma_{\nu]}\,\gamma^{ab}\chi_i - \bar\psi_{[\mu}{}^i\,\gamma_{\nu]}\,R(Q)^{ab}{}_i + \text{h.c.})\,,\\
R(D)_{\mu\nu} &= 2\,\partial_{[\mu}b_{\nu]} - 2f_{[\mu}{}^a e_{\nu]a} - \tfrac{1}{2}\bar\psi_{[\mu}{}^i\phi_{\nu]i} + \tfrac{3}{4}\bar\psi_{[\mu}{}^i\gamma_{\nu]}\chi_i - \tfrac{1}{2}\bar\psi_{[\mu i}\phi_{\nu]}{}^i + \tfrac{3}{4}\bar\psi_{[\mu i}\gamma_{\nu]}\chi^i\,,\\
R(S)_{\mu\nu}{}^i &= 2\,\mathcal{D}_{[\mu}\phi_{\nu]}{}^i - 2f_{[\mu}{}^a\gamma_a\psi_{\nu]}{}^i - \tfrac{1}{8}\not{D}T_{ab}{}^{ij}\gamma^{ab}\gamma_{[\mu}\psi_{\nu]\,j} - \tfrac{3}{2}\gamma_a\psi_{[\mu}{}^i\,\bar\psi_{\nu]}{}^j\gamma^a\chi_j\\
&\quad + \tfrac{1}{4}R(\mathcal{V})_{ab}{}^i{}_j\gamma^{ab}\gamma_{[\mu}\psi_{\nu]}{}^j + \tfrac{1}{2}\mathrm{i}R(A)_{ab}\gamma^{ab}\gamma_{[\mu}\psi_{\nu]}{}^i\,,\\
R(K)_{\mu\nu}{}^a &= 2\,\mathcal{D}_{[\mu}f_{\nu]}{}^a - \tfrac{1}{4}\big(\bar\phi_{[\mu}{}^i\gamma^a\phi_{\nu]i} + \bar\phi_{[\mu i}\gamma^a\phi_{\nu]}{}^i\big)\\
&\quad + \tfrac{1}{4}\big(\bar\psi_\mu{}^i D_b T^{ba}{}_{ij}\psi_\nu{}^j - 3\,e_{[\mu}{}^a\psi_{\nu]}{}^i\not{D}\chi_i\\
&\qquad + \tfrac{3}{2}D\,\bar\psi_{[\mu}{}^i\gamma^a\psi_{\nu]j} - 4\,\bar\psi_{[\mu}{}^i\gamma_{\nu]}D_bR(Q)^{ba}{}_i + \text{h.c.}\big)\,. \qquad \text{(B.2.3)}
\end{aligned}
$$

There are three conventional constraints (which have already been incorporated in (B.2.3),

$$
\begin{aligned}
&R(P)_{\mu\nu}{}^a = 0\,,\\
&\gamma^\mu R(Q)_{\mu\nu}{}^i + \tfrac{3}{2}\gamma_\nu\chi^i = 0\,,\\
&e^\nu{}_b\,R(M)_{\mu\nu a}{}^b - \mathrm{i}\tilde R(A)_{\mu a} + \tfrac{1}{8}T_{abij}T_\mu{}^{bij} - \tfrac{3}{2}D\,e_{\mu a} = 0\,, \qquad \text{(B.2.4)}
\end{aligned}
$$

which are S-supersymmetry invariant. They determine the fields $\omega_\mu{}^{ab}$, $\phi_\mu{}^i$ and $f_\mu{}^a$ as follows,

$$
\begin{aligned}
\omega_\mu^{ab} =&\; -2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\sigma e_\nu{}^c - 2e_\mu{}^{[a}e^{b]\nu}b_\nu\\
&\; - \tfrac{1}{4}(2\bar\psi_\mu^i\gamma^{[a}\psi_i^{b]} + \bar\psi^{ai}\gamma_\mu\psi_i^b + \text{h.c.})\,,\\
\phi_\mu{}^i =&\; \tfrac{1}{2}\left(\gamma^{\rho\sigma}\gamma_\mu - \tfrac{1}{3}\gamma_\mu\gamma^{\rho\sigma}\right)\left(\mathcal{D}_\rho\psi_\sigma{}^i - \tfrac{1}{16}T^{abij}\gamma_{ab}\gamma_\rho\psi_{\sigma j} + \tfrac{1}{4}\gamma_{\rho\sigma}\chi^i\right)\,,\\
f_\mu{}^\mu =&\; \tfrac{1}{6}\mathcal{R}(\omega,e) - D - \big(\tfrac{1}{12}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_\mu{}^i\,\gamma_\nu\mathcal{D}_\rho\psi_{\sigma i}\\
&\qquad - \tfrac{1}{12}\bar\psi_\mu{}^i\psi_\nu{}^j T^{\mu\nu}{}_{ij} - \tfrac{1}{4}\bar\psi_\mu{}^i\gamma^\mu\chi_i + \text{h.c.}\big)\,. \qquad \text{(B.2.5)}
\end{aligned}
$$

We will also need the bosonic part of the expression for the uncontracted connection $f_\mu{}^a$,

$$
f_\mu{}^a = \tfrac{1}{2}\mathcal{R}(\omega,e)_\mu{}^a - \tfrac{1}{4}\big(D + \tfrac{1}{3}\mathcal{R}(\omega,e)\big)e_\mu{}^a - \tfrac{1}{2}\mathrm{i}\tilde R(A)_\mu{}^a + \tfrac{1}{16}T_{\mu b}{}^{ij}T^{ab}{}_{ij}\,, \qquad \text{(B.2.6)}
$$

where $\mathcal{R}(\omega,e)_\mu{}^a = \mathcal{R}(\omega)_{\mu\nu}{}^{ab}e_b{}^\nu$ is the non-symmetric Ricci tensor, and $\mathcal{R}(\omega,e)$ the corresponding Ricci scalar. The curvature $\mathcal{R}(\omega)_{\mu\nu}{}^{ab}$ is associated with the spin connection field $\omega_\mu{}^{ab}$, given in (B.2.5).

The transformations of $\omega_\mu{}^{ab}$, $\phi_\mu{}^i$ and $f_\mu{}^a$ are induced by the constraints (B.2.4). We present their Q- and S-supersymmetry variations, as well as the transformations under conformal boosts, below,

$$\begin{aligned}
\delta\omega_\mu{}^{ab} =& -\tfrac{1}{2}\bar\epsilon^i\gamma^{ab}\phi_{\mu i} - \tfrac{1}{2}\bar\epsilon^i\psi_\mu{}^j\, T^{ab}{}_{ij} + \tfrac{3}{4}\bar\epsilon^i\gamma_\mu\gamma^{ab}\chi_i \\
&+ \bar\epsilon^i\gamma_\mu\, R(Q)^{ab}{}_i - \tfrac{1}{2}\bar\eta^i\gamma^{ab}\psi_{\mu i} + \text{h.c.} + 2\,\Lambda_{\rm K}{}^{[a}e_\mu{}^{b]}\,, \\
\delta\phi_\mu{}^i =& -2\,f_\mu{}^a\gamma_a\epsilon^i + \tfrac{1}{4}R(\mathcal{V})_{ab}{}^i{}_j\gamma^{ab}\gamma_\mu\epsilon^j + \tfrac{1}{2}\mathrm{i}R(A)_{ab}\gamma^{ab}\gamma_\mu\epsilon^i - \tfrac{1}{8}\rlap{\,/}D T^{ab\,ij}\gamma_{ab}\gamma_\mu\epsilon_j \\
&+ \tfrac{3}{2}[(\bar\chi_j\gamma^a\epsilon^j)\gamma_a\psi_\mu{}^i - (\bar\chi_j\gamma^a\psi_\mu{}^j)\gamma_a\epsilon^i] + 2\,\mathcal{D}_\mu\eta^i + \Lambda_{\rm K}{}^a\gamma_a\psi_\mu{}^i\,, \\
\delta f_\mu{}^a =& -\tfrac{1}{2}\bar\epsilon^i\psi_\mu{}^i\, D_bT^{ba}{}_{ij} - \tfrac{3}{4}e_\mu{}^a\bar\epsilon^i\rlap{\,/}D\chi_i - \tfrac{3}{4}\bar\epsilon^i\gamma^a\psi_{\mu i}\,D \\
&+ \bar\epsilon^i\gamma_\mu\, D_bR(Q)^{ba}{}_i + \tfrac{1}{2}\bar\eta^i\gamma^a\phi_{\mu i} + \text{h.c.} + \mathcal{D}_\mu\Lambda_{\rm K}{}^a\,.
\end{aligned} \tag{B.2.7}$$

The transformations under S-supersymmetry and conformal boosts reflect the structure of the underlying SU(2, 2|2) gauge algebra. The presence of curvature constraints and of the non-gauge fields T_{abij}, χ^i and D induce deformations of the Q-supersymmetry algebra, as is manifest in the above results, in particular in (B.2.3) and (B.2.7).

