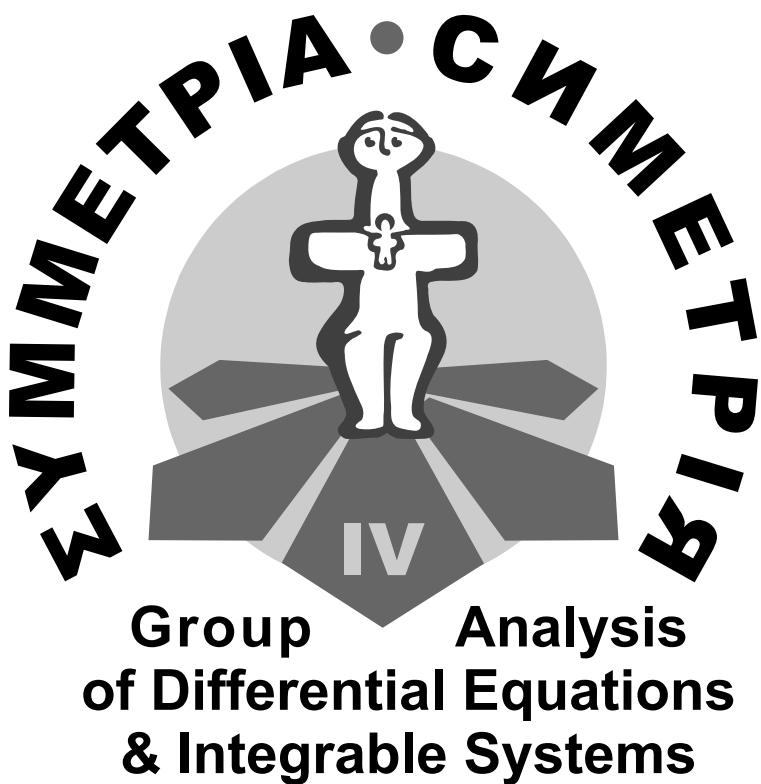


The 4th International Workshop in
**Group Analysis of Differential
Equations and Integrable Systems**



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Preface

The Fourth International Workshop *Group Analysis of Differential Equations and Integrable Systems* GADEIS-IV took place in Protaras, Cyprus, from Sunday, October 26, to Thursday, October 30, 2008.

The aim of the meeting was to bring together leading scientists in group analysis, integrability and mathematical modelling. The main emphasis of the workshop was to be on applications of group methods in investigating nonlinear wave and diffusion phenomena, integrability theory, the modern theory of Lie groups and Lie algebras as well as the classical heritage, historical aspects and new theoretical developments in group analysis.

The series of Workshops is organized by the Department of Mathematics and Statistics of the University of Cyprus and the Department of Applied Research of the Institute of Mathematics of the National Academy of Science of Ukraine. The theme of the series is concentrated on recent developments in Lie theory of differential equations and integrability. It was initiated in 2005 as a meeting for discussion of results obtained due to intensive cooperation between teams of Cyprian and Ukrainian scientists. The workshop is held annually. The three previous Workshops took place in the new Campus of the University of Cyprus near Nicosia, October 27 (2005), September 25-28 (2006) and October 4-5 (2007). Every year the Workshops attract an increasing number of specialists in the themes of the series and related fields. The range of problems discussed on the Workshops also is continuously extended.

Forty scientists from fifteen different countries participated in the Fourth Workshop. Thirty one lectures on recent developments in traditional and modern aspects of group analysis and integrability and their applications were presented.

This book consists of twenty selected papers presented at the conference. All papers have been reviewed by two independent referees. We are grateful to the contributors for preparing their manuscripts promptly and furthermore we express our gratitude to all anonymous referees for their constructive suggestions for improvement of the papers that appear in this book.

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Symmetries in atmospheric sciences

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Selected applications of symmetry methods in the atmospheric sciences are reviewed briefly. In particular the focus is put on the utilisation of the classical Lie symmetry approach to derive classes of closed-form solutions in from atmospheric models. This is illustrated with the barotropic vorticity equation. Moreover the possibility for construction of partially-invariant solutions is discussed for this model. A further point is a discussion of using symmetries for relating different classes of differential equations. This is illustrated with the spherical and the potential vorticity equation. Finally discrete symmetries are used to derive the minimal finite-mode version of the vorticity equation first discussed by E. Lorenz [*Tellus*, 1960, V.12, 243–254] in a sound mathematical fashion.

1 Introduction

Dynamic meteorology is concerned with the mathematical theory of atmospheric motion. It lies the cornerstone for daily weather prediction as it prepares the grounds for numerical computer models, without which reliable forecasts are hardly imaginable. However, since the advent of capable supercomputers and the accompanying shift towards methods for exhausting their capacities, analytical investigations of the governing equations have somewhat taken a back seat.

However, as is evident, numerical models need benchmark tests to check their reliability and hence closed-form solutions of the underlying mathematical models are still of great value. Testing whether a forecast model is able to reproduce a known closed-form solution of the original equation may serve as a first consistency check. It follows that techniques for obtaining such solutions in a systematic way are of rather pretty importance. For this purpose the classical Lie symmetry methods are well-suited.

Many dynamical models in use in the atmospheric sciences are adapted forms of the Navier–Stokes or the ideal Euler equations, taking into account both the rotation of the earth and the anisotropy of the atmosphere (the region of interest for weather prediction extends about 10 km in the vertical but several thousands of kilometers in the horizontal direction). For large-scale dynamics (i.e., horizontal length scale about some 1000 km) or for sake of conceptual simplicity, it is possible to restrict oneself to two-dimensional models. The most relevant example of such a model in dynamic meteorology is the barotropic vorticity equation. It is derived

from the incompressible Euler equations in a rotating reference frame by using the stream function (or vorticity) as dynamic variable.

In this contribution, we discuss symmetries and group-invariant solutions of the barotropic vorticity equation. Moreover the construction of partially-invariant solutions is shown. These issues are addressed in Section 2. Subsequently, in Section 3 the usage of symmetries for finding related differential equations is demonstrated using the potential and the spherical vorticity equation. It is shown that in both cases the rotational term in the equation can be canceled using suitable point transformations. In Section 4 discrete symmetries of the vorticity equation are used systematically to rederive the Lorenz (1960) model. This contribution ends with a short summary and discussion of future research plans, which can be found in the final section.

2 Group-invariant solutions and barotropic vorticity equation

It is worth considering the Lie symmetry problem of the barotropic vorticity equation since to the best of our knowledge this equation has not been investigated thoroughly in the light of symmetries before. Either only the symmetries and some closed-form solutions were computed without reference to classification of inequivalent subgroups [5, 8, 9] or the classification itself was not done in the most complete fashion [1].

We use the stream function–vorticity notation. The barotropic vorticity equation in Cartesian coordinates reads:

$$\zeta_t + \psi_x \zeta_y - \psi_y \zeta_x + \beta \psi_x = 0, \quad \zeta = \psi_{xx} + \psi_{yy}, \quad (1)$$

where ζ stands for the vorticity, ψ is the stream function and $\beta = \text{const}$ is a parameter controlling the North–South variation of the Earth’s angular rotation. It can be expressed as the North–South change of the vertical Coriolis parameter $f = 2\Omega \sin \varphi$ via $\beta = df/dy$, where Ω is the absolute value of the Earth’s angular rotation vector and φ denotes the latitude.

Equation (1) admits the infinite-dimensional maximal Lie invariance algebra \mathcal{B}_β^∞ generated by the operators [5, 9]

$$\begin{aligned} \mathcal{D} &= t\partial_t - x\partial_x - y\partial_y - 3\psi\partial_\psi, \quad \partial_t, \quad \partial_y, \\ \mathcal{X}(f) &= f(t)\partial_x - f'(t)y\partial_\psi, \quad \mathcal{Z}(g) = g(t)\partial_\psi, \end{aligned}$$

where f and g are arbitrary real-valued time-dependent functions. For a systematic group-invariant reduction of the vorticity equation by means of using subalgebras of \mathcal{B}_β^∞ it is necessary first to compute the corresponding optimal system of inequivalent subalgebras [11]. For one-dimensional subalgebras they are [1, 4]

$$\langle \mathcal{D} \rangle, \quad \langle \partial_t + c\partial_y \rangle, \quad \langle \partial_y + \mathcal{X}(f) \rangle, \quad \langle \mathcal{X}(f) + \mathcal{Z}(g) \rangle,$$

where $c = \{0, \pm 1\}$. The optimal system of two-dimensional subalgebras is [4]

$$\begin{aligned} & \langle \mathcal{D}, \partial_t \rangle, \quad \langle \mathcal{D}, \partial_y + a\mathcal{X}(1) \rangle, \quad \langle \mathcal{D}, \mathcal{X}(|t|^a) + c\mathcal{Z}(|t|^{a-2}) \rangle, \quad \langle \mathcal{D}, \mathcal{Z}(|t|^{a-2}) \rangle, \\ & \langle \partial_t + b\partial_y, \mathcal{X}(e^{at}) + \mathcal{Z}((abt + c)e^{at}) \rangle, \quad \langle \partial_t + b\partial_y, \mathcal{Z}((abt + c)e^{at}) \rangle, \\ & \langle \partial_y + \mathcal{X}(f^1), \mathcal{X}(1) + \mathcal{Z}(g^2) \rangle, \quad \langle \partial_y + \mathcal{X}(f^1), \mathcal{Z}(g^2) \rangle, \\ & \langle \mathcal{X}(f^1) + \mathcal{Z}(g^1), \mathcal{X}(f^2) + \mathcal{Z}(g^2) \rangle, \end{aligned}$$

with a, b, c being arbitrary constants and f^i and g^i , $i = 1, 2$, being arbitrary real-valued functions of time. In the fifth and sixth subalgebra additionally the condition $abc = 0$ has to hold. In the last subalgebra the pairs of functions (f^1, g^1) and (f^2, g^2) have to be linearly independent. Using the above two-dimensional subalgebras allows one to reduce (1) to ODEs. Moreover it is straightforward to show that the third and the fourth case of one-dimensional subalgebras lead to completely integrable PDEs. Hence, since all but the first two-dimensional subalgebra are extensions of special forms of the third or fourth one-dimensional subalgebras, only reduction by means of the first two-dimensional subalgebra may give an essentially new result.

Based on the above classification, it is possible to derive classes of inequivalent group-invariant solutions [4]. As a notable example we investigate reduction by means of $\langle \partial_y + \mathcal{X}(f) \rangle$. The invariant functions of this subalgebra are

$$p = x - fy, \quad q = t, \quad v = \psi + \frac{1}{2}f'y^2,$$

which allows one to reduce (1) to the classical Klein-Gordon equation $\tilde{v}_{\tilde{p}\tilde{q}} + \beta\tilde{v} = 0$, where

$$\tilde{q} = \int \frac{dq}{1 + f^2}, \quad \tilde{p} = p, \quad \tilde{v} = v - \frac{f''}{\beta}p + \frac{h(q)}{\beta} + \frac{((1 + f^2)f'')'}{\beta^2}.$$

For the meteorological application the case $f = \text{const}$ and making a harmonic ansatz for \tilde{v} is the most relevant: The corresponding solution represents the classical Rossby wave, which to a large extend governs the weather regimes in the mid-latitudes.

Considering (1) as a system of two equations using ψ and ζ as dependent variables, it is possible to compute partially-invariant solutions [12]. This should be illustrated using the subalgebra $\langle \mathcal{X}(1), \mathcal{Z}(g) \rangle$. Due to the presence of the second basis element, $\mathcal{Z}(g)$, no ansatz for ψ can be chosen. However, it is still possible to make an ansatz for ζ . Because of the joint invariance under $\mathcal{X}(1)$ and $\mathcal{Z}(g)$ we have $\zeta = \zeta(t, y)$ and thus (1) is transformed to

$$\zeta_t + \psi_x(\zeta_y + \beta) = 0, \quad \zeta = \psi_{xx} + \psi_{yy}.$$

Introducing the absolute vorticity $\eta = \zeta + \beta y$ we have to distinguish two cases. For $\eta_y = 0$ the solution of the vorticity equation is

$$\psi = \Psi(t, x, y) - \frac{1}{6}\beta y^3 + \frac{1}{2}\eta(t)y^2,$$

where Ψ satisfies the Laplace equation $\Psi_{xx} + \Psi_{yy} = 0$. For $\eta_y \neq 0$ the solution is

$$\psi = \frac{1}{(g^1)^2} F(\omega) - \frac{1}{6} \beta y^3 - \frac{g_t^1 y + g_t^0}{g^1} x + f^1 y + f^0,$$

where $\omega = g^1 y + g^0$ and g^1, g^0, f^1, f^0 are all real-valued functions of time.

3 Symmetries and the ineffective earth rotation

3.1 The spherical vorticity equation

Although the vorticity equation in Cartesian coordinates allows one to study some prominent features of large-scale geophysical fluid dynamics, it does not take into account the Earth's sphericity. For this purpose it is necessary to study vorticity dynamics in a rotating spherical coordinate system. The corresponding equation then is [14]:

$$\zeta_t + \frac{1}{a^2} (\psi_\lambda \zeta_\mu - \psi_\mu \zeta_\lambda) + \frac{2\Omega}{a^2} \psi_\lambda = 0, \quad (2)$$

where ψ is the (spherical) stream function and ζ the (spherical) vorticity,

$$\zeta = \frac{1}{a^2} \left[\frac{1}{1 - \mu^2} \psi_{\lambda\lambda} + ((1 - \mu^2) \psi_\mu)_\mu \right].$$

Rather than using λ (longitude) and φ (latitude) as spatial variables, it is advantageous to use λ and $\mu = \sin \varphi$. The mean radius of the earth is denoted by a .

The maximal Lie invariance algebra \mathfrak{g}_Ω of (2) is generated by the basis operators

$$\begin{aligned} \mathcal{D} &= t\partial_t - (\psi - \Omega\mu)\partial_\psi - \Omega t\partial_\lambda, & \partial_t, & \mathcal{Z}(g) = g(t)\partial_\psi, & \mathcal{J}_1 = \partial_\lambda, \\ \mathcal{J}_2 &= \mu \frac{\sin(\lambda + \Omega t)}{\sqrt{1 - \mu^2}} \partial_\lambda + \frac{\cos(\lambda + \Omega t)}{\sqrt{1 - \mu^2}} ((1 - \mu^2)\partial_\mu + \Omega\partial_\psi), \\ \mathcal{J}_3 &= \mu \frac{\cos(\lambda + \Omega t)}{\sqrt{1 - \mu^2}} \partial_\lambda - \frac{\sin(\lambda + \Omega t)}{\sqrt{1 - \mu^2}} ((1 - \mu^2)\partial_\mu + \Omega\partial_\psi), \end{aligned}$$

where g runs through the set of smooth functions of t . It is straightforward to map the algebra \mathfrak{g}_Ω with $\Omega \neq 0$ to the algebra \mathfrak{g}_0 by means of the transformation

$$\tilde{t} = t, \quad \tilde{\mu} = \mu, \quad \tilde{\lambda} = \lambda + \Omega t, \quad \tilde{\psi} = \psi - \Omega\mu.$$

Moreover this transformation also maps the equation (2) with $\Omega \neq 0$ to the equation of the same form with $\Omega = 0$, that is, it is possible to disregard the rotational term by setting $\Omega = 0$ for all practical calculations and finally obtain the corresponding results for the case $\Omega \neq 0$ by applying the above transformation. We note in passing that this transformation was already used by Platzman [14] to transform the vorticity equation to a zero angular momentum coordinate system.

For optimal systems of one- and two-dimensional subalgebras of \mathfrak{g}_0 and the computation of group-invariant solutions of (2) in a fashion similar to the previous section, see [4].

3.2 The potential vorticity equation

An extension of the classical barotropic vorticity equation is given through the potential vorticity equation. For flat topography the equation is

$$\zeta_t - F\psi_t + \psi_x\zeta_y - \psi_y\zeta_x + \beta\psi_x = 0, \quad \zeta = \psi_{xx} + \psi_{yy}, \quad (3)$$

where F is the ratio of the characteristic length scale to the Rossby radius of deformation [13]. In the Lagrangian view the above equation can be understood as individual conservation of the potential vorticity $q = \zeta + f - F\psi$. In this light the extension to the barotropic vorticity equation, which in turn states the individual conservation of absolute vorticity $\eta = \zeta + f$, becomes most obvious: While barotropic dynamics takes place solely in two dimensions, the additional term $-F\psi$ accounts for a variation of the fluid height in the vertical. In this respect the potential vorticity equation is especially suited for shallow water theory.

Determining the symmetries for the case $\beta = 0$ and $\beta \neq 0$ shows that both algebras and hence also the two corresponding equations are mapped to each other by applying the transformation [3]

$$\tilde{t} = t, \quad \tilde{x} = x + \frac{\beta}{F}t, \quad \tilde{y} = y, \quad \tilde{\psi} = \psi - \frac{\beta}{F}y.$$

Again this shows that there are models in the atmospheric sciences, in which the rotational terms are apparently not of prime importance.

4 Symmetries and finite-mode models

There is a long history in dynamic meteorology to convert the governing nonlinear PDEs to systems of coupled ODEs by means of series expansions, followed by a reasonable truncation of the series in use. Among one of the first models that was analyzed in this way again was the barotropic vorticity equation in Cartesian coordinates. This was done by Lorenz [10] in an adhoc fashion: He firstly expanded the vorticity in a a double Fourier series on the torus

$$\zeta = \sum_{\mathbf{m}} C_{\mathbf{m}} \exp(i\hat{\mathbf{m}} \cdot \mathbf{x}),$$

where $\mathbf{x} = x\mathbf{i} + y\mathbf{j}$, $\mathbf{m} = m_1\mathbf{i} + m_2\mathbf{j}$, $\hat{\mathbf{m}} = m_1k\mathbf{i} + m_2l\mathbf{j}$, $k = \text{const}$, $l = \text{const}$, $\mathbf{i} = (1, 0, 0)^T$, $\mathbf{j} = (0, 1, 0)^T$, m_1 and m_2 run through the integers and the coefficient C_{00} vanishes. Afterwards he substituted this expansion into (1) for $\beta = 0$ obtaining the spectral form of the vorticity equation, namely

$$\frac{dC_{\mathbf{m}}}{dt} = - \sum_{\mathbf{m}' \neq 0} \frac{c_{\mathbf{m}'} C_{\mathbf{m}-\mathbf{m}'}}{\hat{\mathbf{m}}'^2} (\mathbf{k} \cdot [\hat{\mathbf{m}}' \times \hat{\mathbf{m}}]), \quad (4)$$

where $\mathbf{k} = (0, 0, 1)^T$. Finally, he restricted the indices of the Fourier coefficients $C_{\mathbf{m}} = 1/2(A_{\mathbf{m}} - iB_{\mathbf{m}})$ to only run through the indices $\{-1, 0, 1\}$, leading to an

eight-component initial model. He then noted that, if the imaginary parts $B_{\mathbf{m}}$ of the coefficients at hand vanish initially, they remain zero for all times. Moreover, if the real coefficient $A_{11} = -A_{1,-1}$ initially, this equality is also preserved under the finite-mode dynamics. These observations allow to reduce (4) to the following three-component model

$$\begin{aligned}\frac{dA}{dt} &= -\left(\frac{1}{k^2} - \frac{1}{k^2 + l^2}\right) klFG, \\ \frac{dF}{dt} &= \left(\frac{1}{l^2} - \frac{1}{k^2 + l^2}\right) klAG, \\ \frac{dG}{dt} &= -\frac{1}{2} \left(\frac{1}{l^2} - \frac{1}{k^2}\right) klAF.\end{aligned}\tag{5}$$

where, $A := A_{01}, F := A_{10}, G := A_{1,-1}$.

Now it is instructive to see, whether it is possible to give a more rigorous justification of the two observations by Lorenz [2]. The key for this investigation is to start with the generators of admitted mirror symmetries of (1) for $\beta = 0$

$$\begin{aligned}e_1: \quad (x, y, t, \psi) &\mapsto (x, -y, t, -\psi), \\ e_2: \quad (x, y, t, \psi) &\mapsto (-x, y, t, -\psi), \\ e_3: \quad (x, y, t, \psi) &\mapsto (x, y, -t, -\psi).\end{aligned}$$

and induce them in the space of Fourier coefficients by means of series expansion. The induced symmetries are

$$\begin{aligned}e_1: \quad C_{m_1 m_2} &\mapsto -C_{m_1, -m_2}, \\ e_2: \quad C_{m_1 m_2} &\mapsto -C_{-m_1 m_2}, \\ e_3: \quad C_{m_1 m_2} &\mapsto -C_{m_1 m_2}, \quad t \mapsto -t.\end{aligned}$$

Moreover the translations with the values π/k and π/l in directions of x and y , respectively, induce proper transformations of the Fourier coefficients:

$$p: \quad C_{m_1 m_2} \mapsto (-1)^{m_1} C_{m_1 m_2}, \quad q: \quad C_{m_1 m_2} \mapsto (-1)^{m_2} C_{m_1 m_2}.$$

The transformations e_1, e_2, p and q act on the dependent variable and thus may be useful for reducing the number of Fourier coefficients at hand. Therefore the finite-dimensional symmetry group that is relevant for us is generated by these four elements and has the structure $G \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Selecting various subgroups of G allows one to reduce the eight-component initial model to five-, four- and three-component submodels, respectively. The three-component submodel by Lorenz is derived upon using the subgroup $S = \{1, pqe_1, pqe_2, e_1 e_2\}$. The transformation $e_1 e_2$ accounts for his first observation, i.e. $B_{m_1 m_2} = 0$, while the transformation pqe_1 yields the identification $A_{11} = -A_{1,-1}$, which justifies the second observation. In this way (5) is derived merely using symmetry techniques.

This example shows that it is possible to derive a consistent finite-mode model by means of a sound mathematical method. Having such a method at ones disposal is especially useful since up to now there are only few criteria for the selection of modes in finite-mode models.

5 Conclusion and future plans

In this paper we have reviewed some possible applications of well-established symmetry methods in the atmospheric sciences. The technique of constructing group-invariant solutions allows one to systematically rederive the well-known Rossby wave solution of the barotropic vorticity equation. Moreover relating differential equations by mapping their Lie algebras to each other enables one to cancel the terms due to the earth's angular rotation in both the potential and spherical vorticity equation. From the physical viewpoint this result is somewhat astonishing since these rotational terms are in fact to a great extend responsible for the overall complexity in formation of weather pattern.

In future works the classical method of Lie reduction should be applied to more sophisticated models of geophysical fluid dynamics, including the quasi-geostrophic model [7] and some classical convection model [6]. Moreover the usage of symmetries for a systematic determination of conservation laws in atmospheric science should be examined. This again has potential application in providing consistency checks for schemes of numerical integration.

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Construction of conservation laws: how the direct method generalizes Noether's theorem

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This paper shows how to construct directly the local conservation laws for essentially any given DE system. This comprehensive treatment is based on first finding conservation law multipliers. It is clearly shown how this treatment is related to and subsumes the classical Noether's theorem (which only holds for variational systems). In particular, multipliers are symmetries of a given PDE system only when the system is variational as written. The work presented in this paper amplifies and clarifies earlier work by the first and third authors.

1 Introduction

A conservation law of a non-degenerate DE system is a divergence expression that vanishes on all solutions of the DE system. In general, any such nontrivial expression that yields a local conservation law of a given DE system arises from a linear combination formed by *local multipliers (characteristics)* with each DE in the system, where the multipliers depend on the independent and dependent variables as well as at most a finite number of derivatives of the dependent variables of the given DE system. It turns out that a divergence expression depending on independent variables, dependent variables and their derivatives to some finite order is annihilated by the Euler operators associated with each of its dependent variables; conversely, if the Euler operators, associated with each dependent variable in an expression involving independent variables, dependent variables and their derivatives to some finite order, annihilate the expression, then the expression is a divergence expression. From this it follows that a given DE system has a local conservation law if and only if there exists a set of local multipliers whose scalar product with each DE in the system is identically annihilated *without restricting* the dependent variables in the scalar product to solutions of the DE system, i.e., the dependent variables, as well as each of their derivatives, are treated as *arbitrary* functions.

Thus the problem of finding local conservation laws of a given DE system reduces to the problem of finding sets of local multipliers whose scalar product with each DE in the system is annihilated by the Euler operators associated with each dependent variable where the dependent variables and their derivatives in the given DE system are replaced by arbitrary functions. Each such set of local multipliers yields a local conservation law of the given DE system. Moreover, for any given set of local conservation law multipliers, there is an integral formula to obtain the fluxes of the local conservation law [1–3]. Often it is straightforward to obtain the conservation law by direct calculation after its multipliers are known [4]. What has been outlined here is the *direct method* for obtaining local conservation laws.

For a given DE system, Lie's algorithm yields an over-determined set of linear determining equations whose solutions yield local symmetries. This set of linear PDEs arises from the linearization of the given DE system (*Fréchet derivative*) about an *arbitrary* solution of the given DE system, i.e., the resulting set of linear PDEs must hold for each solution of the given DE system. After the given DE system and its differential consequences are substituted into its linearization, the resulting linear PDE system yielding local symmetries must hold with the remaining dependent variables and their derivatives of the given DE system replaced by *arbitrary* functions.

In contrast, for a given DE system, sets of local conservation law multipliers are solutions of an over-determined set of linear determining equations arising from annihilations by Euler operators. It turns out that the set of linear multiplier determining equations for local conservation law multipliers includes the *adjoint* of the set of linear PDEs arising from the linearization of the given PDE system about an arbitrary solution of the given DE system [1].

It follows that in the situation when the set of linearized equations of a given DE system (Fréchet derivative) is self-adjoint, the set of multiplier determining equations includes the set of local symmetry determining equations. Consequently, here each set of local conservation law multipliers yields a local symmetry of the given DE system. In particular, the local conservation law multipliers are also components of the infinitesimal generators of local symmetries in evolutionary form. However, in the self-adjoint case, the set of linear determining equations for local conservation law multipliers is more over-determined than those for local symmetries since here the set of linear determining equations for local conservation law multipliers includes additional linear PDEs as well as the set of linear PDEs for local symmetries. Consequently, in the self-adjoint case, there can exist local symmetries that do not yield local conservation law multipliers.

Noether [5] showed that if a given system of DEs admits a variational principle, then any one-parameter Lie group of point transformations that leaves invariant the action functional yields a local conservation law. In particular, she gave an explicit formula for the fluxes of the local conservation law. Noether's theorem was extended by Bessel-Hagen [6] to allow the one-parameter Lie group of point transformations to leave invariant the action functional to within a divergence

term. As presented, their results depend on Lie groups of point transformations used in their canonical form, i.e., not in evolutionary form. Boyer [7] showed how all such local conservation laws could be obtained from Lie groups of point transformations used in evolutionary form. From this point of view, it is straightforward to apply Noether's theorem to obtain a local conservation law for any for any one-parameter higher-order local transformation leaving invariant the action functional to within a divergence term. Such a higher-order transformation that leaves invariant an action functional to within a divergence term is called a *variational symmetry*.

As might be expected, Noether's explicit formula for a local conservation law arises from sets of local multipliers that yield components of local symmetries in evolutionary form. From this point of view, it follows that all local conservation laws arising from Noether's theorem are obtained by the direct method. Moreover, one can see that a variational symmetry must map an extremal of the action functional to another extremal. Since an extremal of an action functional is a solution of the DE system arising from the variational principle, it follows that a variational symmetry must be a local symmetry of the given DE system arising from the variational principle.

A system of DEs (as written) has a variational principle if and only if its linearized system (Fréchet derivative) is self-adjoint [8–10]. From this point of view, it also follows that all conservation laws obtained by Noether's theorem must arise from the direct method.

The direct method supersedes Noether's theorem. In particular, for Noether's theorem, including its generalizations by Bessel-Hagen and Boyer, to be directly applicable to a given DE system, the following must hold:

- The linearized system of the given DE system is self-adjoint.
- One has an explicit action functional.
- One has a one-parameter local transformation that leaves the action functional invariant to within a divergence. In order to find such a variational symmetry systematically, one first finds local symmetries (solutions) of the linearized system and then checks whether or not such local symmetries leave the action functional invariant to within a divergence.

On the other hand, the direct method is applicable to any given DE system, whether or not its linearized system is self-adjoint. No functional needs to be determined. Moreover, a set of local conservation law multipliers is represented by *any* solution of an over-determined linear PDE system satisfied by the multipliers and this over-determined linear PDE system is obtained directly from the given DE system. As mentioned above, in the case when the linearized system is self-adjoint, the local symmetry determining equations are a subset of this over-determined linear PDE system.

In the study of DEs, conservation laws have many significant uses. They describe physically conserved quantities such as mass, energy, momentum and an-

gular momentum, as well as charge and other constants of motion. They are important for investigating integrability and linearization mappings and for establishing existence and uniqueness of solutions. They are also used in stability analysis and the global behavior of solutions. In addition, they play an essential role in the development of numerical methods and provide an essential starting point for finding potential variables and nonlocally related systems. In particular, a conservation law is fundamental in studying a given DE in the sense that it holds for any posed data (initial and/or boundary conditions). Moreover, the structure of conservation laws is coordinate-independent since a point (contact) transformation maps a conservation law to a conservation law.

The rest of this paper is organized as follows. In Section 2, the direct method is presented with a nonlinear telegraph system and the Korteweg-de Vries equation used as examples. Noether's theorem is presented in Section 3. In Section 4, there is a discussion of the limitations of Noether's theorem and the consequent advantages of the direct method.

2 The direct method

Consider a system $\mathbf{R}\{x; u\}$ of N differential equations of order k with n independent variables $x = (x^1, \dots, x^n)$ and m dependent variables $u(x) = (u^1(x), \dots, u^m(x))$, given by

$$R^\sigma[u] = R^\sigma(x, u, \partial u, \dots, \partial^k u) = 0, \quad \sigma = 1, \dots, N. \quad (1)$$

Definition 2.1. A *local conservation law* of the DE system (1) is a divergence expression

$$D_i \Phi^i[u] = D_1 \Phi^1[u] + \dots + D_n \Phi^n[u] = 0 \quad (2)$$

holding for all solutions of the DE system (1).

In (2), D_i and $\Phi^i[u] = \Phi^i(x, u, \partial u, \dots, \partial^r u)$, $i = 1, \dots, n$, respectively are total derivative operators and the *fluxes* of the conservation law.

Definition 2.2. A DE system $\mathbf{R}\{x; u\}$ (1) is *non-degenerate* if (1) can be written in Cauchy-Kovalevskaya form [3, 10] after a point (contact) transformation, if necessary.

In general, for a given non-degenerate DE system (1), nontrivial local conservation laws arise from seeking scalar products that involve linear combinations of the equations of the DE system (1) with *multipliers* (factors) that yield nontrivial divergence expressions. In seeking such expressions, the dependent variables and each of their derivatives that appear in the DE system (1) or in the multipliers, are replaced by arbitrary functions. Such divergence expressions vanish on all solutions of the DE system (1) provided the multipliers are non-singular.

In particular a set of multipliers $\{\Lambda_\sigma[U]\}_{\sigma=1}^N = \{\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)\}_{\sigma=1}^N$ yields a divergence expression for the DE system $\mathbf{R}\{x; u\}$ (1) if the identity

$$\Lambda_\sigma[U]R^\sigma[U] \equiv D_i\Phi^i[U] \quad (3)$$

holds for *arbitrary* functions $U(x)$. Then on the solutions $U(x) = u(x)$ of the DE system (1), if $\Lambda_\sigma[u]$ is non-singular, one has a local conservation law

$$\Lambda_\sigma[u]R^\sigma[u] = D_i\Phi^i[u] = 0.$$

[A multiplier $\Lambda_\sigma[U]$ is *singular* if it is a singular function when computed on solutions $U(x) = u(x)$ of the given DE system (1) (e.g., if $\Lambda_\sigma[U] = F[U]/R^\sigma[U]$). One is only interested in non-singular sets of multipliers, since the consideration of singular multipliers can lead to arbitrary divergence expressions that are not conservation laws of a given DE system.]

Definition 2.3. The *Euler operator* with respect to U^μ is the operator defined by

$$E_{U^\mu} = \frac{\partial}{\partial U^\mu} - D_i \frac{\partial}{\partial U^{\mu_i}} + \dots + (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial U_{i_1 \dots i_s}^\mu} + \dots \quad (4)$$

By direct calculation, one can show that the Euler operators (4) annihilate *any* divergence expression $D_i\Phi^i(x, U, \partial U, \dots, \partial^r U)$ for any r . In particular, the following identities hold for arbitrary $U(x)$:

$$E_{U^\mu}(D_i\Phi^i(x, U, \partial U, \dots, \partial^r U)) \equiv 0, \quad \mu = 1, \dots, m.$$

It is straightforward to show that the converse also holds. Namely, the only scalar expressions annihilated by Euler operators are divergence expressions. This establishes the following theorem.

Theorem 2.1. *The equations $E_{U^\mu}F(x, U, \partial U, \dots, \partial^s U) \equiv 0$, $\mu = 1, \dots, m$ hold for arbitrary $U(x)$ if and only if $F(x, U, \partial U, \dots, \partial^s U) \equiv D_i\Psi^i(x, U, \partial U, \dots, \partial^{s-1} U)$ for some functions $\Psi^i(x, U, \partial U, \dots, \partial^{s-1} U)$, $i = 1, \dots, n$.*

From Theorem 2.1, the proof of the following theorem that connects local multipliers and local conservation laws is immediate.

Theorem 2.2. *A set of non-singular local multipliers $\{\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)\}_{\sigma=1}^N$ yields a divergence expression for a DE system $\mathbf{R}\{x; u\}$ (1) if and only if the set of equations*

$$E_{U^\mu}(\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)R^\sigma(x, U, \partial U, \dots, \partial^k U)) \equiv 0, \quad \mu = 1, \dots, m, \quad (5)$$

holds for arbitrary functions $U(x)$.

The set of equations (5) yields the set of *linear determining equations* to find *all* sets of local conservation law multipliers of a given DE system $\mathbf{R}\{x; u\}$ (1) by letting $l = 1, 2, \dots$ in (5). Since the equations (1) hold for arbitrary $U(x)$, it follows that they also hold for *each* derivative of $U(x)$ replaced by an arbitrary function. In particular, since derivatives of $U(x)$ of orders higher than l can be replaced by arbitrary functions, it follows that the linear PDE system (1) splits into an over-determined linear system of determining equations whose solutions are the sets of local multipliers $\{\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)\}_{\sigma=1}^N$ of the DE system $\mathbf{R}\{x; u\}$ (1).

One can show the following [11]: Suppose each DE of a given k th order DE system $\mathbf{R}\{x; u\}$ (1) can be written in a solved form

$$R^\sigma[u] = u_{i_{1\sigma} \dots i_{s\sigma}}^{j_\sigma} - G^\sigma(x, u, \partial u, \dots, \partial^k u) = 0, \quad \sigma = 1, \dots, N, \quad (6)$$

where $1 \leq j_\sigma \leq m$ and $1 \leq i_{1\sigma}, \dots, i_{s\sigma} \leq n$ for each $\sigma = 1, \dots, N$; $\{u_{i_{1\sigma} \dots i_{s\sigma}}^{j_\sigma}\}$ is a set of N linearly independent s th order leading (partial) derivatives, with the property that none of them or their differential consequences appears in $\{G^\sigma[u]\}_{\sigma=1}^N$. Then, to within equivalence, *all* local conservation laws of the DE system $\mathbf{R}\{x; u\}$ (1) arise from sets of local multipliers that are solutions of the determining equations (5). [It should be noted that the assumption that a given DE system $\mathbf{R}\{x; u\}$ (1) can be written in a solved form (6) is the same assumption that is required when one is finding the local symmetries of $\mathbf{R}\{x; u\}$ (1).]

Remark 2.1. In the situation when a given DE system $\mathbf{R}\{x; u\}$ (1) cannot be written in a solved form (6), the multiplier approach still can be used to see local conservation laws of (1). However, here it is possible that some local conservation laws are missed since the corresponding divergence expressions may not satisfy (3), since they could involve differential consequences of $\mathbf{R}\{x; u\}$ (1).

Following from the above, a systematic procedure for the construction of local conservation laws of a given DE system $\mathbf{R}\{x; u\}$ (1), referred to as the *direct method*, is now outlined.

- For a given k th order DE system $\mathbf{R}\{x; u\}$ (1), seek sets of multipliers of the form $\{\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)\}_{\sigma=1}^N$ to some specified order l . Choose the dependence of multipliers on their arguments so that singular multipliers do not arise. [In particular, if the given DE system is written in a solved form (6) and is non-degenerate, the multipliers can be assumed to have no dependence on the leading derivatives $\{u_{i_{1\sigma} \dots i_{s\sigma}}^{j_\sigma}\}$ and their differential consequences.]
- Solve the set of determining equations (5) for arbitrary $U(x)$ to find all such sets of multipliers.
- Find the corresponding fluxes $\Phi^i(x, U, \partial U, \dots, \partial^r U)$ satisfying the identity
$$\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U) R^\sigma(x, U, \partial U, \dots, \partial^k U) \equiv D_i \Phi^i(x, U, \partial U, \dots, \partial^r U).$$
- Each set of multipliers and resulting fluxes yields a local conservation law holding for all solutions $u(x)$ of the given DE system $\mathbf{R}\{x; u\}$ (1).

2.1 Examples

The direct method to obtain local conservation laws is now illustrated through two examples.

2.1.1 A nonlinear telegraph system

As a first example, consider a nonlinear telegraph system ($u^1 = u$, $u^2 = v$) given by

$$R^1[u, v] = v_t - (u^2 + 1)u_x - u = 0, \quad R^2[u, v] = u_t - v_x = 0. \quad (7)$$

This is a first order PDE system with leading derivatives v_t and u_t .

We seek all local conservation law multipliers of the form

$$\Lambda_1 = \xi(x, t, U, V), \quad \Lambda_2 = \phi(x, t, U, V) \quad (8)$$

of the PDE system (7). In terms of Euler operators

$$E_U = \frac{\partial}{\partial U} - D_x \frac{\partial}{\partial U_x} - D_t \frac{\partial}{\partial U_t}, \quad E_V = \frac{\partial}{\partial V} - D_x \frac{\partial}{\partial V_x} - D_t \frac{\partial}{\partial V_t},$$

the determining equations (5) for the multipliers (8) become

$$\begin{aligned} E_U[\xi(x, t, U, V)(V_t - (U^2 + 1)U_x - U) + \phi(x, t, U, V)(U_t - V_x)] &\equiv 0, \\ E_V[\xi(x, t, U, V)(V_t - (U^2 + 1)U_x - U) + \phi(x, t, U, V)(U_t - V_x)] &\equiv 0, \end{aligned} \quad (9)$$

where $U(x, t)$ and $V(x, t)$ are arbitrary differentiable functions. Equations (9) split with respect to U_t, V_t, U_x, V_x to yield the over-determined linear PDE system given by

$$\begin{aligned} \phi_V - \xi_U &= 0, \quad \phi_U - (U^2 + 1)\xi_V = 0, \\ \phi_x - \xi_t - U\xi_V &= 0, \quad (U^2 + 1)\xi_x - \phi_t - U\xi_U - \xi = 0. \end{aligned} \quad (10)$$

The solutions of (10) are the five sets of local conservation multipliers given by

$$\begin{aligned} (\xi_1, \phi_1) &= (0, 1), \quad (\xi_2, \phi_2) = (t, x - \frac{1}{2}t^2), \\ (\xi_3, \phi_3) &= (1, -t), \quad (\xi_4, \phi_4) = (e^{x+\frac{1}{2}U^2+V}, Ue^{x+\frac{1}{2}U^2+V}), \\ (\xi_5, \phi_5) &= (e^{x+\frac{1}{2}U^2-V}, -Ue^{x+\frac{1}{2}U^2-V}). \end{aligned}$$

Each set (ξ, ϕ) determines a nontrivial local conservation law $D_t\Psi(x, t, u, v) + D_x\Phi(x, t, u, v) = 0$ with the characteristic form

$$\begin{aligned} D_t\Psi(x, t, U, V) + D_x\Phi(x, t, U, V) \\ \equiv \xi(x, t, U, V)R^1[U, V] + \phi(x, t, U, V)R^2[U, V]. \end{aligned} \quad (11)$$

In particular, after equating like derivative terms of (11), one has the relations

$$\Psi_U = \phi, \quad \Psi_V = \xi, \quad \Phi_U = -(U^2 + 1)\xi, \quad \Phi_V = -\phi, \quad \Psi_t + \Phi_x = -U\xi. \quad (12)$$

For each set of local multipliers, it is straightforward to integrate equations (12) to obtain the following five linearly independent local conservation laws of the PDE system (7):

$$\begin{aligned} D_t u + D_x[-v] &= 0, \\ D_t[(x - \frac{1}{2}t^2)u + tv] + D_x[(\frac{1}{2}t^2 - x)v - t(\frac{1}{3}u^3 + u)] &= 0, \\ D_t[v - tu] + D_x[tv - (\frac{1}{3}u^3 + u)] &= 0, \\ D_t[e^{x+\frac{1}{2}u^2+v}] + D_x[-ue^{x+\frac{1}{2}u^2+v}] &= 0, \\ D_t[e^{x+\frac{1}{2}u^2-v}] + D_x[ue^{x+\frac{1}{2}u^2-v}] &= 0. \end{aligned}$$

2.1.2 Korteweg-de Vries equation

As a second example, consider the KdV equation

$$R[u] = u_t + uu_x + u_{xxx} = 0. \quad (13)$$

Since PDE (13) can be directly expressed in the solved form $u_t = g[u] = -(uu_x + u_{xxx})$, without loss of generality, it follows that local multipliers yielding local conservation laws of PDE (13) are of the form $\Lambda = \Lambda(t, x, U, \partial_x U, \dots, \partial_x^l U)$, $l = 1, 2, \dots$, i.e., multipliers can be assumed to depend on at most on x -derivatives of U . This follows from the observation that through PDE (13), all t -derivatives of u appearing in the fluxes of any local conservation law $D_t \Psi[u] + D_x \Phi[u] = 0$ of PDE (13) can be expressed in terms of x -derivatives of u . It is then easy to show [3] that the resulting multipliers for the fluxes $\Psi(t, x, U, \partial_x U, \dots, \partial_x^r U)$ and $\Phi(t, x, U, \partial_x U, \dots, \partial_x^r U)$ must have no dependence on U_t and its derivatives. Consequently, $\Lambda(t, x, U, \partial_x U, \dots, \partial_x^l U)$ is a local conservation law multiplier of the PDE (13) if and only if

$$\begin{aligned} E_U(\Lambda(t, x, U, \partial_x U, \dots, \partial_x^l U)(U_t + UU_x + U_{xxx})) &\equiv \\ -D_t \Lambda - UD_x \Lambda - D_x^3 \Lambda + (U_t + UU_x + U_{xxx})\Lambda_U & \\ -D_x((U_t + UU_x + U_{xxx})\Lambda_{\partial_x U}) & \\ + \dots + (-1)^l D_x^l((U_t + UU_x + U_{xxx})\Lambda_{\partial_x^l U}) &\equiv 0 \end{aligned} \quad (14)$$

holds for an *arbitrary* $U(x, t)$ where here the Euler operator

$$E_U = \frac{\partial}{\partial u} - (D_t \frac{\partial}{\partial U_t} + D_x \frac{\partial}{\partial U_x}) + D_x^2 \frac{\partial}{\partial U_{xx}} + \dots$$

truncates after $\max(3, l)$ x -derivatives of U . Note that the linear determining equation (14) is of the form

$$\alpha_1[U] + \alpha_2[U]U_t + \alpha_3[U]\partial_x U_t + \dots + \alpha_{l+2}[U]\partial_x^l U_t \equiv 0 \quad (15)$$

where each $\alpha_i[U]$ depends at most on t , x , U and x -derivatives of U . Since $U(x, t)$ is an arbitrary function, in equation (15) each of $U_t, \partial_x U_t, \dots, \partial_x^l U_t$ can be treated as independent variables, and hence $\alpha_i[U] = 0$, $i = 1, \dots, l + 2$. Furthermore, there is a further splitting of these $l + 2$ determining equations with respect to each x -derivative of U .

Now suppose $\Lambda = \Lambda(t, x, U)$. Then from equations (14) and (15), it follows that

$$\begin{aligned} & (\Lambda_t + U\Lambda_x + \Lambda_{xxx}) + 3\Lambda_{xxU}U_x + 3\Lambda_{xUU}U_x^2 + \Lambda_{UUU}U_x^3 \\ & + 3\Lambda_{xU}U_{xx} + 3\Lambda_{UU}U_xU_{xx} \equiv 0. \end{aligned} \quad (16)$$

Equation (16) is a polynomial identity in the variables U_x, U_{xx} . Hence equation (16) splits into the three equations (the other three equations are differential consequences)

$$\Lambda_t + U\Lambda_x + \Lambda_{xxx} = 0, \quad \Lambda_{xU} = 0, \quad \Lambda_{UU} = 0,$$

whose solution yields the three local conservation law multipliers

$$\Lambda_1 = 1, \quad \Lambda_2 = U, \quad \Lambda_3 = tU - x.$$

It is easy to check that these three multipliers respectively yield the divergence expressions

$$\begin{aligned} U_t + UU_x + U_{xxx} & \equiv D_t U + D_x(\frac{1}{2}U^2 + U_{xx}), \\ U(U_t + UU_x + U_{xxx}) & \equiv D_t(\frac{1}{2}U^2) + D_x(\frac{1}{3}U^3 + UU_{xx} - \frac{1}{2}U_x^2), \\ (tU - x)(U_t + UU_x + U_{xxx}) & \equiv D_t(\frac{1}{2}tU^2 - xU) \\ & + D_x(-\frac{1}{2}xU^2 + tUU_{xx} - \frac{1}{2}tU_x^2 - xU_{xx} + U_x), \end{aligned}$$

and consequently, one obtains the local conservation laws

$$\begin{aligned} D_t u + D_x(\frac{1}{2}u^2 + u_{xx}) & = 0, \\ D_t(\frac{1}{2}u^2) + D_x(\frac{1}{3}u^3 + uu_{xx} - \frac{1}{2}u_x^2) & = 0, \\ D_t(\frac{1}{2}tu^2 - xu) + D_x(-\frac{1}{2}xu^2 + tuu_{xx} - \frac{1}{2}tu_x^2 - xu_{xx} + u_x) & = 0, \end{aligned}$$

of the KdV equation (13).

From equations (14) and (15), it is easy to see that PDE (13) has no additional multipliers of the form $\Lambda = \Lambda(t, x, U, U_x)$ with an essential dependence on U_x . Moreover, one can show that there is only one additional local multiplier of the form $\Lambda = \Lambda(t, x, U, U_x, U_{xx})$, given by

$$\Lambda_4 = U_{xx} + \frac{1}{2}U^2.$$

Furthermore, one can show that in terms of the recursion operator

$$R^*[U] = D_x^2 + \frac{1}{3}U + \frac{1}{3}D_x^{-1} \circ U \circ D_x,$$

the KdV equation (13) has an infinite sequence of local conservation law multipliers given by

$$\Lambda_{2n} = (R^*[U])^n U, \quad n = 1, 2, \dots,$$

with the first two multipliers in this sequence exhibited above.

2.2 Linearizing operators and adjoint equations

Consider a given DE system $\mathbf{R}\{x; u\}$ (1). The *linearizing operator* $L[U]$ associated with the DE system $\mathbf{R}\{x; u\}$ (1) is given by

$$L_\rho^\sigma[U] V^\rho = \left[\frac{\partial R^\sigma[U]}{\partial U^\rho} + \frac{\partial R^\sigma[U]}{\partial U_i^\rho} D_i + \dots + \frac{\partial R^\sigma[U]}{\partial U_{i_1 \dots i_k}^\rho} D_{i_1} \dots D_{i_k} \right] V^\rho, \quad (17)$$

$$\sigma = 1, \dots, N,$$

in terms of an *arbitrary* function $V(x) = (V^1(x), \dots, V^m(x))$. The *adjoint operator* $L^*[U]$ associated with the DE system $\mathbf{R}\{x; u\}$ (1) is given by

$$L_\rho^{*\sigma}[U] W_\sigma = \frac{\partial R^\sigma[U]}{\partial U^\rho} W_\sigma - D_i \left(\frac{\partial R^\sigma[U]}{\partial U_i^\rho} W_\sigma \right) + \dots$$

$$+ (-1)^k D_{i_1} \dots D_{i_k} \left(\frac{\partial R^\sigma[U]}{\partial U_{i_1 \dots i_k}^\rho} W_\sigma \right), \quad \rho = 1, \dots, m, \quad (18)$$

in terms of an *arbitrary* function $W(x) = (W_1(x), \dots, W_N(x))$.

In particular, one can show that the linearizing and adjoint operators, defined respectively through (17) and (18), satisfy the divergence relation

$$W_\sigma L_\rho^\sigma[U] V^\rho - V^\rho L_\rho^{*\sigma}[U] W_\sigma \equiv D_i \Psi^i[U]$$

with

$$D_i \Psi^i[U] = \sum_{q=1}^k \sum_{i_1 \dots i_q} D_{i_m} \left[(-1)^{m-1} (D_{i_{m+1}} \dots D_{i_q} V^\rho) \times \right.$$

$$\left. \times D_{i_1} \dots D_{i_{m-1}} \left(W_\sigma \frac{\partial R^\sigma[U]}{\partial U_{i_1 \dots i_q}^\rho} \right) \right],$$

where the second sum is taken over all ordered sets of indices $1 \leq i_1 \leq \dots \leq i_m \leq \dots \leq i_q \leq n$ of independent variables $x = (x^1, \dots, x^n)$.

Now let $W_\sigma = \Lambda_\sigma[U] = \Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)$, $\sigma = 1, \dots, N$. By direct calculation, in terms of the Euler operators defined by (4), one can show that

$$E_{U^\rho}(\Lambda_\sigma[U] R^\sigma[U]) \equiv L_\rho^{*\sigma}[U] \Lambda_\sigma[U] + F_\rho(R[U]) \quad (19)$$

with

$$\begin{aligned}
F_\rho(R[U]) &= \frac{\partial \Lambda_\sigma[U]}{\partial U^\rho} R^\sigma[U] - D_i \left(\frac{\partial \Lambda_\sigma[U]}{\partial U_i^\rho} R^\sigma[U] \right) + \dots \\
&+ (-1)^l D_{i_1} \dots D_{i_l} \left(\frac{\partial \Lambda_\sigma[U]}{\partial U_{i_1 \dots i_l}^\rho} R^\sigma[U] \right), \quad \rho = 1, \dots, m.
\end{aligned} \tag{20}$$

From expression (19), it immediately follows that $\{\Lambda_\sigma[U]\}_{\sigma=1}^N$ yields a set of local conservation law multipliers of the DE system $\mathbf{R}\{x; u\}$ (1) if and only if the right hand side of (19) vanishes for arbitrary $U(x)$. Now suppose each multiplier is nonsingular for each solution $U(x) = u(x)$ of the DE system (1). Since then the expression (20) vanishes for each solution $U(x) = u(x)$ of DE system $\mathbf{R}\{x; u\}$ (1), it follows that every set of nonsingular multipliers $\{\Lambda_\sigma[U]\}_{\sigma=1}^N$ of $\mathbf{R}\{x; u\}$ is a solution of its adjoint linearizing DE system when $U(x) = u(x)$ is a solution of the DE system $\mathbf{R}\{x; u\}$, i.e.,

$$L_\rho^{*\sigma}[u] \Lambda_\sigma[u] = 0, \quad \rho = 1, \dots, m. \tag{21}$$

In particular, the following two results have been proved.

Theorem 2.3. *For a given DE system $\mathbf{R}\{x; u\}$ (1), each set of local conservation law multipliers $\{\Lambda_\sigma[U] = \Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)\}_{\sigma=1}^N$ satisfies the identity*

$$\begin{aligned}
L_\rho^{*\sigma}[U] \Lambda_\sigma[U] &+ \frac{\partial \Lambda_\sigma[U]}{\partial U^\rho} R^\sigma[U] - D_i \left(\frac{\partial \Lambda_\sigma[U]}{\partial U_i^\rho} R^\sigma[U] \right) \\
&+ \dots + (-1)^l D_{i_1} \dots D_{i_l} \left(\frac{\partial \Lambda_\sigma[U]}{\partial U_{i_1 \dots i_l}^\rho} R^\sigma[U] \right) \equiv 0, \quad \rho = 1, \dots, m,
\end{aligned} \tag{22}$$

holding for arbitrary functions $U(x) = (U^1(x), \dots, U^m(x))$ where the components $\{L_\rho^{*\sigma}[U]\}$ of the adjoint operator of the linearizing operator (Fréchet derivative) for the DE system (1) are given by expressions (18).

Corollary 2.1. *For any solution $U(x) = u(x) = (u^1(x), \dots, u^m(x))$ of a given DE system $\mathbf{R}\{x; u\}$ (1), each set of local conservation law multipliers $\{\Lambda_\sigma[U]\}_{\sigma=1}^N$ satisfies the adjoint linearizing system (21), where $\{L_\rho^{*\sigma}[U]\}$ is given by the components of the adjoint operator (18).*

The identity (22) provides the explicit general form of the multiplier determining system (5) in Theorem 2.2. In general, the adjoint system (21) is strictly a subset of system (5) after one takes into account the splitting of (22) with respect to a set of leading derivatives for $R^\sigma[U]$, $\sigma = 1, \dots, N$.

2.3 Determination of fluxes of conservation laws from multipliers

There are several ways of finding the fluxes of local conservation laws from a known set of multipliers.

A first method is a direct method that has been illustrated through the nonlinear telegraph system considered in Section 2.1.1 where one converts (3) directly into the set of determining equations to be solved for the fluxes $\Phi^i[U]$. This method is easy to implement for simple types of conservation laws.

A second method is another direct method that has been illustrated through the KdV equation considered in Section 2.1.2 where one simply manipulates (3) to find the fluxes $\Phi^i[U]$.

A third method [1–3] is now presented that allows one to find the fluxes in the case of complicated forms of multipliers and/or DE systems through an integral (homotopy) formula:

For each multiplier $\Lambda_\sigma[U] = \Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)$, one introduces the corresponding linearization operator

$$(\mathbf{L}_\Lambda)_{\sigma\rho}[U]\tilde{V}^\rho = \left[\frac{\partial \Lambda_\sigma[U]}{\partial U^\rho} + \frac{\partial \Lambda_\sigma[U]}{\partial U_i^\rho} D_i + \dots + \frac{\partial \Lambda_\sigma[U]}{\partial U_{i_1 \dots i_l}^\rho} D_{i_1} \dots D_{i_l} \right] \tilde{V}^\rho, \quad (23)$$

$\sigma = 1, \dots, N,$

and its adjoint

$$\begin{aligned} (\mathbf{L}_\Lambda^*)_{\sigma\rho}[U]\tilde{W}^\sigma &= \frac{\partial \Lambda_\sigma[U]}{\partial U^\rho} \tilde{W}^\sigma - D_i \left(\frac{\partial \Lambda_\sigma[U]}{\partial U_i^\rho} \tilde{W}^\sigma \right) \\ &+ \dots + (-1)^k D_{i_1} \dots D_{i_l} \left(\frac{\partial \Lambda_\sigma[U]}{\partial U_{i_1 \dots i_l}^\rho} \tilde{W}^\sigma \right), \quad \rho = 1, \dots, m, \end{aligned} \quad (24)$$

acting respectively on arbitrary functions $\tilde{V}(x) = (\tilde{V}^1(x), \dots, \tilde{V}^m(x))$ and $\tilde{W}(x) = (\tilde{W}^1(x), \dots, \tilde{W}^N(x))$.

It is straightforward to show that the operators defined by (17), (18), (23), and (24) satisfy the following divergence identities:

$$W_\sigma \mathbf{L}_\rho^\sigma[U] V^\rho - V^\rho \mathbf{L}_\rho^{*\sigma}[U] W_\sigma \equiv D_i S^i[V, W; R[U]], \quad (25)$$

$$\tilde{W}^\sigma (\mathbf{L}_\Lambda)_{\sigma\rho}[U] \tilde{V}^\rho - \tilde{V}^\rho (\mathbf{L}_\Lambda^*)_{\sigma\rho}[U] \tilde{W}^\sigma \equiv D_i \tilde{S}^i[\tilde{V}, \tilde{W}; \Lambda[U]], \quad (26)$$

with $S^i[V, W; R[U]]$ and $\tilde{S}^i[\tilde{V}, \tilde{W}; \Lambda[U]]$ defined by corresponding terms in the expressions

$$\begin{aligned} D_i S^i[V, W; R[U]] &= \sum_{q=1}^k \sum_{i_1 \dots i_q} D_{i_m} \left[(-1)^{m-1} (D_{i_{m+1}} \dots D_{i_q} V^\rho) \times \right. \\ &\quad \left. \times D_{i_1} \dots D_{i_{m-1}} \left(W_\sigma \frac{\partial R^\sigma[U]}{\partial U_{i_1 \dots i_q}^\rho} \right) \right], \end{aligned} \quad (27)$$

$$\begin{aligned} D_i \tilde{S}^i[\tilde{V}, \tilde{W}; \Lambda[U]] &= \sum_{q=1}^l \sum_{i_1 \dots i_q} D_{i_m} \left[(-1)^{m-1} (D_{i_{m+1}} \dots D_{i_q} \tilde{V}^\rho) \times \right. \\ &\quad \left. \times D_{i_1} \dots D_{i_{m-1}} \left(\tilde{W}^\sigma \frac{\partial \Lambda_\sigma[U]}{\partial U_{i_1 \dots i_q}^\rho} \right) \right]. \end{aligned} \quad (28)$$

In equations (27) and (28), k is the order of the given DE system (1), l is the maximal order of the derivatives appearing in the multipliers, and the second sums are taken over all ordered sets of indices $1 \leq i_1 \leq \dots \leq i_m \leq \dots \leq i_q \leq n$ of independent variables $x = (x^1, \dots, x^n)$.

Let $U_{(\lambda)} = U(x) + (\lambda - 1)V(x)$, where $U(x) = (U^1(x), \dots, U^m(x))$ and $V(x) = (V^1(x), \dots, V^m(x))$ are arbitrary functions, and λ is a scalar parameter. Replacing U by $U_{(\lambda)}$ in the conservation law identity (3), one obtains

$$\frac{\partial}{\partial \lambda}(\Lambda_\sigma[U_{(\lambda)}]R^\sigma[U_{(\lambda)}]) \equiv \frac{\partial}{\partial \lambda}D_i\Phi^i[U_{(\lambda)}] = D_i\left(\frac{\partial}{\partial \lambda}\Phi^i[U_{(\lambda)}]\right). \quad (29)$$

The left-hand side of (29) can then be expressed in terms of the linearizing operators (17) and (23) as follows:

$$\frac{\partial}{\partial \lambda}(\Lambda_\sigma[U_{(\lambda)}]R^\sigma[U_{(\lambda)}]) = \Lambda_\sigma[U_{(\lambda)}]L_\rho^\sigma[U_{(\lambda)}]V^\rho + R^\sigma[U_{(\lambda)}](L_\Lambda)_{\sigma\rho}[U_{(\lambda)}]V^\rho.$$

From (25) and (26) with $W_\sigma = \Lambda_\sigma[U_{(\lambda)}]$ and $\tilde{W}^\sigma = R^\sigma[U_{(\lambda)}]$, respectively, one obtains

$$\begin{aligned} \frac{\partial}{\partial \lambda}(\Lambda_\sigma[U_{(\lambda)}]R^\sigma[U_{(\lambda)}]) &= V^\rho L_\rho^{*\sigma}[U_{(\lambda)}]\Lambda_\sigma[U_{(\lambda)}] + D_i S^i[V, \Lambda[U_{(\lambda)}]; R[U_{(\lambda)}]] \\ &+ V^\rho (L_\Lambda^*)_{\sigma\rho}[U_{(\lambda)}]R^\sigma[U_{(\lambda)}] + D_i \tilde{S}^i[V, R[U_{(\lambda)}]; \Lambda[U_{(\lambda)}]] \\ &= D_i \left(S^i[V, \Lambda[U_{(\lambda)}]; R[U_{(\lambda)}]] + \tilde{S}^i[V, R[U_{(\lambda)}]; \Lambda[U_{(\lambda)}]] \right), \end{aligned} \quad (30)$$

where the last equality follows from the identity (22) holding for local conservation law multipliers in Theorem 2.3.

Comparing (29) and (30), one finds that

$$D_i \left(\frac{\partial}{\partial \lambda} \Phi^i[U_{(\lambda)}] \right) = D_i \left(S^i[V, \Lambda[U_{(\lambda)}]; R[U_{(\lambda)}]] + \tilde{S}^i[V, R[U_{(\lambda)}]; \Lambda[U_{(\lambda)}]] \right),$$

leading to

$$\frac{\partial}{\partial \lambda} \Phi^i[U_{(\lambda)}] = S^i[V, \Lambda[U_{(\lambda)}]; R[U_{(\lambda)}]] + \tilde{S}^i[V, R[U_{(\lambda)}]; \Lambda[U_{(\lambda)}]], \quad (31)$$

up to fluxes of a trivial conservation law. Now let $V(x) = U(x) - \tilde{U}(x)$, for an arbitrary function $\tilde{U}(x) = (\tilde{U}^1(x), \dots, \tilde{U}^m(x))$. Then $U_{(\lambda)} = \lambda U(x) + (1 - \lambda)\tilde{U}(x)$. Integrating (31) with respect to λ from 0 to 1, one finds that

$$\begin{aligned} \Phi^i[U] &= \Phi^i[\tilde{U}] + \int_0^1 (S^i[U - \tilde{U}, \Lambda[\lambda U + (1 - \lambda)\tilde{U}]; R[\lambda U + (1 - \lambda)\tilde{U}]] \\ &+ \tilde{S}^i[U - \tilde{U}, R[\lambda U + (1 - \lambda)\tilde{U}]; \Lambda[\lambda U + (1 - \lambda)\tilde{U}]] d\lambda, \end{aligned} \quad (32)$$

$$i = 1, \dots, n.$$

In summary, the following theorem has been proven.

Theorem 2.4. For a set of local conservation law multipliers $\{\Lambda_\sigma[U]\}_{\sigma=1}^N$ of a DE system $\mathbf{R}\{x; u\}$ (1), the corresponding fluxes are given by the integral formula (32).

In (32), $\tilde{U}(x)$ is an arbitrary function of x , chosen so that the integral converges. Different choices of $\tilde{U}(x)$ yield fluxes of equivalent conservation laws, i.e., conservation laws that differ by trivial divergences. One commonly chooses $\tilde{U}(x) = 0$ (provided that the integral (32) converges). Once $\tilde{U}(x)$ has been chosen, the corresponding fluxes $\{\Phi^i[\tilde{U}]\}_{i=1}^N$ can be found by direct integration through the divergence relation $D_i \Phi^i[\tilde{U}] = \Lambda_\sigma[\tilde{U}] R^\sigma[\tilde{U}] = F(x)$. For example, one may choose $\Phi^1[\tilde{U}] = \int F(x) dx^1, \Phi^2[\tilde{U}] = \dots = \Phi^n[\tilde{U}] = 0$.

Finally, a fourth method [12] replaces the integral formula (32) by a simpler algebraic formula that applies to DE systems $\mathbf{R}\{x; u\}$ that have scaling symmetries.

2.4 Self-adjoint DE systems

An especially interesting situation arises when the linearizing operator (Fréchet derivative) $L[U]$ of a given DE system (1) is self-adjoint.

Definition 2.4. Let $L[U]$, with its components $L_\rho^\sigma[U]$ given by (17), be the linearizing operator associated with a DE system (1). The adjoint operator of $L[U]$ is $L^*[U]$, with its components $L_\rho^{*\sigma}[U]$ given by (18). $L[U]$ is a *self-adjoint* operator if and only if $L[U] \equiv L^*[U]$, i.e., $L_\rho^\sigma[U] \equiv L_\rho^{*\sigma}[U], \sigma, \rho = 1, \dots, m$.

It is straightforward to see that if a DE system, as written, has a self-adjoint linearizing operator, then

- the number of dependent variables appearing in the system must equal the number of equations appearing in the system, i.e., $N = m$;
- if the given DE system is a scalar equation, the highest-order derivative appearing in it must be of even order.

The converse of this statement is false. For example, consider the linear heat equation $u_t - u_{xx} = 0$. The linearizing operator of this PDE is obviously given by $L = D_t - D_x^2$, with adjoint operator $L^* = -D_t - D_x^2 \not\equiv L$.

Most importantly, one can show that a given DE system, as written, has a variational formulation if and only if its associated linearizing operator is self-adjoint [8–10].

If the linearizing operator associated with a given DE system is self-adjoint, then each set of local conservation law multipliers yields a local symmetry of the given DE system. In particular, one has the following theorem.

Theorem 2.5. Consider a given DE system $\mathbf{R}\{x; u\}$ (1) with $N = m$. Suppose its associated linearizing operator $L[U]$, with components (17), is self-adjoint. Suppose $\{\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)\}_{\sigma=1}^m$ is a set of local conservation law multipliers of the DE system (1). Let $\eta^\sigma(x, u, \partial u, \dots, \partial^l u) = \Lambda_\sigma(x, u, \partial u, \dots, \partial^l u)$,

$\sigma = 1, \dots, m$, where $U(x) = u(x)$ is any solution of the DE system $\mathbf{R}\{x; u\}$ (1). Then

$$\eta^\sigma(x, u, \partial u, \dots, \partial^l u) \frac{\partial}{\partial u^\sigma} \quad (33)$$

is a local symmetry of the DE system $\mathbf{R}\{x; u\}$ (1).

Proof. From equations (21) with $L[U] = L^*[U]$, it follows that in terms of the components (17) of the associated linearizing operator $L[U]$, one has

$$L_\rho^\sigma[u] \Lambda_\sigma(x, u, \partial u, \dots, \partial^l u) = 0, \quad \rho = 1, \dots, m, \quad (34)$$

where $u = \Theta(x)$ is any solution of the DE system $\mathbf{R}\{x; u\}$ (1). But the set of equations (34) is the set of determining equations for a local symmetry

$$\Lambda_\sigma(x, u, \partial u, \dots, \partial^l u) \frac{\partial}{\partial u^\sigma}$$

of the DE system $\mathbf{R}\{x; u\}$ (1). Hence, it follows that (33) is a local symmetry of the DE system $\mathbf{R}\{x; u\}$ (1). \blacksquare

The converse of Theorem 2.5 is false. In particular, suppose

$$\eta^\sigma(x, u, \partial u, \dots, \partial^l u) \frac{\partial}{\partial u^\sigma}$$

is a local symmetry of a given DE system $\mathbf{R}\{x; u\}$ (1) with a self-adjoint linearizing operator $L[U]$. Let $\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U) = \eta^\sigma(x, U, \partial U, \dots, \partial^l U)$, $\sigma = 1, \dots, m$, where $U(x) = (U^1(x), \dots, U^m(x))$ is an arbitrary function. Then it does not necessarily follow that $\{\Lambda_\sigma(x, U, \partial U, \dots, \partial^l U)\}_{\sigma=1}^m$ is a set of local conservation law multipliers of the DE system (1). This can be seen as follows: in the self-adjoint case, the set of local symmetry determining equations is a subset of the set of local multiplier determining equations. Here *each* local symmetry yields a set of local conservation law multipliers if and only *each* solution of the set of local symmetry determining equations also solves the remaining set of local multiplier determining equations.

3 Noether's theorem

In 1918, Noether [5] presented her celebrated procedure (Noether's theorem) to find local conservation laws for systems of DEs that admit a variational principle. When a given DE system admits a variational principle, then the extremals of an action functional yield the given DE system (the *Euler–Lagrange equations*). In this case, Noether showed that if one has a point symmetry of the action functional (action integral), then one obtains the fluxes of a local conservation law through an explicit formula that involves the infinitesimals of the point symmetry and the Lagrangian (Lagrangian density) of the action functional.

We now present Noether's theorem and its generalizations due to Bessel-Hagen [6] and Boyer [7].

3.1 Euler–Lagrange equations

Consider a functional $J[U]$ in terms of n independent variables $x = (x^1, \dots, x^n)$ and m arbitrary functions $U = (U^1(x), \dots, U^m(x))$ and their derivatives to order k , defined on a domain Ω ,

$$J[U] = \int_{\Omega} L[U] dx = \int_{\Omega} L[(x, U, \partial U, \dots, \partial^k U)] dx. \quad (35)$$

The function $L[U] = L[(x, U, \partial U, \dots, \partial^k U)]$ is called a *Lagrangian* and the functional $J[U]$ is called an *action integral*. Consider an infinitesimal change of U given by $U(x) \rightarrow U(x) + \varepsilon v(x)$ where $v(x)$ is any function such that $v(x)$ and its derivatives to order $k-1$ vanish on the boundary $\partial\Omega$ of the domain Ω . The corresponding change (variation) in the Lagrangian $L[U]$ is given by

$$\begin{aligned} \delta L &= L[(x, U + \varepsilon v, \partial U + \varepsilon \partial v, \dots, \partial^k U + \varepsilon \partial^k v)] - L[(x, U, \partial U, \dots, \partial^k U)] \\ &= \varepsilon \left(\frac{\partial L[U]}{\partial U^\sigma} v^\sigma + \frac{\partial L[U]}{\partial U_j^\sigma} v_j^\sigma + \dots + \frac{\partial L[U]}{\partial U_{j_1 \dots j_k}^\sigma} v_{j_1 \dots j_k}^\sigma \right) + O(\varepsilon^2). \end{aligned}$$

Then after repeatedly using integration by parts, one can show that

$$\delta L = \varepsilon (v^\sigma E_{U^\sigma}(L[U]) + D_l W^l[U, v]) + O(\varepsilon^2), \quad (36)$$

where E_{U^σ} is the Euler operator with respect to U^σ and

$$\begin{aligned} W^l[U, v] &= v^\sigma \left(\frac{\partial L[U]}{\partial U_l^\sigma} + \dots + (-1)^{k-1} D_{j_1} \dots D_{j_{k-1}} \frac{\partial L[U]}{\partial U_{l j_1 \dots j_{k-1}}^\sigma} \right) \\ &+ v_{j_1}^\sigma \left(\frac{\partial L[U]}{\partial U_{j_1 l}^\sigma} + \dots + (-1)^{k-2} D_{j_2} \dots D_{j_{k-1}} \frac{\partial L[U]}{\partial U_{j_1 l j_2 \dots j_{k-1}}^\sigma} \right) \\ &+ \dots + v_{j_1 \dots j_{k-1}}^\sigma \frac{\partial L[U]}{\partial U_{j_1 j_2 \dots j_{k-1} l}^\sigma}. \end{aligned} \quad (37)$$

The corresponding variation in the action integral $J[U]$ is given by

$$\begin{aligned} \delta J &= J[U + \varepsilon v] - J[U] = \int_{\Omega} \delta L dx \\ &= \varepsilon \int_{\Omega} (v^\sigma E_{U^\sigma}(L[U]) + D_l W^l[U, v]) dx + O(\varepsilon^2) \\ &= \varepsilon \left(\int_{\Omega} v^\sigma E_{U^\sigma}(L[U]) dx + \int_{\partial\Omega} W^l[U, v] n^l dS \right) + O(\varepsilon^2) \end{aligned} \quad (38)$$

where $\int_{\partial\Omega}$ represents the surface integral over the boundary $\partial\Omega$ of the domain Ω with $n = (n^1, \dots, n^n)$ being the unit outward normal vector to $\partial\Omega$. From (37), it is evident that each $W^l[U, v]$ vanishes on $\partial\Omega$, and hence $\int_{\partial\Omega} W^l[U, v] n^l dS = 0$.

Hence if $U = u(x)$ extremizes the action integral $J[U]$, then the $O(\varepsilon)$ term of δJ must vanish so that $\int_{\Omega} v^\sigma E_{U^\sigma}(L[u]) dx = 0$ for an *arbitrary* $v(x)$ defined on the

domain Ω . Thus if $U = u(x)$ extremizes the action integral (35), then $u(x)$ must satisfy the *Euler–Lagrange equations*

$$E_{u^\sigma}(L[u]) = \frac{\partial L[u]}{\partial u^\sigma} + \cdots + (-1)^k D_{j_1} \cdots D_{j_k} \frac{\partial L[u]}{\partial u_{j_1 \cdots j_k}^\sigma} = 0, \quad \sigma = 1, \dots, m. \quad (39)$$

Hence, the following theorem has been proved.

Theorem 3.1. *If a smooth function $U(x) = u(x)$ is an extremum of an action integral $J[U] = \int_{\Omega} L[U]dx$ with $L[U] = L(x, U, \partial U, \dots, \partial^k U)$, then $u(x)$ satisfies the Euler–Lagrange equations (39).*

3.2 Noether's formulation of Noether's theorem

We now present Noether's formulation of her famous theorem. In this formulation, the action integral $J[U]$ (35) is required to be invariant under the one-parameter Lie group of point transformations

$$\begin{aligned} (x^*)^i &= x^i + \varepsilon \xi^i(x, U) + O(\varepsilon^2), \quad i = 1, \dots, n, \\ (U^*)^\mu &= U^\mu + \varepsilon \eta^\mu(x, U) + O(\varepsilon^2), \quad \mu = 1, \dots, m, \end{aligned} \quad (40)$$

with corresponding infinitesimal generator given by

$$X = \xi^i(x, U) \frac{\partial}{\partial x^i} + \eta^\nu(x, U) \frac{\partial}{\partial U^\nu}, \quad (41)$$

Invariance holds if and only if $\int_{\Omega^*} L[U^*]dx^* = \int_{\Omega} L[U]dx$ where Ω^* is the image of Ω under the point transformation (40). The Jacobian J of the transformation (40) is given by $J = \det(D_i(x^*)^j) = 1 + \varepsilon D_i \xi^i(x, U) + O(\varepsilon^2)$. Then $dx^* = Jdx$. Moreover, since (40) is a Lie group of transformations, it follows that $L[U^*] = e^{\varepsilon X^{(k)}} L[U]$ in terms of the k th extension of the infinitesimal generator (41). Consequently, in Noether's formulation, the one-parameter Lie group of point transformations (40) is a point symmetry of $J[U]$ (35) if and only if

$$\int_{\Omega} (J e^{\varepsilon X^{(k)}} - 1) L[U]dx = \varepsilon \int_{\Omega} (L[U] D_i \xi^i(x, U) + X^{(k)} L[U])dx + O(\varepsilon^2) \quad (42)$$

holds for arbitrary $U(x)$ where $X^{(k)}$ is the k th extended infinitesimal generator with U replacing u . Hence, if $J[U]$ (35) has the point symmetry (40), then the $O(\varepsilon)$ term in (42) vanishes, and thus one obtains the identity

$$L[U] D_i \xi^i(x, U) + X^{(k)} L[U] \equiv 0. \quad (43)$$

The one-parameter Lie group of point transformations (40) is equivalent to the one-parameter family of transformations

$$\begin{aligned} (x^*)^i &= x^i, \quad i = 1, \dots, n, \\ (U^*)^\mu &= U^\mu + \varepsilon [\eta^\mu(x, U) - U_i^\mu \xi^i(x, U)] + O(\varepsilon^2), \quad \mu = 1, \dots, m. \end{aligned} \quad (44)$$

Under the transformation (44), the corresponding infinitesimal change $U(x) \rightarrow U(x) + \varepsilon v(x)$ has components $v^\mu(x) = \hat{\eta}^\mu[U] = \eta^\mu(x, U) - U_i^\mu \xi^i(x, U)$ in terms of the transformations (44). Moreover, from the group property of (44), it follows that

$$\delta L = \varepsilon \hat{X}^{(k)} L[U] + O(\varepsilon^2) \quad (45)$$

where $\hat{X}^{(k)}$ is the k th extension of the infinitesimal generator $\hat{X} = \hat{\eta}^\mu[U] \frac{\partial}{\partial U^\mu}$ yielding the transformation (44). Thus

$$\int_{\Omega} \delta L dx = \varepsilon \int_{\Omega} \hat{X}^{(k)} L[U] dx + O(\varepsilon^2). \quad (46)$$

Consequently, after comparing expression (46) and expression (38) with $v^\mu(x) = \hat{\eta}^\mu[U] = \eta^\mu(x, U) - U_i^\mu \xi^i(x, U)$, it follows that

$$\hat{X}^{(k)} L[U] \equiv \hat{\eta}^\mu[U] E_{U^\mu}(L[U]) + D_l W^l[U, \hat{\eta}[U]] \quad (47)$$

where $W^l[U, \hat{\eta}[U]]$ is given by expression (37) with the obvious substitutions.

The proof of the following theorem is obtained by direct calculation.

Theorem 3.2. *Let $X^{(k)}$ be the k th extended infinitesimal generator of the one-parameter Lie group of point transformations (40) and let $\hat{X}^{(k)}$ be the k th extended infinitesimal generator of the equivalent one-parameter family of transformations (44). Let $F[U] = F(x, U, \partial U, \dots, \partial^k U)$ be an arbitrary function of its arguments. Then the following identity holds:*

$$X^{(k)} F[U] + F[U] D_i \xi^i(x, U) \equiv \hat{X}^{(k)} F[U] + D_i(F[U] \xi^i(x, U)). \quad (48)$$

Putting all of the above together, one obtains the following theorem.

Theorem 3.3 (Noether's formulation of Noether's theorem). *Suppose a given DE system $\mathbf{R}\{x; u\}$ (1) is derivable from a variational principle, i.e., the given DE system is a set of Euler-Lagrange equations (39) whose solutions $u(x)$ are extrema $U(x) = u(x)$ of an action integral $J[U]$ (35) with Lagrangian $L[U]$. Suppose the one-parameter Lie group of point transformations (40) is a point symmetry of $J[U]$. Let $W^l[U, v]$ be defined by (37) for arbitrary functions $U(x), v(x)$. Then*

(1) *The identity*

$$\hat{\eta}^\mu[U] E_{U^\mu}(L[U]) \equiv -D_i(\xi^i(x, U) L[U] + W^i[U, \hat{\eta}[U]]) \quad (49)$$

holds for arbitrary functions $U(x)$, i.e., $\{\hat{\eta}^\mu[U]\}_{\mu=1}^m$ is a set of local conservation law multipliers of the Euler-Lagrange system (39);

(2) *The local conservation law*

$$D_i(\xi^i(x, u) L[u] + W^i[u, \hat{\eta}[u]]) = 0 \quad (50)$$

holds for any solution $u = \Theta(x)$ of the Euler-Lagrange system (39).

