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Gauge Invariance of String Fields

THOMAS BANKS[*] [†]

Department of Physics and Astronomy,
Tel Aviv University,
Ramat Aviv, Tel Aviv, Israel

and

MICHAEL E. PESKIN[*]

Stanford Linear Accelerator Center
Stanford University, Stanford, California, 94305

ABSTRACT

We identify the gauge invariances of the linearized field theory of strings which give rise to the Yang-Mills and general coordinate invariance of this theory. We construct a kinetic energy term for string fields which is invariant to these gauge symmetries. By gauge-fixing, we derive from this action the expressions for the free string action in particular gauges found by Kaku and Kikkawa and by Siegel. The structure of Stueckelberg auxiliary fields required to make the gauge-invariant action local is rather intricate; to clarify this structure, we develop a theory of differential forms on the space of strings. We conclude with some remarks on the origin of the dilaton and the appearance in the superstring of local supersymmetry.

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† Present address: Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305.

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1. Introduction

The discovery by Green and Schwarz[1] of consistent theories of supersymmetric strings, endowed with a phenomenologically relevant gauge symmetry and yet free of gauge and gravitational anomalies, has caused an explosion of interest in the subject of string theories. This discovery provides the latest link in a remarkable chain of mathematical properties which these string theories possess. Other such properties are the automatic appearance of gauge particles and gravitons[2,3] and, for the appropriately truncated spinning string, the automatic appearance of supersymmetry[4] and the cancellation of divergences, at least at one loop[5,6]. The whole theory provides a formal structure of great coherence and power, one which it seems important to understand more deeply.

One particularly puzzling aspect of this structure is the appearance of scattering amplitudes which are automatically gauge-invariant. This behavior was first noted by Neveu and Scherk[2], who studied the low-energy limit of the scattering of zero-mass open strings and found exactly the scattering amplitude of gauge bosons in Yang-Mills theory. Scherk and Schwarz[3] performed a similar analysis of the low-energy scattering of closed strings and found the graviton-graviton scattering amplitude. In neither case did the result seem to follow from some higher principle; rather it appeared magically from the string formalism.

A more recent development, however, has provided a clue to the origin of these gauge-theory results. Though a field theory of strings was formulated long ago in the transverse gauge[7–9], the corresponding covariant treatment was discovered only a year ago, when Siegel[10,11] wrote down a transcription in field theory of the covariant and BRST-invariant first quantization of the string[12]. Examining his formulation of the covariant string field theory mass level by mass level, Siegel found a rich structure of BRST-invariant particle theories, including the covariantly gauge-fixed versions of Yang-Mills theory and gravity.

Motivated by this discovery, we set out to understand this structure further by identifing the gauge-invariant string field theory from which Siegel's formulation might arise by gauge-fixing. In this paper, we would like to present a proposal for the linearized version of that theory. The action we will present is invariant under a huge group of gauge symmetries which arise naturally from the mathematical structure of the string. These gauge symmetries contain linearized Yang-Mills and general coordinate invariance as proper subgroups.

The plan of this paper is as follows. The bulk of our analysis will concern the simple, purely bosonic open string theory. In Section 2, we will review some basic formalism and apply this formalism to construct a suitably reparametrization-invariant kinetic energy term for string fields. In Section 3, we will study the symmetries of this action and recognize, in particular, an enormous group of gauge invariances. In Section 4, we will present a relatively explicit form of the kinetic energy term for strings which respects these symmetries. Our construction, however, yields an action which is nonlocal when considered as an action for fields on coordinate space. To define a proper quantum theory, we must remove this nonlocality by introducing Stueckelberg fields. As an introduction to this procedure, we show explcitly how to do this at the spin-2 mass level. We also present a simple construction which brings the action into a local form at all levels.

The set of Stueckelberg fields presented at the end of Section 4 is unsatisfactory, however, for two reasons. First, it leads to an action with 4-derivative terms, and, secondly, it yields a larger number of degrees of freedom than appear in the conventional quantized string theory. To solve these problems, we must seek the minimal set of Stueckelberg fields necessary to make the action local in the critical dimension, $d = 26$. We will present this set of Stueckelberg fields in Section 6. In Section 5, we will present a mathematical development which is useful in this analysis, a theory of differential forms on the space of strings. In Section 7, we will discuss the quantization of this action by gauge-fixing. We will present the quantization using two different gauge-fixing procedures and, in this way, connect our formalism with the earlier string field theories of Kaku and Kikkawa[7] and Siegel[10,11] . The analysis will provide a confirmation of the set of Stueckelberg fields found in Section 6.

The remainder of the paper will discuss some generalizations of this construction. In Section 8, we will discuss the extension of our analysis to closed strings. In Section 9, we will discuss the extension of our analysis to the case of superstrings. (These sections do not depend on the relatively technical arguments of Sections 5 – 7.)

The covariant formulation of the string field theory has also been studied recently by Kaku and Lykken[13] ; working from a rather different viewpoint, these authors have also arrived at an action similar to the one we will present in Section 4. Between the time of the first announcement of our results[14] and the completion of this paper, Friedan[15] has developed our proposal in some new directions. Thorn[16] has discussed the gauge fixing to the transverse gauge.

As we were completing this paper, we received a preprint by Neveu and West[17] in which the structure discussed in Section 6 was built up through the first five excited mass levels by explicit rearrangements of the nonlocal string action. We have also learned that Siegel and Zweibach[18] have derived the complete action which we present in Section 6, using a technique very different from the one explained here.

2. Reparametrization Invariance

We begin our analysis from the case of bosonic open strings. These strings describe two-dimensional world sheets as they move through space-time. The mechanics of strings is defined by the condition that the evolution of these world sheets is determined purely geometrically and does not depend on the coordinate system used to parametrize the sheet. This means that the transformations which generate reparametrizations of an individual sheet must be symmetries of the equations of motion. The quantization of the theory should respect these symmetries. Of course, quantization procedures for a single string which deal properly with the reparametrization invariance are well known[19] . But we would like to address this question at a somewhat different level; we ask, how does the reparametrization invariance of individual world sheets manifest itself in the field theory of strings?

To pose this question more carefully, let us introduce some notation* . We choose units in which the Regge slope is given by $2\alpha' = 1$. For a single string, the coordinate and momentum variables may be expanded in normal modes:

$$\begin{aligned} x^\mu(\sigma) &= x^\mu + \sum_{n>0} \frac{2}{\sqrt{n}} X^\mu_n cos n\sigma \\ p^\mu(\sigma) &= \frac{1}{\pi}\left\{p^\mu + \sum_{n>0} \sqrt{n} P^\mu_n cos n\sigma\right\}; \end{aligned} \tag{2.1}$$

$0 \le \sigma \le \pi$, and $[X_n, P_m] = i\delta_{nm}$. It is convenient to replace

$$X_n = \frac{i}{2\sqrt{n}}(\alpha_n - \alpha_{-n}) \ , \quad P_n = \frac{1}{\sqrt{n}}(\alpha_n + \alpha_{-n}) \ , \tag{2.2}$$

and to set $\alpha_0^\mu = p^\mu$; then the α_n have the commutation relations:

$$[\alpha^\mu_n, \alpha^\nu_m] = n\,\delta(n+m)\,\eta^{\mu\nu}. \tag{2.3}$$

$p(\sigma)$ and $x'(\sigma)$ are especially simple functions of the α_n:

$$\left(\pi p \pm x'\right) = \sum_{n=-\infty}^{\infty} \alpha_n e^{\mp in\sigma}. \tag{2.4}$$

The generators of reparametrizations of the string are the local Hamiltonian and

* For a review of string technology, see ref. 20.

momentum densities:

$$\mathcal{H}(\sigma) = \frac{1}{2\pi}\left(\pi^2 p^2 + (x')^2\right), \quad \mathcal{P}(\sigma) = p \cdot x'. \tag{2.5}$$

These quantities are summarized as:

$$\frac{1}{2}\left(\pi p \pm x'\right)^2 = \sum_{-\infty}^{\infty} L_n e^{\mp in\sigma}, \tag{2.6}$$

where the L_n are the Virasoro operators[21]

$$L_n = \frac{1}{2}\sum_{-\infty}^{\infty} : \alpha^{\mu}_{n+k}\alpha^{\mu}_{-k} : . \tag{2.7}$$

These operators satisfy the algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{d}{12}n(n^2-1)\delta(n+m), \tag{2.8}$$

in which the central charge depends on d, the dimensionality of space.

The Virasoro operators summarize much of the dynamical content of the theory of a single string. L_0 contains the string mass operator, so that

$$2(L_0 - 1) = p^2 + 2\Big\{\sum_{n>0} \alpha_{-n}\cdot\alpha_n - 1\Big\} = p^2 + \mathcal{M}^2 \tag{2.9}$$

gives the equation of motion of physical states. Reparametrizations of the evolving string surface, local shifts of both σ and τ, may be expressed as transformations

$$\delta\left|\Phi\right\rangle = i\sum_{n=-\infty}^{\infty} b_n L_{-n}\left|\Phi\right\rangle \qquad (b_{-n} = b_n^*) \tag{2.10}$$

of the string wave function $|\Phi\rangle$. Note that it is reasonable to discuss local shifts of coordinate time τ on the evolving surface even though the wavefunction depends only on $x^{\mu}(\sigma)$, the string location at one fixed time; this is done here in the same way one discusses the symmetry with respect to local shifts of coordinate time in quantum gravity[22] .

To go from the quantization of a single string to the field theory of strings, we should reinterpret the string wavefunction $|\Phi\rangle$ as a string field functional $\Phi[x(\sigma)]$. Let us write the linearized action for the string field schematically as

$$S_{\mathrm{R}} = -\frac{1}{2}(\Phi \mid K_{\mathrm{R}}\Phi). \tag{2.11}$$

The inner product involves an integral over string configurations $x(\sigma)$. We would like to arrange that S is invariant to the transformations (2.10) and thus inherits the reparametrization invariance of the single string. To see how this might work, insert (2.10) into the expression for S; one obtains;

$$\delta S_{\mathrm{R}} = -\frac{i}{2}\sum_{n} b_n(\Phi \mid [K_{\mathrm{R}}, L_{-n}]\Phi). \tag{2.12}$$

We must, then, construct a kinetic energy operator K_{R} which commutes with all the L_n.

Before beginning that construction, however, it is useful to recall that reparametrization invariance is actually implemented in a rather different way in the standard covariant (first) quantization of the string due to Goddard, Goldstone, Rebbi, and Thorn[19] . In that formalism, one restricts one's attention to the subspace of Φ's which satisfy the condition:

$$L_n\Phi[x(\sigma)] = 0 \qquad (n > 0) \tag{2.13}$$

and implements reparametrization invariance on this subspace. In the literature on dual models, states satisfying (2.13) are called *physical states.* We find it less confusing to call them simply *states at level 0.* (We will define the higher levels in a moment.)

It is possible to set up an analogue of this restricted invariance for the string field theory by the following construction: Let us define a kinetic energy operator K which is proportional to a projector onto the subspace of states at level 0.

Let Φ_0 denote the projection of Φ onto this subspace. The action of a general reparametrization on Φ_0 is:

$$\delta\Phi_0 = i\sum_{n\geq 0} b_n L_{-n}\Phi_0 \tag{2.14}$$

The motions (2.14) will be symmetries of

$$S \;=\; -\frac{1}{2}(\Phi \mid K\Phi). \tag{2.15}$$

if (1) $[K, L_0] = 0$ and (2) $KL_{-n} = 0$ for $n > 0$. Condition (1) is generally straightforward to arrange. Condition (2) is actually implied by the statement that K contains the projector. Thus, S can readily be made invariant to (2.14). A specific choice of K which satisfies these requirements and reduces to (2.9) on states at level 0 is:

$$K \;=\; 2(L_0 - 1)P, \tag{2.16}$$

where P is the projector onto level 0.

At this point, it is worth discussing further the nature of the projection we require. Given a state at level 0, we can form states at higher levels by applying operators L_{-n}. Let us label the products of L_{-n}'s which raise the mass level of the string state by n units as $\mathcal{L}_{-i}^{(n)}$:

$$\mathcal{L}_{-i}^{(n)} \in \left\{L_{-1}{}^{n}, L_{-2}L_{-1}{}^{n-2}, L_{-2}{}^{2}L_{-1}{}^{n-4}, \ldots, L_{-n}\right\} \tag{2.17}$$

Call $(\mathcal{L}_{-i}^{(n)})^\dagger = \mathcal{L}_{i}^{(n)}$. We may then define the states at level n to be the states created from level 0 by the application of the $\mathcal{L}_{-i}^{(n)}$. The whole tower of states arising from a particular level 0 state is called a Verma module[23] . States at different levels are orthogonal; for example, if Φ_0 and Φ_1 are at level 0 and 1, respectively,

$$(\Phi_0 \mid \Phi_1) \;=\; (\Phi_0 \mid L_{-1}\Phi_0') \;=\; (L_1\Phi_0 \mid \Phi_0') \;=\; 0. \tag{2.18}$$

It is known that the $\mathcal{L}_{-i}^{(n)}$ create linearly independent vectors in level n, except at a discrete set of values of p (which enters the L_n as a parameter)[24,25] . Since we will work off-shell, it suffices to establish the properties of $\mathcal{K}$ for generic p. Note that L_{-n} raises the level; hence the projector onto level 0 annihilates L_{-n}.

