The periastron advance in curvature based Extended Gravity and Dark Energy

Antonio Capolupo [1,2], Gaetano Lambiase [1,2] and Antonio Tedesco [1,2]

[1] Dipartimento di Fisica "E.R. Caianiello", Università degli Studi di Salerno, via G. Paolo II, I - 84084 Fisciano, Italy

[2] Istituto Nazionale di Fisica Nucleare (INFN), Gruppo collegato di Salerno, via G. Paolo II, I - 84084 Fisciano, Italy

E-mail: capolupo@sa.infn.it, lambiase@sa.infn.it, atedesco@unisa.it

Abstract. In the context of the periastron advance as a fundamental test for gravity theories, we show an approach to deduce constraints on the sizes of the new forces arising in the weak field limit of the Extended Theories of Gravity (ETG). It is also studied the force motivated by Quintessence Fields deforming the Schwarzschild geometry and associated to the Dark Energy, responsible of the acceleration of the Universe. As for the ETG, we consider the more general Scalar-Tensor Fourth Order Gravity (STFOG) and the NonCommutative Spectral Gravity (NCSG) as a special case. The solutions of the linearized field equations provide corrections to the Newtonian potential in the Yukawa-like form $V(r) = \alpha \frac{e^{-\beta r}}{r}$, where α is the parameter related to the strength of the potential and β to the range of the force. Quintessence Fields lead to a power-law correction with parameters related to the Dark Energy. By analysing the periastron advance with the use of the current data, we find improvements on the parameters of the gravitational models as well as on the bounds of the parameter β by several orders of magnitude.

1. Introduction

We report here the results presented on Ref. [82]. In the last decades, observational investigations concerning the motion of objects in galaxy clusters at extra-galactic scales have clearly highlighted how the matter present in the Universe is dominated by an invisible dark component, denominated *dark matter*. In particular we observe that the effects of dark matter act at galactic scales giving rise to rotation curves characterized by a flat behavior, differently from what we should expect from Newtonian gravity. In addition to the dark matter, the discovery that the Universe is currently accelerating allowed to realize that it is also dominated by an unknown form of energy, supposed to be responsible of this relevant phenomenon, i.e. the *dark energy* [1, 2, 3, 4, 5, 6]. However, up to now there is no result coming from experimental projects finalised to detect particles that might constitute the dark matter. If we relax the assumption that it is only a form of matter not emitting light responsible of the observed gravitational effects on galaxies and clusters, it is possible to take into consideration many other theoretical proposals. Another approach to understand the nature of dark matter is represented by the Extended Theories of Gravity (ETG), which have captured an ever increasing interest in the scientific community. The interest resides in the fact that it is possible to explain the dark matter and dark energy in a pure gravitational setting based on curvature fields. The

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.
Published under licence by IOP Publishing Ltd

Journal of Physics: Conference Series **2533** (2023) 012033 doi:10.1088/1742-6596/2533/1/012033

starting idea is that there is no fundamental reason for assuming the gravitational action (from which we derive the field equations) to be just a linear function of the Ricci scalar minimally coupled to matter, namely the Hilbert Einstein action as it occurs in General Relativity [8]. In this framework, if we introduce in the action higher order terms expressed by scalar curvature invariants, we can literally *extend* the theory of gravity and General Relativity is recovered as a special case of Extended Gravity. Thanks to a more general action, the effects attributed to the presence of dark matter and dark energy are thus interpreted as results of the extra-curvature terms. Linked to this, there is the fundamental property that ETG conduct to a gravitational law which acts differently at different scales. More specifically the results of General Relativity at Solar System scales are preserved, while at galactic and extra-galactic scales the gravitational pull has an growing contribution due to the extra-curvature terms.

We emphasize that modifications of General Relativity are suggested also by the fact that Einstein's theory of gravity breaks down in the UV [9]. The deviations from General Relativity are already present in several frameworks, such as Brans-Dicke and scalar-tensor (ST) theories [10, 11, 12], braneworld theories [13, 14], and finally higher order invariants such as $f(R)$ and $f(\phi, R, R^2, R_{\mu\nu}R^{\mu\nu})$, corresponding to an Einstein's gravity plus one or multiple coupled scalar fields [15, 16, 17, 18]) [19, 20, 21, 22], Non-Commutative Spectral Geometry [23], and compactified extra dimension/Kaluza-Klein models [26]. Moreover, they can also be generated from higher-order terms in the curvature invariants, nonminimal couplings to the background geometry in the Hilbert-Einstein Lagrangian [27]. Additional terms into the action of gravity may also come from string loop effects [28], dilaton fields in string cosmology [29], and nonlocally modified gravity induced by quantum loop corrections [30].

The Newtonian limit of some models of ETG have been studied in [31], while the Minkowskian limit in [32]. Natural candidates for experimentally testing ETG are the galactic rotation curves, stellar systems and gravitational lensing [33, 34, 35] (see also [36]). In this perspective, corrections to GR were already considered by several authors [37, 38, 39, 40]. Due to the large amount of possible models, an important issue is to select viable ones by experiments and observations. Therefore, the new born *multimessenger astronomy* is giving important constraints to admit or exclude gravitational theories (see e.g. [42, 43]). However, also fine experiments can be conceived and realized in order to fix possible deviations and extensions with respect to GR. They can involve space-based setups like satellites and precise electromagnetic measurements [44]. The Yukawa-like potential potentials occur in ETG and in Non-Commutative Spectral Geometry, and Quintessence field surrounding a massive gravitational source provide a Power Law potential. We shall infer the corrections to periastron advance for Solar Planets, referring in particular to Mercury, Mars, Jupiter and Saturn, as well as to S2 star orbiting around Sagittarius A^*.

In Sec. 2 we introduce the Scalar-Tensor Fourth Order Gravity (STFOG), the particular class of the NonCommutative Spectral Geometry (NCSG) and the solutions in the weak field limit of STFOG and NCSG. Then the Quintessence Field reproducing the Dark Energy is discussed. In Sec. 3, we study the effects of these corrections to the periastron advance of Planets as well as the effects of the Quintessence field present around a Schwarzschild Black Hole, and derive a lower bound on the adiabatic index of equation of state. Finally, conclusions are drawn in the last Section.

2. Extended Gravity and Quintessence Fields

The action for the ETG is given by (see for example [7])

$$\mathcal{S} = \int d^4x\sqrt{-g}\Big[f(R, R_{\mu\nu}R^{\mu\nu}, \phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} + \mathcal{X}\mathcal{L}_m\Big], \tag{1}$$

Journal of Physics: Conference Series **2533** (2023) 012033 doi:10.1088/1742-6596/2533/1/012033

where f is a generic function of the invariant R (the Ricci scalar), the invariant $R_{\mu\nu}R^{\mu\nu} = Y$ ($R_{\mu\nu}$ is the Ricci tensor), the scalar field ϕ, g is the determinant of metric tensor $g_{\mu\nu}$ and $\mathcal{X} = 8\pi G$. The Lagrangian density $\mathcal{L}_m$ is the minimally coupled ordinary matter Lagrangian density, $\omega(\phi)$ is a generic function of the scalar field.