Proof. Let $F[U] = L[U]$ in the identity (48). Then from the identity (43), one obtains

$$\hat{X}^{(k)}L[U] + D_i(L[U]\xi^i(x, U)) \equiv 0 \quad (51)$$

holding for arbitrary functions $U(x)$. Substitution for $\hat{X}^{(k)}L[U]$ in (51) through (47) yields (49). If $U(x) = u(x)$ solves the Euler-Lagrange system (39), then the left-hand side of equation (49) vanishes. This yields the conservation law (50). ■

3.3 Boyer's formulation of Noether's theorem

Boyer [7] extended Noether's theorem to enable one to conveniently find conservation laws arising from invariance under higher-order transformations by generalizing Noether's definition of invariance of an action integral $J[U]$ (35). In particular, under the following definition, an action integral $J[U]$ (35) is invariant under a one-parameter higher-order local transformation if its integrand $L[U]$ is invariant to within a divergence under such a transformation.

Definition 3.1. Let

$$\hat{X} = \hat{\eta}^\mu(x, U, \partial U, \dots, \partial^s U) \frac{\partial}{\partial U^\mu} \quad (52)$$

be the infinitesimal generator of a one-parameter higher-order local transformation

$$\begin{aligned} (x^*)^i &= x^i, \quad i = 1, \dots, n, \\ (U^*)^\mu &= U^\mu + \varepsilon \hat{\eta}^\mu(x, U, \partial U, \dots, \partial^s U) + O(\varepsilon^2), \quad \mu = 1, \dots, m, \end{aligned} \quad (53)$$

with its extension to all derivatives denoted by \hat{X}^∞ . Let

$$\hat{\eta}^\mu[U] = \hat{\eta}^\mu(x, U, \partial U, \dots, \partial^s U).$$

The transformation (53) is a *local symmetry* of $J[U]$ (35) if and only if

$$\hat{X}^\infty L[U] \equiv D_i A^i[U] \quad (54)$$

holds for some set of functions $A^i[U] = A^i(x, U, \partial U, \dots, \partial^r U)$, $i = 1, \dots, n$.

Definition 3.2. A local transformation with infinitesimal generator (52) that is a local symmetry of $J[U]$ (35) is called a *variational symmetry* of $J[U]$.

The proof of the following theorem follows from the property of Euler operators annihilating divergences.

Theorem 3.4. *A variational symmetry with infinitesimal generator (52) of the action integral $J[U]$ (35) yields a local symmetry with infinitesimal generator $\hat{X} = \hat{\eta}^\mu(x, u, \partial u, \dots, \partial^s u) \frac{\partial}{\partial u^\mu}$ of the corresponding Euler-Lagrange system (39).*

The following theorem generalizes Noether's formulation of her theorem.

Theorem 3.5 (Boyer's generalization of Noether's theorem). Suppose a given DE system $\mathbf{R}\{x; u\}$ (1) is derivable from a variational principle, i.e., the given DE system is a set of Euler–Lagrange equations (39) whose solutions $u(x)$ are extrema $U(x) = u(x)$ of an action integral $J[U]$ (35) with Lagrangian $L[U]$. Suppose a local transformation with infinitesimal generator (52) yields a variational symmetry of $J[U]$. Let $W^l[U, v]$ be defined by (37) for arbitrary functions $U(x), v(x)$. Then

(1) *The identity*

$$\hat{\eta}^\mu[U]E_{U^\mu}(L[U]) \equiv D_i(A^i[U] - W^i[U, \hat{\eta}[U]]) \quad (55)$$

holds for arbitrary functions $U(x)$, i.e., $\{\hat{\eta}^\mu[U]\}_{\mu=1}^m$ is a set of local conservation law multipliers of the Euler–Lagrange system (39);

(2) *The local conservation law*

$$D_i(W^i[u, \hat{\eta}[u]] - A^i[u]) = 0 \quad (56)$$

holds for any solution $u = \Theta(x)$ of the Euler–Lagrange system (39).

Proof. For a local transformation with infinitesimal generator (52), it follows that the corresponding infinitesimal change $U(x) \rightarrow U(x) + \varepsilon v(x)$ has components $v^\mu(x) = \hat{\eta}^\mu[U]$. Consequently, equation (45) becomes

$$\delta L = \varepsilon \hat{X}^\infty L[U] + O(\varepsilon^2).$$

But from (36) it follows that

$$\delta L = \varepsilon(\hat{\eta}^\mu[U]E_{U^\mu}(L[U]) + D_i(W^i[U, \hat{\eta}[U]])) + O(\varepsilon^2).$$

Hence it immediately follows that

$$\hat{X}^\infty L[U] = \hat{\eta}^\mu[U]E_{U^\mu}(L[U]) + D_i(W^i[U, \hat{\eta}[U]]) \quad (57)$$

holds for arbitrary functions $U(x)$. Since the local transformation with infinitesimal generator (52) is a variational symmetry of $J[U]$ (35), it follows that equation (54) holds. Substitution for $\hat{X}^\infty L[U]$ in (57) through (54) yields the identity (55). If $U(x) = u(x)$ solves the Euler–Lagrange system (39), then the left-hand side of equation (55) vanishes. This yields the conservation law (56). \blacksquare

Theorem 3.6. If a conservation law is obtained through Noether's formulation (Theorem 3.3), then the conservation law can be obtained through Boyer's formulation (Theorem 3.5).

Proof. Suppose the one-parameter Lie group of point transformations (40) yields a conservation law. Then the identity (51) holds. Consequently,

$$\hat{X}^{(k)} L[U] = \hat{X}^\infty L[U] = D_i A^i[U] \quad (58)$$

where $A^i[U] = -D_i(L[U]\xi^i(x, U))$. But equation (58) is just the condition for the one-parameter Lie group of point transformations (40) to be a variational symmetry of $J[U]$ (35). Consequently, one obtains the same conservation law from Boyer's formulation. \blacksquare

4 Limitations of Noether's theorem and consequent advantages of the direct method

There are several limitations inherent in using Noether's theorem to find local conservation laws for a given DE system $\mathbf{R}\{x; u\}$. First of all, it is restricted to variational systems. Consequently, the linearizing operator (Fréchet derivative) for $\mathbf{R}\{x; u\}$, as written, must be self-adjoint, which implies that $\mathbf{R}\{x; u\}$ must be of even order (if it is a scalar PDE), and the number of PDEs must be the same as the number of dependent variables appearing in $\mathbf{R}\{x; u\}$. [In particular, this can be seen from comparing expressions (17) and (18).] In addition, one must find an explicit Lagrangian $L[U]$ whose Euler–Lagrange equations yield $\mathbf{R}\{x; u\}$.

There is also the difficulty of finding the variational symmetries for a given variational DE system $\mathbf{R}\{x; u\}$. First, for the given DE system, one must determine local symmetries depending on derivatives of dependent variables up to some chosen order. Second, one must find an explicit Lagrangian $L[U]$ and check if each symmetry of the given DE system leaves invariant the Lagrangian $L[U]$ to within a divergence, i.e., if a symmetry is indeed a variational symmetry.

Finally, the use of Noether's theorem to find local conservation laws is coordinate dependent since the action of a point (contact) transformation can transform a DE having a variational principle to one that does not have one. On the other hand, it is known that conservation laws are *coordinate-independent* in the sense that a point (contact) transformation maps a conservation law into a conservation law [13], and therefore it follows that an ideal method for finding conservation laws should be coordinate-independent.

Artifices may make a given DE system variational. Such artifices include:

- *The use of multipliers.* As an example, the PDE $u_{tt} + 2u_x u_{xx} + u_x^2 = 0$, as written, does not admit a variational principle since its linearized equation $v_{tt} + 2u_x v_{xx} + (2u_{xx} + 2u_x)v_x = 0$ is not self-adjoint. However, the equivalent PDE $e^x[u_{tt} + 2u_x u_{xx} + u_x^2] = 0$, as written, is self-adjoint!
- *The use of a contact transformation of the variables.* As an example, the PDE

$$e^x u_{tt} - e^{3x} (u + u_x)^2 (u + 2u_x + u_{xx}) = 0, \quad (59)$$

as written, does not admit a variational principle, since its linearized PDE and the adjoint PDE are different. But the point transformation $x^* = x$, $t^* = t$, $u^*(x^*, t^*) = y(x, t) = e^x u(x, t)$, maps the PDE (59) into the self-adjoint PDE $y_{tt} - (y_x)^2 y_{xx} = 0$, which is the Euler–Lagrange equation for an extremum $Y = y$ of the action integral with Lagrangian $L[Y] = \frac{1}{2}Y_t^2 - \frac{1}{12}Y_x^4$.

- *The use of a differential substitution.* As an example, the KdV equation (13) as written, obviously does not admit a variational principle since it is of odd order. But the well-known differential substitution $u = v_x$ yields the related transformed KdV equation $v_{xt} + v_x v_{xx} + v_{xxxx} = 0$, which arises from the Lagrangian $L[V] = \frac{1}{2}V_{xx}^2 - \frac{1}{6}V_x^3 - \frac{1}{2}V_x V_t$.

- *The use of an artificial additional equation.* For example, the linear heat equation $u_t - u_{xx} = 0$ is not self-adjoint since its adjoint equation is given by $w_t + w_{xx} = 0$. However the decoupled PDE system $u_t - u_{xx} = 0$, $\tilde{u}_t + \tilde{u}_{xx} = 0$ is evidently self-adjoint! [In general, the formal system, obtained through appending any given DE system by the adjoint of its linearized system, is self-adjoint.]

The direct method for finding local conservation laws is free of all of the above problems. It is directly applicable to any DE system, whether or not it is variational. Moreover, it does not require the knowledge of a Lagrangian, whether or not one exists. Indeed, under the direct method, variational and non-variational DE systems are treated in the same manner.

The direct method is naturally coordinate-independent. This follows from the fact that a point (contact) transformation maps a conservation law into a conservation law, and hence either form of a conservation law (in original or transformed variables) will arise from corresponding sets of multipliers, which can be found by the direct method in either coordinate system.

Finding conservation laws through the direct method is computationally more straightforward than through Noether's theorem even when a given DE system is variational. One simply writes down the set of linear determining equations (5) holding for arbitrary functions $U(x)$, which in the case of a variational system, include the symmetry determining equations as a subset of the multiplier determining equations. Hence, the resulting linear determining equations for local multipliers are usually not as difficult to solve as those for local symmetries since this determining system is more over-determined in the variational case.

On the other hand, if a given DE system is variational and one has obtained the Lagrangian for the DE system, then it is worthwhile to combine the direct method with Noether's theorem as follows. First, use the direct method to find the local conservation law multipliers and hence the corresponding variational symmetries. Second, for each variational symmetry, find the corresponding divergence term $D_i A^i[U]$ that arises from the use of Boyer's formulation of the extended Noether's theorem. Third, use expression (37) in conjunction with Boyer's formula (56) to find the resulting local conservation law.

Many examples illustrating the use of the direct method to find local conservation laws, including examples that compare the use of Noether's theorem and the direct method (for PDE systems that admit a variational formulation) appear in [11]. A comparison of the local symmetry and local conservation law structure for non-variational PDE systems appears in [11, 14].

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Invariants of Lie Algebras via Moving Frames

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A purely algebraic algorithm for computation of invariants (generalized Casimir operators) of Lie algebras by means of moving frames is discussed. Results on the application of the method to computation of invariants of low-dimensional Lie algebras and series of solvable Lie algebras restricted only by a required structure of the nilradical are reviewed.

1 Introduction

The invariants of Lie algebras are one of their defining characteristics. They have numerous applications in different fields of mathematics and physics, in which Lie algebras arise (representation theory, integrability of Hamiltonian differential equations, quantum numbers etc). In particular, the polynomial invariants of a Lie algebra exhaust its set of Casimir operators, i.e., the center of its universal enveloping algebra. This is why non-polynomial invariants are also called generalized Casimir operators, and the usual Casimir operators are seen as ‘specific’ generalized Casimir operators. Since the structure of invariants strongly depends on the structure of the algebra and the classification of all (finite-dimensional) Lie algebras is an inherently difficult problem (actually unsolvable¹), it seems to be impossible to elaborate a complete theory for generalized Casimir operators in the general case. Moreover, if the classification of a class of Lie algebras is known, then the invariants of such algebras can be described exhaustively. These problems have already been solved for the semi-simple and low-dimensional Lie algebras, and also for the physically relevant Lie algebras of fixed dimensions.

The standard method of construction of generalized Casimir operators consists of integration of overdetermined systems of first-order linear partial differential

¹The problem of classification of Lie algebras is *wild* since it includes, as a subproblem, the problem on reduction of pairs of matrices to a canonical form [10]. For a detailed review on classification of Lie algebras we refer to [17].

equations. It turns out to be rather cumbersome calculations, once the dimension of Lie algebra is not one of the lowest few. Alternative methods use matrix representations of Lie algebras. They are not much easier and are valid for a limited class of representations.

In our recent papers [3–7] we have developed the purely algebraic algorithm for computation of invariants (generalized Casimir operators) of Lie algebras. The suggested approach is simpler and generally valid. It extends to our problem the exploitation of the Cartan’s method of moving frames in Fels–Olver version [9]. (For modern development of the moving frames method and more references see also [14, 15].)

2 Preliminaries

Consider a Lie algebra \mathfrak{g} of dimension $\dim \mathfrak{g} = n < \infty$ over the complex or real field \mathbb{F} (either $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$) and the corresponding connected Lie group G . Let \mathfrak{g}^* be the dual space of the vector space \mathfrak{g} . The map $\text{Ad}^*: G \rightarrow \text{GL}(\mathfrak{g}^*)$ defined for any $g \in G$ by the relation

$$\langle \text{Ad}_g^* x, u \rangle = \langle x, \text{Ad}_{g^{-1}} u \rangle \quad \text{for all } x \in \mathfrak{g}^* \text{ and } u \in \mathfrak{g}$$

is called the *coadjoint representation* of the Lie group G . Here $\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$ is the usual adjoint representation of G in \mathfrak{g} , and the image Ad_G of G under Ad is the inner automorphism group $\text{Int}(\mathfrak{g})$ of the Lie algebra \mathfrak{g} . The image of G under Ad^* is a subgroup of $\text{GL}(\mathfrak{g}^*)$ and is denoted by Ad_G^* .

The maximal dimension of orbits of Ad_G^* is called the *rank of the coadjoint representation* of G (and \mathfrak{g}) and denoted by rank Ad_G^* . It is a basis independent characteristic of the algebra \mathfrak{g} . Orbits of this dimension are called regular ones.

A function $F \in C^\infty(\Omega)$, where Ω is a domain in \mathfrak{g}^* , is called a (global in Ω) *invariant* of Ad_G^* if $F(\text{Ad}_g^* x) = F(x)$ for all $g \in G$ and $x \in \Omega$ such that $\text{Ad}_g^* x \in \Omega$. The set of invariants of Ad_G^* on Ω is denoted by $\text{Inv}(\text{Ad}_G^*)$ without an explicit indication of the domain Ω . Let below Ω is a neighborhood of a point from a regular orbit. It can always be chosen in such a way that the group Ad_G^* acts regularly on Ω . Then the maximal number $N_{\mathfrak{g}}$ of functionally independent invariants in $\text{Inv}(\text{Ad}_G^*)$ coincides with the codimension of the regular orbits of Ad_G^* , i.e., it is given by the difference $N_{\mathfrak{g}} = \dim \mathfrak{g} - \text{rank Ad}_G^*$.

To calculate the invariants explicitly, one should fix a basis $\mathcal{E} = \{e_1, \dots, e_n\}$ of the algebra \mathfrak{g} . It leads to fixing the dual basis $\mathcal{E}^* = \{e_1^*, \dots, e_n^*\}$ in the dual space \mathfrak{g}^* and to the identification of $\text{Int}(\mathfrak{g})$ and Ad_G^* with the associated matrix groups. The basis elements e_1, \dots, e_n satisfy the commutation relations $[e_i, e_j] = c_{ij}^k e_k$, where c_{ij}^k are components of the tensor of structure constants of \mathfrak{g} in the basis \mathcal{E} . Here and in what follows the indices i, j and k run from 1 to n and the summation convention over repeated indices is used. Let $x \rightarrow \check{x} = (x_1, \dots, x_n)$ be the coordinates in \mathfrak{g}^* associated with \mathcal{E}^* .

It is well known that there exists a bijection between elements of the universal enveloping algebra (i.e., *Casimir operators*) of \mathfrak{g} and polynomial invariants of \mathfrak{g}

(which can be assumed defined globally on \mathfrak{g}^*). See, e.g., [1]. Such a bijection is established, e.g., by the symmetrization operator Sym which acts on monomials by the formula

$$\text{Sym}(e_{i_1} \cdots e_{i_r}) = \frac{1}{r!} \sum_{\sigma \in S_r} e_{i_{\sigma_1}} \cdots e_{i_{\sigma_r}},$$

where i_1, \dots, i_r take values from 1 to n , $r \in \mathbb{N}$. The symbol S_r denotes the permutation group consisting of r elements. The symmetrization also can be correctly defined for rational invariants [1]. If $\text{Int}(\text{Ad}_G^*)$ has no a functional basis consisting of only rational invariants, the correctness of the symmetrization needs an additional investigation for each fixed algebra \mathfrak{g} since general results on this subject do not exist. After symmetrized, elements from $\text{Int}(\text{Ad}_G^*)$ are naturally called invariants or *generalized Casimir operators* of \mathfrak{g} . The set of invariants of \mathfrak{g} is denoted by $\text{Inv}(\mathfrak{g})$.

Functionally independent invariants $F^l(x_1, \dots, x_n)$, $l = 1, \dots, N_{\mathfrak{g}}$, forms a *functional basis (fundamental invariant)* of $\text{Inv}(\text{Ad}_G^*)$ since any element from $\text{Inv}(\text{Ad}_G^*)$ can be (uniquely) represented as a function of these invariants. Accordingly the set of $\text{Sym } F^l(e_1, \dots, e_n)$, $l = 1, \dots, N_{\mathfrak{g}}$, is called a basis of $\text{Inv}(\mathfrak{g})$.

In framework of the infinitesimal approach any invariant $F(x_1, \dots, x_n)$ of Ad_G^* is a solution of the linear system of first-order partial differential equations [1,2,16] $X_i F = 0$, i.e., $c_{ij}^k x_k F_{x_j} = 0$, where $X_i = c_{ij}^k x_k \partial_{x_j}$ is the infinitesimal generator of the one-parameter group $\{\text{Ad}_G^*(\exp \varepsilon e_i)\}$ corresponding to e_i . The mapping $e_i \rightarrow X_i$ gives a representation of the Lie algebra \mathfrak{g} .

3 The algorithm

Let $\mathcal{G} = \text{Ad}_G^* \times \mathfrak{g}^*$ denote the trivial left principal Ad_G^* -bundle over \mathfrak{g}^* . The right regularization \widehat{R} of the coadjoint action of G on \mathfrak{g}^* is the diagonal action of Ad_G^* on $\mathcal{G} = \text{Ad}_G^* \times \mathfrak{g}^*$. It is provided by the map

$$\widehat{R}_g(\text{Ad}_h^*, x) = (\text{Ad}_h^* \cdot \text{Ad}_{g^{-1}}^*, \text{Ad}_g^* x), \quad g, h \in G, \quad x \in \mathfrak{g}^*.$$

The action \widehat{R} on the bundle $\mathcal{G} = \text{Ad}_G^* \times \mathfrak{g}^*$ is regular and free. We call \widehat{R} the *lifted coadjoint action* of G . It projects back to the coadjoint action on \mathfrak{g}^* via the Ad_G^* -equivariant projection $\pi_{\mathfrak{g}^*}: \mathcal{G} \rightarrow \mathfrak{g}^*$. Any *lifted invariant* of Ad_G^* is a (locally defined) smooth function from \mathcal{G} to a manifold, which is invariant with respect to the lifted coadjoint action of G . The function $\mathcal{I}: \mathcal{G} \rightarrow \mathfrak{g}^*$ given by $\mathcal{I} = \mathcal{I}(\text{Ad}_g^*, x) = \text{Ad}_g^* x$ is the *fundamental lifted invariant* of Ad_G^* , i.e., \mathcal{I} is a lifted invariant and any lifted invariant can be locally written as a function of \mathcal{I} in a unique way. Using an arbitrary function $F(x)$ on \mathfrak{g}^* , we can produce the lifted invariant $F \circ \mathcal{I}$ of Ad_G^* by replacing x with $\mathcal{I} = \text{Ad}_g^* x$ in the expression for F . Ordinary invariants are particular cases of lifted invariants, where one identifies any invariant formed as its composition with the standard projection $\pi_{\mathfrak{g}^*}$.

Therefore, ordinary invariants are particular functional combinations of lifted ones that happen to be independent of the group parameters of Ad_G^* .

The essence of the normalization procedure by Fels and Olver can be presented in the form of the following statement.

Proposition 1. *Suppose that $\mathcal{I} = (\mathcal{I}_1, \dots, \mathcal{I}_n)$ is a fundamental lifted invariant, for the lifted invariants $\mathcal{I}_{j_1}, \dots, \mathcal{I}_{j_\rho}$ and some constants c_1, \dots, c_ρ the system $\mathcal{I}_{j_1} = c_1, \dots, \mathcal{I}_{j_\rho} = c_\rho$ is solvable with respect to the parameters $\theta_{k_1}, \dots, \theta_{k_\rho}$ and substitution of the found values of $\theta_{k_1}, \dots, \theta_{k_\rho}$ into the other lifted invariants results in $m = n - \rho$ expressions $\hat{\mathcal{I}}_l$, $l = 1, \dots, m$, depending only on x 's. Then $\rho = \text{rank } \text{Ad}_G^*$, $m = N_{\mathfrak{g}}$ and $\hat{\mathcal{I}}_1, \dots, \hat{\mathcal{I}}_m$ form a basis of $\text{Inv}(\text{Ad}_G^*)$.*

The *algebraic algorithm* for finding invariants of the Lie algebra \mathfrak{g} is briefly formulated in the following four steps.

1. *Construction of the generic matrix $B(\theta)$ of Ad_G^* .* $B(\theta)$ is the matrix of an inner automorphism of the Lie algebra \mathfrak{g} in the given basis e_1, \dots, e_n , $\theta = (\theta_1, \dots, \theta_r)$ is a complete tuple of group parameters (coordinates) of $\text{Int}(\mathfrak{g})$, and $r = \dim \text{Ad}_G^* = \dim \text{Int}(\mathfrak{g}) = n - \dim Z(\mathfrak{g})$, where $Z(\mathfrak{g})$ is the center of \mathfrak{g} .

2. *Representation of the fundamental lifted invariant.* The explicit form of the fundamental lifted invariant $\mathcal{I} = (\mathcal{I}_1, \dots, \mathcal{I}_n)$ of Ad_G^* in the chosen coordinates (θ, \dot{x}) in $\text{Ad}_G^* \times \mathfrak{g}^*$ is $\mathcal{I} = \dot{x} \cdot B(\theta)$, i.e.,

$$(\mathcal{I}_1, \dots, \mathcal{I}_n) = (x_1, \dots, x_n) \cdot B(\theta_1, \dots, \theta_r).$$

3. *Elimination of parameters by normalization.* We choose the maximum possible number ρ of lifted invariants $\mathcal{I}_{j_1}, \dots, \mathcal{I}_{j_\rho}$, constants c_1, \dots, c_ρ and group parameters $\theta_{k_1}, \dots, \theta_{k_\rho}$ such that the equations $\mathcal{I}_{j_1} = c_1, \dots, \mathcal{I}_{j_\rho} = c_\rho$ are solvable with respect to $\theta_{k_1}, \dots, \theta_{k_\rho}$. After substituting the found values of $\theta_{k_1}, \dots, \theta_{k_\rho}$ into the other lifted invariants, we obtain $N_{\mathfrak{g}} = n - \rho$ expressions $F^l(x_1, \dots, x_n)$ without θ 's.

4. *Symmetrization.* The functions $F^l(x_1, \dots, x_n)$ necessarily form a basis of $\text{Inv}(\text{Ad}_G^*)$. They are symmetrized to $\text{Sym} F^l(e_1, \dots, e_n)$. It is the desired basis of $\text{Inv}(\mathfrak{g})$.

Our experience on the calculation of invariants of a wide range of Lie algebras shows that the version of the algebraic method, which is based on Proposition 1, is most effective. In particular, it provides finding the cardinality of the invariant basis in the process of construction of the invariants. The algorithm can in fact involve different kinds of coordinate in the inner automorphism groups (the first canonical, the second canonical or special one) and different techniques of elimination of parameters (empirical techniques, with additional combining of lifted invariants, using a floating system of normalization equations etc).

Let us underline that the search of invariants of a Lie algebra \mathfrak{g} , which has been done by solving a linear system of first-order partial differential equations under the conventional infinitesimal approach, is replaced here by the construction of the matrix $B(\theta)$ of inner automorphisms and by excluding the parameters θ from the fundamental lifted invariant $\mathcal{I} = \dot{x} \cdot B(\theta)$ in some way.

4 Illustrative example

The four-dimensional solvable Lie algebra $\mathfrak{g}_{4.8}^b$ has the following non-zero commutation relations

$$[e_2, e_3] = e_1, \quad [e_1, e_4] = (1+b)e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = be_3, \quad |b| \leq 1.$$

Its nilradical is three-dimensional and isomorphic to the Weil–Heisenberg algebra $\mathfrak{g}_{3.1}$. (Here we use the notations of low-dimensional Lie algebras according to Mubarakzyanov's classification [11].)

We construct a presentation of the inner automorphism matrix $B(\theta)$ of the Lie algebra \mathfrak{g} , involving second canonical coordinates on Ad_G as group parameters θ . The matrices $\hat{\text{ad}}_{e_i}$, $i = 1, \dots, 4$, of the adjoint representation of the basis elements e_1, e_2, e_3 and e_4 respectively have the form

$$\begin{aligned} & \begin{pmatrix} 0 & 0 & 0 & 1+b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ & \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1-b & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The inner automorphisms of $\mathfrak{g}_{4.8}^b$ are then described by the triangular matrix

$$\begin{aligned} B(\theta) &= \prod_{i=1}^3 \exp(\theta_i \hat{\text{ad}}_{e_i}) \cdot \exp(-\theta_4 \hat{\text{ad}}_{e_4}) \\ &= \begin{pmatrix} e^{(1+b)\theta_4} & -\theta_3 e^{\theta_4} & \theta_2 e^{b\theta_4} & b\theta_2\theta_3 + (1+b)\theta_1 \\ 0 & e^{\theta_4} & 0 & \theta_2 \\ 0 & 0 & e^{b\theta_4} & b\theta_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore, a functional basis of lifted invariants is formed by

$$\begin{aligned} \mathcal{I}_1 &= e^{(1+b)\theta_4} x_1, \\ \mathcal{I}_2 &= e^{\theta_4}(-\theta_3 x_1 + x_2), \\ \mathcal{I}_3 &= e^{b\theta_4}(\theta_2 x_1 + x_3), \\ \mathcal{I}_4 &= (b\theta_2\theta_3 + (1+b)\theta_1)x_1 + \theta_2 x_2 + b\theta_3 x_3 + x_4. \end{aligned}$$

Further the cases $b = -1$ and $b \neq -1$ should be considered separately.

There are no invariants in case $b \neq -1$ since in view of Proposition 1 the number of functionally independent invariants is equal to zero. Indeed, the system $\mathcal{I}_1 = 1, \mathcal{I}_2 = \mathcal{I}_3 = \mathcal{I}_4 = 0$ is solvable with respect to the whole set of the parameters θ .

It is obvious that in the case $b = -1$ the element e_1 generating the center $Z(\mathfrak{g}_{4.8}^{-1})$ is an invariant. (The corresponding lifted invariant $\mathcal{I}_1 = x_1$ does not depend on the parameters θ .) Another invariant is easily found via combining the lifted invariants: $\mathcal{I}_1\mathcal{I}_4 - \mathcal{I}_2\mathcal{I}_3 = x_1x_4 - x_2x_3$. After the symmetrization procedure we obtain the following polynomial basis of the invariant set of this algebra

$$e_1, \quad e_1e_4 - \frac{e_2e_3 + e_3e_2}{2}.$$

The second basis invariant can be also constructed by the normalization technique. We solve the equations $\mathcal{I}_2 = \mathcal{I}_3 = 0$ with respect to the parameters θ_2 and θ_3 and substitute the expressions for them into the lifted invariant \mathcal{I}_4 . The obtained expression $x_4 - x_2x_3/x_1$ does not contain the parameters θ and, therefore, is an invariant of the coadjoint representation. For the basis of invariants to be polynomial, we multiply this invariant by the invariant x_1 . It is the technique that is applied below for the general case of the Lie algebras under consideration.

Note that in the above example the symmetrization procedure can be assumed trivial since the symmetrized invariant $e_1e_4 - \frac{1}{2}(e_2e_3 + e_3e_2)$ differs from the non-symmetrized version $e_1e_4 - e_2e_3$ (resp. $e_1e_4 - e_3e_2$) on the invariant $\frac{1}{2}e_1$ (resp. $-\frac{1}{2}e_1$). If we take the rational invariant $e_4 - e_2e_3/e_1$ (resp. $e_4 - e_3e_2/e_1$), the symmetrization is equivalent to the addition of the constant $\frac{1}{2}$ (resp. $-\frac{1}{2}$).

Invariants of $\mathfrak{g}_{4.8}^b$ were first described in [16] within the framework of the infinitesimal approach.

5 Review of obtained results

Using the moving frames approach, we recalculated invariant bases and, in a number of cases, enhanced their representation for the following Lie algebras (in additional brackets we cite the papers where invariants bases of the same algebras were computed by the infinitesimal method):

- the complex and real Lie algebras up to dimension 6 [3] ([8, 12, 16]);
- the complex and real Lie algebras with Abelian nilradicals of codimension one [4] ([18]);
- the complex indecomposable solvable Lie algebras with the nilradicals isomorphic to \mathfrak{J}_0^n , $n = 3, 4, \dots$ (the nonzero commutation relations between the basis elements e_1, \dots, e_n of \mathfrak{J}_0^n are exhausted by $[e_k, e_n] = e_{k-1}$, $k = 2, \dots, n-1$) [4] ([13]);
- the nilpotent Lie algebra $\mathfrak{t}_0(n)$ of $n \times n$ strictly upper triangular matrices [4, 5] ([20]);
- the solvable Lie algebra $\mathfrak{t}(n)$ of $n \times n$ upper triangular matrices and the solvable Lie algebras $\mathfrak{st}(n)$ of $n \times n$ special upper triangular matrices [5–7] ([20]);
- the solvable Lie algebras with nilradicals isomorphic to $\mathfrak{t}_0(n)$ and diagonal nilindependent elements [5–7] ([20]).

Note that earlier only conjectures on invariants of two latter families of Lie algebras were known. Moreover, for the last family the conjecture was formulated only for the particular case of a single nilindependent element. Here we present the exhaustive statement on invariants of this series of Lie algebras, obtained in [7].

Consider the solvable Lie algebra $\mathfrak{t}_\gamma(n)$ with the nilradical $\text{NR}(\mathfrak{t}_\gamma(n))$ isomorphic to $\mathfrak{t}_0(n)$ and s nilindependent element f_p , $p = 1, \dots, s$, which act on elements of the nilradical in the way as the diagonal matrices $\Gamma_p = \text{diag}(\gamma_{p1}, \dots, \gamma_{pn})$ act on strictly triangular matrices. The matrices Γ_p , $p = 1, \dots, s$, and the unity matrix are linear independent since otherwise $\text{NR}(\mathfrak{t}_\gamma(n)) \neq \mathfrak{t}_0(n)$. The parameter matrix $\gamma = (\gamma_{pi})$ is defined up to nonsingular $s \times s$ matrix multiplier and homogeneous shift in rows. In other words, the algebras $\mathfrak{t}_\gamma(n)$ and $\mathfrak{t}_{\gamma'}(n)$ are isomorphic iff there exist $\lambda \in M_{s,s}(\mathbb{F})$, $\det \lambda \neq 0$, and $\mu \in \mathbb{F}^s$ such that

$$\gamma'_{pi} = \sum_{p'=1}^s \lambda_{pp'} \gamma_{p'i} + \mu_p, \quad p = 1, \dots, s, \quad i = 1, \dots, n.$$

The parameter matrix γ and γ' are assumed equivalent. Up to the equivalence the additional condition $\text{Tr} \Gamma_p = \sum_i \gamma_{pi} = 0$ can be imposed on the algebra parameters. Therefore, the algebra $\mathfrak{t}_\gamma(n)$ is naturally embedded into $\mathfrak{st}(n)$ as a (mega)ideal under identification of $\text{NR}(\mathfrak{t}_\gamma(n))$ with $\mathfrak{t}_0(n)$ and of f_p with Γ_p .

We choose the union of the canonical basis of $\text{NR}(\mathfrak{t}_\gamma(n))$ and the s -element set $\{f_p, p = 1, \dots, s\}$ as the canonical basis of $\mathfrak{t}_\gamma(n)$. In the basis of $\text{NR}(\mathfrak{t}_\gamma(n))$ we use ‘matrix’ enumeration of basis elements e_{ij} , $i < j$, with the ‘increasing’ pair of indices similarly to the canonical basis $\{E_{ij}^n, i < j\}$ of the isomorphic matrix algebra $\mathfrak{t}_0(n)$.

Hereafter E_{ij}^n (for the fixed values i and j) denotes the $n \times n$ matrix $(\delta_{ii'} \delta_{jj'})$ with i' and j' running the numbers of rows and column correspondingly, i.e., the $n \times n$ matrix with the unit on the cross of the i -th row and the j -th column and the zero otherwise. The indices i , j , k and l run at most from 1 to n . Only additional constraints on the indices are indicated. The subscript p runs from 1 to s , the subscript q runs from 1 to s' . The summation convention over repeated indices p and q is used unless otherwise stated. The number s is in the range $0, \dots, n-1$. In the case $s = 0$ we assume $\gamma = 0$, and all terms with the subscript p should be omitted from consideration. The value s' ($s' < s$) is defined below.

Thus, the basis elements $e_{ij} \sim E_{ij}^n$, $i < j$, and $f_p \sim \sum_i \gamma_{pi} E_{ii}^n$ satisfy the commutation relations $[e_{ij}, e_{i'j}] = \delta_{i'j} e_{ij} - \delta_{ij'} e_{i'j}$, $[f_p, e_{ij}] = (\gamma_{pi} - \gamma_{pj}) e_{ij}$, where δ_{ij} is the Kronecker delta.

The Lie algebra $\mathfrak{t}_\gamma(n)$ can be considered as the Lie algebra of the Lie subgroup $T_\gamma(n) = \{B \in T(n) \mid \exists \varepsilon_p \in \mathbb{F}: b_{ii} = e^{\gamma_{pi} \varepsilon_p}\}$ of the Lie group $T(n)$ of non-singular upper triangular $n \times n$ matrices.

Below $A_{j_1, j_2}^{i_1, i_2}$, where $i_1 \leq i_2$, $j_1 \leq j_2$, denotes the submatrix $(a_{ij})_{j=j_1, \dots, j_2}^{i=i_1, \dots, i_2}$ of a matrix $A = (a_{ij})$. The conjugate value of k with respect to n is denoted by \varkappa , i.e., $\varkappa = n - k + 1$. The standard notation $|A| = \det A$ is used.

Proposition 2. *Up to the equivalence relation on algebra parameters, the following conditions can be assumed satisfied for some $s' \in \{0, \dots, \min(s, [n/2])\}$ and $k_q, q = 1, \dots, s', 1 \leq k_1 < k_2 < \dots < k_{s'} \leq [n/2]$:*

$$\begin{aligned} \gamma_{qk} &= \gamma_{q\kappa}, \quad k < k_q, \quad \gamma_{q\kappa_q} - \gamma_{qk_q} = 1, \quad \gamma_{pk_q} = \gamma_{p\kappa_q}, \quad p \neq q, \quad q = 1, \dots, s', \\ \gamma_{pk} &= \gamma_{p\kappa}, \quad p > s', \quad k = 1, \dots, [n/2]. \end{aligned}$$

We will say that the parameter matrix γ has a *reduced form* if it satisfies the conditions of Proposition 2.

Theorem 1. *Let the parameter matrix γ have a reduced form. A basis of $\text{Inv}(\mathfrak{t}_\gamma(n))$ is formed by the expressions*

$$\begin{aligned} &|\mathcal{E}_{\kappa,n}^{1,k}| \prod_{q=1}^{s'} |\mathcal{E}_{\kappa_q,n}^{1,k_q}|^{\alpha_{qk}}, \quad k \in \{1, \dots, [n/2]\} \setminus \{k_1, \dots, k_{s'}\}, \\ &f_p + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^{k+1}}{|\mathcal{E}_{\kappa,n}^{1,k}|} (\gamma_{pk} - \gamma_{p,k+1}) \sum_{k < i < \kappa} \begin{vmatrix} \mathcal{E}_{i,i}^{1,k} & \mathcal{E}_{\kappa,n}^{1,k} \\ 0 & \mathcal{E}_{\kappa,n}^{i,i} \end{vmatrix}, \quad p = s' + 1, \dots, s, \end{aligned}$$

where $\kappa := n - k + 1$, $\mathcal{E}_{j_1,j_2}^{i_1,i_2}$, $i_1 \leq i_2$, $j_1 \leq j_2$, denotes the matrix $(e_{ij})_{j=j_1, \dots, j_2}^{i=i_1, \dots, i_2}$ and

$$\alpha_{qk} := - \sum_{k'=1}^k (\gamma_{q\kappa'} - \gamma_{qk'}).$$

We use the short ‘non-symmetrized’ form for basis invariants, where it is uniformly assumed that in all monomials elements of $\mathcal{E}_{i,i}^{1,k}$ is placed before (or after) elements of $\mathcal{E}_{\kappa,n}^{i,i}$.

6 Conclusion

The main advantage of the proposed method is in that it is purely algebraic. Unlike the conventional infinitesimal method, it eliminates the need to solve systems of partial differential equations, replaced in our approach by the construction of the matrix $B(\theta)$ of inner automorphisms and by excluding the parameters θ from the fundamental lifted invariant $\mathcal{I} = \dot{x} \cdot B(\theta)$ in some way.

The efficient exploitation of the method imposes certain constraints on the choice of bases of the Lie algebras. See, e.g., Proposition 2 and Theorem 1. That then automatically yields simpler expressions for the invariants. In some cases the simplification is considerable.

Possibilities on the usage of the approach and directions for further investigation were outlined in our previous papers [3–7]. Recently advantages of the moving frames approach for computation of generalized Casimir operators were demonstrated in [19] with a new series of solvable Lie algebras. The problem on optimal ways of applications of this approach to unsolvable Lie algebras is still open.

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Symmetry reductions and exact solutions of Benjamin-Bona-Mahony-Burgers equation

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We consider a generalized Benjamin-Bona-Mahony-Burgers equation. The functional forms, for which the equation can be reduced to ordinary differential equations by classical Lie symmetries, are obtained. In order to obtain travelling wave solutions two procedures are applied: a direct method and the G'/G -expansion method. A catalogue of symmetry reductions and a catalogue of exact solutions are given. A set of solitons, kinks, antikinks and compactons are derived.

1 Introduction

In [11], Khaled, Momani and Alawneh implemented the Adomian's decomposition method for obtaining explicit and numerical solutions of the Benjamin-Bona-Mahony-Burgers (BBMB) equation

$$\Delta \equiv u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + (g(u))_x = 0, \quad (1)$$

where $u(x, t)$ represents the fluid velocity in the horizontal direction x , α is a positive constant, $\beta \in \mathbb{R}$ and $g(u)$ is a C^2 -smooth nonlinear function [11]. Equation (1) is the alternative regularized long-wave equation proposed by Peregrine [17] and Benjamin [2]. In the physical sense, equation (1) with the dissipative term αu_{xx} is proposed if the good predictive power is desired, such problem arises in the phenomena for both the bore propagation and the water waves.

Tari and Ganji, [19], in order to derive approximate explicit solutions for (1) with $g(u) = \frac{u^2}{2}$ have applied two methods for solving nonlinear differential equations. These methods are known as variational iteration method and homotopy perturbation method.

El-Wakil, Abdou and Hendi [8] to obtain the generalized solitary solutions and periodic solutions for (1) with $g(u) = \frac{u^2}{2}$ has used the exp-function method with the aid of symbolic computational system. In [9] Fakhari *et al.* to evaluate the nonlinear equation (1) with $g(u) = \frac{u^2}{2}$, $\alpha = 0$ and $\beta = 1$ solved the resulting nonlinear differential equation by homotopy analysis methods.

By applying the classical Lie method of infinitesimals Bruzón and Gandarias [4] obtained, for a generalization of a family of Benjamin-Bona-Mahony equations, many exact solutions expressed by various single and combined nondegenerative Jacobi elliptic functions. In [5, 6] we studied similarity reductions of the BBMB

equation (1) and a set of new solitons, kinks, antikinks, compactons and Wadati solitons were derived.

The classical method for finding symmetry reductions of partial differential equations is the Lie group method [10, 15, 16]. The fundamental basis of this method is that, when a differential equation is invariant under a Lie group of transformations, a reduction transformation exists. For partial differential equations (PDEs) with two independent variables a single group reduction transforms the PDE into an ordinary differential equations (ODEs), which in general are easier to solve. Since the relevant calculations are usually rather laborious, they can be conveniently carried out by means of symbolic computations. In our work, we used the MACSYMA program `symmgrp.max` [7]. Most of the required theory and description of the method can be found in [15, 16].

In the last years a great progress was made in the development of methods for finding exact solutions of nonlinear differential equations.

In [12] Kudryashov presented a new method to look for exact solutions of nonlinear differential equations. This method is based in two ideas: The first idea is to apply the simplest nonlinear differential equations (the Riccati equation, the equation for the Jacobi elliptic function, the equation for the Weierstrass elliptic function and so on) that have lesser order than the studied equation. The second idea is to use all possible singularities of the studied equation.

Recently Wang *et al* [20] introduced a method which is called the $\frac{G'}{G}$ -expansion method to look for travelling wave solutions of nonlinear evolution equations. The main ideas of the proposed method are that the travelling wave solutions of a nonlinear evolution equation can be expressed by a polynomial in $\frac{G'}{G}$, where $G = G(z)$ satisfies the linear second order ordinary differential equation (ODE) $G''(z) + \omega G'(z) + \zeta G(z) = 0$, the degree of the polynomial can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in a given nonlinear evolution equation, and the coefficients of the polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the proposed method.

In [13] Kudryashov and Loguinova introduced the modified simplest equation method. By this method the exact solutions are expressed by a polynomial in $\frac{\psi'}{\psi}$, where the unknown function $\psi = \psi(z)$ satisfies the third order linear ODE

$$\psi''' + \alpha\psi'' + \beta\psi' + \gamma\psi = 0, \quad \alpha, \beta, \gamma = \text{const.}$$

To determine the degree of the polynomial the authors concentrate their attention on the leading terms of the ODE. The homogeneous balance between the leading terms provides the polynomial degree value.

The aim of this paper is to study the functional forms $g(u)$ for which equation (1) admits a classical symmetry group. By using the symmetry reductions, we derive the reduced form of the original nonlinear PDE as a nonlinear ODE. We determine travelling wave solutions. We also find the functions $g(h) = h^m$ for which we can apply the $\frac{G'}{G}$ -expansion method. For these functions we obtain exact solutions of equation (1).

2 Classical symmetries

To apply the Lie classical method to equation (1) we consider the one-parameter Lie group of infinitesimal transformations in (x, t, u) given by

$$\begin{aligned} x^* &= x + \epsilon \xi(x, t, u) + O(\epsilon^2), & t^* &= t + \epsilon \tau(x, t, u) + O(\epsilon^2), \\ u^* &= u + \epsilon \eta(x, t, u) + O(\epsilon^2), \end{aligned} \quad (2)$$

where ϵ is the group parameter. We require that this transformation leaves invariant the set of solutions of equation (1). This yields to an overdetermined, linear system of equations for the infinitesimals $\xi(x, t, u)$, $\tau(x, t, u)$ and $\eta(x, t, u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \eta(x, t, u) \frac{\partial}{\partial u}. \quad (3)$$

Having determined the infinitesimals, the symmetry variables are found by solving characteristic equation which is equivalent to solving invariant surface condition

$$\eta(x, t, u) - \xi(x, t, u) \frac{\partial u}{\partial x} - \tau(x, t, u) \frac{\partial u}{\partial t} = 0. \quad (4)$$

The set of solutions of equation (1) is invariant under the transformation (2) provided that

$$\text{pr}^{(3)}\mathbf{v}(\Delta) = 0 \quad \text{when} \quad \Delta = 0,$$

where $\text{pr}^{(3)}\mathbf{v}$ is the third prolongation of the vector field (3) given by

$$\begin{aligned} \text{pr}^{(3)}\mathbf{v} &= \mathbf{v} + \sum_J \eta^J(x, t, u^{(3)}) \frac{\partial}{\partial u_J}, \quad \text{where} \\ \eta^J(x, t, u^{(3)}) &= D_J(\eta - \xi u_x - \tau u_t) + \xi u_{Jx} + \eta u_{Jt}, \end{aligned}$$

with $J = (j_1, \dots, j_k)$, $1 \leq j_k \leq 2$ y $1 \leq k \leq 3$. Hence we obtain the following ten determining equations for the infinitesimals:

$$\begin{aligned} \tau_u &= 0, & \tau_x &= 0, & \xi_u &= 0, & \xi_t &= 0, & \eta_{uu} &= 0, & \alpha \tau_t + \eta_{tu} &= 0, \\ 2\eta_{ux} - \xi_{xx} &= 0, & \eta_{uxx} - 2\xi_x &= 0, & \eta_x g_u - \alpha \eta_{xx} + \beta \eta_x - \eta_{txx} + \eta_t &= 0, \\ -\alpha \xi_{xx} - g_u \xi_x - \beta \xi_x - g_u \tau_t - \beta \tau_t - \eta g_{uu} + 2\alpha \eta_{ux} + 2\eta_{tux} &= 0. \end{aligned} \quad (5)$$

From system (5) $\xi = \xi(x)$, $\tau = \tau(t)$ and $\eta = \gamma(x, t)u + \delta(x, t)$ where $\alpha, \beta, \xi, \tau, \gamma, \delta$ and g satisfy

$$\begin{aligned} \gamma_t + \alpha \tau_t &= 0, & 2\gamma_x - \xi_{xx} &= 0, & \gamma_{xx} - 2\xi_x &= 0, \\ 2\alpha \gamma_x + 2\gamma_{tx} - g_{uu} u \gamma - \alpha \xi_{xx} - g_u \xi_x - \beta \xi_x - g_u \tau_t - \beta \tau_t - \delta g_{uu} &= 0, \\ -\alpha u \gamma_{xx} + g_u u \gamma_x + \beta u \gamma_x - u \gamma_{txx} + u \gamma_t + \delta_x g_u - \alpha \delta_{xx} \\ + \beta \delta_x - \delta_{txx} + \delta_t &= 0. \end{aligned} \quad (6)$$

From (6) we obtain

$$\begin{aligned}\gamma &= \frac{e^{-2x}}{8} \left((k_4 + 2k_3) e^{4x} + (4k_1 - 8\alpha\tau) e^{2x} - k_4 + 2k_3 \right), \\ \xi &= \frac{(k_4 + 2k_3) e^{2x}}{8} + \frac{(k_4 - 2k_3) e^{-2x}}{8} - \frac{k_4 - 4k_2}{4},\end{aligned}$$

and $\alpha, \beta, \tau, \delta$ and g are related by the following conditions:

$$\begin{aligned}&((g_u + \beta - 2\alpha) k_4 + (2g_u + 2\beta - 4\alpha) k_3) ue^{4x} + (-4\alpha\tau_t u + \delta_x (4g_u + 4\beta) \\ &- 4\alpha\delta_{xx} - 4\delta_{txx} + 4\delta_t) e^{2x} + ((g_u + \beta + 2\alpha) k_4 \\ &+ (-2g_u - 2\beta - 4\alpha) k_3) u = 0, \\ &((g_{uu} k_4 + 2g_{uu} k_3) u + (2g_u + 2\beta) k_4 + (4g_u + 4\beta) k_3) e^{4x} \\ &+ ((4g_{uu} k_1 - 8\alpha g_{uu} \tau) u + 8g_u \tau_t + 8\beta \tau_t + 8\delta g_{uu}) e^{2x} + (2g_{uu} k_3 - g_{uu} k_4) u \\ &+ (-2g_u - 2\beta) k_4 + (4g_u + 4\beta) k_3 = 0.\end{aligned}$$

Solving system (2) we obtain that if g is an arbitrary function the only symmetries admitted by (1) are

$$\xi = k_1, \quad \tau = k_2, \quad \eta = 0. \quad (7)$$

The generators of this are $\mathbf{v}_1 = \frac{\partial}{\partial x}$ (corresponding to space translational invariance) and $\mathbf{v}_2 = \frac{\partial}{\partial t}$ (time translational invariance). By substituting (7) into the invariant surface condition (4) we obtain the similarity variable and the similarity solution

$$z = \mu x - \lambda t, \quad u(x, t) = h(z). \quad (8)$$

Substituting (8) into (1) we obtain

$$\lambda\mu^2 h''' - \alpha\mu^2 h'' + (\beta\mu - \lambda)h' + \mu h' g_h(h) = 0. \quad (9)$$

Integrating (9) once we get

$$\lambda\mu^2 h'' - \alpha\mu^2 h' + (\beta\mu - \lambda)h + \mu g(h) + k = 0. \quad (10)$$

In the following cases equation (1) have extra symmetries:

$$(i) \text{ If } \alpha = 0, g(u) = -\beta u + \frac{k}{a(n+1)}(au + b)^{n+1}, a \neq 0,$$

$$\xi = k_1, \quad \tau = k_2 t + k_3, \quad \eta = -\frac{k_2}{an}(au + b).$$

Besides \mathbf{v}_1 and \mathbf{v}_2 , we obtain the infinitesimal generator $\mathbf{v}_3 = t\partial_t - \frac{au+b}{an}\partial_u$.

$$(ii) \text{ If } \alpha \neq 0, \beta \neq 0 \text{ and } g(u) = au + b, \xi = k_1, \tau = k_2, \eta = \delta(x, t), \text{ where } \delta \text{ satisfy } \alpha\delta_{xx} - g_u\delta_x - \beta\delta_x + \delta_{txx} - \delta_t = 0.$$

We do not considerer case (ii) because in this case equation (1) is a linear PDE.

In order to determine solutions of PDE (1) that are not equivalent by the group action, we must calculate the one-dimensional optimal system [15]. The generators of the nontrivial one-dimensional optimal system are the set

$$\mu \mathbf{v}_1 + \lambda \mathbf{v}_2, \quad \mathbf{v}_3, \quad \mathbf{v}_1 + \mathbf{v}_3.$$

Since equation (1) has additional symmetries and the reductions that correspond to \mathbf{v}_1 and \mathbf{v}_2 have already been derived, we must determine the similarity variables and similarity solutions corresponding to the generators \mathbf{v}_3 and $\mathbf{v}_1 + \mathbf{v}_3$.

• \mathbf{v}_3 : We obtain the reduction

$$z = x, \quad u = t^{-\frac{1}{n}} h(x) - \frac{b}{a}, \quad (11)$$

where $h(t)$ satisfies the Lienard equation

$$\Delta_1 \equiv h'' + kna^n h^n h' - h = 0. \quad (12)$$

By substituting $w(h) = h'(z)$ into (12) leads to the second kind Abel equation

$$ww' + kna^n h^n w - h = 0. \quad (13)$$

In [18] the authors present a large number of solutions to the Abel equation of the second kind. Consequently, from (11) one can obtain solutions of equation (1).

The problem of finding Lie symmetries for the first-order ODE is equivalent to finding solutions for these equations, and for this reason the direct application of the Lie method is complicated in the general case [3]. In [3] the author applied an approach for description of integrable cases of the Abel equations using the procedure of increasing the order and equivalence transformations for the induced second-order equations.

Let us now prove that the Lienard equation (12) has no nontrivial point symmetries. Indeed, let

$$\mathbf{S} = \xi(z, h) \partial_z + \eta(z, h) \partial_h \quad (14)$$

denote a symmetry vector of (12). The necessary and sufficient condition for this is [10, 15, 16] is

$$\text{pr}^{(2)} \mathbf{v}(\Delta_1) = 0 \quad \text{when} \quad \Delta_1 = 0,$$

where $\text{pr}^{(2)} \mathbf{S}$ is the second prolongation of the vector field \mathbf{S} . This yields to an overdetermined system of equations for the infinitesimals $\xi(x, t, u)$ and $\eta(x, t, u)$. Solving this system we obtain that $\xi = k_1$ and $\eta = 0$ with k_1 arbitrary constant. To conclude this case, let us summarize our findings in a theorem:

Theorem 1. *The Lienard equation (12) admits no nontrivial symmetries, but it is nevertheless reduced to an Abel equation.*

- $\mathbf{v}_1 + \mathbf{v}_3$: The reduction is

$$z = x - \ln |t|, \quad u = t^{-\frac{1}{n}} h(z) - \frac{b}{a}. \quad (15)$$

The reduced ODE is

$$nh''' + h'' - nh' + nka^n h^n h' - h = 0. \quad (16)$$

Invariance of equation (16) under a Lie group of point transformations with infinitesimal generator (14) leads to a set of five determining equations. Solving this system we obtain that $\xi = k_1$ and $\eta = 0$ with k_1 arbitrary constant.

We can observe that, for reduction (15), we have

$$u(x, t) = t^{-\frac{1}{n}} h(x - \ln |t|) - \frac{b}{a}.$$

This solution describes if $n > 0$ a travelling wave with decaying velocity $v = \frac{1}{t}$ and decaying amplitude $t^{-\frac{1}{n}}$.

On the other hand by making $w(h) = h'$ into (16), equation (16) can be reduced to

$$w'' + \frac{(w')^2}{w} + \frac{1}{n} \frac{w'}{w} - \frac{1}{w} + kna^n \frac{h^n}{w} - \frac{h}{w^2} = 0. \quad (17)$$

Let us summarize our findings in a theorem:

Theorem 2. *The third-order differential equation (16) admits no nontrivial symmetries, but it is nevertheless reduced to equation (17) which does not admit any Lie symmetries.*

3 Travelling wave solutions

In the last years a great progress has been made in the development of methods and their applications to nonlinear ODEs for finding exact solutions [13, 20].

In this section, in order to obtain travelling wave solutions we apply two procedures: a direct method and the $\frac{G'}{G}$ -expansion method. These class of solutions has physical interest because they yield to compactons, kins, anti-kinks and solitons.

We write equation (10) in the form

$$h'' + \mathcal{A}h' + F(h) = 0, \quad (18)$$

where $\mathcal{A} = -\frac{\alpha}{\lambda}$ and

$$F(h) = \frac{1}{\mu} \left(\frac{\beta}{\lambda} - \frac{1}{\mu} \right) h + \frac{1}{\lambda\mu} g(h). \quad (19)$$

Let us assume that equation (18) has solution of the form $h = aH^b(z)$, where a, b are parameters and $H(z)$ can be: a solution of the Jacobi equation

$$(H')^2 = r + pH^2 + qH^4, \quad (20)$$

with r, p and q constants; an exponential function or a polynomial function.

Case 1: If H is solution of equation (20) we can distinguish three subcases: (i) H is the Jacobi elliptic sine function, $\text{sn}(z, m)$, (ii) H is the Jacobi elliptic cosine function, $\text{cn}(z, m)$, (iii) H is the Jacobi elliptic function of the third kind $\text{dn}(z, m)$.

Subcase (i): If $H(z) = \text{sn}(z, m)$,

$$h(z) = a \text{sn}^b(z, m). \quad (21)$$

By substituting (21) into (18) we obtain

$$\begin{aligned} & -abmJ_2^2J_1^b + a^2bJ_2J_3J_1^{b-1} + ab^2J_2^2J_3^2J_1^{b-2} - abJ_2^2J_3^2J_1^{b-2} \\ & -abJ_3^2J_1^b + F = 0, \end{aligned} \quad (22)$$

where $J_1 = \text{sn}(z, m)$, $J_2 = \text{cn}(z, m)$ and $J_3 = \text{dn}(z, m)$. Taking into account that $\text{cn}^2(z, m) = 1 - \text{sn}^2(z, m) = 1 - (\frac{h}{a})^{\frac{2}{b}}$ and $\text{dn}^2(z, m) = 1 - m\text{sn}^2(z, m) = 1 - m(\frac{h}{a})^{\frac{2}{b}}$ [1],

$$F(h) = \alpha_1 h^{\frac{2}{b}+1} + \alpha_2 h^{-\frac{2}{b}+1} + \alpha_3 h^{-\frac{1}{b}+1} + \alpha_4 h, \quad (23)$$

where

$$\begin{aligned} \alpha_1 &= -\frac{b(b+1)m}{a^{\frac{2}{b}}}, \quad \alpha_2 = -a^{\frac{2}{b}}(b-1)b, \\ \alpha_3 &= -\frac{b\mathcal{A}\sqrt{a^{\frac{2}{b}}-h^{\frac{2}{b}}}\sqrt{a^{\frac{2}{b}}-h^{\frac{2}{b}}m}}{a^{\frac{1}{b}}}, \quad \alpha_4 = b^2(m+1). \end{aligned} \quad (24)$$

Substituting (23)–(24) into (19) we obtain the function $F(h)$ for which (21) is a solution of equation (18). Consequently, an exact solution of equation (1) is

$$u(x, t) = a \text{sn}^b(\mu x - \lambda t, m), \quad (25)$$

where $g(u) = \lambda\mu F(u) - \left(\beta - \frac{\lambda}{\mu}\right)u$ and $F(u)$ is obtained substituting h by u in (23)–(24).

In the following we show three examples of equations which have solutions with physical interest:

- Setting $m = 0$, $a = 1$ and $b = 4$ in (23)–(24) we get

$$F(h) = -12h^{\frac{1}{2}} + 16h - 4\mathcal{A}\left(1 - h^{\frac{1}{2}}\right)^{\frac{1}{2}}h^{\frac{3}{4}},$$

where $h(z) = \text{sn}^4(z, 0)$ is a solution of (18). Taking into account that $\text{sn}(z, 0) = \sin(z)$, we obtain that for $\mu = \lambda = \frac{k}{2}$ and $k = \sqrt{\frac{5}{12}}$

$$u(x, t) = \begin{cases} \sin^4 \left[\frac{k}{2}(x - t) \right], & |x - t| \leq \frac{2\pi}{k}, \\ 0, & |x - t| > \frac{2\pi}{k} \end{cases} \quad (26)$$

is a solution of equation (1) with

$$g(u) = -\beta u + \frac{8}{3}u + \sqrt{\frac{5}{3}}\alpha u^{\frac{3}{4}}\sqrt{1 - u^{\frac{1}{2}}} - \frac{5}{4}u^{\frac{1}{2}}.$$

In Fig. 1 we plot solution (26) which is a sine-type double compacton.

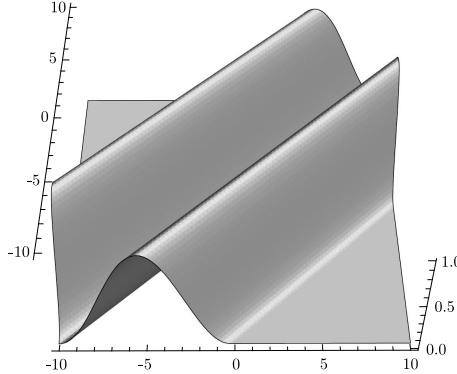


Figure 1. Solution (26)

- Setting $m = 1$, $a = \frac{1}{4}$ and $b = 1$ in (23)–(24) we get

$$F(h) = -32h^3 - 4\mathcal{A} \left(\frac{1}{16} - h^2 \right) + 2h,$$

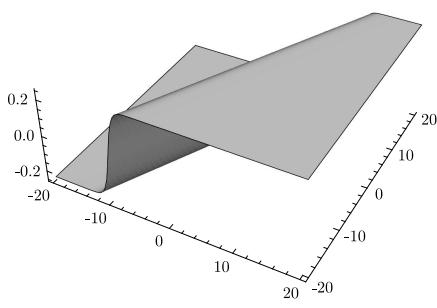
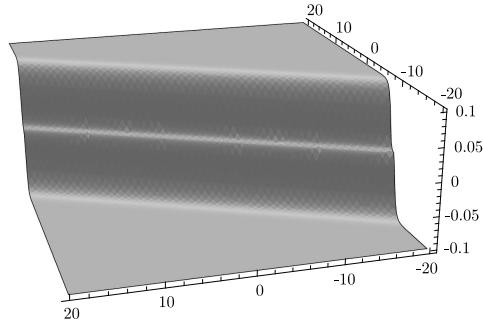
where $h(z) = \frac{1}{4}\text{sn}(z, 1)$ is a solution of (18). Taking into account that $\text{sn}(z, 1) = \tanh(z)$, we obtain that for $\mu = 1$ and $\lambda = \frac{1}{2}$

$$u(x, t) = \frac{1}{4}\tanh \left(x - \frac{t}{2} \right) \quad (27)$$

is a solution of equation (1) with

$$g(u) = -16u^3 + 4\alpha \left(\frac{1}{16} - u^2 \right) + \left(\frac{3}{2} - \beta \right) u.$$

In Fig. 2 we plot solution (27) which describes a kink solution.

**Figure 2.** Solution (27)**Figure 3.** Solution (28)

- Setting $m = 1$, $a = 1$ and $b = 3$ in (23)–(24) we get

$$F(h) = -12h^{\frac{5}{3}} + 18h - 3\mathcal{A}\left(1 - h^{\frac{2}{3}}\right)h^{\frac{2}{3}} - 6h^{\frac{1}{3}},$$

where $h(z) = \text{sn}^3(z, 1)$ is a solution of (18). Taking into account that $\text{sn}(z, 1) = \tanh(z)$, we obtain that for $\mu = 1$ and $\lambda = \frac{1}{2}$

$$u(x, t) = \tanh^3\left(x - \frac{t}{2}\right) \quad (28)$$

is a solution of equation (1) with

$$g(u) = -6u^{\frac{5}{3}} + \frac{19u}{2} + 3\alpha(1 - u^{\frac{2}{3}})u^{\frac{2}{3}} - 3u^{\frac{1}{3}} - \beta u.$$

In Fig. 3 we plot solution (28) which describes an anti-kink solution.

Subcase (ii): If $H(z) = \text{cn}(z, m)$,

$$h(z) = a \text{cn}^b(z, m). \quad (29)$$

By substituting (29) into (18) we obtain

$$\begin{aligned} & ab^2 J_2^{b-2} J_3^2 J_1^2 - ab J_2^{b-2} J_3^2 J_1^2 + abm J_2^b J_1^2 - a^2 b J_2^{b-1} J_3 J_1 \\ & - ab J_2^b J_3^2 + F = 0, \end{aligned} \quad (30)$$

where $J_1 = \text{sn}(z, m)$, $J_2 = \text{cn}(z, m)$ and $J_3 = \text{dn}(z, m)$. Taking into account that $\text{dn}^2(z, m) = 1 - m \text{sn}^2(z, m)$, $\text{sn}^2(z, m) = 1 - \text{cn}^2(z, m)$ and $\text{cn}(z, m) = (\frac{h}{a})^{\frac{1}{b}}$ [1] we obtain

$$F(h) = \beta_1 h^{\frac{2}{b}+1} + \beta_2 h^{-\frac{2}{b}+1} + \beta_3 h^{-\frac{1}{b}+1} + \beta_4 h, \quad (31)$$

where

$$\begin{aligned} \beta_1 &= a^{-\frac{2}{b}} b (b+1) m, \quad \beta_2 = a^{\frac{2}{b}} (b-1) b (m-1), \\ \beta_3 &= a^{-\frac{1}{b}} b \mathcal{A} \sqrt{a^{\frac{2}{b}} - h^{\frac{2}{b}}} \sqrt{\left(h^{\frac{2}{b}} - a^{\frac{2}{b}}\right) m + a^{\frac{2}{b}}}, \quad \beta_4 = -b^2 (2m-1). \end{aligned} \quad (32)$$

Substituting (31)–(32) into (19) we obtain the function $F(h)$ for which (29) is a solution of equation (18). Consequently, an exact solution of equation (1), where $g(u) = \lambda\mu F(u) - \left(\beta - \frac{\lambda}{\mu}\right)u$ and $F(u)$ is obtained via substituting h by u into (31)–(32), is

$$u(x, t) = a \operatorname{cn}^b(\mu x - \lambda t, m). \quad (33)$$

In the following we give two examples of equations which are solutions with physical interest:

- Setting $m = 0$, $a = 1$ and $b = 4$ in (31)–(32) we get

$$F(h) = 16h - 12h^{\frac{1}{2}} + 4\mathcal{A}h^{\frac{3}{4}}\sqrt{1 - h^{\frac{1}{2}}},$$

where $h(z) = \operatorname{cn}^4(z, 0)$ is a solution of (18). Taking into account that $\operatorname{cn}(z, 0) = \cos(z)$, we obtain that for $\mu = \lambda = \frac{k}{2}$ and $k = \sqrt{\frac{5}{12}}$

$$u(x, t) = \begin{cases} \cos^4\left[\frac{k}{2}(x - t)\right], & |x - t| \leq \frac{\pi}{k}, \\ 0, & |x - t| > \frac{\pi}{k}, \end{cases} \quad (34)$$

is a solution of equation (1) with

$$g(u) = \frac{8u}{3} - \sqrt{\frac{5}{3}}\alpha u^{\frac{3}{4}}\sqrt{1 - u^{\frac{1}{2}}} - \frac{5}{4}u^{\frac{1}{2}} - \beta u.$$

In Fig. 4 we plot solution (34) which is a compacton solution with a single peak.

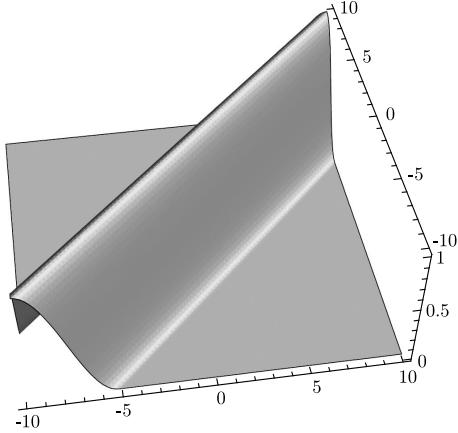


Figure 4. Solution (34)

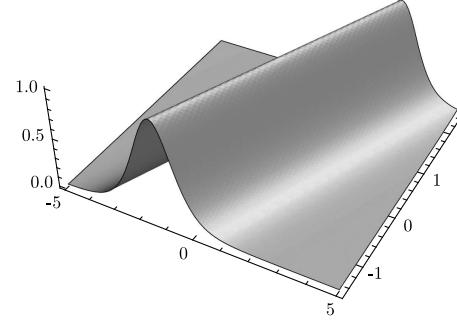


Figure 5. Solution (35)

- Setting $m = 1$, $a = 1$ and $b = 2$ in (31)–(32) we get

$$F(h) = h^2 + (2\mathcal{A}\sqrt{1 - h} - 4)h,$$

where $h(z) = \operatorname{cn}^2(z, 0)$ is a solution of (18). Taking into account that $\operatorname{cn}(z, 0) = \operatorname{sech}(z)$, we obtain that for $\lambda = \mu = 1$,

$$u(x, t) = \operatorname{sech}^2(x - t) \quad (35)$$

is a solution of equation (1) with $g(u) = 6u^2 - (2\alpha\sqrt{1-u} - \beta - 3)u$. In Fig. 5 we plot solution (35) which describes a soliton.

Case 2: If $H(z) = \exp(z)$

$$h(z) = a \exp(bz). \quad (36)$$

By substituting (36) into (18) we obtain $a b^2 e^{b z} + a^2 b e^{b z} + F = 0$. Taking into account that $\exp(bz) = \frac{h}{a}$ we obtain

$$F(h) = -b(b+a)h. \quad (37)$$

Substituting (37) into (19) we obtain the function $g(h)$ for which (36) is solution of equation (10). Consequently, an exact solution of equation (1), where $g(u) = \lambda\mu F(u) - \left(\beta - \frac{\lambda}{\mu}\right)u$ and $F(u)$ is obtained substituting h by u in (37), is

$$u(x, t) = a \exp[b(\mu x - \lambda t)].$$

Case 3: If $H(z) = az + b$,

$$h(z) = (az + b)^n. \quad (38)$$

By substituting (38) into (18) we obtain

$$a^2 n (az + b)^{n-1} + a^2 n^2 (az + b)^{n-2} - a^2 n (az + b)^{n-2} + F = 0.$$

Taking into account that $az + b = h^{\frac{1}{n}}$ we obtain

$$F(h) = -a^2 (n-1) nh^{-\frac{2}{n}+1} - a^2 nh^{-\frac{1}{n}+1}. \quad (39)$$

Substituting (39) into (19) we obtain the function $g(h)$ for which (38) is solution of equation (10). Consequently, an exact solution of equation (1), where $g(u) = \lambda\mu F(u) - \left(\beta - \frac{\lambda}{\mu}\right)u$ and $F(u)$ is obtained substituting h by u in (39), is

$$u(x, t) = [a(\mu x - \lambda t) + b]^n.$$

In the same way, we can obtain functions $g(h)$ for which functions $\text{dn}(z, m)$, $\text{cd}(z, m)$, $\text{sd}(z, m)$, $\text{nd}(z, m)$, $\text{dc}(z, m)$, $\text{nc}(z, m)$, $\text{sc}(z, m)$, $\text{ns}(z, m)$, $\text{ds}(z, m)$ and $\text{cs}(z, m)$ (see [1]), are solutions of equation (1).

3.1 G'/G -expansion method

We consider the generalized Benjamin-Bona-Mahony-Burgers equation (1) and we look for travelling wave solutions of this equation. In this case the reduced equation becomes

$$\lambda\mu^2 h'' - \alpha\mu^2 h' + (\beta\mu - \lambda)h + \mu g(h) + k = 0. \quad (40)$$

To apply the $\frac{G'}{G}$ -expansion method to equation (40) we suppose that the solutions can be expressed by a polynomial in $\frac{G'}{G}$ in the form

$$h = \sum_{i=0}^n a_i \left(\frac{G'}{G} \right)^i, \quad (41)$$

where $G = G(z)$ satisfies the linear second order ODE

$$G''(z) + \omega G'(z) + \zeta G(z) = 0, \quad (42)$$

a_i , $i = 0, \dots, n$, α and β are constants to be determined later, $a_n \neq 0$.

The general solutions of equation (42) are:

- If $\omega^2 - 4\zeta > 0$,

$$G(z) = c_1 \cosh \left(\frac{z\omega}{2} - \frac{1}{2}z\sqrt{\omega^2 - 4\zeta} \right) + c_2 \cosh \left(\frac{\omega z}{2} + \frac{1}{2}\sqrt{\omega^2 - 4\zeta}z \right) \\ - c_1 \sinh \left(\frac{z\omega}{2} - \frac{1}{2}z\sqrt{\omega^2 - 4\zeta} \right) - c_2 \sinh \left(\frac{\omega z}{2} + \frac{1}{2}\sqrt{\omega^2 - 4\zeta}z \right). \quad (43)$$

- If $\omega^2 - 4\zeta < 0$,

$$G(z) = \left[c_2 \cos \left(\frac{1}{2}z\sqrt{4\zeta - \omega^2} \right) + c_1 \sin \left(\frac{1}{2}z\sqrt{4\zeta - \omega^2} \right) \right] \\ \times \left(\cosh \left(\frac{z\omega}{2} \right) - \sinh \left(\frac{z\omega}{2} \right) \right). \quad (44)$$

- If $\omega^2 = 4\zeta$,

$$G(z) = (c_2 + c_1 z) \left(\cosh \left(\frac{z\omega}{2} \right) - \sinh \left(\frac{z\omega}{2} \right) \right). \quad (45)$$

In order to determine the positive number n in (41) we concentrate our attention on the leading terms of (40). These are the terms that lead to the least positive p when substituting a monomial $h = \frac{a}{z^p}$ in all the items of this equation, [14]. The homogeneous balance between the leading terms provides us with the value of n . To find them we substitute $h = \frac{a}{z^p}$ in all the items of this equation. We compare $g(h) = h^m$ and h'' : $p + 2 = mp \Rightarrow m = 2$ or $m = 3$.

3.2 G'/G -expansion method for $g(h) = h^2$

By using (41) and (42) we obtain

$$h = a_n \left(\frac{G'}{G} \right)^n + \dots, \quad h^2 = a_n^2 \left(\frac{G'}{G} \right)^{2n} + \dots, \\ h'' = n(n+1)a_n \left(\frac{G'}{G} \right)^{n+2} + \dots. \quad (46)$$

Considering the homogeneous balance between h'' and h^2 in (40), based on (46), we require that $n + 2 = 2n \Rightarrow n = 2$, we can write (41) as

$$h = a_0 + a_1 \left(\frac{G'}{G} \right) + a_2 \left(\frac{G'}{G} \right)^2, \quad a_2 \neq 0. \quad (47)$$

Substituting the general solutions of (42) in (47), we respectively obtain from (43), (44) and (45):

$$\begin{aligned} h_1(z) &= \frac{a_2 \omega^2}{4} - \frac{a_1 \omega}{2} + a_0 + \frac{\sqrt{\omega^2 - 4\zeta}}{8} (-4a_2 \omega + 4a_1) H_1 \\ &\quad + \frac{\omega^2 - 4\zeta}{8} (2a_2) (H_1)^2, \end{aligned} \quad (48)$$

$$\begin{aligned} h_2(z) &= \frac{a_2 \omega^2}{4} - \frac{a_1 \omega}{2} + a_0 + \frac{a_2}{4} (4\zeta - \omega^2) (H_2)^2 \\ &\quad + \frac{a_1 - a_2 \omega}{2} \sqrt{4\zeta - \omega^2} (H_2)^3, \end{aligned} \quad (49)$$

$$h_3(z) = \frac{a_2 \omega^2}{4} - \frac{a_1 \omega}{2} + a_0 + \frac{a_2 c_1^2}{(c_1 z + c_2)^2} + \frac{(-a_2 \omega + a_1) c_1}{(c_1 z + c_2)}, \quad (50)$$

where

$$\begin{aligned} H_1(z) &= \frac{c_2 \cosh(\frac{1}{2}\sqrt{\omega^2 - 4\zeta}z) + c_1 \sinh(\frac{1}{2}\sqrt{\omega^2 - 4\zeta}z)}{c_1 \cosh(\frac{1}{2}\sqrt{\omega^2 - 4\zeta}z) + c_2 \sinh(\frac{1}{2}\sqrt{\omega^2 - 4\zeta}z)}, \\ H_2(z) &= \frac{c_2 \cos(\frac{1}{2}z\sqrt{4\zeta - \omega^2}) + c_1 \sin(\frac{1}{2}z\sqrt{4\zeta - \omega^2})}{c_1 \cos(\frac{1}{2}z\sqrt{4\zeta - \omega^2}) - c_2 \sin(\frac{1}{2}z\sqrt{4\zeta - \omega^2})}. \end{aligned}$$

In the following we determine a_i , $i = 0, 1, 2$. From (47) we calculate h_j^2 , h_j' and h_j'' , $j = 1, 2, 3$, and we substitute this expression in equation (40). Equating each coefficient of $(G'/G)^i$ to zero, yields a set of simultaneous algebraic equations for a_i , α , β , λ , μ , ω , ζ and k . Solving this system, we obtain the set of solutions:

$$\begin{aligned} a_0 &= \frac{1}{50} \left(\frac{3\mu\alpha^2}{\lambda} - 30\mu\omega\alpha - 75\lambda\mu\omega^2 - 25\beta + \frac{25\lambda}{\mu} \right), \\ a_1 &= -\frac{6}{5}\mu(\alpha + 5\lambda\omega), \quad a_2 = -6\lambda\mu, \quad \zeta = \frac{\omega^2}{4} - \frac{\alpha^2}{100\lambda^2}, \\ k &= -\frac{9\mu^3\alpha^4}{625\lambda^2} - \frac{\beta\lambda}{2} + \frac{\beta^2\mu}{4} + \frac{\lambda^2}{4\mu}, \end{aligned} \quad (51)$$

where α , β , λ , μ and ω are arbitrary constants.

Due to fact that $4\zeta - \omega^2 = \frac{\alpha^2}{25\lambda^2} > 0$, substituting (51) in (48) we obtain the following solution of equation (40)

$$h_1(z) = \frac{25\lambda^2 - 25\beta\mu\lambda + 3\alpha^2\mu^2 - 6\alpha^2\mu^2\lambda F_1(z) - 3\alpha^2\mu^2 F_1^2(z)}{50\lambda\mu}, \quad (52)$$

where $g(h) = h^2$ and k given by (51),

$$F_1(z) = \frac{c_2 \cosh\left(\frac{\alpha}{10\lambda}z\right) + c_1 \sinh\left(\frac{\alpha}{10\lambda}z\right)}{c_1 \cosh\left(\frac{\alpha}{10\lambda}z\right) + c_2 \sinh\left(\frac{\alpha}{10\lambda}z\right)}.$$

Substituting h_1 into (8) we have travelling wave solutions of the generalized Benjamin-Bona-Mahony-Burgers equation (1) with $g(u) = u^2$:

$$u_1(x, t) = \frac{25\lambda^2 - 25\beta\mu\lambda + 3\alpha^2\mu^2 - 6\alpha^2\mu^2\lambda H_1(x, t) - 3\alpha^2\mu^2 H_1^2(x, t)}{50\lambda\mu}, \quad (53)$$

where

$$H_1(x, t) = \frac{c_2 \cosh\left(\frac{\alpha}{10\lambda}(\mu x - \lambda t)\right) + c_1 \sinh\left(\frac{\alpha}{10\lambda}(\mu x - \lambda t)\right)}{c_1 \cosh\left(\frac{\alpha}{10\lambda}(\mu x - \lambda t)\right) + c_2 \sinh\left(\frac{\alpha}{10\lambda}(\mu x - \lambda t)\right)}.$$

From (53) for different values of α and λ we can obtain solutions of equation (1) which do not appear in [8].

