



R-matrix-valued Lax pairs and long-range spin chains

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ABSTRACT

In this paper we discuss R -matrix-valued Lax pairs for sl_N Calogero–Moser model and their relation to integrable quantum long-range spin chains of the Haldane–Shastry–Inozemtsev type. First, we construct the R -matrix-valued Lax pairs for the third flow of the classical Calogero–Moser model. Then we notice that the scalar parts (in the auxiliary space) of the M -matrices corresponding to the second and third flows have form of special spin exchange operators. The freezing trick restricts them to quantum Hamiltonians of long-range spin chains. We show that for a special choice of the R -matrix these Hamiltonians reproduce those for the Inozemtsev chain. In the general case related to the Baxter’s elliptic R -matrix we obtain a natural anisotropic extension of the Inozemtsev chain. Commutativity of the Hamiltonians is verified numerically. Trigonometric limits lead to the Haldane–Shastry chains and their anisotropic generalizations.

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1. Introduction

Integrable systems are known to be actively engaged in high energy physics. For example, the low energy sector of SUSY ($\mathcal{N}=2$) gauge theories is described by the Seiberg–Witten [32] solution in terms of the classical integrable models [11], while their quantum counterparts are described by the supersymmetric vacua of this gauge theory (deformed by the Ω -background) [27]. A link to the conformal field theories is given by the AGT relation [1], which (in the Nekrasov–Shatashvili limit) turns into a certain interrelation between integrable systems known as the spectral duality [26]. On the CFT side integrable systems appear also naturally from the Matsuo–Cherednik construction for the Knizhnik–Zamolodchikov equations [21]. Its classical version – the quantum–classical duality – provides a link between the quantum spin chains and classical many-body integrable systems [12]. In this paper we discuss alternate example of a relation between quantum spin chains and classical integrable systems, which is based on the so-called R -matrix-valued Lax pairs [17,16], [23,13].

The completely integrable Hamiltonian models can be subdivided into two large families. The first one consists of many-body systems including their spin and/or multispin generalizations. A representative example is given by the Calogero–Moser model. The

classical spinless N -body elliptic gl_N model is described by the Hamiltonian

$$H_2 = \sum_{i=1}^N \frac{p_i^2}{2} - v^2 \sum_{i<j}^N \wp(q_i - q_j), \quad (1.1)$$

where $\wp(x)$ is the Weierstrass \wp -function and $v \in \mathbb{C}$ is a coupling constant. Its spin quantum analogue [10] is given by

$$\hat{H} = \sum_{i=1}^N \frac{\hat{p}_i^2}{2} - \sum_{i<j}^N v(v + \hbar P_{ij}) \wp(q_i - q_j), \quad (1.2)$$

where $\hat{p}_i = \hbar \partial_{q_i}$ and P_{ij} is the (spin exchange) permutation operator.

The second family of integrable models is represented by integrable tops, spin chains and/or Gaudin models. In contrast to the previous family these are governed by numerical (non-dynamical) R -matrices. A typical example is given by the (local) XYZ spin chain Hamiltonian [2,7,33]

$$\hat{H} = \sum_{i=1}^{N-1} \hat{h}_{i,i+1} + \hat{h}_{N,1}, \quad \hat{h}_{i,i+1} = \sum_{a=0}^3 \sigma_a^i \sigma_a^{i+1} J_a, \quad (1.3)$$

where J_a are constants (anisotropy parameters) and σ_a^i are the a -th components of the spin operator at i -th site. For the $1/2$ -spin case these are the Pauli matrices acting on the j -th tensor component of $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ – the Hilbert space of the model. When all the

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constants are equal to each other $J_a = J_b$ we have $\hat{h}_{i,i+1} = 2P_{i,i+1}$. Then (1.3) is the isotropic (XXX) Heisenberg Hamiltonian.

In this letter we deal with the integrable models which can be regarded as an intermediate link between the above mentioned families. These are the Haldane–Shastry–Inozemtsev long-range spin chains [14,17]. The Hamiltonian

$$\hat{H} = \sum_{i < j}^N P_{ij} \wp(x_i - x_j), \quad x_i = i/N, \quad i = 1, \dots, N \quad (1.4)$$

describes pairwise interaction of N spins on a unit circle with equidistant positions. It can be shown that the scaling limit of (1.4) provides the XXX Heisenberg model with the nearest-neighbor interaction likewise the (periodic) Toda chain is obtained from the Calogero–Moser model (1.1) [19].¹ On the other hand the Hamiltonian (1.4) can be obtained from the quantum spin Calogero–Moser one (1.2) by the so-called freezing trick [31], when the particles positions are frozen as $q_i \rightarrow x_i$. The models of the Haldane–Shastry–Inozemtsev type found applications in the AdS/CFT correspondence, where the problem of computation of anomalous dimensions of certain $\mathcal{N} = 4$ composite operators emerged. The one-loop anomalous dimensions of these operators were calculated by means of the Bethe ansatz method for the Heisenberg chain [25], and the higher-loop dilatation operator appeared to be expressed through the conserved charges of the long-range chains [20].