Given this structure, we can now explain how to construct $\mathcal{K}_{\mathrm{R}}$. $\mathcal{K}_{\mathrm{R}}$ will commute with all of the generators of the Virasoro algebra if it takes the same value on all states of a Verma module. Therefore, let $\mathcal{K}_{\mathrm{R}}$ be equal to $\mathcal{K}$ on level 0. On higher levels, define $\mathcal{K}_{\mathrm{R}}$ as the value of $\mathcal{K}$ on the level 0 state in the same Verma module. We will give a more explicit form of $\mathcal{K}_{\mathrm{R}}$ in Section 4.

3. Gauge Invariance

We have now sketched the construction of a completely reparametrization-invariant string kinetic energy term. Its construction involved an auxiliary object $\mathcal{K}$, which contained a projector onto level 0 states. This auxilary object is, however, interesting in its own right. In this section, we will study it further. We will present evidence that it is this $\mathcal{K}$, and not $\mathcal{K}_{\mathrm{R}}$, which is in fact the correct kinetic energy term for strings.

The remarkable property of $\mathcal{K}$, not shared by $\mathcal{K}_{\mathrm{R}}$, is its invariance under an enormous group of additional symmetries. We constructed $\mathcal{K}$ to preserve a part of the homogeneous transformations (2.10) which implement reparametrization invariance. However, $\mathcal{K}$ is also invariant to the corresponding inhomogeneous transformations, the shifts:

$$\delta\Phi[x(\sigma)] \;=\; \mathcal{L}^{(n)}_{-i}\Psi_{ni}[x(\sigma)], \tag{3.1}$$

where Ψ_{ni} is at level 0, or, equivalently (with some double-counting),

$$\delta\Phi[x(\sigma)] \;=\; L_{-n}\Psi_{n}[x(\sigma)], \tag{3.2}$$

where Ψ_n is unconstrained. These motions are symmetries of $\mathcal{K}$ because P annihilates the L_{-n}. Eq. (3.2) is very similar in structure to the invariance of the string field theory proposed by Siegel in ref. 10. Transformations of this form stand at a level of a hierarchy above global gauge transformations, in which one shifts by a constant, and local gauge transformation, in which one shifts by a function of x. Here one shifts by a function on the space of strings, and so we should refer to (3.2) as a *chordal* gauge transformation.

What is the content of this huge group of invariances? To analyze this question, it is useful to expand Φ in eigenstates of the mass operator $\mathcal{M}^2$, eq. (2.9). Let $\Phi^{(0)}$ be the state annihilated by all of the α_n, $n > 0$. (Explicitly, $\Phi^{(0)} = \exp(-\sum X_n^2)$.) A basis for the space of functionals of $x(\sigma)$ is formed by applying the α_{-n} to $\Phi^{(0)}$. The center-of-mass position x does not appear in $\Phi^{(0)}$; we will retain the dependence on this variable in the coefficient functions. Then an arbitrary $\Phi[x(\sigma)]$ may be expanded* :

$$\Phi[x(\sigma)] \;=\; \Big\{\phi(x) - iA^\mu(x)\alpha_{-1}^\mu - \frac{1}{2}h^{\mu\nu}(x)\alpha_{-1}^\mu\alpha_{-1}^\nu - iv^\mu\alpha_{-2}^\mu + \dots\Big\}\Phi^{(0)} \tag{3.3}$$

The gauge motion of Φ is given by applying L_{-n} to new string functionals. The first such motion is given by

$$\begin{aligned} L_{-1}\Psi[x(\sigma)] \;&=\; \big(p\cdot\alpha_{-1} + \alpha_{-2}\cdot\alpha_1 + \dots\big)\Big\{\phi_\Psi(x) - iA_\Psi^\mu(x)\alpha_{-1}^\mu + \dots\Big\}\Phi^{(0)} \\ &= \Big\{ - i\partial^\mu\phi_\Psi(x)\cdot\alpha_{-1}^\mu + \dots\Big\}\Phi^{(0)} \end{aligned} \tag{3.4}$$

We find, then, that $\phi(x)$ in (3.3) is gauge-invariant but that $A^\mu(x)$ is translated by:

$$\delta A^\mu \;=\; \partial^\mu\phi_\Psi. \tag{3.5}$$

The shift (3.4) thus contains linearized Yang-Mills gauge invariance.

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* Observe from the definition (2.2) and the representation $P_n = -i\partial/\partial X_n$ that α_n is pure imaginary; we construct Φ as a real string field.

The fields at the second mass level are transformed both by $L_{-1}\Psi[x(\sigma)]$ and by a second transformation $L_{-2}\Xi[x(\sigma)]$. One finds the transformation laws:

$$\begin{aligned} \delta h^{\mu\nu} &= (\partial^\mu A^\nu_\Psi + \partial^\nu A^\mu_\Psi) - \eta^{\mu\nu}\phi_\Xi \\ \delta v^\mu &= A^\mu_\Psi + \partial^\mu \phi_\Xi \end{aligned} \quad . \tag{3.6}$$

At each higher level, one finds a system of fields of increasing spin; the string gauge invariance (3.2) reduces in each system to a gauge invariance of the coupled equations for these fields.

The superiority of the kinetic energy term $\mathcal{K}$ to $\mathcal{K}_\mathrm{R}$ is now apparent. Since the gauge degrees of freedom are fields at higher levels of Verma modules, they are not annilated by $\mathcal{K}_\mathrm{R}$. From the viewpoint of string geometry, the action derived from $\mathcal{K}_\mathrm{R}$ is a particular gauge-fixing of the action derived from $\mathcal{K}$. It is certainly preferable to retain the maximum amount of symmetry in defining the classical string theory, especially when this symmetry has the power apparent in eqs. (3.5) and (3.6). We therefore propose $\mathcal{K}$ as the correct form of the reparametrization-invariant string kinetic energy.

Before continuing, let us note one generalization of this construction. The gauge invariance we have discussed reduces on the first mass level to an Abelian gauge symmetry; however, it is easily generalized to yield the linearized version of a gauge invariance under any of the classical groups, by the standard Chan-Paton procedure of attaching quantum numbers to the ends of the string. If, instead of a scalar string field $\Phi[x(\sigma)]$, we introduce a string field with $SU(n)$ indices $\Phi^a_b[x(\sigma)]$, the vector field A^μ will become an $SU(n)$ gauge field with the correct (linearized) transformation law. Removing the string orientation by a restriction:

$$\Phi^{ab}[x(\sigma)] \;=\; \pm\Phi^{ba}[x(\pi-\sigma)] \tag{3.7}$$

puts A^μ into the correct representation to be an $O(n)$ ($Sp(n)$) gauge field, and, again, the gauge invariances of the action contain the proper linearized gauge transformation.

4. The Gauge-Invariant Action

We have now set up the requirements for the kinetic energy term of the string field theory and solved them formally by requiring:

$$K = 2(L_0 - 1)P, \tag{4.1}$$

where P is the projector onto level 0. In this section, we will find an explicit form for this object and study its properties on the lowest few mass levels.

The projector onto level 0 was actually introduced long ago by Brower and Thorn[26] in their work on ghost elimination in dual resonance models. They also introduced the essential mathematical objects necessary to study the properties of P. This technology was later developed by the mathematicians Kac[24] and Feigin and Fuks[25]. The central object of their study was the *contravariant form* $\mathcal{M}_{ij}^{(n)}$, defined as follows: Let $|h\rangle$ be a state at level 0 which is also an eigenstate of L_0 with eigenvalue h. Then

$$\mathcal{M}_{ij}^{(n)}(h) = \langle h| \mathcal{L}_i^{(n)} \mathcal{L}_{-j}^{(n)} |h\rangle . \tag{4.2}$$

The indicated matrix element can be evaluated by commuting the $\mathcal{L}_i^{(n)}$ to the right and annihilating them against $|h\rangle$; thus $\mathcal{M}^{(n)}$ is completely determined by the commutation relations of the Virasoro algebra and is independent of the detailed properties of $|h\rangle$. Kac and Feigin and Fuks have computed the determinant of $\mathcal{M}^{(n)}$ and show it to be nonvanishing except on a specific set of values of p. For generic p, then, $\mathcal{M}^{(n)}$ is invertible.

Now define

$$\Pi^{(n)} = 1 - \mathcal{L}_{-i}^{(n)} \mathcal{M}_{ij}^{(n)\,-1}(L_0) \mathcal{L}_j^{(n)}. \tag{4.3}$$

If Φ_m denotes a state at level m, $\Pi^{(n)}$ satisfies the identities

$$\Pi^{(n)}\Phi_m = \Phi_m \text{ for } m < n, \quad \Pi^{(n)}\Phi_n = \Pi^{(n)}\mathcal{L}_{-k}^{(n)}\Phi_0 = 0. \tag{4.4}$$

Then the projector onto level 0 is given by

$$P = \Pi^{(1)}\Pi^{(2)}\dots\Pi^{(n)}\dots. \tag{4.5}$$

When P acts on a state at level n, all the projectors to the right of $\Pi^{(n)}$ reduce to 1; then $\Pi^{(n)}$ can project this state away. As defined in eq. (4.5), P is not manifestly Hermitian. However, its Hermiticity is clear from the fact that it is a projector. Note also that $[P, L_0] = 0$. P is, then, exactly the object we need to complete the construction of the kinetic energy operator $\mathcal{K}$, eq. (2.16)* .

We may note parenthetically that the mathematical apparatus we have just presented allows one to write explicitly the construction of $\mathcal{K}_{\mathrm{R}}$ presented intuitively at the end of Section 2. The requirements set out there are satisfied by[27,15]

$$\mathcal{K}_{\mathrm{R}} \;=\; \mathcal{K} - \sum_{ijn} \mathcal{L}^{(n)}_{-i}\mathcal{K}\,\mathcal{M}^{(n)\,-1}_{ij}(L_0)\mathcal{L}^{(n)}_{j}. \tag{4.6}$$

The terms added to $\mathcal{K}$ are explicitly gauge motions.

It is instructive to examine the properties of $\mathcal{K}$ explicitly at the lowest few mass levels. Noting that $\Pi^{(n)}$ reduces to 1 on all states below the nth mass level, one can easily find the explicit formula:

$$\begin{aligned}\mathcal{K} \;=\; 2(L_0-1) \;&-\; L_{-1}L_1 + L_{-1}^2\frac{4L_0+\frac{d}{2}-9}{B(L_0)}L_1^2 \\ &+L_{-1}^2\frac{6L_0+6}{B(L_0)}L_2 + \;h.c.\; - L_{-2}\frac{(4L_0+2)(2L_0+2)}{B(L_0)}L_2+\dots,\end{aligned} \tag{4.7}$$

where d is the dimension of space-time,

$$B(L_0) = 16L_0^2 + (2d-10)L_0 + d, \tag{4.8}$$

and the omitted terms annihilate all states below the third mass level. We can use this formula to the quadratic action $S = \frac{1}{2}(\Phi \mid \mathcal{K}\Phi)$ in terms of component

* A form for S similar to this one has been constructed by Kaku and Lykken[13] . Their action contains an extra term $(\partial/\partial\tau)$, whose origin we do not understand.

fields. Combining (4.7) with (3.3), we find at the zeroth mass level,

$$-\int d^d x \, \frac{1}{2} \phi (p^2 - 2) \phi, \tag{4.9}$$

a Klein-Gordon equation with $m^2 = -2$, as expected. At the first level, we find, using $L_1 = p \cdot \alpha_1 + \ldots$,

$$-\int d^d x \, \frac{1}{2} A_\mu (\eta^{\mu\nu} p^2 - p^\mu p^\nu) A_\nu \; = \; \int d^d x \, (-\frac{1}{4} F_{\mu\nu}^2). \tag{4.10}$$

Since our action S is gauge-invariant, a properly gauge-invariant kinetic-energy term for A_μ should emerge, and it does.

At the second mass level, though, we meet a problem. Evaluating (4.7) on the second mass level turns the denominator $B(L_0)$ into a factor $(4p^4 + (d-5)p^2 + d)^{-1}$ which makes the action S nonlocal. One can check, in fact, that there is no local action second order in derivatives containing only the fields $h^{\mu\nu}$ and v^μ which is invariant to the gauge symmetries (3.6). One can, however, convert the action to a local form by introducing Stueckelberg[28] fields. Let us examine how this works at the second mass level. In a general dimension d we would need two scalar fields, with masses given by the zeros of the denominator of (4.7):

$$m_\pm^2 \; = \; \frac{1}{8} \{ (d-5) \pm ((d-1)(d-25))^{\frac{1}{2}} \}. \tag{4.11}$$

The field corresponding to m_-^2 is a ghost when $d > 26$, decouples at $d = 26$, and can be considered a physical boson when $d < 26$, in accord with the old results of Brower and Thorn[26] .