The field equations obtained by varying the action (1) with respect to $g_{\mu\nu}$ and ϕ, In the metric approach, are[1]:

$$f_R R_{\mu\nu} - \frac{f + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha}}{2} g_{\mu\nu} - f_{R;\mu\nu} + g_{\mu\nu}\Box f_R + 2f_Y R_\mu{}^\alpha R_{\alpha\nu} + \tag{2}$$

$$-2[f_Y R^\alpha{}_{(\mu}]_{;\nu)\alpha} + \Box[f_Y R_{\mu\nu}] + [f_Y R_{\alpha\beta}]^{;\alpha\beta} g_{\mu\nu} + \omega(\phi)\phi_{;\mu}\phi_{;\nu} = \mathcal{X}\, T_{\mu\nu} \,,$$

$$2\omega(\phi)\Box\phi + \omega_\phi(\phi)\phi_{;\alpha}\phi^{;\alpha} - f_\phi = 0 \,. \tag{3}$$

where:

$$f_R = \frac{\partial f}{\partial R}, \quad f_Y = \frac{\partial f}{\partial Y}, \quad \omega_\phi = \frac{d\omega}{d\phi}, \quad f_\phi = \frac{df}{d\phi},$$

and $T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$ is the the energy-momentum tensor of matter. We confine ourselves to the case in which the generic function f can be expanded as follows (notice that the all other possible contributions in f are negligible [31, 40, 41])

$$\begin{aligned} f(R, R_{\alpha\beta}R^{\alpha\beta}, \phi) = \quad & f_R(0,0,\phi^{(0)})\, R + \frac{f_{RR}(0,0,\phi^{(0)})}{2} R^2 + \frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2}(\phi - \phi^{(0)})^2 \\ & + f_{R\phi}(0,0,\phi^{(0)}) R\,\phi + f_Y(0,0,\phi^{(0)}) R_{\alpha\beta}R^{\alpha\beta} \,. \end{aligned} \tag{4}$$

To study the weak-field approximation, we perturb Eqs.(2) and (3) in a Minkowski background $\eta_{\mu\nu}$ [31], i.e. we look for perturbed solutions of the form

$$g_{\mu\nu} \simeq \begin{pmatrix} -1-2\Phi & 2\mathbf{A} \\ 2\mathbf{A} & (1-2\Psi)\delta_{ij} \end{pmatrix} . \tag{5}$$

and

$$\phi \sim \phi^{(0)} + \phi^{(2)} + \ldots = \phi^{(0)} + \varphi.$$

For matter described as a perfect fluid, hence $T_{00} = \rho$ and $T_{ij} = 0$, one gets that, for a ball-like source with radius $\mathcal{R}$, the gravitational potentials $\{\Phi, \Psi, A_i\}$ and the scalar field φ take the form

[1] We use, for the Ricci tensor, the convention $R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu}$, whilst for the Riemann tensor we define $R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} + \cdots$. The affine connections are the Christoffel symbols of the metric, namely $\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma})$, and we adopt the signature is $(-,+,+,+)$.

$(c = 1)$ [40, 41]

$$\begin{aligned}
\Phi(\mathbf{x}) &= -\frac{GM}{|\mathbf{x}|}\left[1+\zeta(|\mathbf{x}|)\right], && (6)\\
\zeta(|\mathbf{x}|) &\equiv g(\xi,\eta)\,F(m_+\mathcal{R})\,e^{-m_+|\mathbf{x}|}+\left[\frac{1}{3}-g(\xi,\eta)\right]F(m_-\mathcal{R})\,e^{-m_-|\mathbf{x}|}-\frac{4\,F(m_Y\mathcal{R})}{3}\,e^{-m_Y|\mathbf{x}|}, && (7)\\
\Psi(\mathbf{x}) &= -\frac{GM}{|\mathbf{x}|}\left[1-\psi(|\mathbf{x}|)\right], && (8)\\
\psi(|\mathbf{x}|) &\equiv g(\xi,\eta)\,F(m_+\mathcal{R})\,e^{-m_+|\mathbf{x}|}+\left[\frac{1}{3}-g(\xi,\eta)\right]F(m_-\mathcal{R})\,e^{-m_-|\mathbf{x}|}+\frac{2\,F(m_Y\mathcal{R})}{3}\,e^{-m_Y|\mathbf{x}|}, && (9)\\
\mathbf{A}(\mathbf{x}) &= -\frac{2\,G\left[1-\mathcal{A}(|\mathbf{x}|)\right]}{|\mathbf{x}|^2}\,\mathbf{x}\times\mathbf{J}, && (10)\\
\mathcal{A}(|\mathbf{x}|) &\equiv (1+m_Y|\mathbf{x}|)\,e^{-m_Y|\mathbf{x}|}, && (11)\\
\varphi(\mathbf{x}) &= \frac{GM}{|\mathbf{x}|}\sqrt{\frac{\xi}{3}}\,\frac{2}{\omega_+-\omega_-}\left[F(m_+\mathcal{R})\,e^{-m_+|\mathbf{x}|}-F(m_-\mathcal{R})\,e^{-m_-|\mathbf{x}|}\right], && (12)
\end{aligned}$$

where $\mathbf{J} = 2M\mathcal{R}^2\mathbf{\Omega}_0/5$ is the angular momentum of the ball, $f_R(0,0,\phi^{(0)}) = 1$, $\omega(\phi^{(0)}) = 1/2$, and

$$\begin{aligned}
g(\xi,\eta) &= \frac{1-\eta^2+\xi+\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}{6\sqrt{\eta^4+(\xi-1)^2-2\eta^2(\xi+1)}}, && (13)\\
F(m\,\mathcal{R}) &= 3\frac{m\,\mathcal{R}\cosh m\,\mathcal{R}-\sinh m\,\mathcal{R}}{m^3\mathcal{R}^3}, && (14)\\
\xi &= 3{f_{R\phi}(0,0,\phi^{(0)})}^2, \quad \eta = \frac{m_\phi}{m_R}, && (15)\\
m_\pm^2 &= m_R^2\,\omega_\pm, && (16)\\
\omega_\pm &= \frac{1-\xi+\eta^2\pm\sqrt{(1-\xi+\eta^2)^2-4\eta^2}}{2}, && (17)\\
{m_R}^2 &\doteq -\frac{f_R(0,0,\phi^{(0)})}{3f_{RR}(0,0,\phi^{(0)})+2f_Y(0,0,\phi^{(0)})}, && (18)\\
{m_Y}^2 &\doteq \frac{f_R(0,0,\phi^{(0)})}{f_Y(0,0,\phi^{(0)})}, \quad {m_\phi}^2 \doteq -\frac{f_{\phi\phi}(0,0,\phi^{(0)})}{2\omega(\phi^{(0)})}. && (19)
\end{aligned}$$

Some ETG models studied in literature and reported in Table 1 (see [40] for further details).