The purpose of the paper is to show that the Haldane–Shastry–Inozemtsev spin chains admit anisotropic integrable extensions such as XYZ model generalizes the XXX Heisenberg chain. That is to say that we are going to define an integrable model with the Hamiltonian of the form

$$\hat{H} = \sum_{i < j}^N \sum_{a=0}^3 \sigma_a^i \sigma_a^j J_a(x_i - x_j). \quad (1.5)$$

To construct such Hamiltonian we use the R -matrix-valued Lax pair [23] for the (spinless) Calogero–Moser model (1.1). It is a generalization of the well-known Lax pair with spectral parameter [22] to the case when the matrix elements are not scalar functions but R -matrices satisfying associative Yang–Baxter equation [8,29] and some additional properties. The Lax equations

$$\mathcal{L} = [\mathcal{L}, \mathcal{M}] \quad (1.6)$$

with the Lax pair (4.1)–(4.3) are equivalent to the classical equations of motion for the model (1.1). The matrix elements of \mathcal{L} , \mathcal{M} are operators on the Hilbert space \mathcal{H} , i.e. $\mathcal{L}, \mathcal{M} \in \text{Mat}(N, \mathbb{C}) \otimes \text{End}(\mathcal{H})$. We will refer to $\text{Mat}(N, \mathbb{C})$ component as the auxiliary space, and to the $\text{End}(\mathcal{H})$ component as the quantum (spin chain) space.

Our strategy is as follows. At the level of the classical Calogero–Moser model (1.1) the above mentioned freezing trick turns into the set of conditions

$$p_i = 0, \quad q_i = x_i, \quad (1.7)$$

understood to be the equilibrium position. Being restricted to the constraints (1.7) the Lax equations (1.6) become $[\mathcal{L}, \mathcal{M}] = 0$ on-shell (1.7). At the same time the \mathcal{M} -matrix (4.2) contains the part

$\Delta\mathcal{M} = 1_{N \times N} \otimes v\mathcal{F}^0$ (4.3), which is a scalar operator in the auxiliary space. Therefore, we may interpret the reduced Lax equations as follows:

$$[v\mathcal{F}^0, \mathcal{L}] = [\mathcal{L}, \mathcal{M} - \Delta\mathcal{M}] \quad \text{on-shell (1.7)}, \quad (1.8)$$

where the commutator in the l.h.s. is in the quantum space only. Thus the $v\mathcal{F}^0$ term is the quantum spin chain Hamiltonian. We will show that it is of the form (1.5), and reproduces the Inozemtsev chain (1.4) for a special choice of the R -matrix.

Unfortunately, in the general (elliptic) case the Lax equations allow to compute the higher Hamiltonians for the classical model (1.1) only but not for the quantum spin chain (1.8), because the Hamiltonian in the latter case appeared as a scalar (in the auxiliary space) part of \mathcal{M} . Nevertheless we may repeat the above-described computation procedure to the higher flow of the Calogero–Moser model. We will construct the R -matrix-valued Lax pair for the third flow. Then restrict it to the equilibrium position (1.7) and find the scalar (in the auxiliary space) part of the corresponding \mathcal{M} -matrix. At last, we verify by numerical calculations that the Hamiltonian obtained in this way indeed commutes with the one related to the second flow (1.8).

2. Classical Calogero–Moser model

In this paper we deal with the classical Calogero–Moser–Sutherland models [5,28]. Equations of motion

$$\dot{q}_i = p_i, \quad \dot{p}_i = v^2 \sum_{k:k \neq i}^N \wp'(q_i - q_k) \quad (2.1)$$

are generated by the Hamiltonian (1.1) (and the canonical Poisson brackets $\{p_i, q_j\} = \delta_{ij}$). In the trigonometric limit $\wp(x) \rightarrow \pi^2/\sin^2(\pi x)$ the classical Sutherland model is reproduced.

The Hamiltonian (1.1) is included into a family of the higher integrals of motion, which are in involution with respect to the canonical Poisson brackets: $\{H_k, H_l\} = 0$. For example, the third Hamiltonian

$$H_3 = \sum_{i=1}^N \frac{p_i^3}{3} - v^2 \sum_{i \neq j}^N p_i \wp(q_i - q_j) \quad (2.2)$$

provides equations of motion

$$\begin{cases} \partial_{t_3} q_i = p_i^2 - v^2 \sum_{k \neq i} \wp(q_i - q_k), \\ \partial_{t_3} p_i = v^2 \sum_{k \neq i} (p_i + p_k) \wp'(q_i - q_k). \end{cases} \quad (2.3)$$

All the flows are described by the Lax equations

$$\partial_{t_k} L(z) \equiv \{H_k, L(z)\} = [L(z), M^{(k)}(z)], \quad (2.4)$$

where $L(z)$ and $M^{(k)}(z)$ are $N \times N$ matrices depending on the spectral parameter z , which does not enter equations of motion. So that (2.4) are identities in z on the equations motions. The Lax matrix is as follows [22]:

$$L_{ij}(z) = \delta_{ij} p_i + v(1 - \delta_{ij}) \phi(z, q_{ij}), \quad (2.5)$$

$$q_{ij} = q_i - q_j, \quad \phi(z, q) = \frac{\vartheta'(0)\vartheta(q+z)}{\vartheta(q)\vartheta(z)},$$

where $\vartheta(z)$ is the odd Riemann theta-function.² The $M^{(2)}$ -matrix is of the form

¹ In fact, the Toda model can be also treated as an intermediate link between the two families since it admits two types of the Lax representations: 2×2 as for the spin chains and $N \times N$ as for the many-body systems [7].