Combining the conventional constraints (B.2.4) with the various Bianchi identities one derives that not all the curvatures are independent. For instance,

$$\varepsilon^{abcd}D_bR(M)_{cd}{}^{ef} = 2\,\varepsilon^{abc[e}\,R(K)_{bc}{}^{f]} + \tfrac{9}{2}\,\eta^{a[e}\bar\chi^i\gamma^{f]}\chi_i + \tfrac{3}{2}\big[\bar\chi^i\gamma^aR(Q)^{ef}_i - \text{h.c.}\big]\,. \tag{B.2.8}$$

Furthermore it is convenient to modify two of the curvatures by including suitable covariant terms,

$$\begin{aligned}
\hat R(M)_{ab}{}^{cd} =& R(M)_{ab}{}^{cd} + \tfrac{1}{16}\big(T_{abij}\,T^{cdij} + T_{ab}{}^{ij}\,T^{cd}{}_{ij}\big)\,, \\
\mathcal{R}(S)_{ab}{}^i =& R(S)_{ab}{}^i + \tfrac{3}{4}T_{ab}{}^{ij}\chi_j\,.
\end{aligned} \tag{B.2.9}$$

where we observe that $\gamma^{ab}\big(\mathcal{R}(S) - \mathcal{R}(S)\big)_{ab}{}^i = 0$. The modified curvature $\hat R(M)_{ab}{}^{cd}$ satisfies the following relations,

$$\begin{aligned}
\hat R(M)_{\mu\nu}{}^{ab}\,e^\nu{}_b =& \mathrm{i}\tilde R(A)_{\mu\nu}e^{\nu a} + \tfrac{3}{2}D\,e_\mu{}^a\,, \\
\tfrac{1}{4}\varepsilon_{ab}{}^{ef}\,\varepsilon^{cd}{}_{gh}\,\hat R(M)_{ef}{}^{gh} =& \hat R(M)_{ab}{}^{cd}\,, \\
\varepsilon_{cdea}\,\hat R(M)^{cd\,e}{}_b =& \varepsilon_{becd}\,\hat R(M)_a{}^{e\,cd} = 2\tilde R(D)_{ab} = 2\mathrm{i}R(A)_{ab}\,.
\end{aligned} \tag{B.2.10}$$

The first of these relations corresponds to the third constraint given in (B.2.4), while the remaining equations follow from combining the curvature constraints with the Bianchi identities. Note that the modified curvature does not satisfy the pair exchange property; instead we have,

$$\hat R(M)_{ab}{}^{cd} = \hat R(M)^{cd}{}_{ab} + 4i\delta^{[c}_{[a}\,\tilde R(A)_{b]}{}^{d]}\,. \tag{B.2.11}$$

We now turn to the fermionic constraint given in (B.2.4) and its consequences for the modified curvature defined in (B.2.9). First we note that the constraint on $R(Q)_{\mu\nu}{}^i$ implies that this curvature is anti-selfdual, as follows from contracting the constraint with $\gamma^\nu\,\gamma_{ab}$,

$$\tilde{R}(Q)_{\mu\nu}{}^i \equiv \tfrac{1}{2}\, e\, \varepsilon_{\mu\nu}{}^{\rho\sigma} R(Q)_{\rho\sigma}{}^i = -R(Q)_{\mu\nu}{}^i\,. \tag{B.2.12}$$

Furthermore, combination of the Bianchi identity and the constraint on $R(Q)_{\mu\nu i}$ yields the following condition on the modified curvature $\mathcal{R}(S)_{ab}{}^i$,

$$\gamma^a \tilde{\mathcal{R}}(S)_{ab}{}^i = 2\, D^a \tilde{R}(Q)_{ab}{}^i = -2\, D^a R(Q)_{ab}{}^i\,. \tag{B.2.13}$$

This identity (upon contraction with $\gamma^b\gamma_{cd}$) leads to the following identity on the anti-selfdual part of $\mathcal{R}(S)_{ab}{}^i$,

$$\mathcal{R}(S)_{ab}{}^i - \tilde{\mathcal{R}}(S)_{ab}{}^i = 2\, \not{D}\big(R(Q)_{ab}{}^i + \tfrac{3}{4}\gamma_{ab}\chi^i\big)\,. \tag{B.2.14}$$

Finally we note the following useful identities for products of (anti)selfdual tensors,

$$\begin{aligned}
G^{\pm}_{[a[c}\, H^{\pm}_{d]b]} &= \pm\, \tfrac{1}{8} G^{\pm}_{ef}\, H^{\pm ef}\, \varepsilon_{abcd} - \tfrac{1}{4}(G^{\pm}_{ab}\, H^{\pm}_{cd} + G^{\pm}_{cd}\, H^{\pm}_{ab})\,,\\
G^{\pm}_{ab}\, H^{\mp cd} + G^{\pm cd}\, H^{\mp}_{ab} &= 4\delta^{[c}_{[a} G^{\pm}_{b]e}\, H^{\mp d]e}\,,\\
\tfrac{1}{2}\varepsilon^{abcd}\, G^{\pm e}_{[c}\, H^{\pm}_{d]e} &= \pm\, G^{\pm[a}{}_e\, H^{\pm b]e}\,,\\
G^{\pm ac}\, H^{\pm b}_c + G^{\pm bc}\, H^{\pm a}_c &= -\, \tfrac{1}{2}\eta^{ab}\, G^{\pm cd}\, H^{\pm}_{cd}\,,\\
G^{\pm ac}\, H^{\mp b}_c &= G^{\pm bc}\, H^{\mp a}_c\,, \qquad G^{\pm ab}\, H^{\mp}_{ab} = 0\,.
\end{aligned} \tag{B.2.15}$$

B.2.2 Chiral multiplets

Chiral multiplets are complex carrying a Weyl weight w and a chiral U(1) weight c, which is opposite to the Weyl weight, i.e. $c = -w$. The weights indicate how the lowest-θ component of the superfield scales under Weyl and chiral U(1) transformations. Anti-chiral multiplets can be obtained from chiral ones by complex conjugation, so that anti-chiral multiplets will have equal Weyl and chiral weights, hence $w = c$.

The components of a generic scalar chiral multiplet are a complex scalar A, a Majorana doublet spinor Ψ_i, a complex symmetric scalar B_{ij}, an anti-selfdual tensor G^-_{ab}, a Majorana doublet spinor Λ_i, and a complex scalar C. The assignment of their Weyl and chiral weights is shown in table B.5. The Q- and S-supersymmetry transformations for

	Chiral multiplet					
field	A	Ψ_i	B_{ij}	G^-_{ab}	Λ_i	C
w	w	$w+\frac{1}{2}$	$w+1$	$w+1$	$w+\frac{3}{2}$	$w+2$
c	$-w$	$-w+\frac{1}{2}$	$-w+1$	$-w+1$	$-w+\frac{3}{2}$	$-w+2$
γ_5		$+$			$+$	

Table B.5: Weyl and chiral weights (w and c) and fermion chirality (γ_5) of the chiral multiplet component fields.

a scalar chiral multiplet of weight w, are as follows

$$\begin{aligned}
\delta A &= \bar\epsilon^i \Psi_i \,, \\
\delta\Psi_i &= 2\,\not{D} A \epsilon_i + B_{ij}\,\epsilon^j + \tfrac{1}{2}\gamma^{ab} G^-_{ab}\,\varepsilon_{ij}\epsilon^j + 2\,w A\,\eta_i \,, \\
\delta B_{ij} &= 2\,\bar\epsilon_{(i}\not{D}\Psi_{j)} - 2\,\bar\epsilon^k \Lambda_{(i}\,\varepsilon_{j)k} + 2(1-w)\,\bar\eta_{(i}\Psi_{j)} \,, \\
\delta G^-_{ab} &= \tfrac{1}{2}\varepsilon^{ij}\,\bar\epsilon_i \not{D}\gamma_{ab}\Psi_j + \tfrac{1}{2}\bar\epsilon^i\gamma_{ab}\Lambda_i - \tfrac{1}{2}(1+w)\,\varepsilon^{ij}\bar\eta_i\gamma_{ab}\Psi_j \,, \\
\delta\Lambda_i &= -\tfrac{1}{2}\gamma^{ab}\not{D} G^-_{ab}\epsilon_i - \not{D} B_{ij}\varepsilon^{jk}\epsilon_k + C\varepsilon_{ij}\,\epsilon^j + \tfrac{1}{4}\big(\not{D} A\,\gamma^{ab} T_{abij} + w\,A\,\not{D}\gamma^{ab} T_{abij}\big)\varepsilon^{jk}\epsilon_k \\
&\quad - 3\,\gamma_a\varepsilon^{jk}\epsilon_k\,\bar\chi_{[i}\gamma^a\Psi_{j]} - (1+w)\,B_{ij}\varepsilon^{jk}\,\eta_k + \tfrac{1}{2}(1-w)\,\gamma^{ab}\,G^-_{ab}\eta_i \,, \\
\delta C &= -2\,\varepsilon^{ij}\bar\epsilon_i\not{D}\Lambda_j - 6\,\bar\epsilon_i\chi_j\,\varepsilon^{ik}\varepsilon^{jl} B_{kl} \\
&\quad - \tfrac{1}{4}\varepsilon^{ij}\varepsilon^{kl}\big((w-1)\,\bar\epsilon_i\gamma^{ab}\not{D} T_{abjk}\Psi_l + \bar\epsilon_i\gamma^{ab} T_{abjk}\not{D}\Psi_l\big) + 2\,w\varepsilon^{ij}\bar\eta_i\Lambda_j \,. \qquad \text{(B.2.16)}
\end{aligned}$$

Products of chiral superfields constitute again a chiral superfield, whose Weyl weight is equal to the sum of the Weyl weights of the separate multiplets. Also functions of chiral superfields may describe chiral superfields, assuming that they can be assigned a proper Weyl weight. In the local supersymmetry setting, we will usually be dealing with homogeneous functions of chiral multiplets with equal Weyl weight w so that a scaling weight equal to the product of w times the degree of homogeneity can be assigned to the function.