Another possibility we want to discuss is related to the Quintessence Field, invoked to explain the speed-up of the present Universe [63]. Quintessence may generate a negative pressure, and, since it is diffuse everywhere in the Universe, it can be the responsible of the observed accelerated phase, as well as it is present around a massive astrophysical object deforming the spacetime around it [64]. The studies of quintessential black holes are also motivated from M-theory/superstring inspired models [65] (see [66, 67, 68, 63] for applications). The solution of Einstein's field equations for a static spherically symmetric quintessence surrounding a black hole in 4 dimension is given by [64, 66]

$$g_{\mu\nu} = \mathrm{diag}\left(-f(r), f^{-1}(r), r^2, r^2\sin^2\theta\right), \tag{20}$$

with

$$f(r) = 1-\frac{2M}{r}-\frac{c}{r^{3\omega_Q+1}}, \tag{21}$$

Table 1. We report different cases of Extended Theories of Gravity including a scalar field and higher-order curvature terms. The free parameters are given as effective masses with their asymptotic behavior. Here, we assume $f_R(0,\,0,\,\phi^{(0)}) = 1$, $\omega(\phi^{(0)}) = 1/2$.

Case	**ETG**	**Parameters**				
		m_R^2	m_Y^2	m_ϕ^2	m_+^2	m_-^2
A	$f(R)$	$-\frac{f_R(0)}{3f_{RR}(0)}$	∞	0	m_R^2	∞
B	$f(R, R_{\alpha\beta}R^{\alpha\beta})$	$-\frac{f(0)}{3f_{RR}(0)+2f_Y(0)}$	$\frac{f_R(0)}{f_Y(0)}$	0	m_R^2	∞
C	$f(R,\phi)+\omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$	$-\frac{f_R(0)}{3f_{RR}(0)}$	∞	$-\frac{f_{\phi\phi}(0)}{2\omega(\phi(0))}$	$m_R^2 w_+$	$m_R^2 w_-$
D	$f(R, R_{\alpha\beta}R^{\alpha\beta},\phi)+\omega(\phi)\phi_{;\alpha}\phi^{;\alpha}$	$-\frac{f(0)}{3f_{RR}(0)+2f_Y(0)}$	$\frac{f_R(0)}{f_Y(0)}$	$-\frac{f_{\phi\phi}(0)}{2\omega(\phi(0))}$	$m_R^2 w_+$	$m_R^2 w_-$

where ω_Q is the adiabtic index (the parameter of equation of state), $-1 \leqslant \omega_Q \leqslant -\frac{1}{3}$, and c the quintessence parameter. The cosmological constant (ΛCMD model) follows from (20) and (21) with $\omega_Q = -1$ and $c = \Lambda/3$,

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\ . \tag{22}$$

3. The periastron advance and inference of the theoretical bounds

In this section, we proceed to evaluate the contraints on the sizes of new gravitational forces (inferred in ETG or other scenarios) by making use the data coming from the precession of Planets. For this purpose, we follow the paper by Adkins and McDonnell [45] (see also [46, 47]), where it is calculated the precession of Keplerian orbits under the influence of arbitrary central-force perturbations. In the limit of nearly circular orbits, the perturbed orbit equation takes the form $(u = 1/r)$

$$\frac{d^2u}{d\varphi^2} + u = \frac{GM}{h^2} - \frac{g(u)}{h^2} \tag{23}$$

where $g(u) = r^2\frac{F(r)}{m}|_{r=1/u}$ $(\frac{F}{m} = -\nabla V)$ and $h^2 = GMa$. $g(u) = 0$ corresponds to the unperturbed solution. We refer to the corrections to the Planets precession induced by the Yukawa-like potential, $V_Y(r) = \alpha\frac{e^{-\beta r}}{r}$, and power law (PL) potentials, $V_{PL}(r) = \alpha_n r^n$.

In GR, the first post-Newtonian correction is a perturbing potential given by

$$V(r)\Big|_{\rm GR} = -\frac{GMh^2}{c^2r^3},$$

which corresponds the precession

$$\Delta\theta_p\Big|_{\rm GR} = \frac{6\pi GM}{c^2a}.$$

This gives the well known 43 arcsec per century when applied to the orbit of Mercury. The correction to the Planet precessions induced by a generic perturbing force $F(z)$ and perturbing potential $V(z)$ is [45]

$$\Delta\theta_p = -\frac{2a^2}{GM\epsilon}\int_{-1}^{1}\frac{dz\,z}{\sqrt{1-z^2}}\frac{F(z)}{(1+\epsilon z)^2} \tag{24}$$

$$= -\frac{2a}{GM\epsilon^2}\int_{-1}^{1}\frac{dz\,z}{\sqrt{1-z^2}}\frac{dV(z)}{dz}\,, \tag{25}$$

where, for the sake of convenience, the correction $\Delta\theta_p$ is written in terms of the dimensionless integration variable z with a fixed range, while ϵ is the eccentricity ($\epsilon < 1$). The perturbing force $F(z)$ and $V(z)$ are evaluated at radius $r = a/(1+\epsilon z)$. In the following we refer to the Yukawa-like and power-law potentials following from different gravitational theories of gravity.

- **The Yukawa force** - The Yukawa potential (as a correction to the Newtonian potential $V_N = GM/r$) is of the form [48, 49]

$$V_Y(r) = \alpha \frac{e^{-r/\lambda}}{r} \equiv \alpha \frac{e^{-\beta r}}{r} \tag{26}$$

where α and $\lambda \equiv 1/\beta$ are the strength and the range of the interaction, respectively. As we will see, such a potential occurs in several modified theories of gravity. The precession due to a Yukawa perturbation depends on two parameters: a range parameter $\kappa = a/\lambda = \beta a$ and the eccentricity ϵ, i.e. $\Delta\theta_p(\kappa, \epsilon)$, where a is the semi-major axis. According to [45], the correction to the precession is of the integral form

$$\Delta\theta_p(\kappa, \epsilon) = -\frac{2\alpha}{GM\epsilon} I_{\epsilon,\beta}\,, \tag{27}$$

where

$$I_{\epsilon,\beta} \equiv \int_{-1}^{1} \frac{dz\, z}{\sqrt{1-z^2}} \left(1 + \frac{\kappa}{1+\epsilon z}\right) e^{-\frac{\kappa}{1+\epsilon z}}\,. \tag{28}$$

The behavior of the integral (28) is represented in Fig. 1 for several Planets.