The form of this action simplifies greatly when $d = 26$. In that case, $(L_0 + 1)$ is a common factor of $B(L_0)$ and all of the numerators shown in (4.7). Dividing through by this factor yields a simpler expression for K. For future reference, we quote the complete expression through the level 3 components:

$$
\begin{aligned}
K \;=\; & 2(L_0-1) - L_{-1}L_1 - \frac{1}{2}L_{-2}L_2 \\
& +\frac{1}{2}(3L_{-2}+2L_{-1}^2)\frac{1}{8L_0+13}(3L_2+2L_1^2) \\
& -\frac{1}{3}L_{-3}L_3 \\
& -(3L_{-2}+2L_{-1}^2)L_{-1}\frac{12L_0+37}{48(3L_0+7)(L_0+4)(8L_0+21)}L_1(3L_2+2L_1^2) \\
& -(3L_{-2}+2L_{-1}^2)L_{-1}\frac{3}{48(3L_0+7)(L_0+4)}(8L_3+3L_1L_2) \;+h.c. \\
& +(8L_{-3}+3L_{-2}L_{-1})\frac{4L_0+7}{48(3L_0+7)(L_0+4)}(8L_3+3L_1L_2) \;+\ldots\,.
\end{aligned}
\tag{4.12}
$$

The denominator of the level 2 term is now a quadratic form, and the numerator has simplified in such a way that one Stueckelberg field with mass $m_+^2 = \frac{13}{4}$ now suffices to remove the nonlocality at this level. (4.12) leads to the following expression for the quadratic action on the fields of the second mass level:

$$
\begin{aligned}
S_{(2)} = \int d^dx \Big\{ & -\frac{1}{4}h_{\mu\nu}\big[(-\partial^2+2)\eta^{\mu\lambda}\eta^{\nu\sigma} + \partial^\mu\partial^\lambda\eta^{\nu\sigma} + \partial^\nu\partial^\sigma\eta^{\mu\lambda} - \frac{1}{4}\eta^{\mu\nu}\eta^{\lambda\sigma}\big]h_{\lambda\sigma} \\
& +h_{\mu\nu}\big[\partial^\mu v^\nu + \partial^\nu v^\mu + \frac{1}{2}\eta^{\mu\nu}\partial^\lambda v_\lambda\big] \\
& -v_\mu\big[-\partial^2\eta^{\mu\nu} + \partial^\mu\partial^\nu\big]v_\nu \;+\; s\big[-\partial^2 + \frac{13}{4}\big]s \\
& +s\big[\partial^\mu\partial^\nu h_{\mu\nu} - \frac{3}{4}h^\mu_\mu - 5\partial^\mu v_\mu\big]\Big\}.
\end{aligned}
\tag{4.13}
$$

In this expression $s(x)$ is the Stueckelberg field. This action is invariant under (3.6), supplemented by the transformation

$$
\delta s \;=\; \partial_\mu A^\mu_\Psi - 3\phi_\Xi. \tag{4.14}
$$

The simplest way to analyze (4.13) is to use the vector and scalar gauge transformations to set both $v_\mu(x)$ and $s(x)$ to zero; then one may use the equations of motion to set the unwanted components of $h_{\mu\nu}$ to zero and arrive at a theory containing only a massive tensor field. This is, of course, the correct content for the string at this level.

At the second mass level, then, one finds nonlocal terms in the string action which force one to introduce an extra scalar field, exactly the field needed to remove the extra scalar gauge invariance found at the end of the previous section. The necessity of adding such additional fields was already noted by Siegel[11] . In his covariant-gauge quantization of the string (restricted to $d = 26$), Siegel also found an extra scalar field at the second mass level. He found, together with the (commuting) ghosts of the ghost fields, one field which he thought natural to include with the physical fields arising directly from the string. He concluded that the fields contained in $\Phi[x(\sigma)]$ are insufficient to describe the full content of the classical string theory. We will see the connection between this viewpoint and ours in Section 7.

At higher mass levels, our formula for $\mathcal{K}$ contains higher-order polynomials in L_0 in the denominator, and, therefore, more formidable nonlocalities. We would like to be able to remove all of these nonlocalities by introducting Stueckelberg auxiliary fields. We will present several different sets of Stueckelberg fields which accomplish this goal, closing in, eventually, on the minimal set which leads to the standard quantum theory of the string in 26 dimensions. Let us begin which the most simple example. It is of interest because it is the only one of these constructions which works in a general space-time dimension, and because its weaknesses will make the requirements for the correct set of Stueckelberg fields more clear.

To render the action derived from $\mathcal{K}$ local, one requires fields of successively higher spin at higher mass levels. It is natural to expect that these fields can be assembled into string fields $S_n[x(\sigma)]$ and, therefore, to search for a string action which contains Stueckelberg string fields in addition to the fundamental string field Φ. We will now prove that our action (2.15), (2.16) is obtained (at the classical level) from the following manifestly local action[29] :

$$S = -\frac{1}{2}(\Phi - \sum_n L_{-n}S_n \mid 2(L_0 - 1) \mid \Phi - \sum_n L_{-n}S_n), \tag{4.15}$$

by integrating out the Stueckelberg string fields S_n.

Define the projector onto levels N and lower as

$$P_N = \Pi^{(N+1)}\Pi^{(N+2)}\ldots, \tag{4.16}$$

and define

$$S_N = -\frac{1}{2}(\Phi_N \mid 2(L_0 - 1)P_N \mid \Phi_N), \tag{4.17}$$

where

$$\Phi_N = \Phi - P_N \sum_n L_{-n}S_n. \tag{4.18}$$

If we can show that S_N is equivalent to S_{N-1}, the equivalence of (4.15) and (2.15) follows by induction. Let us, then, separate out of (4.18) the Stueckelberg fields at level N:

$$\Phi_N = \Phi_{N-1} - \sum_i \mathcal{L}_{-i}^{(N)} S_i^{(0)}, \tag{4.19}$$

where $S_i^{(0)}$ are fields at level 0. Then (4.17) takes the form

$$\begin{aligned}
S_N =& -\frac{1}{2}(\mathcal{L}_{-i}^{(N)} S_i^{(0)} \mid 2(L_0-1) \mid \mathcal{L}_{-j}^{(N)} S_j^{(0)}) \\
&\qquad + (\mathcal{L}_{-i}^{(N)} S_i^{(0)} \mid 2(L_0-1)P_N \mid \Phi_{N-1}) - \frac{1}{2}(\Phi_{N-1} \mid 2(L_0-1) \mid \Phi_{N-1}) \\
=& -\frac{1}{2}(S_i^{(0)} \mid 2(L_0-1+N)\mathcal{M}_{ij}^{(N)} \mid S_j^{(0)}) \\
&\qquad + (S_i^{(0)} \mid 2(L_0-1+N)\mathcal{L}_i^{(N)} P_N \mid \Phi_{N-1}) - \frac{1}{2}(\Phi_{N-1} \mid 2(L_0-1) \mid \Phi_{N-1}) \\
=& -\frac{1}{2}(S_i^{(0)\prime} \mid 2(L_0-1+N)\mathcal{M}_{ij}^{(N)} \mid S_j^{(0)\prime}) \\
&\qquad - \frac{1}{2}(\Phi_{N-1} \mid P_N\, 2(L_0-1)(1-\mathcal{L}_{-i}^{(N)}\mathcal{M}_{ij}^{(N)-1}\mathcal{L}_j^{(N)})P_N \mid \Phi_{N-1}) \\
=& -\frac{1}{2}(S_i^{(0)\prime} \mid 2(L_0-1+N)\mathcal{M}_{ij}^{(N)} \mid S_j^{(0)\prime}) + S_{N-1},
\end{aligned} \tag{4.20}$$

where $S_i^{(0)\prime}$ is a shift of $S_i^{(0)}$. Integrating over $S_i^{(0)}$ removes the first term in the last line and proves the classical equivalence. Our expression for the string action, then, can be made completely local in terms of the component fields, in any space-time dimension, by introducing a sufficient number of Stueckelberg auxiliary fields.

The action (4.15) has, however, several notable defects. In this formalism, we introduce Stueckelberg fields even at the massless level. Maxwell's action arises from integrating out χ in the action $S = \int \frac{1}{2}(A_\mu - \partial_\mu\chi)\partial^2(A^\mu - \partial^\mu\chi)$. This implies that, first, we have more Stueckelberg fields than are strictly necessary to render the original action local. More importantly, it shows explicitly that the component-field action derived from (4.15) contains terms with 4 derivatives.

The example does make clear the necessity of finding the correct set of Stueckelberg fields. Though (4.15) is equivalent to the nonlocal action for Φ at the classical level, it differs at the quantum level precisely by the determinants obtained when the Stueckelberg fields $S_j^{(0)\prime}$ are integrated out. Only one choice of Stueckelberg fields, then, can reproduce the conventional quantum theory of the string, even at the level of the counting of states.

Eq. (4.15) has a further property which one should keep in mind as we discuss more refined sets of Stueckelberg fields. The Stueckelberg fields contained in (4.15) are redundant, in the sense that there exist transformations of the Stueckelberg fields alone, not involving Φ, which leave the action invariant. An example of such a transformation is

$$\delta S_n = -\sum_m L_{-m}\mathcal{G}_{mn} + \frac{1}{2}\sum_{k+\ell=n}(k-\ell)\mathcal{G}_{k\ell}, \tag{4.21}$$

for $\mathcal{G}_{mn}$ antisymmetric in its indices. Because of the non-Abelian nature of the Virasoro algebra, such redundancies are inevitable in any formulation in which Stueckelberg fields are unconstrained. We must try to turn this puzzling feature to our advantage.

5. Differential Forms on the Space of Strings

The presence of redundant gauge transformations of the structure of eq. (4.21) has a natural interpretation in the language of differential forms. In our discussion of chordal gauge transformations, we considered the L_{-n}'s as differential operators on the space of strings. It is not unreasonable, then, to consider the term $\sum L_{-n}S_n$ as the divergence of a 1-form. This quantity must then be zero if S_n is the divergence of a 2-form. That is precisely the structure displayed in eq. (4.21); the only unfamiliar piece is the last term, which compensates the structure constants of the Virasoro algebra. It is clearly of interest to flesh out this analogy, both to discover new invariance principles which generalize (4.21) and, perhaps, to illuminate some of the differential geometry of the space of strings. In this section, then, we will digress to discuss this issue. We apologize that much of our development is purely formal and is not physically well motivated.

We believe it is a very important problem to find a geometrical interpretation for the construction we will now present. We find it remarkable that one of the central identities of our construction is valid only in 26 dimensions.

The algebra of differential forms which we will present is closely connected to the algebra of Siegel's BRST-invariant string quantization[10]. We were, in fact, led to consider relations of this form by our attempts to derive this gauge-fixed action from a gauge-invariant viewpoint. Many of the basic relations we will present—in particular, the gauge transformations and the gauge fixing terms for the basic fields—can be read directly from the BRST charge presented in ref. 10.

Let us begin by defining our notation. We construct differential forms as tensors whose components are string fields, functionals of $x(\sigma)$. We will consider these tensors to have two different types of indices, covariant and contravariant, each of which may appear either raised or lowered. It is most convenient to think of covariant indices as lowered and contravariant indices as raised; however, lowering a contravariant index does not automatically make it transform as a covariant index, or vice versa. An index of either type takes as its value any positive integer. The derivatives L_n and L_{-n} carry n as a covariant index. Differential forms are defined to be completely antisymmetrized in all covariant indices and, separately, in all contravariant indices. Indices of both types are raised and lowered by a metric tensor

$$\eta_{mn} = m\delta_{mn}. \tag{5.1}$$

Define tensors $V_{mn}{}^{p}$ and $W_{mn}{}^{p}$ to contain the structure constants of the Virasoro algebra, by rewriting the basic commutation relations (2.8) as follows:

$$\begin{aligned} [L_m, L_n] &= V_{mn}{}^{p}\, L_p \\ [L_{-m}, L_{-n}] &= -\, V_{mn}{}^{p}\, L_{-p} \\ [L_m, L_{-n}] &= W_{mn}{}^{p}\, L_p \; + \; W_{nm}{}^{p}\, L_{-p} \; + \; \eta_{mn}\mathbf{L}(m) \; , \end{aligned} \tag{5.2}$$

where m,n,p are positive integers. More explicitly,

$$
\begin{aligned}
V_{mn}{}^{p} &= \delta\big(p-(m+n)\big)(m-n)\\
W_{mn}{}^{p} &= \delta\big(m-(n+p)\big)(m+n)\\
\mathbf{L}(m) &= 2L_0+\frac{13}{6}(m^2-1),
\end{aligned}
\tag{5.3}
$$

the last relation evaluated in $d=26$.