If $\omega^2 = 4\zeta$, then $\alpha = 0$. In this case equation (40) is

$$\lambda\mu^2 h'' + (\beta\mu - \lambda)h + \mu h^2 + k = 0. \quad (54)$$

Comparing our results and Kudryashov's results [12] for equation (54) then all solutions of equation (54) can be obtained from solutions obtained by Kudryashov if we use different values of the constant and some additional transformations.

3.3 G'/G -expansion method for $g(h) = h^3$

By using (41) and (42) we obtain

$$\begin{aligned} h &= a_n \left(\frac{G'}{G}\right)^n + \dots, \quad h^3 = a_n^3 \left(\frac{G'}{G}\right)^{3n} + \dots, \\ h'' &= n(n+1)a_n \left(\frac{G'}{G}\right)^{n+2} + \dots \end{aligned} \quad (55)$$

Considering the homogeneous balance between h'' and h^3 in (40), based on (55), we require that $n+2 = 3n \Rightarrow n = 1$, we can write (41) as

$$h = a_0 + a_1 \left(\frac{G'}{G}\right), \quad a_1 \neq 0. \quad (56)$$

Substituting the general solutions of (42) in (56) we respectively obtain from (43), (44) and (45):

$$\begin{aligned} h_4(z) &= -\frac{a_1\omega}{2} + a_0 \\ &+ \frac{\sqrt{\omega^2 - 4\zeta}}{2} a_1 \frac{c_2 \cosh(\frac{1}{2}\sqrt{\omega^2 - 4\zeta}z) + c_1 \sinh(\frac{1}{2}\sqrt{\omega^2 - 4\zeta}z)}{c_1 \cosh(\frac{1}{2}\sqrt{\omega^2 - 4\zeta}z) + c_2 \sinh(\frac{1}{2}\sqrt{\omega^2 - 4\zeta}z)}, \end{aligned} \quad (57)$$

$$h_5(z) = -\frac{a_1\omega}{2} + a_0 + \frac{a_1}{2} \sqrt{4\zeta - \omega^2} (H_2)^3, \quad (58)$$

$$h_6(z) = -\frac{a_1\omega}{2} + a_0 + \frac{a_1 c_1}{(c_1 z + c_2)}, \quad (59)$$

where

$$H_2(z) = \frac{c_2 \cos\left(\frac{1}{2}z\sqrt{4\zeta - \omega^2}\right) + c_1 \sin\left(\frac{1}{2}z\sqrt{4\zeta - \omega^2}\right)}{c_1 \cos\left(\frac{1}{2}z\sqrt{4\zeta - \omega^2}\right) - c_2 \sin\left(\frac{1}{2}z\sqrt{4\zeta - \omega^2}\right)}.$$

In order to determine a_0 and a_1 we consider h_i , $i = 4, 5, 6$, we calculate h_i^3 , h'_i and h''_i and we substitute this expression in equation (40). Equating each coefficient of $\left(\frac{G'}{G}\right)^n$, $n = 0, 1$ to zero, yields a set of simultaneous algebraic equations for $a_i, \alpha, \beta, \lambda, \mu, \omega, \zeta$ and k . Solving this system, we obtain the set of solutions:

$$a_0 = \frac{i\sqrt{\mu}(\alpha + 3\lambda\omega)}{3\sqrt{2}\sqrt{\lambda}}, \quad a_1 = i\sqrt{2}\sqrt{\lambda}\sqrt{\mu}, \quad (60)$$

$$\zeta = \frac{3\mu^2\omega^2\lambda^2 + 6\lambda^2 - 6\beta\mu\lambda + \alpha^2\mu^2}{12\lambda^2\mu^2}, \quad (61)$$

$$k = -\frac{i\sqrt{2}\alpha\sqrt{\mu}(9\lambda^2 - 9\beta\mu\lambda + 2\alpha^2\mu^2)}{27\lambda^{3/2}}, \quad (62)$$

where $\alpha, \beta, \lambda, \mu$ and ω are arbitrary constants.

Substituting (60)–(61) in (57)–(59) we obtain the following solutions for equation (40) with $g(h) = h^3$ and k given by (62):

If $\omega^2 - 4\zeta > 0$:

$$h_4(z) = \frac{i\sqrt{\mu}}{3\sqrt{2}\sqrt{\lambda}} \left(\alpha + \sqrt{3}\lambda \sqrt{\frac{-6\lambda^2 + 6\beta\mu\lambda - \alpha^2\mu^2}{\lambda^2\mu^2}} F_4(z) \right),$$

where

$$F_4(z) = \frac{c_2 \cosh\left(\frac{z}{2\sqrt{3}\lambda} \sqrt{-\alpha^2 - \frac{6\lambda(\lambda - \beta\mu)}{\mu^2}}\right) + c_1 \sinh\left(\frac{z}{2\sqrt{3}\lambda} \sqrt{-\alpha^2 - \frac{6\lambda(\lambda - \beta\mu)}{\mu^2}}\right)}{c_1 \cosh\left(\frac{z}{2\sqrt{3}\lambda} \sqrt{-\alpha^2 - \frac{6\lambda(\lambda - \beta\mu)}{\mu^2}}\right) + c_2 \sinh\left(\frac{z}{2\sqrt{3}\lambda} \sqrt{-\alpha^2 - \frac{6\lambda(\lambda - \beta\mu)}{\mu^2}}\right)}.$$

If $\omega^2 - 4\zeta < 0$:

$$h_5(z) = \frac{i\sqrt{\mu}}{3\sqrt{2}\sqrt{\lambda}} \left(\alpha + \sqrt{3}\lambda \sqrt{\frac{6\lambda^2 - 6\beta\mu\lambda + \alpha^2\mu^2}{\lambda^2\mu^2}} F_5(z) \right),$$

where

$$F_5(z) = \frac{c_1 \cos\left(\frac{z}{2\sqrt{3}\lambda} \sqrt{\alpha^2 + \frac{6\lambda(\lambda - \beta\mu)}{\mu^2}}\right) - c_2 \sin\left(\frac{z}{2\sqrt{3}\lambda} \sqrt{\alpha^2 + \frac{6\lambda(\lambda - \beta\mu)}{\mu^2}}\right)}{c_2 \cos\left(\frac{z}{2\sqrt{3}\lambda} \sqrt{\alpha^2 + \frac{6\lambda(\lambda - \beta\mu)}{\mu^2}}\right) + c_1 \sin\left(\frac{z}{2\sqrt{3}\lambda} \sqrt{\alpha^2 + \frac{6\lambda(\lambda - \beta\mu)}{\mu^2}}\right)}.$$

If $\omega^2 = 4\zeta$, then $6\lambda^2 - 6\beta\lambda\mu + \alpha^2\mu^2 = 0$:

$$h_6(z) = \frac{i(c_2\alpha + c_1z\alpha + 6c_1\lambda)\sqrt{\mu}}{3\sqrt{2}(c_2 + c_1z)\sqrt{\lambda}}.$$

Substituting h_4 , h_5 and h_6 into (8) we have three classes of travelling wave solutions of the generalized Benjamin-Bona-Mahony-Burgers equation (1) with $g(u) = u^3$:

If $\omega^2 - 4\zeta > 0$:

$$u_4(x, t) = \frac{i\sqrt{\mu}}{3\sqrt{2}\sqrt{\lambda}} \left(\alpha + \sqrt{3}\lambda \sqrt{\frac{-6\lambda^2 + 6\beta\mu\lambda - \alpha^2\mu^2}{\lambda^2\mu^2}} H_4(x, t) \right), \quad (63)$$

where

$$H_4(x, t) = \frac{c_2 \cosh \left(\frac{\mu x - \lambda t}{2\sqrt{3}\lambda} \sqrt{-\alpha^2 - \frac{6\lambda(\lambda - \beta\mu)}{\mu^2}} \right) + c_1 \sinh \left(\frac{\mu x - \lambda t}{2\sqrt{3}\lambda} \sqrt{-\alpha^2 - \frac{6\lambda(\lambda - \beta\mu)}{\mu^2}} \right)}{c_1 \cosh \left(\frac{\mu x - \lambda t}{2\sqrt{3}\lambda} \sqrt{-\alpha^2 - \frac{6\lambda(\lambda - \beta\mu)}{\mu^2}} \right) + c_2 \sinh \left(\frac{\mu x - \lambda t}{2\sqrt{3}\lambda} \sqrt{-\alpha^2 - \frac{6\lambda(\lambda - \beta\mu)}{\mu^2}} \right)}.$$

If $\omega^2 - 4\zeta < 0$:

$$u_5(x, t) = \frac{i\sqrt{\mu}}{3\sqrt{2}\sqrt{\lambda}} \left(\alpha + \sqrt{3}\lambda \sqrt{\frac{6\lambda^2 - 6\beta\mu\lambda + \alpha^2\mu^2}{\lambda^2\mu^2}} H_5(x, t) \right),$$

where

$$H_5(x, t) = \frac{c_1 \cos \left(\frac{\mu x - \lambda t}{2\sqrt{3}\lambda} \sqrt{\alpha^2 + \frac{6\lambda(\lambda - \beta\mu)}{\mu^2}} \right) - c_2 \sin \left(\frac{\mu x - \lambda t}{2\sqrt{3}\lambda} \sqrt{\alpha^2 + \frac{6\lambda(\lambda - \beta\mu)}{\mu^2}} \right)}{c_2 \cos \left(\frac{\mu x - \lambda t}{2\sqrt{3}\lambda} \sqrt{\alpha^2 + \frac{6\lambda(\lambda - \beta\mu)}{\mu^2}} \right) + c_1 \sin \left(\frac{\mu x - \lambda t}{2\sqrt{3}\lambda} \sqrt{\alpha^2 + \frac{6\lambda(\lambda - \beta\mu)}{\mu^2}} \right)}.$$

If $6\lambda^2 - 6\beta\lambda\mu + \alpha^2\mu^2 = 0$:

$$u_6(x, t) = \frac{i(c_2\alpha + c_1\alpha(\mu x - \lambda t) + 6c_1\lambda)\sqrt{\mu}}{3\sqrt{2}(c_2 + c_1(\mu x - \lambda t))\sqrt{\lambda}}.$$

4 Conclusions

In this paper, the complete Lie group classification for a generalized Benjamin-Bona-Mahony-Burgers equation (1) has been obtained. The corresponding reduced equations have been derived from the optimal system of subalgebras.

We consider ODE (18) and we determine the functional forms $F(u)$ for which this equation admits solutions in terms of the Jacobi elliptic functions. From this equation, we have derived travelling wave solutions for equation (1). Among them we found solitons, kinks, anti-kinks and compactons.

We also determine the functions $g(h) = h^m$ for which we can apply the $\frac{G'}{G}$ -expansion method. By using this method we obtain more exact solutions of equation (1) with $g(u) = u^2$ and $g(u) = u^3$.

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Special polynomials and exact solutions of the dispersive water wave and modified Boussinesq equations

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Exact solutions of the dispersive water wave and modified Boussinesq equations are expressed in terms of special polynomials associated with rational solutions of the fourth Painlevé equation, which arises as generalized scaling reductions of these equations. Generalized solutions that involve an infinite sequence of arbitrary constants are also derived which are analogues of generalized rational solutions for the Korteweg-de Vries, Boussinesq and nonlinear Schrödinger equations.

1 Introduction

In this paper we are concerned with special polynomials associated with exact solutions of the dispersive water wave (DWW) equation

$$U_{tt} + 2U_tU_{xx} + 4U_xU_{xt} + 6U_x^2U_{xx} - U_{xxxx} = 0, \quad (1)$$

which is a soliton equation solvable by inverse scattering [38,39], sometimes known as “Kaup’s higher-order wave equation” (cf. [50]), and the modified Boussinesq equation

$$\frac{1}{3}U_{tt} - 2U_tU_{xx} - 6U_x^2U_{xx} + U_{xxxx} = 0, \quad (2)$$

which also is a soliton equation solvable by inverse scattering [51]. These equations may be written in the non-local form (by setting $U_x = u$)

$$u_{tt} - 2u_{xx}\partial_x^{-1}(u_t) + 4uu_{xt} + 6u_xu_t + 2(u^3)_{xx} - u_{xxxx} = 0, \quad (3)$$

$$\frac{1}{3}u_{tt} - 2u_tu_x + 2u_{xx}\partial_x^{-1}(u_t) - 2(u^3)_{xx} + u_{xxxx} = 0, \quad (4)$$

where $(\partial_x^{-1}f)(x) = \int_x^\infty f(y) dy$, respectively, which is the form in which they arise in physical applications.

The DWW equation (1) can be derived from the classical Boussinesq system

$$\eta_t + v_x + \eta\eta_x = 0, \quad v_t + (\eta v)_x + \eta_{xxx} = 0, \quad (5)$$

which arise in the description of surface waves propagating in shallow water [14,38, 39,56]. Indeed Broer [14] called the system “the oldest, simplest and most widely known set of equations ... which are the Boussinesq equations proper”. Hirota and Satsuma [32] showed that there is a “Miura type” transformation relating solutions of the modified Boussinesq equation (2) to solutions of the Boussinesq equation

$$u_{tt} + u_{xx} + (u^2)_{xx} + u_{xxxx} = 0. \quad (6)$$

There has been considerable interest in partial differential equations solvable by inverse scattering, the *soliton equations*, since the discovery in 1967 by Gardner, Greene, Kruskal and Miura [29] of the method for solving the initial value problem for the Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0. \quad (7)$$

Clarkson and Ludlow [24] show that the generalized Boussinesq equation

$$U_{tt} + pU_tU_{xx} + qU_xU_{xt} + rU_x^2U_{xx} + U_{xxxx} = 0, \quad (8)$$

with p , q and r constants, satisfies the necessary conditions of the Painlevé conjecture due to Ablowitz, Ramani and Segur [2,3] to be solvable by inverse scattering in two cases: (i), if $q = 2p$ and $r = \frac{3}{2}p^2$, when (8) is equivalent to the DWW equation (1), and (ii), if $q = 0$ and $r = -\frac{1}{2}p^2$, when (8) is equivalent to the modified Boussinesq equation (2).

During the past thirty years or so there has been much interest in rational solutions of the soliton equations. Further applications of rational solutions to soliton equations include the description of explode-decay waves [43] and vortex solutions of the complex sine-Gordon equation [10,47]. The idea of studying the motion of poles of solutions of the KdV equation (7) is attributed to Kruskal [40]. Airault, McKean and Moser [8] studied the motion of the poles of rational solutions of the KdV equation (7) and the Boussinesq equation (6). Further they related the motion to an integrable many-body problem, the Calogero-Moser system with constraints; see also [7, 16]. Ablowitz and Satsuma [4] derived some rational solutions of the KdV equation (7) and the Boussinesq equation (6) by finding a long-wave limit of the known N -soliton solutions of these equations. Studies of rational solutions of other soliton equations include for the Boussinesq equation [22,28,49] and for the nonlinear Schrödinger (NLS) equation [21,33,43]

$$iu_t + u_{xx} \pm 2|u|^2u = 0. \quad (9)$$

Ablowitz and Segur [5] demonstrated a close relationship between completely integrable partial differential equations solvable by inverse scattering and the Painlevé equations. For example the second Painlevé equation (P_{II}),

$$w'' = 2w^3 + zw + \alpha, \quad (10)$$

where $' \equiv d/dz$ and α is an arbitrary constant, arises as a scaling reduction of the KdV equation (7), see [5], and the fourth Painlevé equation (P_{IV}),

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}, \quad (11)$$

where α and β are arbitrary constants, arises as scaling reductions of the Boussinesq equation (6) and the NLS equation (9). Consequently special solutions of these equations can be expressed in terms of solutions of P_{II} and P_{IV} .

The six Painlevé equations ($P_{\text{I}}\text{--}P_{\text{VI}}$) are nonlinear ordinary differential equations, the solutions of which are called the *Painlevé transcedents*, were discovered about a hundred years ago by Painlevé, Gambier and their colleagues whilst studying which second-order ordinary differential equations have the property that the solutions have no movable branch points, i.e. the locations of multi-valued singularities of any of the solutions are independent of the particular solution chosen and so are dependent only on the equation; this is now known as the *Painlevé property*. Painlevé, Gambier *et al.* showed that there were fifty canonical equations with this property, forty four are either integrable in terms of previously known functions, such as elliptic functions or are equivalent to linear equations, or are reducible to one of six new nonlinear ordinary differential equations, which define new transcendental functions (cf. [34]). The Painlevé equations can be thought of as nonlinear analogues of the classical special functions (cf. [20, 26, 35, 53]). Indeed Iwasaki, Kimura, Shimomura and Yoshida [35] characterize the Painlevé equations as “the most important nonlinear ordinary differential equations” and state that “many specialists believe that during the twenty-first century the Painlevé functions will become new members of the community of special functions”. Further Umemura [53] states that “Kazuo Okamoto and his circle predict that in the 21st century a new chapter on Painlevé equations will be added to Whittaker and Watson’s book”.

Vorob’ev [55] and Yablonskii [57] expressed the rational solutions of P_{II} (10) in terms of certain special polynomials, which are now known as the *Yablonskii–Vorob’ev polynomials*. Okamoto [46] derived analogous special polynomials, which are now known as the *Okamoto polynomials*, related to some of the rational solutions of P_{IV} (11). Subsequently Okamoto’s results were generalized by Noumi and Yamada [45] who showed that all rational solutions of P_{IV} can be expressed in terms of logarithmic derivatives of two sets of special polynomials, called the *generalized Hermite polynomials* and the *generalized Okamoto polynomials* (see Section 2 below). Clarkson and Mansfield [25] investigated the locations of the roots of the Yablonskii–Vorob’ev polynomials in the complex plane and showed that these roots have a very regular, approximately triangular structure. The structure of the (complex) roots of the generalized Hermite and generalized Okamoto polynomials is described in [18], which respectively have an approximate rectangular structure and a combination of approximate rectangular and triangular structures. The term “approximate” is used since the patterns are not exact triangles and rectangles as the roots lie on arcs rather than straight lines.

In this paper our interest is in exact solutions and associated polynomials of the special case of the DWW equation (1) and the modified Boussinesq equation (2), both of which have generalized scaling reductions to P_{IV} (11). Consequently solutions of (1) and (2) can be obtained in terms the generalized Hermite and generalized Okamoto polynomials. Further some of these solutions whose derivatives decay as $x \rightarrow \pm\infty$, are generalized to give more general solutions of the DWW equation (1) and the modified Boussinesq equation (2). These solutions are analogues of the rational solutions of the KdV equation [4, 7, 8, 16], the Boussinesq equation [22, 28, 49] and the NLS equation [21, 33]; see also [19]. This paper is organized as follows. In Section 2 we review the special polynomials associated with rational solutions of P_{IV} (11). In Sections 3 and 4 we use the special polynomials discussed in Section 2 to derive special polynomials and associated solutions of the DWW equation (1) and the modified Boussinesq equation (2), respectively. We also derive generalized solutions which involve an infinite number of arbitrary constants. All exact solutions of equations (1) and (2), which are rational solutions of equations (3) and (4), which are described here are expressed as Wronskians of polynomials. Finally in Section 5 we discuss our results.

2 Rational solutions of P_{IV}

Simple rational solutions of P_{IV} (11) are

$$w_1(z; \pm 2, -2) = \pm 1/z, \quad w_2(z; 0, -2) = -2z, \quad w_3(z; 0, -\frac{2}{9}) = -\frac{2}{3}z. \quad (12)$$

It is known that there are three sets of rational solutions of P_{IV} , which have the solutions (12) as the simplest members. These sets are known as the “ $-1/z$ hierarchy”, the “ $-2z$ hierarchy” and the “ $-\frac{2}{3}z$ hierarchy”, respectively (cf. [9]). The “ $-1/z$ hierarchy” and the “ $-2z$ hierarchy” form the set of rational solutions of P_{IV} (11) with parameters given by (13a) and the “ $-\frac{2}{3}z$ hierarchy” forms the set with parameters given by (13b). The rational solutions of P_{IV} (11) with parameters given by (13a) lie at the vertexes of the “Weyl chambers” and those with parameters given by (13b) lie at the centres of the “Weyl chamber” [54].

Theorem 1. P_{IV} (11) has rational solutions if and only if the parameters α and β are given by either

$$\alpha = m, \quad \beta = -2(2n - m + 1)^2, \quad (13a)$$

or

$$\alpha = m, \quad \beta = -2(2n - m + \frac{1}{3})^2, \quad (13b)$$

with $m, n \in \mathbb{Z}$. For each given m and n there exists only one rational solution of P_{IV} with parameters given by (13).

Proof. See Lukashevich [41], Gromak [30] and Murata [42]; also Bassom, Clarkson and Hicks [9], Gromak, Laine and Shimomura [31, §26], Umemura and Watanabe [54]. \blacksquare

In a comprehensive study of properties of solutions of P_{IV} (11), Okamoto [46] introduced two sets of polynomials associated with rational solutions of P_{IV} (11), analogous to the Yablonskii–Vorob’ev polynomials associated with rational solutions of P_{II} . Noumi and Yamada [45] generalized Okamoto’s results and introduced the *generalized Hermite polynomials*, which are defined in Definition 1, and the *generalized Okamoto polynomials*, which are defined in Definition 2; see also [18, 19]. Kajiwara and Ohta [36] expressed the generalized Hermite and generalized Okamoto polynomials in terms of Schur functions in the form of determinants, which is how they are defined below.

Definition 1. The *generalized Hermite polynomial* $H_{m,n}(z)$ is defined by

$$H_{m,n}(z) = \mathcal{W}(H_m(z), H_{m+1}(z), \dots, H_{m+n-1}(z)), \quad (14)$$

and $H_{m,0}(z) = H_{0,n}(z) = 1$, for $m, n \geq 1$, with $H_k(z)$ the k th Hermite polynomial.

The polynomials, $H_{m,n}(z)$, defined by (14) are called the generalized Hermite polynomials since $H_{m,1}(z) = H_m(z)$ and $H_{1,m}(z) = i^{-m} H_m(iz)$, where $H_m(z)$ is the standard Hermite polynomial defined by

$$H_m(z) = (-1)^m \exp(z^2) \frac{d^m}{dz^m} \{ \exp(-z^2) \}$$

or alternatively through the generating function

$$\sum_{m=0}^{\infty} \frac{H_m(z) \xi^m}{m!} = \exp(2\xi z - \xi^2). \quad (15)$$

Examples of generalized Hermite polynomials and plots of the locations of their roots in the complex plane are given by Clarkson [18]; see also [19, 21, 22]. The roots take the form of $m \times n$ “rectangles”, which are only approximate rectangles since the roots lie on arcs rather than straight lines. The generalized Hermite polynomial $H_{m,n}(z)$ can be expressed as the multiple integral

$$\begin{aligned} H_{m,n}(z) = & \frac{\pi^{m/2} \prod_{k=1}^m k!}{2^{m(m+2n-1)/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^n \prod_{j=i+1}^n (x_i - x_j)^2 \\ & \times \prod_{k=1}^n (z - x_k)^m \exp(-x_k^2) dx_1 dx_2 \dots dx_n, \end{aligned}$$

which arises in random matrix theory [13, 27, 37]. The generalized Hermite polynomials also arise in the theory of orthogonal polynomials [15].

Theorem 2. Suppose $H_{m,n}(z)$ is the generalized Hermite polynomial, then

$$w_{m,n}^{[1]}(z) = \frac{d}{dz} \ln \left\{ \frac{H_{m+1,n}(z)}{H_{m,n}(z)} \right\}, \quad (16a)$$

$$w_{m,n}^{[2]}(z) = \frac{d}{dz} \ln \left\{ \frac{H_{m,n}(z)}{H_{m,n+1}(z)} \right\}, \quad (16b)$$

$$w_{m,n}^{[3]}(z) = -2z + \frac{d}{dz} \ln \left\{ \frac{H_{m,n+1}(z)}{H_{m+1,n}(z)} \right\}, \quad (16c)$$

where $w_{m,n}^{[j]} = w(z; \alpha_{m,n}^{[j]}, \beta_{m,n}^{[j]}), j = 1, 2, 3$, are solutions of P_{IV} , respectively for

$$\begin{aligned} \alpha_{m,n}^{[1]} &= 2m + n + 1, & \beta_{m,n}^{[1]} &= -2n^2, \\ \alpha_{m,n}^{[2]} &= -(m + 2n + 1), & \beta_{m,n}^{[2]} &= -2m^2, \\ \alpha_{m,n}^{[3]} &= n - m, & \beta_{m,n}^{[3]} &= -2(m + n + 1)^2. \end{aligned}$$

Proof. See Theorem 4.4 in Noumi and Yamada [45]; also Theorem 3.1 in [18]. ■

The rational solutions of P_{IV} defined by (16) include all solutions in the “ $-1/z$ ” and “ $-2z$ ” hierarchies, i.e. the set of rational solutions of P_{IV} with parameters given by (13a). In fact these rational solutions of P_{IV} (11) are special cases of the special function solutions which are expressible in terms of parabolic cylinder functions $D_\nu(\xi)$.

Definition 2. The generalized Okamoto polynomial $\Omega_{m,n}(z)$ is defined by

$$\Omega_{m,n}(z) = \mathcal{W}(\varphi_1(z), \varphi_4(z), \dots, \varphi_{3m-2}(z), \varphi_2(z), \varphi_5(z), \dots, \varphi_{3n-1}(z)), \quad (17a)$$

$$\Omega_{m,0}(z) = \mathcal{W}(\varphi_1(z), \varphi_4(z), \dots, \varphi_{3m-2}(z)), \quad (17b)$$

$$\Omega_{0,n}(z) = \mathcal{W}(\varphi_2(z), \varphi_5(z), \dots, \varphi_{3n-1}(z)), \quad (17c)$$

for $m, n \geq 1$, and $\Omega_{0,0}(z) = 1$, where $\varphi_k(z) = 3^{k/2} e^{-k\pi i/2} H_k(\frac{1}{3}\sqrt{3}iz)$, with $H_k(z)$ the k th Hermite polynomial.

The generalized Okamoto polynomial $\Omega_{m,n}(z)$ defined here have been reindexed in comparison to the generalized Okamoto polynomial $Q_{m,n}(z)$ defined in [18, 19] by setting $Q_{m,n}(z) = \Omega_{m+n-1,n-1}(z)$ and $Q_{-m,-n}(z) = \Omega_{n-1,m+n}(z)$, for $m, n \geq 1$. The polynomials introduced by Okamoto [46] are given by $Q_m(z) = \Omega_{m-1,0}(z)$ and $R_m(z) = \Omega_{m,1}(z)$. Further the generalized Okamoto polynomial introduced by Noumi and Yamada [45] is given by $\tilde{Q}_{m,n}(z) = \Omega_{m-1,n-1}(z)$.

Examples of generalized Okamoto polynomials and plots of the locations of their roots in the complex plane are given by Clarkson [18, 19]. The roots of the polynomial $Q_{m,n}(z) = \Omega_{m+n-1,n-1}(z)$ with $m, n \geq 1$ take the form of an $m \times n$ “rectangle” with an “equilateral triangle”, which have either $\frac{1}{2}m(m-1)$ or $\frac{1}{2}n(n-1)$ roots, on each of its sides. The roots of the polynomial $Q_{-m,-n}(z) = \Omega_{n-1,m+n}(z)$ with $m, n \geq 1$ take the form of an $m \times n$ “rectangle” with an “equilateral triangle”, which now have either $\frac{1}{2}m(m+1)$ or $\frac{1}{2}n(n+1)$ roots, on each

of its sides. Again these are only approximate rectangles and equilateral triangles since the roots lie on arcs rather than straight lines.

Theorem 3. *Suppose $Q_{m,n}(z)$ is the generalized Okamoto polynomial, then*

$$\tilde{w}_{m,n}^{[1]}(z) = -\frac{2}{3}z + \frac{d}{dz} \ln \left\{ \frac{Q_{m+1,n}(z)}{Q_{m,n}(z)} \right\}, \quad (18a)$$

$$\tilde{w}_{m,n}^{[2]}(z) = -\frac{2}{3}z + \frac{d}{dz} \ln \left\{ \frac{Q_{m,n}(z)}{Q_{m,n+1}(z)} \right\}, \quad (18b)$$

$$\tilde{w}_{m,n}^{[3]}(z) = -\frac{2}{3}z + \frac{d}{dz} \ln \left\{ \frac{Q_{m,n+1}(z)}{Q_{m+1,n}(z)} \right\}, \quad (18c)$$

where $\tilde{w}_{m,n}^{[j]} = w(z; \tilde{\alpha}_{m,n}^{[j]}, \tilde{\beta}_{m,n}^{[j]})$, $j = 1, 2, 3$, are solutions of P_{IV} , respectively for

$$\begin{aligned} \tilde{\alpha}_{m,n}^{[1]} &= 2m + n, & \tilde{\beta}_{m,n}^{[1]} &= -2(n - \frac{1}{3})^2, \\ \tilde{\alpha}_{m,n}^{[2]} &= -(m + 2n), & \tilde{\beta}_{m,n}^{[2]} &= -2(m - \frac{1}{3})^2, \\ \tilde{\alpha}_{m,n}^{[3]} &= n - m, & \tilde{\beta}_{m,n}^{[3]} &= -2(m + n + \frac{1}{3})^2. \end{aligned}$$

Proof. See Theorem 4.3 in Noumi and Yamada [45]; also Theorem 4.1 in [18]. ■

3 Exact solutions of the dispersive water wave equation

3.1 Exact solutions from the scaling reduction

The DWW equation (1) has the generalized scaling reduction [24]

$$U(x, t) = V(z) - \kappa x - \kappa^2 t - \mu \ln t, \quad z = (x + 2\kappa t)/\sqrt{4t}, \quad (19)$$

with κ and μ arbitrary constants, which is a classical symmetry reduction [24], and where $v(z) = V'(z)$ satisfies

$$v''' = 6v^2v' - 12zvv' + (4z^2 - 8\mu)v' - 8v^2 + 12zv + 16\mu. \quad (20)$$

Then letting $v(z) = w(z) + 2z$ in (20) and integrating yields P_{IV} (11) and so we can obtain exact solutions of the DWW equation (1) from the rational solutions of P_{IV} given in Section 2. However, it is possible to generate these solutions directly, i.e. without having to consider the generalized Hermite and generalized Okamoto polynomials, by extending the representations of these polynomials in terms of the determinants given in Definitions 1 and 2.

Theorem 4. *Consider the polynomials $\varphi_n(x, t; \kappa)$ defined by*

$$\sum_{n=0}^{\infty} \frac{\varphi_n(x, t; \kappa) \lambda^n}{n!} = \exp \left\{ (x + 2\kappa t)\lambda - t\lambda^2 \right\}, \quad (21)$$

so $\varphi_n(x, t; \kappa) = t^{n/2} H_n((x + 2\kappa t)/\sqrt{4t})$, with $H_n(z)$ the Hermite polynomial, and then let

$$\Phi_{m,n}(x, t; \kappa) = \mathcal{W}_x(\varphi_m, \varphi_{m+1}, \dots, \varphi_{m+n-1}). \quad (22)$$

where $\mathcal{W}_x(\varphi_m, \varphi_{m+1}, \dots, \varphi_{m+n-1})$ is the Wronskian with respect to x . Then the DWW equation (1) has exact solutions in the form

$$U_{m,n}^{[1]}(x, t; \kappa) = \ln \left\{ \frac{\Phi_{m+1,n}(x, t; \kappa)}{\Phi_{m,n}(x, t; \kappa)} \right\} + \frac{x^2}{4t} - (m + n + \frac{1}{2}) \ln t, \quad (23a)$$

$$U_{m,n}^{[2]}(x, t; \kappa) = \ln \left\{ \frac{\Phi_{m,n}(x, t; \kappa)}{\Phi_{m,n+1}(x, t; \kappa)} \right\} + \frac{x^2}{4t} + (m + n + \frac{1}{2}) \ln t, \quad (23b)$$

$$U_{m,n}^{[3]}(x, t; \kappa) = \ln \left\{ \frac{\Phi_{m,n+1}(x, t; \kappa)}{\Phi_{m+1,n}(x, t; \kappa)} \right\} - \kappa(x + \kappa t), \quad (23c)$$

Proof. The polynomials $\varphi_n(x, t; \kappa)$ defined by (21) are obtained by $z = (x + \kappa t)/\sqrt{4t}$ and $\xi = \lambda\sqrt{t}$ in (15). Then (22) follows from the definition of $H_{m,n}(z)$ given by (14). Finally substitution of these expressions into (19) yields the desired result. \blacksquare

Theorem 5. Consider the polynomials $\psi_n(x, t; \kappa)$ defined by

$$\sum_{n=0}^{\infty} \frac{\psi_n(x, t; \kappa) \lambda^n}{n!} = \exp \left\{ (x + 2\kappa t)\lambda + 3t\lambda^2 \right\}, \quad (24)$$

so $\psi_n(x, t; \kappa) = (3t)^{n/2} e^{-n\pi i/2} H_n(i(x + 2\kappa t)/\sqrt{3t})$, with $H_n(z)$ the Hermite polynomial, and then let

$$\begin{aligned} \Psi_{m,n}(x, t; \kappa) &= \mathcal{W}_x(\psi_1, \psi_4, \dots, \psi_{3m+3n-5}, \psi_2, \psi_5, \dots, \psi_{3n-4}), \\ \Psi_{-m,-n}(x, t) &= \mathcal{W}_x(\psi_1, \psi_4, \dots, \psi_{3n-2}, \psi_2, \psi_5, \dots, \psi_{3m+3n-1}), \end{aligned} \quad (25)$$

for $m, n \geq 1$, where $\mathcal{W}_x(\psi_1, \psi_2, \dots, \psi_m)$ is the Wronskian with respect to x . Then the DWW equation (1) has exact solutions in the form

$$U_{m,n}^{[1]}(x, t; \kappa) = \ln \left\{ \frac{\Psi_{m+1,n}(x, t)}{\Psi_{m,n}(x, t; \kappa)} \right\} + \frac{x^2}{6t} - \frac{1}{3}\kappa(x + \kappa t) - (2m + n) \ln t, \quad (26a)$$

$$U_{m,n}^{[2]}(x, t; \kappa) = \ln \left\{ \frac{\Psi_{m,n}(x, t; \kappa)}{\Psi_{m,n+1}(x, t; \kappa)} \right\} + \frac{x^2}{6t} - \frac{1}{3}\kappa(x + \kappa t) + (m + 2n) \ln t, \quad (26b)$$

$$U_{m,n}^{[3]}(x, t; \kappa) = \ln \left\{ \frac{\Psi_{m,n+1}(x, t; \kappa)}{\Psi_{m+1,n}(x, t; \kappa)} \right\} + \frac{x^2}{6t} - \frac{1}{3}\kappa(x + \kappa t) - (m - n) \ln t. \quad (26c)$$

Proof. The proof is similar to that for Theorem 4 above and so is left to the reader. \blacksquare

3.2 Generalized solutions

Here we discuss possible generalizations of the rational solutions obtained above. Motivated by the structure of the generalized exact solutions of the KdV equation [7, 8], the Boussinesq equation [22, 28], and the NLS equation [21], the idea is to replace the exponent of the exponentials (21) and (24) by the infinite series

$$(x + 2\kappa t)\lambda + bt\lambda^2 + \sum_{j=3}^{\infty} \xi_j \lambda^j, \quad (27)$$

with $b = -1$ and $b = 3$, respectively, where ξ_j , for $j \geq 3$, are arbitrary parameters. These give generalizations of the polynomials (22) and (25) which in turn give generalizations of (23) and (26). However, when we substitute the generalizations of (23) and (26) into the DWW equation (1), it can be shown that necessarily $\kappa = 0$ and $\xi_j = 0$ for $j \geq 3$ except for the generalization of (23c), and is described in the following theorem.

Theorem 6. *Consider the polynomials $\vartheta_n(x, t; \boldsymbol{\xi})$ defined by*

$$\sum_{n=0}^{\infty} \frac{\vartheta_n(x, t; \boldsymbol{\xi}) \lambda^n}{n!} = \exp \left(x\lambda - t\lambda^2 + \sum_{j=3}^{\infty} \xi_j \lambda^j \right), \quad (28)$$

where $\boldsymbol{\xi} = (\xi_3, \xi_4, \dots)$, with ξ_j arbitrary constants and then let

$$\Theta_{m,n}(x, t; \boldsymbol{\xi}) = \mathcal{W}_x(\vartheta_m, \vartheta_{m+1}, \dots, \vartheta_{m+n-1}). \quad (29)$$

where $\mathcal{W}_x(\vartheta_m, \vartheta_{m+1}, \dots, \vartheta_{m+n-1})$ is the Wronskian with respect to x . Then the DWW equation (1) has exact solutions in the form

$$U_{m,n}^{[3]}(x, t; \boldsymbol{\xi}) = \ln \{ \Theta_{m,n+1}(x, t; \boldsymbol{\xi}) / \Theta_{m+1,n}(x, t; \boldsymbol{\xi}) \}, \quad (30)$$

The polynomials (29) and associated solutions (30) are analogous to the polynomials and associated rational solutions of the KdV equation (7) derived by Airault, McKean and Moser [8] and Adler and Moser [7]; see also [4, 16]. Further we conclude that generalized solutions, i.e. solutions which depend on an infinite number of arbitrary parameters, of the DWW equation (1) exist only if the derivative of the solution, which is a decaying rational solution of (3), obtained through the scaling reduction to a Painlevé equation decays as $|x| \rightarrow \infty$. This is analogous to the situation for generalized rational solutions of the KdV equation [7, 8, 16], the Boussinesq equation [22, 28, 49], and the NLS equation [33]; see also [19].

4 Exact solutions of the modified Boussinesq equation

4.1 Exact solutions from the scaling reduction

The modified Boussinesq equation (2) has the generalized scaling reduction [17]

$$U(x, t) = V(z) + \kappa xt - \mu \ln t, \quad z = (x + 3\kappa t^2) / \sqrt{4t}, \quad (31)$$

with κ and μ arbitrary constants and where $v(z) = V'(z)$ satisfies

$$v''' = 6v^2v' + 4zvv' - (\frac{4}{3}z^2 + 8\mu)v' - 4zv + \frac{16}{3}\mu. \quad (32)$$

In the case when $\kappa \neq 0$, the generalized scaling reduction (31) is a nonclassical symmetry reduction that is derived either using the nonclassical method [11] or the direct method [23] — see [17] for details. Letting $V(z) = w(z) + \frac{2}{3}z$ in (32) and integrating yields P_{IV} (11) and so we can obtain exact solutions of the modified Boussinesq equation (2) from the rational solutions of P_{IV} given in Section 2. As for the DWW equation (1), it is possible to generate these solutions directly, i.e. without having to consider the generalized Hermite and generalized Okamoto polynomials, by extending the representations of these polynomials in terms of the determinants given in Definitions 1 and 2.

Theorem 7. *Consider the polynomials $\varphi_n(x, t; \kappa)$ defined by*

$$\sum_{n=0}^{\infty} \frac{\varphi_n(x, t; \kappa) \lambda^n}{n!} = \exp \left\{ (x + 3\kappa t^2) \lambda - t \lambda^2 \right\}, \quad (33)$$

so $\varphi_n(x, t; \kappa) = t^{n/2} H_n((x + 3\kappa t^2)/\sqrt{4t})$, with $H_n(z)$ the Hermite polynomial, and then let

$$\Phi_{m,n}(x, t; \kappa) = \mathcal{W}_x(\varphi_m, \varphi_{m+1}, \dots, \varphi_{m+n-1}). \quad (34)$$

where $\mathcal{W}_x(\varphi_m, \varphi_{m+1}, \dots, \varphi_{m+n-1})$ is the Wronskian with respect to x . Then the modified Boussinesq equation (2) has exact solutions in the form

$$U_{m,n}^{[1]}(x, t; \kappa) = \ln \left\{ \frac{\Phi_{m+1,n}(x, t; \kappa)}{\Phi_{m,n}(x, t; \kappa)} \right\} + \frac{x^2}{12t} + \frac{3}{2}\kappa xt + \frac{3}{4}\kappa^2 t^3 + (m + \frac{1}{2}) \ln t, \quad (35a)$$

$$U_{m,n}^{[2]}(x, t; \kappa) = \ln \left\{ \frac{\Phi_{m,n}(x, t; \kappa)}{\Phi_{m,n+1}(x, t; \kappa)} \right\} + \frac{x^2}{12t} + \frac{3}{2}\kappa xt + \frac{3}{4}\kappa^2 t^3 - (n + \frac{1}{2}) \ln t, \quad (35b)$$

$$U_{m,n}^{[3]}(x, t; \kappa) = \ln \left\{ \frac{\Phi_{m,n+1}(x, t; \kappa)}{\Phi_{m+1,n}(x, t; \kappa)} \right\} - \frac{x^2}{6t} - \frac{3}{2}\kappa^2 t^3 - (m - n) \ln t, \quad (35c)$$

Theorem 8. *Consider the polynomials $\psi_n(x, t; \kappa)$ defined by*

$$\sum_{n=0}^{\infty} \frac{\psi_n(x, t; \kappa) \lambda^n}{n!} = \exp \left\{ (x + 3\kappa t^2) \lambda + 3t \lambda^2 \right\}, \quad (36)$$

so $\psi_n(x, t; \kappa) = (3t)^{n/2} e^{-n\pi i/2} H_n(i(x + 3\kappa t^2)/\sqrt{3t})$, with $H_n(z)$ the Hermite polynomial, and then let

$$\begin{aligned} \Psi_{m,n}(x, t; \kappa) &= \mathcal{W}_x(\psi_1, \psi_4, \dots, \psi_{3m+3n-5}, \psi_2, \psi_5, \dots, \psi_{3n-4}), \\ \Psi_{-m,-n}(x, t; \kappa) &= \mathcal{W}_x(\psi_1, \psi_4, \dots, \psi_{3n-2}, \psi_2, \psi_5, \dots, \psi_{3m+3n-1}), \end{aligned} \quad (37)$$

for $m, n \geq 1$, where $\mathcal{W}_x(\psi_1, \psi_2, \dots, \psi_m)$ is the Wronskian with respect to x . Then the modified Boussinesq equation (2) has exact solutions in the form

$$U_{m,n}^{[1]}(x, t; \kappa) = \ln \{\Psi_{m+1,n}(x, t; \kappa) / \Psi_{m,n}(x, t; \kappa)\} + \kappa x t, \quad (38a)$$

$$U_{m,n}^{[2]}(x, t; \kappa) = \ln \{\Psi_{m,n}(x, t; \kappa) / \Psi_{m,n+1}(x, t; \kappa)\} + \kappa x t, \quad (38b)$$

$$U_{m,n}^{[3]}(x, t; \kappa) = \ln \{\Psi_{m,n+1}(x, t; \kappa) / \Psi_{m+1,n}(x, t; \kappa)\} + \kappa x t. \quad (38c)$$

4.2 Generalized solutions

Here we discuss possible generalizations of the exact solutions obtained above. As for the DWW equation, the idea is to replace the exponent of the exponentials (21) and (36) by the infinite series

$$(x + 3\kappa t^2)\lambda + bt\lambda^2 + \sum_{j=3}^{\infty} \xi_j \lambda^j, \quad (39)$$

with $b = -1$ and $b = 3$, respectively, where ξ_j , for $j \geq 3$, are arbitrary parameters. However, when we substitute the generalizations of (35) and (38) into the modified Boussinesq equation (2), it can be shown that necessarily $\kappa = 0$ and $\xi_j = 0$ for $j \geq 3$ except for the generalization of (38) with $\kappa = 0$, the derivatives of which decay as $|x| \rightarrow \infty$ and are decaying rational solutions of (4), and are described in the following theorem.

Theorem 9. Consider the polynomials $\vartheta_n(x, t; \xi)$ defined by

$$\sum_{n=0}^{\infty} \frac{\vartheta_n(x, t; \xi) \lambda^n}{n!} = \exp \left(x\lambda + 3t\lambda^2 + \sum_{j=3}^{\infty} \xi_j \lambda^j \right), \quad (40)$$

where $\xi = (\xi_3, \xi_4, \dots)$, with ξ_j arbitrary constants and then let

$$\begin{aligned} \Theta_{m,n}(x, t; \xi) &= \mathcal{W}_x(\vartheta_1, \vartheta_4, \dots, \vartheta_{3m+3n-5}, \vartheta_2, \vartheta_5, \dots, \vartheta_{3n-1}), \\ \Theta_{-m,-n}(x, t; \xi) &= \mathcal{W}_x(\vartheta_1, \vartheta_4, \dots, \vartheta_{3n-2}, \vartheta_2, \vartheta_5, \dots, \vartheta_{3m+3n-1}), \end{aligned} \quad (41)$$

for $m, n \geq 1$, where $\mathcal{W}_x(\theta_1, \theta_2, \dots, \theta_m)$ is the Wronskian with respect to x . Then modified Boussinesq equation (2) has exact solutions in the form

$$U_{m,n}^{[1]}(x, t; \xi) = \ln \{\Theta_{m+1,n}(x, t; \xi) / \Theta_{m,n}(x, t; \xi)\}, \quad (42a)$$

$$U_{m,n}^{[2]}(x, t; \xi) = \ln \{\Theta_{m,n}(x, t; \xi) / \Theta_{m,n+1}(x, t; \xi)\}, \quad (42b)$$

$$U_{m,n}^{[3]}(x, t; \xi) = \ln \{\Theta_{m,n+1}(x, t; \xi) / \Theta_{m+1,n}(x, t; \xi)\}. \quad (42c)$$

5 Discussion

In this paper we have studied exact solutions of the DWW equation (1) and the modified Boussinesq equation (2) through rational solutions of PIV (11), which

arises as a scaling reduction of these equations. Further we have derived some generalized solutions of the DWW equation (1) and the modified Boussinesq equation (2) which are analogues of the generalized rational solutions of the KdV equation (7), the Boussinesq equation (6) and the NLS equation (9). The DWW equation (1) and the modified Boussinesq equation (2) also possess the accelerating wave reductions

$$U(x, t) = W(z) + \mu xt - \frac{2}{3}\mu^2 t^3, \quad z = x - \mu t^2, \quad (43)$$

$$U(x, t) = W(z) - \frac{1}{3}\mu xt, \quad z = x - \mu t^2, \quad (44)$$

respectively, where μ is an arbitrary constant and $W(z)$ is solvable in terms of solutions of P_{II} (10). Since P_{II} has rational solutions expressed in terms of the Yablonskii–Vorob’ev polynomials [55, 57], then we can derive another class of exact solutions of (1) and (2) in terms of these special polynomials. We remark that the reduction (43) is obtained using classical Lie group method [12, 48] whereas the reduction (44) is a nonclassical symmetry reduction that is derived either using the nonclassical method [11] or the direct method [23] — see [17, 24] for details.

However, there are further exact solutions of the DWW equation (1) and the modified Boussinesq equation (2), which are not special cases of the solutions discussed in Section 3 and Section 4 above, or through the accelerating wave reductions (43) and (44). For example, the DWW equation (1) has the exact solutions

$$u_1(x, t) = \ln \left(\frac{x+t}{x+t+2} \right) - \frac{t}{2}, \quad (45a)$$

$$u_2(x, t) = \ln \left\{ \frac{(x+t)^3 - 3(x+t)^2 - 6t}{(x+t)^3 + 3(x+t)^2 - 6t} \right\} - \frac{t}{2}, \quad (45b)$$

and the modified Boussinesq equation (2) has the exact solutions

$$u_1(x, t) = \ln \left(\frac{x+t}{x+t-6} \right) - \frac{t}{6}, \quad (46a)$$

$$u_2(x, t) = \ln \left\{ \frac{(x+t)^3 + 9(x+t)^2 + 24x + 42t - 144}{(x+t)^3 - 9(x+t)^2 + 24x + 42t + 144} \right\} - \frac{t}{6}, \quad (46b)$$

which are not obtained by the procedure described above. These are analogous to the rational solutions of the classical Boussinesq equation (5) derived by Sachs [52] by applying the limiting procedure in [4] to N -soliton solutions of (5).

The classical orthogonal polynomials, such as Hermite, Laguerre, Legendre, and Tchebychev polynomials which are associated with rational solutions of the classical special functions, play an important role in a variety of applications (cf. [6]). Hence it seems likely that the special polynomials discussed here which are associated with rational solutions of nonlinear special functions, i.e. the soliton and Painlevé equations, also arise in a variety of applications such as in numerical analysis.

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Hidden and weak symmetries for partial differential equations

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The Type-II hidden symmetries are extra symmetries in addition to the inherited symmetries of the differential equations when the number of independent and dependent variables is reduced by a Lie point symmetry.

In this paper we analyze the connection between one of the methods analyzed in [2] and the weak symmetries of the partial differential equation in order to determine the source of these hidden symmetries. We have considered the shallow water wave (SWW) equation presented in [7], and the second heavenly equation [1], which reduces, in the case of three fields, to a single second order equation of Monge-Ampère type.

1 Introduction

There is no existing general theory for solving nonlinear partial differential equations (PDE's) and it happens that many PDE's of physical importance are nonlinear. Lie classical symmetries admitted by nonlinear PDE's are useful for finding invariant solutions.

If a PDE is invariant under a Lie group, the number of independent variables can be reduced by one. The reduced equation loses the symmetry used to reduce the number of variables and may lose other Lie symmetries depending on the structure of the associated Lie algebra. If a PDE loses (gains) a symmetry in addition to the one used to reduce the number of independent variables of the PDE, the PDE possesses a Type I (Type II) hidden symmetry [2].

It has been noted [3] that these Type II hidden symmetries do not arise from contact symmetries or nonlocal symmetries since the transformations to reduce the number of variables involve only variables. Thus the origin of these hidden symmetries must be in point symmetries [2]. In [2] B. Abraham-Shrauner and K.S. Govinder have identified a common provenance for the Type II hidden symmetries of differential equations reduced from PDE's that covers the PDE's studied. In [2] it was pointed out that the crucial point is that the differential equation that is reduced from a PDE and possesses a Type II hidden symmetry is also a reduced differential equation from one or more other PDE's. The inherited symmetries from these other PDE's are a larger class of Lie point symmetries that includes the Type II hidden symmetries. The Type II hidden symmetries are actually inherited

symmetries from one or more of the other PDE's. The crucial question [2] is whether we can identify the PDE's from which the Type II hidden symmetries are inherited. In [2] two methods were proposed: some PDE's may be constructed by calculating the invariants by reverse transformations and some PDE's may be identified by inspection.

The *weak symmetries* were introduced in Olver and Rosenau [10]. Their approach consists in calculating the symmetries of the basic equation supplemented by certain differential constraints, chosen in order to weaken the invariance criterion of the basic system and to provide us with the larger Lie-point symmetry groups for the augmented system. In this way one obtains an overdetermined nonlinear system of equations and the solution set is, in this case, quite larger than the corresponding to classical symmetries. In [5] and [6] we have analyzed the connection between one of these methods and weak symmetries of the PDE with special differential constraints in order to determine the source of the Type-II hidden symmetries.

In this work we consider weak symmetries of the SWW equation presented in [7], and the second heavenly equation [1], which reduces, in the case of three fields, to a single second order equation of Monge-Ampère type, as well as of the homogeneous Monge-Ampère equation. The weak symmetries of these PDE's with special differential constraint are derived in order to determine the source of the Type II hidden symmetries. The main result is that we can identify the PDE from which the Type II hidden symmetries are inherited by using as differential constraint the *side condition* from which the reduction has been derived.

2 Weak symmetries for a shallow water wave (SWW) equation

We begin by considering the SWW equation presented in [7]

$$u_{xxxx} + \alpha u_x u_{xt} + \beta u_t u_{xxt} - u_{xt} - u_{xx} = 0, \quad (1)$$

where the subscripts denote differentiation with respect to the variable indicated. The Lie point symmetries of (1), which appeared in [4] are represented by the Lie group generators:

$$\mathbf{v}_1 = x\partial_x - \left(u - \frac{2x}{\alpha} - \frac{t}{\beta}\right)\partial_u, \quad \mathbf{v}_2 = \partial_x, \quad \mathbf{v}_3 = \partial_u, \quad \mathbf{v}_4 = g(t) \left(\partial_t + \frac{1}{\beta}\partial_u\right).$$

If we reduce equation (1) by using the generator $\mathbf{v}_2 + \mathbf{v}_4$ we get $u = w(z) + \frac{t}{\beta}$, $z = x - \int \frac{1}{g(t)}$ and the reduced ODE is

$$w_{zzzz} + (\alpha + \beta)w_z w_{zz} - w_{zz} = 0, \quad (2)$$

which admits a three-parameter Lie group. The associated Lie algebra can be represented by the following generators

$$\mathbf{w}_1 = \partial_z, \quad \mathbf{w}_2 = \partial_w, \quad \mathbf{w}_3 = z\partial_z - \left(w - \frac{2z}{\alpha + \beta}\right)\partial_w. \quad (3)$$

The inherited symmetries are $\mathbf{v}_2 \rightarrow \mathbf{w}_1$, $\mathbf{v}_3 \rightarrow \mathbf{w}_2$. The other symmetry \mathbf{w}_3 is a Type II symmetries [7]. In [2] it was shown that hidden symmetries of PDE's arise from point symmetries of another PDE. A PDE that reduce to (2) by using the variables z and w and was proposed, by guessing, in [7] is

$$w_{xxxx} + (\alpha + \beta)w_x w_{xx} - w_{xx} = 0. \quad (4)$$

We propose to have as differential constraint the side condition from which the reduction has been derived and to derive weak symmetries, that is, Lie classical symmetries of the original equation and the side condition. The PDE from which the hidden symmetries are inherited is the original PDE in which we substitute the side condition from which the reduction has been derived.

We are going to derive some weak symmetries of the SWW equation (1), choosing as side condition the differential constraint

$$u_x + g(t)u_t - \frac{g(t)}{\beta} = 0, \quad (5)$$

which is associated to the generator $\mathbf{v}_2 + \mathbf{v}_4$ that has been used to derive the reduction. Applying Lie classical method to equation (1) with the side condition (5) we get the following generators

$$\begin{aligned} \mathbf{u}_1 &= f_1(t)\partial_z, \quad \mathbf{u}_2 = f_2(t)\partial_t, \quad \mathbf{u}_3 = f_3(t)\partial_w, \\ \mathbf{u}_4 &= f_4(t) \left(z\partial_z - \left(w - \frac{2z}{\alpha + \beta} \right) \partial_w \right), \end{aligned} \quad (6)$$

with $f_i(t)$, $i = 1, \dots, 4$, arbitrary functions. However, by appropriate choice of $f_i(t) = 1$ we recover the group generators (3). These generators (2) have been derived in [7] and it was pointed out in [7] that symmetry \mathbf{w}_3 is a hidden symmetry. We prove that \mathbf{w}_3 is inherited as a weak symmetry of equations (1) with the side condition (5). The crucial point is that \mathbf{u}_3 is a Lie symmetry of equation (1) in which we have substituted the side condition (5), and this equation is precisely (4).

3 Monge-Ampère equation

The second heavenly equation is

$$\theta_{xx}\theta_{yy} - \theta_{xy}^2 + \theta_{xw} + \theta_{yz} = 0, \quad (7)$$

where $\theta(x, y, w, z)$ is a holomorphic complex-valued function of four complex variables in some local coordinate system. The Lie point symmetries of (7) have been reported in [9] and [1].

The second heavenly equation (7) has been reduced in [1] by the translation symmetry in w . By using $\mathbf{v} = \partial_w$ the second heavenly equation has been reduced [1] to

$$u_{xx}u_{yy} - u_{xy}^2 + u_{yz} = 0. \quad (8)$$

where $u(x, y, z)$ replaces θ . The Lie point symmetries were computed by the computer program LIE and reported in [1]

$$\begin{aligned}\mathbf{v}_1 &= \partial_z, \quad \mathbf{v}_2 = f_1 \partial_u, \quad \mathbf{v}_3 = \partial_x, \quad \mathbf{v}_4 = f_2 \partial_y + \frac{x^2}{2} f_{2z} \partial_u, \quad \mathbf{v}_5 = y \partial_u, \\ \mathbf{v}_6 &= x f_3 \partial_u, \quad \mathbf{v}_7 = x \partial_x + 2z \partial_z, \quad \mathbf{v}_8 = x f_4 \partial_y + \frac{x^3}{6} f_{4z} \partial_u, \\ \mathbf{v}_9 &= -2z \partial_x + xy \partial_u, \quad \mathbf{v}_{10} = y \partial_y + u \partial_u, \quad \mathbf{v}_{11} = -z \partial_z + u \partial_u \\ \mathbf{v}_{12} &= -4xz \partial_x + 2yz \partial_y - 4z^2 \partial_z + (2zu + x^2 y) \partial_u,\end{aligned}$$

where $f_j(z)$, $j = 1, \dots, 4$ are arbitrary functions. The symmetries represented by \mathbf{v}_1 through \mathbf{v}_{11} are all inherited from the Lie point symmetries of the second heavenly equation which appear in [1], \mathbf{v}_{12} is not an inherited symmetry and is a Type II Lie point hidden symmetry.

The new similarity variables and solution for \mathbf{v}_{12} were derived in [1] and are

$$r = \frac{x}{z}, \quad s = y\sqrt{z}, \quad u = \frac{1}{\sqrt{z}}G(r, s) - \frac{x^2 y}{4z}. \quad (9)$$

The substitution of (9) into (8) leads to the homogeneous Monge-Ampère equation

$$G_{rr}G_{ss} - G_{rs}^2 = 0. \quad (10)$$

The Lie point symmetries of the homogeneous Monge-Ampère equation have been found in [8] and are

$$\begin{aligned}\mathbf{u}_1 &= \partial_r, \quad \mathbf{u}_2 = \partial_s, \quad \mathbf{u}_3 = \partial_G, \quad \mathbf{u}_4 = r \partial_r, \quad \mathbf{u}_5 = s \partial_s, \quad \mathbf{u}_6 = G \partial_G, \\ \mathbf{u}_7 &= s \partial_r, \quad \mathbf{u}_8 = r \partial_s, \quad \mathbf{u}_9 = r \partial_G, \quad \mathbf{u}_{10} = s \partial_G, \quad \mathbf{u}_{11} = G \partial_r, \\ \mathbf{u}_{12} &= G \partial_s, \quad \mathbf{u}_{13} = r^2 \partial_r + rs \partial_s + rG \partial_G, \quad \mathbf{u}_{14} = rs \partial_r + s^2 \partial_s + sG \partial_G, \\ \mathbf{u}_{15} &= rG \partial_r + sG \partial_s + G^2 \partial_G.\end{aligned} \quad (11)$$

In (11) [1] there are nine inherited symmetries from (8) and six Type-II hidden Lie point symmetries. In [1] the provenance of Type II hidden symmetries of the target PDE (8) has been investigated.

In this work we are considering the provenance of Type II hidden symmetries of the homogeneous Monge-Ampère equation (10).

In order to determine the other possible PDE's with three independent variables the inherited symmetries of which include all the symmetries in (10) we consider the PDE equation obtained considering the target PDE (8) and the side condition from which the reduction was derived. This side condition associated to generator \mathbf{v}_{12} is

$$-4xz u_x + 2yz u_y - 4z^2 u_z - 2zu - x^2 y = 0. \quad (12)$$

Applying the classical method to equation (8) with the side condition (12), we get the following generators:

$$\begin{aligned}
\mathbf{w}_1 &= F_1(z) \left(\partial_x - \frac{xy}{2z} \partial_u \right), \quad \mathbf{w}_2 = F_2(z) \left(\partial_y - \frac{x^2}{4z} \partial_u \right), \\
\mathbf{w}_3 &= F_3(z) \left(\partial_x + \frac{x^2}{4z^2} \partial_u \right), \quad \mathbf{w}_4 = F_4(z) \partial_u, \quad \mathbf{w}_5 = F_5(z) \left(u + \frac{x^2 y}{4z} \right) \partial_u, \\
\mathbf{w}_6 &= F_6(z) \left(\left(xu + \frac{x^3 y}{4z} \right) \partial_x + \left(yu + \frac{x^2 y}{4z} \right) \partial_y - \left(u^2 - \frac{x^2 y u}{4z} - \frac{x^4 y^2}{8z^2} \right) \partial_u \right), \\
\mathbf{w}_7 &= F_7(z) \left(\left(u + \frac{x^2 y}{4z} \right) \partial_x - \left(\frac{xyu}{2z} + \frac{x^3 y^2}{8z^2} \right) \partial_u \right), \\
\mathbf{w}_8 &= F_8(z) \left(\left(u + \frac{x^2 y}{4z} \right) \partial_y - \left(\frac{x^2 u}{4z} + \frac{x^4 y}{16z^2} \right) \partial_u \right), \\
\mathbf{w}_9 &= F_9(z) \left(y \partial_x - \frac{xy^2}{2z} \partial_u \right), \\
\mathbf{w}_{10} &= F_{10}(z) \left(xy \partial_x + y^2 \partial_y - \left(\frac{x^2 y^2 u}{2z} - yu \right) \partial_u \right), \\
\mathbf{w}_{11} &= F_{11}(z) \left(x \partial_y - \frac{x^3}{4z} \partial_u \right), \quad \mathbf{w}_{12} = F_{12}(z) \left(y \partial_y - \frac{x^2 y}{4z} \partial_u \right), \\
\mathbf{w}_{13} &= F_{13}(z) \left(x^2 \partial_x + xy \partial_y - \left(\frac{x^3 y}{2z} - xu \right) \partial_u \right), \\
\mathbf{w}_{14} &= F_{14}(z) \left(x \partial_x - \frac{x^2 y}{2z} \partial_u \right), \quad \mathbf{w}_{15} = F_{15}(z) y \partial_u,
\end{aligned}$$

with $F_i(z)$, $i = 1, \dots, 15$, arbitrary functions. The PDE

$$u_{xx}u_{yy} - u_{xy}^2 - \frac{x}{z}u_{xy} + \frac{y}{2z}u_{yy} - \frac{x^2}{4z^2} = 0$$

from which the hidden symmetries are inherited is the original PDE in which we substitute the side condition (12), and some differential consequences.

We show which weak symmetries of the reduced second heavenly equation (8) reduce to the symmetries of (10). Taking into account that

$$\partial_x = \frac{1}{z} \partial_r + \frac{xy}{2\sqrt{z}} \partial_G = \frac{1}{z} \partial_r + \frac{rs}{2} \partial_G, \quad \partial_u = \sqrt{z} \partial_G, \quad \partial_y = \sqrt{z} \partial_s + \frac{x^2}{4\sqrt{z}} \partial_G,$$

the inherited symmetries are $\mathbf{w}_1 \rightarrow \mathbf{u}_1$ if $F_1 = z$, $\mathbf{w}_2 \rightarrow \mathbf{u}_2$ if $F_2 = \frac{1}{\sqrt{z}}$, $\mathbf{w}_4 \rightarrow \mathbf{u}_3$ if $F_4 = \frac{1}{\sqrt{z}}$, $\mathbf{w}_{14} \rightarrow \mathbf{u}_4$, $\mathbf{w}_{12} \rightarrow \mathbf{u}_5$, $\mathbf{w}_5 \rightarrow \mathbf{u}_6$ if $F_5 = 1$, $\mathbf{w}_9 \rightarrow \mathbf{u}_7$ if $F_9 = z^{3/2}$, $\mathbf{w}_{11} \rightarrow \mathbf{u}_8$ if $F_{11} = z^{-3/2}$, $\mathbf{w}_{15} \rightarrow \mathbf{u}_{10}$ if $F_{15} = 1$,

The symmetries \mathbf{w}_3 , \mathbf{w}_6 , \mathbf{w}_7 , \mathbf{w}_8 , \mathbf{w}_{10} and \mathbf{w}_{13} are not inherited symmetries. these are Type II hidden symmetries.

Now we consider the reduction of the homogeneous Monge-Ampère equation (10) through the generator $-a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_9 + d\mathbf{u}_{10}$ that leads to

$$\omega = br + as, \quad G = -\frac{c}{b^2} s\omega + \left(\frac{d}{2b} - \frac{ac}{2b^2} \right) s^2 + \varphi(\omega),$$

where φ satisfies the autonomous and linear ODE $(b^3d - ab^2c)\varphi'' - c^2 = 0$. Setting $\frac{c^2}{b^3d - ab^2c} = k$ the ODE can be written as $\varphi'' - k = 0$ with $b^3d - ab^2c \neq 0$, and admits an eight-parameter Lie group. The associated Lie algebra can be represented by the following generators

$$\begin{aligned}\hat{\mathbf{w}}_1 &= \partial_\omega, \quad \hat{\mathbf{w}}_2 = \partial_\varphi, \quad \hat{\mathbf{w}}_3 = \omega\partial_\omega + \left(\frac{\varphi}{2} + \frac{3k\omega^2}{4}\right)\partial_\varphi, \quad \hat{\mathbf{w}}_4 = \omega\partial_\varphi, \\ \hat{\mathbf{w}}_5 &= \frac{\omega^2}{2}\partial_\omega + \left(\frac{\omega\varphi}{2} + \frac{k\omega^3}{4}\right)\partial_\varphi, \quad \hat{\mathbf{w}}_6 = \left(-\frac{k}{2}\omega^3 + \omega\varphi\right)\partial_\omega - \frac{k^2\omega^4}{4}\partial_\varphi, \\ \hat{\mathbf{w}}_7 &= \varphi\partial_\omega + \left(\frac{3k}{2}\omega\varphi + \frac{k^2\omega^3}{4}\right)\partial_\varphi, \quad \hat{\mathbf{w}}_8 = -\frac{k^2\omega^2}{2}\partial_\varphi.\end{aligned}\quad (13)$$

The inherited symmetries are $\mathbf{u}_1 \rightarrow \hat{\mathbf{w}}_1$, $\mathbf{u}_3 \rightarrow \hat{\mathbf{w}}_2$, $\mathbf{u}_8 \rightarrow \hat{\mathbf{w}}_4$. $\hat{\mathbf{w}}_3$, $\hat{\mathbf{w}}_5$, $\hat{\mathbf{w}}_6$ and $\hat{\mathbf{w}}_7$ are Type II hidden symmetries. We now consider weak symmetries of (10) with the following side condition

$$-aG_r + bG_s = cr + ds \quad (14)$$

corresponding to the generator $-a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_9 + d\mathbf{u}_{10}$. Applying the classical method to equation (10), with the side condition (14) we get:

$$\begin{aligned}\hat{\mathbf{v}}_1 &= f_1(s)\partial_r, \quad \hat{\mathbf{v}}_2 = f_2(s)\partial_G, \quad \hat{\mathbf{v}}_3 = f_3(s)\left(r\partial_r + \left(\frac{G}{2} + \frac{3kr^2}{4}\right)\partial_G\right), \\ \hat{\mathbf{v}}_4 &= f_4(s)r\partial_G, \quad \hat{\mathbf{v}}_5 = f_5(s)\left(\frac{r^2}{2}\partial_r + \left(\frac{rG}{2} + \frac{kr^3}{4}\right)\partial_G\right), \\ \hat{\mathbf{v}}_6 &= f_6(s)\left(\left(-\frac{k}{2}r^3 + rG\right)\partial_r - \frac{k^2r^4}{4}\partial_G\right), \\ \hat{\mathbf{v}}_7 &= f_7(s)\left(G\partial_r + \left(\frac{3k}{2}rG + \frac{k^2r^3}{4}\right)\partial_G\right), \quad \hat{\mathbf{v}}_8 = -f_8(s)\frac{k^2r^2}{2}\partial_G,\end{aligned}$$

with $f_i(s), i = 1, \dots, 8$, arbitrary functions. However, by appropriate choice of polynomials in s for $f_i(s)$ the group generators reduce to the eight generators (13).

The PDEs the inherited symmetries of which include all the symmetries in (13) are $G_{rr} = M$, $G_{ss} = N$, where $M = \frac{c^2}{bd-ac}$ for $b \neq 0$ and $N = \frac{d^2}{bd-ac}$ for $d \neq 0$. These two equations can be easily derived by substituting some differential consequences of the side condition (14) into (10).

4 Conclusions

The Type-II hidden symmetries are extra symmetries that appear when the number of variables of a PDE is reduced by a variable transformation found from a Lie symmetry of the PDE. In [2] two methods were presented for finding one or more PDE's from which the Type-II hidden symmetries are inherited. In [5] and [6] we have analyzed the connection between one of these methods and weak symmetries of the PDE with special differential constraints in order to determine the source

of the Type-II hidden symmetries. In this work we have considered the SWW equation presented in [7], and the second heavenly equation [1], which reduces, in the case of three fields, to a single second order equation of Monge-Ampère type.

We can identify the PDE's, which has been found previously by guessing, by using as differential constraint the *side condition* from which the reduction has been derived. The significance of these Type-II hidden symmetries is that there may be more symmetries in the subsequent reduced differential equations than can be predicted from the Lie algebra of the original PDE.

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On Lie symmetries of a class of reaction-diffusion equations

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Complete group classification of a class of reaction-diffusion equations of form $u_t = u_{xx} + k(x)u^2(1 - u)$ is given. Lie symmetries are used to reduce these reaction-diffusion equations to ordinary differential equations.

We consider reaction-diffusion equation of the form

$$u_t = u_{xx} + k(x)u^2(1 - u), \quad (1)$$

where $k(x) \neq 0$. This equation models many phenomena that occur in different areas of mathematical physics and biology. In particular, it can be used to describe the spread of a recessive advantageous allele through a population in which there are only two possible alleles at the locus in question. Equation (1) is interesting also in the area of nerve axon potentials [5]. For more details about application see [1, 2] and references therein.

Motivated by this, Bradshaw-Hajek *et al* [1, 2] started studying of class (1) from the symmetry point of view. More precisely, they found some cases of equations (1) admitting Lie and/or nonclassical symmetries. Our intention is to complete this analysis of nonclassical symmetries and construct a list of new exact solutions of the equations under consideration. Since classification of nonclassical symmetries is impossible without detailed knowledge of Lie invariance properties, in this short note we restrict ourselves to study classical Lie symmetries of equations from class (1). Detailed investigation of nonclassical symmetries for class (1) will form a subject of a sequel paper.

In the classical Ovsiannikov's formulation [4] exhaustive consideration of the problem of group classification for a class of systems of differential equations includes the following steps: construction of the equivalence group; finding the kernel of maximal invariance groups of local transformations that are symmetries for all equations from the given class and description of all possible inequivalent (with respect to equivalence group) values of parameters that admit maximal invariance groups wider than the kernel group. Following S. Lie, one usually considers infinitesimal transformations instead of finite ones. This approach essentially simplifies the problem of group classification, reducing it to problems for Lie algebras of vector fields.

Group classification of class (1) will be performed up to equivalence transformation group G^\sim containing scaling and translation transformations of independent variables t and x .

We look for infinitesimal generators of maximal Lie groups of equations from class (1) in form

$$Q = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u.$$

Substituting the coefficients of operator Q into the Lie–Ovsiannikov infinitesimal invariance criterium [3, 4] and splitting the obtained equation with respect to unconstrained derivatives u_{xx} and powers of u_x , we obtain a system of determining equations. Integrating them give immediately that

$$\tau = \tau(t), \quad \xi = \xi(t, x), \quad \eta = \eta^1(t, x)u + \eta^0(t, x).$$

Then, the rest of equations take form

$$\begin{aligned} 2\xi_x &= \tau_t, \quad -2\eta_x^1 + \xi_{xx} - \xi_t = 0, \\ \xi k_x + 2\eta^1 k + \tau_t k &= 0, \quad -\xi k_x - \eta^1 k + 3k\eta^0 - \tau_t k = 0, \\ -\eta_{xx}^1 - 2\eta^0 k + \eta_t^1 &= 0, \quad \eta_t^0 - \eta_{xx}^0 = 0. \end{aligned}$$

From the first and second equation we have that $\xi_{xx} = 0$ and η^1 is at most quadratic with respect to x . From the third and fourth equations we obtain that $\eta^0 = -\eta^1/3$. Then, from the last two equations we derive $k\eta^0 = 0$. Since $k \neq 0$, we have $\eta^0 = \eta^1 = 0$. Finally, we get very simple system for coefficients of symmetry generator:

$$2\xi_x = \tau_t, \quad \xi_{xx} = 0, \quad \xi_t = 0, \quad \xi k_x + \tau_t k = 0.$$

The last equation depends explicitly on k and is called classifying equation. With respect to k and x it looks like $(a_1x + a_2)k_x + 2a_1k = 0$. Considering all possible cases of integration of this equation (up to transformations from G^\sim) we obtain the complete group classification of class (1).

Theorem 1. *The kernel Lie algebra of class (7) is $\langle \partial_t \rangle$. There exists two G^\sim -inequivalent cases of extension of the maximal Lie invariance algebra (the values of k are given together with the corresponding maximal Lie invariance algebras):*

- 1 : $k = c, \quad \langle \partial_t, \partial_x \rangle;$
- 2 : $k = cx^{-2}, \quad \langle \partial_t, 2t\partial_t + x\partial_x \rangle.$

Here $c = \text{const.}$

Lie symmetries can be used for construction of exact solutions of the partial differential equations. However, the obtained maximal Lie invariance algebras are not very wide and give rise to limited classes of exact invariant solutions.

Thus, e.g., besides the solution $u = \text{const}$, equation $u_t = u_{xx} + cu^2(1 - u)$ admits only one family of invariant solutions, namely invariant travelling wave solution $u = v(x - \alpha t)$ being solution of the ordinary differential equation $v'' + \alpha v' + cv^2(1 - v) = 0$.

Optimal system of one-dimensional subalgebras of Lie symmetry algebra of equation $u_t = u_{xx} + cx^{-2}u^2(1 - u)$ consists of $\langle \partial_t \rangle$ and $\langle 2t\partial_t + x\partial_x \rangle$. Reduction with respect to $\langle \partial_t \rangle$ gives stationary solution $u = v(x)$, where $v'' + cx^{-2}v^2(1 - v) = 0$. Reduction with respect to $\langle 2t\partial_t + x\partial_x \rangle$ has the form $u = v(\omega)$, $\omega = t^{-1/2}x$ and $2\omega^2v'' + \omega^3v' + 2cv^2(1 - v) = 0$.

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A geometric derivation of KdV-type hierarchies from root systems

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For the root system of each semi-simple complex Lie algebra of rank two and for the associated 2D Toda chain $\mathcal{E} = \{u_{xy} = \exp(K\mathbf{u})\}$, we calculate the first integrals of the characteristic equation $D_y(w) \doteq 0$ on \mathcal{E} . Using the integrals, we reconstruct and make coordinate-independent the (2×2) -matrix operators \square in total derivatives that generate symmetries of the chains. Writing other factorizations that involve the operators \square , we obtain pairs of compatible Hamiltonian operators that produce KdV-type hierarchies of symmetries for \mathcal{E} . Having thus reduced the problem to the Hamiltonian case, we calculate the Lie-type brackets, which are transferred from the commutators of the symmetries in the images of the operators \square onto their domains. With all this, we describe the generators and derive the commutation relations in the symmetry algebras of the 2D Toda chains, which serve here as an illustration for a much more general algebraic and geometric setup.

Introduction

In the paper [12] we introduced a well-defined notion of linear matrix operators in total derivatives, whose images in the Lie algebras of evolutionary vector fields on the jet spaces are closed with respect to the commutation. This yields a generalization for the classical theory of recursion operators and Poisson structures for integrable systems. We explained how each operator transfers the commutation of the vector fields to the Lie brackets with bi-differential structural constants on the quotient of its domain by the kernel.

Second, in [9, 10] we associated such operators with the 2D Toda chains

$$\mathcal{E}_{\text{Toda}} = \left\{ u_{xy}^i = \exp\left(\sum_{j=1}^m K_j^i u^j\right), 1 \leq i \leq m \right\} \quad (1)$$

related to semi-simple complex Lie algebras [14, 15]. We derived an explicit formula for the operators that determine higher symmetries of these chains. Using the auxiliary Hamiltonian matrix operators, we elaborated a procedure that yields all the commutation relations in the symmetry Lie algebras $\text{sym } \mathcal{E}_{\text{Toda}}$ (naturally, these symmetry algebras are not commutative). This solved a long-standing problem in geometry of differential equations and completed previously known results by Leznov, Meshkov, Shabat, Sokolov, and others (see [15, 17, 21, 25] and references therein).

Actually, the general scheme of [10] is applicable to the description of symmetry algebras for a wider class of the Euler–Lagrange hyperbolic systems of Liouville type [21, 25]. Moreover, the group analysis of integrable systems, as a motivation, results in the well-defined concept [12] of operators whose images span involutive distributions on the jet spaces, but not on differential equations, which is of an independent interest.

In this note we illustrate the reasonings of [10] using the root systems of the semi-simple complex Lie algebras of rank two. Among all two-component exponential nonlinear systems (1), these 2D Toda chains with Cartan matrices K admit the largest groups of conservation laws [21] and are integrable in quadratures [14].

The equality of the rank to two means the following:

- The hyperbolic Toda chains (1) upon u^1, u^2 are, we repeat, *two*-component.
- The number of vector fields Y_i that generate the characteristic Lie algebra through commutators (see section 2.1 below and [15]) equals *two*. Also, the numbers of linear independent iterated commutators $Y_{(i_1, \dots, i_k)} = [Y_{i_1}, [\dots [Y_{i_{k-1}}, Y_{i_k}] \dots]]$ for fixed k is at most *two*. This number drops at most *twice*, and $Y_{(\vec{i})} \equiv 0$ for k large. Thence, by the Frobenius theorem, *two* invariants w^1, w^2 appear.¹ Using the characteristic Lie algebras, we introduce *two* finite sequences of the adapted coordinates; this choice simplifies the description of these invariants. On the other hand, we use the *two* invariants to replace the derivatives of the *two* dependent variables at all sufficiently high differential orders.
- Conservation laws for Toda system (1) are differentially generated (up to $x \leftrightarrow y$) by these *two* invariants, which are solutions of the characteristic equation $D_y(w) \doteq 0$ on $\mathcal{E}_{\text{Toda}}$.
- Higher symmetries of the Toda chain (1) have a functional freedom and are parameterized by *two* functions ϕ^1, ϕ^2 that depend on x and any derivatives of the integrals w^i up to a certain differential order.
- The differential operators \square that yield symmetries of (1), when applied to the tuples ${}^t(\phi^1, \phi^2)$, are (2×2) -matrices.
- The Lie algebra structures transferred from $\text{sym } \mathcal{E}_{\text{Toda}}$ to the domains of \square are described by the bi-differential brackets $\{\{, \}\}_{\square}$ that contain *two* components.
- The Hamiltonian structures \hat{A}_k that are defined on the domains of the operators \square but take values elsewhere (in the Lie algebra of velocities of the integrals w^i , see [25] and [10]), are also (2×2) -matrices. Likewise, the brackets $\{\{, \}\}_{\hat{A}_k}$ transferred onto the domains of \hat{A}_k from the commutators of Hamiltonian vector fields in their images are also *two*-component (thence the equality $\{\{, \}\}_{\square} = \{\{, \}\}_{\hat{A}_k}$ makes sense).