- **Power Law potential** - The power law potential is of the form

$$V_{PL}(r) = \alpha_q\, r^q\,, \tag{29}$$

where the parameter q assume arbitrary values. The precession (24) can be exactly integrated, and leads to [45]

$$\Delta\theta_p(q) = \frac{-\pi\alpha_q}{GM} a^{q+1}\sqrt{1-\epsilon^2}\chi_q(\epsilon)\,, \tag{30}$$

where $\chi_q(\epsilon)$ is written in terms of the Hypergeometric function

$$\chi_q(\epsilon) = q(q+1)\,{}_2F_1\left(\frac{1}{2} - \frac{q}{2}, 1 - \frac{q}{2}\,; 2\,; \epsilon^2\right)\,. \tag{31}$$

3.1. Planet precession in Scalar Tensor Fourth Order Gravity

In this Section we study the periastron shift of the orbital period of objects, both astrophysical and Solar System, in Scalar Tensor Fourth Order Gravity (STFOG). As seen before, the linearised STFOG field equations lead to a gravitational potential of the Yukawa-like form ($r = |\mathbf{x}|$)

$$V(r) = \frac{GM}{r}\left(1 + \sum_{i=\pm,Y} F_i e^{-\beta_i r}\right)\,, \tag{32}$$

where F_i and β are the strength and range of the interaction corresponding to each mode $i = +, -, Y$. Comparing (32) with (26), it follows the correspondence (referring to (6) and (7))

$$\alpha \to GMF_i\,, \quad \beta \to \beta_i\,, \quad i = \pm, Y\,. \tag{33}$$

Journal of Physics: Conference Series **2533** (2023) 012033 doi:10.1088/1742-6596/2533/1/012033

Table 2. Values of periastron advance for the first six planets of the Solar System. In the table we present the values of the eccentricity ϵ, semi-major axis a in meters, the orbital period P in years, the periastron advance predicted in General Relativity (GR).

Planet	ϵ	$a\,(10^{11}\,m)$	$P\,(yrs)$	$\Delta\phi_{GR}\,(''/century)$	$\Delta\phi_{obs}$
Mercury	0.205	0.578	0.24	43.125	42.989 ± 0.500
Venus	0.007	1.077	0.62	8.62	8.000 ± 5.000
Earth	0.017	1.496	1.00	3.87	5.000 ± 1.000
Mars	0.093	2.273	1.88	1.36	1.362 ± 0.0005
Jupiter	0.048	7.779	11.86	0.0628	0.070 ± 0.004
Saturn	0.056	14.272	29.46	0.0138	0.014 ± 0.002

Table 3. Bounds on F_i, $i = \pm, Y$ obtained from (36) using the values of periastron advance for planets of the Solar System.

Planet	$\lvert\eta\rvert$	$I^{\max}_{\epsilon,\beta}$	$\beta_i^{\max} \simeq$	$\lvert F_i\rvert \lesssim$
Mercury	0.5	0.18	$4 \times 10^{-11} m^{-1}$	0.28
Mars	5×10^{-4}	0.08	$1.1 \times 10^{-11} m^{-1}$	2.9×10^{-4}
Jupiter	4×10^{-3}	0.04	$2.5 \times 10^{-12} m^{-1}$	2.4×10^{-3}
Saturn	2×10^{-3}	0.05	$2 \times 10^{-13} m^{-1}$	1.1×10^{-3}

with

$$F_+ = g(\xi,\eta)\,F(m_+\mathcal{R})\,, \quad F_- = \left[\frac{1}{3} - g(\xi,\eta)\right] F(m_-\mathcal{R})\,, \quad F_Y = -\frac{4}{3}\,F(m_Y\mathcal{R})\,, \tag{34}$$

$$\beta_\pm = m_R\sqrt{\omega_\pm}\,, \qquad \beta_Y = m_Y\,. \tag{35}$$

We impose that the periastron shift $\Delta\theta_p(\kappa,\epsilon) = -\frac{2\alpha}{GM\epsilon}\,I_{\epsilon,\beta}$ given by (27), where $I_{\epsilon,\beta}$ is defined in (28), is lesser than the error η. Fixing $I_{\epsilon,\beta}$ to the maximum values, one gets the bounds on the parameters F_i:

$$|\Delta\theta_p(\kappa,\epsilon)| \lesssim \eta \quad \to \quad |F_i| \lesssim \frac{\eta\epsilon}{2I_{\epsilon,\beta_i}}\,,\; i = \pm, Y\,. \tag{36}$$

In Fig. 1 are plotted the function $I_{\epsilon,\beta}$ for the Mercury, Mars, Jupiter and Saturn planets. In Table 3 are reported the corresponding bounds on F_i. As an illustrative example, we plot $|F_\pm(\xi,\eta)|$ in Fig. 2, for $m_R = \mathcal{R}^{-1}$. The available values of the parameters $\{\xi,\eta\}$ allow to fix the masses, via Eqs. (13) , (14), (15), (18), of extra modes arising in Scalar Tensor Fourth Order Gravity. The analysis of Yukawa gravitational potential for $f(R)$ has been carried out in [50].

3.2. Non-Commutative Spectral Geometry

Following the previous section, Eqs. (27) (36), the periastron advance in NCSG for planets is given by

$$|\Delta\theta_p(\beta,\epsilon)| \lesssim \eta \quad \to \quad |I_{\epsilon,\beta}| \lesssim I_0\,, \quad I_0 \equiv \frac{3\eta\epsilon}{8}\,, \tag{37}$$

where $I_{\epsilon,\beta}$ is defined in (28). From Eq, (37) one infers the bounds on β, or equivalently an upper bound on λ. Results are reported in Table 4 (see also Fig. 3). These results show that the

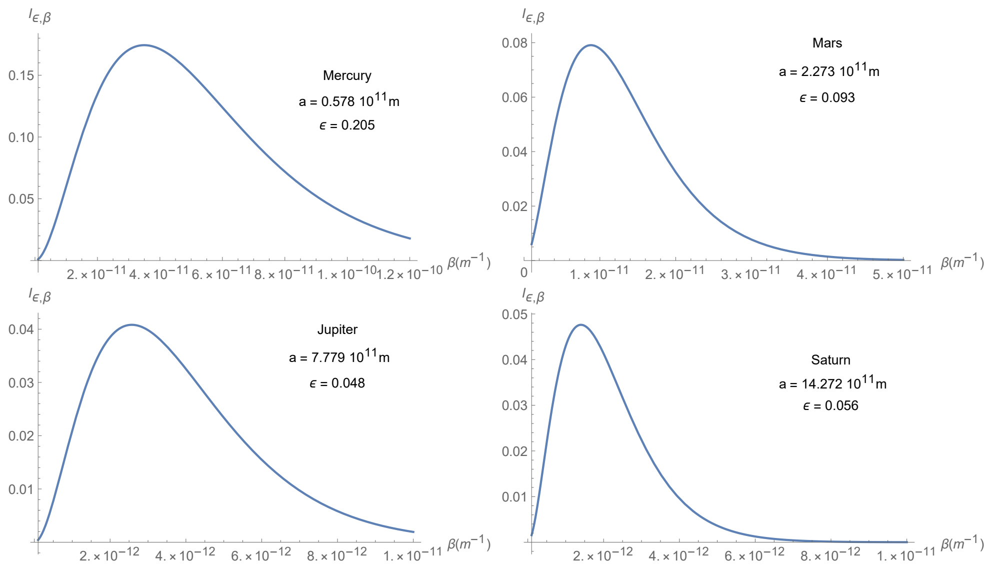

Figure 1. (a) $I_{\epsilon,\beta}$ vs β for Mercury. (b) $I_{\epsilon,\beta}$ vs β for Mars. (c) $I_{\epsilon,\beta}$ vs β for Jupiter. (d) $I_{\epsilon,\beta}$ vs β for Saturn.