² In the trigonometric limit $\phi(z, q) \rightarrow \pi(\cot(\pi z) + \cot(\pi q))$.

$$M_{ij}^{(2)}(z) = \nu d_i \delta_{ij} + \nu(1 - \delta_{ij})f(z, q_{ij}),$$

$$d_i = - \sum_{k \neq i}^N f(0, q_{ik}), \tag{2.6}$$

where $f(z, q) = \partial_q \phi(z, q)$, and $f(0, q)$ coincides with $-\wp(q)$ up to a constant. Namely,

$$f(0, z) = \partial_z^2 \log \vartheta(z) = -\wp(z) + \frac{1}{3} \frac{\vartheta'''(0)}{\vartheta'(0)}. \tag{2.7}$$

For the third flow³ (2.2)–(2.3) the M -matrix is of the form:

$$M_{ij}^{(3)}(z) = -\delta_{ij} \nu \sum_{k \neq i} (p_i + p_k) f(0, q_{ik}) + (1 - \delta_{ij}) \left(\nu(p_i + p_j) f(z, q_{ij}) + \nu^2 \sum_{k \neq i, j} (\phi(z, q_{ik}) f(z, q_{kj}) - \phi(z, q_{ij}) f(0, q_{kj})) \right). \tag{2.8}$$

Verification of the above statements is based on the identities

$$\phi(z, q_{ab}) f(z, q_{ba}) - f(z, q_{ab}) \phi(z, q_{ba}) = \wp'(q_{ab}), \tag{2.9}$$

$$\begin{aligned} \phi(z, q_{ab}) f(z, q_{bc}) - f(z, q_{ab}) \phi(z, q_{bc}) \\ = \phi(z, q_{ac}) (f(0, q_{bc}) - f(0, q_{ab})), \end{aligned} \tag{2.10}$$

which follow from

$$\phi(z, q) \phi(z, -q) = \wp(z) - \wp(q) = f(0, q) - f(0, z) \tag{2.11}$$

and the (genus one) Fay identity for the Kronecker function $\phi(z, w)$ [35]:

$$\begin{aligned} \phi(z, q_{ab}) \phi(w, q_{bc}) \\ = \phi(w, q_{ac}) \phi(z - w, q_{ab}) + \phi(w - z, q_{bc}) \phi(z, q_{ac}). \end{aligned} \tag{2.12}$$

3. Haldane–Shastry–Inozemtsev chain [14,17]

The Hamiltonian of the Inozemtsev chain has form

$$H_2^{\text{Inoz}} = \sum_{i < j} P_{ij} \wp(x_i - x_j). \tag{3.1}$$

In the trigonometric limit it reproduces the Haldane–Shastry model:

$$H_2^{\text{HS}} = \sum_{i < j} \frac{P_{ij}}{\sin^2 \pi(x_i - x_j)}. \tag{3.2}$$

In (3.1)–(3.2) P_{ij} is the permutation (or spin exchange) operator, which acts on the Hilbert space $(\mathbb{C}^2)^{\otimes N}$ of the chain. It interchanges the i -th and j -th components in the tensor product (and keeps unchanged the rest of the components)⁴:

$$P_{12} = \frac{1}{2} \sum_{\alpha=0}^3 \sigma_\alpha \otimes \sigma_\alpha \equiv \frac{1}{2} \sum_{\alpha=0}^3 \sigma_\alpha^1 \sigma_\alpha^2, \tag{3.3}$$

³ While the Hamiltonians H_k are evaluated from $\text{tr} L^k(z)$, the expressions for $M^{(k)}(z)$ corresponding to higher flows can be similarly extracted from $\text{tr}_2(r_{12}(z, w) L_2^{k-1}(w))$, where $r_{12}(z, w)$ is the classical r -matrix.

⁴ In the general case $P_{ij} = \sum_{a,b=1}^{\tilde{N}} E_{ab} E_{ba}$, where $\{E_{ab} \in \text{Mat}_{\tilde{N}}, a, b = 1 \dots \tilde{N}\}$ – is the standard basis in $\text{Mat}_{\tilde{N}}$: $(E_{ab})_{cd} = \delta_{ac} \delta_{bd}$. In (3.1)–(3.3) $\tilde{N} = 2$.