Before introducing differential operators in their full generality, let us write the simplest exterior derivatives and divergences and compute their relations. Define the divergence of a contravariant 1-form C^p, a 2-form $\mathcal{G}^{pq}$, and a 2-form with one index of each type $\Lambda^m{}_n$ as follows:

$$
\begin{aligned}
\delta C &= L_{-p}C^p\ , \quad \delta\mathcal{G}^m \;=\; L_{-p}\mathcal{G}^{pm}-\frac{1}{2}V_{kl}{}^{m}\mathcal{G}^{kl}\\
\delta\Lambda_n &= L_{-p}\Lambda^p{}_n+W_{np}{}^{q}\Lambda^p{}_q.
\end{aligned}
\tag{5.4}
$$

The first two of these have, respectively, the structure of a chordal gauge transformation and of the redundant transformation (4.21). Define the exterior derivative of a 0-form Φ, a contravariant 1-form C^p, and a covariant 1-form $\bar{C}_p$ as follows:

$$
\begin{aligned}
d\Phi_n &= L_n\Phi\ , \quad dC^m{}_n \;=\; L_nC^m+W_{nk}{}^{m}C^k\ ,\\
d\bar{C}_{[n_1n_2]} &= L_{[n_1}\bar{C}_{n_2]}-\frac{1}{2}V_{n_1n_2}{}^{p}\bar{C}_p.
\end{aligned}
\tag{5.5}
$$

Then one can straightforwardly verify the relations

$$
\delta^2\mathcal{G} \;=\; 0\ , \quad d^2\Phi=0.
\tag{5.6}
$$

Using the identity

$$
W_{pm}{}^{n}W_{qn}{}^{m} \;=\; \delta(p-q)\sum_{\substack{m+n=p\\ m,n>0}}(p+m)(p+n) \;=\; p\delta_{pq}\Big(\frac{13}{6}p+\frac{1}{6}\Big)(p-1),
\tag{5.7}
$$

one can also verify (but only in 26 dimensions)

$$(d\delta - \delta d)C_n \;=\; \eta_{np} \cdot 2(L_0 - 1 + p)C^p. \tag{5.8}$$

If we view C^p as a gauge motion, the relation $\delta_C \Phi = L_{-p} C^p$ indicates that C^p should have its lowest components at the pth mass level of Φ. Thus, $2(L_0 - 1 + p)$ is the appropriate kinetic energy operator for C^p, and eq. (5.8) takes the conventional form

$$(d\delta - \delta d) \;=\; \Delta. \tag{5.9}$$

Eq. (5.8) is useful in the following context: A natural choice for a set of gauge fixing conditions for chordal gauge transformations is $\{L_p \Phi = d\Phi_p = 0\}$. However, the variation of this term with respect to C^p is $(d\delta C)$, which is not particularly simple. But if we add to the gauge fixing condition a Stueckelberg field $\Lambda^m{}_n$ with the gauge transformation $\delta_C \Lambda^m{}_n = -dC^m{}_n$, we find

$$\delta_C\big(d\Phi + \delta\Lambda\big) \;=\; (d\delta - \delta d)C = 2(L_0 - 1 + p)C. \tag{5.10}$$

In the Fadde'ev-Popov formalism, this variation gives the kinetic energy of the ghost string fields. Apparently, the Stueckelberg field $\Lambda^m{}_n$ allows this operator to take a simple form.

Let us now generalize the relations (5.6) and (5.8) to forms of arbitrary rank. Let us refer to a form with a contravariant and b covariant indices as an $\binom{a}{b}$-form. Define the exterior derivative and the divergence of an $\binom{a}{b}$-form C by:

$$\begin{aligned}
(dC)^{[m_1 \ldots m_a]}{}_{[n_1 \ldots n_{b+1}]} &= L_{[n_1}\, C^{[m_1 \ldots m_a]}{}_{n_2 \ldots n_{b+1}]} + a\, W_{p[n_1}{}^{[m_1}\, C^{p m_2 \ldots m_a]}{}_{n_2 \ldots n_{b+1}]} \\
&\qquad - \frac{1}{2} b\, V_{[n_1 n_2}{}^{p}\, C^{[m_1 \ldots m_a]}{}_{p n_3 \ldots n_{b+1}]}, \\
(\delta C)^{[m_1 \ldots m_{a-1}]}{}_{[n_1 \ldots n_b]} &= L_{-p}\, C^{[p m_1 \ldots m_{a-1}]}{}_{[n_1 \ldots n_{b+1}]} + b\, W_{[n_1 p}{}^{q}\, C^{[p m_1 \ldots m_{a-1}]}{}_{q n_2 \ldots n_b]} \\
&\qquad - \frac{1}{2}(a-1)\, V_{kl}{}^{[m_1}\, C^{k l m_2 \ldots m_{a-1}]}{}_{[n_1 \ldots n_b]}.
\end{aligned} \tag{5.11}$$

Here and henceforth, we make the convention that raised indices labeled as (m_i) are antisymmetrized together in the indicated order and lowered indices labeled as (n_i) are antisymmetrized together similarly. One may verify that these operators satisfy the fundamental identities of cohomology:

$$d^2 C = 0 , \quad \delta^2 C = 0, \tag{5.12}$$

and

$$(d\delta - \delta d) C^{m_1 \dots m_{a-1}}{}_{n_1 \dots n_{b+1}} = \mathbf{K}\, \eta_{[n_1 p}\, C^{p m_1 \dots m_{a-1}}{}_{n_2 \dots n_{b+1}]}, \tag{5.13}$$

where

$$\mathbf{K} = 2(L_0 - 1 + (\text{sum of indices})) \tag{5.14}$$

is the natural generalization of the kinetic energy operator in (5.8). Note that $\mathbf{K}$ commutes with d, δ, and the L_n's. To prove the relations (5.12), (5.13), one needs the Jacobi identities of the Virasoro algebra

$$\begin{aligned} W_{pm}{}^q W_{qn}{}^k - W_{pn}{}^q W_{qm}{}^k + V_{mn}{}^q W_{pq}{}^k &= 0 \\ V_{[n_1 n_2}{}^q V_{q n_3]}{}^k &= 0 \\ W_{[n_1 p}{}^q V_{q n_2]}{}^k - \frac{1}{2} V_{n_1 n_2}{}^q W_{qp}{}^k - W_{p[n_1}{}^q W_{n_2]q}{}^k &= k\,(\eta_{n_1 p}\eta_{n_2}{}^k - \eta_{n_2 p}\eta_{n_1}{}^k) \end{aligned} \tag{5.15}$$

and the relation (5.7).

Thus far, we have treated covariant and contravariant indices as completely distinct. However, since our formalism does contain a metric η_{pq}, we can, in principle, bring covariant and contravariant indices to the same level and symmetrize or antisymmetrize them in pairs. This procedure decomposes a general $\binom{a}{b}$-form into components with definite permutation symmetry, each component corresponding to a given Young tableau. Because the covariant and contravariant indices are (separately) antisymmetrized among themselves, only Young tableaux with one or two columns appear in this decomposition. For example:

$$\binom{3}{1} = \ydiagram{1,1,1,1} + \ydiagram{2,1,1}. \tag{5.16}$$

Let us refer to a Young tableau of with columns of length k, ℓ as a (k,ℓ)-tableau, or simply as (k,ℓ). In general, an $\binom{a}{b}$-form decomposes as follows:

$$\binom{a}{b} = \begin{cases} \sum_{0\le c\le b} (a+c, b-c) & (a>b) \\ \sum_{0\le c\le a} (b+c, a-c) & (b>a) \end{cases}. \tag{5.17}$$

The case $c = 0$ is clearly special; let us, then define a *maximally symmetrized* $\binom{a}{b}$-form to be one symmetrized according to (a,b) for $a \ge b$ or according to (b,a) for $b \ge a$.

We will indicate the Young symmetrization of a form C by the notation $[C]_{(k,\ell)}$. This symmetrization bracket will be normalized in such a way that if the symmetrization is automatic, the bracket can be ignored and no additional factors need be supplied. For example, if C is a $\binom{3}{1}$-form, as in (5.16),

$$C = [C]_{(4,0)} + [C]_{(3,1)} \tag{5.18}$$

If D is a $\binom{1}{3}$-form, symmetrized according to $(1,3)$,

$$D^{n_1}{}_{m_1m_2m_3} [C]_{(3,1)}{}^{m_1m_2m_3}{}_{n_1} = D^{n_1}{}_{m_1m_2m_3} C^{m_1m_2m_3}{}_{n_1}. \tag{5.19}$$

Young symmetrization must be used sparingly, because raising and lowering of indices changes the transformation law of the form. Inserting a Young symmetrizer between the two operators in the any of the fundamental relations

(5.12), (5.13) generally spoils the relation. This can be easily seen in the following example: Let $\tilde{C}_p$ be a $\binom{0}{1}$-form. Then

$$\begin{aligned} d(\eta^{pq}\tilde{C}_q)^p{}_r = -\eta^{pk}(d\tilde{C})_{kr} \\ - \eta^{pk}\{L_{\{k}\,\tilde{C}_{r\}} + U_{qkr}\,\tilde{C}^q\}, \end{aligned} \tag{5.20}$$

where

$$U_{qkr} \;=\; W_{qkr} + \frac{1}{2}V_{krq} \;=\; \delta(k+r-q)\,\frac{k^2+r^2+4kr}{2}, \tag{5.21}$$

and W_{qkr} and V_{krq} are defined by lowering indices of (5.3) using η_{pq}. A crucial property of U_{qkr} is that it is symmetric in its last two indices. Eq. (5.20) indicates that (at least in this circumstance) raising an index anticommutes with the d operator only if the raised index is then re-antisymmetrized with the remaining covariant indices.

Let us now discuss the generalization of eq. (5.20) to $\binom{a}{b}$-forms. We must first define a suitable generalization of the raising and lowering of a single index. Let $\Uparrow$ denote the operation of raising the first covariant index and antisymmetrizing it, at the front of the line, with the contravariant indices. Let $\Downarrow$ be the corresponding lowering operator. Explicitly, if C is an $\binom{a}{b}$-form:

$$\begin{aligned} \Big(\Uparrow C\Big)^{m_1\ldots m_{a+1}}{}_{n_1\ldots n_{b-1}} &= \eta^{m_1q}\,C^{m_2\ldots m_{a+1}}{}_{qn_1\ldots n_{b-1}} \\ \Big(\Downarrow C\Big)^{m_1\ldots m_{a-1}}{}_{n_1\ldots n_{b+1}} &= \eta_{n_1q}\,C^{qm_1\ldots m_{a-1}}{}_{n2\ldots n_{b+1}}. \end{aligned} \tag{5.22}$$

By our convention, the (m_i) are antisymmetrized together, and the (n_i) are antisymmetrized together. We have encountered $\Downarrow$ implicitly already in our discussion. Using (5.22), we can rewrite the identity between d and δ, eq. (5.13), as

$$d\,\delta \;-\; \delta\,d \;=\; \mathbf{K}\,\Downarrow\;. \tag{5.23}$$

An important property of $\Uparrow$ and $\Downarrow$ is that one or the other of these operators will annihilate a maximally symmetrized $\binom{a}{b}$-form C:

$$\begin{cases} \Uparrow [C]_{(a,b)} = 0 & (a \geq b) \\ \Downarrow [C]_{(b,a)} = 0 & (b \geq a)\,. \end{cases} \tag{5.24}$$

These relations follow from the observation that the index raised or lowered will always be antisymmetrized with an index with which it had previously been symmetrized. Conversely, the validity of (5.24) implies that C is maximally symmetrized. This criterion is, in fact, quite a useful one.

Eq. (5.24) makes clear that $\Uparrow$ and $\Downarrow$ are not in general the inverses of one another. It is not difficult, though, to work out from the definitions (5.22) the following relation between these operators: If C is an $\binom{a}{b}$-form,

$$a(b+1) \Uparrow\Downarrow C \;-\; b(a+1) \Uparrow\Downarrow C \;=\; (a-b)\,C. \tag{5.25}$$

If C is maximally symmetrized, one of the two terms on the left-hand side of this equation vanishes. In this special circumstance, $\Uparrow$ inverts the operation of $\Downarrow$ (or vice versa).

We are now ready to quote the generalizations of eq. (5.20). $\Downarrow$ anticommutes with both d and δ; for example,

$$\Big((\Downarrow d + d \Downarrow)C\Big)^{m_1 \ldots m_{a_1}}{}_{n_1 \ldots n_{b+2}} \;=\; U_{p[n_1 n_2}\, C^{p m_1 \ldots m_{a-1}}{}_{n_3 \ldots n_{b+2}]} \;=\; 0, \tag{5.26}$$

because of the symmetry of $U_{pk\ell}$. The commutation relations of $\Uparrow$ with d and δ are unfortunately quite complicated. However, for the analysis we will present here, the following information will suffice: If C is a maximally symmetrized $\binom{a}{b}$-form, with $a < b$, then

$$\Big\{ \Big(\frac{b}{b-a}\Big)\, d \Uparrow \;+\; \Big(\frac{b+1}{b+1-a}\Big) \Uparrow d \Big\}\, C \text{ is maximally symmetrized,} \tag{5.27}$$

$$\Big\{ \Big(\frac{a+1}{b-a}\Big)\, \delta \Uparrow \;+\; \Big(\frac{a}{b+1-a}\Big) \Uparrow \delta \Big\}\, C \text{ is maximally symmetrized.} \tag{5.28}$$

To prove these identities, operate with $\Downarrow$ on the two quantities displayed; the result is true if $\Downarrow$ annihilates these quantities. To show that this is so, use (5.25) to pass $\Downarrow$ through $\Uparrow$, and apply the relations $\{\Downarrow, d\} = 0$, $\{\Downarrow, \delta\} = 0$, $\Downarrow C = 0$.

The inner product of two forms should include both the Hilbert space scalar product for the string coordinates and a contraction of covariant with contravariant indices. Thus, if two forms C_A and C_B are to have an inner product and C_A is an $\binom{a}{b}$-form, C_B must be a $\binom{b}{a}$-form. With this definition,

$$\left(dC_A \,\middle|\, C_B\right) \;=\; \left(C_A \,\middle|\, \delta C_B\right), \tag{5.29}$$

and $\Uparrow$ and $\Downarrow$ are self-adjoint. In principle, one can take the inner product of C_A with $\binom{c}{d}$-forms such that $c + d = a + b$ by raising and lowering indices appropriately. We will, in fact, find such a generalized inner product useful in the next section. We will always be careful, though, to indicate the required raising and lowering explicitly.