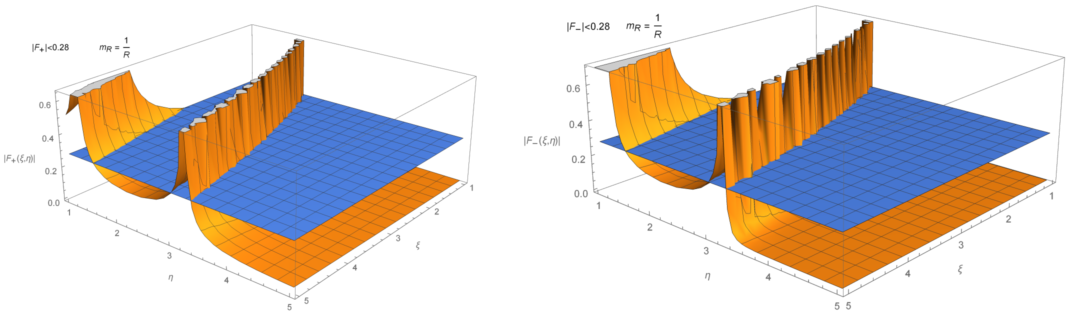

Figure 2. (a) F_+ vs $\{\xi,\eta\}$ for Mercury ($|F_+| \leq 0.28$), with $m_R = \frac{1}{\mathcal{R}}$. (b) F_- vs $\{\xi,\eta\}$ for Mercury ($|F_-| \lesssim 0.28$) with $m_R = \frac{1}{\mathcal{R}}$.

bounds on β improve several order of magnitude as compared with ones obtained using recent observations of pulsar timing, $\beta \geq 7.55 \times 10^{-13}\text{m}^{-1}$ [58, 59]. Bounds on the parameter β have been obtained in different frameworks. From Gravity Probe B experiment one gets $\beta > 10^{-6}\text{m}^{-1}$ [54]. A more stringent constraint on β can be obtained from laboratory experiments designed to test the fifth force, that is, by constraining λ through torsion balance experiments which implies to obtain a stronger lower bound on β (or equivalently an upper bound to the momentum f_0 in NCSG theory). The test masses have a typical size of $\sim$ 10mm and their separation is smaller than their size. As we have already mentioned above, in NCSG one has $|\alpha| \sim \mathcal{O}(1)$, so that the tightest constraint on $\lambda = \beta^{-1}$ provided by Eöt-Wash [60] and Irvine [61] experiments is [62] $\lambda \lesssim 10^{-4}$m, or equivalently $\beta \gtrsim 10^4\text{m}^{-1}$.

3.3. Quintessence Fields related to the Dark Energy
The Quintessential potential reads $V_Q = -\frac{c}{r^{3\omega_Q+1}}$, so that comparing with (29) one gets

$$q \to -(3\omega_Q + 1) \qquad \alpha_q \to c\,.$$

Table 4. Lower bounds on β obtained from (37) using the values of periastron advance for planets of the Solar System.

Planet	η	$I_0 \equiv \frac{3\eta\epsilon}{8}$	$\beta(m^{-1}) >$
Mercury	0.5	0.038	1.0×10^{-10}
Mars	5×10^{-4}	1.36	7.8×10^{-11}
Jupiter	4×10^{-3}	0.0628	2.1×10^{-11}
Saturn	2×10^{-3}	0.0138	8.5×10^{-12}

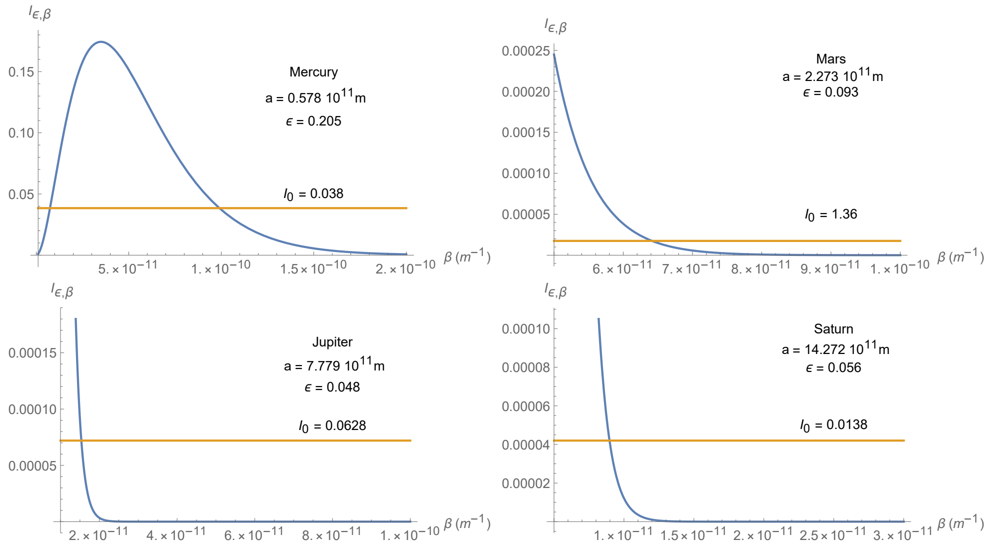

Figure 3. (a) $I_{\epsilon,\beta}$ vs β for Mercury. (b) $I_{\epsilon,\beta}$ vs β for Mars. (c) $I_{\epsilon,\beta}$ vs β for Jupiter. (d) $I_{\epsilon,\beta}$ vs β for Saturn.

The precession (30) leads to

$$|\Delta\theta_p(\omega_Q, \epsilon)| = \frac{\pi c}{GM} a^{-3\omega_Q} \sqrt{1-\epsilon^2} \chi_{\omega_Q}(\epsilon)\,, \tag{38}$$

with

$$\chi_{\omega_Q}(\epsilon) = 3\omega_Q(1+3\omega_Q)\ {}_2F_1\left(\frac{2+3\omega_Q}{2}, \frac{3+3\omega_Q}{2}\,; 2\,; \epsilon^2\right)\,. \tag{39}$$

By requiring $|\Delta\theta_p(\omega_Q, \epsilon)| \lesssim \eta$ one gets the bounds on the parameters $\{\omega_Q, c\}$. Results are reported in Table 5 and Fig. 4 for fixed values of c.