where σ_α are the Pauli matrices. The positions x_j are fixed and equidistant:

$$x_j = \frac{j}{N}, \quad j = 1, \dots, N. \tag{3.4}$$

The model admits the quantum Lax representation [17]:

$$[H_2^{\text{Inoz}}, \mathcal{L}^{\text{Inoz}}(z)] = [\mathcal{L}^{\text{Inoz}}(z), -\mathcal{M}^{\text{Inoz}}(z)] \tag{3.5}$$

with

$$\mathcal{L}^{\text{Inoz}}(z) = \sum_{i,j} E_{ij} \otimes (1 - \delta_{ij}) P_{ij} \phi(z, x_{ij}), \quad x_{ij} = x_i - x_j \tag{3.6}$$

and⁵

$$\begin{aligned} \mathcal{M}^{\text{Inoz}}(z) = \sum_{i,j} E_{ij} \otimes \left(d_i \delta_{ij} + (1 - \delta_{ij}) P_{ij} f(z, x_{ij}) \right), \\ d_i = - \sum_{k \neq i}^N P_{ik} f(0, x_{ik}). \end{aligned} \tag{3.7}$$

So that the Lax matrix is of $N \times N$ size with matrix elements being proportional to the permutation operators $P_{ij} \in \text{Mat}(2^N)$. Therefore, $\mathcal{L}, \mathcal{M} \in \text{Mat}(N2^N)$.

The Lax pair (3.6)–(3.7) owes its origin to the quantum spin Calogero–Moser model [30,15]. The long-range spin chain appears after imposing (3.4), which is treated as the freezing trick in the spin Calogero–Moser model [31]. In classical mechanics (3.4) means that there is an equilibrium position, where the particles coordinates are fixed as $q_j = x_j$ and $p_j = 0$.

In the general case the Lax equation (3.5) does not allow to calculate the higher integrals of motion. This becomes possible in special cases when the sum up to zero condition ($\sum_i \mathcal{M}_{ij} = \sum_j \mathcal{M}_{ij} = 0$) is fulfilled. In the latter case the higher conserved quantities appear from the total sum of elements of powers of \mathcal{L} . In our case the Lax pair is elliptic, and there is no such condition. The receipt for higher integrals was conjectured in [17] and then discussed (and partly proved) in [18]. Two next Hamiltonians commuting with (3.1) are of the form:

$$J_1 = \sum'_{i,j,k} (E_1(x_{ij}) + E_1(x_{jk}) + E_1(x_{ki})) [P_{ij}, P_{jk}], \tag{3.8}$$

$$\begin{aligned} J_2 = \sum'_{i,j,k} \left(2(E_1(x_{ij}) + E_1(x_{jk}) + E_1(x_{ki}))^3 \right. \\ \left. + \wp'(x_{ij}) + \wp'(x_{jk}) + \wp'(x_{ki}) \right) [P_{ij}, P_{jk}], \end{aligned} \tag{3.9}$$

where a prime means that the corresponding summation is over all not coincident values of indices, and $E_1(z) = \partial_z \log \vartheta(z)$.

4. R-matrix-valued Lax pairs

In [23] (see also [13]) the following generalization of the Lax pair (2.5)–(2.6) for the classical Calogero–Moser model was suggested⁶

⁵ In [17] $\mathcal{M}^{\text{Inoz}}$ has different sign.

⁶ In fact, a similar Lax pair was proposed in [16] for the quantum trigonometric spin Calogero–Sutherland model. In that paper the R -matrix was chosen to be the classical trigonometric one (i.e. the corresponding Lax pair was without spectral parameter as it is for the ordinary Lax pair of the Sutherland model) for $\tilde{N} = 2$ case.

$$\mathcal{L}(z) = \sum_{i,j=1}^N E_{ij} \otimes \mathcal{L}_{ij}(z), \quad (4.1)$$

$$\mathcal{L}_{ij}(z) = 1_{\tilde{N}}^{\otimes N} \delta_{ij} p_i + \nu(1 - \delta_{ij}) R_{ij}^z(q_{ij})$$

and similarly

$$\begin{aligned} \mathcal{M}_{ij}^{(2)}(z) &= \nu d_i \delta_{ij} + \nu(1 - \delta_{ij}) F_{ij}^z(q_{ij}) + \nu \delta_{ij} \mathcal{F}^0, \\ d_i &= - \sum_{k:k \neq i}^N F_{ik}^0(q_{ik}), \end{aligned} \quad (4.2)$$

$$\mathcal{F}^0 = \sum_{k < m}^N F_{km}^0(q_{km}) = \frac{1}{2} \sum_{k,m=1}^N F_{km}^0(q_{km}), \quad (4.3)$$

where $F_{ij}^z(q) = \partial_q R_{ij}^z(q)$. By the construction $\mathcal{L}(z), \mathcal{M}^{(2)}(z) \in \text{Mat}(N\tilde{N}^N)$. The Lax equation $\dot{\mathcal{L}} = [\mathcal{L}, \mathcal{M}^{(2)}]$ is equivalent to (2.1) with the coupling constant $\tilde{N}\nu$ instead of ν . For exact matching one should rescale $\nu \rightarrow \nu/\tilde{N}$ in (4.1)–(4.3) but we keep it as it is.