¹For example, the paper [24] contains a brute force classification of integrable one-component hyperbolic equations with respect to the low-dimensional characteristic Lie algebras.

- The KdV-type hierarchies of velocities of w^i and the modified KdV-type hierarchies of commuting Noether symmetries of $\mathcal{E}_{\text{Toda}}$, which are related by two-component Miura's substitutions with the former, are composed by the (right-hand sides of) two-component evolutionary systems.

We use the constructions and follow the notation of [9, 10, 12] except for the characteristic Lie algebras that were introduced in [15] and were discussed in detail in [20]. All notions from the geometry of PDE are standard (see [2, 13, 18]). All extensive calculations were performed using the software [16].

To begin with, we recall that in the fundamental paper [21] A. B. Shabat *et al.* proved the existence of maximal ($r = \bar{r} = m$) sets of conserved densities

$$w_1, \dots, w_r \in \ker D_y|_{\mathcal{E}_{\text{Toda}}}, \quad \bar{w}_1, \dots, \bar{w}_{\bar{r}} \in \ker D_x|_{\mathcal{E}_{\text{Toda}}}. \quad (2)$$

for $\mathcal{E}_{\text{Toda}}$ if and only if the matrix K in (1) is the Cartan matrix of a root system for a semi-simple complex Lie algebra of rank r , which is always the case in what follows with $r = 2$. Note that the integrals (2) allow to replace the derivatives of unknown functions of any sufficiently high order using the derivatives of the integrals. In [20] A. B. Shabat proposed an iterative procedure that specifies an adapted system of the remaining lower-order coordinates and that makes linear the *coefficients* of the linear first-order characteristic equation $D_y(w) \doteq 0$ on $\mathcal{E}_{\text{Toda}}$. That algorithm is self-starting; it considerably simplifies the search for the first integrals of the characteristic equation and gives the estimate for the differential orders of solutions.

Another method (which we do not use here) for finding the first integrals is based on the use of Laplace's invariants, see [6, 25]. The authors of [25] investigated (primarily, in the case of one unknown function and one equation \mathcal{E} upon it) the operators that factor symmetries of \mathcal{E} . Also there, the pioneering idea to study the operators whose images are closed under the commutation was proposed. We indicate further the papers [4, 9, 10, 22, 23] that address the problem of construction of such operators for multi-component hyperbolic systems of the Liouville type.

In [9] and [10] we obtained explicit formulas for the operators \square that yield higher symmetries of the Euler–Lagrange Liouville-type systems and for the bi-differential brackets on their domains, respectively. The former construction yields the generators of the symmetry algebras for such systems and the latter describes the commutation relations. The general concept of total differential operators whose images determine involutive distributions on the infinite jet spaces has been elaborated in [12].

This paper is structured as follows. First we outline our basic concept using the scalar Liouville equation $u_{xy} = \exp(2u)$ as the motivating example. This covers the case of the root system A_1 . The following fact, which holds true for any rank $r \geq 1$, is very convenient in practice: the differential orders of the r integrals w^1, \dots, w^r for the 2D Toda systems (1) associated with the semi-simple complex Lie algebras \mathfrak{g} are equal (up to a shift by +1) to the exponents of \mathfrak{g} , see [21]; the *table* of values for the respective orders is given in [8, §14.2].

Then we realize the geometric scheme

$$D_y(w) \doteq 0 \longmapsto w^1, w^2 \longmapsto \square \longmapsto \hat{A}_k \longmapsto \{\{, \}\}\square; \quad \hat{A}_k \longmapsto \hat{A}_1^{(\cdot)},$$

for the simple complex rank two Lie algebras (for the root systems A_2 , $B_2 \simeq C_2$, and G_2). In other words, we associate the operators to invariants, the brackets to operators, and find the deformations of the Poisson structures. Only once, for the root system A_2 , we calculate the characteristic Lie algebra for the corresponding 2D Toda chain (1) and obtain the integrals w^1, w^2 using the adapted system of coordinates. We modify the scheme of [20] such that, first, the two-component Toda chain is not represented as a reduction of the infinite chain and, second, we do not introduce an excessive third field which is compensated by using a constraint (as in [20]).

Remark 1. We do not of course re-derive the structures for the algebra $D_2 = A_1 \oplus A_1$ that is not simple, since the operator \square and KdV's second Hamiltonian structure \hat{A}_2 are known for each of the two uncoupled components of the chain (1) with $K = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, see Example 1 below. However, the “ x -component” of the full symmetry algebra with the generators

$$\varphi = \begin{pmatrix} \square^{\#1} & 0 \\ 0 & \square^{\#2} \end{pmatrix} \begin{pmatrix} \phi^1(x, [w^{\#1}], [w^{\#2}]) \\ \phi^2(x, [w^{\#1}], [w^{\#2}]) \end{pmatrix}$$

is *not* just the direct sum of the two symmetry subalgebras for the two Liouville equations. Indeed, this formula shows that the integrals can be intertwined in the generators, although the fields are not coupled in system (1) with the choice of K as above. This proves that the symmetries intertwine the fields.

Remark 2. Each of the operators \square , which we obtain from the r integrals, consists of r columns, one column for each integral. In the $r = 2$ case, only the respective first columns were specified in the encyclopaedia [1], see also [17]. Here we complete the description of the symmetry generators.

1 Basic concept

Let us begin with a motivating example.

Example 1 (The Liouville equation). Consider the scalar Liouville equation $\mathcal{E}_{\text{Liou}} = \{u_{xy} = \exp(2u)\}$. The differential generators w, \bar{w} of its conservation laws $[\eta] = \int f(x, [w]) dx + \int \bar{f}(y, [\bar{w}]) dy$ are $w = u_x^2 - u_{xx}$ and $\bar{w} = u_y^2 - u_{yy}$ such that $D_y(w) \doteq 0$ and $D_x(\bar{w}) \doteq 0$ by virtue (\doteq) of $\mathcal{E}_{\text{Liou}}$ and its differential consequences. The operators $\square = u_x + \frac{1}{2}D_x$ and $\bar{\square} = u_y + \frac{1}{2}D_y$ determine higher ($\varphi, \bar{\varphi}$) and Noether (resp., $\varphi_{\mathcal{L}}, \bar{\varphi}_{\mathcal{L}}$) symmetries of the Liouville equation as follows,

$$\begin{aligned} \varphi &= \square(\phi(x, [w])), \quad \varphi_{\mathcal{L}} = \square\left(\frac{\delta \mathcal{H}(x, [w])}{\delta w}\right); \\ \bar{\varphi} &= \bar{\square}(\bar{\phi}(y, [\bar{w}])), \quad \bar{\varphi}_{\mathcal{L}} = \bar{\square}\left(\frac{\delta \bar{\mathcal{H}}(y, [\bar{w}])}{\delta \bar{w}}\right) \end{aligned}$$

for any smooth $\phi, \bar{\phi}$ and $\mathcal{H}, \bar{\mathcal{H}}$. Note that the operator $\square = \frac{1}{2}D_x^{-1} \circ (\ell_w^{(u)})^*$ is obtained using the adjoint linearization of w , and similarly for $\bar{\square}$.

Each of the images of \square and $\bar{\square}$ is closed w.r.t. the commutation such that, in particular,

$$[\square(p), \square(q)] = \square([p, q]_\square), \quad p = p(x, [w]), \quad q = q(x, [w]),$$

where the bracket $[,]_\square$ on the domain of \square admits the standard decomposition (the vector field $\partial_\varphi = \sum_k D_x^k(\varphi) \cdot \partial/\partial u_k$ is the evolutionary derivation),

$$[p, q]_\square = \partial_{\square(p)}(q) - \partial_{\square(q)}(p) + \{\{p, q\}\}_\square.$$

For the operator \square on the Liouville equation, the bi-differential bracket $\{\{, \}\}_\square$ is

$$\{\{p, q\}\}_\square = D_x(p) \cdot q - p \cdot D_x(q),$$

and similar formulas hold for the operator $\bar{\square}$. The symmetry algebra $\text{sym } \mathcal{E}_{\text{Liou}} \simeq \text{im } \square + \text{im } \bar{\square}$ is the sum of images of these operators, and the two summands commute between each other. $[\text{im } \square, \text{im } \bar{\square}] \doteq 0$ on $\mathcal{E}_{\text{Liou}}$.

The operator \square also gives higher symmetries of the potential modified KdV equation $\mathcal{E}_{\text{pmKdV}} = \{u_t = -\frac{1}{2}u_{xxx} + u_x^3 = \square(w)\}$, whose commutative hierarchy is composed by Noether symmetries $\varphi_{\mathcal{L}} \in \text{im}(\square \circ \delta/\delta w)$ of the Liouville equation. The operator \square factors the second Hamiltonian structure $B_2 = \square \circ A_1 \circ \square^*$ for $\mathcal{E}_{\text{pmKdV}}$, here $A_1 = D_x^{-1} = \hat{A}_1^{-1}$ is the first Hamiltonian operator for the potential KdV equation and equals the inverse of the first Hamiltonian operator for KdV.

The generator w of conservation laws for $\mathcal{E}_{\text{Liou}}$ provides the Miura substitution from $\mathcal{E}_{\text{pmKdV}}$ to the Korteweg–de Vries equation $\mathcal{E}_{\text{KdV}} = \{w_t = -\frac{1}{2}w_{xxx} + 3ww_x\}$. The second Hamiltonian structure for \mathcal{E}_{KdV} is factored to the product $\hat{A}_2 = \square^* \circ \hat{B}_1 \circ \square$, where $\hat{B}_1 = D_x$ is the first Hamiltonian structure for the modified KdV. The bracket $\{\{, \}\}_\square$ on the domain of \square is equal to the bracket $\{\{, \}\}_{\hat{A}_2}$ induced on the domain of the operator \hat{A}_2 (which is Hamiltonian and hence its image is closed under commutation) for \mathcal{E}_{KdV} . In what follows, we refer to these correlations as standard, see [9].

Definition 1 ([25]). A *Liouville-type system*² \mathcal{E}_L is a system $\{u_{xy} = F(u, u_x, u_y; x, y)\}$ of hyperbolic equations which admits nontrivial *first integrals*, see (2), for the linear first order characteristic equations $D_y|_{\mathcal{E}_L}(w_i) \doteq 0$ and $D_x|_{\mathcal{E}_L}(\bar{w}_j) \doteq 0$ that hold by virtue (\doteq) of \mathcal{E}_L .

Example 2. The m -component 2D Toda chains (1) associated with semi-simple complex Lie algebras [14] constitute an important class of Liouville-type systems, here $u = (u^1, \dots, u^m)$. Further on, we consider these exactly solvable systems, bearing in mind that the reasonings remain applicable to a wider class of the Euler–Lagrange Liouville-type systems \mathcal{E}_L . We also note that all conservation laws for the 2D Toda systems \mathcal{E}_L at hand are of the form $\int f(x, [w]) dx \oplus \int g(y, [\bar{w}]) dy$.

²There exist other, non-equivalent definitions of the Liouville type systems.

Remark 3. The 2D Toda systems (1) are Euler–Lagrange with the Lagrangian density $L = -\frac{1}{2}\langle \kappa u_x, u_y \rangle - H_L(u; x, y)$. The $(m \times m)$ -matrix κ with the entries $\kappa_{ij} = 2\langle \alpha_i, \alpha_j \rangle \cdot |\alpha_i|^{-2} \cdot |\alpha_j|^{-2} = K_j^i / |\alpha_i|^2$ is determined by the simple roots α_k of the semi-simple Lie algebra. Let $\mathfrak{m} = \partial L / \partial u_y$ be the momenta, then it can be readily seen that the integrals w^1, \dots, w^m of the characteristic equation are differential functions $w^i = w^i[\mathfrak{m}]$ in \mathfrak{m} .

Proposition 1 ([21]). *The differential orders of the integrals w^i with respect to the momenta \mathfrak{m} for the 2D Toda chains (1) associated with semi-simple complex Lie algebras \mathfrak{g} coincide with the exponents of \mathfrak{g} .*

The following theorem is an adaptation of the main result in [10] to the exponential 2D Toda chains \mathcal{E}_L associated with semi-simple complex Lie algebras. The integrals w^i for such nonlinear Liouville-type hyperbolic system can be obtained using an iterative procedure that is illustrated in section 2.1 below. In the meantime, we assume that the integrals are already known. Let them be *minimal*, meaning that $f \in \ker D_y|_{\mathcal{E}_L}$ implies $f = f(x, [w])$.

Theorem 1. *Let the above assumptions and notation hold. Introduce the operator*

$$\square = (\ell_w^{(\mathfrak{m})})^*, \quad (3)$$

which is the operator adjoint to the linearization (the Fréchet derivative) of the integrals w w.r.t. the momenta \mathfrak{m} . Then we claim the following:

- i. *Up to $x \leftrightarrow y$, Noether symmetries $\varphi_{\mathcal{L}}$ of the Lagrangian \mathcal{L} for the 2D Toda chain \mathcal{E}_L are*

$$\varphi_{\mathcal{L}} = \square \left(\frac{\delta \mathcal{H}(x, [w])}{\delta w} \right) \quad \text{for any } \mathcal{H}.$$

- ii. *Up to $x \leftrightarrow y$, symmetries φ of the system \mathcal{E}_L are $\varphi = \square(\phi(x, [w]))$ for any $\phi = {}^t(\phi^1, \dots, \phi^r)$.*

- iii. *Under a diffeomorphism $\tilde{w} = \tilde{w}[w]$, the r -tuples ϕ are transformed by*

$$\phi \mapsto \tilde{\phi} = [(\ell_{\tilde{w}}^{(w)})^*]^{-1}(\phi).$$

Therefore, under any reparametrization $\tilde{u} = \tilde{u}[u]$ of the dependent variables $\vec{u} = {}^t(u^1, \dots, u^m)$ in equation \mathcal{E}_L , and under a simultaneous change $\tilde{w} = \tilde{w}[w]$, the operator \square obeys the transformation rule

$$\square \mapsto \tilde{\square} = \ell_{\tilde{u}}^{(u)} \circ \square \circ (\ell_{\tilde{w}}^{(w)})^* \Big|_{\substack{w=w[u] \\ u=u[\tilde{u}]}}.$$

- iv. *The operator*

$$\hat{A}_k = \square^* \circ (\ell_{\mathfrak{m}}^{(u)})^* \circ \square \quad (4)$$

is Hamiltonian.

- v. The image of the operator \square is closed with respect to the commutation in the Lie algebra $\text{sym } \mathcal{E}_L$. Consequently, the operator \square is a Frobenius operator of the second kind, see [12].
- vi. The bracket $\{\{, \}\}_{\square}$ on the domain of the operator \square satisfies the equality

$$\{\{, \}\}_{\square} = \{\{, \}\}_{\hat{A}_k}. \quad (5)$$

Its right-hand side is calculated explicitly by using the formula ([13], see also [11]) that is valid for Hamiltonian operators $\hat{A}_k = \|\sum_{\tau} A_{\tau}^{ij} \cdot D_{\tau}\|$,

$$\{\{p, q\}\}_{\hat{A}_k}^{\ell} = \sum_{|\sigma| \geq 0} \sum_{i=1}^m (-1)^{\sigma} \left(D_{\sigma} \circ \left[\sum_{|\tau| \geq 0} \sum_{j=1}^m D_{\tau}(p^j) \cdot \frac{\partial A_{\tau}^{ij}}{\partial u_{\sigma}^{\ell}} \right] \right) (q^i). \quad (6)$$

This yields the commutation relations in the Lie algebra $\text{sym } \mathcal{E}_L$.

- vii. All coefficients of the operator \hat{A}_k and of the bracket $\{\{, \}\}_{\square}$ are differential functions of the minimal conserved densities w for \mathcal{E}_L .

The above theorem is our main instrument that describes the symmetry generators for 2D Toda chains and calculates the commutation relations in the symmetry algebras.

2 The root system A_2

Consider the Euler–Lagrange 2D Toda system associated with the simple Lie algebra $\mathfrak{sl}_3(\mathbb{C})$, see [14, 15, 20],

$$\mathcal{E}_{\text{Toda}} = \left\{ u_{xy} = \exp(2u - v), \ v_{xy} = \exp(-u + 2v), \quad K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \right\}. \quad (7)$$

2.1 The characteristic Lie algebra

We first realize two iterations of the self-adaptive method from [20], which is based on the use of the characteristic Lie algebra, and we obtain two integrals w^1, w^2 of the characteristic equation $D_y(w) \doteq 0$ on (7). Our reasonings differ from the original approach of [20]: we do not introduce excessive dependent variables and hence do not need to compensate their presence with auxiliary constraints.

Our remote goal is a choice of three layers of the adapted variables $b_0^1, b_0^2, b_1^1, b_1^2$, and b_2^1 such that all the coefficients of the linear characteristic equation also become linear. Then all the integrals will be found easily, expressed in these variables. The number of the adapted variables is specified by the problem, and we have to admit that, actually, b_1^2 will be redundant *a posteriori* because it will be replaced using the integral w^1 in the end.

Step 1. Regarding the exponential functions $c(i) := \exp\left(\sum_j K_j^i u^j\right)$ in the right-hand sides of the Toda equations (1) as linear independent, collect the coefficients Y_i of $c(i)$ in the total derivative $D_y = \sum_{i=1}^m c(i) \cdot Y_i$. Clearly, the solution of

the characteristic equation $D_y(w) \doteq 0$ on the Toda chain is equivalent to solution of the system $\{Y_i(w) = 0, 1 \leq i \leq m\}$.

For system (7), we obtain the vector fields

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial u_x} + (2u_x - v_x) \frac{\partial}{\partial u_{xx}} + \underline{\left((2u_x - v_x)^2 + (2u_{xx} - v_{xx}) \right)} \frac{\partial}{\partial u_{xxx}} + \dots, \\ Y_2 &= \frac{\partial}{\partial v_x} + (2v_x - u_x) \frac{\partial}{\partial v_{xx}} + \underline{\left((2v_x - u_x)^2 + (2v_{xx} - u_{xx}) \right)} \frac{\partial}{\partial v_{xxx}} + \dots. \end{aligned} \quad (8)$$

The underlined terms are *quadratic* in derivatives of the fields, and it is our task to make them linear by introducing a convenient system of local coordinates (see take 2 of step 3 below).

Calculating the iterated commutators $Y_{(i_1, \dots, i_k)} := [Y_{i_1}, [\dots, [Y_{i_{k-1}}, Y_{i_k}] \dots]]$ of the basic vector fields Y_i , we generate the *characteristic Lie algebra* [15, 20] for the Toda chain. If this algebra is finite dimensional (which is the case here), then the exponential-nonlinear system (1) is exactly solvable in quadratures; if the characteristic algebra admits a finite dimensional representation, system (1) is integrable by the inverse scattering (*ibid*). For any root system and the Chevalley generators $\mathfrak{e}_n, \mathfrak{f}_n$, and \mathfrak{h}_n of the semi-simple Lie algebra \mathfrak{g} , see [7], the characteristic Lie algebra is isomorphic to the Lie subalgebra of \mathfrak{g} generated by the Chevalley generators \mathfrak{f}_n , see [15].

For A_2 , we obtain the commutator

$$Y_{(2,1)} = -\frac{\partial}{\partial u_{xx}} + \frac{\partial}{\partial v_{xx}} - 3u_x \frac{\partial}{\partial u_{xxx}} + 3v_x \frac{\partial}{\partial v_{xxx}} + \dots.$$

(This manifests a general fact that is always true: the leading terms of the $(k+1)$ -st iterated commutators are the derivations w.r.t. some derivatives u_{k+1}^i , whose order is higher than in the leading terms of the preceding, k -th, iterated commutators.) We finally note that both triple commutators $Y_{(1,2,1)}$ and $Y_{(2,2,1)}$ vanish.

By the Frobenius theorem, a drop of the number of linear independent iterated commutators at the i -th step is equal to the number of first integrals of the characteristic equation that appear at this step. The differential order of these new integrals for the Toda chains will be $i+1$.

For the system (7), there appears one ($1 = \dim\langle Y_i \rangle - \dim\langle Y_{(i_1, i_2)} \rangle$) integral, w^1 , of order 2. The second and last one ($1 = \dim\langle Y_{(i_1, i_2)} \rangle - \langle Y_{(i_1, i_2, i_3)} \rangle \equiv 0$), the integral w^2 , has order 3. For arbitrary root systems, the differential orders (shifted by +1) of the integrals are described by the proposition in the previous section.

Step 2. Our remote goal, see above, will be achieved when the expansion

$$D_x = \sum_{i=1}^m b_1^i Y_i + \sum_{i=1}^{m-1} b_2^i Y_{(i+1, i)} + \dots \quad \text{mod } \mathfrak{Z}: \ker D_y|_{\mathcal{E}_L} \rightarrow \ker D_y|_{\mathcal{E}_L}$$

is found for the *other* total derivative, D_x . Here the vector field \mathfrak{Z} contains only the derivations w.r.t. the (yet unknown) integrals and their derivatives, and the

dots stand for finitely many summands provided that the characteristic algebra is finite dimensional. The former idea expresses the replacement of the higher order field derivatives, u_k^i with $k \gg 1$, using the integrals, while the latter assumption is again based on the fact that there are as many integrals as fields whenever K is a Cartan matrix.

By definition, put $b_0^i := u_x^i$. Substituting the vector fields Y_i contained in (8) for $\partial/\partial u_x^i$ in D_x , we obtain the expansion $D_x = u_{xx} Y_1 + v_{xx} Y_2 + \dots$, where the dots stand for the derivations w.r.t. second and higher order derivatives of the dependent variables. Consequently, we set $b_1^i := u_{xx}^i$.

Step 3. Using the four adapted coordinates b_0^i and b_1^j , we rewrite the basic vector fields Y_k as follows,

$$Y_1 = \frac{\partial}{\partial b_0^1} + (2b_0^1 - b_0^2) \frac{\partial}{\partial b_1^1} + \dots \quad \text{and} \quad Y_2 = \frac{\partial}{\partial b_0^2} + (-b_0^1 + 2b_0^2) \frac{\partial}{\partial b_1^2} + \dots$$

Solving now the system $Y_1(w^1) = 0$, $Y_2(w^1) = 0$ for $w^1(b_0^1, b_0^2, b_1^1, b_1^2)$, we obtain the integral $w^1 = u_{xx} + v_{xx} - u_x^2 + u_x v_x - v_x^2$. We note that, from now on, the coordinate v_{xx} and its descendants can be replaced using w^1 , u_{xx} , and first order derivatives.

Step 1, take 2. Within the second iteration of the algorithm, we repeat steps 1 through 3 advancing one term farther in the expansions.

Let us indeed replace v_{xx} (although it remains an adapted coordinate) with w^1 . Therefore we expand the vector field D_y as

$$D_y = c(1) \cdot Y_1 + c(2) \cdot \frac{\partial}{\partial v_x} \quad \text{mod } \mathfrak{Z}: \ker D_y|_{\mathcal{E}_L} \rightarrow 0,$$

where the derivations w.r.t. w^1 and its descendants cut off the ‘ v -part’ of the total derivative D_y . This yields

$$Y_{(2,1)} = -\frac{\partial}{\partial u_{xx}} + \dots,$$

but now the commutator does not contain any derivations w.r.t. the derivatives of v .

Step 2, take 2. Using the three vector fields, Y_1 , Y_2 , and $Y_{(2,1)}$, we rewrite

$$D_x = u_{xx} Y_1 + v_{xx} Y_2 + ((2u_x - v_x)u_{xx} - u_{xxx}) \cdot Y_{(2,1)} + \dots \quad \text{mod } \mathfrak{Z}: \\ \ker D_y|_{\mathcal{E}_L} \rightarrow \ker D_y|_{\mathcal{E}_L}.$$

Consequently, we set $b_2^1 := (2u_x - v_x)u_{xx} - u_{xxx}$.

Step 3, take 2. Calculating the derivative $D_y|_{\mathcal{E}_L}(b_2^1)$, we substitute it in D_y and collect the coefficients Y_i of the exponential nonlinearities $c(i)$ in this total derivative.

The result is beyond all hopes: the coefficients of both fields, Y_1 and Y_2 , are *linear* in the adapted coordinates,

$$Y_1 = \frac{\partial}{\partial b_0^1} + (2b_0^1 - b_0^2) \cdot \frac{\partial}{\partial b_1^1} + b_1^2 \frac{\partial}{\partial b_2^1} + \dots,$$

$$Y_2 = \frac{\partial}{\partial b_0^2} + (-b_0^1 + 2b_0^2) \cdot \frac{\partial}{\partial b_1^2} - b_1^1 \frac{\partial}{\partial b_2^2} + \dots.$$

In other words, the quadratic terms, which were underlined in (8), are transformed into the linear ones. This is due to the quadratic nonlinearity in the new adapted variable b_2^1 .

Finally, we solve the characteristic equation $Y_1(w^2) = 0$, $Y_2(w^1) = 0$ for $w^2(b_0^1, b_0^2, b_1^1, b_1^2, b_2^1)$ under the assumption³ $\partial w^2 / \partial b_2^1 \neq 0$. We find the solution $w^2 = -b_2^1 - b_0^2 b_1^1 + b_0^1 b_2^1 + (b_0^1)^2 b_0^2 - b_0^1 (b_0^2)^2$. Returning to the original notation, we obtain $w^2 = u_{xxx} - 2u_x u_{xx} + u_x v_{xx} + u_x^2 v_x - u_x v_x^2$. Obviously, the integral w^2 can be used to replace the derivative u_{xxx} and its differential consequences.

We conclude that now, at the endpoint of the algorithm, both total derivatives, D_x and D_y , contain finitely many terms modulo the vector fields that preserve (respectively, annihilate) the kernel $\ker D_y|_{\mathcal{E}_L}$.

In what follows, we do not repeat similar iterative reasonings for the root systems B_2 (see (12)) and G_2 (see p. 100), but write down at once the integrals of orders 2, 4 and 2, 6, respectively. The second integral w^2 for B_2 (with a minor misprint in the last term) and the higher order ‘top’ for w^2 for G_2 are available in the encyclopaedia [1].

2.2 The symmetry algebra: operators and brackets

From the previous section we know the minimal integrals

$$w^1 = u_{xx} + v_{xx} - u_x^2 + u_x v_x - v_x^2,$$

$$w^2 = u_{xxx} - 2u_x u_{xx} + u_x v_{xx} + u_x^2 v_x - u_x v_x^2,$$

for the 2D Toda chain (7) associated with the root system A_2 . Hence we are at the starting point for the description of its symmetry algebra and construction of Hamiltonian operators for the corresponding KdV-type hierarchies. Let us introduce the momenta $\mathfrak{m}^1 := 2u_x - v_x$, $\mathfrak{m}^2 := 2v_x - u_x$, whence we express the integrals as follows,

$$w^1 = 3\mathfrak{m}_x^1 + 3\mathfrak{m}_x^2 - (\mathfrak{m}^1)^2 - \mathfrak{m}^1 \mathfrak{m}^2 - (\mathfrak{m}^2)^2,$$

$$w^2 = 2\mathfrak{m}_{xx}^1 + \mathfrak{m}_{xx}^2 - 2\mathfrak{m}^1 \mathfrak{m}_x^1 - \mathfrak{m}^2 \mathfrak{m}_x^1 + \frac{2}{9}(\mathfrak{m}^1)^3$$

$$+ \frac{1}{3}(\mathfrak{m}^1)^2 \mathfrak{m}^2 - \frac{1}{3}\mathfrak{m}^1 (\mathfrak{m}^2)^2 - \frac{2}{9}(\mathfrak{m}^2)^3.$$

³A practically convenient feature of the algorithm is that it allows to fix the ‘top’ (the higher order terms) of the first integrals in advance, whence the redundant freedom in adding derivatives of the previously found lower order solutions is eliminated.

By the general scheme of [9, 10], the symmetries (up to $x \leftrightarrow y$) of (7) are of the form $\varphi = \square(\vec{\phi}(x, [w^1], [w^2]))$, where $\vec{\phi} = {}^t(\phi^1, \phi^2)$ is a pair of arbitrary functions and the (2×2) -matrix operator in total derivatives is given by formula (3),

$$\square = \begin{pmatrix} u_x + D_x & -\frac{2}{3}D_x^2 - u_x D_x - \frac{1}{3}u_x^2 - \frac{2}{3}u_x v_x + \frac{2}{3}v_x^2 + \frac{1}{3}u_{xx} - \frac{2}{3}v_{xx} \\ v_x + D_x & -\frac{1}{3}D_x^2 + \frac{2}{3}u_{xx} - \frac{1}{3}v_{xx} - \frac{2}{3}u_x^2 + \frac{2}{3}u_x v_x + \frac{1}{3}v_x^2 \end{pmatrix}. \quad (9)$$

Next, we calculate the Hamiltonian operator (4), $\hat{A}_k = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where for the root system A_2 we have that⁴

$$\begin{aligned} A_{11} &= 2D_x^3 + 2w^1 D_x + w_x^1, \\ A_{12} &= -D_x^4 - w^1 D_x^2 + (3w^2 - 2w_x^1) \cdot D_x + (2w_x^2 - w_{xx}^1), \\ A_{21} &= D_x^4 + w^1 D_x^2 + 3w^2 D_x + w_x^2 \\ A_{22} &= -\frac{2}{3}D_x^5 - \frac{4}{3}w^1 D_x^3 - 2w_x^1 D_x^2 + (2w_x^2 - 2w_{xx}^1 - \frac{2}{3}(w^1)^2) \cdot D_x \\ &\quad + \frac{1}{3}(3w_{xx}^2 - 2w_{xxx}^1 - 2w^1 w_x^1). \end{aligned}$$

The bracket (6) for \hat{A}_k equals

$$\begin{aligned} \{\{\vec{p}, \vec{q}\}\}_{\hat{A}_k}^1 &= \boxed{p_x^1 q^1 - p^1 q_x^1} + \boxed{p_{xx}^1 q^2 - p^2 q_{xx}^1} \\ &\quad + \frac{2}{3}(p^2 q_{xxx}^2 - p_{xxx}^2 q^2) + \frac{2}{3}w^1(p^2 q_x^2 - p_x^2 q^2), \end{aligned} \quad (10a)$$

$$\{\{\vec{p}, \vec{q}\}\}_{\hat{A}_k}^2 = \boxed{p_x^2 q^1 - p^1 q_x^2} + \boxed{2(p_x^1 q^2 - p^2 q_x^1)} + \boxed{p^2 q_{xx}^2 - p_{xx}^2 q^2}. \quad (10b)$$

Consequently, not only the image of the entire operator (9) is closed under the commutation, but the image of the first column, of first order, is itself closed under commutation (see, e.g., [9] or the encyclopaedia [1], where only the first column of (9) is presented). However, the image of the second column of $\square = (\square_1, \square_2)$ is not closed under the commutation. Indeed, we box the individual bracket $\{\{\cdot, \cdot\}\}_{\square_1}$ for the (2×1) -matrix operator \square_1 , and we underline the couplings of components in the domain of \square_2 ; under commutation, they hit both images of the first and second columns.

Performing the shift $w^2 \mapsto w^2 + \lambda$ of the second integral, and taking the velocity of the operator \hat{A}_k ,

$$\hat{A}_1^{(2)} := \frac{d}{d\lambda} \Big|_{\lambda=0} (\hat{A}_k),$$

we obtain the ‘junior’ Hamiltonian operator $\hat{A}_1^{(2)} = \begin{pmatrix} 0 & 3D_x \\ 3D_x & 0 \end{pmatrix}$ that is compatible with the former. Obviously, the bracket $\{\{\cdot, \cdot\}\}_{\hat{A}_1^{(2)}}$ on the domain of $\hat{A}_1^{(2)}$ vanishes identically. (We note that the analogous operator $\hat{A}_1^{(1)} = \frac{d}{d\mu} \Big|_{\mu=0} (\hat{A}_k)$, where

⁴For any root system of any rank, the upper-left corner A_{11} of the operator (4) is the second Hamiltonian structure for the KdV equation. The leading coefficient β at D_x^3 depends on the root system, see [9].

$w^1 \mapsto w^1 + \mu$, is not Hamiltonian at all.) Reciprocally, this ‘senior’ Hamiltonian operator \hat{A}_2 can be obtained by taking the Lie derivative of the variational Poisson bi-vector that corresponds to $\hat{A}_1^{(2)}$ along a suitable evolutionary vector field [3].

The pair $(\hat{A}_1^{(2)}, \hat{A}_k)$ is the well-known bi-Hamiltonian structure for the Boussinesq equation (e.g., see [18])

$$w_t^1 = 2w_x^2 - w_{xx}^1, \quad w_t^2 = -\frac{2}{3}w_{xxx}^1 - \frac{2}{3}w^1w_x^1 + w_{xx}^2. \quad (11)$$

Indeed, we have that

$$\vec{w}_t = \hat{A}_1^{(2)} \frac{\delta}{\delta \vec{w}} \int \frac{1}{3} \left[w^1 w_x^2 + \frac{1}{3}(w_x^1)^2 - \frac{1}{6}(w^1)^2 + (w^2)^2 \right] dx = \hat{A}_k \frac{\delta}{\delta \vec{w}} \int w^2 dx.$$

Both densities, w^1 and w^2 , are conserved on system (11). The symmetry $\vec{w}_x = \hat{A}_k \frac{\delta}{\delta \vec{w}} \int w^1 dx$ starts the second sequence of Hamiltonian flows in the Boussinesq hierarchy \mathfrak{A} , see [9] and references therein.

The modified Boussinesq hierarchy \mathfrak{B} shares the two sequences of Hamiltonians with the Boussinesq hierarchy itself by virtue of the Miura substitution $w = w[\mathfrak{m}]$ with $\mathfrak{m} = \mathfrak{m}[u]$. Namely, for any Hamiltonian $\mathcal{H}[w]$, the flows

$$u_\tau = \delta \mathcal{H}[\mathfrak{m}] / \delta \mathfrak{m}, \quad \mathfrak{m}_\tau = -\delta \mathcal{H}[\mathfrak{m}[u]] / \delta u$$

belong to the modified hierarchy \mathfrak{B} . The correlation between the two hierarchies, \mathfrak{A} and \mathfrak{B} , and the Hamiltonian structures,

$$\hat{A}_k, \quad \hat{A}_1^{(2)} = (A_1)^{-1}, \quad A_k, \text{ and } \hat{B}_1 = (\ell_{\mathfrak{m}}^{(u)})^* = B_1^{-1}, \quad \hat{B}_{k'}, \quad B_{k'},$$

for their potential and nonpotential components are standard, see the diagram in [9]. The velocities u_τ constitute the commutative subalgebra of Noether symmetries of the 2D Toda chain (7).

3 The root system B_2

The Toda system is specified by the Cartan matrix $K = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$:

$$u_{xy} = \exp(2u - 2v), \quad v_{xy} = \exp(-u + 2v).$$

The integrals for it are of orders 1 and 3 with respect to the momenta:

$$w^1 = u_{xx} + 2v_{xx} - 2v_x^2 + 2v_x u_x - u_x^2, \quad (12a)$$

$$w^2 = v_{4x} + v_x(u_{xxx} - 2v_{xxx}) + u_{xx}v_x(v_x - 2u_x) \quad (12b)$$

$$\begin{aligned} &+ v_{xx}(4v_x u_x - 2v_x^2 - u_x^2) + v_{xx}(u_{xx} - v_{xx}) \\ &+ v_x^4 + v_x^2 u_x^2 - 2v_x^3 u_x. \end{aligned} \quad (12c)$$

Hence the Frobenius operator (3) is

$$\square = (\square^1, \square^2) = \begin{pmatrix} \square_{11} & \square_{12} \\ \square_{21} & \square_{22} \end{pmatrix}, \quad \text{where} \quad \square^1 = \begin{pmatrix} u_x + 2D_x \\ v_x + \frac{3}{2}D_x \end{pmatrix} \quad \text{and}$$

$$\begin{aligned}\square_{12} &= D_x^3 + (u_{xx} - u_x^2) \cdot D_x + (2v_x^2 u_x - 4v_{xx} v_x + 2u_{xx} v_x - 2v_x u_x^2 + 2v_{xxx}), \\ \square_{22} &= D_x^3 + v_x D_x^2 + (2v_x u_x - v_x^2 - u_x^2 + v_{xx} + u_{xx}) \cdot D_x \\ &\quad + (4v_x^2 u_x - 2v_{xx} v_x + 2u_{xx} v_x - 2v_x u_x^2 + 2v_{xxx} - 2v_x^3).\end{aligned}$$

The Hamiltonian operator $\hat{A}_k = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ has the components

$$\begin{aligned}A_{11} &= 10D_x^3 + 4w^1 D_x + 2w_x^1, \\ A_{12} &= 3D_x^5 + 3w^1 D_x^3 + 6w_x^1 D_x^2 + (3w_{xx}^1 + 8w^2) \cdot D_x + 6w_x^2, \\ A_{21} &= 3D_x^5 + 3w^1 D_x^3 + 3w_x^1 D_x^2 + 8w^2 D_x + 2w_x^2, \\ A_{22} &= D_x^7 + 2w^1 D_x^5 + 5w_x^1 D_x^4 + (6w_{xx}^1 + 6w^2 + (w^1)^2) \cdot D_x^3 \\ &\quad + (4w_{xxx}^1 + 3w^1 w_x^1 + 9w_x^2) \cdot D_x^2 \\ &\quad + (w_{4x}^1 + 7w_{xx}^2 + (w_x^1)^2 + 4w^1 w^2 + w^1 w_{xx}^1) \cdot D_x \\ &\quad + 2 \cdot (w_x^1 w^2 + w_{xxx}^2 + w_x^2 w^1).\end{aligned}$$

Therefore the components of the brackets (5) for both \hat{A}_k and \square are

$$\begin{aligned}\{\{\vec{p}, \vec{q}\}\}_{\square}^1 &= 2(p^1 q_x^1 - p_x^1 q^1) + 3(p_{xx}^1 q_x^2 - p_x^2 q_{xx}^1) + (p_{4x}^2 q_x^2 - p_x^2 q_{4x}^2) \\ &\quad + w^1(p_{xx}^2 q_x^2 - p_x^2 q_{xx}^2) + 2w^2(p^2 q_x^2 - p_x^2 q^2), \\ \{\{\vec{p}, \vec{q}\}\}_{\square}^2 &= 6(p^2 q_x^1 - p_x^1 q^2) + 2(p^1 q_x^2 - p_x^2 q^1) + 2(p^2 q_{xxx}^2 - p_{xxx}^2 q^2) \\ &\quad + (p_{xx}^2 q_x^2 - p_x^2 q_{xx}^2) + 2w^1(p^2 q_x^2 - p_x^2 q^2).\end{aligned}$$

Similar to the case of (10), the commutation of symmetries that belong to the image of the second column \square^2 of the operator $\square = (\square^1, \square^2)$ for B_2 hits the image of the first column \square^1 .

The ‘junior’ Hamiltonian operator $\hat{A}_1^{(2)} = \frac{d}{d\lambda}|_{\lambda=0} \hat{A}_k$ is again obtained by taking the shift $w^2 \mapsto w^2 + \lambda$ in \hat{A}_k :

$$\hat{A}_1^{(2)} = \begin{pmatrix} 0 & 8D_x \\ 8D_x & 6D_x^3 + 4w^1 D_x + 2w_x^1 \end{pmatrix}.$$

The new Hamiltonian operator is compatible with \hat{A}_k . The bracket on the domain of $\hat{A}_1^{(2)}$ is given by $\{\{\vec{p}, \vec{q}\}\}_{\hat{A}_1^{(2)}}^1 = 2(p_x^2 q^2 - p^2 q_x^2)$, $\{\{\vec{p}, \vec{q}\}\}_{\hat{A}_1^{(2)}}^2 = 0$. Likewise to the root system A_2 , the shift $w^1 \mapsto w^1 + \mu$ produces the operator $\hat{A}_1^{(1)} = \frac{d}{d\mu}|_{\mu=0} \hat{A}_k$ which is not Hamiltonian.

The pair $(\hat{A}_1^{(2)}, \hat{A}_k)$ determines the hierarchy of the KdV-type system

$$w_t^1 = 6w_x^2, \quad w_t^2 = D_x(2w_{xx}^2 + 2w^1 w^2).$$

Here we have again that $\vec{w}_t = \hat{A}_k \frac{\delta}{\delta \vec{w}} \int w^2 dx$, and the translation

$$\vec{w}_x = \hat{A}_k \frac{\delta}{\delta \vec{w}} \int w^1 dx$$

starts the auxiliary sequence of flows. The construction of the modified hierarchy is analogous to the previous case of the root system A_2 , see [9] for details.

4 The root system G_2

The Toda system for $K = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ is

$$u_{xy} = \exp(2u - v), \quad v_{xy} = \exp(-3u + 2v). \quad (13)$$

The differential orders of the integrals w.r.t. \mathfrak{m}^j equal 1 and 5, respectively:

$$\begin{aligned} w^1 &= u_{xx} + \frac{1}{3}v_{xx} - u_x^2 + u_x v_x - \frac{1}{3}v_x^2, \\ w^2 &= u_{6x} - 2u_x u_{5x} + u_x v_{5x} + 10u_{4x} u_x v_x - 8u_{4x} u_x^2 - \frac{7}{3}u_{4x} v_x^2 + \frac{7}{3}u_{4x} v_{xx} \\ &\quad - \frac{5}{3}v_{4x} u_x v_x + \frac{2}{3}v_{4x} u_x^2 - \frac{1}{9}v_{4x} v_x^2 + \frac{1}{9}v_{4x} v_{xx} + \frac{10}{3}v_{4x} u_{xx} + \frac{46}{3}u_{xxx} v_{xx} u_x \\ &\quad + 20u_{xxx} u_{xx} v_x - 40u_{xxx} u_{xx} u_x + \frac{10}{3}u_{xxx} v_{xxx} - \frac{19}{3}u_{xxx} v_{xx} v_x \\ &\quad + 16u_{xxx} u_x^3 - 18u_{xxx} u_x^2 v_x - \frac{1}{3}u_{xxx} v_x^3 + \frac{14}{3}u_{xxx} u_x v_x^2 - \frac{19}{3}v_{xxx} u_{xx} v_x \\ &\quad + \frac{40}{3}v_{xxx} u_{xx} u_x - 8v_{xxx} u_x^3 - \frac{4}{9}v_{xxx} v_{xx} v_x - \frac{13}{3}v_{xxx} v_{xx} u_x + \frac{26}{3}v_{xxx} u_x^2 v_x \\ &\quad - 2v_{xxx} u_x v_x^2 + \frac{2}{9}v_x^3 v_{xxx} + \frac{1}{18}v_{xxx}^2 - \frac{17}{3}u_{xx} v_{xx}^2 + 40u_{xx}^2 u_x^2 - 28u_{xx}^2 u_x v_x \\ &\quad - 2u_{xx} v_{xx} v_x^2 + \frac{25}{6}u_{xx}^2 v_x^2 - 16u_{xx}^3 + \frac{1}{3}u_{xx} v_x^4 - 5u_{xx} u_x v_x^3 + 15u_{xx} u_x^2 v_x^2 \\ &\quad - 12u_{xx} u_x^3 v_x + \frac{58}{3}u_{xx} v_{xx} u_x v_x - 34u_{xx} v_{xx} u_x^2 + \frac{49}{3}u_{xx}^2 v_{xx} + \frac{10}{3}v_{xx} v_x^3 u_x \\ &\quad - \frac{8}{27}v_{xx}^3 + 12v_{xx} v_x u_x^3 - \frac{34}{3}v_{xx} v_x^2 u_x^2 - \frac{2}{9}v_{xx} v_x^4 + \frac{25}{3}v_{xx}^2 u_x^2 - 4v_{xx}^2 u_x v_x \\ &\quad + \frac{2}{3}v_{xx}^2 v_x^2 - 3v_{xx} u_x^4 - \frac{2}{3}u_x v_x^5 + \frac{2}{27}v_x^6 + \frac{3}{2}u_x^4 v_x^2 + \frac{13}{6}u_x^2 v_x^4 - 3u_x^3 v_x^3. \end{aligned}$$

The Frobenius operator \square is

$$\square = (\square^1, \square^2) = \begin{pmatrix} \square_{11} & \square_{12} \\ \square_{21} & \square_{22} \end{pmatrix}, \text{ where } \square^1 = \begin{pmatrix} u_x + 3D_x \\ v_x + 5D_x \end{pmatrix} \text{ and} \quad (14)$$

$$\begin{aligned} \square_{12} &= 2D_x^5 + u_x D_x^4 + \left(15u_x v_x - 14u_x^2 - 5v_x^2 + 14u_{xx} + 5v_{xx} \right) \cdot D_x^3 \\ &\quad + \left(8v_{xxx} - 16v_{xx} v_x + 10u_x^2 v_x + \frac{82}{3}v_{xx} u_x - 44u_{xx} u_x \right. \\ &\quad \left. - \frac{10}{3}u_x v_x^2 + 24u_{xx} v_x - 8u_x^3 + 26u_{xxx} \right) \cdot D_x^2 \\ &\quad + \left(12u_x^3 v_x - 16v_{xxx} v_x - 38u_{xxx} u_x - 4u_x^2 v_x^2 + 4u_{xx} v_x^2 + 24u_{xxx} v_x \right. \\ &\quad \left. + 44u_{xx} v_{xx} + 2v_{xx} u_x^2 + \frac{70}{3}v_{xxx} u_x + 10u_{xx} u_x^2 + 21u_{4x} + 8v_{4x} \right. \\ &\quad \left. - 16v_{xx}^2 - 14u_{xx} u_x v_x + \frac{4}{3}v_{xx} u_x v_x - 51u_{xx}^2 - 9u_x^4 \right) \cdot D_x \\ &\quad + \left(15u_x^3 v_x^2 - 23u_{xx} u_x v_x^2 - \frac{44}{3}v_{xxx} u_x v_x + \frac{4}{3}u_x v_x^4 + 18u_{xx} u_x^2 v_x \right. \\ &\quad \left. - 12v_{xx} u_x^2 v_x + \frac{46}{3}v_{xx} u_x v_x^2 + 28u_{xx} v_{xx} u_x - 4u_{xx} v_{xx} v_x + 10u_{xxx} u_x v_x \right. \\ &\quad \left. + 12u_{5x} + 3v_{5x} + u_{xxx} v_x^2 - 20u_{4x} u_x + 4v_{xx}^2 v_x + \frac{40}{3}v_{4x} u_x - 6v_{4x} v_x \right. \\ &\quad \left. + 9u_{4x} v_x - 72u_{xxx} u_{xx} - 20v_{xxx} v_{xx} + 26v_{xxx} u_{xx} - 8u_x^2 v_x^3 + 6u_{xx} v_x^3 \right. \\ &\quad \left. - 8u_{xx}^2 u_x - 4v_{xx} v_x^3 + 26u_{xxx} v_{xx} + 2v_{xxx} v_x^2 - 8u_{xxx} u_x^2 - \frac{40}{3}v_{xx}^2 u_x \right. \\ &\quad \left. - 9u_x^4 v_x - 3u_{xx}^2 v_x + 14v_{xxx} u_x^2 \right), \\ \square_{22} &= 3D_x^5 + \left(20u_x v_x - 20u_x^2 + \frac{23}{3}v_{xx} - \frac{23}{3}v_x^2 + 20u_{xx} \right) \cdot D_x^3 \end{aligned}$$

$$\begin{aligned}
& + \left(40u_{xxx} - 2u_x v_x^2 - 80u_{xx}u_x - \frac{73}{3}v_{xx}v_x + 2u_x^2v_x + 40v_{xx}u_x \right. \\
& \quad \left. + \frac{37}{3}v_{xxx} + 38u_{xx}v_x - \frac{1}{3}v_x^3 \right) \cdot D_x^2 \\
& + \left(38u_{xxx}v_x - 25v_{xx}^2 - \frac{73}{3}v_{xxx}v_x - 68u_{xxx}u_x - 23u_x^2v_x^2 + 36u_x^3v_x \right. \\
& \quad \left. + 9u_{xx}v_x^2 + 66u_{xx}v_{xx} + 2v_{xx}u_x^2 + 34v_{xxx}u_x + 36u_{xx}u_x^2 - 86u_{xx}^2 \right. \\
& \quad \left. - 18u_x^4 + 34u_{4x} + \frac{37}{3}v_{4x} - 44u_{xx}u_xv_x + 2v_{xx}u_xv_x \right. \\
& \quad \left. - \frac{1}{3}v_x^4 + 5u_xv_x^3 \right) \cdot D_x \\
& + \left(36u_{xx}u_x^2v_x - 22v_{xxx}u_xv_x - \frac{2}{3}v_x^5 + 36u_x^3v_x^2 + 7u_xv_x^4 - 46u_{xx}u_xv_x^2 \right. \\
& \quad \left. - 18v_{xx}u_x^2v_x + 23v_{xx}u_xv_x^2 + 40u_{xx}v_{xx}u_x - 6u_{xx}v_{xx}v_x + 4u_{xxx}u_xv_x \right. \\
& \quad \left. + 20u_{5x} + \frac{14}{3}v_{5x} + 3u_{xxx}v_x^2 - 40u_{4x}u_x + 6v_{xx}^2v_x + 20v_{4x}u_x - 9v_{4x}v_x \right. \\
& \quad \left. + 18u_{4x}v_x - 120u_{xxx}u_{xx} - \frac{95}{3}v_{xxx}v_{xx} + 40v_{xxx}u_{xx} - 25u_x^2v_x^3 \right. \\
& \quad \left. + 12u_{xx}v_x^3 - 6v_{xx}v_x^3 + 40u_{xxx}v_{xx} + 3v_{xxx}v_x^2 - 20v_{xx}^2u_x - 18u_x^4v_x \right. \\
& \quad \left. - 14u_{xx}^2v_x + 20v_{xxx}u_x^2 \right).
\end{aligned}$$

Using the fact that the coefficients A_{ij} of the Hamiltonian operator

$$\hat{A}_k = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

are differential functions of the integrals w , we deduce for G_2 that⁵

$$\begin{aligned}
A_{11} &= \frac{14}{3}D_x^3 + 2w^1D_x + w_x^1, \\
A_{12} &= 3D_x^7 + \frac{68}{3}w^1D_x^5 + \frac{245}{3}w_x^1D_x^4 + ((w^1)^2 + \frac{395}{3}w_{xx}^1) \cdot D_x^3 \\
&\quad + (6w^2 + \frac{191}{3}w_{4x}^1 - 62w_{xx}^1w^1 - 62(w_x^1)^2) \cdot D_x \\
&\quad + (\frac{368}{3}w_{xxx}^1 - 28w^1w_x^1) \cdot D_x^2 + (5w_x^2 + \frac{41}{3}w_{5x}^1 - 32w^1w_{xxx}^1 - 96w_{xx}^1w_x^1), \\
A_{21} &= 3D_x^7 + \frac{68}{3}w^1D_x^5 + \frac{95}{3}w_x^1D_x^4 + ((w^1)^2 + \frac{95}{3}w_{xx}^1) \cdot D_x^3 \\
&\quad + (9w_{xxx}^1 + 34w^1w_x^1) \cdot D_x^2 + 6w^2D_x + w_x^2, \\
A_{22} &= 2D_x^{11} + 30w^1D_x^9 + 135w_x^1D_x^8 + (414w_{xx}^1 + \frac{338}{3}(w^1)^2) \cdot D_x^7 \\
&\quad + (819w_{xxx}^1 + \frac{2366}{3}w^1w_x^1) \cdot D_x^6 \\
&\quad + (1119w_{4x}^1 + 4w^2 + \frac{4870}{3}w_{xx}^1w^1 + \frac{1846}{3}(w_x^1)^2 + 8(w^1)^3) \cdot D_x^5 \\
&\quad + (1065w_{5x}^1 + 10w_x^2 + \frac{6260}{3}w^1w_{xxx}^1 + 1220w_{xx}^1w_x^1 + 60w_x^1(w^1)^2) \cdot D_x^4 \\
&\quad + (699w_{6x}^1 + 26w_{xx}^2 + \frac{5402}{3}w_{4x}^1w^1 + 36w^1w^2 + \frac{1096}{3}w_{xxx}^1w_x^1 \\
&\quad - \frac{652}{3}(w_{xx}^1)^2 - 428w_{xx}^1(w^1)^2 - 88(w_x^1)^2w^1 + 54(w^1)^4) \cdot D_x^3 \\
&\quad + (303w_{7x}^1 + 29w_{xxx}^2 + \frac{3026}{3}w_{5x}^1w^1 + 54w_x^2w^1 - \frac{518}{3}w_{4x}^1w_x^1 + 54w_x^1w^2
\end{aligned}$$

⁵Here part (vii) of our main theorem reveals its true power: a verification for G_2 that the bracket $\{\cdot, \cdot\}_\square$, which depends on the fields u through the integrals w^i , satisfies equality (15), see below, results in a 50 Mb size expression.

$$\begin{aligned}
& -\frac{5722}{3}w_{xxx}^1w_{xx}^1 - 702w_{xxx}^1(w^1)^2 - 1908w_{xx}^1w_x^1w^1 - 252(w_x^1)^3 \\
& + 324w_x^1(w^1)^3 \Big) \cdot D_x^2 \\
& + \left(78w_{8x}^1 + 15w_{4x}^2 + 328w_{6x}^1w^1 + 58w_{xx}^2w^1 - \frac{560}{3}w_{5x}^1w_x^1 + 36w_x^2w_x^1 \right. \\
& - \frac{4030}{3}w_{4x}^1w_{xx}^1 - 514w_{4x}^1(w^1)^2 + 54w^2w_{xx}^1 - 36w^2(w^1)^2 - 1086(w_{xxx}^1)^2 \\
& - 2000w_{xxx}^1w_x^1w^1 - 1652(w_{xx}^1)^2w^1 - 1120w_{xx}^1(w_x^1)^2 + 396w_{xx}^1(w^1)^3 \\
& \left. + 540(w_x^1)^2(w^1)^2 \right) \cdot D_x \\
& + \left(9w_{9x}^1 + 3w_{5x}^2 + \frac{140}{3}w_{7x}^1w^1 + 20w_{xxx}^2w^1 - \frac{148}{3}w_{6x}^1w_x^1 + 20w_{xx}^2w_x^1 \right. \\
& - \frac{994}{3}w_{5x}^1w_{xx}^1 - 138w_{5x}^1(w^1)^2 + 18w_x^2w_{xx}^1 - 18w_x^2(w^1)^2 - 678w_{4x}^1w_{xxx}^1 \\
& - 708w_{4x}^1w^1w_x^1 + 18w^2w_{xxx}^1 - w_{xxx}^1 \cdot (1424w_{xx}^1w^1 + 524(w_x^1)^2 \\
& - 144(w^1)^3) - 36w^2w_x^1w^1 - 680(w_{xx}^1)^2w_x^1 + 648w_{xx}^1w_x^1(w^1)^2 \\
& \left. + 216(w_x^1)^3w^1 \right).
\end{aligned}$$

Next, we calculate the bracket (5) on the domain of (14) using formula (6):

$$\begin{aligned}
\{\{\vec{p}, \vec{q}\}\}_{\square}^1 = & [p_x^1q^1 - p^1q_x^1] + \left[\frac{41}{3}(p_{5x}^1q^2 - p^2q_{5x}^1) + \frac{14}{3}(p_{4x}^1q_x^2 - p_x^2q_{4x}^1) \right. \\
& + \frac{14}{3}(p_{xxx}^1q_{xx}^2 - p_{xx}^2q_{xxx}^1) + 9(p_{xxx}^2q_{xx}^1 - p_{xx}^1q_{xxx}^2) + 32w^1(p^2q_{xxx}^1 - p_{xxx}^1q^2) \\
& + 34w^1(p_x^2q_{xx}^1 - p_{xx}^1q_x^2) \Big] + 9(p_{9x}^2q^2 - p^2q_{9x}^2) + 3(p_{8x}^2q_x^2 - p_x^2q_{8x}^2) \\
& + \frac{140}{3}w^1(p_{7x}^2q^2 - p^2q_{7x}^2) + 3(p_{7x}^2q_{xx}^2 - p_{xx}^2q_{7x}^2) + 6(p_{xxx}^2q_{6x}^2 - p_{6x}^2q_{xxx}^2) \\
& + 376w_x^1(p_{6x}^2q^2 - p^2q_{6x}^2) + \frac{4}{3}w^1(p_x^2q_{6x}^2 - p_{6x}^2q_x^2) + \frac{62}{3}w^1(p_{5x}^2q_{xx}^2 - p_{xx}^2q_{5x}^2) \\
& + \frac{304}{3}w_x^1(p_{5x}^2q_x^2 - p_x^2q_{5x}^2) + \frac{2834}{3}w_{xx}^1(p_{5x}^2q^2 - p^2q_{5x}^2) \\
& + 138(w^1)^2(p^2q_{5x}^2 - p_{5x}^2q^2) + \frac{248}{3}w_x^1(p_{4x}^2q_{xx}^2 - p_{xx}^2q_{4x}^2) \\
& + \frac{640}{3}w_{xx}^1(p_{4x}^2q_x^2 - p_x^2q_{4x}^2) + \frac{4184}{3}w_{xxx}^1(p_{4x}^2q^2 - p^2q_{4x}^2) \\
& + 44w^1(p_{xxx}^2q_{4x}^2 - p_{4x}^2q_{xxx}^2) + 672w_x^1w_x^1(p^2q_{4x}^2 - p_{4x}^2q^2) \\
& + 176(w^1)^2(p_x^2q_{4x}^2 - p_{4x}^2q_x^2) + \frac{830}{3}w_{xx}^1(p_{xxx}^2q_{xx}^2 - p_{xx}^2q_{xxx}^2) \\
& + \frac{1060}{3}w_{xxx}^1(p_{xxx}^2q_x^2 - p_x^2q_{xxx}^2) + \frac{4022}{3}w_{4x}^1(p_{xxx}^2q^2 - p^2q_{xxx}^2) \\
& + 452(w_x^1)^2(p^2q_{xxx}^2 - p_{xxx}^2q^2) + 144(w^1)^3(p_{xxx}^2q^2 - p^2q_{xxx}^2) \\
& + 18w^2(p_{xxx}^2q^2 - p^2q_{xxx}^2) + 26(w^1)^2(p_{xx}^2q_{xxx}^2 - p_{xxx}^2q_{xx}^2) \\
& + 1352w_{xx}^1w^1(p^2q_{xxx}^2 - p_{xxx}^2q^2) + 576w_x^1w_x^1(p_x^2q_{xxx}^2 - p_{xxx}^2q_x^2) \\
& + \frac{368}{3}w_{4x}^1(p_x^2q_x^2 - p_x^2q_{xx}^2) + 1360w^1w_{xxx}^1(p^2q_{xx}^2 - p_{xx}^2q^2) \\
& + 1592w_{xx}^1w_x^1(p^2q_{xx}^2 - p_{xx}^2q^2) + 648w_x^1(w^1)^2(p_{xx}^2q^2 - p^2q_{xx}^2) \\
& + 68(w_x^1)^2(p_x^2q_{xx}^2 - p_{xx}^2q_x^2) + 772w_{5x}^1(p_{xx}^2q^2 - p^2q_{xx}^2) \\
& + 36w_x^2(p_{xx}^2q^2 - p^2q_{xx}^2) + 36(w^1)^3(p_{xx}^2q_x^2 - p_x^2q_{xx}^2) \\
& + 584w_{xx}^1w^1(p_x^2q_{xx}^2 - p_{xx}^2q_x^2) + \frac{772}{3}w_{6x}^1(p_x^2q^2 - p^2q_x^2) \\
& + 1020(w_{xx}^1)^2(p^2q_x^2 - p_x^2q^2) + 38w_{xx}^2(p_x^2q^2 - p^2q_x^2) \\
& + 1360w_{xxx}^1w_x^1(p^2q_x^2 - p_x^2q^2) + 680w_{4x}^1w^1(p^2q_x^2 - p_x^2q^2)
\end{aligned}$$

$$\begin{aligned}
& + 36w^1w^2(p^2q_x^2 - p_x^2q^2) + 648w_{xx}^1(w^1)^2(p_x^2q^2 - p^2q_x^2) \\
& + 648(w_x^1)^2w^1(p_x^2q^2 - p^2q_x^2); \\
\{\vec{p}, \vec{q}\}_\square^2 &= p_x^2q^1 - p^1q_x^2 + 5(p_x^1q^2 - p^2q_x^1) + 3(p_{5x}^2q^2 - p^2q_{5x}^2) \\
& + (p_{xx}^2q_{xxx}^2 - p_{xxx}^2q_{xx}^2) + 20w^1(p_{xxx}^2q^2 - p^2q_{xxx}^2) + 40w_x^1(p_{xx}^2q^2 - p^2q_{xx}^2) \\
& + 2w^1(p_{xx}^2q_x^2 - p_x^2q_{xx}^2) + 38w_{xx}^1(p_x^2q^2 - p^2q_x^2) + 18(w^1)^2(p^2q_x^2 - p_x^2q^2).
\end{aligned}$$

The equalities

$$[\hat{A}_k(\vec{p}), \hat{A}_k(\vec{q})] = \hat{A}_k \left(\partial_{\hat{A}_k(\vec{p})}(\vec{q}) - \partial_{\hat{A}_k(\vec{q})}(\vec{p}) + \{\{\vec{p}, \vec{q}\}\}_{\hat{A}_k} \right)$$

and

$$[\square(\vec{p}), \square(\vec{q})] = \square \left(\partial_{\square(\vec{p})}(\vec{q}) - \partial_{\square(\vec{q})}(\vec{p}) + \{\{\vec{p}, \vec{q}\}\}_\square \right) \quad (15)$$

hold, where $\{\{\vec{p}, \vec{q}\}\}_\square = \{\{\vec{p}, \vec{q}\}\}_{\hat{A}_k}$ for any $\vec{p}, \vec{q}(x, [w])$. This yields the commutation relations between symmetries $\varphi = \square(\cdot)$ of the 2D Toda chain (13) associated with the root system G_2 .

Finally, we pass to the KdV-type hierarchy. The deformation $\frac{d}{d\lambda}|_{\lambda=0}\hat{A}_k$ under $w^2 \mapsto w^2 + \lambda$ determines the ‘junior’ Hamiltonian operator

$$\hat{A}_1^{(2)} = \begin{pmatrix} 0 & 6D_x \\ 6D_x & 4D_x^5 + 36w^1D_x^3 + 54w_x^1D_x^2 + (54w_{xx}^1 - 36(w^1)^2) \cdot D_x \\ & + 18w_{xxx}^1 - 36w^1w_x^1 \end{pmatrix}.$$

It is compatible with the ‘senior’ operator \hat{A}_k ; the bracket on its domain is given through

$$\{\{\vec{p}, \vec{q}\}\}_{\hat{A}_1^{(2)}}^1 = 36w^1(p_x^2q^2 - p^2q_x^2) + 18(p^2q_{xxx}^2 - p_{xxx}^2q^2), \quad \{\{\vec{p}, \vec{q}\}\}_{\hat{A}_1^{(2)}}^2 = 0.$$

Applying the Hamiltonian operator \hat{A}_k to $\frac{\delta}{\delta w} \int w^2 dx = \binom{0}{1}$, we obtain the KdV-type system

$$w_t^1 = D_x \left(\frac{41}{3}w_{4x}^1 - 32w^1w_{xx}^1 - 32(w_x^1)^2 + 5w^2 \right), \quad (16a)$$

$$\begin{aligned}
w_t^2 = D_x & \left(3w_{4x}^2 + 9w_{8x}^1 + 20w^1w_{xx}^2 + 18w_{xx}^1w^2 - 18(w^1)^2w^2 + \frac{140}{3}w_{6x}^1w^1 \right. \\
& - 96w_{5x}^1w_x^1 - \frac{706}{3}w_{4x}^1w_{xx}^1 - \frac{664}{3}(w_{xxx}^1)^2 - 138w_{4x}^1(w^1)^2 - 432w_{xxx}^1w_x^1w^1 \\
& \left. - 496(w_{xx}^1)^2w^1 - 92w_{xx}^1(w_x^1)^2 + 144w_{xx}^1(w^1)^3 + 108(w_x^1)^2(w^1)^2 \right). \quad (16b)
\end{aligned}$$

(The second equation in this system can be simplified by adding to w^2 a scaling-homogeneous differential polynomial in w^1 and thus cancelling some irrelevant terms.) The Hamiltonian $\int w^1 dx$ starts the auxiliary sequence of flows by the translation along x . The corresponding modified KdV-type hierarchy, and the Hamiltonian structures $\hat{B}_1 = B_1^{-1}$, $\hat{B}_{k'}$, and $B_{k'}$ for it, are introduced in a standard way [9].

Discussion

The geometric method for the derivation of completely integrable KdV-type hierarchies, which is illustrated in this note, is the most straightforward and efficient, to the best of our knowledge. Of course, the three two-component KdV-type systems which we re-derived in this paper are well known from [5] (e.g., the Boussinesq equation (11), see [18], although the system (16) does not often appear in the literature). At the same time, we emphasize that the use of Liouville-type systems is the generator of infinitely many completely integrable bi-Hamiltonian hierarchies. In [10] we specified the natural requirements on the Liouville-type Euler–Lagrange systems \mathcal{E}_L and their integrals that ensure the existence of the characteristic Lie algebras and the applicability of Theorem 1. Since the 2D Toda chains are only particular examples of such hyperbolic systems, an overwhelming majority of the arising KdV-type equations have never been explored.

Unlike in the fundamental paper [5], which is based on algebraic considerations, the KdV-type systems are derived here, from the very beginning, in the bi-Hamiltonian but not in the Lax form. Second, the formalism of pseudodifferential operators is not required. Indeed, we arrive from the root systems directly to the 2D Toda chains and then to the Poisson structures, bypassing the matrix representations of the semi-simple Lie algebras \mathfrak{g} . *A posteriori* the Lax form $R_t + [R, \ell_F] = 0$ of the KdV-type system $\vec{w}_t = F$, which is bi-Hamiltonian⁶ w.r.t. the operators \hat{A}_1 and \hat{A}_k , is obtained using the recursion $R = \hat{A}_k \circ \hat{A}_1^{-1}$ and the linearization ℓ_F of the right-hand side F .

On the other hand, the use of Liouville-type systems allows to regard the auxiliary linear problems, which are specified by the Lax pairs, from a nontrivial viewpoint. Namely, for \mathfrak{g} semi-simple, the ambient r -component 2D Toda chain is *exactly solvable*. Therefore, in principle, the modified KdV-type flows should be lifted first to the bundles with $2r$ -dimensional fibres, whose sections $\mathbf{f}(x) = (f_1, \dots, f_r)$, $\mathbf{g}(y) = (g_1, \dots, g_r)$ determine the general solutions of the 2D Toda chains. We see that the liftings of the mKdV-type hierarchies determine the evolution of these Cauchy data. It is well known that the Krichever–Novikov equation appears in this context for the root system A_1 . Simultaneously, the linear problem $\psi_t = (L^{3/2})_+(\psi)$ for the KdV leads to the evolution equation close to KN (there is a Bäcklund transformation between them). In our opinion, further analysis of such symmetry liftings for the 2D Toda chains will contribute to the IST theory of the associated KdV-type systems constructed in [5].

The Lax approach of [5] becomes inevitable if \mathfrak{g} is a Kac–Moody algebra and its Cartan matrix K becomes degenerate. In this case our cut-through does not work without serious modifications. This will be discussed elsewhere.

There is one more thing that we lack. Namely, it is an explanation of the way the ‘junior’ Hamiltonian operators are obtained through the deformations of the canonical operators \hat{A}_k , see (4). In all the cases the operators \hat{A}_1 are

⁶Let us finally remark that k may be greater than 2: for instance, we have that $k = 3$ for the Kaup–Boussinesq system, see [10], and two ‘junior’ structures precede it.

obtained by using the Lie derivatives $L_\varphi(\hat{A}_k)$ of the ‘senior’ variational Poisson bi-vectors \hat{A}_k along certain evolutionary vector fields ∂_φ (and even *vice versa* [3], $\hat{A}_k = [\hat{A}_1, \partial_{\varphi'}]^{Sch}$ for some φ'), see [19] and references therein. However, the Poisson cohomology theory for Hamiltonian operators on jet spaces is still far from complete, although its finite-dimensional counterpart is well-established. To begin with, the preference of certain shift directions for the operator (4) is hard to predict [B. A. Dubrovin, private communication]. We conjecture that it has an explanation in terms of the cohomology of the W -algebras for the KdV-type equations at hand.

Moreover, the resolvability of the Magri schemes, that is, the existence of the next Hamiltonian at each step (equivalently, the vanishing of the first Poisson cohomology with respect to the differential $[\hat{A}_1, \cdot]^{Sch}$) is not proved. This is a difficulty of the theory because the differential Hamiltonian operators \hat{A}_1 are higher order and thus exceed the frames of rigorous results for the Dubrovin–Novikov structures. Besides, the symbols of the junior operators \hat{A}_1 are degenerate in most cases. Therefore the *nondegeneracy* assumptions are needed to prove the uniqueness of the trivial solutions for linear homogeneous equations that arise in the deformation cohomology theory for total differential operators. Two known and two new non-equivalent definitions of the nondegeneracy of the operators have been formulated in [10]. Using them *simultaneously*, we found the natural conditions upon \mathcal{E}_L that guarantee the validity of our main Theorem 1.

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Is a nonclassical symmetry a symmetry?

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Various versions of the definition of nonclassical symmetries existing in the literature are analyzed. Comparing properties of Lie and nonclassical symmetries leads to the conclusion that in fact a nonclassical symmetry is not a symmetry in the usual sense. Hence the term “reduction operator” is suggested instead of the name “operator of nonclassical symmetries”. It is shown that in contrast to the case of single partial differential equations a satisfactory definition of nonclassical symmetries for systems of such equations has not been proposed up to now. Moreover, the cardinality of essential nonclassical symmetries is discussed, taking into account equivalence relations on the entire set of nonclassical symmetries.

1 Introduction

The “nonclassical” method of finding similarity solutions was introduced by Bluman and Cole in 1969 [3]. In fact, the method was first appeared in [2] in terms of “nonclassical group” but the terminology was changed in [3]. Over the years the “nonclassical” method began to be associated with the term nonclassical symmetry [13] (also called Q -conditional [8] or, simply, conditional symmetry [6, 10]). In the past two decades, the theoretical background of nonclassical symmetry was intensively investigated and nonclassical symmetry techniques were effectively applied to finding exact solutions of many partial differential equations arising in physics, biology, financial mathematics, etc. See, e.g., the review on investigations of nonclassical symmetries in [18].

Here we mention only works which are directly connected with the subject of our paper. In the pioneering paper [3] the “nonclassical” method was described by means of the example of the $(1+1)$ -dimensional linear heat equation. It was emphasized that any solution of the corresponding (nonlinear) determining equations gives the coefficients of an operator such that an ansatz based on it reduces the heat equation to an ordinary differential equation. A veritable surge of interest in nonclassical symmetry was triggered by the papers [9, 16, 17]. In [16] the “nonclassical” method was considered in the course of a comprehensive analysis of a wide range of methods for constructing exact solutions. The concept of weak symmetry of a system of partial differential equations, generalizing the “nonclas-

sical” method, was introduced in [17], where also the reduction procedure was discussed. Moreover, fundamental identities [17, eq. (23)] crucially important for the theory of nonclassical symmetries were derived (see Myth 3 below). The first version of the conditional invariance criterion explicitly taking into account differential consequences was proposed in [9]. Generalizing results of [7,9] and other previous papers, in [6] Fushchych introduced the notion of general conditional invariance. From the collection of papers containing [6] it becomes apparent that around this time a number of authors began to regularly use the terms “conditional invariance” and “ Q -conditional invariance” in connection with the method of Bluman and Cole. The direct (ansatz) method closely related to this method was explicitly formulated in [4]. To the best of our knowledge, the name “nonclassical symmetry” was first used in [13]. Before this, there was no special name for operators calculated in this approach and the existing terminology on the subject emphasized characteristics of the method or invariance. The involution condition for families of operators was first considered in the formulation of the conditional invariance criterion in [10,25]. The relations between nonclassical symmetries, reduction and formal compatibility of the combined system consisting of the initial equation and the invariant surface equation were discovered in [23] and were also studied in [15]. The problem of the algorithmization of calculating nonclassical symmetries was posed in [5]. Furthermore, the equivalence of the non-classical (conditional symmetry) and direct (ansatz) approaches to the reduction of partial differential equations was established in general form in [26], making use of the precise definition of reduction of differential equations.

In spite of the long history of nonclassical symmetry and the encouraging results in its applications, a number of basic problems of this theory are still open. Moreover, there exists a variety of non-rigorous definitions of related key notions and heuristic results on fundamental properties of nonclassical symmetry in the literature, which are used up to now and form what we would like to call the “mythology” of nonclassical symmetry. These definitions and results require particular care and presuppose the tacit assumption of a number of conventions in order to correctly apply them. Otherwise, certain contradictions and inaccurate statements may be obtained. Note that mythology interpreted in the above sense is an unavoidable and necessary step in the development of any subject.

Basic myths on nonclassical symmetries presented in the literature are discussed in this paper. We try to answer, in particular, the following questions.

- Is a nonclassical symmetry a Lie symmetry of the united system of the initial equation and the corresponding invariant surface condition? Can a nonclassical symmetry be viewed as a conditional symmetry of the initial equation if the corresponding invariant surface condition is taken as the additional constraint? Is nonclassical symmetry a kind of symmetry in general? Does there exist a more appropriate name for this notion?
- What is a rigorous definition of nonclassical symmetry for systems of differential equations? Can such a definition be formulated as a straightforward

extension of the definition of nonclassical symmetry for single partial differential equations?

- Is the number of nonclassical symmetries essentially greater than the number of classical symmetries?

2 Definition of nonclassical symmetry

Following [9, 10, 22, 26], in this section we briefly recall some basic notions and results on nonclassical (conditional) symmetries of partial differential equations. This will form the basis for our discussion of myths in the next sections.

Consider an involutive family $Q = \{Q^1, \dots, Q^l\}$ of l ($l \leq n$) first order differential operators (vector fields)

$$Q^s = \xi^{si}(x, u)\partial_i + \eta^s(x, u)\partial_u, \quad s = 1, \dots, l,$$

in the space of the variables x and u , satisfying the condition $\text{rank } \|\xi^{si}(x, u)\| = l$.

Here and in what follows x is the n -tuple of independent variables (x_1, \dots, x_n) , $n > 1$, and u is treated as the unknown function. The indices i and j run from 1 to n , the indices s and σ run from 1 to l , and we use the summation convention for repeated indices. Subscripts of functions denote differentiation with respect to the corresponding variables, $\partial_i = \partial/\partial x_i$ and $\partial_u = \partial/\partial u$. Any function is considered as its zero-order derivative. All our considerations are in the local setting.

The requirement of involution for the family Q means that the commutator of any pair of operators from Q belongs to the span of Q over the ring of smooth functions of the variables x and u , i.e.,

$$\forall s, s' \exists \zeta^{ss'\sigma} = \zeta^{ss'\sigma}(x, u): [Q^s, Q^{s'}] = \zeta^{ss'\sigma}Q^\sigma.$$

The set of such families will be denoted by \mathfrak{Q}^l .

Consider an r th-order differential equation \mathcal{L} of the form $L[u] := L(x, u_{(r)}) = 0$ for the unknown function u of the independent variables x . Here, $u_{(r)}$ denotes the set of all derivatives of the function u with respect to x of order not greater than r , including u as the derivative of order zero. Within the local approach the equation \mathcal{L} is treated as an algebraic equation in the jet space J^r of the order r and is identified with the manifold of its solutions in J^r . Denote this manifold by the same symbol \mathcal{L} and the manifold defined by the set of all the differential consequences of the characteristic system $Q[u] = 0$ in J^r by $\mathcal{Q}_{(r)}$, i.e.,

$$\mathcal{Q}_{(r)} = \{(x, u_{(r)}) \in J^r \mid D_1^{\alpha_1} \dots D_n^{\alpha_n} Q^s[u] = 0, \alpha_i \in \mathbb{N} \cup \{0\}, |\alpha| < r\},$$

where $D_i = \partial_{x_i} + u_{\alpha+\delta_i} \partial_{u_\alpha}$ is the operator of total differentiation with respect to the variable x_i , $Q^s[u] := \eta^s - \xi^{si}u_i$ is the characteristic of the operator Q , $\alpha = (\alpha_1, \dots, \alpha_n)$ is an arbitrary multi-index, $|\alpha| := \alpha_1 + \dots + \alpha_n$, δ_i is the multiindex whose i th entry equals 1 and whose other entries are zero. The variable u_α of the jet space J^r corresponds to the derivative $\partial^{|\alpha|}u/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$.

Definition 1. The differential equation \mathcal{L} is called *conditionally invariant* with respect to the involutive family Q if the relation

$$Q_{(r)}^s L(x, u_{(r)})|_{\mathcal{L} \cap \mathcal{Q}_{(r)}} = 0 \quad (1)$$

holds, which is called the *conditional invariance criterion*. Then Q is called an *involutive family of conditional symmetry* (or Q -conditional symmetry, nonclassical symmetry, etc.) operators of the equation \mathcal{L} .

Here the symbol $Q_{(r)}^s$ stands for the standard r th prolongation of the operator Q^s [14, 19]:

$$Q_{(r)}^s = Q^s + \sum_{0 < |\alpha| \leq r} (D_1^{\alpha_1} \dots D_n^{\alpha_n} Q^s[u] + \xi^{s_i} u_{\alpha+\delta_i}) \partial_{u_\alpha}.$$

3 Myths on name and definition

We restrict our consideration mainly to the case of families consisting of single operators ($l = 1$) for simplicity and since mostly this case is investigated in the literature. Then the involution condition degenerates to an identity and we can omit the words “involutive family” and talk only about operators.