4. Test on S2 Star

Finally, we shortly conclude our analysis testing the modified gravity predictions for S2 Star orbiting around Sagittarius A*, the Supermassive Black Hole at the center of the Milky Way, which has got a mass equal to $M = (4.5 \pm 0.6) \times 10^6 M_\odot$ and a Schwarzschild radius $R_S = 2GM = 1.27 \times 10^{10}\, m$. The S2 Star orbit has an eccentrity $\epsilon = 0.88$ and a semi-major axis $a = 1.52917 \times 10^{14}$m. According to Ref. [75], the periastron advance is (0.2 ± 0.57) deg, hence $\eta = 0.57$ (it is expected that the interferometer GRAVITY may improve such an accuracy level). We discuss the periastron advance for the gravitational models above discussed:

Table 5. Values of the parameter ω_Q obtained from (38) using the values of periastron advance for planets of the Solar System.

Planet	η	$c(m^{3\omega_Q+1}) \sim$	$\omega_Q \gtrsim$
Mercury	0.5	10^{-25}	-0.86
Mars	5×10^{-4}	10^{-30}	-0.88
Jupiter	4×10^{-3}	10^{-30}	-0.84
Saturn	2×10^{-3}	10^{-30}	-0.82

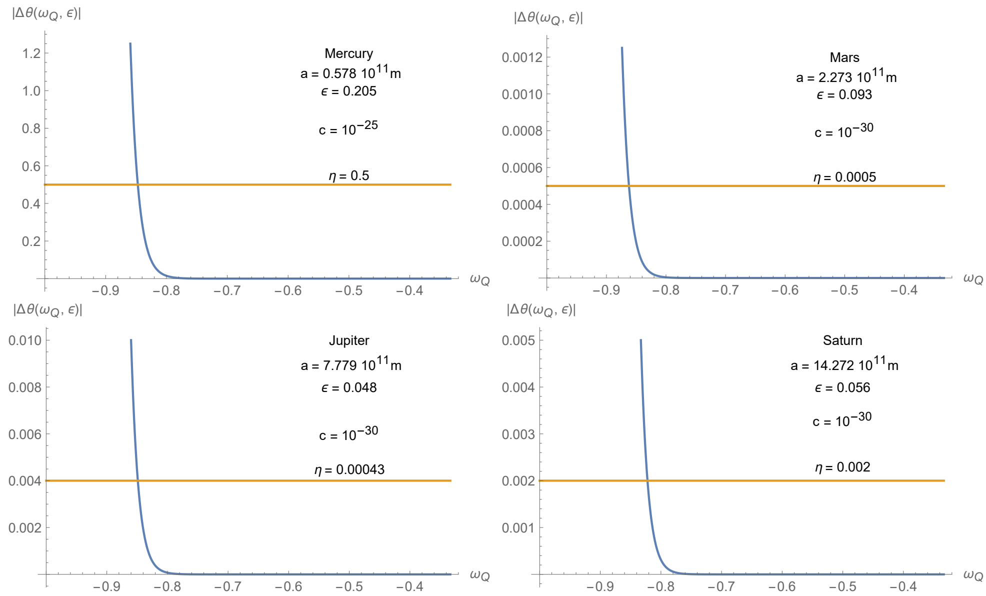

Figure 4. (a) $|\Delta\theta(\omega_Q,\epsilon)|$ vs ω_Q for Mercury. (b) $|\Delta\theta(\omega_Q,\epsilon)|$ vs ω_Q for Mars. (c) $|\Delta\theta(\omega_Q,\epsilon)|$ vs ω_Q for Jupiter. (d) $|\Delta\theta(\omega_Q,\epsilon)|$ vs ω_Q for Saturn.

- **STFOG** - Referring to Scalar-Tensor Fourth Order Gravity, from Eq. (36) one gets

$$|\Delta\theta_p(\kappa,\epsilon)| \lesssim \eta \quad \to \quad |F_i| \lesssim \frac{\eta\epsilon}{2I_{\epsilon,\beta_i}} \sim 0.36\,, \; i = \pm, Y\,. \tag{40}$$

where in Fig. 5(a) is plotted the function $I_{\epsilon,\beta}$ for the S2 star. We ave taken the maximum value of $I_{\epsilon,\beta}$ corresponding to $\beta \sim 2\times10^{-14} m^{-1}$ (see Fig. 5(a)). The analysis of S2 star orbit around the Galactic Centre in $f(\phi, R)$ and $f(R, \Box R)$ has been investigated in [76].

- **NCSG** - The S2 star values $\{\epsilon, \eta, a\}$ imply that, from (37),

$$|\Delta\theta_p(\beta,\epsilon)| \lesssim \eta \quad \to \quad |I_{\epsilon,\beta}| \lesssim I_0\,, \quad I_0 \equiv \frac{3\eta\epsilon}{8} \simeq 0.19\,. \tag{41}$$

Results are reported in Fig. 5(b). We can see that the lower bound on β is $\beta \gtrsim 1.1\times10^{-13} m^{-1}$. These bounds are compatible with the astrophysical bounds [58, 59].

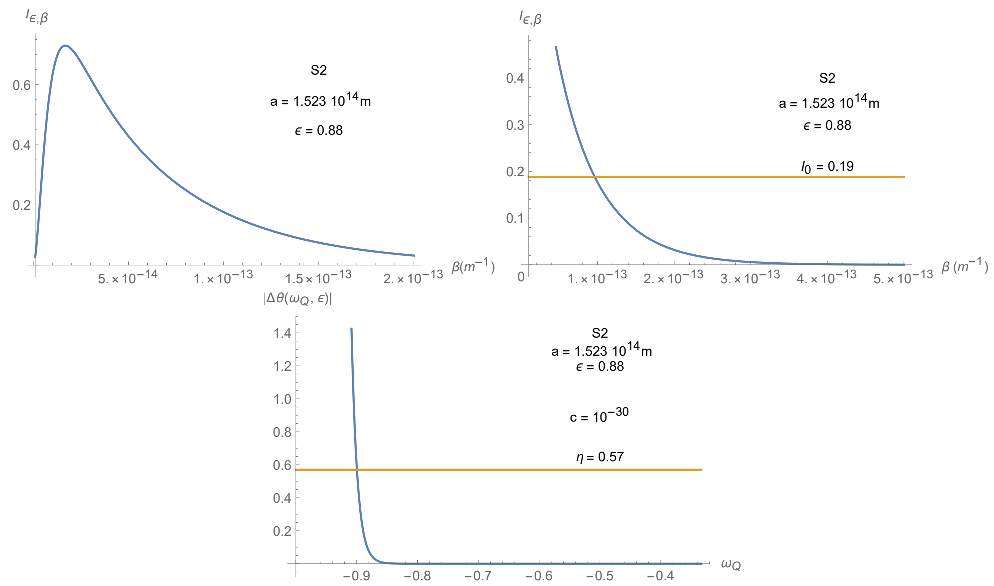

Figure 5. (a) $I_{\epsilon,\beta}$ vs β for S2 star in FOG theories. (b) $I_{\epsilon,\beta}$ vs β for S2 star. (c) $|\Delta\theta\omega_Q,\epsilon|$ vs ω_Q for S2 star. (c is in $m^{3\omega_Q+1}$ units).