The product of two chiral multiplets, with components $\big(A, \Psi_i, B_{ij}, G^-_{ab}, \Lambda_i, C\big)$ and $\big(a, \psi_i, b_{ij}, g^-_{ab}, \lambda_i, c\big)$, respectively, leads to the following decomposition,

$$\begin{aligned}
&\big(A, \Psi_i, B_{ij}, G^-_{ab}, \Lambda_i, C\big) \otimes \big(a, \psi_i, b_{ij}, g^-_{ab}, \lambda_i, c\big) = \\
&\quad \big(A\,a\,,\ A\,\psi_i + a\,\Psi_i, A\,b_{ij} + a\,B_{ij} - \bar\Psi_{(i}\psi_{j)} \,, \\
&\quad A\,g^-_{ab} + a\,G^-_{ab} - \tfrac{1}{4}\varepsilon^{ij}\bar\Psi_i\gamma_{ab}\psi_j \,, \\
&\quad A\,\lambda_i + a\,\Lambda_i - \tfrac{1}{2}\varepsilon^{kl}(B_{ik}\,\psi_l + b_{ik}\,\Psi_l) - \tfrac{1}{4}(G^-_{ab}\gamma^{ab}\psi_i + g^-_{ab}\gamma^{ab}\Psi_i) \,, \\
&\quad A\,c + a\,C - \tfrac{1}{2}\varepsilon^{ik}\varepsilon^{jl} B_{ij}\,b_{kl} + G^-_{ab}\,g^{-ab} + \varepsilon^{ij}(\bar\Psi_i\lambda_j + \bar\psi_i\Lambda_j)\big) \,. \qquad \text{(B.2.17)}
\end{aligned}$$

	Vector multiplet			
field	X	Ω_i	W_μ	Y_{ij}
w	1	$\frac{3}{2}$	0	2
c	-1	$-\frac{1}{2}$	0	0
γ_5		$+$		
	Hypermultiplet			
field	$A_i{}^\alpha$		ζ^α	
w	1		$\frac{3}{2}$	
γ_5			$-$	

Table B.6: Weyl and chiral weights (w and c) and fermion chirality (γ_5) of the vector multiplet and hypermultiplet component fields.

Using this, one can show that a function $\mathcal{G}(\Phi)$ of chiral superfields Φ^I defines a chiral superfield, whose component fields take the following form,

$$\begin{aligned}
A|_{\mathcal{G}} &= \mathcal{G}(A)\,, \\
\Psi_i|_{\mathcal{G}} &= \mathcal{G}(A)_I\,\Psi_i{}^I\,, \\
B_{ij}|_{\mathcal{G}} &= \mathcal{G}(A)_I\,B_{ij}{}^I - \tfrac{1}{2}\mathcal{G}(A)_{IJ}\,\bar\Psi_{(i}{}^I\Psi_{j)}{}^J\,, \\
G^-_{ab}|_{\mathcal{G}} &= \mathcal{G}(A)_I\,G^-_{ab}{}^I - \tfrac{1}{8}\mathcal{G}(A)_{IJ}\,\varepsilon^{ij}\bar\Psi_i{}^I\gamma_{ab}\Psi_j{}^J\,, \\
\Lambda_i|_{\mathcal{G}} &= \mathcal{G}(A)_I\,\Lambda_i{}^I - \tfrac{1}{2}\mathcal{G}(A)_{IJ}\big[B_{ij}{}^I\varepsilon^{jk}\Psi_k{}^J + \tfrac{1}{2}G^-_{ab}{}^I\gamma^{ab}\Psi_k{}^J\big] \\
&\quad + \tfrac{1}{48}\mathcal{G}(A)_{IJK}\,\gamma^{ab}\Psi_i{}^I\,\varepsilon^{jk}\bar\Psi_j{}^J\gamma_{ab}\Psi_k{}^K\,, \\
C|_{\mathcal{G}} &= \mathcal{G}(A)_I\,C^I - \tfrac{1}{4}\mathcal{G}(A)_{IJ}\big[B_{ij}{}^I B_{kl}{}^J\,\varepsilon^{ik}\varepsilon^{jl} - 2\,G^-_{ab}{}^I G^{-abJ} + 4\,\varepsilon^{ik}\bar\Lambda_i{}^I\Psi_j{}^J\big]\,, \\
&\quad + \tfrac{1}{4}\mathcal{G}(A)_{IJK}\big[\varepsilon^{ik}\varepsilon^{jl}B_{ij}{}^I\Psi_k{}^J\Psi_l{}^K - \tfrac{1}{2}\varepsilon^{kl}\bar\Psi_k{}^I G^-_{ab}{}^J\gamma^{ab}\Psi_l{}^K\big] \\
&\quad + \tfrac{1}{192}\mathcal{G}(A)_{IJKL}\,\varepsilon^{ij}\bar\Psi_i{}^I\gamma_{ab}\Psi_j{}^J\,\varepsilon^{kl}\bar\Psi_k{}^K\gamma_{ab}\Psi_l{}^L\,.
\end{aligned} \tag{B.2.18}$$

B.2.3 Reduced chiral multiplets

Chiral multiplets of $w = 1$ are special, because they are reducible upon imposing a reality constraint [186, 187]. The two cases that are relevant are the vector multiplet, which arises upon reduction from a scalar chiral multiplet, and the Weyl multiplet, which is a reduced anti-selfdual chiral tensor multiplet. Both reduced multiplets require weight $w = 1$.

We will denote the components of the $w = 1$ multiplet that describes the vector multiplet by $(A, \Psi, B, G^-, \Lambda, C)|_{\text{vector}}$. The constraint for a scalar chiral superfield reads,

$\varepsilon^{ij}\,\bar{D}_i\gamma_{ab}D_j\Phi = [\varepsilon^{ij}\,\bar{D}_i\gamma_{ab}D_j\Phi]^*$, where D_i denotes the covariant derivative in superspace. This implies that $C|_{\text{vector}}$ and $\Lambda_i|_{\text{vector}}$ are expressed in terms of the lower components of the multiplet, and imposes a reality constraint on $B|_{\text{vector}}$ and a Bianchi identity on $G^-|_{\text{vector}}$ [46, 186, 187]. The latter implies that $G^-|_{\text{vector}}$ can be expressed in terms of a gauge field A_μ.

The reduced scalar chiral multiplet thus describes the covariant fields and field strength of a *vector multiplet*, which encompasses $8+8$ bosonic and fermionic components. Table B.6 summarizes the Weyl and chiral weights of the various fields belonging to the vector multiplet: a complex scalar X, a Majorana doublet spinor Ω_i, a vector gauge field A_μ, and a triplet of auxiliary fields Y_{ij}. The identification with the chiral multiplet components is as follows,

$$\begin{aligned}
A|_{\text{vector}} &= X\,,\\
\Psi_i|_{\text{vector}} &= \Omega_i\,,\\
B_{ij}|_{\text{vector}} &= Y_{ij} = \varepsilon_{ik}\varepsilon_{jl}Y^{kl}\,,\\
G^-_{ab}|_{\text{vector}} &= F^-_{ab} + \tfrac{1}{4}\big[\bar{\psi}_\rho{}^i\gamma_{ab}\gamma^\rho\Omega^j + \bar{X}\,\bar{\psi}_\rho{}^i\gamma^{\rho\sigma}\gamma_{ab}\psi_\sigma{}^j - \bar{X}\,T_{ab}{}^{ij}\big]\varepsilon_{ij}\,,\\
\Lambda_i|_{\text{vector}} &= -\,\varepsilon_{ij}\,\not{D}\Omega^j\\
C|_{\text{vector}} &= -\,2\,\Box_{\text{c}}\bar{X} - \tfrac{1}{4}G^+_{ab}\,T^{ab}{}_{ij}\varepsilon^{ij} - 3\,\bar{\chi}_i\Omega^i\,,
\end{aligned} \tag{B.2.19}$$

where $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$ is the field strength of the gauge field. The corresponding Bianchi identity on G_{ab} can be written as,

$$D^b\left(G^+_{ab} - G^-_{ab} + \tfrac{1}{4}XT_{abij}\varepsilon^{ij} - \tfrac{1}{4}\bar{X}T_{ab}{}^{ij}\varepsilon_{ij}\right) + \tfrac{3}{4}\left(\bar{\chi}_i\gamma_a\Omega_j\varepsilon^{ij} - \bar{\chi}^i\gamma_a\Omega^j\varepsilon_{ij}\right) = 0\,, \tag{B.2.20}$$

and the reality constraint on Y_{ij} is included in (B.2.19).