Myth 1. *A nonclassical symmetry operator Q of an equation \mathcal{L} is a vector field Q which is a Lie symmetry operator of the united system of the equation \mathcal{L} and the invariant surface condition $Q[u] = 0$ corresponding to Q .*

This is the conventional non-rigorous way in order to quickly define nonclassical symmetry (see, e.g., [10, 11]). It becomes rigorous only after a special interpretation of the notions of system of differential equations and Lie symmetry. Otherwise, using the empiric definition leads to a number of inconsistencies.

A closer look reveals that the above definition is a tautology. Indeed, the invariant surface condition $Q[u] = 0$ means that the function u is a fixed point of the one-parametric local group G_Q of local transformations generated by the operator Q . Therefore, we can reformulate the definition in the following way.

Reformulation. If the set of those solutions of the equation \mathcal{L} which are fixed points of G_Q , is invariant with respect to G_Q , then Q is called a nonclassical symmetry operator Q of the equation \mathcal{L} .

The tautology of the reformulation is obvious. If each element of the set is invariant then the whole set is necessarily invariant. The definition of nonclassical symmetry according to Myth 1 leads to the conclusion that *any differential equation is invariant, in the nonclassical sense, with respect to any vector field in the corresponding space of dependent and independent variables*.

The case when the equation \mathcal{L} has no Q -invariant solutions fits well into the non-rigorous approach in the sense that the empty set is a particularly symmetric set.

Therefore, uncritically following the non-rigorous approach, we would get no effective methods for constructing exact solutions and no information on the partial differential equations under consideration.

There exist a number reformulations of Myth 1 in the literature in different terms. The first one is in terms of conditional symmetry.

Myth 2. *A nonclassical symmetry operator Q of an equation \mathcal{L} is a conditional symmetry operator of the equation \mathcal{L} under the auxiliary condition $Q[u] = 0$.*

The association of nonclassical symmetries (under the name Q -conditional symmetries) with conditional ones can be traced back to [7] (see also [8] and earlier papers of the same authors). Here the term conditional symmetry is understood in the following sense [6] (it can easily be defined for the case of a general system of differential equations).

Definition 2. A vector field Q is called a *conditional symmetry operator* of a system \mathcal{L} of differential equations under an auxiliary condition \mathcal{L}' (which is another system of differential equations in the same variables) if Q is a Lie symmetry operator of the united system of \mathcal{L} and \mathcal{L}' .

Conditional symmetries defined in this way essentially differ from nonclassical symmetries. In particular, auxiliary conditions for conditional symmetries do not involve any associated conditional symmetry operators. The conditional symmetry operators of a system \mathcal{L} under an auxiliary condition \mathcal{L}' form a Lie algebra. Conditional symmetry indeed is a kind of symmetry and can be applied to generate new solutions from known ones. At the same time, in contrast to the case of nonclassical symmetries, finding auxiliary conditions associated with nontrivial conditional symmetries is an art rather than an algorithmic procedure. This is why sometimes nonclassical symmetries are called either *Q -conditional symmetries*, where the prefix “ Q ” is used to emphasize the differences between nonclassical and conditional symmetries, or *conditional symmetries* without any connection with Definition 2.

The second reformulation of Myth 1 is in infinitesimal terms. Note that infinitesimal criteria lie at the basis of Lie symmetry theory since they allow one to study linear problems for infinitesimal transformations instead of nonlinear problems for finite transformations.

Myth 3. *The conditional invariance criterion for an equation \mathcal{L} and an operator Q coincides with the infinitesimal Lie invariance criterion for the united system $\{\mathcal{L}, Q[u] = 0\}$ with respect to the same operator, i.e.,*

$$Q_{(r)}L[u] = 0 \quad \text{if} \quad L[u] = 0 \quad \text{and} \quad Q[u] = 0.$$

The infinitesimal Lie invariance criterion for the invariant surface condition $Q[u] = 0$ with respect to the operator Q is identically satisfied as an algebraic consequence of this condition since

$$Q_{(r)}Q[u] = Q_{(1)}Q[u] = (\eta_u - \xi_u^j u_j)Q[u] \equiv 0 \quad \text{if} \quad Q[u] = 0.$$

We also have

$$Q_{(r)}L[u] = \xi^i D_i L[u] + \sum_{|\alpha| \leq r} L_{u_\alpha}[u] D_1^{\alpha_1} \dots D_n^{\alpha_n} Q[u], \quad (2)$$

i.e., the equation $Q_{(r)}L[u] = 0$ is a differential consequence of the equations $L[u] = 0$ and $Q[u] = 0$ and, therefore, becomes an identity on the set of their common solutions. This tautology was first observed in [17].

In the local approach to group analysis of differential equations, a system of differential equations is associated with the infinite tuple of systems of algebraic equations defined by this system and its differential consequences in the infinite tower of the corresponding jet spaces. The exclusion of the differential consequence $Q_{(r)}L[u]$ when considering the system $L[u] = 0$ and $Q[u] = 0$ seems unnatural from the viewpoint of group analysis.

A variation of Myth 3 is to replace, due to the Hadamard lemma, the “invariance condition” holding on the solution set of the system $L[u] = 0$ and $Q[u] = 0$ by the associated multiplier-condition, to be satisfied on the entire jet space J^r .

Myth 4. *An operator Q is a nonclassical symmetry of an equation \mathcal{L} if there exist λ^1 and λ^2 such that*

$$Q_{(r)}L[u] = \lambda^1 L[u] + \lambda^2 Q[u]. \quad (3)$$

The problem is to precisely define the nature of the multipliers λ^1 and λ^2 . A number of different conditions on the multipliers have been put forward in the literature. The simplest version is to prescribe no conditions at all on λ^1 and λ^2 , which is obviously unacceptable.

Sometimes λ^1 and λ^2 are assumed to be differential functions. This condition is natural for λ^1 but overly restrictive for λ^2 . In fact, if only such λ^2 are allowed, the equivalence relation of nonclassical symmetries up to nonvanishing functional multipliers will be broken. Moreover, in this case the associated invariance criterion will become merely a sufficient condition for an ansatz constructed with the operator Q to reduce the equation \mathcal{L} . As a result, a number of well-defined reductions may be lost.

On the other hand, requiring that both the multipliers λ^1 and λ^2 are polynomials of total differentiation operators with respect to the independent variables, whose coefficients are differential functions, is too weak an assumption. It arises from the association of nonclassical symmetries with conditional symmetries for which such multipliers are admissible. If we choose

$$\lambda^1 = \xi^i D_i \quad \text{and} \quad \lambda^2 = \sum_{|\alpha| \leq r} L_{u_\alpha}[u] D_1^{\alpha_1} \dots D_n^{\alpha_n},$$

condition (3) obviously becomes an identity for any operator Q . In other words, condition (3) reduces to the tautology (2) if both λ^1 and λ^2 are treated as differential operators of the above kind.

Comparing Definition 1 and Myth 4 shows that λ^1 should be a differential function (i.e., a zeroth order operator) and λ^2 should be an order $(r-1)$ operator. These conditions for the multipliers can be weakened. Thus, bounding the order of total differentiations in λ^2 is not essential. If λ^1 is a differential function, condition (3) implies that λ^2 cannot include total differentiations of orders greater than $r-1$. At the same time, explicitly prescribing the bound allows one to fix the order of the jet space under consideration.

Myth 5 (The main myth of the theory). *Nonclassical symmetry is a kind of symmetry of differential equations.*

Any kind of symmetry of differential equations (Lie, contact, hidden, conditional, approximate, generalized, potential, nonlocal etc.) has the *invariance* property, i.e., symmetries transform solutions to solutions in an appropriate sense.

The basic prerequisite of the definition of nonclassical symmetry is the consideration of only the set of solutions invariant under the associated finite transformations. It is impossible to use nonclassical symmetries in order to generate new solutions from known ones. A nonclassical symmetry operator Q of \mathcal{L} represents only a symmetry of

- each Q -invariant solution of \mathcal{L} (as a weak symmetry [17]) and
- the manifold $\mathcal{L} \cap \mathcal{Q}_{(r)}$ in J^r , where $r = \text{ord } \mathcal{L}$.

The manifold $\mathcal{L} \cap \mathcal{Q}_{(r)}$ is properly related to the joint system $L[u] = 0$ and $Q[u] = 0$ of differential equations only if the operator Q and the equation \mathcal{L} satisfy the conditional invariance criterion.

At the same time, properties of the set of nonclassical symmetries and properties of the set of Q -invariant solutions for each nonclassical symmetry operator Q characterize the equation \mathcal{L} .

Since a nonclassical symmetry is not in fact a kind of symmetry of differential equations, it is of utmost importance to discuss possibilities for replacing the name by one not involving the word “symmetry”.

4 Nonclassical symmetry, compatibility and reduction

To understand the real nature of nonclassical symmetry, we discuss properties and applications of Lie symmetries and single out those of them which carry over to nonclassical symmetries.

Properties of Lie symmetries:

Invariance. Any Lie symmetry (in the form of a parameterized family of finite transformations) locally maps the solution set of the corresponding system of differential equations onto itself. This is the main characteristic of any kind of symmetry. It gives rise to the possibility of generating new solutions from known ones.

Formal compatibility. Attaching the invariant surface conditions associated with a Lie invariance algebra to the initial system of differential equations results in a system having no nontrivial differential consequences. In other words, the invariant surface conditions forms a class of proper universal differential constraints and, therefore, is appropriate for finding subsets of solutions of the initial system.

Reduction. Each Lie invariance algebra satisfying the infinitesimal transversality condition leads to an ansatz reducing the initial system to a system with a smaller number of independent variables, i.e., the reduced system is more easily solvable than the initial one.

Conditional compatibility. There exists a bijection between solutions of the initial system which satisfy the invariant surface conditions, and solutions of the corresponding reduced system. This means that all solutions of the initial system invariant with respect to a Lie invariance algebra, can be constructed via solving the corresponding reduced system.

For nonclassical symmetries, the property of invariance is broken but the other properties (formal compatibility, reduction, conditional compatibility) are preserved. In fact, the conditional invariance criterion (1) is the condition of formal compatibility of the joint system $L[u] = 0$ and $Q[u] = 0$ [23]. We can identify *nonclassical symmetries of \mathcal{L} with first-order quasilinear differential constraints which are formally compatible with \mathcal{L}* .

Definition 3. The differential equation \mathcal{L} is called *conditionally invariant* with respect to the involutive family of operators Q if the joint system of \mathcal{L} with the characteristic system $Q[u] = 0$ is formally compatible.

What is the main property that adequately represents the essence of nonclassical symmetry?

The fact that the characteristic equations $Q^s[u] = 0$ are quasilinear and of first order implies the possibility of integrating them explicitly, i.e., an ansatz associated with the characteristic system $Q[u] = 0$ can be constructed. In view of the Frobenius theorem, the involution and transversality conditions for the family Q (together with the fact that the operators from Q are of first order) imply that the ansatz involves one new unknown function of $n - l$ new independent variables. Then the formal compatibility of the joint system $L[u] = 0$ and $Q[u] = 0$ guarantees the reduction of \mathcal{L} by the ansatz to a single differential equations \mathcal{L}' in $n - l$ independent variables. Thus, the number of dependent variables and equations are preserved under the reduction with Q and the number of independent variables decreases by the cardinality of Q , i.e., similarly to Lie symmetries nonclassical symmetries lead to the conventional reduction of the number of independent variables.

There exist integrable differential constraints which are not formally compatible with the initial system. Differential constraints can be formally compatible with the initial system and, at the same time, non-integrable in an explicit form.

An ansatz constructed with a general integrable differential constraint may involve a number of new unknown functions depending on different variables. Therefore, only all the above properties combined (first order, quasilinearity, formal compatibility, transversality and involution) result in the classical reduction procedure.¹

The conditional invariance of the equation \mathcal{L} with respect to the family Q is equivalent to the ansatz constructed with this family reducing \mathcal{L} to a differential equation with $n - l$ independent variables [26]. Moreover, reducing the number of independent variables in partial differential equations is the main goal in the study of nonclassical symmetries. Since the reduction by the associated ansatz is the quintessence of nonclassical symmetries, it was proposed in [21, 22, 24] to call involutive families of nonclassical symmetry operators *families of reduction operators* of \mathcal{L} .

Another important property holding for Lie symmetries is broken for nonclassical symmetries. Let the equation \mathcal{L} be of order r and

$$L_{(k)} = \{D_1^{\alpha_1} \dots D_n^{\alpha_n} L[u] = 0, |\alpha| \leq k - r\}.$$

Denote by $L_{(k)}$ a maximal set of algebraically independent differential consequences of \mathcal{L} that have, as differential equations, orders not greater than k . We identify $L_{(k)}$ with the corresponding system of algebraic equations in $J^k(x|u)$ and associate it with the manifold $\mathcal{L}_{(k)}$ determined by this system. For Lie symmetries we have the following properties.

1. If Q is a Lie symmetry operator of $\mathcal{L}_{(r)}$ then Q is a Lie symmetry operator of $\mathcal{L}_{(\rho)}$ for any $\rho > r$.
2. If Q is a Lie symmetry operator of $\mathcal{L}_{(\rho)}$ for some $\rho > r$ then Q is a Lie symmetry of $\mathcal{L}_{(r)}$.

The first of these properties extends to nonclassical symmetries but this is not the case for the second one. In fact:

1. If Q is a Lie symmetry operator of $\mathcal{L}_{(r)} \cap \mathcal{Q}_{(r)}$ then Q is a Lie symmetry operator of $\mathcal{L}_{(\rho)} \cap \mathcal{Q}_{(\rho)}$ for any $\rho > r$.
2. The fact that Q is a Lie symmetry operator of $\mathcal{L}_{(\rho)} \cap \mathcal{Q}_{(\rho)}$ for some $\rho > r$ does not imply that $\mathcal{L}_{(r)} \cap \mathcal{Q}_{(r)}$ admits the operator Q .

Example 1. Let $L[u] = u_t + u_{xx} + tu_x$, $\mathcal{L}: L[u] = 0$ and $Q = \partial_t$. Then the manifold $\mathcal{L}_{(2)} \cap \mathcal{Q}_{(2)}$ is determined in J^2 by the equations $u_t = u_{tt} = u_{tx} = 0$ and $u_{xx} = -tu_x$. Since $Q_{(2)}L|_{\mathcal{L}_{(2)} \cap \mathcal{Q}_{(2)}} = u_x \neq 0$, the operator ∂_t is not a reduction operator of \mathcal{L} . Substituting the corresponding ansatz $u = \varphi(\omega)$, where the invariant independent variable is $\omega = x$, into \mathcal{L} results in the equation $\varphi_{\omega\omega} + t\varphi_\omega = 0$,

¹Extended notions of reduction are also used. Thus, weak symmetries imply reductions decreasing the number of independent variables, preserving the number of unknown functions and increasing the number of equations [17]. The reduced system can be much more overdetermined than the initial one. The reductions associated with higher-order nonclassical symmetries preserve the determinacy type of systems, simultaneously increasing the numbers of unknown functions and equations [15].

in which the “parametric” variable t cannot be excluded via multiplying by a nonvanishing differential function. As expected, the ansatz does not reduce the equation \mathcal{L} .

Consider the same operator Q and the first prolongation $\mathcal{L}_{(3)}$ of \mathcal{L} , which is determined by the equations $L[u] = 0$, $D_t L[u] = 0$ and $D_x L[u] = 0$. The manifold $\mathcal{L}_{(3)} \cap \mathcal{Q}_{(3)}$ is singled out from J^3 by the equations

$$u_t = u_{tt} = u_{tx} = u_{ttt} = u_{ttx} = u_{txx} = 0, \quad u_x = u_{xx} = u_{xxx} = 0.$$

The conditional invariance criterion is satisfied for the prolonged system $\mathcal{L}_{(3)}$ and the operator Q :

$$Q_{(2)} L|_{\mathcal{L}_{(3)} \cap \mathcal{Q}_{(3)}} = Q_{(3)} D_t L|_{\mathcal{L}_{(3)} \cap \mathcal{Q}_{(3)}} = Q_{(3)} D_x L|_{\mathcal{L}_{(3)} \cap \mathcal{Q}_{(3)}} = 0,$$

i.e., Q is a nonclassical symmetry operator of the system $\mathcal{L}_{(3)}$ and the above ansatz reduces $\mathcal{L}_{(3)}$ to the system of three ordinary differential equations $\varphi_\omega = 0$, $\varphi_{\omega\omega} = 0$ and $\varphi_{\omega\omega\omega} = 0$ since

$$\begin{pmatrix} \varphi_{\omega\omega} + t\varphi_\omega \\ \varphi_\omega \\ \varphi_{\omega\omega\omega} + t\varphi_{\omega\omega} \end{pmatrix} = \begin{pmatrix} t & 1 & 0 \\ 1 & 0 & 0 \\ 0 & t & 1 \end{pmatrix} \begin{pmatrix} \varphi_{\omega\omega} \\ \varphi_{\omega\omega} \\ \varphi_{\omega\omega\omega} \end{pmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} t & 1 & 0 \\ 1 & 0 & 0 \\ 0 & t & 1 \end{vmatrix} \neq 0.$$

Note 1. In general, for any system \mathcal{L} and any involutive family Q there exists an order r such that $\mathcal{L}_{(r)} \cap \mathcal{Q}_{(r)}$ is invariant with respect to $Q_{(r)}$. This gives the theoretical background of the notion of weak symmetry [17].

5 Definition of nonclassical symmetries for systems

Myth 6. *The definition of nonclassical symmetry for systems of differential equations is a simple extension of the definition of nonclassical symmetry for single partial differential equations to the case of systems.*

Example 1 and Note 1 indicate problems arising in attempts of defining nonclassical symmetries for systems of partial differential equations.

Let \mathcal{L} denote a system $L(x, u_{(r)}) = 0$ of l differential equations $L^1 = 0, \dots, L^l = 0$ for m unknown functions $u = (u^1, \dots, u^m)$ of n independent variables $x = (x_1, \dots, x_n)$. It is always assumed that the set of differential equations forming the system under consideration canonically represents this system and is minimal. The minimality of a set of equations means that no equation from this set is a differential consequence of the other equations. By $L_{(k)}$ we will denote a maximal set of algebraically independent differential consequences of \mathcal{L} that have, as differential equations, orders not greater than k . We identify $L_{(k)}$ with the corresponding system of algebraic equations in the jet space J^k and associate it with the manifold $\mathcal{L}_{(k)}$ determined by this system. Let $L_{(r)} = \{\hat{L}^\nu, \nu = 1, \dots, \hat{l}\}$. Note that the general system includes equations of different orders.

What is the correct conditional invariance criterion for the system \mathcal{L} ?

$$Q_{(r)} L^\mu|_{\mathcal{L} \cap \mathcal{Q}_{(r)}} = 0, \quad \mu = 1, \dots, l?$$

$$Q_{(r)} L^\mu|_{\mathcal{L}_{(r)} \cap \mathcal{Q}_{(r)}} = 0, \quad \mu = 1, \dots, l?$$

$$Q_{(r)} \hat{L}^\nu|_{\mathcal{L}_{(r)} \cap \mathcal{Q}_{(r)}} = 0, \quad \nu = 1, \dots, \hat{l}?$$

All of the above candidates for the criterion are not satisfactory. The second candidate is not a good choice since it neglects the equations having lower orders than the order of the whole system. Taking the third candidate, we obtain nonclassical symmetries of a prolongation of the system. As shown by Example 1, these may be weakly related to nonclassical symmetries of the system. It is not well understood what differential consequences are really essential. Thus, elements of $\mathcal{L}_{(r)}$ whose trivial differential consequences also belong to $\mathcal{L}_{(r)}$ are neglected by this candidate.

Although all operators satisfying the first of the above criteria give proper reductions, it is overly restrictive and in fact is only a sufficient condition for nonclassical symmetries. Even Lie symmetries can be lost when employing it.

The above discussion is illustrated by the following example.

Example 2. Consider the system

$$\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u} - \Delta \vec{u} + \nabla p + \vec{x} \times \nabla \operatorname{div} \vec{u} = \vec{0}, \quad \operatorname{div} \vec{u} = 0. \quad (4)$$

which is obviously equivalent to the system of Navier–Stokes equations describing the motion of an incompressible fluid. (The additional term $\vec{x} \times \nabla(\operatorname{div} \vec{u})$ vanishes if $\operatorname{div} \vec{u} = 0$.) If we do not take into account differential consequences of system (4), we derive the unnatural claim that this system is not conditionally invariant with respect to translations of the space variables x_i . At the same time, the infinitesimal generators of these translations belong to the maximal Lie invariance algebra of the Navier–Stokes equations. A maximal set $L_{(2)}$ of algebraically independent differential consequences of \mathcal{L} that have, as differential equations, orders not greater than 2 is formed by the equations

$$\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u} - \Delta \vec{u} + \nabla p = \vec{0}, \quad \operatorname{div} \vec{u} = 0,$$

$$\operatorname{div} \vec{u}_t = \vec{0}, \quad \nabla \operatorname{div} \vec{u} = \vec{0}, \quad u_j^i u_i^j + \Delta p = 0.$$

Here the indices i and j run from 1 to 3. The equation $Q_{(2)} \operatorname{div} \vec{u} = \vec{0}$ is identically satisfied on the set $\mathcal{L}_{(2)} \cap \mathcal{Q}_{(2)}$. Therefore, the application of the second or third candidate for the conditional invariance criterion to the equation $\operatorname{div} \vec{u} = \vec{0}$ gives no equations for nonclassical symmetries of the system (4).

Definition 3 can also not be extended to the case of systems in an easy way. The problem again is to define what set of differential consequences of the initial system should be chosen for testing formal compatibility with the appropriate characteristic system.

The notions of nonclassical symmetry and reduction are strongly related in the case of single partial differential equations. It therefore seems natural for these notions to also be closely related in the case of systems. Hence the problem of rigorously defining nonclassical symmetries for systems is additionally complicated by the absence of a canonical extension of the classical reduction to the case of systems. A chain of simple examples can be presented to illustrate possible features of such an extension.

6 Myths on number of nonclassical symmetries

Myth 7. *The number of nonclassical symmetries is essentially greater than the number of classical symmetries.*

At first sight this statement seems obviously true. There exist classes of partial differential equations whose maximal Lie invariance algebra is zero and which admit large sets of reduction operators. This is the case, e.g., for general $(1+1)$ -dimensional evolution equations. At the same time, certain circumstances significantly reduce the number of essential nonclassical symmetries. We briefly list them below.

- The usual equivalence of families of reduction operators. Involutive families Q and \tilde{Q} of l operators are called equivalent if $\tilde{Q}^s = \lambda^{s\sigma} Q^\sigma$ for some $\lambda^{s\sigma} = \lambda^{s\sigma}(x, u)$ with $\det \|\lambda^{s\sigma}\| \neq 0$.
- Nonclassical symmetries equivalent to Lie symmetries.
- The equivalence of nonclassical symmetries with respect to Lie symmetry groups of single differential equations [13, 20] and equivalence groups of classes of such equations [22].
- No-go cases. The problem of finding certain wide subsets of reduction operators may turn out to be equivalent to solving the initial equation [12, 21].
- Non-Lie reductions leading to Lie invariant solutions.

Thus, the existence of a wide Lie symmetry group for a partial differential equation \mathcal{L} complicates, in a certain sense, finding nonclassical symmetries of \mathcal{L} . Indeed, any subalgebra of the corresponding maximal Lie invariance algebra, satisfying the transversality condition, generates a class of equivalent Lie families of reduction operators. If a non-Lie family of reduction operators exists, the action of symmetry transformations on it results in a series of non-Lie families of reduction operators, which are inequivalent in the usual sense. Therefore, for any fixed value of l the system of determining equations for the coefficients of operators from the set $\mathfrak{Q}^l(\mathcal{L})$ of families of l reduction operators is not sufficiently overdetermined to be completely integrated in an easy way, even after factorizing with respect to the equivalence relation in $\mathfrak{Q}^l(\mathcal{L})$. To produce essentially different non-Lie reductions, one has to exclude the solutions of the determining equations which give Lie families of reduction operators and non-Lie families which are equivalent to others

with respect to the Lie symmetry group of \mathcal{L} . As a result, the ratio of efficiency of such reductions to the expended efforts can become vanishingly small.

7 Conclusion

Although the name “nonclassical symmetry” and other analogous names for reduction operators, which refer to symmetry or invariance, do not reflect actual properties of these objects, the usage of such names is justified by historical conventions and additionally supported by the terminology of related fields of group analysis of differential equations. It is a quite common situation for different fields of human activity that a modifier completely changes the meaning of the initial notion (think of terms like “negative growth”, “military intelligence”, etc.). Empiric definitions of nonclassical symmetry can be used in a consistent way if all involved terms and notions are properly interpreted. Nevertheless, as we have argued, the term reduction operator more adequately captures the underlying mathematical content.

In this paper we discussed certain basic myths of the theory of nonclassical symmetries, pertaining to different versions of their definition and the estimation of their cardinality. Over and above these, there are a number of more sophisticated myths concerning, among others, the factorization of sets of nonclassical symmetries, involutive families of reduction operators in the multidimensional case, and singular sets of reduction operators. A discussion of such myths requires a careful theoretical analysis substantiated by nontrivial examples and will be the subject of a forthcoming paper.

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Quasi-linear elliptic type equations invariant under five-dimensional solvable Lie algebras

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The problem of group classification of quasi-linear elliptic type equations in two-dimensional space is considered. The list of all equations of this type, which admit solvable Lie algebras of symmetry operators of dimension up to five is obtained.

1 Introduction

The problem of group classification of quasi-linear elliptic type equations

$$\Delta u = f(x, y, u, u_x, u_y) \quad (1)$$

in two-dimensional space is considered. The list of all equations of this type, which admit solvable Lie algebras of symmetry operators is obtained.

Note that the problem of classification of equations of form (1), which admit Lie algebras of symmetry operators with non-trivial Levi factor, is solved completely in [1]. Furthermore, we are interested in essentially nonlinear equations that can not be reduced to linear ones with use of local or nonlocal transformations. For example, the first of these equations

$$\Delta u = f(u)(u_x^2 + u_y^2), \quad f \neq 0; \quad \Delta u = \lambda e^{\gamma u}, \quad \lambda, \gamma \in \mathbb{R}, \quad \lambda \cdot \gamma \neq 0,$$

is equivalent to the linear equation $\Delta u = \lambda u$ by the local transformation, and the second one is equivalent to Laplace equation by the Bäcklund transformation [2].

Theorem 1. *The invariance group of the equation (1) is generated by the infinitesimal operator*

$$v = a(x, y)\partial_x + b(x, y)\partial_y + c(x, y, u)\partial_u, \quad (2)$$

where functions a, b, c, F satisfy the following system of equations:

$$\begin{aligned} a_y + b_x &= 0, \quad a_x - b_y = 0, \\ c_{xx} + c_{yy} + 2u_x c_{xu} + 2u_y c_{yu} + (u_x^2 + u_y^2)c_{uu} + (c_u - 2a_x)F &= \\ = aF_x + bF_y + cF_u + [c_x + u_x(c_u - a_x) - u_y b_x]F_{u_x} & \quad (3) \\ + [c_y + u_y(c_u - b_y) - u_x a_y]F_{u_y}. \end{aligned}$$

It is easy to see that the two first equations are Cauchy–Riemann conditions, that means that functions a and b are harmonic functions.

The equivalence group \mathcal{E} is formed by those transformations

$$\bar{x} = \alpha(x, y, u), \quad \bar{y} = \beta(x, y, u), \quad v = \gamma(x, y, u), \quad \frac{D(\bar{x}, \bar{y}, v)}{D(x, y, u)} \neq 0,$$

that preserve differential structure of the equation (1), i.e. transform it to an equation of the form

$$v_{\bar{x}\bar{x}} + v_{\bar{y}\bar{y}} = \Phi(\bar{x}, \bar{y}, v, v_{\bar{x}}, v_{\bar{y}}).$$

Theorem 2. *The group \mathcal{E} of the equation (1) is formed by the transformations*

$$\bar{x} = \alpha(x, y), \quad \bar{y} = \beta(x, y), \quad v = \gamma(x, y, u), \quad (4)$$

where

$$\alpha_x = \epsilon\beta_y, \quad \alpha_y = -\epsilon\beta_x \quad (\epsilon = \pm 1), \quad \alpha_x^2 + \alpha_y^2 = \beta_x^2 + \beta_y^2 \neq 0, \quad \gamma_u \neq 0.$$

Theorem 3. *There exist such transformations from the group \mathcal{E} that reduce operator (2) to one of the following operators*

$$v = \partial_x, \quad v = \partial_u.$$

The corresponding classes of invariant equations are

$$\begin{aligned} \Delta u &= F(y, u, u_x, u_y) : A_1^1 = \langle \partial_x \rangle, \\ \Delta u &= F(x, y, u_x, u_y) : A_1^2 = \langle \partial_u \rangle. \end{aligned} \quad (5)$$

2 General method of classification

Let $A_n = \langle e_1, \dots, e_n \rangle$ be a Lie algebra of dimension n , with basic elements

$$e_i = a_i(x, y)\partial_x + b_i(x, y)\partial_y + c_i(x, y, u)\partial_u, \quad i = \overline{1, n}, \quad (6)$$

where each pair of functions $a_i(x, y)$, $b_i(x, y)$ satisfies the Cauchy–Riemann conditions (3) (first two non-classifying equations).

Definition 1. Representation (6) of the algebra A_n is called *admissible* if there exist such functions $F(x, y, u, u_x, u_y)$ for which the three functions $a_i(x, y)$, $b_i(x, y)$, $c_i(x, y, u)$ are solutions of the determining equations (3) and corresponding invariant equations (1) are nonlinear and can not be linearized.

Definition 2. Two representations of the algebra A_n with functions $a_i(x, y)$, $b_i(x, y)$, $c_i(x, y, u)$ and $\tilde{a}_i(x, y)$, $\tilde{b}_i(x, y)$, $\tilde{c}_i(x, y, u)$ are called *equivalent* if any operator e_i of form (6) can be reduced to the operator $\tilde{e}_i = \tilde{a}_i(x, y)\partial_x + \tilde{b}_i(x, y)\partial_y + \tilde{c}_i(x, y, u)\partial_u$ using the equivalence group \mathcal{E} . Denote the set of all inequivalent and admissible representations (possibly, isomorphic) of k -dimensional algebra $A_{k,l}$ as $\mathfrak{N}A_{k,l}$ (l is numbering index of an algebra).

Definition 3. Two representations of an algebra are called *isomorphic* if the set of basis operators of one algebra can be obtained as linear combination of basis operators of the second algebra.

Definition 4. Two representations of an algebra are called *different* if operators of one of them can not be obtained with use of superposition of equivalence transformations \mathcal{E} and linear transformations of basis elements of the second algebra. Denote the set of all different admissible representations of k -dimensional algebra $A_{k,l}$ as $\mathfrak{D}A_{k,l}$.

Examples for the algebra $A_{2,1} = \langle e_1, e_2 \rangle$, $[e_1, e_2] = 0$:

1. The representations $\langle \partial_x, \partial_y \rangle$ and $\langle \partial_y, \partial_x \rangle$ are isomorphic and equivalent.
2. The representations $\langle \partial_x, \partial_u \rangle$ and $\langle \partial_u, \partial_x \rangle$ are isomorphic but not equivalent.
3. The representations $\langle \partial_u, x\partial_u \rangle$ and $\langle \partial_u, y\partial_u \rangle$ are equivalent but not isomorphic.
4. The representations $\langle \partial_y, \partial_x \rangle$ and $\langle \partial_u, x\partial_u \rangle$ are different.

The general inductive algorithm of construction of representations $\mathfrak{D}A_{k,l}$, and of corresponding classes of invariant equations is the following (see also [3, 4]).

- According to Theorem 3 for 1-dimensional algebra $A_{1,1}$ we have $\mathfrak{N}A_{1,1} = \mathfrak{D}A_{1,1} = \{A_1^1, A_1^2\} = \{\langle \partial_x \rangle, \langle \partial_u \rangle\}$ and obtain corresponding classes of invariant equations (5).
- Suppose we constructed the sets of representations $\mathfrak{N}A_{k,l}$, $\mathfrak{D}A_{k,l}$ and corresponding classes of invariant equations for all algebras $A_{k,l}$ of dimension k .
- Consider $k+1$ -dimensional algebras $A_{k+1,l}$. Each of them has a k -dimensional ideal $A_{k,m}$ (this is true for any solvable algebra). Then, using sequentially a constructed representation, from the set $\mathfrak{N}A_{k,m}$ for algebra $A_{k,m} = \langle e_1, \dots, e_k \rangle$ we obtain an additional operator e_{k+1} so that all operators satisfy commutating relationships of the algebra $A_{k+1,l}$. Using equivalence transformations group \mathcal{E} we simplify a form of the operators (4).
- We verify if the obtained representation is *admissible*, namely if the corresponding class of nonlinear equations exists. If so, then we include it to the set $\mathfrak{N}A_{k+1,l}$, otherwise we exclude it from the further consideration. From the constructed set $\mathfrak{N}A_{k+1,l}$ we choose the maximal set $\mathfrak{D}A_{k+1,l}$ of non-isomorphic representations.
- When the set $\mathfrak{D}A_{4,l}$ of all different representations of 4-dimensional algebras and corresponding classes of invariant equations are constructed we use the direct Ovsiannikov's classification method [5] to obtain a list of concrete equations invariant under 5-dimensional Lie algebras.

For illustration of the above algorithm we construct a set of different representations of the 3-dimensional Abelian algebra $A_{3,1} = \langle e_1, e_2, e_3 \rangle$. So, the ideal

of this algebra is $A_{2.1} = \langle e_1, e_2 \rangle$. We found that $\mathfrak{N}A_{2.1} = \{A_{2.1}^1, A_{2.1}^2, A_{2.1}^3, A_{2.1}^4\}$, where

$$\begin{aligned} A_{2.1}^1 &= \langle \partial_x, \lambda_1 \partial_x + \lambda_2 \partial_y \rangle \ (\lambda_2 \neq 0), \quad A_{2.1}^2 = \langle \partial_u, f(x, y) \partial_u \rangle, \\ A_{2.1}^3 &= \langle \partial_x, \lambda_1 \partial_x + \partial_u \rangle, \quad A_{2.1}^4 = \langle \partial_u, \partial_x \rangle. \end{aligned} \quad (7)$$

Here λ_1, λ_2 are constants and $f(x, y)$ is arbitrary function. It is clear that inequivalent representations (7) are isomorphic to one of the following:

$$\mathfrak{D}A_{2.1} = \{\langle \partial_x, \partial_y \rangle; \langle \partial_x, \partial_u \rangle; \langle \partial_u, f(x, y) \partial_u \rangle, f \neq \text{const}\}.$$

However, for construction of representations of 3-dimensional algebras including $A_{2.1}$ as an ideal it is necessary to consider all representations (7) with parameters λ_1, λ_2 (see below representations $A_{3.1}^3$ and $A_{3.1}^4$, for example).

Then each representation (7) is expanded by operator e_3 so that it commutes with others. After simplification all three operators e_1, e_2, e_3 by means of equivalence transformations group we get all inequivalent representations:

$$\begin{aligned} A_{3.1}^1 &= \langle \partial_x, \lambda_1 \partial_x + \lambda_2 \partial_y, \lambda_3 \partial_x + \lambda_4 \partial_y + \partial_u \rangle, \ (\lambda_2 \neq 0), \\ A_{3.1}^2 &= \langle \partial_x, \lambda_1 \partial_x + \partial_u, \lambda_3 \partial_x + \lambda_4 \partial_y \rangle, \ (\lambda_4 \neq 0), \\ \mathbf{A}_{3.1}^3 &= \langle \partial_x, \lambda_1 \partial_x + \partial_u, \lambda_3 \partial_x + f(y) \partial_u \rangle, \ (f'(y) \neq 0), \\ \mathbf{A}_{3.1}^4 &= \langle \partial_u, \partial_x, \lambda_3 \partial_x + \lambda_4 \partial_y \rangle, \ (\lambda_4 \neq 0), \\ A_{3.1}^5 &= \langle \partial_u, \partial_x, \lambda_3 \partial_x + f(y) \partial_u \rangle, \ (f'(y) \neq 0), \\ A_{3.1}^6 &= \langle \partial_u, f(y) \partial_u, \partial_x \rangle, \ (f'(y) \neq 0), \\ A_{3.1}^7 &= \langle \partial_u, f(x, y) \partial_u, g(x, y) \partial_u \rangle. \end{aligned}$$

It is easily to see that for the representations $\mathbf{A}_{3.1}^3$ and $\mathbf{A}_{3.1}^4$ the operators of its 2-dimensional ideals $A_{2.1} = \langle e_1, e_2 \rangle$ are isomorphic, but the resulting algebras are different. Note, that the opposite case is also possible, namely that representations of ideals are different, but its extensions are not different. Finally, it is easy to show that representations $A_{3.1}^i$ ($i = \overline{1, 7}$) are isomorphic to one of the following:

$$\langle \partial_x, \partial_y, \partial_u \rangle; \quad \langle \partial_x, \partial_u, f(y) \partial_u \rangle, \ (f'(y) \neq 0); \quad \langle \partial_u, f(x, y) \partial_u, g(x, y) \partial_u \rangle.$$

Here the last representation is not admissible (see below). So,

$$\mathfrak{D}A_{3.1} = \{\langle \partial_x, \partial_y, \partial_u \rangle; \langle \partial_x, \partial_u, f(y) \partial_u \rangle, \ (f'(y) \neq 0)\}.$$

Below we give the results of classification obtained using the above algorithm. Hereinafter, the notations of algebras $A_{k,l}$ (k and l are dimension and numbering index of an algebra respectively) are given in accordance with Mubarakzjanov's classification [6] of solvable Lie algebras.

3 Invariance with respect to 2-dimensional Lie algebras

We obtained the following representations and the corresponding forms F of the right hand side of equation (1).

Algebra $A_{2.1}$:

1. $A_{2.1}^1 = \langle \partial_x, \partial_y \rangle : F = G(u, u_x, u_y);$
2. $A_{2.1}^2 = \langle \partial_x, \partial_u \rangle : F = G(y, u_x, u_y);$
3. $A_{2.1}^3 = \langle \partial_u, f(x, y) \partial_u \rangle : F = \frac{f_x u_x + f_y u_y}{f_x^2 + f_y^2} (f_{xx} + f_{yy}) + G(x, y, \omega),$
 $\omega = f_y u_x - f_x u_y, \quad f_x^2 + f_y^2 \neq 0.$

Algebra $A_{2.2}$:

1. $A_{2.2}^1 = \langle -x \partial_x - y \partial_y, \partial_x \rangle : F = (u_x^2 + u_y^2) G(u, \omega_1, \omega_2), \quad \omega_1 = y u_x, \quad \omega_2 = y u_y;$
2. $A_{2.2}^2 = \langle \partial_x - u \partial_u, \partial_u \rangle : F = e^{-x} G(y, \omega_1, \omega_2), \quad \omega_1 = e^x u_x, \quad \omega_2 = e^x u_y;$
3. $A_{2.2}^3 = \langle -u \partial_u, \partial_u \rangle : F = (u_x + u_y) G(x, y, \omega), \quad \omega = u_x u_y^{-1}.$

Let us note that for arbitrary forms of functions G the respective representations are maximal invariance algebras of equations.

4 Invariance with respect to 3-dimensional Lie algebras

The solvable algebras $A_{3.i} = \langle e_1, e_2, e_3 \rangle$, $i = \overline{1, 9}$ contain the 2-dimensional subalgebra $A_{2.1}$. Furthermore, it can be assumed for all algebras, except $A_{3.2}$, that $A_{2.1} = \langle e_1, e_2 \rangle$. For the algebra $A_{3.2}$ we assume that $A_{2.1} = \langle e_2, e_3 \rangle$. We obtained all admissible representations.

The number of different admissible representations for algebras:

- each of the algebras $A_{3.i}$ ($i = 1, 3, 4, 6, 7$) has two representations,
- algebras $A_{3.8}$ and $A_{3.9}$ have three representations,
- algebras $A_{3.2}$ and $A_{3.5}$ have four representations.

So, we have 24 representations for three-dimensional Lie algebras.

The appearance of the representations and the corresponding forms of the right hand side of equation (1) are the following. Below the parameter q parametrizes the algebras $A_{3.7}^q$ and $A_{3.9}^q$.

Algebra $A_{3.1}$:

1. $A_{3.1}^1 = \langle \partial_x, \partial_y, \partial_u \rangle : F = G(u_x, u_y);$
2. $A_{3.1}^2 = \langle \partial_x, \partial_u, f(y) \partial_u \rangle, \quad f'(y) \neq 0 : F = \frac{f''}{f'} u_y + G(y, u_x);$
3. $A_{3.1}^3 = \langle \partial_u, f(x, y) \partial_u, g(x, y) \partial_u \rangle.$

The last representation is not admissible, because either the corresponding invariant equation is linear or the infinitesimal operators are linear-dependent.

Algebra $A_{3.2}$:

1. $A_{3.2}^1 = \langle -x\partial_x - y\partial_y, \partial_x, \partial_u \rangle : F = u_x^2 G(yu_x, yu_y);$
2. $A_{3.2}^2 = \langle -u\partial_u, \partial_u, \partial_x \rangle : F = (u_x + u_y)G\left(y, \frac{u_x}{u_y}\right);$
3. $A_{3.2}^3 = \langle \partial_y - u\partial_u, \partial_u, \partial_x \rangle : F = (u_x + u_y)G(e^y u_x, e^y u_y);$
4. $A_{3.2}^4 = \langle \partial_x - u\partial_u, \partial_u, f(y)e^{-x}\partial_u \rangle, f(y) \neq 0 : F = -\frac{f + f''}{f}u_x + e^{-x}G(y, e^x f'u_x + e^x f u_y).$

Algebra $A_{3.3}$:

1. $A_{3.3}^1 = \langle \partial_u, \partial_x, \partial_y + x\partial_u \rangle, F = G(u_x - y, u_y);$
2. $A_{3.3}^2 = \langle \partial_u, \partial_x, (f(y) + x)\partial_u \rangle, F = f''u_x + G(y, u_y - f'u_x).$

Algebra $A_{3.4}$:

1. $A_{3.4}^1 = \langle \partial_u, \partial_x, x\partial_x + y\partial_y + (u + x)\partial_u \rangle, F = e^{-u_x}G(u_y, ye^{-u_x});$
2. $A_{3.4}^2 = \langle \partial_u, (f(y) - x)\partial_u, \partial_x + u\partial_u \rangle, F = \frac{f'u_y - u_x}{1 + (f')^2}f'' + e^xG(y, f'e^{-x}u_x + e^{-x}u_y).$

Algebra $A_{3.5}$:

1. $A_{3.5}^1 = \langle \partial_x, \partial_y, x\partial_x + y\partial_y \rangle, F = (u_x + u_y)^2 G\left(u, \frac{u_x}{u_y}\right);$
2. $A_{3.5}^2 = \langle \partial_x, \partial_y, x\partial_x + y\partial_y + \partial_u \rangle, F = e^{-2u}G\left(e^u u_x, \frac{u_x}{u_y}\right);$
3. $A_{3.5}^3 = \langle \partial_x, \partial_u, x\partial_x + y\partial_y + u\partial_u \rangle, F = y^{-1}G(u_x, u_y);$
4. $A_{3.5}^4 = \langle \partial_u, f(y)\partial_u, \partial_x + u\partial_u \rangle, f' \neq 0, F = \frac{u_y}{f'}f'' + e^xG(y, f'e^{-x}u_x);$
5. $A_{3.5}^5 = \langle \partial_u, f(x, y)\partial_u, u\partial_u \rangle.$

The further analysis shows that for the last representation the corresponding invariant equation is linear.

Algebra $A_{3.6}$:

1. $A_{3.6}^1 = \langle \partial_x, \partial_u, x\partial_x + y\partial_y - u\partial_u \rangle, F = y^{-3}G(y^2 u_x, y^2 u_y);$
2. $A_{3.6}^2 = \langle \partial_u, e^{2x}f(y)\partial_u, \partial_x + u\partial_u \rangle, f(y) \neq 0, F = \frac{2f u_x + f' u_y}{4f^2 + (f')^2}(f'' + 4f) + e^xG(y, f'e^{-x}u_x - 2f e^{-x}u_y).$

Algebra $A_{3.7}^q$:

1. $A_{3.7}^1 = \langle \partial_x, \partial_u, x\partial_x + y\partial_y + qu\partial_u \rangle, q \neq 0, \pm 1,$
 $F = y^{q-2}G(y^{1-q}u_x, y^{1-q}u_y);$
2. $A_{3.7}^2 = \langle \partial_u, e^{(1-q)x}f(y)\partial_u, \partial_x + u\partial_u \rangle, 0 < |q| < 1,$
 $F = \frac{(1-q)fu_x + f'u_y}{(1-q)^2f^2 + (f')^2}(f'' + f(1-q)^2) +$
 $e^xG(y, f'e^{-x}u_x + (q-1)fe^{-x}u_y).$

Algebra $A_{3.8}$:

1. $A_{3.8}^1 = \langle \partial_x, \partial_y, y\partial_x - x\partial_y \rangle, F = G(u, u_x^2 + u_y^2);$
2. $A_{3.8}^2 = \langle \partial_x, \partial_y, y\partial_x - x\partial_y + \partial_u \rangle, F = G\left(u_x^2 + u_y^2, \arctan\left(\frac{u_x}{u_y}\right) - u\right);$
3. $A_{3.8}^3 = \langle \partial_u, \tan(f(y) - x)\partial_u, \partial_x - u\tan(f(y) - x)\partial_u \rangle,$
 $F = \frac{f'u_y - u_x}{(f')^2 + 1}f'' + 2(f'u_y - u_x)\tan(f - x) +$
 $(f'u_x + u_y)G(y, \cos(f - x)(f'u_x + u_y)).$

Algebra $A_{3.9}^q$:

1. $A_{3.9}^1 = \langle \partial_x, \partial_y, (qx + y)\partial_x + (qy - x)\partial_y \rangle, q > 0,$
 $F = (u_x^2 + u_y^2)G\left(u, \ln(u_x^2 + u_y^2) + 2q\arctan\frac{u_x}{u_y}\right);$
2. $A_{3.9}^2 = \langle \partial_x, \partial_y, (qx + y)\partial_x + (qy - x)\partial_y + \partial_u \rangle, q > 0,$
 $F = e^{-2qu}G\left(\arctan\left(\frac{u_x}{u_y}\right) - u, \ln(u_x^2 + u_y^2) + 2q\arctan\frac{u_x}{u_y}\right);$
3. $A_{3.9}^3 = \langle \partial_u, \tan(f(y) - x)\partial_u, \partial_x + (q - \tan(f(y) - x))u\partial_u \rangle, q > 0,$
 $F = \frac{f'u_y - u_x}{(f')^2 + 1}f'' + 2(f'u_y - u_x)\tan(f - x) +$
 $(f'u_x + u_y)G(y, \cos(f - x)(f'u_x + u_y)e^{-qx}).$

Summary:

1. Operators of certain representations contain arbitrary functions $f(y)$, which can not be reduced using the equivalence transformations group.
2. All representations of the algebra $A_{3.7}$ with $q = -1$ coincide with the representations of the algebra $A_{3.6}$, as well as the corresponding classes of invariant equations.
3. The representation $A_{3.7}^1$ with $q = 1$ coincides with the representation $A_{3.5}^3$.
4. The representations of the algebra $A_{3.9}$ with $q = 0$ coincide with the representations of the algebra $A_{3.8}$.

5 Invariance with respect to 4-dimensional Lie algebras

According to our algorithm we use the obtained inequivalent and admissible representations $\mathfrak{N}A_{3,i}$ ($(i = 1, 2, \dots, 9)$) and $\mathfrak{N}A_{2,2}$ of the algebras $A_{3,i}$ and $A_{2,2}$, which are subalgebras of decomposable 4-dimensional algebras $\tilde{A}_{4,i} = A_{3,i} \oplus A_1$ and $\tilde{A}_{4,10} = A_{2,2} \oplus A_{2,2}$, correspondingly. For the indecomposable algebras the following subalgebraic structure exists:

1. $A_{3,1} \subset A_{4,i}$, ($i = \overline{1,6}$);
2. $A_{3,3} \subset A_{4,i}$, ($i = 7, 8, 9$);
3. $A_{3,5} \subset A_{4,10}$.

Thus, we obtained all different and admissible representations of 4-dimensional Lie algebras.

The number of different admissible representations for algebras:

- each of the algebras $A_{4,i}$ ($i = 1, 4, 7, 9$) and $\tilde{A}_{4,3}$ has one representation;
- algebras $\tilde{A}_{4,i}$ ($i = 2, \overline{4,10}$) and $A_{4,i}$ ($i = 3, 6$) have two representations;
- algebras $A_{4,i}$ ($i = 2, 5, 10$) have three representations;
- algebra $A_{4,8}$ has four representations.

So, we have 38 representations for 4-dimensional Lie algebras.

The appearance of the representations and the corresponding forms of the right hand side of equation (1) are listed bellow. Here q and p parameterize the certain algebras, while λ, λ_1 and λ_2 parameterize the certain representations of algebras. In this case we can not reduce these three parameters using the equivalence transformations group (4).

Decomposable algebras

Algebra $\tilde{A}_{4,2} = A_{3,2} \oplus A_1$:

1. $\tilde{A}_{4,2}^1 = \langle -u\partial_u, \partial_u, \partial_x \rangle \oplus \langle \partial_y \rangle$, $F = (u_x + u_y)G\left(\frac{u_x}{u_y}\right)$;
2. $\tilde{A}_{4,2}^2 = \langle \partial_y - u\partial_u, \partial_u, \partial_x \rangle \oplus \langle e^{-y}\partial_u \rangle$, $F = e^{-y}G(e^y u_x) - (u_x + u_y)$.

Algebra $\tilde{A}_{4,3} = A_{3,3} \oplus A_1$:

1. $\tilde{A}_{4,3}^1 = \langle \partial_u, \partial_x, x\partial_u \rangle \oplus \langle \partial_y \rangle$, $F = G(u_y)$;
2. $\tilde{A}_{4,3}^2 = \langle \partial_u, \partial_x, \partial_y + x\partial_u \rangle \oplus \langle \partial_x + y\partial_u \rangle$, $F = G(u_x - y)$.

Using the change of variables $u = \tilde{u} + xy$, $\tilde{x} = y$, $\tilde{y} = -x$ this representation is reduced to the case 1. (This is given to illustrate the fact that expanding different representations we can get not only different ones).

Algebra $\tilde{A}_{4.4} = A_{3.4} \oplus A_1$:

1. $\tilde{A}_{4.4}^1 = \langle \partial_u, \partial_x, x\partial_x + y\partial_y + (u+x)\partial_u \rangle \oplus \langle y\partial_u \rangle, F = e^{-u_x}G(ye^{-u_x});$
2. $\tilde{A}_{4.4}^2 = \langle \partial_u, -x\partial_u, \partial_x + u\partial_u \rangle \oplus \langle \partial_y \rangle, F = e^xG(e^{-x}u_y).$

Algebra $\tilde{A}_{4.5} = A_{3.5} \oplus A_1$:

1. $\tilde{A}_{4.5}^1 = \langle \partial_x, \partial_y, x\partial_x + y\partial_y \rangle \oplus \langle \partial_u \rangle, F = (u_x + u_y)^2G\left(\frac{u_x}{u_y}\right);$
2. $\tilde{A}_{4.5}^2 = \langle \partial_x, \partial_u, x\partial_x + y\partial_y + u\partial_u \rangle \oplus \langle y\partial_u \rangle, F = y^{-1}G(u_x);$

Algebra $\tilde{A}_{4.6} = A_{3.6} \oplus A_1$:

1. $\tilde{A}_{4.6}^1 = \langle \partial_x, \partial_u, x\partial_x + y\partial_y - u\partial_u \rangle \oplus \langle y^{-1}\partial_u \rangle,$
 $F = -2y^{-1}u_y + y^{-3}G(y^2u_x);$
2. $\tilde{A}_{4.6}^2 = \langle \partial_u, e^{2x}\partial_u, \partial_x + u\partial_u \rangle \oplus \langle \partial_y \rangle, F = 2u_x + e^xG(e^{-x}u_y).$

Algebra $\tilde{A}_{4.7}^q = A_{3.7}^q \oplus A_1$:

1. $\tilde{A}_{4.7}^1 = \langle \partial_x, \partial_u, x\partial_x + y\partial_y + qu\partial_u \rangle \oplus \langle y^q\partial_u \rangle, q \neq 0, \pm 1,$
 $F = (q-1)y^{-1}u_y + y^{q-2}G(y^{1-q}u_x);$
2. $\tilde{A}_{4.7}^2 = \langle \partial_u, e^{(1-q)x}\partial_u, \partial_x + u\partial_u \rangle \oplus \langle \partial_y \rangle, 0 < |q| < 1,$
 $F = (1-q)u_x + e^xG(e^{-x}u_y).$

Algebra $\tilde{A}_{4.8} = A_{3.8} \oplus A_1$:

1. $\tilde{A}_{4.8}^1 = \langle \partial_x, \partial_y, y\partial_x - x\partial_y \rangle \oplus \langle \partial_u \rangle, F = G(u_x^2 + u_y^2);$
2. $\tilde{A}_{4.8}^2 = \langle \partial_u, -\tan x\partial_u, \partial_x + u\tan x\partial_u \rangle \oplus \langle \partial_y \rangle,$
 $F = 2u_x \tan x + u_y G(u_y \cos x).$

Algebra $\tilde{A}_{4.9}^q = A_{3.9}^q \oplus A_1$:

1. $\tilde{A}_{4.9}^1 = \langle \partial_x, \partial_y, (qx+y)\partial_x + (qy-x)\partial_y \rangle \oplus \langle \partial_u \rangle, q > 0,$
 $F = (u_x^2 + u_y^2)G\left((u_x^2 + u_y^2) \exp\left(2q \arctan \frac{u_x}{u_y}\right)\right);$
2. $\tilde{A}_{4.9}^2 = \langle \partial_u, -\tan x\partial_u, \partial_x + u(q + \tan x)\partial_u \rangle \oplus \langle \partial_y \rangle, q > 0,$
 $F = 2u_x \tan x + u_y G(u_y e^{-qx} \cos x)$

Algebra $\tilde{A}_{4.10} = A_{2.2} \oplus A_{2.2}$:

1. $\tilde{A}_{4.10}^1 = \langle -x\partial_x - y\partial_y, \partial_x \rangle \oplus \langle -u\partial_u, \partial_u \rangle, F = \frac{(u_x^2 + u_y^2)}{yu_x}G\left(\frac{u_x}{u_y}\right);$
2. $\tilde{A}_{4.10}^2 = \langle \partial_x - u\partial_u, \partial_u \rangle \oplus \langle \lambda_1\partial_x + \lambda_2\partial_y, e^{-x}e^{\frac{(1+\lambda_1)y}{\lambda_2}}\partial_u \rangle, \lambda_2 \neq 0,$
 $F = -\left(1 + \left(\frac{1+\lambda_1}{\lambda_2}\right)^2\right)u_x + e^{-x}e^{\frac{\lambda_1 y}{\lambda_2}}G\left(e^x e^{-\frac{\lambda_1 y}{\lambda_2}}\left(\frac{1+\lambda_1}{\lambda_2}u_x + u_y\right)\right).$

Non-decomposable algebras

Algebra $A_{4.1}$:

$$A_{4.1}^1 = \langle \partial_y - xy\partial_u \rangle \oplus \langle \partial_u, -y\partial_u, \partial_x \rangle, \quad F = G(y^2 + 2u_x).$$

Algebra $A_{4.2}^q$:

1. $A_{4.2}^1 = \langle x\partial_x + y\partial_y + (u + y)\partial_u \rangle \oplus \langle \partial_x, \partial_u, \partial_y \rangle, \quad q = 1, \quad F = e^{-u_y}G(u_x);$
2. $A_{4.2}^2 = \langle qx\partial_x + qy\partial_y + u\partial_u \rangle \oplus \langle \partial_x, \partial_u, -q^{-1}\ln y\partial_u \rangle, \quad q \neq 0,$

$$F = -\frac{u_y}{y} + \frac{u_x}{y}G\left(y^{\frac{q-1}{q}}u_x\right);$$
3. $A_{4.2}^3 = \langle x\partial_x + y\partial_y + (qu + y^{q-1}x)\partial_u \rangle \oplus \langle \partial_u, y^{q-1}\partial_u, \partial_x \rangle \quad (q \neq 0; 1),$

$$F = \frac{q-2}{y}u_y + y^{q-2}G(u_xy^{1-q} - \ln y).$$

Algebra $A_{4.3}$:

1. $A_{4.3}^1 = \langle x\partial_x + y\partial_y \rangle \oplus \langle \partial_x, \partial_u, -\ln y\partial_u \rangle, \quad F = -\frac{u_y}{y} + u_x^2G(yu_x);$
2. $A_{4.3}^2 = \langle \partial_y + (u + xe^y)\partial_u \rangle \oplus \langle \partial_u, e^y\partial_u, \partial_x \rangle, \quad F = u_y + e^yG(e^{-y}u_x - y).$

Algebra $A_{4.4}$:

$$A_{4.4}^1 = \langle x\partial_x + y\partial_y + (u - x\ln y)\partial_u \rangle \oplus \langle \partial_u, -\ln y\partial_u, \partial_x \rangle,$$

$$F = -\frac{u_y}{y} + \frac{1}{y}G(2u_x + \ln^2 y).$$

Algebra $A_{4.5}^{p,q}$:

1. $A_{4.5}^1 = \langle x\partial_x + y\partial_y + pu\partial_u \rangle \oplus \langle \partial_x, \partial_y, \partial_u \rangle, \quad p \neq 0; 1, \quad q = 1,$

$$F = (u_xu_y)^rG\left(\frac{u_x}{u_y}\right), \quad r = \frac{p-2}{2(p-1)};$$
2. $A_{4.5}^2 = \langle x\partial_x + y\partial_y + qu\partial_u \rangle \oplus \langle \partial_x, \partial_u, y^{q-p}\partial_u \rangle, \quad -1 \leq p < q \leq 1, \quad pq \neq 0,$

$$F = \frac{q-p-1}{y}u_y + y^{q-2}G(u_xy^{1-q}); \quad (8)$$
3. $A_{4.5}^3 = \langle qx\partial_x + qy\partial_y + u\partial_u \rangle \oplus \langle \partial_u, \partial_x, y^{(1-p)/q}\partial_u \rangle, \quad (9)$

$$-1 \leq p < 1, \quad -1 \leq q \leq 1, \quad pq \neq 0,$$

$$F = \frac{1-q-p}{qy}u_y + y^{(1-2q)/q}G(u_xy^{(q-1)/q}).$$

Algebra $A_{4.6}^{p,q}$:

1. $A_{4.6}^1 = \langle (px + y)\partial_x + (py - x)\partial_y + qu\partial_u \rangle \oplus \langle \partial_u, \partial_x, \partial_y \rangle, \quad q \neq 0, \quad p \geq 0,$
 $F = \exp \left((q - 2p) \arctan \frac{u_x}{u_y} \right) G \left((u_x^2 + u_y^2) \exp \left(2(p - q) \arctan \frac{u_x}{u_y} \right) \right);$
2. $A_{4.6}^2 = \langle qx\partial_x + qy\partial_y + (p + \tan(q^{-1} \ln y))u\partial_u \rangle \oplus$
 $\langle \partial_x, \partial_u, -\tan(q^{-1} \ln y)\partial_u \rangle, \quad q \neq 0, \quad p \geq 0,$
 $F = \left(\frac{2 \tan(q^{-1} \ln y)}{qy} - \frac{1}{y} \right) u_y + \frac{G \left(u_x \cos(q^{-1} \ln y) y^{\frac{q-p}{q}} \right)}{\cos(q^{-1} \ln y) y^{\frac{2q-p}{q}}}.$

Algebra $A_{4.7}$:

$$A_{4.7}^1 = \langle x\partial_x + y\partial_y + \left(2u - \frac{x^2}{2} + \lambda xy \right) \partial_u \rangle \oplus \langle \partial_u, (\lambda y - x)\partial_u, \partial_x \rangle,$$

$$F = -\ln y + G \left(\frac{\lambda u_x + u_y}{y} - \lambda^2 \ln y \right).$$

Algebra $A_{4.8}^q$:

1. $A_{4.8}^1 = \langle x\partial_x + y\partial_y + 2u\partial_u \rangle \oplus \langle \partial_u, \partial_x, \partial_y + x\partial_u \rangle \quad (q = 1),$
 $F = G \left(\frac{u_x - y}{u_y} \right);$
2. $A_{4.8}^2 = \langle x\partial_x + y\partial_y + (1 + q)u\partial_u \rangle \oplus \langle \partial_u, \partial_x, (\lambda y + x)\partial_u \rangle, \quad |q| \leq 1,$
 $F = y^{q-1} G(y^{-q}(u_y - \lambda u_x));$
3. $A_{4.8}^3 = \langle \partial_y + u\partial_u \rangle \oplus \langle \partial_u, -x\partial_u, \partial_x + \lambda\partial_y \rangle, \quad (q = 0),$
 $F = \exp(y - \lambda x) G(\exp(\lambda x - y) u_y);$
4. $A_{4.8}^4 = \langle qx\partial_x + qy\partial_y + (1 + q)u\partial_u \rangle \oplus \langle \partial_u, (\lambda y - x)\partial_u, \partial_x \rangle, \quad 0 < |q| \leq 1,$
 $F = y^{(1-q)/q} G \left(\frac{\lambda u_x + u_y}{y^{1/q}} \right).$

Algebra $A_{4.9}^q$:

$$A_{4.9}^1 = \langle (qx + y)\partial_x + (qy - x)\partial_y + \left(2qu + \frac{y^2}{2} - \frac{x^2}{2} \right) \partial_u \rangle \oplus$$

$$\langle \partial_u, \partial_x, \partial_y + x\partial_u \rangle, \quad q \geq 0,$$

$$F = G \left(((u_x - y)^2 + u_y^2) \exp \left(-2q \arctan \left(\frac{u_x - y}{u_y} \right) \right) \right).$$

Algebra $A_{4.10}$:

1. $A_{4.10}^1 = \langle y\partial_x - x\partial_y + \epsilon\partial_u \rangle \oplus \langle \partial_x, \partial_y, x\partial_x + y\partial_y \rangle, \epsilon = 0, 1,$
 $F = (u_x^2 + u_y^2)G \left(\epsilon \arctan \frac{u_x}{u_y} - u \right);$
2. $A_{4.10}^2 = \langle \lambda_1\partial_x + \lambda_2\partial_y + u \tan(\lambda_2^{-1}y)\partial_u \rangle \oplus \langle \partial_u, -\tan(\lambda_2^{-1}y)\partial_u, \partial_x + u\partial_u \rangle,$
 $F = 2\lambda_2^{-1} \tan(\lambda_2^{-1}y)u_y + u_x G(\exp(\lambda_1y/\lambda_2 - x) \cos(\lambda_2^{-1}y)u_x), \lambda_2 \neq 0;$
3. $A_{4.10}^3 = \langle y\partial_x - x\partial_y + \lambda\partial_u \rangle \oplus \langle \partial_x, \partial_y, x\partial_x + y\partial_y + \partial_u \rangle,$
 $F = (u_x^2 + u_y^2)G \left((u_x^2 + u_y^2)e^{2u} \exp \left(-2\lambda \arctan \left(\frac{u_x}{u_y} \right) \right) \right).$

6 Invariance with respect to 5-dimensional Lie algebras. Direct method

Below we apply the direct method (but essentially modified) to above obtained 38 classes of equations invariant under 4-dimensional solvable Lie algebras. As a result, only 12 representations have additional operator e_5 such that the corresponding invariant equation can not be linearized, i.e. its 5-dimensional invariance algebra is admissible. Here we give the complete list of such equations and five operators spanning their invariance algebras.

Below, for example, notation “extension $\tilde{A}_{4.3}^1 \subset A_{5.20}^p, p = 0$ ” means that for operators $\langle e_1, e_2, e_3, e_4 \rangle$ of the representation $\tilde{A}_{4.3}^1$ a fifth operator e_5 exists, such that the operators $\langle e_1, e_2, e_3, e_4, e_5 \rangle$ generate a representation of the algebra $A_{5.20}^p$ with $p = 0$.

1. **Extension** $\tilde{A}_{4.2}^1 \subset A_{2.2} \oplus A_{3.9}^q, q > 0$:

$$\Delta u = \sqrt{u_x^2 + u_y^2} \exp \left(-q \arctan \frac{u_x}{u_y} \right),$$

$$\langle -u\partial_u, \partial_u \rangle \oplus \langle \partial_x, \partial_y, (qx + y)\partial_x + (qy - x)\partial_y \rangle$$

2. **Extension** $\tilde{A}_{4.3}^1 \subset A_{5.20}^p, p = 0$

$$\Delta u = \exp(u_y), \langle -\partial_u, \partial_x, -x\partial_u, \partial_y, x\partial_x + y\partial_y + (u - y)\partial_u \rangle$$

3. **Extension** $\tilde{A}_{4.3}^1 \subset A_{5.23}^p, p = 1$

$$\Delta u = \ln(u_y), \langle \partial_u, \partial_x, x\partial_u, \partial_y, x\partial_x + y\partial_y + \left(2u + \frac{x^2}{2} \right) \partial_u \rangle$$

4. **Extension** $\tilde{A}_{4.3}^1 \subset A_{5.19}^{p,q}, p \neq 0; 1, q = 1$

$$\Delta u = u_y^{\frac{p-1}{p}}, \langle \partial_u, \partial_x, x\partial_u, \partial_y, x\partial_x + y\partial_y + (p+1)u\partial_u \rangle$$

5. **Extension** $\tilde{A}_{4.5}^1 \subset A_{5.35}^{p,q}$, $p = 0$, $q \neq 0$

$$\Delta u = (u_x^2 + u_y^2) \exp \left(-q \arctan \frac{u_x}{u_y} \right),$$

$$\langle \partial_u, \partial_x, \partial_y, x\partial_x + y\partial_y, y\partial_x - x\partial_y + qu\partial_u \rangle$$

6. **Extension** $\tilde{A}_{4.6}^1 \subset A_{5.19}^{p,q}$, $p = q = -1$

$$\Delta u = yu_x^2 - 2y^{-1}u_y, \langle -y^{-1}\partial_u, \partial_u, \partial_y - \frac{u}{y}\partial_u, \partial_x, -x\partial_x - y\partial_y + u\partial_u \rangle$$

7. **Extension** $\tilde{A}_{4.8}^1 \subset A_{2.2} \oplus A_{3.8}$

$$\Delta u = \sqrt{u_x^2 + u_y^2}, \langle -u\partial_u, \partial_u \rangle \oplus \langle \partial_x, \partial_y, y\partial_x - x\partial_y \rangle$$

8. **Extension** $\tilde{A}_{4.8}^1 \subset A_{5.35}^{p,q}$, $p \neq 0; 1$, $q = 0$

$$\Delta u = (u_x^2 + u_y^2)^{(2-p)/2(1-p)}, \langle \partial_u, \partial_x, \partial_y, x\partial_x + y\partial_y + pu\partial_u, y\partial_x - x\partial_y, \rangle$$

9. **Extension** $\tilde{A}_{4.9}^1 \subset A_{5.35}^{p,\tilde{q}}$, $\tilde{q} = -pq$, $p \neq 0; 1$, $q > 0$

$$\Delta u = (u_x^2 + u_y^2)^{(2-p)/2(1-p)} \exp \left(\frac{qp}{1-p} \arctan \frac{u_x}{u_y} \right),$$

$$\langle \partial_u, \partial_x, \partial_y, x\partial_x + y\partial_y + pu\partial_u, y\partial_x - x\partial_y - qpu\partial_u, \rangle$$

10. **Extension** $\tilde{A}_{4.10}^1 \subset A_{2.2} \oplus sl(2, \mathbb{R})$

$$\Delta u = \lambda y^{-1} \sqrt{u_x^2 + u_y^2},$$

$$\langle -u\partial_u, \partial_u \rangle \oplus \langle 2x\partial_x + 2y\partial_y, -(x^2 - y^2)\partial_x - 2xy\partial_y + y\partial_u, \partial_x \rangle$$

11. **Extension** $A_{4.5}^3 \subset A_{5.30}^p$, $p = 1$

$$\Delta u = \lambda \sqrt{2u_x \pm y^2}, \langle \partial_u, -y\partial_u, \pm\partial_x, \partial_y \mp xy\partial_u, x\partial_x + y\partial_y + 3u\partial_u \rangle$$

12. **Extension** $A_{4.8}^4 \subset A_{5.30}^p$, $p = 1$

$$\Delta u = (\lambda u_x + u_y)^{1/2} + \mu y, \left\langle \partial_u, (\lambda y - x)\partial_u, -\frac{1 + \lambda^2}{\mu} \left(\partial_y + \frac{\mu}{2(1 + \lambda^2)} (\lambda y - x)^2 \partial_u \right), \partial_x, x\partial_x + y\partial_y + 3u\partial_u \right\rangle$$

Summary:

1. We can not reduce the parameters λ and μ in the cases 10, 11 and 12 using the equivalence transformations group (4);
2. The extension $\tilde{A}_{4.10}^1$ (Case 10) leads to the non-solvable 5-dimensional algebra (with nontrivial Levi factor).

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Analysis of a model of shallow-water waves based upon an approach using symmetry

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A number of papers have been devoted to the differential equations describing the motion of waves in shallow water as modelled by Whitham, Broer and Kaup from some forty to thirty years ago. We examine these equations from the viewpoint of symmetry with the explicit intention to carry those considerations to the end whereby we see that some interesting relationships between special functions are naturally revealed. As a subsequent exploration we look at the model equations when two of the critical parameters are zero and explore if there is any solution of interest.

1 Introduction

Whitham, Broer and Kaup [7, 15, 22] in several papers presented a system of two equations for the motion of waves in shallow water as the 1+1 evolution equations

$$u_t = uu_x + v_x + \beta u_{xx}, \quad v_t = u_x v + uv_x + \beta v_{xx} + \alpha u_{xxx}, \quad (1)$$

where $u(t, x)$ is the horizontal velocity, $v(t, x)$ is the vertical displacement of the fluid from its equilibrium position and α and β are parameters related to the degree of diffusion.

The system, (1), has been studied by a variety of methods [1, 13, 15, 20, 21, 23–25]. Of particular relevance to the present work is the paper of Zhang *et al* [25] which treats system (1) in terms of the optimal system of subgroups of the Lie point symmetries of the system. Curiously the authors of that paper present a closed-form solution in only one case. However, they do pay considerable attention to the case that α and β are both zero, *ie* the problem of one-dimensional shallow-water equations over an horizontal base.

In this paper we supplement the results in [25] with a deeper analysis of the systems of ordinary differential equations which result from the reduction of system (1) using the several optimal subgroups for nonzero α and β . This leads to a number of equations which are well known in the general literature on ordinary differential equations. We conclude our study with a different approach to the case of zero α and β than that presented in [25].

2 The case of nonzero α and β

Without loss of generality the variables in system (1) can be rescaled so that there is only one essential parameter. We write the system as

$$u_t = uu_x + kv_x + u_{xx}, \quad v_t = u_xv + uv_x + v_{xx} + u_{xxx}, \quad (2)$$

where $k = \alpha/\beta^2$. The rescaling of the variables does not affect the Lie point symmetries which are¹

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = t\partial_x - \partial_u, \quad \Gamma_4 = 2t\partial_t + x\partial_x - u\partial_u - 2v\partial_v$$

for which the optimal system of one-dimensional subalgebras is composed of these four symmetries plus

$$\Gamma_5 = \Gamma_1 - \Gamma_3 = \partial_t - t\partial_x + \partial_u.$$

We consider the different possibilities for reduction in turn.

Γ_1 and Γ_2 . Reduction under Γ_2 leads to the trivial result that both u and v are constants.

Although reduction under Γ_1 implies a steady state, the reduced system is not without interest in the light of results to be demonstrated below. Obviously u and v are functions only of x , $U(x)$ and $V(x)$ respectively. The reduced system is

$$0 = UU' + kV' + U'' \quad (3)$$

$$0 = U'V + UV' + V'' + U''', \quad (4)$$

where the prime denotes differentiation with respect to x . We integrate (3) to obtain $V = (A - U' - U^2/2)/k$, where A is a constant of integration. We substitute for V into (4) to obtain

$$(1 - k)U''' + 2(UU'' + U'^2) + \frac{3}{2}U^2U' - AU = 0 \quad (5)$$

which can be integrated once to give

$$(1 - k)U'' + 2UU' + \frac{1}{2}U^3 - AU + B = 0, \quad (6)$$

where B is a further constant of integration. We recognise (6) as a generalisation of the well-known Painlevé-Ince equation². In the case that $k = 1/9$ (6) is linearisable to $w''' - \frac{9A}{8}w' + \frac{27}{32}Bw = 0$ by means of the Riccati transformation $U = 4w'/3w$ and is trivially integrable. For the same value of k the equation has eight Lie

¹Courtesy of the Mathematica add-on Sym [5, 9, 10]. Note that we adopt a different ordering to that in [25].

²The equation, $y'' + 3yy' + y^3 = 0$, appears in a bewildering array of contexts – theory of univalent functions [14]; astrophysics [16, 19]; fusion of pellets [11]; mechanics [6, 17]; motion on a geodesic in a space of constant curvature [8]; a painful paradigm for some singularity analysts [3] – not to mention the context of the present paper.

point symmetries independently of the values of the constants of integration, A and B . In the case of the Painlevé-Ince equation for general values of k there are always two Lie point symmetries [18]. However, in the case of (6) this is not the case if both of A and B not zero. When one or both are nonzero, there is only the obvious symmetry, ∂_x .

From (5) it is obvious that the case $k = 1$ is special. The solution can be reduced to the inversion of the quadrature

$$\int \frac{4UdU}{U^3 - 2AU + 2B} + x = x_0, \quad (7)$$

where x_0 is the third constant of integration. Indeed the degeneracy is obvious by a comparison of (3) and (4) when $k = 1$ for then the system reduces from a fourth-order system to a third-order system. In general the inversion of the quadrature depends upon the coefficients of the expansion in terms of partial fractions of the integrand in (7). The division into partial fractions gives

$$\begin{aligned} & \frac{4\lambda_1}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_1)} \frac{1}{U + \lambda_1} + \frac{4\lambda_2}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)} \frac{1}{U + \lambda_2} \\ & + \frac{4\lambda_3}{(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)} \frac{1}{U + \lambda_3}, \end{aligned}$$

where the denominator factors as $(U + \lambda_1)(U + \lambda_2)(U + \lambda_3)$, and it is highly unlikely that the integral could be inverted to give U as an explicit function of $x - x_0$ for general values of the constants of integration A and B . An obvious exception to this occurs when the initial conditions are such that $B = 0$. For then the integral is easily inverted to give the solution

$$U(x) = \sqrt{2A} \tanh \left[\frac{1}{4} \sqrt{2A} (x - x_0) \right].$$

A more profitable line of investigation is to take the combination $\Gamma_1 + c\Gamma_2$, where c is an arbitrary constant³, to obtain a travelling-wave solution. In terms of the independent variable, $y = x - ct$, and the dependent variables $W(y) = U(y) + c$ and $V(y)$ system (2) becomes

$$kV' + WW' + W'' = 0 \quad (8)$$

$$W'V + WV' + V'' + W''' = 0, \quad (9)$$

where now the prime denotes differentiation with respect to y . In terms of the current dependent variables, $V(y)$ and $W(y)$, system (8)–(9) is the same as system (3,4) and consequently the treatment is the same. One integrates (8) and uses the integral to eliminate V from the second equation and integrates again to obtain a second-order equation for W , namely

$$W''(1 - k) + 2WW' + \frac{1}{2}W^3 - AW + B = 0 \quad (10)$$

³In [25] the constant, c , is written as the coefficient of ∂_t which gives it the unusual dimensions of $L^{-1}T$.

which has the same structure as (6). Consequently there is no need to repeat the discussion already given above.

As a final point to be made in the treatment of this case one should note that the transformation $U = 4w'/3w$ is not the only linearising transformation. The analysis to find the numerical coefficient in the Riccati transformation also gives the possibility $U = 8w'/3w$. In this case the third-order equation is

$$(w^2)''' - \frac{9}{8}A(w^2)' + \frac{27}{32}(w^2) = 0.$$

When one looks at the transformations, the resultant equation after the second transformation is not surprising. However, if one examines (6) with $k = 1/9$ in terms of the singularity analysis the former transformation, which is consistent with the solution of (6) in terms of a Right Painlevé Series, whereas the latter transformation, which generates a Left Painlevé Series, is not consistent with the nondominant terms in (6). Nevertheless one must bear in mind that (6) possesses eight Lie point symmetries and so is linearisable by means of a point transformation to the simplest second-order equation, namely $q''(p) = 0$, which certainly satisfies the requirements of the singularity analysis. One recalls that the singularity analysis is very much dependent upon representation and the present situation may simply be a reflection of that. However, this is a curious result and may be worthy of further investigation from a theoretical approach.

Γ₁ – Γ₃. In terms of the independent variable $y = x + t^2/2$ and dependent variables $U = u - t$ and $V = v$ system (2) becomes

$$UU' + kV' + U'' - 1 = 0, \quad (UV)' + V'' + U''' = 0.$$

Both equations can be integrated and V eliminated using the integral of the first equation. We obtain

$$(k - 1)U'' - 2UU' - \frac{1}{2}U^3 + (y - A)U + B + 1 = 0, \quad (11)$$

where A and B/k are the respective constants of integration.