- **Quintessence** - In the case of Quintessence field deforming the Schwarzschild geometry, Eq. (38) implies

$$|\Delta\theta_p(\omega_Q,\epsilon)| = \frac{\pi c}{GM} a^{-3\omega_Q}\sqrt{1-\epsilon^2}\chi_{\omega_Q}(\epsilon) \lesssim 0.57\,, \tag{42}$$

$$\chi_{\omega_Q}(\epsilon) = 3\omega_Q(1+3\omega_Q)\;{}_2F_1\left(\frac{2+3\omega_Q}{2},\frac{3+3\omega_Q}{2};2;\epsilon^2\right). \tag{43}$$

Results are reported in Fig. 5(c), from which it follows that for Quintessence $|\Delta\theta_p(\omega_Q,\epsilon)| \lesssim 0.57$ provided $\omega_Q \gtrsim 0.9$. Therefore, the exact value $\omega_Q = -1$ corresponding to the cosmological constant is excluded in this range of values.

5. Conclusions

In this conference we have reported the results presented on Ref. [82]. The analysis is relative to the periastron advance of Solar system Planets in the case in which the gravitational interactions between massive bodies is described by modified theories of gravity. In these classes of theories, the corrections to the Newtonian gravitational interaction is of the Yukawa-like form, $V(r) = V_N(1+\alpha e^{-\beta r})$, where $V_N = GM/r$ is the Newtonian potential, or the power-law form, $V(r) = V_N + \alpha_q r^q$. To compute the corrections to the periastron advance, we have used results of Ref. [45] in which the general integrals are provided in terms of the central body's mass M, and the orbital parameters a and ϵ, the semi-major axis and eccentricity of the orbit, respectively. This two-body system constitutes a good model for many astrophysical scenarios, such as those at the scale of Solar System, constituted by the Sun and a planet, as well as binary system composed by a Super Massive Black Hole and an orbiting star, which are both the most suitable candidates to test a gravitational theory.

In the case of Scalar Tensor Fourth Order Gravity, we find that the parameters of the model are given by (see Eqs. (35, 34, 33) $\alpha \sim F_i$, $\beta \sim \beta_i$, with $i = \pm, Y$, $F_+ = g(\xi,\eta)\,F(m_+\mathcal{R})$, $F_- = \left[\frac{1}{3} - g(\xi,\eta)\right] F(m_-\mathcal{R})$, $F_Y = -\frac{4}{3}\,F(m_Y\mathcal{R})$, $\beta_\pm = m_R\sqrt{\omega_\pm}$, $\beta_Y = m_Y$. The greatest value of β_i is $\beta_i \sim 5 \times 10^{-11} m^{-1}$, which leads to the constraint on F_i is $F_i < 10^{-4}$. This allows to get a bound on the massive modes m_i, $i = \pm, Y$, corresponding to the extra modes presents in ETG.

For the Non-Commutative Spectral Gravity, we have found that the perihelion's shift of planets allows to constrain the parameter β at $\beta > (10^{-11} - 10^{-10}) m^{-1}$ (in this theory the parameter α is given and is of the order $\alpha \sim \mathcal{O}(1)$). Such a constraint on the parameter β improves several order of magnitude ones derived by using pulsar timing $\beta \geq 7.55 \times 10^{-13} m^{-1}$ [58, 59]. These constraints, however, are weaker compared to the ones obtained from terrestrial experimental data, Eöt-Wash [60] and Irvine [61] experiments is [62], which give $\beta \gtrsim 10^4 m^{-1}$ (a bound on β has been derived from Gravity Probe B experiment, giving $\beta > 10^{-6} m^{-1}$ [54]).

We also studied the Quintessence field surrounding a massive gravitational source. Here the parameter characterizing the gravitational field are the adiabatic index ω_Q and the quintessence parameter c. The analysis shows that c assumes tiny values, as expected, being essentially related to the cosmological constant, while $\omega_Q \gtrsim -(0.9 - 0.8)$, that is it never assumes the value -1 corresponding to the pure cosmological constant.

The case of the S2 Star around Sagittarius A*, the Super Massive Black Hole at the center of the Milky Way, has been also studied. In such a case, we have found that for STFOG and NCSG $\beta > 10^{-13} m^{-1}$, a this bound is compatible with astrophysical constraints. For the Quintessence Field we have inferred $\omega_Q \gtrsim -0.9$.

6. Acknowledgments

We acknowledge the INFN (*Istituto Nazionale di Fisica Nucleare*) for the support.

References

[1] Riess A G *et al.* 1998 *Astron. J.* **116** 1009-1038.
[2] Perlmutter S *et al.* 1999 *Astrophys. J.* **517** 565-586.
[3] Cole S *et al.* 2005 *Mon. Notices Royal Astron. Soc.* **362** 505-534.
[4] Spergel S D *et al.* 2007 *Astrophys. J., Suppl. Ser.* **170** 377-408.
[5] Carroll S M, Press W H and Turner E L 1992 *Annu. Rev. Astron. Astrophys.* **30** 499-542.
[6] Sahni V and Starobinski A A 2000 *Int. J. Mod. Phys. D* 2000, **9**, 373.
[7] Capozziello S and De Laurentis M 2011 *Phys. Rep.* **509** 167.
[8] Magnano G, Ferraris M and Francaviglia M 1987 *Gen. Rel. Grav.* **19** 465.
[9] Capozziello S and De Laurentis M 2011 *Phys. Rep.* **509** 167.
[10] Perivolaropoulos L 2010 *Phys. Rev.D* **81** 047501.
[11] Hohmann M, Jarv L, Kuusk P and Randla E 2013 *Phys. Rev. D* **88** 084054.
[12] Järv L, Kuusk P, Saal M and Vilson O 2015 *Phys. Rev. D* **91** 024041.
[13] Nojiri S and Odintsov S D 2002 *Phys. Lett. B* **548** 215.
[14] Bronnikov K A Kononogov S A and Melnikov V N 2006 *Gen. Rel. Grav* **38** 1215.
[15] Teyssandier P and Tourrenc J. 1983 *Math. Phys.* **24** 2793.
[16] Maeda K I 1989 *Phys. Rev. D* **39** 3159.
[17] Wands D 1994 *Class. Quant. Grav.* **11** 56.
[18] Schmidt H J 1987 *Astron. Nachr.* **308** 34.
[19] Berry C P L and Gair J R 2011 *Phys. Rev. D* **83** 104022.
[20] Capozziello S, Lambiase G, Sakellariadou M and Stabile A 2015 *Phys. Rev. D* **91** 044012.
[21] Schellstede G O 2016 *Gen. Rel. Grav.* **48** 118.
[22] Kaczmarek A Z and Szczesniak D 2021 *Sci. Rep.* **11** 18363.
[23] G. Lambiase, M. Sakellariadou and A. Stabile 2013 *J. Cosmol. Astropart. Phys.* 1312 020.
[24] Arkani-Hamed N, Dimopoulos S and Dvali G R 1998 *Phys. Lett. B* **429** 263.
[25] Arkani-Hamed N, Dimopoulos S and Dvali G R 1998 *Phys. Rev. D* **59** 086004.
[26] Antoniadis I, Arkani-Hamed N, Dimopoulos S and Dvali G R 1998 *Phys. Lett. B* **436** 257.
[27] Gasperini M and Veneziano G 1992 *Phys. Lett. B* **277** 256.
[28] Damour T and Polyakov A M 1994 *Gen. Rel. Grav.* **26** 1171.
[29] Gasperini M and Veneziano G 1994 *Phys. Rev. D* **50** 2519.
[30] Deser S and Woodard R, *Phys. Rev. Lett.* **99** 111301.
[31] Stabile A 2010 *Phys. Rev. D* **82** 064021.