6. Stueckelberg Fields for the String in 26 Dimensions

Let us now apply this formalism to the problem of finding a more compact set of Stueckelberg fields which render the gauge-invariant string action local. We will advance toward our final answer in stages. We will first examine the string at low mass levels, and generalize the structure we find there by introducing Stueckelberg fields which are differential forms. This will lead us to a gauge-invariant local action, but with a set of Stueckelberg fields which is still not minimal. Clarification of this issue will then lead us to the correct set of Stueckelberg fields.

To begin, let us return to the action of the fields at the second mass level, which we wrote in a local form in eq. (4.13). At this level, we required only one scalar Stueckelberg field $s(x)$. To generalize this construction, we must identify $s(x)$ as a component of some higher-level string functional. If $s(x)$ is considered as the scalar component of a string field S, its gauge transformation property, eq. (4.14), may be identified as the scalar component of

$$\delta S = -L_1 \Psi - 3\Xi, \tag{6.1}$$

where Ψ and Ξ are chordal gauge parameters defined above eq. (3.6). In the previous section, we found it reasonable to consider Ψ, Ξ, and the higher gauge functions as the components of a differential form C^p. Then S must also be a component of a form, and it is not hard to see that (6.1) is the $m = 1, n = 1$ component of the transformation

$$\delta_C \Lambda^m{}_n = -(d\,C)^m{}_n. \tag{6.2}$$

(In this and the next section, transformations will be denoted by δ with a subscript; δ without a subscript will denote the divergence operator on forms).

The identification of S as the $(1,1)$ component of a $\binom{1}{1}$-form is readily confirmed by noting that the action, up to the second mass level, can be rewritten in terms of string fields as:

$$\begin{aligned} \mathcal{S} = -\frac{1}{2}\Big\{ & \left(\Phi \mid 2(L_0 - 1)\,\Phi\right) - \left(S \mid 2(L_0+1)\,S\right) \\ & - \left(L_1\Phi + L_{-1}S \mid L_1\Phi + L_{-1}S\right) - \frac{1}{2}\left(L_2\Phi + 3S \mid L_2\Phi + 3S\right)\Big\}, \end{aligned} \tag{6.3}$$

(To check this, eliminate S by rearranging L_1's to simplify the quadratic term in S and then completing the square. One recovers by this procedure the first two lines of eq. (4.12).) Note that the two terms in the first line of (6.3) contain the correct values of $\mathbf{K}$, eq. (5.14), for a scalar and a $\binom{1}{1}$-form. The two more mysterious terms in the second line may be recognized as the $m = 1$ and $m = 2$ components of

$$\left(d\,\Phi + \delta\,\Lambda\right)_m, \tag{6.4}$$

truncated to the second mass level.

It is now natural to guess that the complete local action is given by

$$
\begin{aligned}
S \;=\; -\frac{1}{2}\Big\{ & \big(\Phi \mid \mathbf{K}\,\Phi\big) - \big(\Lambda^m{}_n \mid \mathbf{K}\,\Lambda^n{}_m\big) \\
& - \big((d\Phi + \delta\Lambda)_p \mid \eta^{pq}\,(d\Phi + \delta\Lambda)_q\big)\Big\}.
\end{aligned}
\tag{6.5}
$$

This choice for S is gauge-invariant to the chordal transformation parametrized by a 1-form C^p:

$$
\delta_C\,\Phi \;=\; \delta C, \quad \delta_C\,\Lambda \;=\; -dC, \tag{6.6}
$$

since, using (5.10),

$$
\begin{aligned}
-\delta_C\,S \;=\; \Big\{ & \big(\Phi \mid \mathbf{K}\,\delta C\big) - \big(\Lambda^m{}_n \mid \mathbf{K}\,(-dC^n{}_m)\big) \\
& - \big((d\Phi + \delta\Lambda)_p \mid (\mathbf{K}C)^p\big)\Big\},
\end{aligned}
\tag{6.7}
$$

and (5.29) implies that this is zero. Since the symmetry (6.6) contains the desired gauge motion of Φ, and since S reduces to $\big(\Phi \;\big|\; 2(L_0 - 1)\,\Phi\big)$ on the level-0 components of Φ, this action must reduce to the nonlocal action constructed in Section 4 when the Stueckelberg fields Λ are eliminated. Thus, eq. (6.5) provides a second set of Stueckelberg fields which render the string action local. This set is considerably smaller than the one presented at the end of Section 4, and its action is no more than second order in derivatives.

Should we then consider eq. (6.5) to embody the minimal set of Stueckelberg fields? To answer this question, it is useful to study explicitly the way in which (6.5) leads to the complete expression (4.12) after elimination of the Stueckelberg fields up to the third mass level. The new terms in (4.12) contain two new poles in their denominators. (The term $(8L_0 + 21)$ is not a new structure, since it obviously arises from passing an L_1 through $(8L_0 + 13)$.) This would seem to accord well with the appearance at this level of two new Stueckelberg fields, $\Lambda^2{}_1$

and $\Lambda^1{}_2$. However, that is not how the two new poles arise. Explicit evaluation shows that, up to the third mass level, (6.5) depends only on the combination $(\Lambda^1{}_2 + 2\Lambda^2{}_1)$. The second pole arises in the following way: To eliminate $\Lambda^1{}_1$, one must first project it into its level-0 and level-1 components, by the replacement

$$\Lambda^1{}_1 \;\rightarrow\; (1 - L_{-1}\frac{1}{2L_0}L_1)\Lambda^1{}_1 + (L_{-1}\frac{1}{2L_0}L_1)\Lambda^1{}_1. \tag{6.8}$$

The level-1 component, a scalar field, mixes with the surviving combination of the new Λ's; the quadratic term involving these two fields is a 2×2 matrix whose determinant is proportional to $(3L_0+7)(L_0+4)$. Thus, only one new Stueckelberg field is necessary to reproduce the full structure displayed in eq. (4.12).

The combination $\Lambda^1{}_2 + 2\Lambda^2{}_1$ is simply the symmetrization of Λ_{mn} after index lowering. It is tempting to guess that (6.5) contains only the symmetrized form $\Lambda_{\{mn\}}$; however, this guess can be seen to fail at the fourth mass level. We must, then, ask what principle forces Λ to appear symmetrized at low mass levels and whether this principle can possibly be generalized.

The required principle is, as one might have suspected, the invariance of S with respect to redundant gauge transformations. The motion of the gauge parameters given by

$$\delta_{\mathcal{G}}\, C^p \;=\; (\delta\, \mathcal{G})^p\,, \tag{6.9}$$

where $\mathcal{G}$ is a $\binom{2}{0}$-form, is necessarily a symmetry of S, since any possible C^p is a symmetry. This transformation leaves Φ invariant, since $\delta^2 = 0$. One can check that it leaves the symmetric part of Λ invariant, up through the third mass level. Above this level, one can preserve the condition of symmetry by changing the transformation law of Λ under $\mathcal{G}$. Relabel the original gauge parameter C as C_1, let C_3 be a $\binom{2}{1}$-form, and let Λ transform under C according to

$$\delta_C\, \Lambda \;=\; -d\, C_1 \;+\; \delta\, C_3. \tag{6.10}$$

Assign to C_1 and C_3 the $\mathcal{G}$ transformation laws

$$\delta_{\mathcal{G}} C_1 = \delta \mathcal{G} , \quad \delta_{\mathcal{G}} C_3 = d \mathcal{G} . \tag{6.11}$$

Then Λ transforms under $\mathcal{G}$ according to

$$\delta_{\mathcal{G}} \Lambda = (d\delta - \delta d) \mathcal{G} = \mathbf{K} \mathcal{G}. \tag{6.12}$$

Since $\mathcal{G}$ is antisymmetric in its two indices, the symmetric part of Λ is now manifestly invariant to $\mathcal{G}$ at all levels.

One is thus led to modify the action (6.5) in the following way: First, replace Λ everywhere by its symmetric part. Then add terms to the action so that it remains C_1-invariant and becomes also C_3-invariant. A natural term to add is the higher rank generalization of the second line of (6.5):

$$(d\Lambda + \delta \Lambda_4)^2, \tag{6.13}$$

where Λ_4 is a $\binom{2}{2}$-form. But then, just as happened with Λ, Λ_4 will appear to be subjected to a symmetry condition. This condition is the result of a new redundant gauge symmetry

$$\delta_{\mathcal{G}} C_3 = \delta \mathcal{G}_4 \tag{6.14}$$

which leaves Λ invariant. To give Λ_4 a simple transformation law under $\mathcal{G}_4$, introduce a new gauge symmetry C_5, and assign the transformation laws

$$\begin{aligned} \delta_C \Lambda_4 &= -d\, C_3 + \delta\, C_5, \\ \delta_{\mathcal{G}} C_3 = \delta\, \mathcal{G}_4 , &\quad \delta_{\mathcal{G}} C_5 = d\, \mathcal{G}_4 . \end{aligned} \tag{6.15}$$

Then

$$\delta_{\mathcal{G}} \Lambda_4 = (d\delta - \delta d) \mathcal{G}_4 = \mathbf{K} \mathcal{G}_4. \tag{6.16}$$

We must now determine the symmetry of the various fields in this equation. C_3 is a $\binom{2}{1}$-form and so could contain components symmetrized according to the Young tableaux $(2,1)$ or $(3,0)$. However, only the $(2,1)$ component will give a nontrivial shift of the symmetric part of Λ when inserted into (6.10). We may thus restrict C_3 to this symmetry. But then $\mathcal{G}_4$, a $\binom{3}{1}$-form, must be symmetrized according to the Young tableau $(3,1)$. Λ_4, a $\binom{2}{2}$-form, could, in principle, be symmetrized according to $(4,0)$, $(3,1)$, or $(2,2)$. However, if the term (6.13) is to couple Λ_4 to the symmetrized Λ, Λ_4 cannot belong to $(4,0)$. (6.16) implies that Λ_4 will be $\mathcal{G}_4$-invariant if it contains no component with the symmetry of $\mathcal{G}_4$, $(3,1)$. Thus, Λ_4 should contain only the component $(2,2)$.

To insure the symmetry of Λ_4, however, one must add a new $\binom{3}{3}$-form Stueckelberg field. A set of arguments similar to those just given restrict the symmetry of this form to the Young tableau $(3,3)$. The process continues indefinitely. We find, then, that the minimal set of Stueckelberg fields for the bosonic open string is a set of $\binom{k}{k}$-forms, one for each integer k, symmetrized according to

$$\ydiagram{2} + \ydiagram{2,2} + \ydiagram{2,2,2} + \ydiagram{2,2,2,2} + \cdots . \tag{6.17}$$

We have now established the field content of the local, gauge invariant linearized string action; we have also motivated the general form of the terms by which these fields couple to one another. Let us, then, present the final form of this action and verify that it satisfies our requirements. To prepare for this, let us regularize our notation in the following way. The fields which appear in the gauge-invariant string action are all elementary components of the string, in the sense that they participate in the classical equations of motion. Let us label all of these fields as Φ's. Let $\Phi_0 = \Phi$, the original string field, and label the $\binom{k}{k}$-form Stueckelberg field, symmetrized according to (k,k), as Φ_{2k}. The action should be gauge-invariant under the chordal transformations

$$\delta_C\,\Phi_{2k} \;=\; -d\,C_{2k-1} \;+\; \delta\,C_{2k+1}, \tag{6.18}$$

where C_{2k+1} is a $\binom{k+1}{k}$-form symmetrized according to $(k+1,k)$. These gauge transformations are themselves invariant under the redundant transformations

$$\delta_{\mathcal{G}}\, C_{2k+1} \;=\; d\,\mathcal{G}_{2k} \;+\; \delta\,\mathcal{G}_{2k+2}, \tag{6.19}$$

where $\mathcal{G}_{2k+2}$ is a $\binom{k+2}{k}$-form symmetrized according to $(k+2,k)$. The $\mathcal{G}$ transformation law is in turn left invariant by transformations parametrized by $\binom{2k+3}{k}$-forms C', symmetrized according to $(k+3,k)$, and so on. Each form that we have introduced has the total number of indices denoted by its subscript and cannot appear at a mass level lower than this number. Thus, at any given mass level, the proliferation of Stueckelberg fields and their successive gauge transformations eventually terminates. The full string field theory, however, contains an infinite number of levels.