The Lax pair (4.1)–(4.3) is called *R-matrix-valued Lax pair* since it can be viewed as $N \times N$ matrices which matrix elements are quantum $\text{GL}_{\tilde{N}}$ *R-matrices* (or its derivatives), satisfying the associative Yang–Baxter equation [8]

$$R_{ab}^z R_{bc}^w = R_{ac}^w R_{ab}^{z-w} + R_{bc}^{w-z} R_{ac}^z, \quad R_{ab}^z = R_{ab}^z(q_a - q_b). \quad (4.4)$$

It was observed in [29] that (4.4) is fulfilled by the elliptic Baxter–Belavin [2,3] *R-matrix* (written in proper normalization):

$$\begin{aligned} R_{12}^z(q) &= \sum_a T_a \otimes T_{-a} \exp\left(2\pi i \frac{a_2}{\tilde{N}} q\right) \phi\left(q, z + \frac{a_1 + a_2 \tau}{\tilde{N}}\right), \\ a &= (a_1, a_2) \in \mathbb{Z}_{\tilde{N}} \times \mathbb{Z}_{\tilde{N}}, \end{aligned} \quad (4.5)$$

where the basis T_a is defined in terms of the finite dimensional representation of the Heisenberg group

$$\begin{aligned} T_a &= T_{a_1 a_2} = \exp\left(\frac{\pi i}{\tilde{N}} a_1 a_2\right) Q^{a_1} \Lambda^{a_2}, \quad a = (a_1, a_2) \in \mathbb{Z}_{\tilde{N}} \times \mathbb{Z}_{\tilde{N}} \\ Q_{kl} &= \delta_{kl} \exp\left(\frac{2\pi i}{\tilde{N}} k\right), \quad \Lambda_{kl} = \delta_{k-l+1=0 \bmod \tilde{N}}, \\ Q^{\tilde{N}} &= \Lambda^{\tilde{N}} = 1_{\tilde{N}}. \end{aligned} \quad (4.6)$$

Equations (4.4) are matrix analogues of the Fay identities (2.12) (they coincide for $\tilde{N} = 1$). In the same way the unitarity property⁷

$$R_{12}^z(q_{12}) R_{21}^z(q_{21}) = 1_{\tilde{N}} \otimes 1_{\tilde{N}} \tilde{N}^2 (\wp(\tilde{N}z) - \wp(q_{12})) \quad (4.7)$$

is similar to (2.11). Together with the skew-symmetry (likewise $\phi(z, q) = -\phi(-z, -q)$)

$$R_{12}^z(q) = -R_{21}^{-z}(-q) \quad (4.8)$$

equations (4.4) and (4.7) results to the quantum Yang–Baxter equation

$$R_{ab}^z(q_{ab}) R_{ac}^z(q_{ac}) R_{bc}^z(q_{bc}) = R_{bc}^z(q_{bc}) R_{ac}^z(q_{ac}) R_{ab}^z(q_{ab}). \quad (4.9)$$

Coming back to the Lax pair (4.1)–(4.2) it must be emphasized that it is a straightforward generalization of the Krichever's Lax pair (2.5)–(2.6) except the last term (4.3), which is not necessary in (2.6) since for $\tilde{N} = 1$ case it is proportional to the identity matrix.

The matrix function $F^0(q)$ entering this term is simply related to the classical *r-matrix*⁸:

$$F_{ij}^0(q) = F_{ij}^z(q)|_{z=0} = \partial_q r_{ij}(q) = F_{ji}^0(-q). \quad (4.10)$$

From the above it follows that we also have *R-matrix* analogues for identities (2.9), (2.10):

$$R_{ab}^z F_{ba}^z - F_{ab}^z R_{ba}^z = \tilde{N}^2 \wp'(q_{ab}), \quad (4.11)$$

$$R_{ab}^z F_{bc}^z - F_{ab}^z R_{bc}^z = F_{bc}^0 R_{ac}^z - R_{ac}^z F_{ab}^0. \quad (4.12)$$

The latter identities underly the Lax equations for (4.1)–(4.3). The role of the \mathcal{F}^0 term is to correct the order of multipliers

$$[R_{ac}^z, \mathcal{F}^0] + \sum_{b \neq a, c} R_{ab}^z F_{bc}^z - F_{ab}^z R_{bc}^z = \sum_{b \neq c} R_{ac}^z F_{bc}^0 - \sum_{b \neq a} F_{ab}^0 R_{ac}^z. \quad (4.13)$$

It is natural to expect existence of higher *R-matrix-valued M-matrices* related to higher Hamiltonians (2.4). Here we propose *R-matrix-valued generalization* of the *M-matrix* for the third flow (2.8). It is of the form:

$$\begin{aligned} \mathcal{M}_{ij}^{(3)}(z) &= -\delta_{ij} \nu \sum_{k \neq i} (p_i + p_k) F_{ik}^0(q_{ik}) + \\ &+ (1 - \delta_{ij}) \left(\nu(p_i + p_j) F_{ij}^z(q_{ij}) + \right. \\ &+ \nu^2 \sum_{k \neq i, j} (R_{ik}^z(q_{ik}) F_{kj}^z(q_{kj}) - R_{ij}^z(q_{ij}) F_{kj}^0(q_{kj})) \left. + \right) \\ &+ \delta_{ij} \left(\nu^2 \sum'_{b, c} [F_{bc}^0(q_{bc}), r_{ic}(q_{ic})] + \right. \\ &+ \left. \nu \sum'_{b, c} p_b F_{bc}^0(q_{bc}) - \frac{\nu^2}{3} \sum'_{a, b, c} [F_{ab}^0(q_{ab}), r_{cb}(q_{cb})] \right). \end{aligned} \quad (4.14)$$