In (11) we recognise a generalisation of (10). The possibility that $k = 1$ reduces the order of the system from four to three and gives an Abel's equation of the second kind which is not variables separable. For $k = 1/9$ the same Riccati transformation gives the linear third-order equation

$$w''' + \frac{9}{8}(y - A)w' + \frac{9}{8}(B + 1)w = 0.$$

For general values of the parameters the solution of this equation can be expressed in terms of hypergeometric functions as

$$\begin{aligned} w = & C_0 F \left[\left\{ \frac{m}{3} \right\}, \left\{ \frac{1}{3}, \frac{2}{3} \right\}, -\frac{x^3}{9} \right] + C_1 x F \left[\left\{ \frac{1}{3} + \frac{m}{3} \right\}, \left\{ \frac{2}{3}, \frac{4}{3} \right\}, -\frac{x^3}{9} \right] \\ & + C_2 x^2 F \left[\left\{ \frac{2}{3} + \frac{m}{3} \right\}, \left\{ \frac{4}{3}, \frac{5}{3} \right\}, -\frac{x^3}{9} \right], \end{aligned}$$

where C_0 , C_1 and C_2 constants of integration, and in the specific case that $B = -1/2$ (A remains a free parameter) the solution can be written in terms of products of Airy functions as

$$w = K_1 Ai \left[\frac{(-1)^{1/3}x}{2^{2/3}} \right]^2 + K_2 Ai \left[\frac{(-1)^{1/3}x}{2^{2/3}} \right] Bi \left[\frac{(-1)^{1/3}x}{2^{2/3}} \right] \\ + K_3 Bi \left[\frac{(-1)^{1/3}x}{2^{2/3}} \right]^2,$$

where the K_1 , K_2 and K_3 are constants of integration. The difference in the expressions for the solution can be attributed to the fact that for general values of the parameters the equation has only four Lie point symmetries whereas, when the parameters are related by the specific expression, the equation possesses the full seven. Then it can be integrated using the integrating factor w to give an Ermakov-Pinney equation, the solutions of which are given by Pinney's formula from the solutions of the equivalent linear equation, *ie* Airy's equation.

Γ₃. This is a somewhat trivial case. The invariants of $Γ_3$ are t , $u + x/t$ and v . The solutions are easily found to be

$$u = \frac{1}{t}(C_1 - x) \quad \text{and} \quad v = \frac{C_2}{t}, \quad (12)$$

where C_1 and C_2 are the constants of integration.

Γ₄. The invariants of $Γ_4$ are $y = x^2/t$, $U = xu$ and $V = tv$. System (2) becomes⁴

$$0 = 4y^2U'' + 2yUU' + y(y-2)U' + 2U - U^2 + 2ky^2V' \quad (13)$$

$$0 = 8y^3U''' - 6yU' - 6U + 4y^3V'' + (2U + y + 2)y^2V' \\ + (2yU' - U + y)yV, \quad (14)$$

where the prime denotes differentiation with respect to y . As Zhang *et al* [25] observe, this system is quite nonlinear. A modicum of simplification is obtained when one observes that (13) becomes exact on the introduction of the integrating factor y^{-2} . From this integral it follows that

$$V = (my + 2U - yU - U^2 - 4yU')/(2ky),$$

where m is the constant of integration. Then (14) becomes a nonlinear third-order ordinary differential equation for the single variable $U(y)$ given by

$$16(k-1)y^3U''' - 16y^2UU'' - 8y^3U'' - 16y^2U'^2 - 6yU^2U' + 6y(4-y)UU' \\ - 12[y^3 - 2(m-2)y^2 + 12(k+1)y]U' + 3U^3 + (y-12)U^2 \\ - (y^2 + my + 12k)U + my^2 = 0. \quad (15)$$

⁴Note that the coefficient of U'' in (14) differs from that to be found in the corresponding equation in [25] by a factor of 4.

Although some parts of (15) are reminiscent of structures found in elements of the Riccati sequence [4,12], there does not appear to be sufficient similarity in the different parts of the equation for a linearising transformation such as one had for the special cases above for which $k = 1/9$. It is evident that, when $k = 1$, the equation does reduce to a second-order equation.

3 The case $\alpha = 0$ and $\beta = 0$

In this case the system (1) becomes

$$u_t = uu_x + v_x \quad (16)$$

$$v_t = u_x v + uv_x. \quad (17)$$

If we define u in terms of a potential function, $w(t, x)$, as $u = w_x$, (16) may be integrated to give

$$v = w_t - \frac{1}{2}w_x^2 \quad (18)$$

in which the arbitrary function of time of integration can be incorporated into w without loss of generality. We substitute for u and v into (17) to obtain a single equation for w . It is

$$w_{tt} - 2w_x w_{tx} + w_{xx} \left(\frac{3}{2}w_x^2 - w_t \right) = 0 \quad (19)$$

which has the Lie point symmetries

$$\begin{aligned} \Gamma_1 &= \partial_x, & \Gamma_2 &= \partial_w, & \Gamma_3 &= \partial_t, & \Gamma_4 &= x\partial_x - 2w\partial_w, \\ \Gamma_5 &= t\partial_x - x\partial_w, & \Gamma_6 &= t\partial_t - w\partial_w. \end{aligned}$$

The similarity solutions of (19) corresponding to Γ_1 and Γ_3 are trivial being $A+Bt$ and $A+Bx$ respectively.

A travelling-wave solution of system (16,17), equally (19), is even more trivial in that both u and v are constants.

Γ₄. The invariants for Γ_4 are t and w/x^2 . We write $w = x^2 f(t)$. The resulting ordinary differential equation for f is

$$\ddot{f} - 10f\dot{f} + 12f^3 = 0, \quad (20)$$

where the overdot denotes differentiation with respect to the variable t . Equation (20) is an instance of the generalised Painlevé-Ince equation with the generic two symmetries of invariance under translation in t and rescaling. If one investigates (20) using the techniques of singularity analysis, one finds that the leading-order term is a simple pole with coefficient either $-\frac{1}{2}$ or $-\frac{1}{3}$. For the former both resonances are at -1 and the analysis fails. For the latter the second resonance is at $\frac{2}{3}$ and the expansion about the movable singularity is in the

powers of $(t - t_0)^{2/3}$, where t_0 is the location of the movable singularity. Since the equation has the two Lie point symmetries, it can be reduced to a variables separable first-order equation which can be integrated. In terms of the variables $x = \log f$ and $y = \dot{f}/f^2$ the solution can be written as

$$\frac{(y-3)^3}{(y-2)^2} = K \exp[-2x].$$

Γ5. The invariants are t and $w + x^2/2t$. We write $w = f(t) - x^2/2t$ and obtain the equation for f to be $t\ddot{f} - \dot{f} = 0$ which has the easily obtained solution

$$f = A + Bt^2.$$

Γ6. For this symmetry the invariants are x and tw . In terms of $w = f(x)/t$ the reduced ordinary differential equation is

$$3f'^2 f'' + 2ff'' + 4f'^2 + 4f = 0. \quad (21)$$

This also possesses the two symmetries of invariance under translation in the independent variable and rescaling. It is not in a satisfactory form for singularity analysis since the exponent of the leading-order term is 2. This is rectified by means of the replacement $f \rightarrow 1/q$ so that the exponent is now -2 . The coefficient of the leading-order term is either -2 or -3 . The former gives a double -1 resonance whereas the latter gives -1 and 2 and thereby constitutes a viable principal branch. In terms of the variables $x = \log f$ and $y = f'/f$ the separable first-order equation has the solution

$$\frac{3y+4}{(y+2)^2} = K e^x.$$

An alternate route is to use a potential representation of (17) by setting $v = w_x$. Then (17) can be solved to give $u = w_t/w_x$ and (16) becomes

$$w_x^2 w_{tt} - 2w_x w_t w_{tx} + w_{xx} (w_t^2 - w_x^3) = 0, \quad (22)$$

which apart from the ultimate term is a two-dimensional Bateman equation. The Lie point symmetries of (22) are

$$\begin{aligned} \Sigma_1 &= \partial_t, & \Sigma_2 &= \partial_x, & \Sigma_3 &= \partial_w, & \Sigma_4 &= t\partial_x, \\ \Sigma_5 &= t\partial_t - 2w\partial_w, & \Sigma_6 &= x\partial_x + 3w\partial_w. \end{aligned}$$

Reduction using the first four symmetries leads to no result.

Σ5. We write (22) in terms of the reduction $w = f(x)/t^2$ as $f''(4f^2 - f'^3) - 2f f'^2 = 0$ which has two Lie point symmetries and so can be reduced to a variables separable first-order equation. This can be integrated to give

$$f'^2(f' - 6) = C_1 \exp[-3f].$$

This can be integrated to give x in terms of f , but the expression is so complicated that it is pointless to list it here and one could not expect to be able to invert it.

Σ6. The invariants are t and w/x^3 so that we write $w = x^3 f(t)$. Equation (22) becomes $\ddot{f} - \frac{4}{3}\dot{f}^2 - 18f^3 = 0$ which takes the more transparent form

$$\ddot{h} + \frac{6}{h^2} = 0 \quad (23)$$

under the change of dependent variable $f \rightarrow h^{-3}$. Equation (23) is an Emden-Fowler equation of index $(0, -2)$ and the solution can be expressed in terms of Jacobi elliptic functions.

4 Discussion

We have considered in some detail the two cases, $\alpha \neq 0$ and $\beta \neq 0$, and $\alpha = 0$ and $\beta = 0$. For completeness we provide a brief summary of the results for the two cases in which one of the parameters is zero and the other is nonzero.

$\alpha \neq 0$ and $\beta = 0$. Without loss of generality the value of α can be taken as unity. System (1) is now

$$u_t = uu_x + v_x \quad (24)$$

$$v_t = u_x v + uv_x + u_{xxx}. \quad (25)$$

We write u in terms of a potential function as w_x . As in the previous section we may integrate (24) to obtain (18) and substitute for v in (25). The equation for w is

$$w_{tt} - 2w_x w_{xt} + \frac{3}{2}w_x^2 w_{xx} - w_t w_{xx} - w_{xxxx} = 0 \quad (26)$$

which has the Lie point symmetries

$$\Delta_1 = \partial_t, \quad \Delta_2 = \partial_x, \quad \Delta_3 = \partial_w, \quad \Delta_4 = 2t\partial_t + x\partial_x$$

so that there has been a noticeable decrease in symmetry by comparison with systems (2) and (16)–(17).

We note that the alternate reduction using the second equation as for (16)–(17) is not feasible for this system.

Δ₁ and Δ₂. In the case of Δ₁ we can write $w = f(x)$ so that (26) becomes $f'''' - \frac{3}{2}f'^2 f'' = 0$ which can be integrated to give $f''(x)$ in terms of an elliptic function,

the precise nature of which depends upon the precise values of the constants of integration. We note that the equation in $f'(x)$ passes the Painlevé Test.

Reduction by Δ_2 leads to the trivial solution $A + Bt$.

Of greater potential interest is the possibility of a travelling-wave solution generated by reduction using $\Delta_1 + c\Delta_2$. We write $w = f(x - ct)$. Then (26) is

$$f'''' = \left(\frac{3}{2}f'^2 + 3cf' + c^2\right)f''. \quad (27)$$

Despite the fact that (27) has only two Lie point symmetries its solution can be reduced to the integral of an elliptic function containing three constants of integration for general values of those parameters. Simpler solutions are available for specific values of the parameters. For example, if the two constants of integration from the double integration of (27) written as an equation in $g = f'$ are set to zero, the solution of (27) is

$$f(x) = \sqrt{\frac{2}{c}} \arcsin \left[\sqrt{K_3} \exp[c(x - x_0)] \right] + K_4,$$

where K_3 and K_4 are the constants of integration from the third and fourth quadratures respectively. Equation (27) possesses just three Lie point symmetries and so its reducibility to a quadrature in general indicates the existence of nonlocal symmetries of the useful variety⁵.

Δ_4 . This is the worst symmetry of the four to use to perform a reduction of (26) as both Δ_1 and Δ_2 are lost as point symmetries of the reduced equation. Although it is possible for a partial differential equation to gain point symmetries on reduction due to parallel equations leading to the same reduced equation [2], this is not one of them and the reduced equation is

$$16p^4f'''' + 48p^3f''' + 12p^2f'' - 24p^4f'^2f'' - 12p^4f'f'' - p^4f'' - 12p^3f'^3 - 10p^3f'^2 - 2p^3f' = 0,$$

where $p = x^2/t$ and $f(p) = w(t, x)$. There seems to be no further obvious route to reduction apart from the one consequent upon the existence of the symmetry ∂_f .

$\alpha = 0$ and $\beta \neq 0$. Now system (1) is

$$u_t = uu_x + v_x + u_{xx}, \quad v_t = u_xv + uv_x + v_{xx}, \quad (28)$$

where β may be set at unity without loss of generality.

The Lie point symmetries of system (28) are

$$\Sigma_1 = \partial_t, \quad \Sigma_2 = \partial_x, \quad \Sigma_3 = t\partial_x - \partial_u, \quad \Sigma_4 = 2t\partial_t + x\partial_x - u\partial_u - 2v\partial_v.$$

⁵An ordinary differential equation possesses an infinite number of symmetries when there is no restriction placed upon the variable dependence in the coefficient functions. Such symmetries as can be used for reduction of the equation are termed useful.

$\Sigma_1 + c\Sigma_2 = \partial_t + c\partial_x$. One seeks a travelling-wave solution by setting $y = x - ct$, $u(t, x) = U(y)$ and $v(t, x) = V(y)$. The resulting pair of ordinary differential equations may each be integrated once to give the first-order system

$$U' + V + \frac{1}{2}U^2 + cU + A = 0, \quad V' + UV + cV + B = 0.$$

So far no useful further reduction of this system has been found.

$\Sigma_3 = t\partial_x - \partial_u$. System (28) can be reduced to a pair of elementary quadratures in terms of the variables $u = U(t) - x/t$ and $v = V(t)$. The solutions

$$u = \frac{A + x}{t} \quad \text{and} \quad v = \frac{B}{t}$$

are perhaps not the most exciting.

$\Sigma_4 = 2t\partial_t + x\partial_x - u\partial_u - 2v\partial_v$. In terms of the variables $u = x^{-1}U(y)$ and $v = t^{-1}V(y)$, where $y = x^2/t$, system (28) reduces to

$$\begin{aligned} 4y^2U'' + y(y-1)U' + \frac{1}{2}yUU' - U^2 + 2U + 2y^2V' &= 0, \\ 4y^2V'' + y(y+2)V' + 2y(UV)' - UV + yV &= 0. \end{aligned}$$

So far no further reduction of this system has been obtained.

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An application of Hermite functions

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We study the equivalence of linear differential operators on the Schwartz space using the isomorphism between this space and the sequence space of rapidly decreasing sequences.

1 Introduction

Sequences of polynomials play a fundamental role in applied mathematics and physics. An important class is formed by the Hermite polynomials, $h_n(x)$, which are a classical orthogonal polynomial sequence that arise in physics, as the eigenstates of the quantum harmonic oscillator are Hermite functions, associated with Hermite polynomials; in probability, such as the Edgeworth series; in combinatorics, as an example of an Appell sequence. Also they play a key role in the Brownian motion and the Schrödinger wave equation.

They can be described in various ways, for example,

1. as solutions to the series of differential equations $y'' - 2xy' + 2ny = 0$
2. by the generating function $\exp(2tx - t^2)$ expanded about zero as a Taylor series in t
3. by a differential recurrence solution, in fact $h'_n = 2nh_{n-1}$
4. by the formulae

$$h_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx} \right)^n e^{-x^2}$$

It is very well-known that this class of polynomials is a powerful tool in the solution of many problems. The Hermite functions, intimately related to Hermite polynomials, given by

$$H_n(x) = (2^n n! \sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_n(x)$$

form an orthonormal basis for the space $L^2(\mathbb{R})$ and this fact allows us to use a method that would help to solve, at least in some concrete examples, a problem which is described below.

One problem that has long been of interest is that of the “equivalence” of differential operators in any of the meanings of that word. In the sense adopted in the present paper the idea is due to Delsarte, in [1], where he introduced the notion of “opérateur de transmutation”.

Given two differential operators A and B on a space H , an operator X is called an “opérateur de transmutation” (transformation operator) if X is an isomorphism and $BX = XA$. This notion depends, obviously, on the two operators and the space. The first result in this direction is due to Delsarte, taking H to be a space of functions of one variable defined for $x \geq 0$ and A and B two differential operators of order two on H . Several generalizations and applications can be found in [5, 6]. If A and B are of order greater than two, with infinitely differentiable coefficients, there are not, in general, transformation operators and the problem for spaces of functions with domain in the real line seems to be a difficult one.

The picture, though, changes completely when the domain of the functions is complex. Then, it is always possible to transform A in B if both differential operators are of the same order and $H = \mathcal{H}(\mathbb{C})$, the space of entire functions of one complex variable [2]. Other results in the same direction can be found in [9, 10].

In section 3 of this paper we present our approach in the context of the space $\mathcal{C}_{2\pi}^\infty(\mathbb{R})$ of all 2π -periodic \mathcal{C}^∞ -functions on \mathbb{R} . This example has been studied for linear differential operators with constant coefficients in [7].

In section 4 we apply the method to the space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions (Schwartz class of functions) which is a subspace of $L^1(\mathbb{R})$ that is invariant under the Fourier transform, differentiation and multiplication by polynomials, the first-named property leading to the introduction of the subspace of tempered distributions. The Schwartz space plays a fundamental role in many branches of science, for instance, in signal processing where Fourier transform and convolution are essential notions.

2 The method

Assume that H is a complex linear topological space and (h_n) a Schauder basis. Then any element $h \in H$ can be represented as a series

$$h = \sum_n a_n h_n,$$

where (a_n) is a sequence of complex numbers. If there is a sequence space Λ with a topology such that the mapping

$$\begin{aligned} H &\xrightarrow{F} \Lambda \\ h &\longmapsto (a_n) \end{aligned}$$

is an isomorphism, that is, F is linear, bijective and bicontinuous, then we say that both spaces can be identified.

Let A and B be two linear differential operators mapping the space H on H . The following diagram

$$\begin{array}{ccc} H & \xrightarrow{A, B} & H \\ F \downarrow & & \downarrow F \\ \Lambda & \xrightarrow{A, B} & \Lambda \end{array}$$

shows that the mappings A and B induce on Λ two mappings \mathbb{A} and \mathbb{B} such that $\mathbb{A} = FAF^{-1}$, $\mathbb{B} = FBF^{-1}$. As F is an isomorphism, A and B are equivalent if and only if \mathbb{A} and \mathbb{B} are.

Thus we see that the transformation operator X between A and B can be obtained by determining the transformation operator between \mathbb{A} and \mathbb{B} . The advantage is that the new operators are, in many cases, much easier to deal with than the original ones and besides the structure of sequence spaces is simple and well-understood.

3 The space $\mathcal{C}_{2\pi}^\infty(\mathbb{R})$ and the sequence space s

Let us begin by explaining our approach by a simple example in the space $\mathcal{C}_{2\pi}^\infty(\mathbb{R})$ of all 2π -periodic \mathcal{C}^∞ -functions on \mathbb{R} .

The space s of rapidly decreasing sequences is defined by

$$s = \left\{ x \in \mathbb{C}^{\mathbb{N}} : \|x\|_k^2 = \sum_{j \in \mathbb{N}} |x_j|^2 j^{2k} < \infty \text{ for all } k \in \mathbb{N} \right\}.$$

We have also

$$s = \left\{ x \in \mathbb{C}^{\mathbb{N}} : \lim_{j \rightarrow \infty} |x_j| j^k = 0 \text{ for all } k \in \mathbb{N} \right\}.$$

The topological structure of s is given by the seminorms

$$\|x\|_k^2 = \sum_{j \in \mathbb{N}} |x_j|^2 j^{2k} \text{ for all } k \in \mathbb{N},$$

or equivalently

$$\|x\|_k^2 = \sup_j (|x_j|^2 j^{2k}) \text{ for all } k \in \mathbb{N}.$$

As is well known, $\mathcal{C}_{2\pi}^\infty(\mathbb{R})$ can be identified with the sequence space s by means of the Fourier series, that is the mapping $F: \mathcal{C}_{2\pi}^\infty(\mathbb{R}) \rightarrow s$ given by $f \mapsto (\tilde{f}_0, \tilde{f}_1, \tilde{f}_{-1}, \tilde{f}_2, \tilde{f}_{-2}, \dots)$, where

$$f(x) = \sum_{n \in \mathbb{N}} \tilde{f}_n e^{inx}$$

is the Fourier series of f , is an isomorphism [8].

Given the linear differential operators, $A_1 = D + I$ and $A_2 = D$ on $\mathcal{C}_{2\pi}^\infty(\mathbb{R})$, consider the linear operators on s , $\mathbb{A}_1 = FA_1F^{-1}$ and $\mathbb{A}_2 = FA_2F^{-1}$. If A_1 and A_2 are “equivalent” in the sense indicated above, so they are \mathbb{A}_1 and \mathbb{A}_2 (in fact, $XA_1 = A_2X$ if and only if $\mathbb{X}\mathbb{A}_1 = \mathbb{A}_2\mathbb{X}$, $\mathbb{X} = F^{-1}XF$).

The operators \mathbb{A}_1 and \mathbb{A}_2 are given by the matrices

$$\begin{pmatrix} 1 & & & & \\ & i+1 & & & \\ & & -i+1 & & \\ & & & 2i+1 & \\ & & & & -2i+1 \\ & & & & & 3i+1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & & & & \\ & i & & & \\ & & -i & & \\ & & & 2i & \\ & & & & -2i \\ & & & & & 3i \end{pmatrix},$$

respectively.

It is easily seen, from the algebraic equations given by $\mathbb{A}_1\mathbb{X} = \mathbb{X}\mathbb{A}_2$, that \mathbb{A}_1 and \mathbb{A}_2 are not equivalent.

4 The Schwartz space $\mathcal{S}(\mathbb{R})$

The space $\mathcal{S}(\mathbb{R})$ is defined as

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}) : \sum_{\alpha+\beta \leq k} \int |x|^{2\alpha} |f^{(\beta)}(x)|^2 dx < \infty, \forall k \in \mathbb{N} \right\}.$$

All the functions in $\mathcal{S}(\mathbb{R})$, with all their derivatives, decrease faster than each polynomial and therefore the space $\mathcal{S}(\mathbb{R})$ is called, too, the space of rapidly decreasing functions.

The topological structure of $\mathcal{S}(\mathbb{R})$ is given by the countable sequence of semi-norms

$$\|f\|_k^2 = \sum_{\alpha+\beta \leq k} \int |x|^{2\alpha} |f^{(\beta)}(x)|^2 dx, \quad k \in \mathbb{N}$$

or by the equivalent system of seminorms

$$\|f\|_k = \sup \left\{ |x^\alpha f^{(\beta)}(x)| : x \in \mathbb{R}, \alpha + \beta \leq k \right\}, \quad k \in \mathbb{N}.$$

The Hermite functions $H_n(x)$, $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ are elements of $\mathcal{S}(\mathbb{R})$ and form an orthonormal basis for $L^2(\mathbb{R})$. Consequently the linear map

$$H: \mathcal{S}(\mathbb{R}) \longrightarrow s$$

defined by $H(f) = (\langle f, H_n \rangle)_{n \in \mathbb{N}_0}$ is bijective and bicontinuous (H and H^{-1} are continuous), that is an isomorphism [8].

Recalling that the Hermite functions satisfy the following equations (with $H_{-1} = 0$)

$$H'_n = \frac{1}{\sqrt{2}}(\sqrt{n}H_{n-1} - \sqrt{n+1}H_{n+1}),$$

$$xH_n = \frac{1}{\sqrt{2}}(\sqrt{n}H_{n-1} + \sqrt{n+1}H_{n+1}),$$

the operator D (respectively, xI) from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$ can be identified with the operator \mathbb{D} (respectively, $x\mathbb{I}$) from s to s given by

$$\mathbb{D}\delta_n = \frac{1}{\sqrt{2}}(\sqrt{n}\delta_{n-1} - \sqrt{n+1}\delta_{n+1}), \quad (x\mathbb{I})\delta_n = \frac{1}{\sqrt{2}}(\sqrt{n}\delta_{n-1} + \sqrt{n+1}\delta_{n+1}),$$

where $\delta_n = \left(\delta_j^n\right)_{j=0}^{\infty}$.

Therefore a linear differential operator of the form

$$A = p_0(x)I + p_1(x)D + \cdots + p_{m-2}(x)D^{m-2} + p_{m-1}(x)D^{m-1} + D^m,$$

where $p_j(x)$ are polynomials, can be identified with a linear operator from s to s . The formula for differential operators of order greater than two is really cumbersome but a simple example shows how the procedure works.

Let A be a linear differential operator of second order, precisely $A = I + xD + x^2D^2$. Then

$$AH_n = H_n + xH'_n + x^2H''_n.$$

As

$$xH'_n = \frac{\sqrt{n(n-1)}}{2}H_{n-2} - \frac{1}{2}H_n - \frac{\sqrt{(n+1)(n+2)}}{2}H_{n+2},$$

$$x^2H''_n = x^2 \left(\frac{\sqrt{n(n-1)}}{2}H_{n-2} - \frac{2n+1}{2}H_n + \frac{\sqrt{(n+1)(n+2)}}{2}H_{n+2} \right),$$

$$x^2H_n = \frac{\sqrt{n(n-1)}}{2}H_{n-2} + \frac{2n+1}{2}H_n + \frac{\sqrt{(n+1)(n+2)}}{2}H_{n+2},$$

it follows that

$$AH_n = \frac{\sqrt{n(n-1)(n-2)(n-3)}}{4}H_{n-4} - \frac{\sqrt{n(n-1)}}{2}H_{n-2} - \frac{2n^2 + 2n - 3}{4}H_n$$

$$+ \frac{\sqrt{(n+1)(n+2)}}{2}H_{n+2} + \frac{\sqrt{(n+1)(n+2)(n+3)(n+4)}}{4}H_{n+4}$$

and A can be identified with the operator \mathbb{A}

$$\mathbb{A}\delta_n = \frac{\sqrt{n(n-1)(n-2)(n-3)}}{4}\delta_{n-4} - \frac{\sqrt{n(n-1)}}{2}\delta_{n-2} - \frac{2n^2 + 2n - 3}{4}\delta_n +$$

$$+ \frac{\sqrt{(n+1)(n+2)}}{2}\delta_{n+2} + \frac{\sqrt{(n+1)(n+2)(n+3)(n+4)}}{4}\delta_{n+4}.$$

Example 1

We consider as our first example two operators A and B from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$ such that

$$A = I + pD, \quad B = I + qD,$$

where p and q are complex numbers, different from zero. Then we have two operators \mathbb{A} and \mathbb{B} from s to s given by

$$\mathbb{A}(\delta_0) = \delta_0 - \frac{p}{\sqrt{2}} \delta_1, \quad \mathbb{A}(\delta_n) = \sqrt{\frac{n}{2}} p \delta_{n-1} + \delta_n - \sqrt{\frac{n+1}{2}} p \delta_{n+1}, \quad n \geq 1$$

and

$$\mathbb{B}(\delta_0) = \delta_0 - \frac{q}{\sqrt{2}} \delta_1, \quad \mathbb{B}(\delta_n) = \sqrt{\frac{n}{2}} q \delta_{n-1} + \delta_n - \sqrt{\frac{n+1}{2}} q \delta_{n+1}, \quad n \geq 1.$$

We are looking for a transformation operator \mathbb{X} (represented by a matrix $\mathbb{X} = (x_{j,n})_{j,n=0}^{\infty}$, that is $\mathbb{X}(\delta_n) = \sum_{j=0}^{\infty} x_{j,n} \delta_j$) such that $\mathbb{X}\mathbb{A} = \mathbb{B}\mathbb{X}$ and $x_{0,0} = 1$, $x_{0,n} = 0$, $n \geq 1$.

From the two previous conditions (using the program Mathematica) we get the matrix \mathbb{X} whose elements are

$$x_{j,n} = \begin{cases} 0 & j < n, \\ 0 & j - n = 2m - 1, m = 0, 1, 2, \dots, \\ \sqrt{\binom{j}{n} \binom{j-n-1}{m} \frac{1}{2^{j-n-1}} \frac{p^n}{q^j}} (q^2 - p^2)^m & j - n = 2m, m = 1, 2, \dots, \\ \frac{p^n}{q^n} & j = n. \end{cases}$$

This matrix \mathbb{X} is a lower triangular matrix and so invertible. Therefore, from the algebraic point view, the transformation operator between \mathbb{A} and \mathbb{B} exists and is a linear operator from φ to φ , where

$$\varphi = \{(x_n) : x_n \in \mathbb{C} \text{ and } x_n = 0 \text{ for all } n, \text{ except a finite number}\}.$$

To ensure that \mathbb{X} is a linear continuous operator from s to s it is enough to prove the following condition: $\forall k \in \mathbb{N}, \exists N(k) \in \mathbb{N}, \exists C(k) > 0$ such that

$$\sup_{j \geq n} \frac{|x_{j,n}| j^k}{n^{N(k)}} \leq C(k), \text{ for all } n \in \mathbb{N}. \quad (1)$$

Write the formula for $j = n + 2m$, $m = 0, 1, 2, \dots$, $n = 0, 1, 2, \dots$. Then

$$\begin{aligned} & \sup_{n \in \mathbb{N}, m \in \mathbb{N}} \left(|x_{n+2m,n}| \frac{(n+2m)^k}{n^{N(k)}} \right) \\ &= \sup_{n \in \mathbb{N}, m \in \mathbb{N}} \left\{ \frac{1}{2^{\frac{2m-1}{2}} (2m)!} \binom{2m-1}{m} \left| \frac{(q^2 - p^2)^m}{q^{2m}} \right| \cdot \right. \\ & \quad \left. (n+1) \dots (n+2m) \frac{(n+2m)^k}{n^{N(k)}} \left| \frac{p}{q} \right|^n \right\}. \end{aligned} \quad (2)$$

Assume that given k , there exists $N(k)$ such that (2) is finite and take $m = N(k) + 1 - k$. As

$$(n+1)\dots(n+2m)\frac{(n+2m)^k}{n^{N(k)}} \geq \frac{n^{m+k}}{n^{N(k)}},$$

it follows that for such an m

$$(n+1)\dots(n+2m)\frac{(n+2m)^k}{n^{N(k)}} \longrightarrow \infty \text{ with } n$$

and there is a contradiction unless $q^2 - p^2 = 0$ or $\left|\frac{p}{q}\right| < 1$.

Suppose $q^2 - p^2 = 0$. When $p = q$, $\mathbb{A} = \mathbb{B}$ ($A = B$) and the matrix $\mathbb{X} = I$. If $p = -q$, \mathbb{X} is

$$x_{j,n} = \begin{cases} 0 & j \neq n, \\ 1 & j = n, n \text{ even,} \\ -1 & j = n, n \text{ odd,} \end{cases}$$

that is

$$\mathbb{X}\delta_n = \begin{cases} \delta_n & n \text{ even,} \\ -\delta_n & n \text{ odd.} \end{cases}$$

Obviously \mathbb{A} and \mathbb{B} (so A and B) are equivalent.

Assume now that $\left|\frac{p}{q}\right| < 1$ and $p \neq \pm q$.

If \mathbb{A} and \mathbb{B} were equivalent the operator \mathbb{X} would be an isomorphism. So \mathbb{X} and \mathbb{X}^{-1} are continuous. As the elements of the matrix of \mathbb{X}^{-1} are obtained from the expressions for elements of \mathbb{X} by the permutation of p and q , the continuity of \mathbb{X}^{-1} implies that $\left|\frac{q}{p}\right| < 1$ (because $p \neq \pm q$). Therefore $|p| = |q|$ and if $p \neq \pm q$ neither \mathbb{X} nor \mathbb{X}^{-1} are continuous.

Example 2

We consider now two operators A and B from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$ such that

$$A = pI + D, \quad B = qI + D,$$

where p and q are complex numbers, different from zero. Then we have two operators \mathbb{A} and \mathbb{B} from s to s given by

$$\mathbb{A}(\delta_0) = p\delta_0 - \frac{1}{\sqrt{2}}\delta_1, \quad \mathbb{A}(\delta_n) = \sqrt{\frac{n}{2}}\delta_{n-1} + p\delta_n - \sqrt{\frac{n+1}{2}}\delta_{n+1}, \quad n \geq 1$$

and

$$\mathbb{B}(\delta_0) = q\delta_0 - \frac{1}{\sqrt{2}}\delta_1,$$

$$\mathbb{B}(\delta_n) = \sqrt{\frac{n}{2}}\delta_{n-1} + q\delta_n - \sqrt{\frac{n+1}{2}}\delta_{n+1}, \quad n \geq 1.$$

The matrix $\mathbb{X} = (x_{jn})$ such that $\mathbb{X}\mathbb{A} = \mathbb{B}\mathbb{X}$ and $x_{0,0} = 1$, $x_{0,n} = 0$, $n \geq 1$ is

$$x_{j,n} = \begin{cases} 0 & j < n, \\ 1 & j = n, \\ \sqrt{\binom{j}{n} \frac{2^{j-n}}{(j-n)!}} (p-q)^{j-n} & j > n, \end{cases}$$

\mathbb{X} is a lower triangular matrix and so invertible. Therefore, from the algebraic point view, the transformation operator between \mathbb{A} and \mathbb{B} exists and is a linear operator from φ to φ .

For the continuity of the operator \mathbb{X} from s to s it is enough to prove the condition (1) which, in this case, reads

$$\sup_{j>n} \left(\sqrt{\binom{j}{n} \frac{2^{j-n}}{(j-n)!}} |(p-q)^{j-n}| \frac{j^k}{n^{N(k)}} \right) \leq C(k), \quad \text{for all } n \in \mathbb{N}. \quad (3)$$

Writing the formula (3) for $j = n+m$, $m = 1, 2, \dots$, $n = 0, 1, 2, \dots$, it is easy to see that

$$\begin{aligned} & \sup_{n \in \mathbb{N}, m \in \mathbb{N}} \left(\sqrt{\binom{n+m}{n} \frac{2^m}{m!}} |(p-q)^m| \frac{(n+m)^k}{n^{N(k)}} \right) \\ & \geq \sup_{n \in \mathbb{N}, m \in \mathbb{N}} \left(\frac{2^{\frac{m}{2}}}{m!} |p-q|^m \frac{(n+1)^{\frac{m}{2}+k}}{n^{N(k)}} \right). \end{aligned}$$

Assume that $\forall k$, $\exists N(k)$, $C(k)$ such that the previous condition is true. Then taking $m \in \mathbb{N}$ such that $\frac{m}{2} + k = N(k) + 2$ it follows that

$$\frac{2^{\frac{m}{2}}}{m!} |p-q|^m \frac{(n+1)^{\frac{m}{2}+k}}{n^{N(k)}} \geq \frac{2^{\frac{m}{2}}}{m!} |p-q|^m n^2 \longrightarrow \infty \text{ with } n$$

and a contradiction appears unless $p = q$.

Example 3

Consider the linear operators of order one whose coefficients are polynomial functions $A = xD$ and $B = D$.

The induced operators are

$$\begin{aligned} \mathbb{A}\delta_0 &= -\frac{1}{2} \delta_0 - \frac{\sqrt{2}}{2} \delta_2, \quad \mathbb{A}\delta_1 = -\frac{1}{2} \delta_1 - \frac{\sqrt{2 \cdot 3}}{2} \delta_3, \\ \mathbb{A}\delta_n &= \frac{\sqrt{n(n-1)}}{2} \delta_{n-2} - \frac{1}{2} \delta_n - \frac{\sqrt{(n+1)(n+2)}}{2} \delta_{n+2}, \quad n \geq 2 \end{aligned}$$

and

$$\mathbb{B}\delta_0 = -\sqrt{\frac{1}{2}} \delta_1, \quad \mathbb{B}\delta_n = \sqrt{\frac{n}{2}} \delta_{n-1} - \sqrt{\frac{n+1}{2}} \delta_{n+1}, \quad n \geq 1.$$

If $\mathbb{X} = (x_{jn})$ is a matrix such that $\mathbb{X}\mathbb{A} = \mathbb{B}\mathbb{X}$ and

$$\begin{aligned} x_{0,0} &= x_{1,1} = 1, & x_{0,n} &= 0, & n &\geq 1, \\ x_{1,0} &= 0, & x_{1,n} &= 0, & n &> 1, \end{aligned}$$

then its elements are given by the recurrence formula

$$x_{j+2,n} = \begin{cases} \frac{\sqrt{(j-1)j}x_{j-2,n} + x_{j,n} + \sqrt{2n}x_{j,n-1} - \sqrt{2(n+1)}x_{j,n+1}}{\sqrt{(j+1)(j+2)}} & n \leq j+2, j \geq 0, \\ 0, & n > j+2, j \geq 0. \end{cases}$$

A careful look to the formula gives immediately that $x_{j+2,n} = 0$, when $j+2 = n$, $j \geq 0$. \mathbb{X} is not invertible and there is not an algebraic solution.

Remark 1. The solution of the problem for differential operators of order greater than one involves quite complicated calculations but we hope to give some answers in the future.

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On orbit functions, their properties and applications

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Recently introduced families of special functions are recalled and their most valuable properties are pointed out. In particular it is shown that each function is an eigenfunction of the Laplace operator appropriate for G and their eigenvalues are known explicitly, therefore orbit functions can be applied to the solution of the corresponding Neumann and Dirichlet boundary-value problems on the fundamental domains of the Weyl groups.

1 Introduction

We recall three infinite families of spacial functions depending on n variables ($1 \leq n < \infty$) and emphasize some of their applications. These functions are named C - and S - and E -functions [17] in recognition of the fact that they can be understood as generalizations of the cosine, sine, and exponent. Corresponding n -dimensional C -, S - and E -transforms were recently described in [7–9]. Each transform is based on a compact semisimple Lie group of rank n [1, 4, 6] and comes in three versions: analogs of Fourier series, Fourier integrals, and Fourier transforms on an n -dimensional lattice. Orbit functions and transforms in low-dimensional cases were investigated, e.g. in [5, 15].

The orbit functions are defined in \mathbb{R}^n and have continuous derivatives of all degrees. Their orthogonality, when integrated over the finite region F appropriate for each Lie group, and the discrete orthogonality was shown in [14]. Some important aspects are also studied in a forthcoming paper [3]. The completeness of these systems of functions directly follows from the completeness of the system of exponential functions.

Within each family, orbit functions are described in a uniform way for semisimple Lie groups of any type and rank. The price to pay for the uniformity of methods is having to work with non-orthogonal bases, but in some cases (see Section 4 and [16]) one can work with orbit functions in orthogonal basis too.

The functions have a number of other useful properties, which can be found in [7, 8, 10]. For example, the decomposition of their products into sums, the splitting of functions into as many mutually exclusive congruence classes as is the order of the center of the Lie group. These properties are exposed in more details and in a more general setting in [2].

The first aim of this paper is to show the relation between orbit functions and classical problems of mathematical physics (n -dimensional Neumann and Dirichlet boundary-value problems for the Laplace operator on fundamental regions of semisimple Lie groups). This connection is studied in Section 5.

Another motivation for this paper is the application of orbit functions to processing of the rapidly increasing amount of multi-dimensional digital data gathered today. Special functions, which serve as the kernel of our transform, have simple symmetry property under the action of the corresponding affine Weyl group. The affine group contains as a subgroup the group of translations in \mathbb{R}^n , which underlies the common Fourier transform. This is the primary reason for the superior performance of our transforms, although detailed comparisons, rather than examples, will have to provide quantitative content to substantiate such a claim. The necessary information on such transforms is presented in Sections 3 and 4.

2 Definitions of orbit functions

In this section, we define what we mean by C -, S - and E -functions, specified by a given point $\lambda \in \mathbb{Z}^n$ and a chosen semisimple Lie group G with the corresponding Weyl group W , its even subgroup W^e and fundamental region F . We suppose the reader to be familiar with theory of semisimple Lie groups [1, 4, 6] and, in particular, with the discretization process on a fundamental region [7–9].

The C -function $C_\lambda(x)$ is defined as

$$C_\lambda(x) := \sum_{\mu \in W_\lambda} e^{2\pi i \langle \mu, x \rangle}, \quad x \in \mathbb{R}^n, \quad \lambda \in \mathbb{Z}^{\geq 0} \omega_1 + \cdots + \mathbb{Z}^{\geq 0} \omega_n, \quad (1)$$

where W_λ is the Weyl group orbit generated from λ and $\omega_1, \dots, \omega_n$ is the basis of fundamental weights.

If in (1) we restrict ourselves to the orbit of the even subgroup W_λ^e , then we define E -function $E_\lambda(x)$, $\lambda \in \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$ as

$$E_\lambda(x) := \sum_{\mu \in W_\lambda^e} e^{2\pi i \langle \mu, x \rangle}, \quad x \in \mathbb{R}^n. \quad (2)$$

The definition of an S -function $S_\lambda(x)$, $\lambda \in \mathbb{Z}^{>0} \omega_1 + \cdots + \mathbb{Z}^{>0} \omega_n$ is almost identical, but the sign of each summand is determined by the number $p(\mu)$ of the elementary Weyl group reflections necessary to obtain μ from λ

$$S_\lambda(x) := \sum_{\mu \in W_\lambda} (-1)^{p(\mu)} e^{2\pi i \langle \mu, x \rangle}, \quad x \in \mathbb{R}^n. \quad (3)$$

In the 1-dimensional case, C -, S - and E -functions are respectively a cosine, a sine and an exponential functions up to a constant. In general, C -, S - and E -functions are the finite sums of exponential functions, therefore they are continuous and have continuous derivatives of all orders in \mathbb{R}^n .

All three families of orbit functions are based on compact semisimple Lie groups and the number of variables coincides with the rank of the corresponding Lie algebra.

The S -functions are antisymmetric with respect to $(n - 1)$ -dimensional boundary of F . Hence they are zero on the boundary of fundamental region F . The C -functions are symmetric with respect to $(n - 1)$ -dimensional boundary of F and their normal derivative at the boundary is equal to zero (because the normal derivative of a C -function is an S -function). A number of other properties of orbit functions are presented in [7–9].

The orbit functions are described in a uniform way for different semisimple Lie groups but we have to work with non-orthogonal bases which are not normalized. There is another useful property of orbit functions due to the isomorphism of the Weyl group $W(A_{n-1})$ of the Lie algebra A_{n-1} and the group S_n of permutations of n elements (see [10–13]). The corresponding orbit functions are in one-to-one correspondence although the function based on S_n are naturally defined using an orthonormal basis in \mathbb{R}^n while those of $W(A_{n-1})$ are defined in terms of non-orthogonal basis of simple roots.

The transformation between two these families of special functions depending on any number of variables ($1 < n < \infty$) allows one to work in orthonormal basis and is made explicit in a forthcoming paper [16].

3 Continuous and discrete orthogonality

For any two squared integrable functions $\phi(x)$ and $\psi(x)$ defined on the fundamental region F , we define a continuous scalar product

$$\langle \phi(x), \psi(x) \rangle := \int_F \phi(x) \overline{\psi(x)} dx. \quad (4)$$

Here, integration is carried out with respect to the Euclidean measure, the bar means complex conjugation and $x \in F$, where F is the fundamental region of a semisimple group G with respect to the action of either W or W^e .

Any pair of orbit functions from the same family is orthogonal on the corresponding fundamental region with respect to the scalar product (4), namely

$$\langle C_\lambda(x), C_{\lambda'}(x) \rangle = |W_\lambda| \cdot |F| \cdot \delta_{\lambda\lambda'}, \quad (5)$$

$$\langle S_\lambda(x), S_{\lambda'}(x) \rangle = |W| \cdot |F| \cdot \delta_{\lambda\lambda'}, \quad (6)$$

$$\langle E_\lambda(x), E_{\lambda'}(x) \rangle = |W_\lambda^e| \cdot |F_e| \cdot \delta_{\lambda\lambda'}, \quad (7)$$

where $\delta_{\lambda\lambda'}$ is the Kronecker delta, $|W|$ is the size of Weyl group, $|W_\lambda|$ and $|W_\lambda^e|$ are the sizes of Weyl group orbits, and $|F|$ and $|F_e|$ are volumes of fundamental regions.

Proof of the relations (5), (6), (7) follows from the orthogonality of the usual exponential functions and from the fact that a given weight μ in the definitions (1), (2), and (3) belongs to precisely one orbit function.

The families of C -, S - and E -functions are complete on the fundamental domain. The completeness of these systems is directly follows from the completeness of the system of exponential functions, i.e. there does not exist a function $\phi(x)$ in the system of function under consideration, such that $\langle \phi(x), \phi(x) \rangle > 0$ and at the same time $\langle \phi(x), \psi(x) \rangle = 0$ for all functions $\psi(x)$ form the same system.

Let us denote by F_M the lattice fragment $F_M = F \cap (L/M)$ of the weight lattice L refined by $M \in \mathbb{N}$. For non-simple (semisimple) G it may be determined by more than one positive integer M

$$F_M = \begin{cases} F_M & \text{for } C\text{-functions,} \\ F_M \setminus \partial F & \text{for } S\text{-functions,} \\ F_{eM} & \text{for } E\text{-functions.} \end{cases}$$

A discrete scalar product of two functions $\phi(x)$ and $\psi(x)$ given on F_M (including C -, S - and E -functions) depends on this grid and it is defined by the bilinear form

$$\langle \phi(x), \psi(x) \rangle_M = \sum_{i=1}^{|F_M|} \varepsilon(x_i) \phi(x_i) \overline{\psi(x_i)}, \quad x_i \in F_M. \quad (8)$$

Here $\varepsilon(x_i)$ is the number of points conjugate to x_i under the action of Weyl group W on the maximal torus of the Lie group (or even subgroup $W^e \subset W$ in the case of E -functions).

Again, as in the continuous case, the C -, S - and E -functions are pairwise orthogonal, i.e.

$$\langle C_\lambda(x), C_{\lambda'}(x) \rangle_M = \sum_{i=1}^N \varepsilon(x_i) C_\lambda(x_i) \overline{C_{\lambda'}(x_i)} = h_\lambda^\vee \cdot \frac{|W_\lambda|^2}{|W|} \cdot |A_M| \cdot \delta_{\lambda\lambda'}, \quad (9)$$

$$\langle S_\lambda(x), S_{\lambda'}(x) \rangle_M = |W| \sum_{i=1}^N S_\lambda(x_i) \overline{S_{\lambda'}(x_i)} = |W| \cdot |A_M| \cdot \delta_{\lambda\lambda'}, \quad (10)$$

$$\langle E_\lambda(x), E_{\lambda'}(x) \rangle_M = \sum_{i=1}^N \varepsilon(x_i) E_\lambda(x_i) \overline{E_{\lambda'}(x_i)} = |W_\lambda^e| \cdot |A_M| \cdot \delta_{\lambda\lambda'}. \quad (11)$$

Here $h_\lambda^\vee = |\text{Stab}^\vee(\lambda)|$ with $\text{Stab}^\vee(\lambda) = \{w \in W | w\lambda = \lambda\}$, and A_M denotes W -invariant Abelian subgroup of the grid points on the maximal torus \mathbb{T}

$$|A_M| = \sum_{i=1}^{|F_M|} \varepsilon(x_i), \quad x_i \in F_M.$$

For S -functions the values of $\varepsilon(x_i)$ are equal to $|W|$. The coefficients $\varepsilon(x_i)$ for other functions and the proof of the orthogonality relations can be found in [3].

For different fixed $m \in \mathbb{R}^n$ the set of exponential functions $\{e^{2\pi i \langle m, x \rangle}, x \in \mathbb{R}^n\}$ determines continuous and discrete Fourier transforms on \mathbb{R}^n . In much the same way, the orbit functions (which are a symmetrized version of exponential functions) determine an analogue of the Fourier transform.

4 Orbit functions transforms

It follows from Section 3 that each family of orbit functions forms an orthogonal basis in the Hilbert space of squared integrable functions $\mathcal{L}^2(F)$. Hence functions given on F can be expanded in terms of linear combinations of C -, S - or E -functions.

In this section, we introduce the essentials of the continuous and discrete C -, S - and E -transforms. The discrete transform can be used for the continuous interpolation of values of a function $f(x)$ between its given values on a grid F_M .

Each continuous function on the fundamental region with continuous derivatives can be expanded as the sum of C -, S - or E -functions. Let $f(x)$ be a function defined on F (or F_e for E -functions), then it may be written that

$$f(x) = \sum_{\lambda \in P^+} c_\lambda C_\lambda(x), \quad c_\lambda = |W_\lambda|^{-1} |F|^{-1} \langle f(x), C_\lambda(x) \rangle; \quad (12)$$

$$f(x) = \sum_{\lambda \in P^{++}} c_\lambda S_\lambda(x), \quad c_\lambda = |W|^{-1} |F|^{-1} \langle f(x), S_\lambda(x) \rangle; \quad (13)$$

$$f(x) = \sum_{\lambda \in P_e} c_\lambda E_\lambda(x), \quad c_\lambda = |W_\lambda^e|^{-1} |F_e|^{-1} \langle f(x), E_\lambda(x) \rangle. \quad (14)$$

Here $\langle \cdot, \cdot \rangle$ denotes the continuous scalar product of (4), P^+ and P^{++} are dominant and strictly dominant weights respectively and $P_e = P^+ \cup wP^{++}$, where $w \in W$. Direct and inverse C -, S - and E -transforms of the function $f(x)$ are in (12), (13) and (14) respectively.

Let Λ_M be the maximal set of points, such that for any two $\lambda, \lambda' \in \Lambda_M$ the condition of discrete orthogonality holds for any of the families of orbit functions in (9), (10) or (11).

Then we have the following discrete transforms for the function $f(x)$:

$$f(x) = \sum_{\lambda \in \Lambda_M} b_\lambda C_\lambda(x), \quad x \in F_M, \quad b_\lambda = \frac{\langle f, C_\lambda \rangle_M}{\langle C_\lambda, C_\lambda \rangle_M}; \quad (15)$$

$$f(x) = \sum_{\lambda \in \Lambda_M} b_\lambda S_\lambda(x), \quad x \in F_M, \quad b_\lambda = \frac{\langle f, S_\lambda \rangle_M}{\langle S_\lambda, S_\lambda \rangle_M}; \quad (16)$$

$$f(x) = \sum_{\lambda \in \Lambda_M} b_\lambda E_\lambda(x), \quad x \in F_e, \quad b_\lambda = \frac{\langle f, E_\lambda \rangle_M}{\langle E_\lambda, E_\lambda \rangle_M}. \quad (17)$$

Here $\langle \cdot, \cdot \rangle_M$ denotes the discrete scalar product given of (8).

Once the coefficients b_λ of the expansions (15), (16) and (17) are calculated, discrete variables x_i in F_M may be replaced by continuous variables x in F

$$\begin{aligned} f_{\text{cont}}(x) &:= \sum_{\lambda \in \Lambda_M} b_\lambda C_\lambda(x), \quad x \in F; \\ f_{\text{cont}}(x) &:= \sum_{\lambda \in \Lambda_M} b_\lambda S_\lambda(x), \quad x \in F; \\ f_{\text{cont}}(x) &:= \sum_{\lambda \in \Lambda_M} b_\lambda E_\lambda(x), \quad x \in F_e. \end{aligned}$$

The function $f_{\text{cont}}(x)$ smoothly interpolates the values of $f(x_i)$, $i = 1, 2, \dots, |F_M|$. At the points x_i , we have the equality $f_{\text{cont}}(x_i) = f(x_i)$.

5 Orbit functions as eigenfunctions of the Laplace operator and boundary value problems

Consider the functions $C_\lambda(x)$, $E_\lambda(x)$ and $S_\lambda(x)$ and suppose that the continuous variable x is given relative to the orthogonal basis. In the case of Lie algebra A_n we use orthogonal coordinates x_1, x_2, \dots, x_{n+1} and coordinates x_1, x_2, \dots, x_n for B_n , C_n and D_n (the orthogonal bases for these algebras are well known, see e.g. [1, 4, 6]).

The Laplace operator in orthogonal coordinates has the form

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_k^2}, \quad \text{where } k = n \text{ (or } k = n + 1 \text{ for } A_n\text{).}$$

For the algebras A_n , B_n , C_n and D_n , the Laplace operator gives the same eigenvalues on every exponential function summand of an orbit function with eigenvalue $-4\pi\langle\lambda, \lambda\rangle$.

Hence, the functions $C_\lambda(x)$, $E_\lambda(x)$ and $S_\lambda(x)$ are eigenfunctions of the Laplace operator:

$$\Delta \begin{pmatrix} C_\lambda(x) \\ E_\lambda(x) \\ S_\lambda(x) \end{pmatrix} = -4\pi^2 \langle \lambda, \lambda \rangle \begin{pmatrix} C_\lambda(x) \\ E_\lambda(x) \\ S_\lambda(x) \end{pmatrix}.$$

Thereby, a Laplace operator for each Lie group is given in a different set of coordinates. The C - and S -functions are its eigenfunctions with known eigenvalues. On the boundary of F , the C -functions have a vanishing normal derivative, while S -functions reach zero at the boundary. this features allow one to solve the followig boundary value problems.

C -function is a solution of the Neumann boundary value problem on n -dimensional simplex F

$$\Delta f(x) = \Lambda f(x), \quad \frac{\partial f(x)}{\partial \nu} = 0 \quad \text{for } x \in \partial F.$$

S -function is a solution of the Dirichlet boundary value problem on n -dimensional simplex F

$$\Delta f(x) = \Lambda f(x), \quad f(x) = 0 \quad \text{for } x \in \partial F.$$

Let the continuous variable x is given relative to the ω -basis and Δ denotes the Laplace operator, where the differentiation ∂_{x_i} is made with respect to the direction given by ω_i then

$$\Delta = \sum_{i,j=1}^n \frac{C_{ij}}{\langle \alpha_i, \alpha_i \rangle} \partial_{x_i} \partial_{x_j}, \quad \text{where } C_{ij} \text{ are elements of the Cartan matrix.}$$

It is known in Lie theory that the matrix of scalar products of the simple roots is positive definite, moreover our definition makes matrix $\frac{C_{ij}}{\langle \alpha_i, \alpha_i \rangle}$ symmetric, hence it can be diagonalized and the Laplace operator could be transformed to the sum of second derivatives by an appropriate change of variables.

We can write the explicit form of the Laplace operator given in the ω -basis for any semisimple Lie algebra. In particular, on figures 1, 2 and 3 we adduce domains (fundamental regions) and the corresponding Laplace operators for three-dimensional Neumann and Dirichlet boundary value problems are presented in the captures.

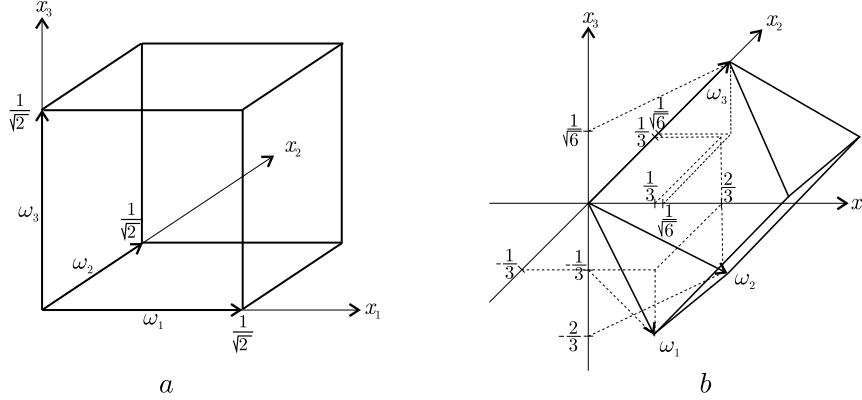


Figure 1. a) the fundamental region of the Lie algebra $A_1 \times A_1 \times A_1$ and the corresponding Laplace operator has the form $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$; b) the fundamental region of the Lie algebra $A_2 \times A_1$ and the corresponding Laplace operator has the form $\Delta = \partial_{x_1}^2 - \partial_{x_1} \partial_{x_2} + \partial_{x_2}^2 + \partial_{x_3}^2$.

6 Conclusions

It was shown that recently introduced families of special functions have a number of valuable properties. Those are the following:

- 1) The functions are defined for each compact semisimple Lie group G . There are

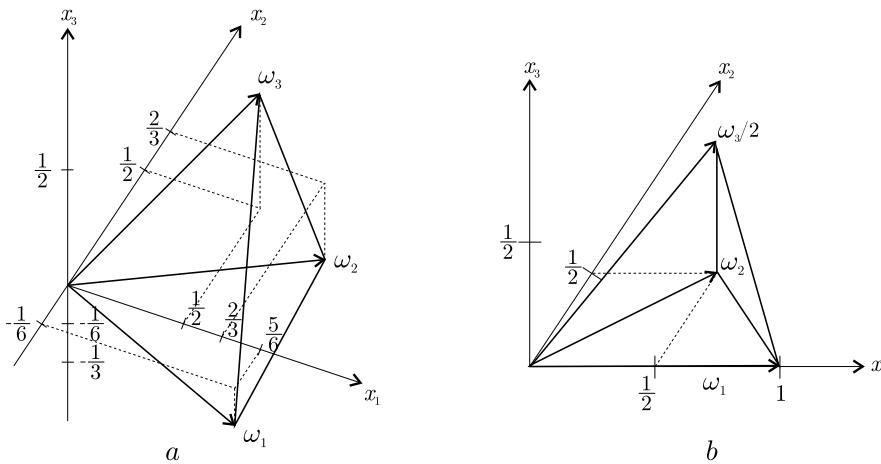


Figure 2. a) the fundamental region of the Lie algebra A_3 , the corresponding Laplace operator has the form $\Delta = \partial_{x_1}^2 - \partial_{x_1}\partial_{x_2} + \partial_{x_2}^2 - \partial_{x_2}\partial_{x_3} + \partial_{x_3}^2$; b) the fundamental region of the Lie algebra B_3 , the corresponding Laplace operator has the form $\Delta = \partial_{x_1}^2 - \partial_{x_1}\partial_{x_2} + \partial_{x_2}^2 - 2\partial_{x_2}\partial_{x_3} + 2\partial_{x_3}^2$.

several infinite families of functions per G . The number of continuous variables on which the functions depend, is equal to the rank of G .

- 2) Orbit functions have well defined symmetries with respect to the affine Weyl group of G and these functions (within each family) are orthogonal when integrated over the fundamental region of F of the maximal torus \mathbb{T} of G .
- 3) The functions can be sampled on the lattice fragment $F_M = F \cap (L/M)$ of the weight lattice L refined by $M \in \mathbb{N}$. There is a finite subset Λ_M of such ‘digital’ functions that are pairwise orthogonal when summed up over the points of F_M .
- 4) Each function is an eigenfunction of the Laplace operator appropriate for G and their eigenvalues are known explicitly. Therefore orbit functions can be applied to the solution of the corresponding Neumann and Dirichlet boundary-value problems on the fundamental domains of the Weyl groups.

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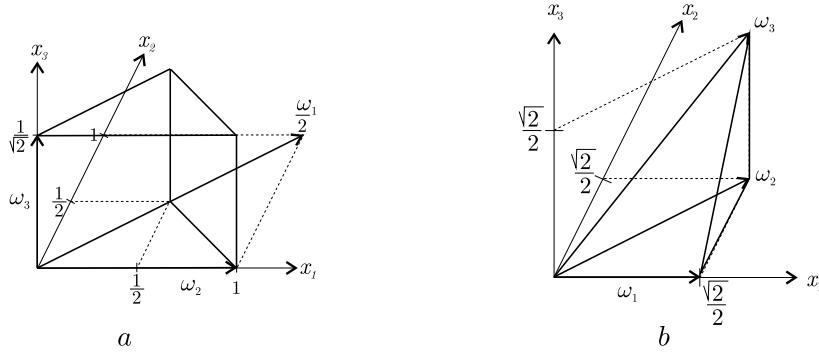


Figure 3. a) the fundamental region of the Lie algebra $C_2 \times A_1$, the corresponding Laplace operator has the form $\Delta = 2\partial_{x_1}^2 - 2\partial_{x_1}\partial_{x_2} + \partial_{x_2}^2 + \partial_{x_3}^2$; b) the fundamental region of the Lie algebra C_3 , the corresponding Laplace operator has the form $\Delta = 2\partial_{x_1}^2 - 2\partial_{x_1}\partial_{x_2} + 2\partial_{x_2}^2 - 2\partial_{x_2}\partial_{x_3} + \partial_{x_3}^2$.

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Galilean massless fields

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Galilei-invariant equations for massless vector fields are obtained with using indecomposable representations of the homogeneous Galilei group found by us earlier. It is shown that the collection of non-equivalent Galilei-invariant wave equations for massless fields with spin equal 1 and 0 is very rich. It describes many physically consistent systems, e.g., those of electromagnetic fields in various media and Galilean Carroll-Field-Jackiw models. Finally, classification of all linear and an extended group of non-linear Galilei-invariant equations for massless fields is presented.

1 Introduction

In physics there are specific symmetries and fundamental ones. The specific symmetries (like the rotation symmetry of the Hydrogen atom) are valid for particular systems while the fundamental ones are integral features of any physical system.

The most important examples of fundamental symmetries are relativistic invariance and Galilei invariance. It is the invariance w.r.t. Lorentz transformations or Galilei ones which is *a priori* required in a consistent physical theory.

Relativistic invariance is treated as a more fundamental one, since Galilei-invariant theories can be obtained as limiting case of relativistic ones. But there are reasons to study just Galilei-invariant theories:

- The majority of physical effects are non-relativistic (i.e., they are characterized by velocities much smaller than the velocity of light). In fact, we never observe directly a macroscopic body whose velocity is compatible with the velocity of light;
- Non-relativistic models in principle are simpler and more convenient than the relativistic ones. In particular the basic equation of non-relativistic quantum mechanics, i.e., the Schrödinger equation, includes a first order derivation w.r.t. the time variable while the relativistic Klein-Gordon equation is a second-order partial differential equation w.r.t. all independent variables;
- A correct definition of non-relativistic limit is by no means a simple problem, in general, and in the case of massless fields in particular. This limit is not unique and some of limits are physically meaningless;

- The very existence of a good non-relativistic approximation can serve as a selection rule for consistent relativistic theories.

Relativistic theories in principle are more complicated than non-relativistic ones. On the other hand, the structure of subgroups of the Galilei group and of its representations are in many respects more complex than those of the Poincaré group and therefore it is perhaps not so surprising that the representations of the Poincaré group were described by Wigner in 1939, almost 15 years earlier than the representations of the Galilei group (Bargman, 1954) in spite of the fact that the relativity principle of classical physics was formulated by Galilei about three centuries prior to that of relativistic physics formulated by Einstein.

It appears that, as opposed to the Poincaré group, the Galilei group has the exact as well as the projective representations. Moreover, finite-dimensional indecomposable representations ¹ of the homogeneous Galilei group $HG(1, 3)$ are not classifiable. And they are the representations which play a key role in formulation of physical models satisfying the Galilei relativity principle!

In paper [2] an important class the indecomposable finite dimensional representations of the homogeneous Galilei group $HG(1, 3)$ was derived. Namely, all such representations were found which when restricted to representations of the rotation subgroup of the group $HG(1, 3)$, are decomposed to the spin 0 and spin 1 representations.

In the present paper we present a complete description of Galilei invariant equations for vector and scalar massless fields. Namely, a complete list of relative differential invariants for the representations found in [2] and [3] is presented and the corresponding invariant equations are discussed. Among them there are equations for fields with more or less components than in the Maxwell equations.

2 Galilei group

The Galilei group is a group of transformations in $R_3 \oplus R_1$:

$$t \rightarrow t' = t + a, \quad \mathbf{x} \rightarrow \mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{v}t + \mathbf{b}, \quad (1)$$

where a, \mathbf{b} and \mathbf{v} are real parameters, \mathbf{R} is a rotation matrix.

The homogeneous Galilei group $HG(1, 3)$ is a subgroup of the group $G(1, 3)$ leaving invariant the point $\mathbf{x} = (0, 0, 0)$ at time $t = 0$. It is formed by space rotations and pure Galilei transformations, i.e., by the transformations (1) with $a = 0$ and $\mathbf{b} = 0$.

Lie algebra $hg(1, 3)$ of the homogeneous Galilei group includes six basis elements: 3 rotation generators $S_a, a = 1, 2, 3$ and three generators of Galilean

¹Let us remind that a representation of a group \mathfrak{G} in a normalized vector space \mathfrak{C} is *irreducible* if its carrier space \mathfrak{C} does not include subspaces invariant w.r.t. \mathfrak{G} . The representation is called *indecomposable* if \mathfrak{C} does not include invariant subspaces $\mathfrak{C}_{\mathfrak{I}}$ which are orthogonal to $\mathfrak{C} \setminus \mathfrak{C}_{\mathfrak{I}}$. Irreducible representations are indecomposable too but indecomposable representations can be reducible in the sense that their carrier spaces can include (non-orthogonal) invariant subspaces.

boosts η_a with the commutation relations

$$[S_a, S_b] = i\varepsilon_{abc} S_c, \quad [\eta_a, S_b] = i\varepsilon_{abc} \eta_c, \quad [\eta_a, \eta_b] = 0.$$

This algebra is a semidirect product of a simple algebra $o(3)$ spanned on basis elements S_1, S_2, S_3 and an Abelian algebra whose basis elements are η_1, η_2, η_3 .

3 Vector representations

All indecomposable representations of $HG(1, 3)$ which, when restricted to the rotation subgroup, are decomposed to direct sums of vector and scalar representations, were found in [2], [3]. These indecomposable representations (denoted as $D(m, n, \lambda)$) are labeled by triplets of numbers: n, m and λ . These numbers take the values

$$\begin{aligned} -1 \leq (n - m) \leq 2, \quad n \leq 3, \\ \lambda = \begin{cases} 0 & \text{if } m = 0, \\ 1 & \text{if } m = 2 \text{ or } n - m = 2, \\ 0, 1 & \text{if } m = 1, n \neq 3. \end{cases} \end{aligned} \tag{2}$$

In accordance with (2) there exist ten non-equivalent indecomposable representations $D(m, n, \lambda)$. Their carrier spaces can include three types of rotational scalars A, B, C and five types of vectors $\mathbf{R}, \mathbf{U}, \mathbf{W}, \mathbf{K}, \mathbf{N}$ whose transformation laws with respect to the Galilei boost are:

$$\begin{aligned} A \rightarrow A' &= A, \quad B \rightarrow B' = B + \mathbf{v} \cdot \mathbf{R}, \\ C \rightarrow C' &= C + \mathbf{v} \cdot \mathbf{U} + \frac{1}{2}\mathbf{v}^2 A, \quad \mathbf{R} \rightarrow \mathbf{R}' = \mathbf{R}, \\ \mathbf{U} \rightarrow \mathbf{U}' &= \mathbf{U} + \mathbf{v}A, \quad \mathbf{W} \rightarrow \mathbf{W}' = \mathbf{W} + \mathbf{v} \times \mathbf{R}, \\ \mathbf{K} \rightarrow \mathbf{K}' &= \mathbf{K} + \mathbf{v} \times \mathbf{R} + \mathbf{v}A, \\ \mathbf{N} \rightarrow \mathbf{N}' &= \mathbf{N} + \mathbf{v} \times \mathbf{W} + \mathbf{v}B + \mathbf{v}(\mathbf{v} \cdot \mathbf{R}) - \frac{1}{2}\mathbf{v}^2 \mathbf{R}, \end{aligned} \tag{3}$$

where \mathbf{v} is a vector whose components are parameters of the considered Galilei boosts, $\mathbf{v} \cdot \mathbf{R}$ and $\mathbf{v} \times \mathbf{R}$ are scalar and vector products of vectors \mathbf{v} and \mathbf{R} respectively.

Carrier spaces of these indecomposable representations of the group $HG(1, 3)$ include such sets of scalars A, B, C and vectors $\mathbf{R}, \mathbf{U}, \mathbf{W}, \mathbf{K}, \mathbf{N}$ which transform among themselves w.r.t. transformations (3) but cannot be split to a direct sum of invariant subspaces. There exist exactly ten such sets:

$$\begin{aligned} \{A\} &\iff D(0, 1, 0), \quad \{\mathbf{R}\} \iff D(1, 0, 0), \quad \{B, \mathbf{R}\} \iff D(1, 1, 0), \\ \{A, \mathbf{U}\} &\iff D(1, 1, 1), \quad \{A, \mathbf{U}, C\} \iff D(1, 2, 1), \quad \{\mathbf{W}, \mathbf{R}\} \iff D(2, 0, 0), \\ \{\mathbf{R}, \mathbf{W}, B\} &\iff D(2, 1, 0), \quad \{A, \mathbf{K}, \mathbf{R}\} \iff D(2, 1, 1), \\ \{A, B, \mathbf{K}, \mathbf{R}\} &\iff D(2, 2, 1), \quad \{B, \mathbf{N}, \mathbf{W}, \mathbf{R}\} \iff D(3, 1, 1). \end{aligned} \tag{4}$$

Thus, in contrary to the relativistic case, where are only three Lorentz covariant quantities which transform as vectors or scalars under rotations (i.e., as a

relativistic four-vector, antisymmetric tensor of the second order and a scalar), there are ten indecomposable sets of the Galilei vectors and scalars which we have enumerated in equation (4). The corresponding vectors of carrier spaces can be one-, three-, four-, five-, six-, eight- and ten-dimensional.

4 Equations for massless fields

We say a partial differential equation is a Galilei invariant equation for massless field if it is invariant with respect to transformations (1) and the related representation of the inhomogeneous Galilei group is exact. Notice that in the case of non exact but projective representations the corresponding invariant equations describe massive fields [5].

To construct Galilei invariant equations for massless fields it is possible to use at least three approaches:

- To start with equations invariant w.r.t. group $P(1, 4)$, i.e., the Poincaré group in (1+4)-dimensional space. This group includes the Galilei group $HG(1, 3)$ as a subgroup and so making reduction $P(1, 4) \rightarrow HG(1, 3)$ we obtain Galilei-invariant equations [4, 5].
- To start with equations invariant w.r.t. the Poincaré group $P(1, 3)$ and to make a contraction to the Galilei group.
- To use our knowledge of indecomposable representations of $HG(1, 3)$ and deduce Galilei-invariant equations using tools of Lie analysis, i.e., calculating absolute and relative differential invariants of an appropriate order.

The first and second approaches were used by many authors. However, in this way it is possible to obtain particular results only. The third approach is the most powerful and just this approach is used in the present paper to describe all Galilei-invariant equations for vector and scalar fields.

5 Covariant differential forms

To start our analysis of Galilei-invariant linear wave equations for vector and scalar fields we present a full list of first order differential forms which transform as indecomposable vectors sets under the Galilei transformations. In this way we describe general linear Galilean equations of first order for scalar and vector fields.