The Q- and S-supersymmetry transformations for the vector multiplet take the form,

$$\begin{aligned}
\delta X &= \bar{\epsilon}^i\Omega_i\,,\\
\delta\Omega_i &= 2\not{D}X\epsilon_i + \tfrac{1}{2}\varepsilon_{ij}G_{\mu\nu}\gamma^{\mu\nu}\epsilon^j + Y_{ij}\epsilon^j + 2X\eta_i\,,\\
\delta A_\mu &= \varepsilon^{ij}\bar{\epsilon}_i(\gamma_\mu\Omega_j + 2\,\psi_{\mu j}X) + \varepsilon_{ij}\bar{\epsilon}^i(\gamma_\mu\Omega^j + 2\,\psi_\mu{}^j\bar{X})\,,\\
\delta Y_{ij} &= 2\,\bar{\epsilon}_{(i}\not{D}\Omega_{j)} + 2\,\varepsilon_{ik}\varepsilon_{jl}\,\bar{\epsilon}^{(k}\not{D}\Omega^{l)}\,,
\end{aligned} \tag{B.2.21}$$

and, for $w=1$, are in clear correspondence with the supersymmetry transformations of generic scalar chiral multiplets given in (B.2.16).

Subsequently we turn to the Weyl multiplet, which is a chiral anti-selfdual tensor multiplet subject to $\bar{D}_i\gamma^{ab}D_j\,\Phi_{ab}{}^{ij} = [\bar{D}_i\gamma^{ab}D_j\,\Phi_{ab}{}^{ij}]^*$. Its chiral superfield components take the following form,

$$
\begin{aligned}
A_{ab}|_W &= T_{ab}{}^{ij}\varepsilon_{ij}\,, \\
\Psi_{abi}|_W &= 8\,\varepsilon_{ij}R(Q)^j_{ab}\,, \\
B_{abij}|_W &= -\,8\,\varepsilon_{k(i}R(\mathcal{V})^{-}_{ab}{}^{k}{}_{j)}\,, \\
\left(G^-_{ab}\right)^{cd}|_W &= -\,8\,\hat{R}(M)^{-cd}_{ab}\,, \\
\Lambda_{abi}|_W &= 8\left(\mathcal{R}(S)^-_{abi} + \tfrac{3}{4}\gamma_{ab}\not{D}\chi_i\right)\,, \\
C_{ab}|_W &= 4D_{[a}\,D^cT_{b]c\,ij}\varepsilon^{ij} - \text{dual}\,.
\end{aligned} \tag{B.2.22}
$$

We give the Q- and S-supersymmetry variations for the first few components,

$$
\begin{aligned}
\delta T_{ab}{}^{ij} &= 8\,\bar{\epsilon}^{[i}R(Q)_{ab}{}^{j]}\,, \\
\delta R(Q)_{ab}{}^{i} &= -\,\tfrac{1}{2}\not{D}T_{ab}{}^{ij}\,\epsilon_j + R(\mathcal{V})^-{}_{ab}{}^{i}{}_{j}\,\epsilon^j - \tfrac{1}{2}\hat{R}(M)_{ab}{}^{cd}\,\gamma_{cd}\epsilon^i + \tfrac{1}{8}T_{cd}{}^{ij}\,\gamma^{cd}\gamma_{ab}\,\eta_j\,, \\
\delta R(\mathcal{V})^-{}_{ab}{}^{i}{}_{j} &= 2\bar{\epsilon}_j\not{D}R(Q)_{ab}{}^{i} - 2\bar{\epsilon}^i\big(\mathcal{R}(S)^-_{abj} + \tfrac{3}{4}\gamma_{ab}\not{D}\chi_j\big) \\
&\quad + \bar{\eta}_j(2R(Q)_{ab}{}^{i} + 3\gamma_{ab}\chi^i) - (\text{traceless})\,, \\
\delta\hat{R}(M)^{-cd}_{ab} &= \tfrac{1}{2}\bar{\epsilon}_i\not{D}\gamma^{cd}R(Q)_{ab}{}^{i} - \tfrac{1}{2}\bar{\epsilon}^i\gamma^{cd}\big(\mathcal{R}(S)^-_{abi} + \tfrac{3}{4}\gamma_{ab}\not{D}\chi_i\big) \\
&\quad - \bar{\eta}_i\gamma_{ab}R(Q)^{cdi} - \tfrac{1}{2}\bar{\eta}_i\gamma^{cd}R(Q)_{ab}{}^{i} - \tfrac{3}{4}\bar{\eta}_i\gamma_{ab}\gamma^{cd}\chi^i\,.
\end{aligned} \tag{B.2.23}
$$

A scalar chiral multiplet with $w = 2$ is obtained by squaring the Weyl multiplet. The various scalar chiral multiplet components are given by,

$$
\begin{aligned}
\hat{A} &= (T_{ab}{}^{ij}\varepsilon_{ij})^2\,, \\
\hat{\Psi}_i &= 16\,\varepsilon_{ij}R(Q)^j_{ab}\,T^{klab}\,\varepsilon_{kl}\,, \\
\hat{B}_{ij} &= -\,16\,\varepsilon_{k(i}R(\mathcal{V})^{k}{}_{j)ab}\,T^{lmab}\,\varepsilon_{lm} - 64\,\varepsilon_{ik}\varepsilon_{jl}\,\bar{R}(Q)_{ab}{}^{k}\,R(Q)^{l\,ab}\,, \\
\hat{G}^{-ab} &= -\,16\,\hat{R}(M)_{cd}{}^{ab}\,T^{klcd}\,\varepsilon_{kl} - 16\,\varepsilon_{ij}\,\bar{R}(Q)^i_{cd}\gamma^{ab}R(Q)^{cd\,j}\,, \\
\hat{\Lambda}_i &= 32\,\varepsilon_{ij}\,\gamma^{ab}R(Q)^j_{cd}\,\hat{R}(M)^{cd}{}_{ab} + 16\,(\mathcal{R}(S)_{ab\,i} + 3\gamma_{[a}D_{b]}\chi_i)\,T^{klab}\,\varepsilon_{kl} \\
&\quad - 64\,R(\mathcal{V})_{ab}{}^{k}{}_{i}\,\varepsilon_{kl}\,R(Q)^{ab\,l}\,, \\
\hat{C} &= 64\,\hat{R}(M)^{-cd}{}_{ab}\,\hat{R}(M)^{-ab}_{cd} + 32\,R(\mathcal{V})^{-ab\,k}{}_{l}\,R(\mathcal{V})^{-l}_{ab\,k} \\
&\quad - 32\,T^{ab\,ij}\,D_a\,D^cT_{cb\,ij} + 128\,\bar{\mathcal{R}}(S)^{ab}{}_{i}\,R(Q)_{ab}{}^{i} + 384\,\bar{R}(Q)^{ab\,i}\gamma_aD_b\chi_i\,.
\end{aligned} \tag{B.2.24}
$$

These components can straightforwardly be substituted in the expression for the higher-derivative couplings.

B.2.4 Hypermultiplets

Hypermultiplets in four dimensions [34, 184] are again on-shell supermultiplets and have the same structure as in five dimensions, described in B.1.4. In terms of the sections $A_i{}^{\alpha}(\phi)$ ($\alpha = 1, 2, \ldots, 2r$) of the Sp($r$)$\times$Sp(1) bundle, the supersymmetry transformations read

$$\begin{aligned} \delta A_i{}^\alpha &= 2\,\bar{\epsilon}_i \zeta^\alpha + 2\,\varepsilon_{ij}\, G^{\alpha\bar{\beta}} \Omega_{\bar{\beta}\bar{\gamma}}\, \bar{\epsilon}^j \zeta^{\bar{\gamma}} \,, \\ \delta \zeta^\alpha &= \not{D}\, A_i{}^\alpha \epsilon^i + A_i{}^\alpha\, \eta^i \,. \end{aligned} \tag{B.2.25}$$

Again, one can show the existence of a covariantly constant skew-symmetric tensor $\Omega_{\alpha\beta}$ and the covariant derivatives contain the Sp(r) connection $\Gamma_A{}^\alpha{}_\beta$. The hyperkähler potential reads

$$\varepsilon_{ij}\, \chi = \Omega_{\alpha\beta}\, A_i{}^\alpha A_j{}^\beta \,. \tag{B.2.26}$$

For the local scale transformations we refer again to the weights shown in table B.6.

Appendix C

The Noether potential and conserved charges

In order to deal with black holes in higher derivative theories, it is important to use generic definitions of quantities such as the entropy. We therefore give a short summary of the Wald formalism and the derivation of the first law of black hole thermodynamics [82, 98, 99]. The following discussion is applicable for any theory of gravity described by a Lagrangian that contains arbitrary combinations of the Riemann tensor, as the Lagrangians that appear throughout this thesis.