Two upper lines of (4.14) are straightforward generalizations of (2.8), while the last line is non-trivial for $\tilde{N} > 1$ only (more precisely, for $\tilde{N} = 1$ it is proportional to the identity matrix). Its role is similar to the \mathcal{F}^0 term in (4.2). As in the case of the second flow here the Lax equations $\dot{\mathcal{L}} = [\mathcal{L}, \mathcal{M}^{(3)}]$ is equivalent to equations of motion (2.3), where the coupling constant ν is replaced by $\tilde{N}\nu$. The proof is direct and somewhat technical. It uses (4.11), (4.12) together with the classical Yang–Baxter equation

$$[r_{ij}(q_{ij}), r_{ik}(q_{ik})] + [r_{ij}(q_{ij}), r_{jk}(q_{jk})] + [r_{ik}(q_{ik}), r_{jk}(q_{jk})] = 0 \quad \forall i, j, k \quad (4.15)$$

or, to be exact, with its derivative

$$[F_{ij}^0(q_{ij}), r_{ki}(q_{ki}) + r_{kj}(q_{kj})] = [F_{ik}^0(q_{ik}), r_{jk}(q_{jk}) + r_{ji}(q_{ji})] = [F_{jk}^0(q_{jk}), r_{ij}(q_{ij}) + r_{ik}(q_{ik})]. \quad (4.16)$$

Details of the proof will be given elsewhere.

5. R-matrix-valued Lax pairs and spin chains

We are now in a position to describe relationship between *R-matrix-valued Lax pairs* and long-range spin chains. For this pur-

⁷ Different properties and identities of the Baxter–Belavin *R-matrix* similar to the elliptic function identities can be found also in [24,36].

⁸ The classical limit is of the form $R_{12}^z(q) = \frac{1 \otimes 1}{z} + r_{12}(q) + O(z)$, and the classical *r-matrix* is skew-symmetric $r_{12}(q) = -r_{21}(-q)$.

pose we restrict ourself to the equilibrium position (3.4). Then the Lax matrix (4.1) turns into⁹

$$\mathcal{L}^{\text{chain}}(z) = \sum_{i,j=1}^N E_{ij} \otimes (1 - \delta_{ij}) R_{ij}^z(x_{ij}). \quad (5.1)$$

The restriction of the M -matrix (4.2) is subdivided into two parts as $\mathcal{M}^{(2)} = (\mathcal{M}^{(2)} - \mathcal{F}^0) + \mathcal{F}^0$. The restriction of the first term is

$$\mathcal{M}^{\text{chain}}(z) = \sum_{i,j=1}^N E_{ij} \otimes \left(-\delta_{ij} \sum_{k \neq i} F_{ik}^0(x_{ik}) + (1 - \delta_{ij}) F_{ij}^z(x_{ij}) \right), \quad (5.2)$$

while the restriction of the second term is denoted as

$$\mathcal{H}_2^{\text{chain}} = \sum_{k>m}^N F_{km}^0(x_{km}). \quad (5.3)$$

Then the restriction of the (classical) Lax equation for Calogero–Moser model gives the quantum Lax equation for the spin chain with the Hamiltonian $\mathcal{H}_2^{\text{chain}}$:

$$[\mathcal{H}_2^{\text{chain}}, \mathcal{L}^{\text{chain}}(z)] = [\mathcal{L}^{\text{chain}}(z), \mathcal{M}^{\text{chain}}(z)]. \quad (5.4)$$

This equation holds for an arbitrary R -matrix (entering \mathcal{L} and $\mathcal{M}^{(2)}$) which satisfies associative Yang–Baxter equation (4.4) together with the unitarity and skew-symmetry properties (4.7), (4.8).

The Inozemtsev chain (3.1), (3.5)–(3.7) is reproduced from (5.1)–(5.4) as follows. Consider the following R -matrix:

$$R_{ij}^z(x_{ij}) = P_{ij} \phi(z, x_{ij}) \quad (5.5)$$

It satisfies all necessary conditions. In particular, the associative Yang–Baxter equation (4.4) for this R -matrix follows from $P_{ab}P_{bc} = P_{ac}P_{ab} = P_{bc}P_{ac}$ and the scalar Fay identity (2.12). Therefore, we may substitute it into (5.1)–(5.3). The corresponding analogues of the classical r -matrix and its derivative are given by

$$r_{ij}(x_{ij}) = P_{ij} E_1(x_{ij}), \quad F_{ij}^0(x_{ij}) = P_{ij} f(0, x_{ij}). \quad (5.6)$$