The gauge-invariant action for the Φ_{2k} can be written compactly using the notation introduced in Section 5:

$$S \;=\; -\frac{1}{2}\Big\{(-1)^k\,(\Phi_{2k}\mid \mathbf{K}\Phi_{2k}) \\ -\;(-1)^k\,(k+1)^2\,\big(d\Phi_{2k}+\delta\Phi_{2k+2}\,\big|\!\Uparrow\,(d\Phi_{2k}+\delta\Phi_{2k+2})\big)\Big\}. \tag{6.20}$$

(The coefficient of the second term must grows with k to compensate the fact that, in our conventions, as k increases, $\Uparrow$ comes increasingly closer to annihilating the highly symmetrized forms.) As with our previous choices for a local string action, this expression is set up so that Φ decouples from its Stueckelberg fields if $L_n\Phi = 0$ for all positive n. In this case, the action reduces, at the classical level, to $-\frac{1}{2}\big(\Phi\mid 2(L_0-1)\Phi\big)$. If this action were also invariant to all C-transformations, that would imply that the action which results from eliminating the Stueckelberg fields must be invariant to the basic chordal gauge transformations (3.2). These two requirements imply that S reduces to exactly the form $-\frac{1}{2}(\Phi\mid 2(L_0-1)P\,\Phi)$ presented in Section 4. To establish the classical equivalence of (6.20) and our original nonlocal action, then, we need only prove that (6.20) is invariant to C-transformations.

Let us, then, study the variation of S under a C-transformation. Since the Φ_{2k} are projected onto representations of definite symmetry, their C transformation laws must be explicitly symmetrized by operators. We indicate this symmetrization using the bracket defined above eq. (5.18). Then

$$
\begin{aligned}
\delta_C S \;=\; & (-1)^k \left(\Phi_{2k} \,\middle|\, \mathbf{K}\left[\delta C_{2k+1} - dC_{2k-1}\right]_{(k,k)}\right) \\
& + (-1)^k k^2 \left(\delta\Phi_{2k} \,\middle|\Uparrow \left(d\left[\delta C_{2k-1} - dC_{2k-3}\right]_{(k-1,k-1)}\right.\right. \\
& \qquad \left.\left. + \;\delta\left[\delta C_{2k+1} - dC_{2k-1}\right]_{(k,k)}\right)\right) \\
& - (-1)^k (k+1)^2 \left(d\Phi_{2k} \,\middle|\Uparrow \left(d\left[\delta C_{2k+1} - dC_{2k-1}\right]_{(k,k)}\right.\right. \\
& \qquad \left.\left. + \;\delta\left[\delta C_{2k+3} - dC_{2k+1}\right]_{(k+1,k+1)}\right)\right).
\end{aligned}
\tag{6.21}
$$

We will prove that (6.21) equals 0 in two stages. First, we will assume that every symmetrization bracket in (6.21) can be dropped and evaluate $\delta_C S$ with this simplification. Then we will prove that it is valid to ignore the explicit symmetrization in this way.

Setting $\left[\delta C_{2k-1}\right]_{(k-1,k-1)} = \delta C_{2k-1}$, etc., and then using $d^2 = \delta^2 = 0$, we can rearrange (6.21) as follows:

$$
\begin{aligned}
\delta_C S \;=\; & (-1)^k \left(d\Phi_{2k} \,\middle|\, \mathbf{K}\, C_{2k+1}\right) \;-\; (-1)^k \left(\delta\Phi_{2k} \,\middle|\, \mathbf{K}\, C_{2k-1}\right) \\
& + (-1)^k k^2 \left(\delta\Phi_{2k} \,\middle|\Uparrow \left(d\,\delta C_{2k-1} \;-\; \delta\, dC_{2k-1}\right)\right) \\
& - (-1)^k (k+1)^2 \left(d\Phi_{2k} \,\middle|\Uparrow \left(d\,\delta C_{2k+1} \;-\; \delta\, dC_{2k+1}\right)\right).
\end{aligned}
\tag{6.22}
$$

Now note that

$$
\Uparrow\, (d\delta - \delta d) C_{2k-1} \;=\; \mathbf{K} \,\Uparrow\Downarrow C_{2k-1} \;=\; \mathbf{K}\,\frac{1}{k^2} C_{2k-1}; \tag{6.23}
$$

we have used the representation (5.23) in the first step, and (5.25) and the

$(k, k-1)$ (maximal) symmetrization of C_{2k-1} in the second step. Inserting (6.23) and the corresponding identity for C_{2k+1} into (6.22), one finds that everything cancels.

Now we must prove that it was valid to ignore the symmetrizers. The bracket in the first term of (6.21) is obviously superfluous, since it contracts directly with Φ_{2k}. In the remaining terms, we must integrate by parts, moving $\Uparrow$ and d or δ to the left side of the inner product. In the piece of the second term involving δC_{2k-1}, we find the structure $\delta \Uparrow (\delta\Phi_{2k})$. Let us rewrite this using

$$k^2 \delta \Uparrow = \left\{k^2 \delta \Uparrow + \frac{(k-1)^2}{2} \Uparrow \delta\right\} - \frac{(k-1)^2}{2} \Uparrow \delta \,. \tag{6.24}$$

The last term on the right leads to $\delta^2\Phi_{2k} = 0$. The term in braces is the combination of δ and $\Uparrow$ which appears in eq. (5.28). $\delta\Phi_{2k}$ is annihilated by $\Downarrow$ (since $\Downarrow$ anticommutes with δ) and is therefore maximally symmetrized; then, by (5.28), the term in braces acting on $\delta\Phi_{2k}$ is maximally symmetrized. We have thus proved that $\delta \Uparrow \delta\Phi_{2k}$ has the symmetrization $(k-1, k-1)$; thus the symmetrization bracket on the right-hand side of the inner product is superfluous and can be dropped. A parallel argument, using eq. (5.27), allows us to drop the symmetrization bracket in the last term of (6.21), the term involving δC_{2k+3}. Finally, we may apply this argument to the two remaining terms. After passing d and δ through $\Uparrow$ by the use of (5.27) and (5.28), we find two terms which combine into the structure

$$(-1)^k \frac{k^2}{2} \left(\Uparrow (d\delta - \delta d)\Phi_{2k} \,\middle|\, [\ldots]_{(k,k)}\right) \,. \tag{6.25}$$

But $(d\delta - \delta d)\Phi_{2k} = \mathbf{K} \Downarrow \Phi_{2k} = 0$, by eq. (5.24). We have now rearranged (6.21) in such a way that every nonzero term is automatically projected onto the appropriate Young symmetrization. This completes the proof of the gauge-invariance of (6.20).

Eq. (6.20) is our final result for the action of the free string field theory, made local by the addition of Stueckelberg fields. Its field content is highly restricted by a web of gauge invariances. We must now check that this content reproduces that of more conventional approaches to the string theory.

7. Gauge-Fixing and Quantization

Having now constructed a plausible form for the quadratic action of string fields, we must demonstrate its equivalence to other forms of this action which have been presented previously. In this section, we will present gauge-fixing prescriptions for our action which reduce it to the forms constructed by Kaku and Kikkawa[7] and by Siegel[10] .

Kaku and Kikkawa made their construction in the transverse gauge. To enter this gauge, let us specialize our fields $\Phi[x(\sigma)]$ on the whole of string configuration space to their values on the subspace for which $x^+(\sigma) = \tau$, independently of σ, and to that subset of functions annihilated by nonzero Fourier components of $p^+(\sigma)$:

$$P_n^+ \Phi[\tau, x^-(\sigma), \vec{x}(\sigma)] = 0 \quad (n \neq 0). \tag{7.1}$$

On such functions, the L_n take the form:

$$L_n \;=\; L_n^{\mathrm{tr}} - p^+ \alpha_n^-, \tag{7.2}$$

where L_n^{tr} is given by eq. (2.7), with μ summed over transverse directions only. We can then solve the level 0 condition explicitly: The action of α_n^- on Φ must be exactly that of L_n^{tr}/p^+. We can restrict the space of Φ's to those which satisfy this condition; such Φ's depend only on the transverse coordinates of the string $\vec{x}(\sigma)$, since the dependence on $x^-(\sigma)$ is specified through the action of α_n^-. On this subspace, K simplifies to $2(L_0 - 1)$. Further,

$$2(L_0 - 1) \;=\; 2(L_0^{\rm tr} - 1) - 2p^+p^- . \qquad (7.3)$$

Now $p^- = -i\partial/\partial\tau$, so we find, finally,

$$S \;=\; -\frac{1}{2}\left(\Phi[\vec{x}(\sigma)]\;\middle|\;\left[2p^+ i\frac{\partial}{\partial\tau} + |\vec{p}|^2 + \sum_{n>0}\vec{\alpha}_{-n}\cdot\vec{\alpha}_n\right]\Phi[\vec{x}(\sigma)]\right). \qquad (7.4)$$

This is precisely the quadratic term in the action of Kaku and Kikkawa.

It is not obvious from this discussion that the gauge symmetries of our action (6.20) suffice to eliminate two coordinate degrees of freedom plus all of the Stueckelberg auxilliary fields. Thorn[16] has studied the counting of degrees of freedom in the transverse gauge, but only far enough to show that two coordinate degrees of freedom can be gauged away. It is possible, by refining this argument, to shown that all of the Stueckelberg fields can also be removed, leaving precisely the states associated with 24 transverse degrees of freedom. This argument requires, however, some additional technical methods; it will be given elsewhere[30] . Here, we will argue to this conclusion in another way, by verifying that the counting of states in our formulation reproduces exactly that which Siegel has found in the covariant gauge.

Siegel[10] discovered a gauge-fixed form of the string field theory in which every component field has as its free-field action precisely $\mathbf{K}$, with no gauge or spin projection. It is appropriate to call this the Feynman-Siegel gauge. To accomplish this, Siegel introduced a string field which depends also on two anticommuting ghost coordinates: $\Phi[x(\sigma), \Theta(\sigma), \hat{\Theta}(\sigma)]$. For the open string, $\Theta(\sigma)$ has a zero mode, but the coefficients of this zero mode are auxiliary fields with trivial kinetic energy terms which we may ignore in this discussion. The expansion of Siegel's field in the nonzero modes of $\Theta(\sigma)$ and $\hat{\Theta}(\sigma)$ yields a sum of terms of the form

$$\Theta_{m_1}\cdots\Theta_{m_a}\hat{\Theta}^{n_1}\cdots\hat{\Theta}^{n_b}\cdot C^{[m_1\ldots m_a]}{}_{[n_1\ldots n_b]}\;; \tag{7.5}$$

the coefficient functions in this expansion, which are the component fields of Siegel's theory, are in 1-to-1 correspondence with our string-field differential forms, before Young symmetrization has been performed. These fields are commuting or anticommuting according to whether the total number of indices is even or odd. Siegel has shown that adding two additional ghost coordinates in this way produces a number of new states which, if anticommuting fields are counted with a negative sign, is exactly equal and opposite to the number of states in the original string theory which involve excitations of oscillators in the two last spatial directions[31] . Thus, the counting of states in Siegel's formulation reproduces exactly that of the transverse gauge, with no additional degrees of freedom.

Our gauge-invariant action (6.20) contains only a small subset of Siegel's fields, the commuting $\binom{k}{k}$-forms with indices symmetrized according to the Young tableaux shown in (6.17). The rest of Siegel's fields must then appear as ghost fields in the Fadde'ev-Popov gauge-fixing procedure. Let us now explain how that procedure works here.

Notice that the action (6.20) has the form:

$$S \;=\; S_{\mathrm{FS}}[\Phi] \;+\; (-1)^k\, k^2 \left(\mathcal{F}_{2k-1}\;|\!\Uparrow\, \mathcal{F}_{2k-1}\right), \tag{7.6}$$

where S_{FS} is the Feynman-Siegel gauge action for the physical fields Φ_{2k} and $\mathcal{F}_{2k-1}$ is a gauge-fixing term:

$$\mathcal{F}_{2k-1} \;=\; \left(d\Phi_{2k-2} + \delta\Phi_{2k}\right)\,. \tag{7.7}$$

We may thus convert (7.6) to $S_{\mathrm{FS}}[\Phi]$ by subtracting the squares of the $\mathcal{F}_{2k-1}$; the price of this is that we must add an appropriate ghost Lagrangian. The ghosts C

must transform as the gauge parameters of (6.20); that is, they must be $\binom{k}{k-1}$-forms symmetrized according to $(k, k-1)$. The antighosts $\bar{\mathcal{C}}$ will be $\binom{k-1}{k}$-forms with the same symmetrization.