[32] Capozziello S and Stabile A 2015 *Astrophys. Space Sci.* 358, 27.
[33] Boehmer C G, Harko T and Lobo F S N 2008 *Astropart. Phys.* **29** 386-392.
[34] Boehmer C G, Harko T and Lobo F S N 2008 *J. Cosmol. Astropart. Phys.* **0803**, 024.
[35] Stabile A and Scelza G 2012 *Phys. Rev. D* **84** 124023.
[36] Lambiase G, Mohanty S and Stabile A 2018 *Eur. Phys. J. C* **78**, 350.
[37] Weyl H 1921 *Raum zeit Materie: Vorlesungen uuber allgemeine Relativitatstheorie* Springer Berlin.
[38] Eddington A S 1924 *The Mathematical Theory of Relativity*, Cambridge University Press London.
[39] Lanczos C 1932 *Z. Phys. A: Hadrons Nucl.* **73**, 147-168.
[40] Capozziello S, Lambiase G, Sakellariadou M, Stabile A and Stabile An 2015 *Phys. Rev. D* **91** 044012.
[41] A. Stabile and S. Capozziello 2013 *Phys. Rev. D* **87** 064002.
[42] Lombriser L and Taylor A 2016 *J. Cosmol. Astropart. Phys.* **03** 031.
[43] De Laurentis M, Porth O, Bovard L, Ahmedov B and Abdujabbarov A *Phys. Rev. D* **94** 124038.
[44] Capozziello S, Lambiase G, Stabile A and Stabile A 2021 *Eur. Phys. J. Plus* **136** 144.
[45] Adkins G S and McDonnell J 2007 *Phys. Rev. D* **75** 082001.
[46] Xu F 2011 *Phys. Rev. D* **83** 084008.
[47] Chashchina O I and Silagadze Z K 2008 Phys. Rev. D **77** 107502.
[48] Yukawa H 1935 *Proc. Phys. Math Soc. Japan* **17** 48.
[49] Nieto M M and Goldman T 1991 *Phys. Rep.* **205** 221.
[50] De Laurentis M, De Martino I and Lazkoz R *Phys. Rev. D* **97** 104068.
[51] Connes A 1994 *Noncommutative Geometry* Academic Press New York.
[52] Connes and Marcolli M 2008 *Noncommutative Geometry, Quantum Fields and Motives* Hindustan Book Agency India.
[53] Sakellariadou M, Stabile A and Vitiello G 2011 *Phys. Rev. D* **84** 045026.
[54] Lambiase G, Sakellariadou M and Stabile A 2013 *J. Cosmol. Astropart. Phys.* 12 020.
[55] Stelle K S 1978 *Gen. Rel. & Grav.* **9** 353.
[56] Donoghue J F 1994 *Phys. Rev.* D **50**, 3874.
[57] Fischbach E and Talmadge C L 1999 *The search for non-newtonian gravity*, Springer Verlag.
[58] Nelson W, Ochoa J and Sakellariadou M 2010 *Phys. Rev. D* **82** 085021.
[59] Nelson W, Ochoa J and Sakellariadou M 2010 *Phys. Rev. Lett.* **105** 101602.
[60] Hoyle C D *et al.* 2001 *Phys. Rev. Lett.* **86** 1418.
[61] Hoskin J K *et al.* 1985 *Phys. Rev. D* **32**, 3084.
[62] Kapner D J *et al.* 2007 *Phys. Rev. Lett.* **98** 021101.
[63] Jamil M, Hussain S and Majeed B 2015 *Eur. Phys. J. C* **75** 24.
[64] V. Kiselev 2003 *Class. Quant. Grav.* **20** 1187.
[65] Belhaj A, El Balali A, El Hadri W, Hassouni Y and Torrente-Lujan E 2021 *Int. J. Mod. Phys. A* **36** 2150057.
[66] Chen S, Wang B and Su R 2008 *Phys. Rev. D* **77** 124011.
[67] Toshmatov B, Stuchlik Z and Ahmedov B 2017 *Eur. Phys. J. Plus* **132** 98.
[68] Ghosh S G 2016 *Eur. Phys. J. C* **76** 222.
[69] Belhaj A, Balali A E, Hadri W E, Hassouni Y and Torrente-Lujan E 2020 *Class. Quant. Grav.* **37** 215004.
[70] Israr Ali Khan S I, Amir Sultan Khan and Ali F 2020 *Int. J. Mod. Phys.* **29** 2050095.
[71] Khan S U and Ren J 2020 *Phys. Dark Univ.* **30** 100644.
[72] Abbas G, Mahmood A and Zubair M 2020 *Chin. Phys. C* **44** 095105.
[73] Javed W, Abbas J and Ovgun A 2020 *Annals Phys.* **418** 168183.
[74] Uniyal R, Chandrachani Devi N, Nandan H and Purohit K D 2015 *Gen. Rel. Grav.* **47** 16.
[75] Iorio L 2017 *Mon. Notices Royal Astron. Soc.* **472** 2249.
[76] Capozziello S, Borka D, Jovanovic P and Borka Jovanovic V 2014 *Phys. Rev. D* **90** 044052.
[77] Khoury J and Weltman A 2004 *Phys. Rev. Lett.* **93** 171104.
[78] Hinterbichler K and Khoury J 2010 *Phys. Rev. Lett.* **104** 231301.
[79] Sakstein J 2018 *Phys. Rev. D* **97** 064028.
[80] Zhang X, Zhao W, Huang H and Y. Cai 2016 *Phys. Rev. D* **93** 124003.
[81] Brax P, van de Bruck C, Davies C, Khoury J, Weltman A 2004 *Phys. Rev D* **70** 123518.
[82] Capolupo A, Lambiase G and Tedesco A 2022 *Eur. Phys. J. C* **82** 286.