As a result we obtain for the case (5.5): $\mathcal{L}^{\text{chain}}(z) = \mathcal{L}^{\text{Inoz}}(z)$, $\mathcal{M}^{\text{chain}}(z) = -\mathcal{M}^{\text{Inoz}}(z)$ and

$$\begin{aligned} \mathcal{H}_2^{\text{chain}} &\stackrel{(2.7)}{=} -\sum_{i<j} P_{ij} \wp(x_{ij}) + \frac{1}{3} \frac{\wp'''(0)}{\wp'(0)} \sum_{i<j} P_{ij} \\ &= -H_2^{\text{Inoz}} + \frac{1}{3} \frac{\wp'''(0)}{\wp'(0)} \sum_{i<j} P_{ij}. \end{aligned} \quad (5.7)$$

Notice that the spin chain Hamiltonian (5.3) appeared as a part (\mathcal{F}^0 -term) of the $\mathcal{M}^{(2)}$ -matrix restricted to $q_j = x_j$. The \mathcal{F}^0 -term enters $\mathcal{M}^{(2)}$ as $\nu 1_N \otimes \mathcal{F}^0$, hence it is (up to a number factor) equal to trace of $\mathcal{M}^{(2)}$ -matrix over the auxiliary space, which is the first (Mat_N -valued) tensor component in its definition:

$$\mathcal{H}_2^{\text{chain}} \propto \text{tr}_{\text{aux}} \mathcal{M}^{(2)}|_{q_j=x_j} \propto \mathcal{F}^0|_{q_j=x_j} \quad (5.8)$$

The above mentioned arguments can be applied to the higher flows of the Calogero–Moser model as well. For example, for the third flow $\mathcal{M}^{(3)}$ (4.14) we have

$$\begin{aligned} \text{tr}_{\text{aux}} \mathcal{M}^{(3)} \\ = N\nu \sum'_{b,c} p_b F_{bc}^0(q_{bc}) + \nu^2 \left(1 - \frac{N}{3}\right) \sum'_{a,b,c} [F_{ab}^0(q_{ab}), r_{cb}(q_{cb})] \end{aligned} \quad (5.9)$$

Recall that the prime means summation over all pairwise distinct values of indices. After imposing constraints $p_i = 0$, $q_i = x_i$ we are left with

$$\mathcal{H}_3^{\text{chain}} = \sum'_{a,b,c} [F_{ab}^0(x_{ab}), r_{cb}(x_{cb})]. \quad (5.10)$$

Then let us define the third Hamiltonian as

$$\mathcal{H}_3^{\text{chain}} = \sum_{i<j<k} [F_{ij}^0(x_{ij}), r_{ik}(x_{ik}) + r_{jk}(x_{jk})]. \quad (5.11)$$

We conjecture that the obtained in this way spin chain Hamiltonians commute for some non-trivial (anisotropic) R -matrices. First, consider the Inozemtsev case (5.6). A comparison of the poles and residues shows that in this case

$$\mathcal{H}_3^{\text{chain}} = -\frac{1}{36} \left(J_2 - \frac{1}{3} \frac{\wp'''(0)}{\wp'(0)} J_1 \right), \quad (5.12)$$

where J_1, J_2 are given by (3.8), (3.9).

For the following R -matrices the commutativity $[\mathcal{H}_2^{\text{chain}}, \mathcal{H}_3^{\text{chain}}] = 0$ can be verified numerically:

1. Baxter’s elliptic XYZ R -matrix (τ is the elliptic moduli)

$$\begin{aligned} R_{12}^z(q) &= 1 \otimes 1 \phi(q, z) + \sigma_1 \otimes \sigma_1 e^{\pi i q} \phi(q, z + \frac{\tau}{2}) \\ &\quad + \sigma_2 \otimes \sigma_2 e^{\pi i q} \phi(q, z + \frac{\tau+1}{2}) \\ &\quad + \sigma_3 \otimes \sigma_3 \phi(q, z + \frac{1}{2}). \end{aligned} \quad (5.13)$$

Then the classical r -matrix

$$\begin{aligned} r_{12}(q) &= 1 \otimes 1 E_1(q) + \sigma_1 \otimes \sigma_1 e^{\pi i q} \phi(q, \frac{\tau}{2}) \\ &\quad + \sigma_2 \otimes \sigma_2 e^{\pi i q} \phi(q, \frac{\tau+1}{2}) \\ &\quad + \sigma_3 \otimes \sigma_3 \phi(q, \frac{1}{2}). \end{aligned} \quad (5.14)$$

For the three functions $\varphi_1(q) = e^{\pi i q} \phi(q, \frac{\tau}{2})$, $\varphi_2(q) = e^{\pi i q} \phi(q, \frac{\tau+1}{2})$ and $\varphi_3(q) = \phi(q, \frac{1}{2})$ the derivative of a one is given by the minus product of two others: $\partial_q \varphi_\alpha(q) = -\varphi_\beta(q) \varphi_\gamma(q)$. Therefore, the second Hamiltonian (5.3) acquires the form (1.5):