The ghost action is given by:

$$
\begin{aligned}
S_C &= (-1)^k k^2 \left(\bar{\mathcal{C}}_{2k-1} \,|\!\Uparrow\, \delta_C \mathcal{F}_{2k-1}\right) \\
&= (-1)^k k^2 \left(\bar{\mathcal{C}}_{2k-1} \,|\!\Uparrow\, \Big(d\left[\delta C_{2k-1} - dC_{2k-3}\right]_{(k-1,k-1)}\right. \\
&\qquad\qquad \left. + \,\delta\left[\delta C_{2k+1} - dC_{2k-1}\right]_{(k,k)}\Big)\right).
\end{aligned}
\tag{7.8}
$$

The factor $(-1)^k k^2$ in front of each term is arbitrarily chosen and can be absorbed into the normalization of $\bar{\mathcal{C}}_{2k-1}$. To simplify this, note that dC_{2k-1} has only two possible Young-symmetrized components, those with (k, k) and $(k+1, k-1)$ symmetrization. Thus

$$
d\,C_{2k-1} = \left[dC_{2k-1}\right]_{(k,k)} + \left[dC_{2k-1}\right]_{(k+1,k-1)} \,. \tag{7.9}
$$

Similarly δC_{2k-1} contains only $(k-1, k-1)$ and $(k, k-2)$, and so a similar identity holds for this quantity. Use these identities, and $d^2 = \delta^2 = 0$, to write the eq. (7.8) as

$$
\begin{aligned}
S_C &= (-1)^k k^2 \Big(\bar{\mathcal{C}}_{2k-1} \,|\!\Uparrow\, (d\delta - \delta d) C_{2k-1} \\
&\qquad - \Big(d\left[\delta C_{2k-1} - dC_{2k-3}\right]_{(k,k-2)} + \delta\left[\delta C_{2k+1} - dC_{2k-1}\right]_{(k+1,k-1)}\Big)\Big).
\end{aligned}
\tag{7.10}
$$

Rearrange the first term on the right using (5.23) and (5.25):

$$
\Uparrow (d\delta - \delta d) C_{2k-1} = \Uparrow\Downarrow\, C_{2k-1} = \frac{1}{k^2}\,\mathbf{K}\, C_{2k-1}. \tag{7.11}
$$

To rearrange the last two terms, note that the quantities

$$
\left(k\, d \Uparrow + \frac{(k+1)^2}{2} \Uparrow d\right) \bar{\mathcal{C}}_{2k-1} \,, \quad \left((k+1)\delta \Uparrow + \frac{k}{2} \Uparrow \delta\right) \bar{\mathcal{C}}_{2k-1} \tag{7.12}
$$

are restricted by (5.27) and (5.28), respectively, to belong to the fully symmetrized Young tableaux (k,k) and $(k-1),(k-1)$; thus, they are annihilated by the explicit Young symmetrizers. Use this property to integrate these two terms by parts, passing d and δ through $\Uparrow$. After this manipulation has been performed, the explicit Young symmetrizers are superfluous, because $\Uparrow$ annihilates the only alternative structures which could appear on the right, (k,k) and $(k-1,k-1)$. The ghost action has now become:

$$\begin{aligned} \mathcal{S}_C = (-1)^k \Big\{ & (\bar{C}_{2k-1} \mid \mathbf{K} C_{2k-1}) \\ & - \frac{k(k+1)}{2} (\delta\bar{C}_{2k+1} - d\bar{C}_{2k-1} \mid \Uparrow (\delta C_{2k+1} - dC_{2k-1})) \Big\}. \end{aligned} \tag{7.13}$$

This is now exactly of the form

$$\mathcal{S}_C = \mathcal{S}_{\mathrm{FS}} - (-1)^k \frac{k(k+1)}{2} (\bar{\mathcal{H}}_{2k} \mid \Uparrow \mathcal{H}_{2k}) , \tag{7.14}$$

where $\mathcal{S}_{\mathrm{FS}}[C]$ is the Feynman-Siegel gauge action for the ghosts and antighosts, and

$$\mathcal{H}_{2k} = \delta C_{2k-1} - dC_{2k+1} . \tag{7.15}$$

The action (7.13) is invariant to the second-level gauge symmetries (6.19). This must be true on general principles, because the gauge-fixing term which we added to the original action was $\mathcal{G}$-invariant. The invariance can also be checked directly by the method we used to verify the gauge-invariance of (6.20). The proof requires the identity

$$\Uparrow\Downarrow \mathcal{G}_{2k} = \frac{2}{k(k+1)} \mathcal{G}_{2k}, \tag{7.16}$$

which follows from (5.25) by the $(k+1,k-1)$ maximal symmetrization of $\mathcal{G}_{2k}$.

$\mathcal{H}_{2k}$ has the form of gauge-fixing term for the $\mathcal{G}$-symmetry. By subtracting the square of $\mathcal{H}_{2k}$ appropriately, we can convert $\mathcal{S}_C$ to $\mathcal{S}_{\mathrm{FS}}[C]$, at the price of adding ghost-of-ghost fields $\mathcal{G}$. One can work out the action for these fields by following, step by step, the methods used to derive (7.13). The result is

$$
\begin{aligned}
S_{\mathcal{G}} &= (-1)^k \Bigg\{ \; (\bar{\mathcal{G}}_{2k} \mid \mathbf{K}\mathcal{G}_{2k}) \\
&\qquad - \frac{k(k+2)}{3} \left(d\bar{\mathcal{G}}_{2k} + \delta \bar{\mathcal{G}}_{2k+2} \,|\!\Uparrow\, (d\mathcal{G}_{2k} + \delta\mathcal{G}_{2k+2}) \right) \Bigg\},
\end{aligned} \tag{7.17}
$$

which is again of the form of a Feynman-Siegel gauge action and sum of squares of gauge-fixing terms. (7.13) can in turn be gauge-fixed, at the price of introducing higher level ghosts. The process continues indefinitely. For example, labeling the $\mathcal{G}_{2k}$ as the first level of commuting ghosts, the action at the nth level of commuting ghosts is

$$
\begin{aligned}
S^{(n)}_{\mathcal{G}} &= (-1)^k \Bigg\{ \; (\bar{\mathcal{G}}^{(n)}_{2k} \mid \mathbf{K}\mathcal{G}^{(n)}_{2k}) \\
&\qquad - \frac{(k+1-n)(k+1+n)}{2n+1} \left(d\bar{\mathcal{G}}^{(n)}_{2k} + \delta \bar{\mathcal{G}}^{(n)}_{2k+2} \,|\!\Uparrow\, (d\mathcal{G}^{(n)}_{2k} + \delta\mathcal{G}^{(n)}_{2k+2}) \right) \Bigg\}.
\end{aligned} \tag{7.18}
$$

This action is invariant to gauge transformations generated by C's which are $\binom{k+n+1}{k-n}$-forms; as before, the proof follows exactly the method set out at the end of Section 6. At each level of the hierarachy of gauge transformations, one finds the Feynman-Siegel gauge action for the (ghost-of-)^{n}ghost fields, plus a gauge-fixing term for the residual gauge symmetry at that level.

We have now shown how the form of the Feynman-Siegel gauge action arises for each component field. It still remains to count the various component fields and confirm that each field generated by our procedure corresponds to a component of Siegel's master field. To do this, we must recall a result[32–34] from the theory of antisymmetric tensor fields, the simplest context in which gauge symmetries have gauge symmetries. Naively, one might suspect that one needs four ghosts-of-ghosts: Since the higher-level gauge transformation may be applied either to the ghost or of the antighost, we have two symmetries and thus we require two ghosts-of-ghosts and two antighosts-of-ghosts (all commuting fields). However, when proper account is taken of the fact that the gauge-fixing term at the first level has its own gauge invariance, one finds at the second level an extra square root of the Fadde'ev-Popov determinant for each gauge-fixing condition at the first level. This effect (called by Siegel[32] 'hidden ghosts') causes one anticommuting ghost to be added, or one commuting ghost to be subtracted, at the second level. Continuing in this way, one finds that the quantum theory of a p-form requires 2 ghosts, 3 ghosts-of-ghosts, 4 (ghosts-of-)^{2}ghosts, ..., $(n+2)$ (ghosts-of-)^{n}ghosts; these fields are commuting when n is even and anticommuting when n is odd.

Using this method of counting, we can work out the content of our gauge-fixed theory. Let us first count the fields which are $(2k-1)$-forms. We require $2k$ fields which are the (ghosts-of-)$^{2k-2}$ghosts of Φ_0; these are symmetrized according to $(2k-1,0)$. We require $(2k-2)$ fields which are the (ghosts-of-)$^{2k-4}$ghosts of Φ_2; these are symmetrized according to $(2k-3,2)$. There are two fewer (ghosts-of-)$^{2k-6}$ghosts of Φ_6, and these have the next higher Young symmetrization $(2k-5,4)$. The process continues in this way until we reach the simple ghosts C_{2k-1} of Φ_{2k}. This content can be partitioned as follows:

$$
\begin{aligned}
&(2k-1,0)\\
&+(2k-1,0)\ +\ (2k-3,2)\\
&+\qquad \cdots\\
&+(2k-1,0)\ +\ (2k-3,2)\ +\ \ldots\ +\ (k+1,k-3)\\
&+(2k-1,0)\ +\ (2k-3,2)\ +\ \ldots\ +\ (k+1,k-3)\ +\ (k,k-1)\\
&+(2k-1,0)\ +\ (2k-3,2)\ +\ \ldots\ +\ (k+1,k-3)\ +\ (k,k-1)\\
&+(2k-1,0)\ +\ (2k-3,2)\ +\ \ldots\ +\ (k+1,k-3)\\
&+\qquad \cdots\\
&+(2k-1,0)\ +\ (2k-3,2)\\
&+(2k-1,0)\ .
\end{aligned}
\tag{7.19}
$$

The nth line of this display gives the decomposition of a general $\binom{2k-n}{n-1}$-form into Young-symmetrized components. Thus, the full content of (7.19) can be assembled into a set of $\binom{2k-n}{n-1}$-forms of general symmetry, one such form for every n. This is precisely the content found by Siegel at the anticommuting levels.

At the commuting levels, the counting of ghosts works in the same way. Considering fields with $2k$ indices, the ghosts account for the entire content of Siegel's theory except for one component of the $\binom{k}{k}$-form which is symmetrized according to (k,k). But this is precisely the physical field Φ_{2k}. Thus our formulations agree exactly in the form of the action and in the counting of states. The field which Siegel originally noticed must be added to the content of Φ_0 to define the classical string theory was the lowest component of Φ_2. We have realized his conjecture that the classical free string theory can be completed by adding this and a set of additional compensating fields.

8. Closed-String Fields

Now that we have worked out the full structure of the gauge-invariant quadratic action for open strings, we should indicate how this analysis generalizes to closed strings. We will work only up to the first excited level, the one which contains the graviton. We will find that the dilaton arises as a Stueckelberg field, in close correspondence to the way that this field arises in Siegel's formalism[11].

Let us first review the basic kinematics. The closed string has twice as many modes as the open string. These can be parametrized by separate sets of α_n corresponding to right- and left-moving modes on the string. For example, $p^\mu(\sigma)$ should now be expanded as:

$$p^\mu(\sigma) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \left[\alpha_n e^{in\sigma} + \bar{\alpha}_n e^{-in\sigma}\right], \tag{8.1}$$

where the α_n and $\bar{\alpha}_n$ commute with one another and have, among themselves, the commutation relations (2.3). The $n = 0$ components must be given by $\alpha_0 = \bar{\alpha}_0 = \frac{1}{2}p$. Virasoro operators L_n and $\bar{L}_n$ can be defined from the α_n and $\bar{\alpha}_n$ according to (2.7). The operator giving the equation of motion of free strings is:

$$4\Big\{(L_0 - 1) + (\bar{L}_0 - 1)\Big\} = p^2 + 4\Big\{\sum_{n>0}(\alpha_{-n}\cdot\alpha_n + \bar{\alpha}_{-n}\cdot\bar{\alpha}_n) - 2\Big\}. \tag{8.2}$$

To generalize the operator (8.2) to a reparametrization-invariant form, we should multiply it by the projector onto level 0. Now, however, we have two independent Virasoro algebras, generated by the L_n's and the $\bar{L}_n$'s, so we must make two level 0 projections, corresponding to the conditions

$$L_n\Phi = 0 \qquad \bar{L}_n\Phi = 0 \qquad (n > 0). \tag{8.3}$$

The two projectors onto level 0, which we will call P and $\bar{P}$, are built from the corresponding L's according to the prescription (4.5). The reparametrization-invariant action for closed string fields must then be

$$S = -\frac{1}{2}(\Phi \mid 4\left[(L_0 - 1) + (\bar{L}_0 - 1)\right] P\bar{P}\,\Phi). \tag{8.4}$$

We must also impose from outside the constraint that the coordinate system on the string not undergo an overall rotation:

$$(L_0 - \bar{L}_0)\Phi = 0. \tag{8.5}$$

Let us now consider a string field, subject to the constraint (8.5), expanded in eigenstates of the mass operator. If $\Phi^{(0)}$ is the state annihilated by the α_n and $\bar{\alpha}_n$, for $n > 0$, we may expand

$$\Phi[x(\sigma)] = \left\{\phi(x) - t^{\mu\nu}(x)\alpha^{\mu}_{-1}\bar{\alpha}^{\nu}_{-1} + + \ldots\right\}\Phi^{(0)}. \tag{8.6}$$

$t^{\mu\nu}$ is a tensor field of indefinite symmetry. The action of the kinetic energy operator on Φ can be represented as

$$K = 4\left[(L_0 - 1) + (\bar{L}_0 - 1)\right]\left[1 - L_{-1}\frac{1}{2L_0}L_1\right]\left[1 - \bar{L}_{-1}\frac{1}{2\bar{L}_0}\bar{L}_1\right] + \ldots; \tag{8.7}$$

the omitted terms annihilate the first mass level.