C.1 The Noether potential

Consider a generic Lagrangian $\mathcal{L}$ in D dimensions that depends on fields that we collectively call ϕ and their derivatives. For the applications in this thesis it will always be assumed to contain Maxwell-Einstein theory as a subsector, in terms of a vielbein e_μ^a and one or more gauge fields A_μ. We assume the theory defined by the corresponding action to admit a number of local symmetries and we collectively denote the associated transformation parameters by Ξ. In particular, we use a vector ξ^μ for diffeomorphisms, an antisymmetric tensor ε^{ab} for local Lorentz and a function ξ for abelian gauge symmetry.

In general, under a symmetry transformation $\delta_\Xi \phi$ of the fields, the Lagrangian will transform as

$$\delta_\Xi \mathcal{L} = \partial_\mu N^\mu_\Xi \,, \tag{C.1.1}$$

where N^μ is a vector linear in Ξ. As the theory contains gravity this always includes $\xi^\mu \mathcal{L}$, but if the Lagrangian contains e.g. gauge non-invariant terms it will also contain ξ and similarly for other cases. On the other hand, one can perform a general variation of the action to obtain the equations of motion E:

$$\delta\mathcal{L} = E\delta\phi + \partial_\mu\theta^\mu(\delta\phi)\,, \tag{C.1.2}$$

up to a boundary term linear in the field variations $\delta\phi$, that we indicate by θ^μ. Of course, when the generic variation is assumed to be a symmetry variation, the two expressions must coincide:

$$\partial_\mu N^\mu_\Xi = E\delta\phi + \partial_\mu\theta^\mu(\delta_\Xi\phi)\,. \tag{C.1.3}$$

It then follows that there exists a current associated with any field configuration:

$$J^\mu = \theta^\mu(\delta_\Xi\phi) - N^\mu_\Xi \quad \Rightarrow \quad \partial_\mu J^\mu = -E\delta\phi\,, \tag{C.1.4}$$

which is conserved when the configuration is a solution to the equations of motion. This is known as the Noether current associated to the symmetry Ξ. For a rigid symmetry, that is when some of the parameters Ξ are necessarily constant and do not appear in N^μ_Ξ, this definition coincides with the conserved current originally introduced by Noether and can be used to define conserved charges.

For the local case the situation is more subtle, and one has to consider the action of symmetries generated by Ξ on the full phase space, or in other words on the space of all solutions viewed as a manifold. The generator of symmetries on this space is expressed through another conserved current:

$$\Omega^\mu(\delta\phi, \delta_\Xi\phi) = \delta\theta^\mu(\delta_\Xi\phi) - \delta_\Xi\theta^\mu(\delta\phi) \tag{C.1.5}$$

and is identified with the variation of the corresponding Hamiltonian associated with the symmetries. As we expect a conserved charge to appear whenever there is a symmetry of the solution at hand, $\delta_\Xi\phi$ and consequently Ω should vanish. This can be computed by variation of (C.1.4), as

$$\Omega^\mu(\delta\phi, \delta_\Xi\phi) = \delta J^\mu - \Pi^\mu_\Xi\,, \tag{C.1.6}$$

$$\Pi^\mu_\Xi \equiv \delta_\Xi\theta^\mu(\delta\phi) - \delta N^\mu_\Xi\,. \tag{C.1.7}$$

As shown in [188], any conserved current locally constructed from fields can be written as the divergence of an antisymmetric tensor, using the equations of motion. It follows that one can locally define the so called Noether potential through

$$J^\mu = \partial_\nu Q^{\mu\nu}\,, \tag{C.1.8}$$

and similarly for Π_Ξ

$$\Pi^\mu_\Xi = \partial_\nu \delta C^{\mu\nu}_\Xi\,, \tag{C.1.9}$$

which also depend linearly on Ξ. The existence of these tensors allows for a definition of the charge associated with symmetric backgrounds, for which Ω vanishes, in the following way. One can compute the integral of (C.1.6) over the total spacial manifold Σ as

$$\int_\Sigma \Omega = \delta \int_{S_1} (Q - C_\Xi) - \delta \int_{S_2} (Q - C_\Xi) = 0\,, \tag{C.1.10}$$

where we used the Gauss theorem and $S_{1,2}$ are $D-2$-dimensional spacial hypersurfaces. The conserved charge can then be defined through

$$\mathcal{Q} = \int_S (Q - C_\Xi)\ , \tag{C.1.11}$$

which is independent of the hypersurface. This is the central result on which all applications are based.

It is worthwhile commenting on the ambiguities introduced in the process of defining the quantities above. Firstly, both θ and N_Ξ are defined up to the divergence of a rank-two antisymmetric tensor, as in (C.1.8), and these ambiguities affect the Noether current and potential. However, one can easily see that the ambiguity of θ leads to terms in (C.1.6) proportional to $\delta_\Xi \phi$, which vanishes on symmetric backgrounds, whereas the shift on N_Ξ drops out. Furthermore, there is an ambiguity in the definition of $Q^{\mu\nu}$ and $C_\Xi^{\mu\nu}$ by the divergence of a rank-three antisymmetric tensor. Such ambiguities drop out from the integral in (C.1.11), provided that S is closed and the fields are well defined on it.

C.2 Applications

One can write concrete expressions for the formal quantities above for specific cases. Here we concentrate on gauge symmetry and diffeomorphisms. For later convenience, consider a five-dimensional Lagrangian that contains two parts, one that depends on the Riemann tensor and an abelian gauge field strength and its derivatives and a Chern-Simons part:

$$\mathcal{L} = \mathcal{L}_0(R_{\mu\nu\rho\sigma}, F_{\mu\nu}, \nabla_\rho F_{\mu\nu}) + \mathcal{L}_{CS}\ , \tag{C.2.1}$$

$$\mathcal{L}_{CS} = \alpha\, \varepsilon^{\mu\nu\rho\sigma\tau} A_\mu F_{\nu\rho} F_{\sigma\tau} + \beta\, \varepsilon^{\mu\nu\rho\sigma\tau} A_\mu R_{\nu\rho}{}^{ab}(M)\, R_{\sigma\tau ab}(M)\,, \tag{C.2.2}$$

where α, β are constants and the spin connection curvature $R_{\mu\nu}{}^{ab}(M)$ can be replaced by the Riemann tensor.

C.2.1 Gauge symmetry

The first part of the Lagrangian (C.2.1) is invariant under gauge transformations, but the Chern-Simons terms vary by a total derivative, so that we have

$$N^\mu = \alpha\, \xi \varepsilon^{\mu\nu\rho\sigma\tau} F_{\nu\rho} F_{\sigma\tau} + \beta\, \xi \varepsilon^{\mu\nu\rho\sigma\tau} R_{\nu\rho}{}^{ab}(M)\, R_{\sigma\tau ab}(M) \tag{C.2.3}$$

$$\theta^\mu(\delta\phi) = 2\,\mathcal{L}_0^{\mu\nu} \delta A_\nu - 2\,\nabla_\rho \mathcal{L}_0^{(\mu,\rho)\nu} \delta A_\nu + 2\,\mathcal{L}_0^{(\mu,\rho)\nu} \nabla_\rho \delta A_\nu + 4\alpha\, \varepsilon^{\mu\nu\rho\sigma\tau} \delta A_\nu A_\rho F_{\sigma\tau}\,, \tag{C.2.4}$$

where

$$\mathcal{L}_0^{\mu\nu} = \frac{\partial \mathcal{L}_0}{\partial F_{\mu\nu}}\,, \quad \mathcal{L}_0^{\rho,\mu\nu} = \frac{\partial \mathcal{L}_0}{\partial \nabla_\rho F_{\mu\nu}}\,. \tag{C.2.5}$$

Straightforward application of the above formulae shows that the only term contributing to Π_Ξ is the cubic Chern-Simons term. The final expression for the conserved electric charge reads

$$q = \int_S dS_{\mu\nu} \Big[2\,\xi\, \mathcal{L}_0^{\mu\nu} - 2\,\xi \nabla_\rho \mathcal{L}_0^{\mu,\rho\nu} + \mathcal{L}_0^{\rho,\mu\nu} \partial_\rho \xi + 6\alpha\, \xi \varepsilon^{\mu\nu\rho\sigma\tau} A_\rho F_{\sigma\tau} \\ + 4\,\beta\, \varepsilon^{\mu\nu\rho\sigma\tau}\, \omega_\rho{}^{ab} \left(\partial_\sigma \omega_{\tau ab} - \tfrac{2}{3} \omega_{\sigma ac}\, \omega_\tau{}^c{}_b \right) \Big]\,, \tag{C.2.6}$$

which should be evaluated on a symmetric background ($\partial\xi = 0$) and $dS_{\mu\nu}$ denotes the surface element on S. The last term can be replaced by the Chern-Simons term involving the Christoffel symbol, depending on the application. This is not an ambiguity of the Noether charge, since one can convert the corresponding term in the Lagrangian into a gauge invariant term proportional to either the Christoffel or spin connection Chern-Simons form by partial integration and combine it with $\mathcal{L}_0$. Then, (C.2.6) formally gives the same result, even though such manipulations may not be well defined for backgrounds containing magnetic and/or Taub-NUT charge.