$$\begin{aligned} \mathcal{H}_2^{\text{chain}} &= \sum_{i<j} \left(\dot{\sigma}_0^i \dot{\sigma}_0^j E_1'(x_{ij}) - \sum_{\alpha=1}^3 \dot{\sigma}_\alpha^i \dot{\sigma}_\alpha^j \varphi_\beta(x_{ij}) \varphi_\gamma(x_{ij}) \right) = \\ &= \sum_{i<j} \left(\dot{\sigma}_0^i \dot{\sigma}_0^j E_1'(x_{ij}) + \sum_{\alpha=1}^3 \dot{\sigma}_\alpha^i \dot{\sigma}_\alpha^j \varphi_\alpha(x_{ij}) (E_1(x_{ij} + \omega_\alpha) \right. \\ &\quad \left. - E_1(x_{ij}) - E_1(\omega_\alpha)) \right), \end{aligned} \quad (5.15)$$

where ω_α is the half-period (the second argument of $\varphi_\alpha(q)$). One more useful form for $\mathcal{H}_2^{\text{chain}}$ is as follows:

$$\begin{aligned} \mathcal{H}_2^{\text{chain}} &= \frac{N(N-1)}{6} \frac{\wp'''(0)}{\wp'(0)} \sigma_0^{\otimes N} \\ &\quad - \frac{1}{2} \sum_{i<j} \left(\sum_{\alpha=0}^3 \dot{\sigma}_\alpha^i \dot{\sigma}_\alpha^j \wp\left(\frac{x_{ij}}{2} + \omega_\alpha\right) \right) P_{ij}, \end{aligned} \quad (5.16)$$

where $\omega_0 = 0$. The third Hamiltonian is evaluated through (5.11).

⁹ We may put $\nu = 1$ since it is a common factor in the Lax equations.

2. Trigonometric XXZ 6-vertex R -matrix

$$R_{12}^z(q) = (\pi \cot \pi z + \pi \cot \pi q) \cdot (\sigma_0 \otimes \sigma_0 + \sigma_3 \otimes \sigma_3) + \frac{\pi}{\sin \pi z} \cdot (\sigma_0 \otimes \sigma_0 - \sigma_3 \otimes \sigma_3) + \frac{\pi}{\sin \pi q} \cdot (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2). \quad (5.17)$$

Then

$$r_{12}(q) = \pi \cot \pi q \cdot (1 \otimes 1 + \sigma_3 \otimes \sigma_3) + \frac{\pi}{\sin \pi q} (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) \quad (5.18)$$

and

$$F_{12}^0(q) = -\frac{\pi^2}{\sin^2 \pi q} \cdot (1 \otimes 1 + \sigma_3 \otimes \sigma_3) + \cos \pi q (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2). \quad (5.19)$$

This gives

$$\mathcal{H}_2^{\text{chain}} = -\pi^2 \sum_{i < j} \frac{\cos(\pi x_{ij}) (\sigma_1^i \sigma_1^j + \sigma_2^i \sigma_2^j) + \sigma_3^i \sigma_3^j}{\sin^2(\pi x_{ij})} + C_N \sigma_0^{\otimes N}, \quad (5.20)$$

$$C_N = -\pi^2 \sum_{i < j} \frac{1}{\sin^2(\pi x_{ij})}.$$

The third Hamiltonian (5.11) has compact form in this case:

$$\mathcal{H}_3^{\text{chain}} = -\frac{\pi^3}{2} \sum_{i < j < k} \frac{\cos \pi x_{ij} (\sigma_1^i \sigma_2^j - \sigma_1^j \sigma_2^i) \sigma_3^k + \cos \pi x_{jk} (\sigma_1^j \sigma_2^k - \sigma_1^k \sigma_2^j) \sigma_3^i + \cos \pi x_{ki} (\sigma_1^k \sigma_2^i - \sigma_1^i \sigma_2^k) \sigma_3^j}{\sin \pi x_{ij} \sin \pi x_{jk} \sin \pi x_{ki}}. \quad (5.21)$$

The spin exchange operator entering (5.20) was obtained in [16] in their study of the spin Calogero–Moser models, and the spin chains of this type were considered in [9,4].

6. Conclusion

The purpose of the paper is two-fold. First, we study R -matrix-valued Lax pairs for the classical Calogero–Moser model and describe its third flow. Then we mention that the scalar part (in the auxiliary space) of the M -matrices provides spin exchange operators entering the Hamiltonians of the long range spin chains. We conjecture commutativity $[\mathcal{H}_2^{\text{chain}}, \mathcal{H}_3^{\text{chain}}] = 0$ for this Hamiltonians restricted to the equilibrium position $p_i = 0$, $q_i = x_i = i/N$. Such hypothesis is based on the coincidence of these Hamiltonians with those for Inozemtsev chain for a special choice of the R -matrix. For the Baxter’s elliptic R -matrix we verify numerically that these Hamiltonians commute. In this way the anisotropic extension of the Inozemtsev chain is described.

Let us also mention that the conjecture on commutativity $[\mathcal{H}_2^{\text{chain}}, \mathcal{H}_3^{\text{chain}}] = 0$ does not hold true for any R -matrix satisfying associative Yang–Baxter equation (and other properties). For instance, it is not true for the 7-vertex R -matrix presented in [6]. Another remark is that we study R -matrices depending on the spectral parameter only. To include the rest of R -matrices into the construction of R -matrix-valued Lax pairs is a challenging task, since the Haldane–Shastry type chains are known to exist for the quantum group like R -matrices [34].

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