Inserting (8.6) and (8.7) into (8.4) and extracting the term involving $t^{\mu\nu}$, we find

$$S_{(2)} = -\frac{1}{2}\int d^dx\; t_{\mu\nu}\left[-\partial^2\left(\eta^{\mu\lambda} - \frac{\partial^\mu\partial^\lambda}{\partial^2}\right)\left(\eta^{\nu\sigma} - \frac{\partial^\nu\partial^\sigma}{\partial^2}\right)\right]t_{\lambda\sigma}. \tag{8.8}$$

To understand this expression, it is useful to divide t into its symmetric and antisymmetric parts:

$$t^{\mu\nu} = \frac{1}{\sqrt{2}}(h^{\mu\nu} + a^{\mu\nu}). \tag{8.9}$$

For the antisymmetric field, (8.8) reduces to

$$\begin{aligned} S_{(2)} &= \frac{1}{4}\int d^dx\ a_{\mu\nu}\Big(-\partial^2\eta^{\mu\lambda}\eta^{\nu\sigma} + \partial^\mu\partial^\lambda\eta^{\nu\sigma} + \partial^\nu\partial^\sigma\eta^{\mu\lambda}\Big)a_{\lambda\sigma} \\ &= -\frac{1}{12}\int d^dx\ \left(H^{\eta\lambda\sigma}\right)^2, \end{aligned} \tag{8.10}$$

where $H^{\eta\lambda\sigma} = \partial^{[\eta}a^{\lambda\sigma]}$ is the gauge-invariant field strength associated with $a^{\lambda\sigma}$. For the symmetric part of t, this action may be written in the form

$$\begin{aligned} S_{(2)} = \frac{1}{8}\int d^dx\ \Bigg[h_{\mu\nu}\Bigg\{ &-\partial^2\left(\eta^{\mu\lambda} - \frac{\partial^\mu\partial^\lambda}{\partial^2}\right)\left(\eta^{\nu\sigma} - \frac{\partial^\nu\partial^\sigma}{\partial^2}\right) + (\lambda \leftrightarrow \sigma) \\ &-2(-\partial^2)\left(\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\partial^2}\right)\left(\eta^{\lambda\sigma} - \frac{\partial^\lambda\partial^\sigma}{\partial^2}\right)\Bigg\}h_{\lambda\sigma} \\ &+2h_{\mu\nu}\Bigg\{\left(\eta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\partial^2}\right)(-\partial^2)\left(\eta^{\lambda\sigma} - \frac{\partial^\lambda\partial^\sigma}{\partial^2}\right)\Bigg\}h_{\lambda\sigma}\Bigg]. \end{aligned} \tag{8.11}$$

The first two lines of this expression may be recognized as the quadratic term in the expansion of the Einstein-Hilbert action

$$\int \sqrt{-g}R \tag{8.12}$$

obtained by replacing $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Thus, the linearized theory of gravity comes directly out of this formalism. The last line can be written, using

$$R = \partial^\mu\partial^\nu h_{\mu\nu} - \partial^2 h^\mu_\mu + \ldots, \tag{8.13}$$

as a nonlocal curvature-curvature interaction, one which would result from eliminating a massless Stueckelberg field φ. If we introduce this field to render the action local, we find a Lagrangian involving a massless graviton, an antisymmetric tensor field, and a massless scalar—exactly the conventional content of the closed string at this level. Our action is invariant to linearized general coordinate transformations and gauge motions of $a^{\mu\nu}$:

$$\begin{aligned} \delta h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \\ \delta a_{\mu\nu} &= \partial_\mu \xi'_\nu - \partial_\nu \xi'_\mu; \end{aligned} \tag{8.14}$$

this gauge invariance arises naturally as the zero-mass level component of the chordal gauge motion

$$\delta\Phi \;=\; L_{-1}\Psi_1 + \bar{L}_{-1}\bar{\Psi}_1. \tag{8.15}$$

Because we have obtained our action only at the linearized level, it is not clear how to complete it to a geometrically invariant form. Recently, however Callan, Martinec, Perry, and Friedan[35] , have studied the constraints which conformal invariance places on the first-quantized string theory and have shown that these constraints take the form of the equations of motion which follow from the following action principle:

$$S \;=\; \int d^d x \,\sqrt{-g}e^{-2\varphi}\big[R + 4(\partial_\mu\varphi)^2 - \frac{1}{12}H^2\big]. \tag{8.16}$$

Our action for the massless closed string fields agrees with this one up to the linearized level. The consistency of the string theory requires that the constraints on background fields necessary for conformal invariance be consistent with the equations of motion of the string component fields. Nevertheless, the agreement between our results seems quite miraculous, considering the very different routes by which these results were obtained.

9. Superstrings

The analysis we have described may be generalized in a natural way to open and closed superstrings* . The formalism one finds is not a completely satisfactory one; in particular, it does not possess manifest supersymmetry. However, it does possess chordal gauge invariances which (at the linearized level) contain the expected local symmetries, including local supersymmetry. In this section, we will present that construction in enough detail to make its features and its problems clear.

Because our analysis depends on the implementation of general reparametrization invariance, we will work in the original Neveu-Schwarz-Ramond [36,37] formulation of the superstring. In this formulation, the operators of the first-quantized string theory are bosonic and fermionic coordinate operators carrying space-time vector indices [20] . The string equations of motion are invariant to a 2-dimensional local supersymmetry. The two possible boundary conditions for the fermionic coordinates define two sectors, the Ramond and Neveu-Schwarz sectors, whose particle states are, respectively, fermions and bosons. In each sector, one must impose that states be invariant to local reparametrizations and local supersymmetry motions. The local supersymmetry generators are called F_n in the Ramond sector (n is an integer) and G_k in the Neveu-Schwarz sector (k is a half-integer). They obey an algebra which is given, for example, in Scherk's review article [20] .

To extend our construction to this context, define projection operators for the reparametrization algebra in each sector. P_R should satisfy

$$P_R L_{-n} = 0 , \qquad P_R F_{-n} = 0, \tag{9.1}$$

for $n > 0$; P_{NS} should satisfy

$$P_{NS} L_{-n} = 0 , \qquad P_{NS} G_{-k} = 0, \tag{9.2}$$

for $n, k > 0$. These projectors may be constructed by following exactly the

* This generalization has also been discussed by Friedan, ref. 15, and Kaku, ref. 13.

prescription given in eqs. (4.3), (4.5) if one takes the $\mathcal{L}_i^{(n)}$ to contain all (nonredundant) combinations of the L_n and F_n (or L_n and G_k) which raise the mass level by n units. In the Neveu-Schwarz sector, one should also include projectors $\Pi^{(k)}$ which remove states at half-integer mass levels.

From these projectors, we can form gauge-invariant kinetic energy operators:

$$\mathcal{K}_R = \sqrt{2}F_0 P_R \, , \qquad \mathcal{K}_{NS} = (2L_0 - 1)P_{NS} \tag{9.3}$$

Let d_n^μ (b_n^μ) denote fermion coordinate operators in the Ramond (Neveu-Schwarz) sector; the zero mode d_0^μ is represented by

$$d_0^\mu = \frac{1}{\sqrt{2}}\gamma^\mu, \tag{9.4}$$

so that $F_0 = \sum d_{n+k}\cdot\alpha_{-k} = \gamma\cdot p/\sqrt{2} + \ldots$. Then one can see that the operators in (9.3) reduce to the kinetic energy terms $(\gamma\cdot p + \mathcal{M})$ and $p^2 + \mathcal{M}^2$, respectively, when acting on states at level 0.

The fields of the string theory should be general functions of the bosonic and fermionic coordinates; we need a scalar and a spinor string field for the Neveu-Schwarz and Ramond string states. To recover a supersymmetric spectrum in 10 dimensions, we must restrict these fields according to the prescription of Gliozzi, Scherk, and Olive[4] :

$$\begin{aligned} \text{Neveu-Schwarz sector:} \qquad & \left(1 + (-1)^{N_f}\right)\Phi = 0, \\ \text{Ramond sector:} \qquad & \left(1 - (-1)^{N_f}\gamma^{11}\right)\Psi = 0; \end{aligned} \tag{9.5}$$

Φ should be real and Ψ Majorana. These string fields may be expanded in normal modes:

$$\begin{aligned} \Phi &= \Big\{ - iA^\mu(x) b^\mu_{-\frac{1}{2}} + \frac{i}{3!} c^{\mu\nu\lambda}(x) b^\mu_{-\frac{1}{2}} b^\nu_{-\frac{1}{2}} b^\lambda_{-\frac{1}{2}} \\ &\qquad - t^{\mu\nu}(x)\alpha^\mu_{-1} b^\nu_{-\frac{1}{2}} - iv^\mu(x) b^\mu_{-\frac{3}{2}} + \ldots \Big\}\Phi^{(0)} \\ \Psi &= \Big\{ \psi(x) - i\xi^\mu(x)\alpha^\mu_{-1} - i\beta^\mu(x) d^\mu_{-1} + \ldots \Big\}\Psi^{(0)} \end{aligned} \tag{9.6}$$

As in eq. (3.3), the coefficient functions belong to the Hilbert space on which the zero mode operators act; thus, in the Ramond case, they carry the spinor index of Ψ. The component fields of Ψ are Majorana-Weyl, with chirality given by (9.5).

We may now write the free-field action of the superstring theory as

$$S = -(\Psi \mid \sqrt{2}F_0 P_R \Psi) - \frac{1}{2}(\Phi \mid (2L_0 - 1)P_{NS}\,\Phi). \tag{9.7}$$

Using

$$P_{NS} = 1 - G_{-\frac{1}{2}}\frac{1}{2L_0}G_{\frac{1}{2}} + \ldots, \tag{9.8}$$

one can easily see that (9.7) reduces on the lowest mass level to the form

$$S = \int \Big(\bar{\psi}(i\gamma\cdot\partial)\psi - \frac{1}{4}(F_{\mu\nu})^2\Big), \tag{9.9}$$

where $F_{\mu\nu}$ is the field strength of $A_\mu(x)$. The gauge invariance of this action is one component of the chordal gauge symmetry

$$\delta\Phi = G_{-\frac{1}{2}}\Lambda. \tag{9.10}$$

At the lowest mass level, then, we recover precisely the linearized action of 10-dimensional supersymmetric Yang-Mills theory. Unfortunately, the higher mass levels of the action (9.7) are not manifestly supersymmetric. As Friedan [15] has already noted, one can see the problem even in the positions of the poles of P_R and P_{NS}, or, equivalently, in the spectrum of Stueckelberg fields necessary to render (9.7) local. At the second level, for example, the Stueckelberg fields required in 10 dimensions are a scalar of mass $m^2 = 5$ and a spinor of mass $m^2 = 25/8$. Perhaps, though, one can cast the action into a supersymmetric form by adding additional Stueckelberg fields.

The closed superstring, like the closed bosonic string, possesses two commuting sets of coordinate operators and, correspondingly, two commuting sets of reparametrization generators. The maximal theory, with oriented closed strings, contains four string fields, corresponding to the choice of Ramond or Neveu-Schwarz boundary conditions for each of the two sets of fermionic coordinates. If we constrain these fields to be annihilated by $(L_0 - \bar{L}_0)$, their expansions in normal modes begin with:

$$\begin{aligned}
\Phi &= \Big\{ - t^{\mu\nu} b^{\mu}_{-\frac{1}{2}} \bar{b}^{\nu}_{-\frac{1}{2}} \Big\} \Phi^{(0)} \\
\Psi_a &= \Big\{ - i\psi^{\mu}_a \bar{b}^{\mu}_{-\frac{1}{2}} + \dots \Big\} \Psi^{(0)} \\
\bar{\Psi}_a &= \Big\{ - i\bar{\psi}^{\mu}_a b^{\mu}_{-\frac{1}{2}} + \dots \Big\} \bar{\Psi}^{(0)} \\
X_{ab} &= \Big\{ \chi_{ab} + \dots \Big\} X^{(0)},
\end{aligned} \tag{9.11}$$

which is the content of the massless level of the type II closed string. The chordal gauge transformations relevant to the massless level are:

$$\delta\Phi = G_{-\frac{1}{2}}\Xi + \bar{G}_{-\frac{1}{2}}\bar{\Xi}, \tag{9.12}$$

which, in precise analogy to eq. (8.15), contains linearized general coordinate invariance and the gauge invariance of the antisymmetric tensor field, and

$$\delta\Psi_a = \bar{G}_{-\frac{1}{2}} E_a \qquad \delta\bar{\Psi}_a = G_{-\frac{1}{2}} \bar{E}_a, \tag{9.13}$$

which contain the linearized $N = 2$ local supersymmetry transformations

$$\delta\psi^{\mu}_a = \partial^{\mu}\epsilon_a, \qquad \delta\bar{\psi}^{\mu}_a = \partial^{\mu}\bar{\epsilon}_a. \tag{9.14}$$

10. Conclusions

In this paper, we have presented a formulation of string field theory which preserves the basic reparametrization invariance of the string. To implement this symmetry, we were led to an action with an enormously enlarged gauge group, one whose motions are parametrized by functionals on the space of strings. We have shown, both for the bosonic string and for the superstring, that these more general gauge transformations contain, at the linearized level, the local gauge invariances expected from the analysis of scattering amplitudes at low energy.

Our analysis leaves many questions unanswered. There are, in particular, three questions which seem to us most pressing and which must be answered to complete and extend this formalism. The first is that of finding an interaction term and a nonlinear chordal gauge transformation which leaves it invariant* The second is that of finding a manifestly supersymmetric form of the action for the superstring. The third is that of finding a derivation of our action directly in the string field theory, from some principle which arises from the geometry of the space of strings and gives an interpretation to the formalism of differential forms which we have presented. These questions are obviously deep and difficult, but they point temptingly toward a new realm of mathematical physics beyond that describable by local fields.

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⋆ Some progress in this direction has been made by Neveu and West, ref. 17.

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