C.2.2 Diffeomorphisms and the first law

We now turn to the Noether potential associated to diffeomorphisms for the same Lagrangian in (C.2.1). For the moment we consider the mixed Chern-Simons term in the metric formulation, assuming the gauge field is globally defined. After a diffeomorphism and a general variation of the Lagrangian, one finds $N^\mu = \xi^\mu \mathcal{L}$ and

$$\theta^\mu(\delta\phi) = 2\left(\mathcal{L}^{\mu\nu\rho\sigma} \nabla_\rho \delta g_{\sigma\nu} - \nabla_\rho \mathcal{L}^{\rho\nu\mu\sigma} \delta g_{\sigma\nu} \right) \\ + 2\,\mathcal{L}^{\mu\nu} \delta A_\nu - 2\,\nabla_\rho \mathcal{L}^{(\mu,\rho)\nu} \delta A_\nu + 2\,\mathcal{L}^{(\mu,\rho)\nu} \nabla_\rho \delta A_\nu\,, \tag{C.2.7}$$

where now

$$\mathcal{L}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}\,, \quad \mathcal{L}^{\rho,\mu\nu} = \frac{\partial \mathcal{L}}{\partial \nabla_\rho F_{\mu\nu}}\,, \quad \mathcal{L}^{\mu\nu\rho\sigma} = \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\,. \tag{C.2.8}$$

Using these results, the vector Π_ξ takes the form:

$$\Pi^\mu = \delta_\xi \theta^\mu(\delta\phi) - \xi^\mu \delta\mathcal{L} = \delta_\xi \theta^\mu(\delta\phi) - \xi^\mu \partial_\nu \theta^\nu(\delta\phi) = 2\,\partial_\nu \left(\xi^{[\mu}\,\theta^{\nu]}(\delta\phi)\right) , \qquad \text{(C.2.9)}$$

whereas the Noether potential reads

$$\begin{aligned} Q^{\mu\nu} \;&=\; 2\,\mathcal{L}^{\mu\nu\rho\sigma}\nabla_\rho\xi_\sigma - 4\,\nabla_\rho\mathcal{L}^{\mu\nu\rho\sigma}\xi_\sigma \\ &\quad + 2\,(\xi^\sigma A_\sigma)\,\mathcal{L}^{\mu\nu} - 2\,(\xi^\sigma A_\sigma)\,\nabla_\rho\mathcal{L}^{\mu,\rho\nu} + \mathcal{L}^{\rho,\mu\nu}\partial_\rho(\xi^\sigma A_\sigma)\,. \end{aligned} \qquad \text{(C.2.10)}$$

Note that these formulae are generic and no special care is taken for the Chern-Simons terms in (C.2.1), which are fully diffeomorphism covariant, but contribute with terms proportional to explicit gauge fields. This is unavoidable for the gauge Chern-Simons term, but not for the mixed term, as we will see later.

The last three terms in (C.2.10) are clearly related to the gauge symmetry Noether potential in (C.2.6), if one omits the Chern-Simons term. The reason is the following decomposition of the diffeomorphisms on a gauge field,

$$\delta_\xi W_\mu{}^I = -\partial_\mu \xi^\nu W_\nu{}^I - \xi^\nu \partial_\nu W_\mu{}^I = \xi^\nu F_{\mu\nu}{}^I + \partial_\mu(-\xi^\nu W_\nu{}^I)\,. \qquad \text{(C.2.11)}$$

which is transferred to the Noether potential, by linearity. This holds for all covariant terms in the action, but not for Chern-Simons terms, which must necessarily be treated using (C.2.10). The relative factor $\xi^\mu W_\mu$ is necessarily constant near infinity [108] and is identified with the electrostatic potential Φ in (1.1.7).

It follows that the integral of (C.1.6) takes the form

$$\int_\Sigma \left[\delta\partial_\nu\, Q^{\mu\nu} - 2\,\partial_\nu \left(\xi^{[\mu}\,\theta^{\nu]}(\delta\phi)\right)\right] = 0\,, \qquad \text{(C.2.12)}$$

so that if one can find a C_ξ such that $\delta C_\xi^{\mu\nu} = 2\xi^{[\mu}\theta^{\nu]}(\delta\phi)$, the following relation holds for variations of the fields:

$$\delta \int_{S_1} dS_{\mu\nu} \left(Q^{\mu\nu} - C_\xi^{\mu\nu}\right) = \delta \int_{S_2} dS_{\mu\nu} \left(Q^{\mu\nu} - C_\xi^{\mu\nu}\right)\,. \qquad \text{(C.2.13)}$$

Evaluated on a symmetric background, which possesses a Killing vector ξ, the quantity under variation is identified with the corresponding conserved charge. In the presence of a Killing horizon, evaluation of the two sides of this relation on the horizon H and at infinity lead to the first law of black hole mechanics as follows. Let the horizon be generated by the Killing vector $\xi = \xi_t + \Omega\xi_\phi$, where ξ_t, ξ_ϕ are the generators of time translation and rotations and Ω is the angular velocity of the horizon. Using the comments below (C.2.11) and the charge in (C.2.6), we find

$$\delta \int_H (Q_\xi - C_\xi) = \delta \int_\infty (Q_{\xi_t} - C_{\xi_t}) + \Omega\,\delta \int_\infty \left(Q_{\xi_\phi} - C_{\xi_\phi}\right) + \Phi\,\delta q\,, \qquad \text{(C.2.14)}$$

where the two integrals in the right hand side are identified with the total mass M and angular momentum J. Generically, there exists a special cross section of the horizon, the bifurcation surface, on which $\xi = 0$ and $\nabla_\mu \xi_\nu = \kappa \epsilon_{\mu\nu}$, where κ is the surface gravity and $\epsilon_{\mu\nu}$ is the binormal on this surface. Using this surface in the integral on the left hand side of (C.2.14), one finds

$$\kappa \delta \mathcal{S} = \delta M + \Omega\, \delta J + \Phi\, \delta q\,. \tag{C.2.15}$$

This relation is identified as the first law of black hole mechanics, with the entropy given by

$$\mathcal{S} = \int_H \epsilon_{\mu\nu} \left(2\, \mathcal{L}^{\mu\nu\rho\sigma} \epsilon_{\rho\sigma} + \mathcal{L}^{\rho,\mu\nu} \epsilon_{\rho\sigma} A^\sigma\right)\,. \tag{C.2.16}$$

The result is always proportional to the area of the horizon, but details vary depending on the theory. In particular, for pure General Relativity only the first term is present and it is proportional to the metric, so that the result is exactly equal to the area, up to normalisation. More generally, this result holds for all theories involving any power of the Riemann tensor and the derivatives of the field strength, which is the case for Lagrangians considered in this thesis.

C.2.3 Local Lorentz symmetry

Until now we have been cavalier about the presence of explicit gauge fields in the expressions for the Noether potential, silently assuming they were globally defined. This is not true for backgrounds that contain magnetic charges and appear in applications. In some cases, such terms are unavoidable and one has to carefully perform the corresponding final integration on different gauge patches, perhaps adding boundary terms. One example where one can avoid them is the mixed gauge/gravitational Chern-Simons term in (C.2.1). By partial integration, this can be written as

$$\mathcal{L}_{\rm CS} = -2\beta\, \varepsilon^{\mu\nu\rho\sigma\tau}\, F_{\mu\nu} \omega_\rho{}^{ab} \left(\partial_\sigma \omega_{\tau ab} - \tfrac{2}{3} \omega_{\sigma ac}\, \omega_\tau{}^c{}_b\right)\,, \tag{C.2.17}$$

where we chose to use the spin connection because this is the natural object in supergravity, unlike the Christoffel connection. This form of the Lagrangian is no longer invariant under local Lorentz transformations, but transforms into a boundary term as a result of the explicit spin-connection, but is manifestly gauge invariant. In this setting one takes the vielbein as the fundamental variable, whereas diffeomorphisms are extended by local Lorentz transformations that act on the flat indices.

As the algorithm for constructing the Noether potential is linear, we concentrate here on the contribution from the term (C.2.17). The relevant variations result in

$$
\begin{aligned}
N^\mu_\xi =& -2\beta\,\xi^\mu\,\varepsilon^{\lambda\nu\rho\sigma\tau}\,F_{\lambda\nu}\omega_\rho{}^{ab}\left(\partial_\sigma\omega_{\tau ab}-\tfrac{2}{3}\omega_{\sigma ac}\,\omega_\tau{}^c{}_b\right)+2\beta\,\varepsilon^{\mu\nu\rho\sigma\tau}F_{\nu\rho}\partial_\sigma\varepsilon^{ab}\,\omega_{\tau ab} \qquad \text{(C.2.18)}\\
\theta^\mu =& -4\beta\,\varepsilon^{\mu\nu\rho\sigma\tau}\delta A_\nu\omega_\rho{}^{ab}(\partial_\sigma\omega_{\tau ab}-\tfrac{2}{3}\omega_{\sigma ac}\,\omega_\tau{}^c{}_b)+2\beta\,\varepsilon^{\mu\nu\rho\sigma\tau}F_{\nu\rho}\,\omega_\sigma{}^{ab}\,\delta\omega_{\tau ab}\\
&-2\beta\,\varepsilon^{\lambda\nu\rho\sigma\tau}F_{\nu\rho}R_{\sigma\tau}{}^{ab}(e_{\lambda c}e_a{}^\mu e_b{}^\beta\,\delta e_\beta{}^c+2e_a{}^\mu\,\delta e_{\lambda b})\,. \qquad \text{(C.2.19)}
\end{aligned}
$$

In this case, $C^{\mu\nu}$ acquires a novel term:

$$
\delta C^{\mu\nu}_\xi = 2\,\xi^{[\mu}\,\theta^{\nu]}(\delta_\xi\phi)+16\,\beta\,\varepsilon^{\mu\nu\rho\sigma\tau}\,\delta A_\rho\,\partial_\sigma\varepsilon^{ab}\,\omega_{\tau\,ab}\,, \qquad \text{(C.2.20)}
$$

whereas the Noether potential reads

$$
\begin{aligned}
\mathcal{Q}^{\mu\nu} \;=\;& 4\beta\,\varepsilon^{\mu\nu\rho\sigma\tau}\,F_{\rho\sigma}\,\omega_\tau{}^{ab}(\varepsilon_{ab}-\tfrac{1}{2}\xi^\lambda\omega_{\lambda ab})\\
&-4\beta\,\varepsilon^{\mu\nu\rho\sigma\tau}\,\xi^\lambda A_\lambda\,\omega_\rho{}^{ab}\left(\partial_\sigma\omega_{\tau ab}-\tfrac{2}{3}\omega_{\sigma ac}\,\omega_\tau{}^c{}_b\right)\\
&+4\beta\,\varepsilon^{\rho\sigma\tau\kappa[\mu}\,F_{\rho\sigma}\,R_{\tau\kappa}{}^{\nu]\lambda}\,\xi_\lambda-2\,\varepsilon^{\rho\sigma\tau\kappa\lambda}\,F_{\rho\sigma}\,R_{\tau\kappa}{}^{\mu\nu}\,\xi_\lambda\,. \qquad \text{(C.2.21)}
\end{aligned}
$$

The integral of these quantities over a closed surface as in (C.1.11) leads to the conserved Noether charge for diffeomorphisms extended by local Lorentz transformations, provided that the gauge parameters are taken to be those of a symmetric background. These are again described by a Killing vector ξ^μ, and a Lorentz parameter given by

$$
\xi^{ab} = -\nabla^{[a}\xi^{b]}+\xi^\lambda\omega_\lambda{}^{ab}\,. \qquad \text{(C.2.22)}
$$

It is simple to verify that the covariant terms in the Noether potential agree with the corresponding ones in the metric formulation. On the other hand, this result is gauge invariant but features explicit spin connections, therefore it is applicable in cases where the gauge fields are not globally defined but the spin connections are.

Finally, we should briefly comment on the ambiguity in (4.3.21) related to the fact that the extraction of the derivative ∂_μ in (C.2.18) is not well motivated for the second term, as we could have also left the derivative on the spin connection field $\omega_\tau{}^{ab}$ and extracted the derivative from the transformation parameter ε^{ab}. The choice made above can be justified along the lines of [100], which is consistent with the original description of Wald [82, 99]. Due to the nonstandard term (C.2.20), the generic variation (C.1.6) equals,

$$
\begin{aligned}
8\pi^2\delta J^\mu(\phi,\xi,\varepsilon) \;=\;& \partial_\nu\left[\xi^\mu\theta^\nu(\phi,\delta\phi)-\xi^\nu\theta^\mu(\phi,\delta\phi)\right]+\Omega^\mu(\phi;\delta_\xi\phi,\delta\phi)\\
&+16\partial_\nu\left[\varepsilon^{\mu\nu\rho\sigma\tau}\delta A_\rho\,\partial_\sigma\varepsilon^{ab}\,\omega_{\tau ab}\right], \qquad \text{(C.2.23)}
\end{aligned}
$$

where δ_ξ denotes the combined effect of both diffeomorphisms and Lorentz transformation. The variations $\delta\phi$ and δA_μ connect two nearby solutions. At this point the diffeomorphism and the Lorentz transformation are arbitrary and do not have to constitute an invariance of the field configuration.

The first term on the right-hand side is generic, as seen in (C.2.20), and does not change the entropy because it does not involve derivatives of ξ^μ. Furthermore, it gives no contribution to the variations of the angular momenta at spatial infinity [99]. Actually, the form of this term ensures that the angular momenta can be determined from the Noether potential and remain constant as a function of the distance from the horizon [107].

The hope is that the third term in (C.2.23) will behave in the same way. This term will also lead to modifications of the Noether potential, and since it depends on ξ^μ as well as on its derivatives, these modifications may affect the entropy. However, it is easy to see that this will not be the case, because the relevant ε^{ab} at the horizon is precisely the bi-normal tensor, whose derivatives vanish. Therefore the third term in (C.2.23) will not lead to extra terms in the entropy. For the angular momenta, the situation is similar but more subtle. In that case the combination $\partial_\sigma \varepsilon^{ab}\,\omega_{\tau ab}$ vanishes at the horizon due to kinematical reasons, except for $\partial_\theta \varepsilon^{ab}\,\omega_{\varphi ab} \propto \sin\theta\cos\theta$. Therefore this term vanishes upon integration over the horizon for all $\delta W_\rho{}^I$ that are allowed. Hence the angular momenta at the horizon are not modified and can be determined from the Noether potential obtained earlier. An obvious question is, whether the angular momentum whose variation appears in the first law, and which is measured at spatial infinity, will coincide with the angular momenta determined at the horizon. The answer to this question is not known, but the results presented in section 4.5 indicate that the answer is affirmative. Obviously a full derivation of the first law for the ring geometry is subtle in the presence of higher-derivative couplings. Without the latter, the derivation of the first law has already been pursued in [108] in connection with the presence of the dipole charges.

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Samenvatting

In de afgelopen jaren is er hernieuwde belangstelling ontstaan voor de studie van zwarte gaten als resultaat van een aantal boeiende ontwikkelingen. Tegenwoordig wordt algemeen aanvaard dat de meeste, zo niet alle, centra van sterrenstelsels zwarte gaten bevatten. Deze zwarte gaten zijn verantwoordelijk voor enkele van de meest boeiende verschijnselen die waargenomen worden in deze gebieden. Bovendien is het bekend dat sterren met een massa groter dan een bepaalde limiet aan het einde van hun levenscyclus onder invloed van de zwaartekracht ineenstorten tot een zwart gat. Dit maakt een eind aan een lange periode van speculaties over het bestaan van dergelijke objecten die begon kort na de ontdekking van Newtons gravitatiewet en een hoogtepunt bereikte met de komst van de algemene relativiteitstheorie.

Beide theorieën voorspellen dat als een object van een gegeven massa klein genoeg is, men met een snelheid groter dan die van licht zou moeten reizen om te ontsnappen aan de aantrekkingskracht op het oppervlak van het object. Een dergelijk object zou volledig donker lijken voor een externe waarnemer. Dit werd lange tijd beschouwd als een wiskundige curiositeit, totdat Einsteins relativiteitstheorie aantoonde dat de snelheid van het licht de hoogst mogelijke snelheid is in de natuur. Eén van de implicaties van deze limiet is, dat als licht niet kan ontsnappen aan een object, niets dat kan. Deze unieke eigenschap is zo interessant en verreikend, dat het onderzoek aan zwarte gaten, zoals deze objecten later werden genoemd, een van de meest actieve gebieden in de natuurkunde werd gedurende de laatste eeuw.

Vanuit een theoretisch oogpunt zijn verschillende aspecten van zwarte gaten van belang. Niet alleen zijn zwarte gaten alomtegenwoordig in alle theorieën van de zwaartekracht, maar volgens de zogenaamde uniciteitstellingen, worden ze over het algemeen beschreven door een klein aantal parameters. Dit betekent dat als men van een afstand de globale eigenschappen van een zwart gat meet, zoals de massa, het impulsmoment en de totale lading, de volledige structuur van het object uniek is vastgelegd. Algemene relativiteitstheorie voorspelt dat voorbij het punt van waaruit licht niet meer kan ontsnappen, genaamd de waarnemingshorizon, elke waarnemer steevast het centrum zal bereiken, een punt van oneindige dichtheid, bekend als de singulariteit.

De aanwezigheid van een singulariteit geeft aan dat de klassieke algemene relativiteitstheorie van Einstein niet meer toereikend is. Hierdoor ontstaat de behoefte aan een theorie die zwaartekracht op de allerkleinste schaal kan beschrijven. In dit domein zijn kwantumeffecten onvermijdelijk, zodat een kwantumtheorie van de zwaartekracht benodigd is. De zoektocht naar deze theorie is nog niet bëindigd, omdat er verschillende kandidaten zijn die elk hun voor- en nadelen hebben, en een volledig bevredigende theorie die algemene relativiteitstheorie met kwantummechanica verenigd ontbreekt nog steeds. Eén van de eisen op een kwantumtheorie van de zwaartekracht is dat het de globale eigenschappen van een zwart gat moet kunnen reproduceren en alle singulariteiten moet oplossen door middel van kwantumeffecten. Met behulp van de hierboven vermeldde eigenschappen, kan men stellen dat een zwart gat het eenvoudigste, niet-triviale, interacterende systeem is in een theorie van kwantumzwaartekracht, want het zou volledig moeten worden vastgelegd door de globale parameters.

Om ervoor te zorgen dat een oplossing die een zwart gat beschrijft zinnig is, zijn er meestal beperkingen op de hierboven genoemde parameters. De belangrijkste restrictie komt uit de eis dat er een eindige horizon moet zijn, die de singulariteit in het centrum van het zwarte gat verbergt. Dit staat ook wel bekend als kosmische censuur. Voor een zwart gat met massa M en elektrische lading Q, vindt men dat de massa moet voldoen aan de relatie,

$$M \geq |Q| \,, \tag{i}$$

in de juiste eenheden. Hieruit volgt dat er een positief minimum is in het massaspectrum van fysische, geladen zwarte gaten. In de aanwezigheid van impulsmoment en/of andere ladingen bestaan soortgelijke begrenzingen. Zwarte gaten die gelijke massa en lading hebben, en dus (i) verzadigen, worden extremaal genoemd en zijn het onderwerp van dit proefschrift. Deze oplossingen zijn interessant vanuit een theoretisch perspectief, omdat het onmogelijk zou moeten zijn dat deze zwarte gaten massa/energie verliezen door een fysisch proces, zolang de singulariteit in het centrum verborgen blijft door de horizon. Met andere woorden, ze vertegenwoordigen een soort grondtoestand voor alle zwarte gaten, iets wat naar verwachting ook kwantummechanisch geldt.

Een hint naar een microscopische theorie die zwarte gaten beschrijft wordt gegeven door de zogenaamde vier wetten van de mechanica van zwarte gaten, ontdekt in de jaren ’70. Het blijkt dat onder bepaalde algemene aannames, het mogelijk is om beperkingen af te leiden op de variaties van behouden grootheden die een continue verscheidenheid van zwarte gaten parametriseren. De resulterende vergelijkingen zijn in opvallende gelijkenis met de conventionele wetten van de thermodynamica als men een temperatuur en een entropie toekent aan het zwarte gat. Zodoende moet de microscopische beschrijving van een zwart gat begrepen worden in termen van een statistisch systeem met een groot aantal vrijheidsgraden, analoog aan de atomen van een gas. Een goede test op een kandidaat voor een kwantumzwaartekrachttheorie is daarom dat de microscopische vrijheidsgraden tot de macroscopische entropie kunnen leiden, als de wetten van de statistische mechanica worden toegepast.

Snaartheorie is een kandidaat voor een kwantumtheorie van de zwaartekracht, en voorziet ons van theoretische modellen van de kwantumstructuur van een klasse van extremale zwarte gaten. Volgens de snaartheorie, kan alle materie worden beschreven door middel van bepaalde oscillaties van ruimtelijk verlengde objecten, zogenaamde snaren. Bij lage energieën en grote afstanden ten opzichte van hun lengte, kunnen snaren worden gezien als puntdeeltjes, zodat de snaartheorie kan worden benaderd door een klasse van effectieve veldentheorieën van de zwaartekracht gekoppeld aan materie. Deze theorieën zijn supersymmetrisch, net zoals de snaartheorie, en worden meestal supergravitatietheorieën genoemd. Supersymmetrische theorieën zijn een interessant onderwerp op zichzelf, omdat bosonen en fermionen in hetzelfde multiplet voorkomen en ze onderling roteren. Hieruit volgt dat de parameters van supersymmetrietransformaties zelf spin moeten dragen. Ze worden meestal gekozen in de kleinst mogelijke spinorrepresentatie. Het aantal van deze onafhankelijke parameters, $\mathcal{N}$, wordt gebruikt om de hoeveelheid supersymmetrie te karakteriseren in een theorie.

Dit proefschrift is gewijd aan het vergelijken van de macroscopische eigenschappen van extremale zwarte gaten met theoretische voorspellingen vanuit de snaartheorie. De focus ligt op een bepaalde klasse van supersymmetrische zwaartekrachttheorieën in vier en vijf ruimtetijd dimensies, die bekend staan als $\mathcal{N} = 2$ supergravitatietheorieën. De aanwezigheid van supersymmetrie in deze theorieën vergemakkelijkt de analyse op verschillende manieren. Men kan bijvoorbeeld de aandacht beperken tot de deelverzameling van alle oplossingen van zwarte gaten die een deel van de supersymmetrie behouden - de zogenaamde BPS oplossingen. Dit lijdt tot zeer sterke beperkingen op de geometrie en de materievelden, tot op het punt dat de supersymmetrische oplossingen volledig gekenmerkt kunnen worden, zonder gebruik te maken van de volledige bewegingsvergelijkingen. In hoofdstuk 2 introduceren we $\mathcal{N} = 2$ supergravitatie in vier en vijf dimensies en geven we een beknopte samenvatting van supersymmetrische oplossingen in beide theorieën.

Supersymmetrie impliceert de verdere vereenvoudiging dat de theorie gecodeerd is in een klein aantal willekeurige functies. Bijvoorbeeld, de $\mathcal{N} = 2$ supersymmetrische uitbreiding van de Einstein-Maxwell theorie is volledig vastgelegd door een enkele functie die alle koppelingen bepaald. Deze eigenschap is cruciaal voor de twee belangrijkste toepassingen beschouwd in dit proefschrift, namelijk de effecten van hogere afgeleide termen op BPS oplossingen en de constructie van niet-supersymmetrische oplossingen. In de hoofdstukken 3 en 4 bespreken we de vierde orde afgeleide termen toegestaan door $\mathcal{N} = 2$ supersymmetrie in vier en vijf dimensies, respectievelijk. Elk van deze invarianten is geparametriseerd door een willekeurige functie, die kan worden vastgelegd wanneer men een inbedding in een microscopische theorie, zoals snaartheorie, aanneemt.

Door te beperken tot de klasse van theorieën verkregen door snaartheoretische modellen, kan men verband maken met microscopische constructies van zwarte gaten.

De constructie van exacte oplossingen van hogere afgeleide theorieën is een ingewikkeld probleem, dat niettemin met succes is bestudeerd in het verleden, althans voor eenvoudige supersymmetrische oplossingen. Desondanks kan men algemene uitspraken doen over BPS zwarte gaten gebaseerd op de regio dicht bij de horizon (de attractorregio), waarvan is aangetoond dat deze de volledige $\mathcal{N}=2$ supersymmetrie behoudt. Hieruit volgt dat supersymmetrische attractoren direct kunnen worden geconstrueerd, gebaseerd op de vergrote symmetrie. In de hoofdstukken 3 en 4 worden de vier- en vijfdimensionale theorieën uitgebreid behandeld, met toelichting op de relatie tussen de twee theorieën. We vergelijken onze resultaten met de corresponderende voorspellingen vanuit de snaartheorie in hoofdstuk 2 en vinden volledige overeenkomst tussen de macroscopische en microscopische voorspellingen.

De zeer beperkte vrijheid in supergravitatietheorieën met $\mathcal{N}>1$ wordt teruggezien in de oplossingen, zelfs als deze supersymmetrie volledig breken. In hoofdstuk 5 bespreken we enkele recente resultaten over de constructie van niet-BPS oplossingen in $\mathcal{N}=2$ theorieën. Het blijkt dat een groot deel van de eigenschappen van BPS zwarte gaten volgt uit het feit dat ze extremaal zijn, en niet uit supersymmetrie. Om deze eigenschappen te onderzoeken, presenteren we eerst een nogal speciale klasse van vijfdimensionale oplossingen. Deze zijn supersymmetrisch op lokale delen van de ruimtetijd, maar breken niettemin globaal alle supersymmetrie. Deze nieuwe oplossingen blijken een aantal eigenschappen te hebben die aanwezig zijn in generieke extremale oplossingen, maar verboden zijn in het supersymmetrische geval. Op basis van dit feit, stellen we vervolgens een schema voor om algemenere extremale oplossingen te construeren, met inbegrip van roterende zwarte gaten.

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Curriculum Vitae

The author was born on 13 February 1983 in Thessaloniki, Greece and grew up in nearby Giannitsa, where he completed his school studies. In the period between 2000 and 2004, he studied Physics at the Aristotle University of Thessaloniki, graduating in July 2004 with honors. In August 2005, he moved to Utrecht to follow the Master Program in Theoretical Physics, after being awarded an Utrecht University Excellence Scholarship. His Master thesis, "The AdS_3/CFT_2 correspondence in black hole physics", under the supervision of prof. dr. B. de Wit, was on the application of holography in the microscopic study of black holes in string theory. In June 2007 he was awarded the Master's degree (cum laude), and in September of the same year he started his PhD studies at the Institute for Theoretical Physics and Spinoza Institute, Utrecht University, under the supervision of prof. dr. B. de Wit. The research carried out during the last four years is presented in this thesis.