Using exact transformation laws given by equations (1) and (3) it is not difficult to find the corresponding transformations for derivatives of vector fields. The differential operators $\frac{\partial}{\partial t}$ and ∇ transform as components of four-vector from a carrier space of the representation $D(1, 1, 0)$ of the $HG(1, 3)$, thus to describe transformation properties of these derivatives it is sufficient to describe tensor products of this representation with all representations enumerated in equation (4). It is evident that the derivatives of vector fields can transform as scalars, vectors or

second rank tensors under rotations. Restricting ourselves to those forms which transform as vectors or scalars we obtain the following indecomposable sets of them:

```

For  $D(0, 1, 0)$  :  $\{\mathcal{R}_1 = \nabla A\};$ 
For  $D(1, 0, 0)$  :  $\{\mathcal{R}_2 = -\nabla \times \mathbf{R}\}$  and  $\{\mathcal{A}_1 = \nabla \cdot \mathbf{R}\}$ 
For  $D(1, 1, 0)$  :  $\{\mathcal{R}_2, \mathcal{W}_2 = \frac{\partial \mathbf{R}}{\partial t} - \nabla B\} \supset \{\mathcal{R}_2\}$ , and  $\{\mathcal{A}_1\}$ ;
For  $D(1, 1, 1)$  :  $\{\mathcal{B}_1 = \frac{1}{2}(\frac{\partial A}{\partial t} + \nabla \cdot \mathbf{U}), \mathcal{W}_1 = \nabla \times \mathbf{U}, \mathcal{R}_1\} \supset (\{\mathcal{B}_1, \mathcal{R}_1\},$ 
 $\{\mathcal{W}_1, \mathcal{R}_1\}, \{\mathcal{R}_1\}),$  and  $\{\mathcal{A}_2 = \frac{\partial A}{\partial t} - \nabla \cdot \mathbf{U}\};$ 
For  $D(2, 0, 0)$  :  $\{\mathcal{U}_1 = \frac{\partial \mathbf{R}}{\partial t} + \nabla \times \mathbf{W}, \mathcal{A}_1\} \supset \{\mathcal{A}_1\},$ 
 $\text{and } \{\mathcal{B}_2 = \nabla \cdot \mathbf{W}, \mathcal{R}_2\} \supset \{\mathcal{R}_2\};$ 
For  $D(1, 2, 1)$  :  $\{\mathcal{N}_1 = \frac{\partial \mathbf{U}}{\partial t} - \nabla C, \mathcal{W}_1, \mathcal{R}_1, \tilde{\mathcal{B}}_1 = \frac{\partial A}{\partial t}\}$ 
 $\supset \{\mathcal{W}_1, \mathcal{R}_1, \tilde{\mathcal{B}}_1 = \frac{\partial A}{\partial t}\} \supset$ 
 $(\{\tilde{\mathcal{B}}_1, \mathcal{R}_1\}, \{\mathcal{W}_1, \mathcal{R}_1\}, \{\mathcal{R}_1\}), \text{ and } \{\mathcal{A}_2\};$  (5)
For  $D(2, 1, 0)$  :  $\{\mathcal{W}_2, \mathcal{R}_2, \mathcal{B}_2\} \supset (\{\mathcal{B}_2, \mathcal{R}_2\}, \{\mathcal{W}_2, \mathcal{R}_2\}, \{\mathcal{R}_2\}), \text{ and } \{\mathcal{A}_1\};$ 
For  $D(2, 1, 1)$  :  $\{\mathcal{K}_1 = \frac{\partial \mathbf{R}}{\partial t} + \nabla \times \mathbf{K}, \tilde{\mathcal{R}}_1 = -\nabla A, \mathcal{A}_1\} \supset (\{\tilde{\mathcal{R}}_1\}, \{\mathcal{A}_1\})$ 
 $\text{and } \{\tilde{\mathcal{B}}_2 = \nabla \cdot \mathbf{K} - \frac{\partial A}{\partial t}, \mathcal{R}_2\} \supset \{\mathcal{R}_2\};$ 
For  $D(2, 2, 1)$  :  $\{\mathcal{K}_1 = \frac{\partial \mathbf{R}}{\partial t} + \nabla \times \mathbf{K}, \mathcal{R}_1, \mathcal{A}_1\} \supset (\{\tilde{\mathcal{R}}_1\}, \{\mathcal{A}_1\})$ 
 $\text{and } \{\tilde{\mathcal{B}}_2, \mathcal{W}_2, \mathcal{R}_2\} \supset (\{\tilde{\mathcal{B}}_2, \mathcal{R}_2\}, \{\mathcal{W}_2, \mathcal{R}_2\}, \{\mathcal{R}_2\});$ 
For  $D(3, 1, 1)$  :  $\{\mathcal{N}_2 = \frac{\partial \mathbf{W}}{\partial t} + \nabla \times \mathbf{N}, \mathcal{W}_2, \mathcal{R}_2, \mathcal{B}_2\}$ 
 $\supset \{\mathcal{W}_2, \mathcal{R}_1, \mathcal{B}_2 = \frac{\partial A}{\partial t}\} \supset (\{\mathcal{B}_2, \mathcal{R}_2\}, \{\mathcal{W}_2, \mathcal{R}_2\}, \{\mathcal{R}_2\}),$ 
 $\text{and } \{\mathcal{C}_1 = \frac{\partial B}{\partial t} - \nabla \cdot \mathbf{N}, \mathcal{U}_1, \mathcal{A}_1\} \supset \{\mathcal{U}_1, \mathcal{A}_1\} \supset \{\mathcal{A}_1\}.$ 

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Transformation properties of the forms presented in equations (5) are described by relations (3) where capital letters should be replaced by calligraphic ones. The forms given in brackets are closed w.r.t. the Galilei transformations.

Notice that there are also tensorial differential forms, namely

$$\begin{aligned}
Y_{ab} &= \nabla_a R_b + \nabla_b R_a, & L_{ab} &= \nabla_a N_b + \nabla_b N_a, \\
Z_{ab}^1 &= \nabla_a U_b + \nabla_b U_a, & R_{ab} &= \nabla_a W_b + \nabla_b W_a, \\
Z_{ab}^2 &= \nabla_a K_b + \nabla_b K_a - R_{ab}, & T_{ab} &= \nabla_a K_b + \nabla_b K_a
\end{aligned} \tag{6}$$

which transform in a covariant manner under the Galilei transformations provided $\mathbf{R}, \mathbf{U}, \mathbf{W}, \mathbf{K}$ and \mathbf{N} are transformed in accordance with (3). To present invariant sets which include (6) we need the forms given in (5) and also the following scalar and vector forms:

$$\begin{aligned}
G &= \frac{\partial B}{\partial t}, & D &= \frac{\partial C}{\partial t}, & \mathbf{G} &= \frac{\partial \mathbf{W}}{\partial t}, & \mathbf{F} &= \frac{\partial \mathbf{N}}{\partial t}, & \mathbf{P} &= \frac{\partial \mathbf{R}}{\partial t}, & \mathbf{T} &= \frac{\partial \mathbf{K}}{\partial t}, \\
\mathbf{X} &= \nabla \times \mathbf{W}, & \mathbf{S} &= \frac{\partial \mathbf{K}}{\partial t} - \mathbf{G}, & \mathbf{M} &= \frac{\partial \mathbf{R}}{\partial t} + \nabla B, & \mathbf{J} &= \frac{\partial \mathbf{U}}{\partial t} + \nabla C.
\end{aligned} \tag{7}$$

The related sets indecomposable w.r.t. the Galilei transformations are enu-

merated in the following formula:

$$\begin{aligned}
& \{Y_{ab}\}, \{Z_{ab}^1, \mathcal{R}_1\}, \{R_{ab}, Y_{ab}, \mathcal{R}_2\}, \{Z_{ab}^2, \mathcal{R}_2\}, \\
& \{\mathbf{M}, Y_{ab}\}, \{\mathbf{P}, Y_{ab}, \mathcal{R}_2\}, \{G, \mathbf{M}, Y_{ab}\}, \{D, Z_{ab}^1, \mathcal{R}_1, \mathbf{J}, \tilde{\mathcal{B}}\}, \\
& \{\mathbf{J}, \tilde{\mathcal{B}}_1, \mathcal{R}_1, Z_{ab}^1\}, \{\mathbf{G}, R_{ab}, Y_{ab}, \mathcal{R}_2, \mathbf{P}, \mathcal{U}_1\}, \{\mathbf{S}, Z_{ab}^2, \mathcal{R}_1, \mathcal{U}_2 - \mathcal{K}_2, \tilde{\mathcal{B}}_1\}, \quad (8) \\
& \{T_{ab}, R_{ab}, \mathbf{X}, \mathcal{R}_1\}, \{T_{ab}, R_{ab}, \mathbf{X}, \mathcal{R}_1\}, \{T_{ab}, R_{ab}, \mathbf{X}, \mathbf{S}, \mathcal{R}_1, \mathcal{K}_2 - \mathbf{P}, \tilde{\mathcal{B}}\}, \\
& \{Y_{ab}, R_{ab}, L_{ab}, \mathcal{R}_2, \mathbf{P}, \mathbf{X}\}, \{Y_{ab}, R_{ab}, L_{ab}, \mathcal{R}_2, \mathbf{P}, \mathbf{X}, \mathbf{F}, \mathbf{M}, \mathbf{G}, G\}.
\end{aligned}$$

Notice that the functional invariants for the vector representations of the Galilei group have been found in paper [3].

6 Extended Galilei electromagnetism

Equating differential forms given in (5) to vectors with the same transformation properties or to zero we obtain systems of linear first order equations for Galilei vector fields. Thus, starting with representation $D(3, 1, 1)$, equating $\mathcal{N}_1, \mathcal{W}_1, \mathcal{R}_1$ and \mathcal{B} to zero and $\mathcal{C}, \mathcal{U}_1, \mathcal{A}$ to components of five-current j^0, \mathbf{j}, j^4 we obtain the system

$$\begin{aligned}
\mathcal{C} & \equiv \nabla \cdot \mathbf{N} - \frac{\partial}{\partial t} B - ej^0 = 0, \quad \mathcal{U} \equiv \nabla \times \mathbf{W} + \nabla B - e\mathbf{j} = 0, \\
\mathcal{A} & \equiv \nabla \cdot \mathbf{R} - ej^4 = 0, \quad \mathcal{N} \equiv \frac{\partial}{\partial t} \mathbf{W} + \nabla \times \mathbf{N} = 0, \quad \mathcal{W} \equiv \frac{\partial}{\partial t} \mathbf{R} - \nabla B = 0, \quad (9) \\
\mathcal{R} & \equiv -\nabla \times \mathbf{R} = 0, \quad \mathcal{B} \equiv \nabla \cdot \mathbf{W} = 0.
\end{aligned}$$

The system of equations (9) includes ten dependent variables, i.e., four more than the Maxwell's equations. It is the most extended system of first order partial differential equations which can be defined in the carrier space of indecomposable vector representation of the Galilei group.

Let us notice two things. First, equations (9) can be reduced to systems of Galilei-invariant equations for the electromagnetic field discussed in paper [1]. For example, imposing a Galilei-invariant additional conditions $\mathbf{R} = 0$ and $B = j_4 = 0$ equation is reduced to the form

$$\nabla \times \mathbf{E}_m - \frac{\partial \mathbf{H}_m}{\partial t} = 0, \quad \nabla \cdot \mathbf{E}_m = ej_m^0, \quad \nabla \times \mathbf{H}_m = e\mathbf{j}_m, \quad \nabla \cdot \mathbf{H}_m = 0, \quad (10)$$

where $\mathbf{E}_m = \mathbf{N}|_{\mathbf{R}=B=0}$ and $\mathbf{H}_m = \mathbf{W}|_{\mathbf{R}=B=0}$.

Equations (10) are Galilei invariant and represent the so called “magnetic limit” of the Maxwell equations [1].

Secondly, the coupled system of equations (10) can be obtained starting with a decoupled system of relativistic wave equations and making the Inönü-Wigner contraction [6]. The detailed analysis of this and other contractions of relativistic equations for vector and scalar fields to Galilei-invariant equations can be found in [7].

Using the complete list of relative differential invariants given in (5), (6) and (7) it is possible to find a big variety of other Galilei invariant systems. An extended list of them can be found in paper [7].

7 Galilean Born-Infeld equations

Starting with indecomposable vector representations of the group $HG(1,3)$ it is possible to find out various classes of partial differential equations invariant w.r.t. the Galilei group. In the above we have restricted ourselves to linear Galilean equations for vector and scalar fields and now we shall present nonlinear equations, namely, a Galilei-invariant analogue of the relativistic Born-Infeld equations [8].

The relativistic Born-Infeld equations include system

$$\begin{aligned}\frac{\partial \mathbf{D}}{\partial t} &= \nabla \times \mathbf{H}, \quad \nabla \cdot \mathbf{D} = 0, \\ \frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E}, \quad \nabla \cdot \mathbf{B} = 0\end{aligned}\tag{11}$$

and the constitutive equations

$$\mathbf{D} = \frac{1}{L}(\mathbf{E} + (\mathbf{B} \cdot \mathbf{E})\mathbf{B}), \quad \mathbf{H} = \frac{1}{L}(\mathbf{B} - (\mathbf{B} \cdot \mathbf{E})\mathbf{E}),\tag{12}$$

where $L = (1 + \mathbf{B}^2 - \mathbf{E}^2 - \mathbf{B} \cdot \mathbf{E})^{1/2}$. Equations (11), (12) are Lorentz-invariant. Making the Inönü-Wigner contraction of the related representation of the Lorentz group we can reduce this system to the following form:

$$\begin{aligned}\frac{\partial \mathbf{D}'}{\partial t} &= \nabla \times \mathbf{H}', \quad \nabla \cdot \mathbf{D}' = 0, \\ \nabla \times \mathbf{E}' &= 0, \quad \nabla \cdot \mathbf{B}' = 0\end{aligned}\tag{13}$$

with the constitutive equations

$$\mathbf{D}' = \frac{\mathbf{E}'}{\sqrt{1 - \mathbf{E}'^2}}, \quad \mathbf{H}' = \frac{\mathbf{B}'}{\sqrt{1 - \mathbf{E}'^2}} - \frac{(\mathbf{B}' \cdot \mathbf{E}')\mathbf{E}'}{\sqrt{1 - \mathbf{E}'^2}}.\tag{14}$$

Equations (13), (14) are Galilei-invariant. Moreover, under Galilei boosts vectors $\mathbf{D}', \mathbf{H}', \mathbf{B}'$ and \mathbf{E}' cotransform as

$$\begin{aligned}\mathbf{D}' &\rightarrow \mathbf{D}', \quad \mathbf{H}' \rightarrow \mathbf{H}' + \mathbf{v} \times \mathbf{D}', \\ \mathbf{B}' &\rightarrow \mathbf{B}' + \mathbf{v} \times \mathbf{E}', \quad \mathbf{E}' \rightarrow \mathbf{E}'.\end{aligned}\tag{15}$$

One more contracted version of the Born-Infeld equations looks as the system:

$$\begin{aligned}\nabla \times \mathbf{H} &= 0, \quad \nabla \cdot \mathbf{D} = 0, \\ \frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E}, \quad \nabla \cdot \mathbf{B} = 0,\end{aligned}\tag{16}$$

which is supplemented with the Galilei-invariant constitutive equations

$$\mathbf{D}' = \frac{\mathbf{E}'}{\sqrt{1 + \mathbf{B}^2}} + \frac{(\mathbf{B}' \cdot \mathbf{E}')\mathbf{B}'}{\sqrt{1 + \mathbf{B}^2}}, \quad \mathbf{H}' = \frac{\mathbf{B}'}{\sqrt{1 + \mathbf{B}^2}}.\tag{17}$$

The corresponding transformation laws read

$$\mathbf{D}' \rightarrow \mathbf{D}' - \mathbf{v} \times \mathbf{H}', \quad \mathbf{H}' \rightarrow \mathbf{H}', \quad \mathbf{B}' \rightarrow \mathbf{B}', \quad \mathbf{E}' \rightarrow \mathbf{E}' - \mathbf{v} \times \mathbf{B}'.\tag{18}$$

Thus there exist two Galilei limits for the Maxwell equations in media which we present in the above.

8 Quasilinear wave equations

Finally, let us present a quasilinear equation for Galilean 10-vector:

$$\begin{aligned} \frac{\partial}{\partial t} B - \nabla \cdot \mathbf{N} + \nu \mathbf{W} \cdot \mathbf{N} + \lambda \mathbf{R} \cdot \mathbf{W} + \sigma (B^2 - \mathbf{R} \cdot \mathbf{N}) + \omega \mathbf{R}^2 + \mu B &= ej^0, \\ \frac{\partial \mathbf{R}}{\partial t} + \nabla \times \mathbf{W} + \nu (B \mathbf{W} + \mathbf{R} \times \mathbf{N}) + \sigma (\mathbf{R} \times \mathbf{W} + B \mathbf{R}) + \mu \mathbf{R} &= ej, \\ \nabla \cdot \mathbf{R} + \nu \mathbf{R} \cdot \mathbf{W} + \sigma \mathbf{R}^2 &= ej^4, \quad \frac{\partial}{\partial t} \mathbf{W} + \nabla \times \mathbf{N} + \rho \mathbf{N} = 0, \\ \frac{\partial}{\partial t} \mathbf{R} - \nabla B + \rho \mathbf{W} &= 0, \quad -\nabla \times \mathbf{R} + \rho \mathbf{R} = 0, \quad \nabla \cdot \mathbf{W} + \rho B = 0. \end{aligned} \quad (19)$$

Formula (19) presents the most general Galilei-invariant quasilinear system which can be obtained from “the most extended” linear system (9) by adding linear terms and quadratic non-linearities. It is rather interesting since includes a number of important systems corresponding to special value of arbitrary parameters which are denoted by Greek letters. In particular it includes a Galilean version of the Carroll-Field-Jackiw model [9].

9 Discussion

Thus, in addition to found in [2] indecomposable representations of the homogeneous Galilei group for vector and scalar fields we present the complete list of relative first order differential invariants. Using this list (given by relations (5)–(8)) it is easy to write systems of first order partial differential equations for vector fields, invariant with respect to the Galilei group. Since higher order equations can be reformulated as systems of first order ones, in fact we present an essential element for description of arbitrary order Galilei invariant equations.

The number of indecomposable vector representations and of the related invariants as well is much more extended than in the case of Lorentz group. On the other hand we have proved [3] that any of these representations can be obtained via the Inönü-Wigner contraction from representations of Lorentz group. And there is not any contradiction between these two statements since to obtain an indecomposable representation of the Galilei group we should contract in general a direct sum of indecomposable representations of the Lorentz one.

Galilei-invariant equations for massless fields can find direct applications in various physical models which are characterized by velocities much smaller than the velocity of light. In particular, such equations are an essential part of Galilean supersymmetric models.

Let us summarize some open questions connected with the presented results:

- Description of differential invariants of any order;
- Definition of non-equivalent Lagrangians for vector fields;
- Application of the presented results to physical models (in particular, hydrodynamical ones);

- Construction of Galilean supersymmetric models for massless and massive fields.

We believe that some of these questions will be answered in our future publications.

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Ternary Poisson algebra for the non degenerate three dimensional Kepler–Coulomb potential

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In the three dimensional flat space any classical Hamiltonian, which has five functionally independent integrals of motion, including the Hamiltonian, is characterized as superintegrable. Kalnins, Kress and Miller [1] have proved that, in the case of non degenerate potentials, i.e potentials depending linearly on four parameters, with quadratic symmetries, posses a sixth quadratic integral, which is linearly independent of the other integrals. The existence of this sixth integral imply that the integrals of motion form a ternary parafermionic-like quadratic Poisson algebra with five generators. The Kepler–Coulomb potential that was introduced by Verrier and Evans [2] is a special case of superintegrable system, having two independent integrals of motion of fourth order among the remaining quadratic ones. The corresponding Poisson algebra of integrals is a quadratic one, having the same special form, characteristic to the non degenerate case of systems with quadratic integrals.

1 Introduction

In classical mechanics, a superintegrable or completely integrable is a Hamiltonian system with a maximum number of integrals. Two well known examples are the harmonic oscillator and the Coulomb potential. In the N -dimensional space the superintegrable system has $2N - 1$ integrals, one among them is the Hamiltonian.

Several cases of three dimensional superintegrable systems with quadratic integrals of motion are described and analyzed by Kalnins, Kress and Miller [1, 3]. Specifically, Kalnins, Kress and Miller studied a special case of superintegrable systems in which the potentials depend of four free parameters, these systems are referred as *non degenerate potentials*. In the case that one three dimensional potential have fewer arbitrary constants than four the potential is called *degenerate*. The generate and non degenerate potentials have been studied by N.W. Evans [4]. One among the degenerate systems is the so called *Generalized Kepler–Coulomb* system [4, 5]

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - \frac{k}{\sqrt{x^2 + y^2 + z^2}} + \frac{k_1}{x^2} + \frac{k_2}{y^2}. \quad (1)$$

This potential has four integrals of motion quadratic in momenta plus the Hamiltonian. These quadratic integrals of motion do not appear to close under repeated commutation and they do not satisfy a quadratic algebra [5] as it happens for the superintegrable two dimensional systems [6]. Therefore one of the open problems is to find the Poisson algebra of the integrals of motion for the superintegrable system (1).

One of the results of the Kalnins, Kress, Miller paper [1] is the so called “5 to 6” Theorem, which states that any three dimensional non degenerate superintegrable system with quadratic integrals of motion has always a sixth quadratic integral F that is linearly independent but not functionally independent regarding the set of five integrals A_1, A_2, B_1, B_2, H . The last statement leads to the result that any three dimensional superintegrable non degenerate system form a, quadratic, ternary Poisson algebra of special character [7], whose the definition is given in Section 2.

Verier and Evans [2] introduced a new superintegrable Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - \frac{k}{\sqrt{x^2 + y^2 + z^2}} + \frac{k_1}{x^2} + \frac{k_2}{y^2} + \frac{k_3}{z^2}, \quad (2)$$

which is the non degenerate version of the potential of the generalized Kepler–Coulomb system (1). The above potential is indeed superintegrable with quadratic and quartic in momenta integrals of motion. The quartic integrals are generalizations of the Laplace RungeLenz vectors of the ordinary Kepler–Coulomb potential [2].

In this paper we prove that the “5 to 6” theorem of Kalnins, Kress, Miller [1] can be applied and the associate Poisson algebra is a ternary quadratic algebra of the constants of motion, which are different of the Hamiltonian. This algebra is similar to the ternary parafermionic-like algebra for the three dimensional non degenerate potentials [7]. Therefore the algebra of the generalized Kepler–Coulomb system is also a ternary parafermionic-like algebra.

2 Ternary Parafermionic-like Poisson Algebra

The definition of the Lie algebra \mathfrak{g} with generators x_1, x_2, \dots, x_n leads to the following relations $[x_i, x_j] = \sum_m c_{ij}^m x_m$, where c_{ij}^m the structure constants. The generators satisfy the obvious ternary (trilinear) relations

$$T(x_i, x_j, x_k) \stackrel{\text{def}}{=} [x_i, [x_j, x_k]] = \sum_n d_{i;jk}^n x_n, \quad \text{where} \quad d_{i;jk}^n = \sum_m c_{im}^n c_{jk}^m. \quad (3)$$

Generally a ternary algebra is an associative algebra \mathcal{A} whose generators satisfy relations like the following one

$$T(x_i, x_j, x_k) = \sum_n d_{i;jk}^n x_n,$$

where $T : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ is a trilinear map. If this trilinear map is defined as in eq. (3) the corresponding algebra is an example of the triple Lie algebras, which were introduced by Jacobson [8] in 1951. At the same time Green [9] has introduced the parafermionic algebra as an associative algebra, whose operators f_i^\dagger, f_i satisfy the ternary relations:

$$\begin{aligned} \left[f_k, \left[f_\ell^\dagger, f_m \right] \right] &= 2\delta_{k\ell} f_m, \quad \left[f_k, \left[f_\ell^\dagger, f_m^\dagger \right] \right] = 2\delta_{k\ell} f_m^\dagger - 2\delta_{km} f_\ell^\dagger, \\ \left[f_k, [f_\ell, f_m] \right] &= 0. \end{aligned}$$

We call parafermionic Poisson algebra the Poisson algebra satisfying the ternary relations:

$$\{x_i, \{x_j, x_k\}_P\}_P = \sum_m c_{i;jk}^m x_m,$$

which is the classical Poisson analogue of the Lie triple algebra (3).

The quadratic parafermionic Poisson algebra is a Poisson algebra satisfying the relations:

$$\{x_i, \{x_j, x_k\}_P\}_P = \sum_{m,n} d_{i;jk}^{mn} x_m x_n + \sum_m c_{i;jk}^m x_m.$$

A classical superintegrable system with quadratic integrals of motion on a two dimensional manifold possesses two functionally independent integrals of motion A and B , which are in involution with the Hamiltonian H of the system:

$$\{H, A\}_P = 0, \quad \{H, B\}_P = 0,$$

the Poisson bracket $\{A, B\}_P$ is different to zero and it is generally an integral of motion cubic in momenta, therefore it could not be in general a linear combination of the integrals H, A, B . Generally if we the Poisson brackets of the integrals of motion $\{A, \{A, B\}_P\}_P, \{\{A, B\}_P, B\}_P$ are not linear functions of the integrals of motion, therefore they don't close in a Lie Poisson algebra with three generators. Considering all the nested Poisson brackets of the integrals of motion, generally they don't close in an Lie Poisson algebra structure.

All the known two dimensional superintegrable systems with quadratic integrals of motion have a common structure [6, 10–13]:

$$\begin{aligned} \{H, A\}_P &= 0, \quad \{H, B\}_P = 0, \quad \{A, B\}_P^2 = 2F(A, H, B), \\ \{A, \{A, B\}_P\}_P &= \frac{\partial F}{\partial B}, \quad \{B, \{A, B\}_P\}_P = -\frac{\partial F}{\partial A}, \end{aligned} \tag{4}$$

where $F = F(A, B, H)$ is a cubic function of the integrals of motion

$$\begin{aligned} F(A, B, H) &= \alpha A^3 + \beta B^3 + \gamma A^2 B + \delta A B^2 + (\epsilon_0 + \epsilon_1 H) A^2 \\ &+ (\zeta_0 + \zeta_1 H) B^2 + (\eta_0 + \eta_1 H) A B + (\theta_0 + \theta_1 H + \theta_2 H^2) A \\ &+ (\kappa_0 + \kappa_1 H + \kappa_2 H^2) B + (\lambda_0 + \lambda_1 H + \lambda_2 H^2 + \lambda_3 H^3), \end{aligned} \tag{5}$$

where the greek letters are constants.

3 Non degenerate three dimensional Kepler–Coulomb ternary parafermionic-like Poisson algebra

The non degenerate Kepler–Coulomb Hamiltonian (2) has three quadratic integrals which in agreement with [4] are:

$$A_1 = \frac{1}{2}J^2 + \frac{k_1(x^2 + y^2 + z^2)}{x^2} + \frac{k_2(x^2 + y^2 + z^2)}{y^2} + \frac{k_3(x^2 + y^2 + z^2)}{z^2},$$

$$A_2 = \frac{1}{2}J_3^2 + \frac{k_1(x^2 + y^2)}{x^2} + \frac{k_2(x^2 + y^2)}{y^2}, \quad B_2 = \frac{1}{2}J_2^2 + \frac{k_1 z^2}{x^2} + \frac{k_3 x^2}{z^2},$$

where $J_1 = y p_z - z p_y$, $J_2 = z p_x - x p_z$, $J_3 = x p_y - y p_x$, $J^2 = J_1^2 + J_2^2 + J_3^2$.

The Coulomb potential differs from the other non degenerate potentials that described in [1] since posses one integral of fourth order in addition to above, quadratic one which denoted by B_1 and have the following form:

$$B_1 = \left(J_1 p_y - J_2 p_x - 2z \left(\frac{-k}{2\sqrt{x^2 + y^2 + z^2}} + \frac{k_1}{x^2} + \frac{k_2}{y^2} + \frac{k_3}{z^2} \right) \right)^2$$

$$+ \frac{2k_3}{z^2} (x p_x + y p_y + z p_z)^2.$$

One of the general results in [1] is the so called “5 to 6” theorem:

5→6 Theorem. *Let V be a nondegenerate potential (depending on 4 parameters) corresponding to a conformally flat space in 3 dimensions*

$$ds^2 = g(x, y, z)(dx^2 + dy^2 + dz^2),$$

that is superintegrable and there are 5, quadratic in momenta, functionally independent constants of the motion $\mathcal{L} = \{S_\ell : \ell = 1, \dots, 5\}$. There is always a 6th quadratic integral S_6 that is functionally dependent on \mathcal{L} , but linearly independent

In case of Coulomb potential the sixth integral exist and is an integral fourth order in momenta as well the B_1 with general form given by the next expression:

$$F = \left(-J_1 p_z + J_3 p_x - 2y \left(\frac{-k}{2\sqrt{x^2 + y^2 + z^2}} + \frac{k_1}{x^2} + \frac{k_2}{y^2} + \frac{k_3}{z^2} \right) \right)^2$$

$$+ \frac{2k_2}{y^2} (x p_x + y p_y + z p_z)^2.$$

By studying all the known non degenerate potentials given by Kalnins, Kress, Miller [1], we can show that:

Proposition. *In case of a three dimensional, non degenerate, superintegrable system with quadratic integrals of motion, on a conformally flat manifold, the integrals of motion satisfy a parafermionic-like quadratic Poisson Algebra with 5 generators which described from the following:*

$$\{S_i, \{S_j, S_k\}_P\}_P = \sum_{mn} d_{i;jk}^{m,n} S_m S_n + \sum_m c_{i;jk}^m S_m. \quad (6)$$

A detailed study of all the cases of non degenerate superintegrable systems on a flat space can be found in ref. [7]. In all the cases (with one exception) the Poisson algebra of the integrals of motion is a ternary quadratic parafermionic-like Poisson algebra (6), which has a specific form.

In all the known cases (with only one exception) the non degenerate systems we can choose beyond the Hamiltonian H four functionally independent integrals of motion A_1, B_1, A_2, B_2 , and one additional quadratic integral of motion F , such that all the integrals of motion are linearly independent. These integrals satisfy a Poisson parafermionic-like algebra (6). The “special” form of the algebra defined by the integrals A_1, B_1, A_2, B_2 is characterized by two cubic functions

$$F_1 = F_1(A_1, A_2, B_1, H), \quad F_2 = F_2(A_1, A_2, B_2, H)$$

and satisfy the relations:

$$\begin{aligned} \{A_1, A_2\}_P &= \{A_1, B_2\}_P = \{A_2, B_1\}_P = 0, \\ \{A_1, B_1\}_P^2 &= 2F_1(A_1, A_2, H, B_1) = \text{cubic function}, \\ \{A_2, B_2\}_P^2 &= 2F_2(A_1, A_2, H, B_2) = \text{cubic function}, \\ \{A_i, \{A_i, B_i\}_P\}_P &= \frac{\partial F_i}{\partial B_i}, \quad \{B_i, \{A_i, B_i\}_P\}_P = -\frac{\partial F_i}{\partial A_i}, \\ \{\{A_1, B_1\}_P, B_2\}_P &= \{A_1, \{B_1, B_2\}_P\}_P, \\ \{\{A_2, B_2\}_P, B_1\}_P &= -\{A_2, \{B_1, B_2\}_P\}_P. \end{aligned} \tag{7}$$

If we put $C_1 = \{A_1, B_1\}_P$, $C_2 = \{A_2, B_2\}_P$, $D = \{B_1, B_2\}_P$, the relations (7) imply the following ones:

$$\begin{aligned} \{C_1, B_2\}_P C_1 - \frac{\partial F_1}{\partial A_2} C_2 - \frac{\partial F_1}{\partial B_1} D &= \{C_2, B_1\}_P C_2 - \frac{\partial F_2}{\partial A_1} C_1 + \frac{\partial F_2}{\partial B_2} D = 0, \\ \{C_1, C_2\}_P &= \frac{\begin{vmatrix} \{A_1, D\}_P & -\frac{\partial F_1}{\partial A_2} \\ \frac{\partial F_2}{\partial A_1} & \{A_2, D\}_P \end{vmatrix}}{D} = \frac{\begin{vmatrix} -\frac{\partial F_1}{\partial B_1} & \{A_1, D\}_P \\ -\frac{\partial F_2}{\partial B_2} & \frac{\partial F_2}{\partial A_1} \end{vmatrix}}{C_2} \\ &= \frac{\begin{vmatrix} -\frac{\partial F_1}{\partial A_2} & -\frac{\partial F_1}{\partial B_1} \\ \{A_2, D\}_P & -\frac{\partial F_2}{\partial B_2} \end{vmatrix}}{C_1} \quad \text{and} \end{aligned} \tag{8}$$

$$\{C_1, C_2\}_P = \frac{\frac{\partial F_1}{\partial B_1} \frac{\partial F_2}{\partial A_1} C_1 + \frac{\partial F_1}{\partial A_2} \frac{\partial F_2}{\partial B_2} C_2 + \frac{\partial F_1}{\partial B_1} \frac{\partial F_2}{\partial B_2} D}{C_1 C_2}. \tag{9}$$

Schematically the structure of the above algebra is described by the following “II” shape

$$\begin{array}{ccc} A_1 & \text{---} & A_2 \\ | & & | \\ B_2 & & B_1 \end{array}$$

where with line represented the vanishing of Poisson bracket whereas the other brackets between the integrals are non vanishing Poisson brackets.

It is important to notice that the integrals A_1, B_1 satisfy a parafermionic-like quadratic Poisson algebra similar to the algebra as in two dimensional case (4). The corresponding structure function to the two dimensional one (5) can be written as:

$$\begin{aligned}
 F_1(A_1, B_1, H, A_2) = & \alpha_1 A_1^3 + \beta_1 B_1^3 + \gamma_1 A_1^2 B_1 + \delta A_1 B_1^2 \\
 & + (\epsilon_{01} + \epsilon_{11} H + \epsilon_{21} A_2) A_1^2 + (\zeta_{01} + \zeta_{11} H + \zeta_{21} A_2) B_1^2 \\
 & + (\eta_{01} + \eta_{11} H + \eta_{21} A_2) A_1 B_1 + \\
 & + (\theta_{01} + \theta_{11} H + \theta_{21} H^2 + \theta_{31} A_2 + \theta_{41} A_2^2 + \theta_{51} A_2 H) A_1 + \\
 & + (\kappa_{01} + \kappa_{11} H + \kappa_{21} H^2 + \kappa_{31} A_2 + \kappa_{41} A_2^2 + \kappa_{51} A_2 H) B_1 + \\
 & + \lambda_{01} + \lambda_{11} H + \lambda_{21} H^2 + \lambda_{31} H^3 + \lambda_{41} A_2 + \lambda_{51} A_2^2 + \lambda_{61} A_2^3 + \\
 & + \lambda_{71} A_2 H + \lambda_{81} A_2^2 H + \lambda_{91} A_2 H^2. \tag{10}
 \end{aligned}$$

The pair A_2, B_2 forms also a parafermionic-like algebra with the corresponding structure function $F_2(A_2, B_2, H, A_1)$, which has a similar form as in (10).

The non degenerate Coulomb potential obey to the above parafermionic-like “ Π ” structure that characterize, almost all three dimensional superintegrable systems. Precisely the algebra needs some modifications due to the difference of the integrals order and to one extra symmetry that appears in system. In fact, the extra symmetries on a superintegrable system cause a number of changes to the default “ Π ” structure always respecting the “ Π ” shape. The extra symmetry that Coulomb potential have is $\{B_2, F\} = 0$ and the general structure can be schematically represented by the following figure:

$$\begin{array}{ccc}
 A_2 & \text{---} & A_2 \\
 | & & | \\
 B_2 & & B_2 \\
 | & & | \\
 & & F
 \end{array}$$

The existence of one extra symmetry cause a basic difference to the general structure; one new subalgebra arising that expands in terms of A_1, B_2, F integrals that also imply the presence of one extra function $F_3(A_1, B_2, F, H)$ which is a fourth order function. Particularly, the “special” form of the algebra defined by the integrals A_1, B_1, A_2, B_2, F is characterized by two fourth order functions and one cubic:

$$F_1 = F_1(A_1, A_2, B_1, H), \quad F_2 = F_2(A_1, A_2, B_2, H), \quad F_3 = F_3(A_1, B_2, F, H)$$

and satisfy the relations:

$$\begin{aligned}
\{A_1, A_2\}_P &= \{A_1, B_2\}_P = \{A_2, B_1\}_P = 0, \{B_2, F\}_P = 0, \\
\{A_1, B_1\}_P^2 &= 2F_1(A_1, A_2, H, B_1) = \text{fourth order function}, \\
\{A_2, B_2\}_P^2 &= 2F_2(A_1, A_2, H, B_2) = \text{cubic function}, \\
\{A_1, F\}_P^2 &= 2F_3(A_1, F, H, B_2) = \text{fourth order function}, \\
\{A_i, \{A_i, B_i\}_P\}_P &= \frac{\partial F_i}{\partial B_i}, \quad \{B_i, \{A_i, B_i\}_P\}_P = -\frac{\partial F_i}{\partial A_i}, \\
\{A_1, \{A_1, F\}\}_P &= \frac{\partial F_3}{\partial F}, \quad \{F, \{A_1, F\}\}_P = -\frac{\partial F_3}{\partial A_1}, \\
\{\{A_1, B_1\}_P, B_2\}_P &= \{A_1, \{B_1, B_2\}_P\}_P, \\
\{\{A_2, B_2\}_P, B_1\}_P &= -\{A_2, \{B_1, B_2\}_P\}_P.
\end{aligned}$$

The structure functions of the above algebra are:

$$\begin{aligned}
F_1 &= 4k^2 A_1 B_1 - 4k^2 A_2 B_1 + 4k_3 k^2 B_1 - 32k_3 k^2 A_1 H - 4A_1 B_1^2 + 16A_1^2 B_1 H \\
&\quad - 16A_1 A_2 B_1 H - 64k_3 A_1^2 h^2 + 16k_3 A_1 B_1 H - 4k_3 k^4, \\
F_2 &= -4k_1 A_1^2 + 4A_2 B_2 A_1 + 8k_1 A_2 A_1 + 4k_1 B_2 A_1 - 4k_2 B_2 A_1 + 8k_1 k_3 A_1 \\
&\quad - 4A_2 B_2^2 - 4k_3 A_2 B_2 - 4A_2^2 B_2 - 4k_1 A_2^2 - 4k_1 A_2 B_2 + 4k_2 A_2 B_2 - 4k_3 A_2^2 \\
&\quad - 4k_1 k_3 B_2 + 8k_2 k_3 A_2 + 4k_2 k_3 B_2 + 8k_1 k_2 k_3 - 4k_1 k_3^2 - 4k_1^2 k_3 - 4k_2^2 k_3, \\
F_3 &= 4k^2 A_1 F - 4k^2 B_2 F - 4k_1 k^2 F + 4k_2 k^2 - 32k_2 k^2 F A_1 H - 4k_3 k^2 F \\
&\quad - 4A_1 F^2 + 16A_1^2 F H - 16A_1 B_2 F H - 16k_1 A_1 F H - 64k_2 A_1^2 H^2 \\
&\quad + 16k_2 A_1 F H - 16k_3 A_1 F H - 4k_2 k^4.
\end{aligned}$$

The full algebra is:

$$\begin{aligned}
\{\{A_1, B_1\}, A_2\} &= \{A_1, \{A_2, B_2\}\} = \{\{A_1, F\}, B_2\} = 0, \\
\{A_1, B_1\}, B_1\} &= \frac{\partial F_1}{\partial A_1} = -4B_1^2 + 4k^2 B_1 + 32A_1 H B_1 - 16A_2 H B_1 + 16H k_3 B_1 \\
&\quad - 128A_1 H^2 k_3 - 32H k^2 k_3, \\
\{A_1, \{A_1, B_1\}\} &= \frac{\partial F_1}{\partial B_1} = 16H A_1^2 + 4k^2 A_1 - 8B_1 A_1 - 16A_2 H A_1 + 16H k_3 A_1 \\
&\quad - 4A_2 k^2 + 4k^2 k_3, \\
\{\{A_2, B_2\}, B_2\} &= \frac{\partial F_2}{\partial A_2} = -4B_2^2 + 4A_1 B_2 - 8A_2 B_2 - 4k_1 B_2 + 4k_2 B_2 \\
&\quad - 4k_3 B_2 + 8A_1 k_1 - 8A_2 k_1 - 8A_2 k_3 + 8k_2 k_3, \\
\{A_2, \{A_2, B_2\}\} &= \frac{\partial F_2}{\partial B_2} = -4A_2^2 + 4A_1 A_2 - 8B_2 A_2 - 4k_1 A_2 + 4k_2 A_2 \\
&\quad - 4k_3 A_2 + 4A_1 k_1 - 4A_1 k_2 - 4k_1 k_3 + 4k_2 k_3, \\
\{\{F, A_1\}, A_1\} &= \frac{\partial F_3}{\partial F} = 16H A_1^2 + 4k^2 A_1 - 8F A_1 - 16B_2 H A_1 - 16H k_1 A_1 \\
&\quad + 16H k_2 A_1 - 16H k_3 A_1 - 4B_2 k^2 - 4k^2 k_1 + 4k^2 k_2 - 4k^2 k_3, \\
\{F, \{F, A_1\}\} &= \frac{\partial F_3}{\partial A_1} = -4F^2 + 4k^2 F + 32A_1 H F - 16B_2 H F - 16H k_1 F \\
&\quad + 16H k_2 F - 16H k_3 F - 128A_1 H^2 k_2 - 32H k^2 k_2,
\end{aligned}$$

$$\begin{aligned}
\{A_1, \{B_1, B_2\}\} &= \{\{A_1, B_1\}, B_2\} = -16HA_1^2 - 4k^2A_1 + 4B_1A_1 + 4FA_1 \\
&+ 16A_2HA_1 - 16Hk_3A_1 + 4A_2k^2 + 4B_1B_2 - 4A_2F + 4B_1k_1 - 4B_1k_2 \\
&- 4k^2k_3 + 4B_1k_3 + 4Fk_3, \\
\{\{A_2, B_2\}, B_1\} &= \{\{B_1, B_2\}, A_2\} = 16HA_1^2 + 4k^2A_1 - 4B_1A_1 - 4FA_1 \\
&- 16A_2HA_1 - 16B_2HA_1 - 16Hk_3A_1 - 4A_2k^2 - 4B_2k^2 + 4A_2B_1 + 4B_1B_2 \\
&+ 4A_2F - 4k^2k_3 + 4B_1k_3 + 4Fk_3, \\
\{A_2, \{F, A_1\}\} &= \{A_1, \{F, A_2\}\} = -16HA_1^2 - 4k^2A_1 + 4B_1A_1 + 4FA_1 \\
&+ 16B_2HA_1 + 16Hk_1A_1 - 16Hk_2A_1 + 16Hk_3A_1 + 4B_2k^2 - 4B_1B_2 + 4A_2F \\
&+ 4k^2k_1 - 4B_1k_1 - 4k^2k_2 + 4B_1k_2 + 4k^2k_3 - 4B_1k_3 - 4Fk_3, \\
\{\{B_1, F\}, A_1\} &= -16A_1B_1H + 16B_1B_2H + 16A_1FH - 16A_2FH \\
&+ 16B_1k_1H - 16B_1k_2H + 16B_1k_3H + 16Fk_3H, \\
\{\{A_1, B_1\}, F\} &= -64A_1^2H^2 + 64A_1A_2H^2 + 64A_1B_2H^2 + 64A_1k_3H^2 \\
&- 16A_1k^2H + 16A_2k^2H + 16B_2k^2H + 16A_1B_1H - 16B_1B_2H - 16B_1k_1H \\
&+ 16B_1k_2H + 16k^2k_3H - 16B_1k_3H - 4B_1F, \\
\{\{A_1, F\}, B_1\} &= -4(16A_1^2H^2 - 16A_1A_2H^2 - 16A_1B_2H^2 - 16A_1k_3H^2 \\
&+ 4A_1k^2H - 4A_2k^2H - 4B_2k^2H - 4A_1FH + 4A_2FH \\
&- 4k^2k_3H - 4Fk_3H + B_1F), \\
\{B_1, \{F, A_1\}\} &= \{\{B_1, F\}, A_2\} = -64A_1^2H^2 + 64A_1A_2H^2 + 64A_1B_2H^2 \\
&+ 64A_1k_3H^2 - 16A_1k^2H + 16A_2k^2H + 16B_2k^2H + 16A_1B_1H - 16A_2B_1H \\
&- 16B_1B_2H + 16A_1FH - 16A_2FH + 16k^2k_3H - 16B_1k_3H - 16Fk_3H, \\
\{\{[-0.2ex]A_2, B_2\}, F\} &= \{B_2, M\} = -16HA_1^2 - 4k^2A_1 + 4B_1A_1 + 4FA_1 \\
&+ 16A_2HA_1 + 16B_2HA_1 + 16Hk_3A_1 + 4A_2k^2 + 4B_2k^2 - 4B_1B_2 - 4A_2F \\
&- 4B_2F - 4B_1k_1 - 4B_1k_2 + 4k^2k_3 - 4B_1k_3 - 4Fk_3, \\
\{\{B_1, B_2\}, F\} &= \{\{B_1, F\}, B_2\} = 64A_1^2H^2 - 64A_1A_2H^2 - 64A_1B_2H^2 \\
&- 64A_1k_3H^2 + 16A_1k^2H - 16A_2k^2H - 16B_2k^2H - 16A_1B_1H + 16B_1B_2H \\
&- 16A_1FH + 16A_2FH + 16B_2FH + 16B_1k_1H + 16B_1k_2H - 16k^2k_3H \\
&+ 16B_1k_3H + 16Fk_3H, \\
\{\{B_1, F\}, F\} &= 64A_1FH^2 - 64A_2FH^2 - 64B_2FH^2 - 128B_1k_2H^2 \\
&- 64Fk_3H^2 + 16B_1FH, \\
\{\{B_1, F\}, B_1\} &= -64A_1B_1H^2 + 64A_2B_1H^2 + 64B_1B_2H^2 \\
&+ 64B_1k_3H^2 + 128Fk_3H^2 - 16B_1FH, \\
\{F, \{F, A_2\}\} &= -4F^2 + 4k^2F - 4B_1F + 16A_1HF + 16A_2HF - 16Hk_1F \\
&+ 16Hk_2F - 128A_1H^2k_2 - 32Hk^2k_2 + 32B_1Hk_2, \\
\{A_2, \{F, A_2\}\} &= 4A_2k^2 - 4k_1k^2 + 4k_2k^2 - 4A_2B_1 - 8A_2F + 16A_1A_2H \\
&+ 4B_1k_1 - 16A_1Hk_1 - 4B_1k_2 + 16A_1Hk_2, \\
\{B_1, \{B_1, B_2\}\} &= -4B_1^2 + 4k^2B_1 - 4FB_1 + 16A_1HB_1 + 16B_2HB_1 \\
&+ 32Hk_3B_1 - 128A_1H^2k_3 - 32Hk^2k_3 + 32FHK_3, \\
\{B_2, \{B_1, B_2\}\} &= -4B_2k^2 - 8k_3k^2 + 8B_1B_2 + 4B_2F - 16A_1B_2H \\
&+ 8B_1k_1 + 8B_1k_3 + 8Fk_3 - 32A_1Hk_3.
\end{aligned}$$

4 Conclusions

The three dimensional non degenerate Kepler–Coulomb potential [2] satisfy a ternary parafermionic-like fourth order Poisson algebra with quartic and quadratic integrals. The systems have three subalgebras forming a special “ Π ” structure. Each subalgebra corresponds to classical superintegrable system possessing two Hamiltonians. The example of the non degenerate Kepler–Coulomb system indicates that probably the known degenerate three dimensional potentials are related to non degenerate systems with integrals of motion of order greater than two.

There is no results yet about the quantum superintegrable systems as also there is not a compact general classification theory for three dimensional superintegrable potentials. The structure of the corresponding Poisson algebras for the degenerate systems is under investigation.

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Group classification of nonlinear fourth-order parabolic equations

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The symmetry classification of a nonlinear fourth-order parabolic equation is performed. It is showed that this equation does not admit potential symmetries and some exact solutions are found.

1 Introduction

In this paper we consider the following fourth-order nonlinear parabolic equation

$$u_t + f(u)u_{xxxx} + f'(u)u_xu_{xxx} - g(u)u_{xx} - g'(u)u_x^2 = 0 \quad (1)$$

where f and g are smooth functions of u with $f \neq 0$. The subscripts t and x represent the partial derivatives with respect t and x respectively. Here and afterwards we use prime to denote derivatives when the function depends on the only one independent variable (e.g. $f'(u) = f_u$).

The equation (1) is the one-dimensional version of

$$u_t = -\nabla \cdot (f(u)\nabla \Delta u) - \nabla \cdot (g(u)\nabla u) \quad (2)$$

which is used to model the dynamics of a thin film of viscous liquid (for an overview see [1] and the references therein). The air/liquid interface is at height $z = u(x, y, t)$ and the liquid/solid interface is at $z = 0$. The one-dimensional equation applies if the liquid film is uniform in the y direction. The fourth-order term of (1) reflects effects of surface tension and also incorporates any slippage at the liquid/solid interface. A typical form is $f(u) = u^3 + \lambda u^p$ where $0 < p < 3$ and $\lambda \geq 0$ determines a slip length. The coefficient of the second-order term can reflect additional forces such as gravity ($g(u) = u^3$), Van der Waals interactions ($g(u) = u^m$, $m < 0$) or thermocapillary effects ($g(u) = u^2/(1 \pm cu)^2$).

Also equation (1) arises in the gravity-driven Hele-Shaw cell, for which $f(u) = g(u) = u$ [2, 3].

When $f(u) = 1$, the equation (1) is the extensively studied Cahn-Hilliard equation

$$u_t = -u_{xxxx} + (g(u)u_x)_x$$

that describes phase separation in binary alloys and is very important in materials science [4–6]. Symmetry analysis for the Cahn-Hilliard equation has been performed in [7–10].

Setting $f(u) = 1$ and $g(u) = -1 - 2u$ the equation (1) becomes the Childress-Spiegel equation

$$u_t = -(u_{xx} + u + u^2)_{xx},$$

that arises as an interface model in biofluids [11], solar convection [12] and binary alloys [13].

When $f(u) = u^p$ and $g(u) = 0$, equation (1) becomes

$$u_t = -(u^p u_{xxx})_x, \quad (3)$$

that appears in numerous applications, for example in the description of the motion of a very thin layer of viscous in compressible fluids along an inclined plane where the variable u represents thickness of the film [14, 15]. For $p = 1$ this equation arises in the modeling of breakup of a droplet in a Hele-Shaw cell, where the variable u describes the thickness of a neck between two masses of fluid. Symmetry reductions of a generalization of equation (3), that is of equation (1) with $g(u) = 0$, can be found in [16].

Equation (1) can be written in a conserved form

$$u_t + (f(u)u_{xxx} - g(u)u_x)_x = 0,$$

and the associated auxiliary system is given by

$$v_x = u, \quad v_t = -f(u)u_{xxx} + g(u)u_x. \quad (4)$$

If $(u(t, x), v(t, x))$ satisfies system (4), then $u(t, x)$ solves equation (1) and $v(t, x)$ solves an integrated equation of (1)

$$v_t + f(v_x)v_{xxxx} - g(v_x)v_{xx} = 0.$$

The paper is set out as follows: in the first part of Section 2, by using the infinitesimal method, we get the group classification of equation (1) that it is possible to find also in [17, 18]. In the second part of Section 2, in similar way, we obtain the group classification of system (4). Symmetry reductions and some exact solutions for equation (1) are presented in Section 3. The conclusions are made in Section 4.

2 Symmetry classifications

2.1 Symmetry classifications of equation (1)

We apply the classical Lie method in order to look for the infinitesimal generator Y of Lie group of point transformations with the form

$$Y = \xi^1(x, t, u)\partial_x + \xi^2(x, t, u)\partial_t + \eta(x, t, u)\partial_u. \quad (5)$$

Then we require that equation (1) be invariant with respect the fourth prolongation of the operator (5)

$$Y^{(4)} = Y + \zeta_1 \partial_{u_x} + \zeta_{11} \partial_{u_{xx}} + \zeta_{111} \partial_{u_{xxx}} + \zeta_{1111} \partial_{u_{xxxx}} + \zeta_2 \partial_{u_t}, \quad (6)$$

where

$$\begin{aligned} \zeta_1 &= D_x(\eta) - u_x D_x(\xi^1) - u_t D_x(\xi^2), \\ \zeta_2 &= D_t(\eta) - u_x D_t(\xi^1) - u_t D_t(\xi^2), \\ \zeta_{11} &= D_x(\zeta_1) - u_{xx} D_x(\xi^1) - u_{xt} D_x(\xi^2), \\ \zeta_{111} &= D_x(\zeta_{11}) - u_{xxx} D_x(\xi^1) - u_{xxt} D_x(\xi^2), \\ \zeta_{1111} &= D_x(\zeta_{111}) - u_{xxxx} D_x(\xi^1) - u_{xxxx} D_x(\xi^2), \end{aligned}$$

and D_x , D_t denote the total derivatives with respect to x and t . For additional details see e.g. [19–21].

When we apply the operator (6) to equation (1), the invariance condition is

$$Y^{(4)} (u_t + (f(u)u_{xxx})_x - (g(u)u_x)_x) = 0$$

under the constraint that the variable u must satisfy equation (1). In the usual way we get the determining system in the unknown ξ^1 , ξ^2 and η .

In order to write the classifying equations we must distinguish the case $f' = 0$ and $f' \neq 0$. In the first case, if $f' = 0$, that is $f(u) = f_0$, we have

$$\xi^1 = \frac{x\xi_t^2}{4} + \beta_1(t), \quad \xi^2 = \xi^2(t), \quad \eta = u\alpha_1(t) + \alpha_2(x, t)$$

and the following classifying equations

$$\begin{aligned} 2\alpha_2 g' + 2u\alpha_1 g' + \xi_t^2 g &= 0, & 8\alpha_2 x g' + x\xi_{tt}^2 + 4\beta_1 t &= 0, \\ \alpha_2 x x g - \alpha_2 x x x x f_0 - \alpha_2 t - u\alpha_1 t &= 0. \end{aligned}$$

In the case $f' \neq 0$ we have $\xi^1 = c_3 x + c_4$, $\xi^2 = c_5 t + c_6$, $\eta = c_1 u + c_2$ and the following classifying equations

$$\begin{aligned} c_2 f' + c_1 u f' + c_5 f - 4c_3 f &= 0, \\ c_2 f f'' + c_1 u f f'' - c_2 f'^2 - c_1 u f'^2 + c_1 f f' &= 0, \\ c_2 f g' + c_1 u f g' - c_2 g f' - c_1 u g f' + 2c_3 f g &= 0, \\ c_2 f g'' + c_1 u f g'' - c_2 f' g' - c_1 u f' g' + 2c_3 f g' + c_1 f g' &= 0. \end{aligned}$$

The principal Lie algebra \mathcal{L} for the class (1) is obtained when we consider f and g arbitrary functions, it is two-dimensional and is spanned by

$$Y_1 = \partial_t, \quad Y_2 = \partial_x.$$

Only when the functional forms of f and g are

$$\begin{aligned} f(u) &= f_0(u+k)^{k_1}, & g(u) &= g_0(u+k)^{k_2}, & \text{or} \\ f(u) &= f_0 e^{f_1 u}, & g(u) &= g_0 e^{g_1 u} \end{aligned} \quad (7)$$

with $f_0, f_1, g_0, g_1, k, k_1$ and k_2 constants we obtain extensions of the principal Lie algebra. Without loss of generality we can take in (7) $k = 0$.

For these forms of f and g we obtain that the admitted group of symmetries have generators:

$$1. \quad f(u) = f_0 u^{k_1} \text{ and } g(u) = g_0 u^{k_2}$$

$$Y_1, Y_2, Y_3 = \frac{k_1 - k_2}{2} x \partial_x + (k_1 - 2k_2) t \partial_t + u \partial_u. \quad (8)$$

$$2. \quad f(u) = f_0 e^{k_1 u} \text{ and } g(u) = g_0 e^{k_2 u}$$

$$Y_1, Y_2, Y_3 = \frac{k_1 - k_2}{2} x \partial_x + (k_1 - 2k_2) t \partial_t + \partial_u. \quad (9)$$

$$3. \quad f(u) = f_0 \text{ and } g(u) = g_0$$

$$Y_1, Y_2, Y_3 = u \partial_u, Y_\alpha = \alpha(t, x) \partial_u,$$

where the function $\alpha(t, x)$ satisfies equation $\alpha_t + f_0 \alpha_{xxxx} - g_0 \alpha_{xx} = 0$.

In all cases, when $g(u) = 0$, the corresponding algebra is expanded by the operator

$$Y_0 = x \partial_x + 4t \partial_t.$$

2.2 Symmetry classifications of system (4)

A Lie point symmetry admitted by system (4) is a symmetry characterized by an infinitesimal generator of the form

$$X = \tau^1(x, t, u, v) \partial_x + \tau^2(x, t, u, v) \partial_t + \mu^1(x, t, u, v) \partial_u + \mu^2(x, t, u, v) \partial_v. \quad (10)$$

This group maps any solution of system (4) to another solution of (4) and hence induces a mapping of any solution of equation (1) to another solution of (1). Thus (10) defines a symmetry group of equation (1).

If $(\tau_v^1)^2 + (\tau_v^2)^2 + (\mu_v^1)^2 \neq 0$ then (10) yields a nonlocal symmetry of (1). Such a nonlocal symmetry is called a potential symmetry of equation (1).

In the usual way we get the determining system in the unknown $\tau^1, \tau^2, \mu^1, \mu^2$.

If $f' = 0$, that is $f(u) = f_0$, we have

$$\tau^1 = \frac{c_2 x}{4} + c_1, \quad \tau^2 = c_2 t + c_3, \quad \mu^1 = u \left(c_0 - \frac{c_2}{4} \right) + \beta_x, \quad \mu^2 = c_0 v + \beta(t, x)$$

and the following classifying equations

$$(c_2u - 4c_0u - 4\beta_x)g' - 2c_2g = 0, \beta_{xx}g - \beta_{xxxx}f_0 - \beta_t = 0.$$

In the case $f' \neq 0$ we have

$$\tau^1 = c_1x + c_3, \tau^2 = c_2t + c_6, \mu^1 = (c_0 - c_1)u + c_4, \mu^2 = vc_0 + c_4x + c_5$$

and the following classifying equations

$$\begin{aligned} (c_4 - uc_1 + uc_0)f' + (c_2 - 4c_1)f &= 0, \\ (c_4 - c_1u + c_0u)fg' + (uc_1 - c_0u - c_4)gf' + 2c_1fg &= 0. \end{aligned}$$

Then we obtain the following group classification of the system (4):

1. $f(u)$ and $g(u)$ arbitrary

$$X_1 = \partial_x, X_2 = \partial_t, X_3 = \partial_v.$$

2. $f(u) = f_0u^{k_1}$ and $g(u) = g_0u^{k_2}$

$$X_1, X_2, X_3, X_4 = (k_1 - k_2)x\partial_x + 2(k_1 - 2k_2)t\partial_t + 2u\partial_u + (k_1 - k_2 + 2)v\partial_v.$$

3. $f(u) = f_0e^{k_1u}$ and $g(u) = g_0e^{k_2u}$

$$X_1, X_2, X_3, X_4 = \frac{k_1 - k_2}{2}x\partial_x + (k_1 - 2k_2)t\partial_t + \partial_u + \left(\frac{k_1 - k_2}{2}v + x\right)\partial_v.$$

4. $f(u) = f_0$ and $g(u) = g_0$

$$X_1, X_2, X_3, X_4 = u\partial_u + v\partial_v, X_\beta = \beta_x\partial_u + \beta\partial_v,$$

where the function $\beta(t, x)$ satisfies:

$$\beta_t + f_0\beta_{xxxx} - g_0\beta_{xx} = 0.$$

In all cases, when $g(u) = 0$, the corresponding algebra is expanded by the operator

$$X_0 = x\partial_x + 4t\partial_t + v\partial_v.$$

We can deduce easily that the point symmetries of the system (4) do not produce any nonlocal symmetry of equation (1), that is, equation (1) does not admit potential symmetries.

3 Symmetry reductions and exact solutions

When $f(u)$ and $g(u)$ are arbitrary, the only symmetries admitted by (1) are the group of space and time translations. In this case we obtain travelling wave reductions,

$$\sigma = x - \lambda t, \quad u = \phi(\sigma),$$

where λ is an arbitrary constant and the function $\phi(\sigma)$, after integrating once with respect to σ , satisfies the following ODE

$$f(\phi)\phi''' - g(\phi)\phi' - \lambda\phi = a_1 \quad (11)$$

where a_1 is a constant of integration.

Now we consider the only functional forms of $f(u)$ and $g(u)$ for which (1) has extra symmetries, but we do not consider the cases where $f(u) = f_0$ or $g(u) = 0$ because in these cases similarity solutions can be found in [7–10] and [16] respectively.

3.1 Case 1: $f(u) = f_0 u^{k_1}$ and $g(u) = g_0 u^{k_2}$

We suppose $k_1 \neq 0$ and $g_0 \neq 0$. In this case equation (11) becomes

$$f_0\phi^{k_1}\phi''' - g_0\phi^{k_2}\phi' - \lambda\phi = a_1.$$

If we choose $a_1 = 0$, in the case $k_1 = k_2 = 1$ we obtain the solution

$$\phi(\sigma) = a_2 \sqrt{\frac{f_0}{g_0}} e^{\sqrt{\frac{g_0}{f_0}}\sigma} + a_3 \sqrt{\frac{f_0}{g_0}} e^{-\sqrt{\frac{g_0}{f_0}}\sigma} - \frac{\lambda\sigma}{g_0} + a_4,$$

where a_2 , a_3 and a_4 are arbitrary constants. Then we obtain the following travelling wave solution

$$u(t, x) = a_2 \sqrt{\frac{f_0}{g_0}} e^{\sqrt{\frac{g_0}{f_0}}(x - \lambda t)} + a_3 \sqrt{\frac{f_0}{g_0}} e^{-\sqrt{\frac{g_0}{f_0}}(x - \lambda t)} - \frac{\lambda(x - \lambda t)}{g_0} + a_4$$

as solution of equation

$$u_t + (f_0 u u_{xxx})_x - (g_0 u u_x)_x = 0. \quad (12)$$

From the generator Y_3 of the algebra (8), that is

$$Y_3 = \frac{k_1 - k_2}{2} x \partial_x + (k_1 - 2k_2) t \partial_t + u \partial_u,$$

in order to obtain the similarity variable and the similarity solutions we distinguish two cases:

1. In the case $k_1 \neq 2k_2$:

$$\sigma = xt^{-\frac{1}{2}\frac{k_1-k_2}{k_1-2k_2}}, \quad u = t^{\frac{1}{k_1-2k_2}}\phi(\sigma),$$

where $\phi(\sigma)$ satisfies the ODE

$$\frac{2\phi - \sigma(k_1 - k_2)\phi'}{2(k_1 - 2k_2)} + (f_0\phi^{k_1}\phi''')' - (g_0\phi^{k_2}\phi')' = 0. \quad (13)$$

If $k_1 = k_2 - 2$ ($k_2 \neq -2$), this equation can be integrated

$$\frac{\sigma\phi}{-k_2 - 2} + f_0\phi^{k_2-2}\phi''' - g_0\phi^{k_2}\phi' + a_1 = 0$$

where a_1 is a constant of integration.

If $k_1 = k_2$ equation (13) becomes

$$-\frac{\phi}{k_2} + (f_0\phi^{k_2}\phi''')' - (g_0\phi^{k_2}\phi')' = 0$$

and in the case that $k_1 = k_2 = 1$ a particular solution is

$$\phi(\sigma) = -\frac{3}{2}g_0a_1^2 + a_1\sigma - \frac{1}{6g_0}\sigma^2$$

with a_1 arbitrary constant. Then we obtain

$$u(t, x) = \left[-\frac{3}{2}g_0a_1^2 + a_1x - \frac{1}{6g_0}x^2 \right] \frac{1}{t}$$

as solution of equation (12).

2. In the case $k_1 = 2k_2$, $\sigma = t$, $u = x^{\frac{2}{k_2}}\phi(\sigma)$, where $\phi(\sigma)$ satisfies the ODE

$$\phi' + \frac{4f_0\phi^{2k_2+1}(k_2-2)(k_2-1)(k_2+2)}{k_2^4} - \frac{2g_0\phi^{k_2+1}(k_2+2)}{k_2^2} = 0.$$

This is a first-order equation the solution of which can be found in implicit form. For particular cases we obtain solutions in explicit form. As examples

(a) If $k_2 = 1$, we obtain $u(t, x) = \frac{x^2}{a_1-6g_0t}$, where a_1 is an arbitrary constant, as solution of the equation

$$u_t + (f_0u^2u_{xxx})_x - (g_0uu_x)_x = 0.$$

(b) If $k_2 = 2$, we obtain $u(t, x) = \pm\frac{x}{\sqrt{a_1-4g_0t}}$, where a_1 is an arbitrary constant, as solution of the equation

$$u_t + (f_0u^4u_{xxx})_x - (g_0u^2u_x)_x = 0.$$

(c) If $k_2 = -2$, we obtain $u(t, x) = \frac{a_1}{x}$, with a_1 an arbitrary constant, as solution of the equation

$$u_t + (f_0u^{-4}u_{xxx})_x - (g_0u^{-2}u_x)_x = 0.$$

3.2 Case 2: $f(u) = f_0 e^{k_1 u}$ and $g(u) = g_0 e^{k_2 u}$

We suppose that $k_1 \neq 0$ and $g_0 \neq 0$. From the generator Y_3 of the algebra (9), that is

$$Y_3 = \frac{k_1 - k_2}{2} x \partial_x + (k_1 - 2k_2) t \partial_t + \partial_u,$$

we have the similarity variable and the similarity solutions of the form:

1. In the case $k_1 \neq 2k_2$, $\sigma = xt^{-\frac{1}{2} \frac{k_1 - k_2}{k_1 - 2k_2}}$, $u = \frac{\ln(t)}{k_1 - 2k_2} + \phi(\sigma)$, where $\phi(\sigma)$ satisfies the ODE

$$\frac{2 - \sigma(k_1 - k_2)\phi'}{2(k_1 - 2k_2)} + (f_0 e^{k_1 \phi} \phi''')' - (g_0 e^{k_2 \phi} \phi')' = 0.$$

If $k_1 = k_2$, this equation can be integrated

$$\frac{\sigma}{k_1} - f_0 e^{k_1 \phi} \phi''' + g_0 e^{k_1 \phi} \phi' + a_1 = 0$$

where a_1 is a constant of integration.

2. In the case $k_1 = 2k_2$, $\sigma = t$, $u = \frac{2 \ln(x)}{k_2} + \phi(\sigma)$, where $\phi(\sigma)$ satisfies the ODE

$$\phi' + \frac{4f_0 e^{2k_2 \phi}}{k_2} - \frac{2g_0 e^{k_2 \phi}}{k_2} = 0. \quad (14)$$

By setting $\phi(s) = \ln(\psi(s))$ equation (14) becomes

$$\psi' + \frac{4f_0}{k_2} \psi^{2k_2+1} - \frac{2g_0}{k_2} \psi^{k_2+1} = 0,$$

the solution of which can be found in implicit form.

4 Conclusions

In this paper we have performed the symmetry group classifications of the one-dimensional equation (1) to model the dynamics of a thin film of viscous liquid. Equation (1) can be written in a conserved form. Then by considering the associated auxiliary system (4) we proved that equation (1) does not admit potential symmetries.

Further by using the symmetry reductions we obtain some exact solutions for particular forms of the arbitrary functions that appear.

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Reduction operators of variable coefficient semilinear diffusion equations with a power source

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Reduction operators (called often nonclassical symmetries) of variable-coefficient semilinear reaction-diffusion equations with power nonlinearity $f(x)u_t = (g(x)u_x)_x + h(x)u^m$ ($m \neq 0, 1, 2$) are investigated using the algorithm suggested in [O.O. Vaneeva, R.O. Popovych and C. Sophocleous, *Acta Appl. Math.*, 2009, V.106, 1–46; arXiv:0708.3457].

1 Introduction

As early as in 1969 Bluman and Cole introduced a new method for finding group-invariant (called also similarity) solutions of partial differential equations (PDEs) [3]. It was called by the authors “non-classical” to emphasize the difference between it and the “classical” Lie reduction method described, e.g., in [16,17]. A precise and rigorous definition of nonclassical invariance was firstly formulated in [11] as “a generalization of the Lie definition of invariance” (see also [26]). Later operators satisfying the nonclassical invariance criterion were called, by different authors, nonclassical symmetries, conditional symmetries and Q -conditional symmetries [8,10,15]. Until now all names are in use. Following [19] we call nonclassical symmetries *reduction operators*. The necessary definitions, including ones of equivalence of reduction operators, and relevant statements on this subject are collected in [22].

The problem of finding reduction operators for a PDE is more complicated than the similar problem on Lie symmetries because the first problem is reduced to the integration of an overdetermined system of nonlinear PDEs, whereas in the case of Lie symmetries one deals with a more overdetermined system of linear

PDEs. The complexity increases in times in the case of classification problem of reduction operators for a class of PDEs having nonconstant arbitrary elements.

Often the usage of equivalence and gauging transformations can essentially simplify the group classification problem. Moreover, their implementation can appear to be a crucial point in solving the problem. This observation is justified by a number of examples [12, 21, 22]. The above transformations are of major importance for studying reduction operators since under their classification one needs to surmount much more essential obstacles then those arising under the classification of Lie symmetries.

In [22] simultaneous usage of equivalence transformations and mappings between classes allowed us to carry out group classification of the class of variable coefficient semilinear reaction–diffusion equations with power nonlinearity

$$f(x)u_t = (g(x)u_x)_x + h(x)u^m, \quad (1)$$

where $f = f(x)$, $g = g(x)$ and $h = h(x)$ are arbitrary smooth functions of the variable x , $f(x)g(x)h(x) \neq 0$, m is an arbitrary constant ($m \neq 0, 1$).

In the same paper an algorithm for finding reduction operators of class (1) involving mapping between classes was proposed. Here using this algorithm we investigate reduction operators of the equations from class (1) with $m \neq 2$. The case $m = 2$ will not be systematically considered since it is singular from the Lie symmetry point of view and needs an additional mapping between classes (see [22] for more details). Nevertheless, all the reduction operators constructed for the general case $m \neq 0, 1, 2$ are also fit for the values $m = 0, 1, 2$.

The structure of this paper is as follows. For convenience of readers sections 2–4 contain a short review of results obtained in [22] and used here. Namely, in section 2 all necessary information concerning equivalence transformations and mapping of class (1) to the so-called “imaged” class is collected. Results on group classification and additional equivalence transformations of the imaged class are also presented. Section 3 describes the algorithm for finding reduction operators of class (1) using mapping between classes. In section 4 known reduction operators of constant-coefficient equations from the imaged class are considered. Their preimages are obtained. The results of sections 5 and 6 are completely original and concern the investigation of reduction operators for equations from the imaged class which have at least one nonconstant arbitrary element. It appears that application of the reduction method to equations from the imaged class with $m = 3$ leads in some cases to necessity of solving first-order nonlinear ODEs of a special form related to Jacobian elliptic functions. The table with solutions of the ordinary differential equations (ODEs) of this kind is placed in Appendix.

2 Lie symmetries and equivalence transformations

To produce group classification of class (1), it is necessary to gauge arbitrary elements of this class with equivalence transformations and subsequent mapping of it to a simpler class [22].

Theorem 1. *The generalized extended equivalence group \hat{G}^\sim of class (1) is formed by the transformations*

$$\begin{aligned}\tilde{t} &= \delta_1 t + \delta_2, \quad \tilde{x} = \varphi(x), \quad \tilde{u} = \psi(x)u, \\ \tilde{f} &= \frac{\delta_0 \delta_1}{\varphi_x \psi^2} f, \quad \tilde{g} = \frac{\delta_0 \varphi_x}{\psi^2} g, \quad \tilde{h} = \frac{\delta_0}{\varphi_x \psi^{m+1}} h, \quad \tilde{m} = m,\end{aligned}$$

where φ is an arbitrary smooth function of x , $\varphi_x \neq 0$ and ψ is determined by the formula $\psi(x) = (\delta_3 \int \frac{dx}{g(x)} + \delta_4)^{-1}$. δ_j ($j = 0, 1, 2, 3, 4$) are arbitrary constants, $\delta_0 \delta_1 (\delta_3^2 + \delta_4^2) \neq 0$.

The usual equivalence group G^\sim of class (1) is the subgroup of the generalized extended equivalence group \hat{G}^\sim , which is singled out with the condition $\delta_3 = 0$.

The presence of the arbitrary function $\varphi(x)$ in the equivalence transformations allows us to simplify the group classification problem of class (1) via reducing the number of arbitrary elements and making its more convenient for mapping to another class.

Thus, the transformation from the equivalence group G^\sim

$$t' = \text{sign}(f(x)g(x))t, \quad x' = \int |f(x)g(x)|^{-1}^{\frac{1}{2}} dx, \quad u' = u \quad (2)$$

maps class (1) onto its subclass $f'(x')u'_{t'} = (f'(x')u'_{x'})_{x'} + h'(x')u'^{m'}$ with the new arbitrary elements $m' = m$, $f'(x') = g'(x') = \text{sign}(g(x))|f(x)g(x)|^{\frac{1}{2}}$ and $h'(x') = |g(x)f(x)|^{-1}^{\frac{1}{2}} h(x)$. Without loss of generality, we can restrict ourselves to study the class

$$f(x)u_t = (f(x)u_x)_x + h(x)u^m, \quad (3)$$

since all results on symmetries and exact solutions for this class can be extended to class (1) with transformation (2).

It is easy to deduce the generalized extended equivalence group for class (3) from theorem 1 by setting $\tilde{f} = \tilde{g}$ and $f = g$. See theorem 4 in [22].

The next step is to make the change of the dependent variable

$$v(t, x) = \sqrt{|f(x)|}u(t, x) \quad (4)$$

in class (3). As a result, we obtain the class of related equations of the form

$$v_t = v_{xx} + H(x)v^m + F(x)v, \quad (5)$$

where the new arbitrary elements F and H are connected with the old ones via the formulas

$$F(x) = -\frac{(\sqrt{|f(x)|})_{xx}}{\sqrt{|f(x)|}}, \quad H(x) = \frac{h(x) \text{sign } f(x)}{(\sqrt{|f(x)|})^{m+1}}. \quad (6)$$

Since class (5) is an image of class (3) with respect to the family of transformations (4) parameterized by the arbitrary element f , we call them the *imaged class* and the *initial class*, respectively.

Theorem 2. *The generalized extended equivalence group \hat{G}_{FH}^\sim of class (5) coincides with the usual equivalence group G_{FH}^\sim of the same class and is formed by the transformations*

$$\tilde{t} = \delta_1^2 t + \delta_2, \quad \tilde{x} = \delta_1 x + \delta_3, \quad \tilde{v} = \delta_4 v,$$

$$\tilde{F} = \frac{F}{\delta_1^2}, \quad \tilde{H} = \frac{H}{\delta_1^2 \delta_4^{m-1}}, \quad \tilde{m} = m,$$

where δ_j , $j = 1, \dots, 4$, are arbitrary constants, $\delta_1 \delta_4 \neq 0$.

The following important proposition is proved in [22].

Proposition 1. *The group classification in class (1) with respect to its generalized extended equivalence group \hat{G}^\sim is equivalent to the group classification in class (5) with respect to the usual equivalence group G_{FH}^\sim of this class. A classification list for class (1) can be obtained from a classification list for class (5) by means of taking a single preimage for each element of the latter list with respect to the resulting mapping from class (1) onto class (5).*

All possible G_{FH}^\sim -inequivalent values of the parameter-functions F and H for which equations (5) admit extension of Lie symmetry are listed in table 1 together with bases of the corresponding maximal Lie invariance algebras.

Table 1. The group classification of the class $v_t = v_{xx} + H(x)v^m + F(x)v$.
 $m \neq 0, 1, 2$; $H(x) \neq 0$.

N	$H(x)$	$F(x)$	Basis of A^{\max}
0	\forall	\forall	∂_t
1	δe^{qx}	a_1	$\partial_t, \partial_x + \alpha v \partial_v$
2	δe^{qx}	$-\alpha^2$	$\partial_t, \partial_x + \alpha v \partial_v, 2t\partial_t + (x - 2\alpha t)\partial_x + \left(\alpha(x - 2\alpha t) + \frac{2}{1-m}\right)v\partial_v$
3	δx^k	$a_2 x^{-2}$	$\partial_t, 2t\partial_t + x\partial_x + \frac{k+2}{1-m}v\partial_v$
4	$\delta x^k e^{px^2}$	$-\beta^2 x^2 + a_2 x^{-2} + \gamma$	$\partial_t, e^{4\beta t}\partial_t + 2\beta x e^{4\beta t}\partial_x - 2\beta e^{4\beta t}\left(\beta x^2 - \frac{k+2}{1-m}\right)v\partial_v$
5	δe^{px^2}	$-\beta^2 x^2 + \beta a_3$	$\partial_t, e^{2\beta t}[\partial_x - \beta x v \partial_v]$
6	δe^{px^2}	$-\beta^2 x^2 + \beta \frac{5-m}{1-m}$	$\partial_t, e^{2\beta t}[\partial_x - \beta x v \partial_v], e^{4\beta t}\left[\partial_t + 2\beta x \partial_x - 2\beta\left(\beta x^2 - \frac{2}{1-m}\right)v\partial_v\right]$

Here $\alpha, \beta, \gamma, \delta, k, p, q, a_1, a_2, a_3$ are constants satisfying the conditions: $\alpha = \frac{q}{1-m}$, $\beta = \frac{2p}{m-1}$, $\gamma = \beta \frac{2k+5-m}{1-m}$, $\delta = \pm 1 \bmod G_{FH}^\sim$; $p \neq 0$, $a_1 \neq -\alpha^2$, $k^2 + a_2^2 \neq 0$, $q^2 + a_1^2 \neq 0$, $a_3 \neq \frac{5-m}{1-m}$.

The results on group classification of class (3) can be found in table 3 of [22].

Additional equivalence transformations between G_{FH}^\sim -inequivalent cases of Lie symmetry extension are also constructed. The independent pairs of point-equivalent cases from table 1 and the corresponding transformations are exhausted by the following:

$$\begin{aligned} 1 &\mapsto \tilde{1}|_{\tilde{q}=0, \tilde{a}_1=a_1+\alpha^2}, \quad 2 \mapsto \tilde{2}|_{\tilde{q}=0}: \quad \tilde{t} = t, \quad \tilde{x} = x + 2\alpha t, \quad \tilde{v} = e^{-\alpha x} v; \\ 4 &\mapsto \tilde{3}: \quad \tilde{t} = -\frac{1}{4\beta} e^{-4\beta t}, \quad \tilde{x} = e^{-2\beta t} x, \quad \tilde{v} = \exp\left(\frac{\beta}{2} x^2 + 2\beta \frac{k+2}{m-1} t\right) v; \quad (7) \\ 6 &\mapsto \tilde{2}|_{\tilde{q}=0}: \quad \text{the previous transformation with } k=0. \end{aligned}$$

The whole set of form-preserving [13] (also called admissible [18]) transformations of the imaged class for the case $m \neq 0, 1, 2$ is described in [22].

3 Construction of reduction operators using mappings between classes

Here we adduce the algorithm of application of equivalence transformations, gauging of arbitrary elements and mappings between classes of equations to classification of reduction operators.

1. Similarly to the group classification, at first we gauge class (1) to subclass (3) constrained by the condition $f = g$. Then class (3) is mapped to the imaged class (5) by transformation (4).

2. Since nonclassical symmetries of constant coefficient equations from the imaged classes are well investigated (see below for more details), they should be excluded from the consideration. It also concerns variable coefficient equations from class (5) which are point-equivalent to constant coefficient ones, namely equations associated with cases $1|_{q \neq 0}$, $2|_{q \neq 0}$ and 6 of table 1 and equations reduced to them by transformations from the corresponding equivalence groups. As a result, only equations from class (5) which are inequivalent with respect to all point transformations to constant coefficient ones should be studied.

3. Reduction operators should be classified up to the equivalence relations generated by the equivalence group or even by the whole set of admissible transformations. Only the nonsingular case $\tau \neq 0$ (reduced to the case $\tau = 1$) should be considered. Operators equivalent to Lie symmetry ones should be neglected.

4. Preimages of the obtained nonclassical symmetries and of equations admitting them should be found using backward gauging transformations and mappings induced by these transformations on the sets of operators.

Reduction operators of equations from class (3) are easily found from reduction operators of corresponding equations from (5) using the formula

$$\tilde{Q} = \tau \partial_t + \xi \partial_x + \left(\frac{\eta}{\sqrt{|f|}} - \frac{\xi f_x}{2f} u \right) \partial_u. \quad (8)$$

Here τ , ξ and η are coefficients of ∂_t , ∂_x and ∂_v , respectively, in the reduction operators of equations from class (5). The substitution $v = \sqrt{|f|}u$ is assumed.

There exist two ways to use mappings between classes of equations in the investigation of nonclassical symmetries. Suppose that nonclassical symmetries of equations from the imaged class are known. The first way is to take the preimages of both the constructed operators and the equations possessing them. Then we can reduce the preimaged equations with respect to the corresponding preimaged operators to find non-Lie solutions of equations from the initial class. The above way seems to be non-optimal since the ultimate goal of the investigation of nonclassical symmetries is the construction of exact solutions. This observation is confirmed by the fact that the equations from the imaged class and the associated nonclassical symmetry operators have, as a rule, a simpler form and therefore, are more suitable than their preimages. Reduced equations associated with equations from the imaged class are also simpler to be integrated. Moreover, it happens that preimages of uniformly parameterized similar equations do not have similar forms and belong to different parameterized families. As a result, making reductions in the initial class, we have to deal with a number of different ansätze and reduced equations although this is equivalent to the consideration of a single ansatz and the corresponding reduced equation within the imaged classes. This is why the second way based on the implementation of reductions in the imaged classes and preimaging of the obtained exact solutions instead of preimaging the corresponding reduction operators is preferable.

4 The case of constant \mathbf{F} and \mathbf{H}

Constant coefficient equations from the imaged class belong to the wider class of a quasilinear heat equations with a source of the general form $v_t = v_{xx} + q(v)$. Lie and nonclassical symmetries of these equations were investigated in [5, 6] and [2, 4, 9, 20], respectively. Their non-Lie exact solutions were constructed by the reduction method in [2, 4], see also their collection in [22]. The nonlinear equation $v_t = v_{xx} + q(v)$ possesses pure nonclassical symmetry operators with nonvanishing coefficients of ∂_t if and only if q is a cubic polynomial in v . Thus, in the case $q = \delta v^3 + \varepsilon v$, where $\delta \neq 0$, such operators are exhausted, up to the equivalence with respect to the corresponding Lie symmetry groups, by the following:

$$\begin{aligned} \delta < 0: \quad & \partial_t \pm \frac{3}{2}\sqrt{-2\delta}v\partial_x + \frac{3}{2}(\delta v^3 + \varepsilon v)\partial_v, \\ \varepsilon = 0: \quad & \partial_t - \frac{3}{x}\partial_x - \frac{3}{x^2}v\partial_v, \\ \varepsilon < 0: \quad & \partial_t + 3\mu \tan(\mu x)\partial_x - 3\mu^2 \sec^2(\mu x)v\partial_v, \\ \varepsilon > 0: \quad & \partial_t - 3\mu \tanh(\mu x)\partial_x + 3\mu^2 \operatorname{sech}^2(\mu x)v\partial_v, \\ & \partial_t - 3\mu \coth(\mu x)\partial_x - 3\mu^2 \operatorname{cosech}^2(\mu x)v\partial_v, \end{aligned}$$

where $\mu = \sqrt{|\varepsilon|/2}$. Note that the last operator was missed in [2, 4].

Finding the preimages of equations with such values of q with respect to transformation (4) and the preimages of the corresponding reduction operators according to formula (8), we obtain the cases presented in table 2. In this table $c_1^2 + c_2^2 \neq 0$, $\nu > 0$, $\mu = \nu/\sqrt{2}$; $\varepsilon = 0$, $\varepsilon = \nu^2 > 0$ and $\varepsilon = -\nu^2 < 0$ in cases 1, 2 and 3, respectively.

Table 2. Nonclassical symmetries of equations of the form
 $f(x)u_t = (f(x)u_x)_x + \delta f(x)^2 u^3$, $f(x) = \zeta(x)^2$.

N	$\zeta(x)$	Reduction operators
1	$c_1 x + c_2$	$\partial_t \pm \frac{3}{2}\sqrt{-2\delta}\zeta u\partial_x + \frac{3}{2}(\delta\zeta^2 u \mp c_1\sqrt{-2\delta})u^2\partial_u,$ $\partial_t - \frac{3}{x}\partial_x - \frac{3c_2}{x^2\zeta}u\partial_u$
2	$c_1 \sin(\nu x) + c_2 \cos(\nu x)$	$\partial_t \pm \frac{3}{2}\sqrt{-2\delta}\zeta u\partial_x + \frac{3}{2}(\delta\zeta^2 u^2 \mp \sqrt{-2\delta}\zeta_x u + \nu^2)u\partial_u,$ $\partial_t - 3\mu \tanh(\mu x)\partial_x + 3\mu \left(\frac{\zeta_x}{\zeta} \tanh(\mu x) + \mu \operatorname{sech}^2(\mu x) \right) u\partial_u,$ $\partial_t - 3\mu \coth(\mu x)\partial_x + 3\mu \left(\frac{\zeta_x}{\zeta} \coth(\mu x) - \mu \operatorname{cosech}^2(\mu x) \right) u\partial_u$
3	$c_1 \sinh(\nu x) + c_2 \cosh(\nu x)$	$\partial_t \pm \frac{3}{2}\sqrt{-2\delta}\zeta u\partial_x + \frac{3}{2}(\delta\zeta^2 u^2 \mp \sqrt{-2\delta}\zeta_x u - \nu^2)u\partial_u,$ $\partial_t + 3\mu \tan(\mu x)\partial_x - 3\mu \left(\frac{\zeta_x}{\zeta} \tan(\mu x) + \mu \sec^2(\mu x) \right) u\partial_u$

5 Reduction operators for general values of m

In this section we look for G_{FH}^\sim -inequivalent reduction operators of the imaged class (5). Here reduction operators have the general form $Q = \tau\partial_t + \xi\partial_x + \eta\partial_v$, where τ , ξ and η are functions of t , x and v , and $(\tau, \xi) \neq (0, 0)$. Since (5) is an evolution equation, there are two principally different cases of finding Q : $\tau \neq 0$ and $\tau = 0$ [10, 14, 25]. The singular case $\tau = 0$ was exhaustively investigated for general evolution equation in [14, 25].

Consider the case $\tau \neq 0$. We can assume $\tau = 1$ up to the usual equivalence of reduction operators. Then the determining equations for the coefficients ξ and η have the form

$$\begin{aligned}
 \xi_{vv} &= 0, & \eta_{vv} &= 2(\xi_{xv} - \xi\xi_v), \\
 \eta_t - \eta_{xx} + 2\xi_x\eta &= \\
 \xi(H_x v^m + F_x v) + (2\xi_x - \eta_v)(Hv^m + Fv) + \eta(F + Hv^{m-1}m), & & (9) \\
 3\xi_v(Hv^m + Fv) + 2\xi_x\xi + \xi_t + 2\eta_{vx} - \xi_{xx} - 2\xi_v\eta &= 0.
 \end{aligned}$$

Integration of first two equations of system (9) gives us the following expressions for ξ and η

$$\begin{aligned}\xi &= av + b, \\ \eta &= -\frac{1}{3}a^2v^3 + (a_x - ab)v^2 + cv + d,\end{aligned}\tag{10}$$

where $a = a(t, x)$, $b = b(t, x)$, $c = c(t, x)$ and $d = d(t, x)$.

Substituting ξ and η from (10) into the third and forth equations of (9), we obtain the classifying equations which include both the residuary uncertainties in coefficients of the operator and the arbitrary elements of the class under consideration.

Since the functions a , b , c , d , F and H do not depend on the variable v , the classifying equations should be split with respect to different powers of v .

Two principally different cases $a = 0$ and $a \neq 0$ should be considered separately.

If $a = 0$ then for any $m \neq 0, 1, 2$ the splitting results in the system of five equations

$$\begin{aligned}mHd &= 0, \quad d_t - d_{xx} + 2b_xd - Fd = 0, \\ b_t - b_{xx} + 2bb_x + 2c_x &= 0, \\ bH_x + (c(m-1) + 2b_x)H &= 0, \\ bF_x + 2b_xF + c_{xx} - c_t - 2b_xc &= 0.\end{aligned}\tag{11}$$

Since $mH \neq 0$ then $d = 0$ and the second equation of (11) becomes identity.

Finding the general solution of the other three equations from (11) appears to be a very difficult problem. But it is easy to construct certain particular solutions setting, e.g., $b_t = 0$. This supposition implies that $c_t = 0$. Then the integration of (11) gives the expressions of c , F and H via the function $b(x) \neq 0$

$$\begin{aligned}c &= -\frac{1}{2}b^2 + \frac{1}{2}b_x + k_1, \\ F &= -\frac{1}{4}b^2 + k_1 + k_2b^{-2} + b_x + \frac{1}{4}\left(\frac{b_x}{b}\right)^2 - \frac{1}{2}\frac{b_{xx}}{b},\end{aligned}\tag{12}$$

$$H = k_3b^{-\frac{m+3}{2}} \exp \left[(m-1) \int \left(\frac{b}{2} - \frac{k_1}{b} \right) dx \right],\tag{13}$$

where k_1 , k_2 and k_3 are arbitrary constants, $k_3 \neq 0$.

Theorem 3. *The equations from class (5) with the arbitrary elements given by formulas (12) and (13) admit reduction operators of the form*

$$Q = \partial_t + b\partial_x + \left(-\frac{1}{2}b^2 + \frac{1}{2}b_x + k_1 \right) v\partial_v,\tag{14}$$

where $b = b(x)$ is an arbitrary smooth function and k_1 is an arbitrary constant.

Note 1. Theorem 3 holds for any $m \in \mathbb{R}$, including $m \in \{0, 1, 2\}$.

We present illustrative examples, by considering various forms of the function $b(x)$.

Example 1. We take $b = x^2$ and substitute it in formulas (12)–(14) to find that the equations

$$v_t = v_{xx} + \frac{k_3}{x^{m+3}} e^{\frac{1}{6}(m-1)(x^3+6k_1x^{-1})} v^m + \left(-\frac{1}{4}x^4 + \frac{k_2}{x^4} + 2x + k_1 \right) v, \quad (15)$$

admit the reduction operator

$$Q = \partial_t + x^2 \partial_x + \left(-\frac{1}{2}x^4 + x + k_1 \right) v \partial_v.$$

The corresponding ansatz $v = xe^{k_1 t - \frac{1}{6}x^3} z(\omega)$, where $\omega = t + x^{-1}$, gives the reduced ODE

$$z_{\omega\omega} + k_3 e^{k_1(m-1)\omega} z^m + k_2 z = 0. \quad (16)$$

For $k_1 = 0$ the general solution of (16) is written in the implicit form

$$\int (Z - k_2 z^2 + C_1)^{-\frac{1}{2}} dz = \pm \omega + C_2, \quad Z = \begin{cases} -\frac{2k_3}{m+1} z^{m+1}, & m \neq -1, \\ -2k_3 \ln z, & m = -1. \end{cases} \quad (17)$$

If $k_2 = 0$ and $m \neq -1$, we are able to integrate (17). Setting $C_1 = 0$, we obtain a partial solution of the reduced equation in an explicit form:

$$z = \left(\pm \frac{m-1}{2} \sqrt{-\frac{2k_3}{m+1}} \omega + C \right)^{\frac{2}{1-m}}, \quad (18)$$

where C is an arbitrary constant. Note that the constant C can be canceled via translations of ω induced by translations of t in the initial variables.

In the case $k_1 = k_2 = 0$ and $m \neq -1$ we construct the exact solution

$$v = xe^{-\frac{1}{6}x^3} \left(\pm \frac{m-1}{2} \sqrt{-\frac{2k_3}{m+1}} (t + x^{-1}) \right)^{\frac{2}{1-m}}$$

of the corresponding equation (15). Preimages of them with respect to transformation (4) are the equation

$$e^{-\frac{1}{3}x^3} x^2 u_t = (e^{-\frac{1}{3}x^3} x^2 u_x)_x + k_3 x^{-2} e^{-\frac{1}{3}x^3} u^m \quad (19)$$

and its exact solution

$$u = \left(\pm \frac{m-1}{2} \sqrt{-\frac{2k_3}{m+1}} (t + x^{-1}) \right)^{\frac{2}{1-m}}.$$

Analogously, if $k_2 = C_1 = 0$ and $m = -1$ then integration of (17) gives $\operatorname{erf}(\sqrt{-\ln z}) = \pm \sqrt{\frac{2k_3}{\pi}} \omega + C$, where $\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-t^2} dt$ is the error function and C is an arbitrary constant which can be canceled by translations of ω . Therefore,

$$z = \exp \left\{ - \left[\operatorname{erf}^{-1} \left(\pm \sqrt{\frac{2k_3}{\pi}} \omega \right) \right]^2 \right\}, \quad (20)$$

where erf^{-1} is the inverse error function, represented by the series

$$\operatorname{erf}^{-1} y = \sum_{k=0}^{\infty} \frac{c_k}{2k+1} \left(\frac{\sqrt{\pi}}{2} y \right)^{2k+1}, \quad \text{where } c_0 = 1,$$

$$c_k = \sum_{m=0}^{k-1} \frac{c_m c_{k-1-m}}{(m+1)(2m+1)} = \left\{ 1, 1, \frac{7}{6}, \frac{127}{90}, \dots \right\}.$$

The corresponding exact solution of equation (19) with $m = -1$ is

$$u = \exp \left\{ - \left[\operatorname{erf}^{-1} \left(\pm \sqrt{\frac{2k_3}{\pi}} (t + x^{-1}) \right) \right]^2 \right\}.$$

Example 2. Consider $b = x^{-1}$. In view of theorem 3 the equations from class (5) with the arbitrary elements

$$F = k_1 + k_2 x^2 - 2x^{-2}, \quad H = k_3 x^{m+1} e^{\frac{1}{2}(1-m)k_1 x^2} \quad (21)$$

admit the reduction operator

$$Q = \partial_t + x^{-1} \partial_x + (k_1 - x^{-2}) v \partial_v.$$

The ansatz constructed with this operator is $v = x^{-1} e^{k_1 t} z(\omega)$, where $\omega = x^2 - 2t$, and the reduced equation reads

$$4z_{\omega\omega} + k_3 e^{\frac{1}{2}(1-m)k_1 \omega} z^m + k_2 z = 0.$$

If $k_1 = k_2 = 0$, the reduced equation is integrated analogously to equation (16) and has the similar particular solution

$$z = \begin{cases} \left(\pm \frac{m-1}{2} \sqrt{-\frac{k_3}{2(m+1)}} \omega \right)^{\frac{2}{1-m}}, & m \neq -1, \\ \exp \left\{ - \left[\operatorname{erf}^{-1} \left(\pm \frac{\sqrt{2}}{2} \sqrt{\frac{k_3}{\pi}} \omega \right) \right]^2 \right\}, & m = -1. \end{cases}$$

Substituting the obtained z to the ansatz, we construct exact solutions of equations from class (5) with the arbitrary elements (21) for the values $k_1 = k_2 = 0$.

The preimaged equation $x^4 u_t = (x^4 u_x)_x + k_3 x^{3(m+1)} u^m$ has the exact solution

$$u = \begin{cases} x^{-3} \left(\pm \frac{m-1}{2} \sqrt{-\frac{k_3}{2(m+1)}} (x^2 - 2t) \right)^{\frac{2}{1-m}}, & m \neq -1, \\ x^{-3} \exp \left\{ - \left[\operatorname{erf}^{-1} \left(\pm \frac{\sqrt{2}}{2} \sqrt{\frac{k_3}{\pi}} (x^2 - 2t) \right) \right]^2 \right\}, & m = -1. \end{cases}$$

In the following two examples we assume that $k_1 = k_2 = 0$ in the formulas (12)–(14) since this supposition allows us to find preimages in class (3) with arbitrary elements being elementary functions.

Example 3. Let $b = e^{-x}$ and $k_1 = k_2 = 0$. The equations of the form

$$v_t = v_{xx} + k_3 e^{\frac{1}{2}((1-m)e^{-x} + (m+3)x)} v^m - \frac{1}{4} (e^{-2x} + 4e^{-x} + 1) v, \quad (22)$$

admit the reduction operator

$$Q = \partial_t + e^{-x} \partial_x - \frac{1}{2} (e^{-x} + e^{-2x}) v \partial_v.$$

The corresponding ansatz $v = e^{\frac{1}{2}(e^{-x}-x)} z(\omega)$, where $\omega = e^x - t$, gives the reduced ODE

$$z_{\omega\omega} + k_3 z^m = 0.$$

It coincides with the equation (16) with $k_1 = k_2 = 0$, which has the particular solution (18) (resp. (20)) for $m \neq -1$ (resp. $m = -1$). Substituting these solutions to the ansatz, we obtain exact solutions of equation (22).

A preimage of (22) with respect to transformation (4) is the equation

$$e^{e^{-x}-x} u_t = (e^{e^{-x}-x} u_x)_x + k_3 e^{e^{-x}+x} u^m$$

having the solution

$$u = \begin{cases} \left(\pm \frac{m-1}{2} \sqrt{-\frac{2k_3}{m+1}} (e^x - t) \right)^{\frac{2}{1-m}}, & m \neq -1, \\ \exp \left\{ - \left[\operatorname{erf}^{-1} \left(\pm \sqrt{\frac{2k_3}{\pi}} (e^x - t) \right) \right]^2 \right\}, & m = -1. \end{cases}$$

Example 4. Substituting $b = \sin x$ and $k_1 = k_2 = 0$ to formulas (12)–(14) and making the reduction procedure, we obtain the following results: The equation

$$v_t = v_{xx} + k_3 (\sin x)^{-\frac{1}{2}(m+3)} e^{\frac{1-m}{2} \cos x} v^m + \frac{1}{4} (\cos^2 x + 4 \cos x + \operatorname{cosec}^2 x) v,$$

has the exact solution

$$v = \begin{cases} e^{\frac{1}{2} \cos x} \sqrt{\sin x} \left(\pm \frac{m-1}{2} \sqrt{-\frac{2k_3}{m+1}} (t - \ln |\tan \frac{x}{2}|) \right)^{\frac{2}{1-m}}, & m \neq -1, \\ \sqrt{\sin x} \exp \left\{ \frac{\cos x}{2} - \left[\operatorname{erf}^{-1} \left(\pm \sqrt{\frac{2k_3}{\pi}} (t - \ln |\tan \frac{x}{2}|) \right) \right]^2 \right\}, & m = -1. \end{cases}$$

The corresponding equation from class (3) is

$$e^{\cos x} \sin x u_t = (e^{\cos x} \sin x u_x)_x + k_3 \operatorname{cosec} x e^{\cos x} u^m$$

whose exact solution is easy to be constructed from the above one using formula (4).

We have shown the applicability of theorem 3 for construction of non-Lie exact solutions of equations from classes (5) and (3). Moreover, using these solutions one can find exact solutions for other equations from (5) and (3) with the help of equivalence transformations from the corresponding equivalence groups.

Note 2. In the case $m = 3$ we are able to construct more exact solutions of equations from class (5) whose coefficients are given by (12)–(13) with $k_1 = 0$, namely, for the equations

$$v_t = v_{xx} + k_3 b^{-3} e^{\int b dx} v^3 + \left(\frac{k_2}{b^2} - \frac{1}{4} b^2 + b_x + \frac{1}{4} \left(\frac{b_x}{b} \right)^2 - \frac{1}{2} \frac{b_{xx}}{b} \right) v, \quad (23)$$

where $b = b(x)$, $k_3 \neq 0$.

According to theorem 3, equation (23) admits the reduction operator (14) (with $k_1 = 0$). An ansatz constructed with this operator has the form

$$v = z(\omega) \sqrt{|b|} e^{-\frac{1}{2} \int b dx}, \quad \text{where } \omega = t - \int \frac{dx}{b},$$

and reduces (23) to the second-order ODE

$$z_{\omega\omega} = -k_3 z^3 - k_2 z.$$

It is interesting that the reduced ODE does not depend on the function $b(x)$. Multiplying this equation by z_ω and integrating once, we obtain the equation

$$z_\omega^2 = -\frac{k_3}{2} z^4 - k_2 z^2 + C_1.$$

Its general solution is expressed via Jacobian elliptic functions depending on values of the constants k_2 , k_3 and C_1 . See Appendix for more details.

For example, if $k_2 = 1 + \mu^2$, $k_3 = -2\mu^2$ and $C_1 = 1$ ($0 < \mu < 1$) we find two exact solutions of equation (23)

$$v = \operatorname{sn} \left(t - \int \frac{dx}{b}, \mu \right) \sqrt{|b|} e^{-\frac{1}{2} \int b dx}, \quad v = \operatorname{cd} \left(t - \int \frac{dx}{b}, \mu \right) \sqrt{|b|} e^{-\frac{1}{2} \int b dx},$$

where $\operatorname{sn}(\omega, \mu)$, $\operatorname{cd}(\omega, \mu)$ are Jacobian elliptic functions [24].

The second case to be considered is $a \neq 0$. Then after substitution of ξ and η from (10) to system (9) its last equation takes the form

$$\begin{aligned} \frac{2}{3} a^3 v^3 + 2a(ab - 2a_x)v^2 + (a_t + 3a_{xx} + 3aF - 2(ab)_x - 2ac)v + \\ b_t + 2b_x b - b_{xx} - 2ad + 2c_x + 3aHv^m = 0. \end{aligned} \quad (24)$$

It is easy to see that $a \neq 0$ if and only if $m = 3$. The investigation of this case is the subject of the next section.

6 Specific reduction operators for the cubic nonlinearity

Splitting equation (24) in the case $m = 3$ and $a \neq 0$ with respect to u , we obtain that the functions a, b, c and d do not depend on the variable t and are expressed via the functions F and H in the following way

$$\begin{aligned} a &= \frac{3}{2}\sqrt{2}\varepsilon\sqrt{-H}, \quad b = \frac{H_x}{H}, \quad c = \frac{1}{8}\left(12F - 2\left(\frac{H_x}{H}\right)_x - \left(\frac{H_x}{H}\right)^2\right), \\ d &= \frac{\sqrt{2}\varepsilon}{2\sqrt{-H}}\left(F_x + \frac{1}{2}\frac{H_x}{H}\left(\frac{H_x}{H}\right)_x - \frac{1}{2}\left(\frac{H_x}{H}\right)_{xx}\right), \end{aligned} \quad (25)$$

where $\varepsilon = \pm 1$. If $H < 0$ the corresponding reduction operators have real coefficients.

Then splitting of the third equation of system (9) for $m = 3$ results in the system of two ordinary differential equations

$$\begin{aligned} &H^3H_{xxxx} - 13H_x^4 + 2F_xH^3H_x + 22HH_x^2H_{xx} - 4FH^2H_x^2 - \\ &4H^2H_{xx}^2 - 6H^2H_xH_{xxx} + 4FH^3H_{xx} - 6F_{xx}H^4 = 0, \\ &16F_{xxx}H^5 + 16H^2H_xH_{xx}^2 + 3H^2H_x^2H_{xxx} - 4F_xH^4H_{xx} - \\ &6H^3H_{xx}H_{xxx} - 18HH_x^3H_{xx} - 8FF_xH^5 + 2F_xH^3H_x^2 - \\ &20FH^2H_x^3 - 12FH^4H_{xxx} + 5H_x^5 + 32FH^3H_xH_{xx} = 0. \end{aligned} \quad (26)$$

The following statement is true.

Theorem 4. *The equations from class (5) with $m = 3$ and the arbitrary elements satisfying system (26) admit reduction operators of the form*

$$\begin{aligned} Q = \partial_t + &\left(\frac{3}{2}\sqrt{2}\varepsilon\sqrt{-H}v + \frac{H_x}{H}\right)\partial_x + \\ &\left[\frac{3}{2}Hv^3 + \frac{3}{4}\sqrt{2}\varepsilon\frac{H_x}{\sqrt{-H}}v^2 + \frac{1}{8}\left(12F - 2\left(\frac{H_x}{H}\right)_x - \left(\frac{H_x}{H}\right)^2\right)v + \right. \\ &\left.\frac{\sqrt{2}\varepsilon}{2\sqrt{-H}}\left(F_x + \frac{1}{2}\frac{H_x}{H}\left(\frac{H_x}{H}\right)_x - \frac{1}{2}\left(\frac{H_x}{H}\right)_{xx}\right)\right]\partial_v, \end{aligned} \quad (27)$$

where $\varepsilon = \pm 1$.

Let us note that system (26) can be rewritten in the simpler form in terms of the functions F and b

$$\begin{aligned} b_{xxx} &= 6F_{xx} - 2b_xb^2 + b_x^2 + 2bb_{xx} - 2F_xb - 4Fb_x, \\ 16F_{xxx} &= 4b_xF_x + 2b_x^2b + 6b_xb_{xx} + 2b^2F_x + b^3b_x + \\ &3b^2b_{xx} + 12Fb_{xx} + 8FF_x + 4Fbb_x. \end{aligned} \quad (28)$$

System (26) consists of two nonlinear fourth- and third-order ODEs. Unfortunately we were not able to find its general solution. Nevertheless, we tested the six pairs of functions F and H appearing in table 1 in order to check whether they satisfy system (26). In the case of positive answer the corresponding reduction operator is easily constructed via formula (27). It appears that system (26) is satisfied by F and H from cases 1, 2 and 6 and by those from cases 3 and 4 for special values of the constants k and a_2 , namely, $(k, a_2) \in \{(-3, \frac{9}{4}), (-\frac{3}{2}, \frac{3}{16})\}$.

So, we can construct preimages of these equations using formulas (6). Below we list the pairs of the coefficients f and h for which the corresponding equations from class (3) with $m = 3$ admit nontrivial reduction operators.

Hereafter $b_1^2 + b_2^2 \neq 0$. The numbers of cases coincide with the numbers of the corresponding cases from table 1. (Case 5 does not appear below since the functions F and H from this case of table 1 do not satisfy system (26).)

1. $a_1 = 0$: $f = (b_1 x + b_2)^2$, $h = \delta e^{qx} (b_1 x + b_2)^4$, $q \neq 0$;
 $a_1 > 0 \Rightarrow a_1 = 1 \bmod G_{FH}^\sim$: $f = (b_1 \sin x + b_2 \cos x)^2$,
 $h = \delta e^{qx} (b_1 \sin x + b_2 \cos x)^4$;
 $a_1 < 0 \Rightarrow a_1 = -1 \bmod G_{FH}^\sim$: $f = (b_1 \sinh x + b_2 \cosh x)^2$,
 $h = \delta e^{qx} (b_1 \sinh x + b_2 \cosh x)^4$, $q \neq \pm 2$.
2. $q = 0$: $f = (b_1 x + b_2)^2$, $h = \delta (b_1 x + b_2)^4$;
 $q \neq 0 \Rightarrow q = -2 \bmod G_{FH}^\sim$: $f = (b_1 \sinh x + b_2 \cosh x)^2$,
 $h = \delta e^{-2x} (b_1 \sinh x + b_2 \cosh x)^4$.
3. $(k, a_2) = (-3, \frac{9}{4})$: $f = x(b_1 \sin(\sqrt{2} \ln |x|) + b_2 \cos(\sqrt{2} \ln |x|))^2$,
 $h = \delta x^{-1} (b_1 \sin(\sqrt{2} \ln |x|) + b_2 \cos(\sqrt{2} \ln |x|))^4$;
 $(k, a_2) = (-\frac{3}{2}, \frac{3}{16})$: $f = x(b_1 |x|^{\frac{1}{4}} + b_2 |x|^{-\frac{1}{4}})^2$,
 $h = \delta \sqrt{|x|} (b_1 |x|^{\frac{1}{4}} + b_2 |x|^{-\frac{1}{4}})^4$.
4. $f = x^{-1} (b_1 M_{\kappa, \mu}(px^2) + b_2 W_{\kappa, \mu}(px^2))^2$,
 $h = \delta x^{k-2} e^{px^2} (b_1 M_{\kappa, \mu}(px^2) + b_2 W_{\kappa, \mu}(px^2))^4$,
where $\kappa = -\frac{k+1}{4}$, $\mu = \frac{\sqrt{|1-4a_2|}}{4}$, $(k, a_2) \in \{(-3, \frac{9}{4}), (-\frac{3}{2}, \frac{3}{16})\}$. $M_{\kappa, \mu}$ and $W_{\kappa, \mu}$ are the Whittaker functions [24].
6. $f = x^{-1} (b_1 M_{-\frac{1}{4}, \frac{1}{4}}(px^2) + b_2 W_{-\frac{1}{4}, \frac{1}{4}}(px^2))^2$,
 $h = \delta x^{-2} e^{px^2} (b_1 M_{-\frac{1}{4}, \frac{1}{4}}(px^2) + b_2 W_{-\frac{1}{4}, \frac{1}{4}}(px^2))^4$.

Note that in the case $p > 0$ the above Whittaker function is expressed via the error function: $M_{-\frac{1}{4}, \frac{1}{4}}(px^2) = \frac{1}{2} \sqrt{\pi} \sqrt[4]{px^2} e^{\frac{p}{2}x^2} \operatorname{erf}(\sqrt{px^2})$ [1].

Since the cases $1|_{q \neq 0}$, $2|_{q \neq 0}$ and 6 are reduced to constant-coefficient ones we do not consider them.

Example 5. Class (5) contains equations with cubic nonlinearity, which are not reduced to constant-coefficient ones by point transformations and admit reduction operators of the form (27). One of them is the equation with the coefficients F and H presented by case 3 of table 1 with $k = -3$, $a_2 = \frac{9}{4}$ and $\delta = -1$, namely,

$$v_t = v_{xx} - x^{-3}v^3 + \frac{9}{4}x^{-2}v. \quad (29)$$

According to theorem 4 this equation admits two similar reduction operators ($\varepsilon = \pm 1$)

$$\begin{aligned} Q_{\pm} = \partial_t + \frac{3}{2}\sqrt{2}\left(\varepsilon x^{-\frac{3}{2}}v - \sqrt{2}x^{-1}\right)\partial_x - \\ \frac{3}{4}\sqrt{2}\left(\sqrt{2}x^{-3}v^3 - 3\varepsilon x^{-\frac{5}{2}}v^2 - \sqrt{2}x^{-2}v + 4\varepsilon x^{-\frac{3}{2}}\right)\partial_v. \end{aligned}$$

They lead to the solutions differing only in their signs. Since equation (29) is invariant with respect to the transformation $v \mapsto -v$, we consider in detail only the case $\varepsilon = 1$. For all expressions to be correctly defined, we have to restrict ourselves with values $x > 0$. (Another way is to replace x by $|x|$.)

For convenient reduction we apply the hodograph transformation

$$\tilde{t} = v, \quad \tilde{x} = x, \quad \tilde{v} = t$$

which maps equation (29) and the reduction operator Q_+ to the equation

$$\tilde{v}_{\tilde{t}}^2 \tilde{v}_{\tilde{x}\tilde{x}} + \tilde{v}_{\tilde{x}}^2 \tilde{v}_{\tilde{t}\tilde{t}} - 2\tilde{v}_{\tilde{t}} \tilde{v}_{\tilde{x}} \tilde{v}_{\tilde{t}\tilde{x}} + \tilde{v}_{\tilde{t}}^2 + \frac{\tilde{t}^3}{\tilde{x}^3} \tilde{v}_{\tilde{t}}^3 - \frac{9}{4} \frac{\tilde{t}}{\tilde{x}^2} \tilde{v}_{\tilde{t}}^3 = 0 \quad (30)$$

and its reduction operator

$$\begin{aligned} \tilde{Q}_+ = -\frac{3}{4}\sqrt{2}\left(\sqrt{2}\tilde{x}^{-3}\tilde{t}^3 - 3\tilde{x}^{-\frac{5}{2}}\tilde{t}^2 - \sqrt{2}\tilde{x}^{-2}\tilde{t} + 4\tilde{x}^{-\frac{3}{2}}\right)\partial_{\tilde{t}} + \\ \frac{3}{2}\sqrt{2}\left(\tilde{x}^{-\frac{3}{2}}\tilde{t} - \sqrt{2}\tilde{x}^{-1}\right)\partial_{\tilde{x}} + \partial_{\tilde{v}}, \end{aligned}$$

respectively. An ansatz constructed with the operator \tilde{Q}_+ has the form

$$\tilde{v} = \frac{1}{24}\tilde{x}^2 \frac{\tilde{t} + \sqrt{2}\tilde{x}}{\tilde{t} - \sqrt{2}\tilde{x}} - \frac{1}{12}\tilde{x}^2 + z(\omega), \quad \text{where } \omega = \tilde{x}^2 \frac{\tilde{t} - \sqrt{2}\tilde{x}}{\tilde{t} + \sqrt{2}\tilde{x}},$$

and reduces (30) to the simple linear ODE $\omega z_{\omega\omega} + 2z_{\omega} = 0$ whose general solution $z = \tilde{c}_1 + \tilde{c}_2\omega^{-1}$ substituted to the ansatz gives the exact solution

$$\tilde{v} = \frac{\tilde{x}^4 + 24\tilde{c}_2}{24\tilde{x}^2} \frac{\tilde{t} + \sqrt{2}\tilde{x}}{\tilde{t} - \sqrt{2}\tilde{x}} - \frac{1}{12}\tilde{x}^2 + \tilde{c}_1$$

of equation (30). Applying the inverse hodograph transformation and canceling the constant \tilde{c}_1 by translations with respect to t , we construct the non-Lie solution

$$v = \sqrt{2x} \frac{3x^4 + 24tx^2 + c_2}{x^4 + 24tx^2 - c_2} \quad (31)$$

of equation (29). The solution (31) with $c_2 = 0$ is a Lie solution invariant with respect to the dilatation operator $D = 4t\partial_t + 2x\partial_x + \partial_v$ from the maximal Lie invariance algebra of equation (29). However, it is much harder to find this solution by the reduction with respect to the operator D . The corresponding ansatz $v = \sqrt{xz}(\omega)$, where $\omega = t^{-1}x^2$, has a simple form but the reduced ODE $4\omega^2 z_{\omega\omega} + \omega(\omega + 4)z_\omega + 2z - z^3 = 0$ is nonlinear and complicated.

This example justifies the observation made by W. Fushchych [7] that “ansatzes generated by conditional symmetry operators often reduce an initial nonlinear equation to a linear one. As a rule, a Lie reduction does not change the nonlinear structure of an equation.” We can also formulate the more general similar observation that a complicated non-Lie ansatz may lead to a simple reduced equation while a simple Lie ansatz may give a complicated reduced equation which is difficult to be integrated.

One of the preimages of equation (29) with respect to transformation (4) is the equation

$$x \sin^2(\sqrt{2} \ln x) u_t = \left(x \sin^2(\sqrt{2} \ln x) u_x \right)_x - x^{-1} \sin^4(\sqrt{2} \ln x) u^3,$$

having the non-Lie exact solution

$$u = \sqrt{\frac{2}{x}} |\operatorname{cosec}(\sqrt{2} \ln x)| \frac{3x^4 + 24tx^2 + c_2}{x^4 + 24tx^2 - c_2}.$$

Example 6. Consider the equation from the imaged class (5)

$$v_t = v_{xx} - x^{-\frac{3}{2}} v^3 + \frac{3}{16} \frac{v}{x^2} \quad (32)$$

for the values $x > 0$ (case 3 of table 1 with $m = 3$, $k = -\frac{3}{2}$, $a_2 = \frac{3}{16}$ and $\delta = -1$). It admits the reduction operator of form (27)

$$Q_+ = \partial_t + \frac{3}{2} \left(\sqrt{2} x^{-\frac{3}{4}} v - x^{-1} \right) \partial_x - \frac{3}{8} \left(4x^{-\frac{3}{2}} v^3 - 3\sqrt{2} x^{-\frac{7}{4}} v^2 + x^{-2} v \right) \partial_v.$$

Usage of the same technique as in the previous example gives the non-Lie exact solution of (32)

$$v = \frac{1}{2} 5\sqrt{2} x^{\frac{1}{4}} \frac{3t + x^2}{\sqrt{x}(15t + x^2) + c_2}. \quad (33)$$

Applying the transformation $v = \sqrt{x}(b_1 x^{\frac{1}{4}} + b_2 x^{-\frac{1}{4}}) u$ to solution (33), we obtain a non-Lie solution of the equation

$$x(b_1 x^{\frac{1}{4}} + b_2 x^{-\frac{1}{4}})^2 u_t = \\ \left(x(b_1 x^{\frac{1}{4}} + b_2 x^{-\frac{1}{4}})^2 u_x \right)_x - \sqrt{x}(b_1 x^{\frac{1}{4}} + b_2 x^{-\frac{1}{4}})^4 u^3$$

from class (3), where b_1 and b_2 are arbitrary constants, $b_1^2 + b_2^2 \neq 0$.

The equivalence of cases 3 and 4 from table 1 with respect to the point transformation (7) allows us to use solutions (31) and (33) for finding non-Lie exact solutions of the equations

$$v_t = v_{xx} - x^{-3} e^{px^2} v^3 + (-p^2 x^2 + \frac{9}{4} x^{-2} + 2p) v, \quad (34)$$

$$v_t = v_{xx} - x^{-\frac{3}{2}} e^{px^2} v^3 + (-p^2 x^2 + \frac{3}{16} x^{-2} + \frac{p}{2}) v \quad (35)$$

(case 4 of table 1, where $m = 3$, $(k, a_2) \in \{(-3, \frac{9}{4}), (-\frac{3}{2}, \frac{3}{16})\}$ and $\delta = -1$). The obtained solutions of (34) and (35) are respectively

$$v = e^{-\frac{p}{2}x^2} \sqrt{2x} \frac{3px^4 - 6x^2 + c_2 e^{8pt}}{px^4 - 6x^2 - c_2 e^{8pt}}$$

and

$$v = 5\sqrt{2} e^{-\frac{p}{2}x^2} x^{\frac{1}{4}} \frac{4px^2 - 3}{2\sqrt{x}(4px^2 - 15) + c_2 e^{5pt}}.$$

Appendix

After reducing an equation from class (5) with the coefficients given by (12), (13) ($m = 3$, $k_1 = 0$) by means of operator (14), we need to integrate an ODE of the form $z_\omega^2 = Pz^4 + Qz^2 + R$, where P , Q and R are real constants (see note 1). By scale transformations, this equation can be transformed to one from those with righthand sides adduced in the fourth column of table 3. The corresponding solutions are Jacobian elliptic functions [1, 24]. Below

$$\begin{aligned} \text{cd}(\omega, \mu) &= \frac{\text{cn}(\omega, \mu)}{\text{dn}(\omega, \mu)}, & \text{ns}(\omega, \mu) &= \frac{1}{\text{sn}(\omega, \mu)}, & \text{dc}(\omega, \mu) &= \frac{1}{\text{cd}(\omega, \mu)}, \\ \text{nc}(\omega, \mu) &= \frac{1}{\text{cn}(\omega, \mu)}, & \text{nd}(\omega, \mu) &= \frac{1}{\text{dn}(\omega, \mu)}, & \text{sc}(\omega, \mu) &= \frac{\text{sn}(\omega, \mu)}{\text{cn}(\omega, \mu)}, \\ \text{sd}(\omega, \mu) &= \frac{\text{sn}(\omega, \mu)}{\text{dn}(\omega, \mu)}, & \text{cs}(\omega, \mu) &= \frac{1}{\text{sc}(\omega, \mu)}, & \text{ds}(\omega, \mu) &= \frac{1}{\text{sd}(\omega, \mu)}. \end{aligned}$$

The parameter μ is a real number. Without loss of generality μ is supposed to be in the closed interval $[0, 1]$ since elliptic functions whose parameter is real can be made to depend on elliptic functions whose parameter lies between 0 and 1 [1, §16]. If μ is equal to 0 or 1, the Jacobian elliptic functions degenerate to elementary ones.

Table 3 ([23]). Relations between values of (P, Q, R) and corresponding solutions $z(\omega)$ of the ODE $z_\omega^2 = Pz^4 + Qz^2 + R$.

P	Q	R	$Pz^4 + Qz^2 + R$	$z(\omega)$
μ^2	$-(1 + \mu^2)$	1	$(1 - z^2)(1 - \mu^2 z^2)$	$\text{sn}(\omega, \mu),$ $\text{cd}(\omega, \mu)$
$-\mu^2$	$2\mu^2 - 1$	$1 - \mu^2$	$(1 - z^2)(\mu^2 z^2 + 1 - \mu^2)$	$\text{cn}(\omega, \mu)$
-1	$2 - \mu^2$	$\mu^2 - 1$	$(1 - z^2)(z^2 + \mu^2 - 1)$	$\text{dn}(\omega, \mu)$
1	$-(1 + \mu^2)$	μ^2	$(1 - z^2)(\mu^2 - z^2)$	$\text{ns}(\omega, \mu),$ $\text{dc}(\omega, \mu)$
$1 - \mu^2$	$2\mu^2 - 1$	$-\mu^2$	$(1 - z^2)((\mu^2 - 1)z^2 - \mu^2)$	$\text{nc}(\omega, \mu)$
$\mu^2 - 1$	$2 - \mu^2$	-1	$(1 - z^2)((1 - \mu^2)z^2 - 1)$	$\text{nd}(\omega, \mu)$
$1 - \mu^2$	$2 - \mu^2$	1	$(1 + z^2)((1 - \mu^2)z^2 + 1)$	$\text{sc}(\omega, \mu)$
$\mu^2(\mu^2 - 1)$	$2\mu^2 - 1$	1	$(1 + \mu^2 z^2)((\mu^2 - 1)z^2 + 1)$	$\text{sd}(\omega, \mu)$
1	$2 - \mu^2$	$1 - \mu^2$	$(1 + z^2)(z^2 + 1 - \mu^2)$	$\text{cs}(\omega, \mu)$
1	$2\mu^2 - 1$	$\mu^2(\mu^2 - 1)$	$(z^2 + \mu^2)(z^2 + \mu^2 - 1)$	$\text{ds}(\omega, \mu)$

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Compacton-like solutions of some nonlocal hydrodynamic-type models

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We show the existence of compacton-like solutions within two hydrodynamic-type systems taking into account effects of spatial and temporal non-localities.

1 Introduction

In this paper solutions to evolutionary equations are studied, describing wave patterns with compact support. Different kinds of wave patterns play key roles in natural processes [1–4]. The most complete mathematical theory dealing with wave patterns' formation and evolution is the soliton theory [5]. The origin of this theory goes back to Scott Russell's description of the solitary wave movement in the surface of channel filled with water. It was the ability of the wave to move quite a long distance without any change of shape which stroke the imagination of the first chronicler of this phenomenon. In 1895 Kortevég and de Vries put forward their famous equation

$$u_t + \beta u u_x + u_{xxx} = 0, \quad (1)$$

describing long waves' evolution on a shallow water. They also obtained the analytical solution to this equation, corresponding to the solitary wave:

$$u = \frac{12V^2}{\beta} \operatorname{sech}^2 [V(x - 4V^2 t)]. \quad (2)$$

Both the already mentioned report by Scott Russell as well as the model suggested to explain his observation did not involve a proper impact till the second half of the XX century when there have been established a number of unusual features of equation (1) finally becoming aware as the consequences of its complete integrability [5].

In 1993 Philip Rosenau and James M. Hyman [6] put forward the following generalization of KdV equation:

$$K(m, n) = u_t + \alpha (u^m)_x + \beta (u^n)_{xxx} = 0, \quad m \geq 2, \quad n \geq 2. \quad (3)$$

The above family, called $K(m, n)$ hierarchy, occurs to possess the generalized traveling wave (TW) solutions with compact support. For $n = m = 2$, $\alpha = \beta = 1$ compactly supported solution is given by the following formula:

$$u = \begin{cases} \frac{4V}{3} \cos^2 \frac{\xi}{4} & \text{if, } |\xi| \leq 2\pi, \\ 0, & \text{if } |\xi| > 2\pi, \end{cases} \quad \xi = x - Vt. \quad (4)$$

Solutions of this type (referred to as *compactons*) attracted attention of many scientists since they occur to inherit the main features of solitons. Thus, like the solitons, compacton-like solutions form a one-parameter family parameterized by the wave pack velocity V . Numerical experiments show [6–8] that any sufficiently smooth initial data with compact support creates an ordered chain of compactons (in agreement with the fact that the maximal amplitude of compacton is proportional to its velocity). The most amazing observation concerning dynamics of compactons is that they collide almost elastically in spite of the fact that a common member of $K(m, n)$ hierarchy is not completely integrable and does not possess an infinite number of conservation laws.

A big progress in understanding properties of compacton-supporting equations have been attained in recent decades [7–12]. Yet most papers dealing with this subject are concerned with the compactons being the solutions to either completely integrable equations, or those which produce a completely integrable ones when being reduced to equations or systems describing a set of TW solutions.

In this paper compacton-like solutions to the hydrodynamic-type model taking into account non-local effects are studied. One of the peculiarities of the models in question is that compacton-like TW solutions supporting by them exist merely for selected values of the parameters. In spite of such restriction, the mere existence of these solutions is significant for they appear in presence of both nonlinear and nonlocal effects and rather cannot occur in any local hydrodynamic model. Besides, compacton-like solutions associated with relaxing hydrodynamic-type model manifest attractive features and can be treated as some universal mechanism of the energy transfer in heterogeneous media.

The structure of the paper is the following. In Section 2 we give a geometric insight into the soliton and compacton TW solutions, discussing the mechanism of appearance of generalized solutions with compact supports. In Section 3 we introduce the modeling systems taking into account non-local effects. In the following sections we show that compacton-like TW solutions do exist within non-local hydrodynamic-type models.

2 Solitons and compactons from the geometric viewpoint

Let us discuss how the solitary wave solution to (1) can be obtained. Since the function $u(\cdot)$ in the formula (2) depends on the specific combination of the independent variables, we can use for this purpose the ansatz $u(t, x) = U(\xi)$, with $\xi =$

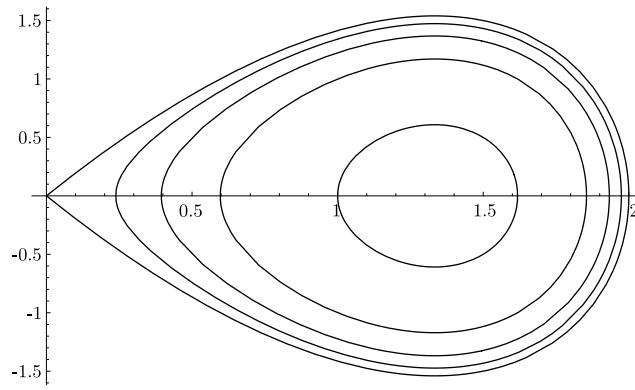


Figure 1. Level curves of the Hamiltonian (6), representing periodic solutions and limiting to them homoclinic solution

$x - V t$. Inserting this ansatz into equation (1) we get, after one integration, the following system of ODEs:

$$\dot{U}(\xi) = -W(\xi), \quad \dot{W}(\xi) = \frac{\beta}{2}U(\xi) \left(U(\xi) - \frac{2V}{\beta} \right). \quad (5)$$

System (5) is a Hamiltonian system describing by the Hamilton function

$$H = \frac{1}{2} \left(W^2 + \frac{\beta}{3}U^3 - VU^2 \right). \quad (6)$$

Every solutions of (5) can be identified with some level curve $H = C$. Already mentioned solution (2) corresponds to the value $C = 0$ and is represented by the homoclinic trajectory shown in Fig. 1 (the only trajectory in the right half-plane going through the origin). Since the origin is an equilibrium point of system (5) and penetration of the homoclinic loop takes an infinite “time”, then the beginning of this trajectory corresponds to $\xi = -\infty$ while its end to $\xi = +\infty$. This assertion is equivalent to the statement that solution (2) is nonzero for all finite values of the argument ξ .

Now let us discuss the geometric structure of compactons. Like in the previous case, we are looking for the solutions of the form $u(t, x) = U(\xi)$, $\xi = x - V t$. Inserting this ansatz into equation (3) we obtain, after one integration, the following dynamical system:

$$\frac{dU}{dT} = -n\beta U^2 W, \quad \frac{dW}{dT} = U^{3-n} \left[-VU + \frac{\alpha}{m+1} U^{m+1} - n\beta U^{n-2} W \right], \quad (7)$$

where $\frac{d}{dT} = n\beta U^2 \frac{d}{d\xi}$. All the trajectories of this system are given by its first integral

$$\frac{\alpha}{(m+1)(5+m-n)} U^{5+m-n} - \frac{V}{5-n} U^{5-n} + \frac{\beta n}{2} (UW)^2 = H = \text{const.} \quad (8)$$

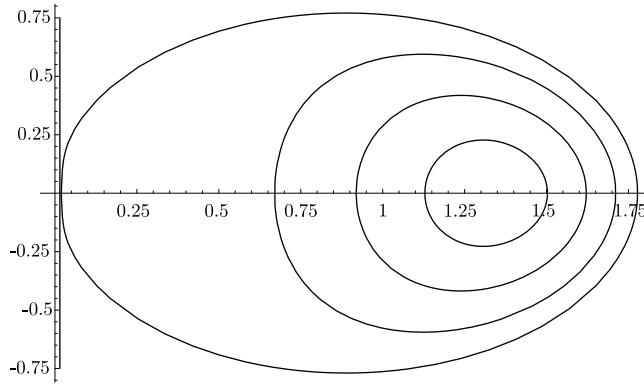


Figure 2. Level curves of the Hamiltonian (8). Dashed line indicates the set of singular points $U = 0$.

Phase portrait of system (7), shown in Fig. 2, is similar to some extent to that corresponding to system (5). Yet the critical point $U = W = 0$ of system (7) lies on the line of singular points $U = 0$. And this implies that modulus of the tangent vector field along the homoclinic trajectory is bounded from below by a positive constant. Consequently the homoclinic trajectory is penetrated in a finite “time” and the corresponding generalized solution to the initial system (3) is the compound of a function corresponding to the homoclinic loop (which now has a compact support) and zero solution corresponding to the rest point $U = W = 0$.

It is quite obvious that similar mechanism of creating the compacton-like solutions can be realized in case of non-Hamiltonian systems as well, but in contrast to the Hamiltonian systems, the homoclinic solution is no more expected to form a one-parameter family. In fact, in the modeling systems described in the following sections, homoclinic solution appear at selected values of the parameters.

Let us note in conclusion that we do not distinguish solutions having the compact supports and those which can be made so by proper change of variables. In particular, the solutions we deal with in the following are realized as compact perturbations evolving in a self-similar mode on the background of the stationary (steady or inhomogeneous) solutions to corresponding systems of PDEs.

3 Non-local hydrodynamic-type models

We are going to analyze the existence of compacton-like solutions within the hydrodynamic-type models taking into account non-local effects. These effects are manifested when an intense pulse loading (impact, explosion, etc.) is applied to media possessing internal structure on mesoscale. Description of the non-linear waves propagation in such media depends in essential way on the ratio of a characteristic size d of elements of the medium structure to the characteristic length λ of the wave pack. If d/λ is of the order $O(1)$ then the basic concepts of

continuum mechanics are not applicable and one should use the description based e.g. on the element dynamics methods. The models studied in this paper apply when the ratio d/λ is much less than unity and therefore the continual approach is still valid, but it is not as small that we can ignore the presence of internal structure.

As it is shown in [13], in the long wave approximation the balance equations for momentum and mass retain their classical form, which in the one-dimensional case can be written as follows:

$$u_t + p_x = 0, \quad \rho_t + \rho^2 u_x = 0. \quad (9)$$

Here u is mass velocity, ρ is density, p is pressure, t is time, x is mass (Lagrangean) coordinate. So the whole information about the presence of structure in this approximation is contained in dynamic equation of state (DES) which should be added to system (9) in order to make it closed.

It is of common knowledge that a unified dynamic equation of state well enough describing the behavior of condensed media in a wide range of changes of pressure, density and regimes of load (unload) actually does not exist. Various particular equations of state describing dynamical behavior of structural media are derived by means of different techniques. There exist, for example, several generally accepted DES in mechanics of heterogeneous media, derived on pure mechanical ground (see e.g. [14, 15]). Contrary, in papers [16–19] derivations of DES are based on the non-equilibrium thermodynamics methods. Both, mechanical and thermodynamical approaches under the resembling assumptions give rise to similar DES, stating the functional dependencies between p , ρ and their partial derivatives.

There is also a number of works aimed at deriving the DES on the basis of the statistical theory of irreversible processes (see [20, 21] and references therein). It is rather firmly stated within this approach that DES for complicated condensed media, being far from the state of thermodynamic equilibrium, takes the form of integral equations, linking together generalized thermodynamical fluxes I_n and generalized thermodynamical forces L_m , causing these fluxes:

$$I_n = f_n(L_k) + \int_{-\infty}^t dt' \int_{-\infty}^{+\infty} K_{mn}(t, t'; x, x') g_m[L_k(t', x')] dx'.$$

Here $K_{mn}(t, t'; x, x')$ is the kernel of non-locality, which can be calculated, in principle, by solving dynamic problem of structure's elements interaction. Yet, such calculations are extremely difficult and very seldom are seen through to the end. Therefore we follow a common practice [22, 23] and use some model kernels describing well enough the main properties of the non-local effects, in particular, the fact that these effects vanish rapidly as $|t - t'|$ and $|x - x'|$ grow.

DES derived here are based on the following relation between the pressure and density:

$$p(t, x) = f[\rho(t, x)] + \int_{-\infty}^t \left\{ \int_{-\infty}^{+\infty} K(t - t', x - x') g[\rho(t', x')] dx' \right\} dt'. \quad (10)$$

Let us first assume that effects of spatial non-locality are unimportant. In this case the kernel of non-locality can be presented as $K(t - t')\delta(x - x')$ and the flux-force relation (10) takes the following form:

$$p(t, x) = f[\rho(t, x)] + \int_{-\infty}^t K(t - t') g[\rho(t', x)] dt'. \quad (11)$$

Since we rather do not want dealing with systems of integro-differential equations, our next step is to extract some acceptable kernels enabling to pass from (11) to pure differential relations. Differentiating equation (11) with respect to the temporal variable we obtain:

$$p_t = \dot{f}[\rho]\rho_t + K(0)g[\rho] + \int_{-\infty}^t K_t(t - t') g[\rho(t', x)] dt'. \quad (12)$$

Equation (12) is equivalent to a pure differential one provided that function $K(z)$ satisfies equation $\dot{K}(z) = cK(z)$. In this case we obtain from (11), (12) the following differential equation:

$$p_t = \dot{f}[\rho]\rho_t + Ag[\rho] + c\{p - f[\rho]\}. \quad (13)$$

In case when $c = -1/\tau < 0$, this equation corresponds to fading memory kernel $K(z) = A \exp[-\frac{z}{\tau}]$. For $A = 1$, $f[\rho] = \chi\rho^{n+1}$, $g[\rho] = -\sigma\rho^{n+1}$, we get the following DES:

$$\tau\{p_t - \chi(n+1)\rho^n\rho_t\} = \kappa\rho^{n+1} - p, \quad (14)$$

where $\kappa = \chi - \sigma\tau$. Equations coinciding with (14) under certain additional assumptions are widely used to describe nonlinear processes in multi-component media with one relaxing process in the elements of structure. Such constitutive equations was obtained by V.A. Danylenko with co-workers [18, 19] by means of non-equilibrium thermodynamics methods and somewhat earlier by G.M. Lyakhov [14] on pure mechanical ground. Assuming that $F = \gamma = \text{const}$, $n = 1$ and passing from the density to specific volume $V = \rho^{-1}$ one is able to obtain from (9) and (14) the following closed system of PDEs:

$$u_t + p_x = \gamma, \quad V_t - u_x = 0, \quad \tau p_t + \frac{\chi}{V^2} u_x = \frac{\kappa}{V} - p. \quad (15)$$

In system (15) parameter γ stands for acceleration of the external force, while $\sqrt{\kappa}$ and $\sqrt{\chi/\tau}$ are interpreted as equilibrium and “frozen” sound velocities, respectively [18].

Now let us address the case of pure spatial non-locality. Following [23], we shall use the kernel of the form $K[t - t', x - x'] = \hat{\sigma} \exp\left[-\left(\frac{x-x'}{l}\right)^2\right] \cdot \delta[t - t']$ giving the equation

$$p = f(\rho) + \hat{\sigma} \int_{-\infty}^{+\infty} e^{-\left[\frac{x-x'}{l}\right]^2} g[\rho(t, x')] dx'. \quad (16)$$

Here parameter l , plays the role of characteristic length of the non-local effects, is introduced. Since the function $\exp\left[-\left(\frac{x-x'}{l}\right)^2\right]$ extremely quickly approaches zero as $|x-x'|$ grows, then for $0 < l << |x-x'| << 1$ (which is rather a common case) we can use the following approximation for function $g[\rho(t, x')]$ inside the inner integral:

$$\begin{aligned} g[\rho(t, x')] &= g[\rho(t, x)] + \{g[\rho(t, x)]\}_x \frac{(x'-x)}{1!} \\ &+ \{g[\rho(t, x)]\}_{xx} \frac{(x'-x)^2}{2!} + O(|x'-x|^3). \end{aligned} \quad (17)$$

Dropping out the term $O(|x'-x|^3)$ and integrating over dx' , we get

$$p = f[\rho(t, x)] + \sigma_0 g[\rho(t, x)] + \sigma_2 \{g[\rho(t, x)]\}_{xx} \quad (18)$$

where

$$\sigma_0 = \hat{\sigma} \int_{-\infty}^{+\infty} e^{-\left(\frac{x-x'}{l}\right)^2} dx', \quad \sigma_2 = \hat{\sigma} \int_{-\infty}^{+\infty} \frac{(x'-x)^2}{2!} e^{-\left(\frac{x-x'}{l}\right)^2} dx'.$$

In the simplest case analyzed e.g. by Peerlings [23] $f[\rho(t, x)] \sim g[\rho(t, x)] \sim \rho^{n+2}$. System of balance equations (9) closed by (18) with such a choice of functions f, g was analyzed in [24]. It occurs to possess merely a solitary wave solution.

In order to obtain a compacton-supporting system we stop on another choice, assuming that $g(\rho) = \hat{\sigma}(\rho - \rho_0)^{n+1}$, and using some unspecified function $f(\bar{\rho}) \equiv f(\rho - \rho_0)$, where $0 < \rho_0 = \text{const}$. This way we obtain system

$$u_t + \dot{f}(\bar{\rho})\rho_x + \sigma(\bar{\rho}^n \bar{\rho}_x)_{xx} = 0, \quad \rho_t + \rho^2 u_x = 0. \quad (19)$$

In the following sections we analyze the conditions leading in both cases to the appearance of compactly supported solutions.

3.1 Compactons in relaxing hydrodynamic-type model

We perform the factorization [25] of system (15) (or, in other words, passage to an ODE system describing TW solutions), using its symmetry properties summarized in the following statement.

Lemma 1. *System (15) is invariant with respect to one-parameter groups of transformations generated by the infinitesimal operators*

$$\hat{X}_0 = \frac{\partial}{\partial t}, \quad \hat{X}_1 = \frac{\partial}{\partial x}, \quad \hat{X}_2 = p \frac{\partial}{\partial p} + x \frac{\partial}{\partial x} - V \frac{\partial}{\partial V}. \quad (20)$$

The above symmetry generators are composed on the following combination:

$$\hat{Z} = \frac{\partial}{\partial t} + \xi \left[(x - x_0) \frac{\partial}{\partial x} + p \frac{\partial}{\partial p} - V \frac{\partial}{\partial V} \right].$$

It is obvious that operator \hat{Z} belongs to the Lie algebra of the symmetry group of (15). Therefore expressing the old variables in terms of four independent solutions of equation $\hat{Z}J(t, x) = 0$, we gain the reduction of the initial system [25]. Solving the equivalent system

$$\frac{dt}{1} = \frac{d(x_0 - x)}{\xi(x_0 - x)} = \frac{dp}{\xi p} = \frac{dV}{-\xi V} = \frac{du}{0},$$

we get the following ansatz

$$\begin{aligned} u &= U(\omega), \quad p = \Pi(\omega)(x_0 - x), \quad V = R(\omega)/(x_0 - x), \\ \omega &= \xi t + \log \frac{x_0}{x_0 - x}, \end{aligned} \quad (21)$$

leading to the reduction. In fact, inserting (21) into the second equation of system (15), we get the quadrature

$$U = \xi R + \text{const}, \quad (22)$$

and the following dynamical system:

$$\begin{aligned} \xi \Delta(R) R' &= -R [\sigma R \Pi - \kappa + \tau \xi R \gamma] = F_1, \\ \xi \Delta(R) \Pi' &= \xi \{\xi R (R \Pi - \kappa) + \chi (\Pi + \gamma)\} = F_2, \end{aligned} \quad (23)$$

where $(\cdot)' = d(\cdot)/d\omega$, $\Delta(R) = \tau(\xi R)^2 - \chi$, $\sigma = 1 + \tau \xi$.

In case when $\gamma < 0$, system (23) has three stationary points in the right half-plane. One of them, having the coordinates $R_0 = 0$, $\Pi_0 = -\gamma$, lies in the vertical coordinate axis. Another one having the coordinates $R_1 = -\kappa/\gamma$, $\Pi_1 = -\gamma$ is the only stationary point lying in the physical parameters range. The last one having the coordinates

$$R_2 = \sqrt{\frac{\chi}{\tau \xi^2}}, \quad \Pi_2 = \frac{\kappa - \tau \xi \gamma R_2}{\sigma R_2},$$

lies on the line of singular points $\tau(\xi R)^2 - \chi = 0$.

As was announced earlier, we are looking for the homoclinic trajectory arising as a result of a limit cycle destruction. So in the first step we should assure the fulfillment of the Andronov–Hopf theorem statements in some stationary point. Since the only good candidate for this purpose is the point $A(R_1, \Pi_1)$, we put the origin into this point by making the following change of the coordinates $X = R - R_1$, $Y = \Pi - \Pi_1$ which gives us the system

$$\xi \Delta(R) \begin{pmatrix} X \\ Y \end{pmatrix}' = \begin{bmatrix} -\kappa & -R_1^2 \sigma \\ \kappa \xi^2 & (\xi R_1)^2 + \chi \xi \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \quad (24)$$

where

$$\begin{aligned} H_1 &= -(\Pi_1 X^2 + 2\sigma R_1 XY + \sigma X^2 Y), \\ H_2 &= \xi^2 (\Pi_1 X^2 + 2R_1 XY + X^2 Y). \end{aligned}$$

A necessary condition for the limit cycle appearance reads as follows [26]:

$$\operatorname{sp} \hat{M} = 0 \Leftrightarrow (\xi R_1)^2 + \chi \xi = \kappa, \quad (25)$$

$$\det \hat{M} > 0 \Leftrightarrow \Omega^2 = \kappa \xi \Delta(R_1) > 0, \quad (26)$$

where \hat{M} is the linearization matrix of system (24). The inequality (26) will be fulfilled if $\xi < 0$ and the coordinate R_1 lies inside the set $(0, \sqrt{\chi/(\tau \xi^2)})$. Note that another option, i.e. when $\xi > 0$ and $\Delta > 0$ is forbidden because of physical reason [27]. On view of that, the critical value of ξ is expressed by the formula

$$\xi_{\text{cr}} = -\frac{\chi + \sqrt{\chi^2 + 4\kappa R_1^2}}{2R_1^2}. \quad (27)$$

Remark 1. Note that as a by-product of the inequality (26) we get the following relations:

$$-1 < \tau \xi < 0. \quad (28)$$

To accomplish the study of Andronov–Hopf bifurcation, we are going to calculate the real part of the first Floquet index C_1 [26]. For this purpose we use the transformation

$$y_1 = X, \quad y_2 = -\frac{\kappa}{\Omega} X - \frac{\sigma R_1^2}{\Omega} Y,$$

enabling to pass from system (24) to the canonical one having the anti-diagonal linearization matrix $\hat{M}_{ij} = \Omega(\delta_{2i}\delta_{1j} - \delta_{1i}\delta_{2j})$. For this representation formulae from [26, 28] are directly applied and using them we obtain the expression:

$$16R_1^2\Omega^2 \operatorname{Re} C_1 = -\kappa \left\{ 3\kappa^2 + (\xi R_1)^2 (3 - \xi\tau) - \kappa (\xi R_1)^2 (6 + \tau\xi) \right\}.$$

Employing (25), we get after some algebraic manipulation the following formula:

$$\operatorname{Re} C_1 = \frac{\kappa}{16\Omega^2 R_1^2} \left\{ 2\kappa\xi\tau (\xi R_1)^2 - \chi\tau (\xi^2 R_1)^2 - 3(\chi\xi)^2 \right\}.$$

Since for $\xi = \xi_{\text{cr}} < 0$ expression in braces is negative, the following statement is true:

Lemma 2. *If $R_1 < R_2$ then in vicinity of the critical value $\xi = \xi_{\text{cr}}$ given by the formula (27) a stable limit cycle appears in system (23).*

We have formulated conditions assuring the appearance of periodic orbit in proximity of stationary point $A(R_1, \Pi_1)$ yet in order that the required homoclinic bifurcation would ever take place, another condition should be fulfilled, namely that, on the same restrictions upon the parameters, critical point $B(R_2, \Pi_2)$ is a saddle. Besides, it is necessary to pose the conditions on the parameters assuring that the stationary point $B(R_2, \Pi_2)$ lies in the first quadrant of the phase plane for otherwise corresponding stationary solution which is needed to compose the compacton would not have the physical interpretation. Below we formulate the statement addressing both of these questions.

Lemma 3. *Stationary point $B(R_2, \Pi_2)$ is a saddle point lying in the first quadrant for any $\xi > \xi_{\text{cr}}$ if the following inequalities hold:*

$$-\tau\xi_{\text{cr}}R_2 < R_1 < R_2. \quad (29)$$

Proof. First we are going to show that the eigenvalues $\lambda_{1,2}$ of the system's (23) Jacobi matrix

$$\hat{M} = \frac{\partial(F_1, F_2)}{\partial(R, \Pi)} = \begin{bmatrix} \frac{\kappa}{\sigma} & \frac{\sigma\chi}{\tau\xi^2} \\ \frac{\xi^2[\kappa(\sigma-2)+2\gamma R_2(\sigma-1)]}{\sigma} & -\frac{\chi\sigma}{\tau} \end{bmatrix}, \quad (30)$$

taken at the rest point (R_2, Π_2) , are real and have different signs. Since the eigenvalues of \hat{M} are expressed by the formula

$$\lambda_{1,2} = \frac{\text{sp}\hat{M} \pm \sqrt{[\text{sp}\hat{M}]^2 - 4 \det \hat{M}}}{2},$$

it is sufficient to show that $\det \hat{M} < 0$. In fact, we have

$$\begin{aligned} \det \hat{M} &= -\frac{\chi\sigma\kappa}{\tau} - \frac{\chi}{\tau} [\kappa(\sigma-2) + 2\gamma R_2(\sigma-1)] = \\ &= -\frac{\chi}{\tau} 2\gamma\tau\xi \left(\frac{\kappa}{\gamma} + R_2 \right) = 2\chi\xi\gamma(R_1 - R_2) < 0. \end{aligned}$$

To finish the proof, we must show that stationary point $B(R_2, \Pi_2)$ lies in the first quadrant. This is equivalent to the statement that $\kappa - \tau\xi_{\text{cr}}\gamma R_2 > 0$. Carrying the first term into the RHS and dividing the inequality obtained by $\gamma < 0$, we get the inequality $-\tau\xi_{\text{cr}}R_2 < R_1$. The latter implies inequalities $-\tau\xi R_2 < R_1 < R_2$ which are valid for any $\xi > \xi_{\text{cr}}$. \blacksquare

Numerical studies of system's (23) behavior reveal the following changes of regimes (cf. Fig. 3). When $\xi < \xi_{\text{cr}}$, $A(R_1, \Pi_1)$ is a stable focus; above the critical value a stable limit cycle softly appears. Its radius grows with the growth of parameter ξ until it gains the second critical value $\xi_{\text{cr}_2} > \xi_{\text{cr}}$ for which the homoclinic loop appears in place of the periodic trajectory. The homoclinic trajectory is based upon the stationary point $B(R_2, \Pi_2)$ lying on the line of singular points $\Delta(R) = 0$ so it corresponds to the generalized compacton-like solution to system (15). We obtain this solution sewing up the TW solution corresponding to homoclinic loop with stationary inhomogeneous solution

$$u = 0, \quad p = \Pi_2(x_0 - x), \quad V = R_2/(x_0 - x), \quad (31)$$

corresponding to critical point $B(R_2, \Pi_2)$. So, strictly speaking it is different from the “true” compacton, which is defined as a solution with compact support. Note, that we can pass to the compactly supported function by the following change of variables:

$$\pi(t, x) = p(t, x) - \Pi_2(x_0 - x), \quad \nu(t, x) = V(t, x) - R_2/(x_0 - x).$$

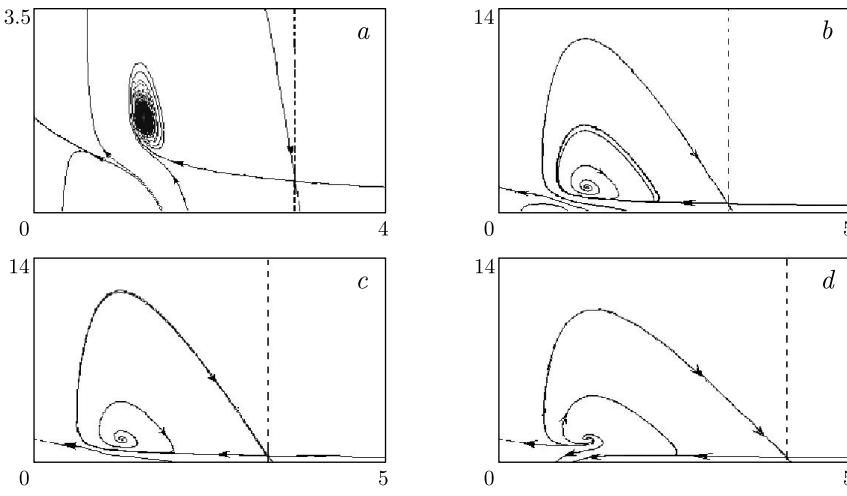


Figure 3. Changes of phase portrait of system (23): (a) $A(R_1, \Pi_1)$ is the stable focus; (b) $A(R_1, \Pi_1)$ is surrounded by the stable limit cycle; (c) $A(R_1, \Pi_1)$ is surrounded by the homoclinic loop; (d) $A(R_1, \Pi_1)$ is the unstable focus.

4 Compactons in the spatially nonlocal hydrodynamic-type system

Now let us consider system (19) and use the TW ansatz

$$u(t, x) = U(x - Dt) \equiv U(\xi), \quad \bar{\rho}(t, x) = R(\xi). \quad (32)$$

Inserting (32) into the second equation of system (19), we get the following quadrature:

$$U = \frac{D}{\rho_0} - \frac{D}{R[\xi] + \rho_0}. \quad (33)$$

Constant of integration have been chosen in such a way that $u(t, \pm\infty) = 0$.

Inserting the ansatz (32) into the first equation of the system, using the formula (33) and integrating once the expression obtained this way we pass, after some algebraic manipulation, to the second order dynamical system

$$\frac{D^2}{R + \rho_0} + f[R] + \kappa R^{n+2} + \sigma [R^{n+1} R']' = E = \frac{D^2}{\rho_0} + f[0]. \quad (34)$$

It is obvious that above equation can be re-written as an equivalent dynamical system. To do this, we define a new function $W = -R'$. Next we introduce new independent variable T such that $\frac{d}{dT} = \sigma R^{n+1} \varphi[R] \frac{d}{d\omega}$, where $\varphi[R]$ is an integrating factor which is incorporated in order to make the system Hamiltonian. With this

notation we get the following dynamical system equivalent to (34):

$$\begin{aligned}\frac{dR}{dT} &= -\sigma\varphi[R]R^{n+1}W = -\frac{\partial H[R, W]}{\partial W}, \\ \frac{dW}{dT} &= \varphi[R] \left[\sigma(n+1)R^nW^2 + f(R) - f(0) + \kappa R^{n+2} - \frac{D^2R}{\rho_0(R+\rho_0)} \right] \\ &= \frac{\partial H[R, W]}{\partial R}.\end{aligned}\quad (35)$$

Solving the first equation of system (35) with respect to function $H[R, W]$ we obtain:

$$H[R, W] = \sigma\varphi[R]R^{n+1}\frac{W^2}{2} + \theta[R]. \quad (36)$$

Now, comparing the RHS of second equation of system (35) with partial derivative of (36) with respect to R , we get the system

$$\begin{aligned}R\varphi'[R] &= (n+1)\varphi[R], \\ \theta'[R] &= \varphi[R] \left\{ f(R) - f(0) + \kappa R^{n+2} - \frac{D^2R}{\rho_0(R+\rho_0)} \right\}.\end{aligned}$$

The first equation is satisfied by the function $\varphi[R] = CR^{n+1}$. For convenience we put $C = 2$. Inserting $\varphi[R]$ in the second equation we obtain the quadrature

$$\theta(R) = 2 \int R^{n+1} \left\{ f(R) - f(0) + \kappa R^{n+2} - \frac{D^2R}{\rho_0(R+\rho_0)} \right\} dR.$$

Since the Hamiltonian function

$$\begin{aligned}H[R, W] &= \sigma R^{2(n+1)}W^2 \\ &+ 2 \int R^{n+1} \left\{ f(R) - f(0) + \kappa R^{n+2} - \frac{D^2R}{\rho_0(R+\rho_0)} \right\} dR\end{aligned}$$

is a first integral of the system (35), the set of its phase trajectories can be presented in the form

$$\begin{aligned}W^2 &= \frac{1}{\sigma R^{2(n+1)}} \left\{ K - 2 \int R^{n+1} \left[f(R) - f(0) \right. \right. \\ &\left. \left. + \kappa R^{n+2} - \frac{D^2R}{\rho_0(R+\rho_0)} \right] dR \right\},\end{aligned}\quad (37)$$

where K is the constant value of the Hamiltonian on a particular trajectory. Now let us analyze formula (37). If we want to have a closed trajectory approaching the origin, then we must properly choose the constant K and "suppress" all the singular terms by the proper choice of function $f(R)$. It can be easily shown by

induction that the following decomposition for the last term inside the integral takes place:

$$\begin{aligned} \frac{R^{n+2}}{R + \rho_0} &= R^{n+1} - \rho_0 R^n + \dots + (-1)^k \rho_0^k R^{n+1-k} + \dots \\ &+ (-1)^{n+1} \rho_0^{n+1} + (-1)^{n+2} \frac{\rho_0^{n+2}}{R + \rho_0}. \end{aligned}$$

Hence

$$\begin{aligned} W^2 &= \frac{1}{\sigma R^{2(n+1)}} \left\{ K - 2 \int R^{n+1} [f(R) - f(0) + \kappa R^{n+2}] dR - \right. \\ &\left. - \frac{2D^2}{\rho_0} \left(\frac{R^{n+2}}{n+2} - \rho_0 \frac{R^{n+1}}{n+1} + \dots + (-\rho_0)^{n+1} R + (-\rho_0)^{n+2} \log(R + \rho_0) \right) \right\}. \end{aligned}$$

From this we conclude that the last term in (37) always produces singularities. More precisely, singularities are connected with monomials R^m when $m < 2(n+1)$ and with the logarithmic term. Therefore the last term in (37) should be rather removed by the proper choice of $f(R)$.

A simple analysis shows that function

$$f(R) = f_0 + \frac{AR}{\rho_0(R + \rho_0)} + g_1 R^{n+1} + g_2 R^{n+2}$$

with $A > 0$ will suppress singularity provided that $D = \pm\sqrt{A}$. In fact, in this case

$$W^2 = \frac{1}{\sigma R^{2(n+1)}} \left\{ K - R^{2(n+1)} \left[\frac{2g_1 R}{2n+3} + \frac{\bar{g}_2 R^2}{n+2} \right] \right\}, \quad (38)$$

where $\bar{g}_2 = g_2 + \kappa$. With such a choice the only trajectory approaching the origin corresponds to $K = 0$. If in addition $g_1 = -\alpha_1 < 0$ and $\bar{g}_2 = \alpha_2 > 0$, i.e.

$$f(R) = f_0 + \frac{AR}{\rho_0(R + \rho_0)} - \alpha_1 R^{n+1} + (\alpha_2 - \kappa) R^{n+2}, \quad (39)$$

then

$$W = \pm \frac{1}{\sqrt{\sigma}} \sqrt{\frac{2\alpha_1}{2n+3} R - \frac{\alpha_2}{n+2} R^2} \quad (40)$$

and there is the point $R_* = \frac{2(n+2)\alpha_1}{(2n+3)\alpha_2}$ in which the trajectory intersects the horizontal axis.

In fact, under the above assumptions we get the geometry which is similar to that obtained when the member of $K(m, n)$ hierarchy is reduced to an ODE

describing the set of TW solutions. To show that, let us consider the system arising from (34) under the above assumptions:

$$\begin{aligned}\frac{dR}{dT} &= -2\sigma WR^{2(n+1)}, \\ \frac{dW}{dT} &= 2(n+1)\sigma W^2 R^{2n+1} + 2\alpha_2 R^{2n+3} - 2\alpha_1 R^{2n+2}.\end{aligned}\quad (41)$$

System (41) possesses two stationary points lying on the horizontal axis: the point $(0, 0)$, and the point $(R_1, 0)$, where $R_1 = \alpha_1/\alpha_2$. Analysis of the Jacobi matrix shows that $(R_1, 0)$ is a center. Moreover, $R_1 < R_*$ when $n > -2$ so the closed trajectory corresponding to $K = 0$ encircles the domain filled with the periodic trajectories. Using the asymptotic decomposition of the solution represented by the homoclinic trajectory near the origin, it can be shown that the trajectory reaches the stationary point $(0, 0)$ in finite “time” so it does correspond to the compacton.

Besides, we can get more direct evidence of the existence of compacton-like solutions by integrating equation (40). Taking in mind that $W = -dR/d\xi$, we obtain the following equation:

$$\frac{1}{\gamma} \frac{dR}{\sqrt{1 - \left(\frac{R}{\gamma} - 1\right)^2}} = \delta d\xi,$$

where $\gamma = \frac{\alpha_1(n+2)}{\alpha_2(2n+3)}$, $\delta = \sqrt{\frac{\alpha_2}{\sigma(n+2)}}$. Solving this equation we get:

$$R[\xi] = \begin{cases} \gamma [1 + \sin(\delta(\xi - \xi_0))], & \text{if } -\pi \leq 2\delta(\xi - \xi_0) \leq 3\pi, \\ 0, & \text{otherwise.} \end{cases}\quad (42)$$

5 Conclusions and discussion.

In this work we have shown that hydrodynamic system of balance equations (9) closed by DES accounting for non-local effects possesses the compacton-like solutions. In contrast to analogous solutions to most of the compacton-supporting equations, the presented solutions do not form a one-parameter family. Concerning (39), for fixed values of the parameters A , α_1 and α_2 there exists exactly one pair of compactons moving with velocity \sqrt{A} in the opposite directions. Whether these solutions are of interest from the point of view of applications or not, depends on their stability and behavior during the mutual collisions. Discussion of these topics goes beyond the scope of this paper. Let us mention, however, that the mere fact of existence of compactons is the consequence of the non-local effects incorporation. To our best knowledge, this type of solutions does not exist in any local hydrodynamic-type model, i.e. the system of balance equations (9) closed by the functional state equation $p = \Phi(\rho)$. Let us note that invariant TW solutions very often play rule of intermediate asymptotics [29, 30], attracting near-by, not

necessarily invariant, solutions. This feature demonstrates compacton-like solutions of another non-local model obtained when the system of balance equations (9) is closed by the DES (14), describing relaxing effects. Solutions with compact supports appear in this model merely in presence of the mass force. Exactly one compacton-like solution occurs to exist for the given set of the parameters, yet this solution serves as an attractor for the wave packs created by the wide class of the initial value problems [31]. In contrast to the above mentioned relaxing model, incorporation of the effects of spatial non-locality leads to the existence of a pair of compacton-like solutions in absence of an external force. In fact, it is shown for the first time that a non-local hydrodynamic-type model possesses more than one compacton-like solution and from now on there exists the opportunity to investigate their behavior during the collisions.

There are some evidences in favour of the stable behavior of these compactons and their elastic collisions. The first is connected with the fact that by proper choice of parameters A , α_1 and α_2 , a necessary condition for stable evolution taking the form $\partial p / \partial \rho > 0$, $\partial^2 p / \partial^2 \rho > 0$ [27] can be fulfilled for DES employed in system (19). Besides, this system possesses at least two conservation laws. Though actually it is not known how many conserved quantities assure the stability of the wave patterns during the collisions, it is almost certain that this number should not necessarily be infinite. For example, the members of the Rosenau–Hyman $K(m, n)$ hierarchy, possessing with certain four conserved quantities, collide almost elastically and so is with the compacton-compacton and kovaton-compacton interactions in the Pikovsky–Rosenau model [8].

In conclusion, let us touch upon the interpretation of TW with compact support (the question which was asked during the IV Workshop and was not addressed there). We'll discuss this issue by comparing the KdV equation with a member of $K(m, n)$ hierarchy. So, dispersion of a wave pack is very often interpreted as the manifestation of parabolic effects in the hyperbolic-type model [32]. And it is well known that dispersion is caused by the presence of the terms with higher-order spatial derivatives. In case of KdV equation this is the third-order linear term u_{xxx} while in case of the $K(m, n)$ -equation the term $(u^n)_{xxx}$ is responsible for dispersion. Comparing these terms we can see that in the first case the effective “transport coefficient” is equal to unity, whereas in the second one it is proportional to u^{n-1} . So in case of a compact initial perturbation the $K(m, n)$ -equation's “conductivity” is nonzero merely at the finite domain containing the wave pack which remains thus compact forever.

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Integrability and symmetries of difference equations: the Adler–Bobenko–Suris case

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The integrability aspects and the symmetry analysis of the Adler–Bobenko–Suris difference equations are presented. Using their multidimensional consistency, auto-Bäcklund transformations and Lax pairs are constructed for each one of them. Employing the symmetry analysis of Viallet’s equation, infinite hierarchies of generalized symmetries for the Adler–Bobenko–Suris equations are explicitly given, along with corresponding hierarchies of integrable differential–difference equations.

1 Introduction

It is well known that, integrable differential equations, like the Korteweg-de Vries (KdV), the sine-Gordon and the nonlinear Schrödinger, have many properties in common. They can be written as the compatibility condition of a Lax pair, which plays a crucial role in solving the initial value problem by the inverse scattering transform. They admit auto-Bäcklund transformations, which allow us to construct new solutions from known ones [1].

Other properties arise from the symmetries of these equations. Specifically, they admit infinite hierarchies of generalized symmetries and, consequently, infinite conservation laws [23]. Also, they reduce to Painlevé equations and result from reductions of the Yang-Mills equations [15]. All the above mentioned characteristics may be considered as criteria establishing the integrability of a differential equation.

Analogous characteristics seem to be in common among integrable difference equations defined on an elementary square of the lattice. In the most well known cases, these equations are characterized by their “multidimensional consistency”. For a two dimensional equation, this means that, the equation may be imposed in a consistent way on a three dimensional lattice, and, consequently, on a multidimensional one. This property incorporates some of the above mentioned integrability aspects. Specifically, the consistency property provides the means to derive algorithmically a Bäcklund transformation and a Lax pair of the difference equation under consideration [9, 16].

Adler, Bobenko and Suris (ABS) used multidimensional consistency as the key property characterizing integrable difference equations to classify integrable scalar

difference equations [3, 4]. The equations emerged from the classification [3] split into the **H** and **Q** list and comprise, apart from known ones, a number of new cases. The subsequent study of the resulting equations has led to construction of exact solutions [7, 8], Bäcklund transformations [6], symmetries [24, 30] and conservation laws [25].

In this paper, the ABS equations will be used as the illustrative example to investigate the similarities between discrete and continuous integrable equations, as described above. For this purpose, we will generalize some recent results and, subsequently, we will use them to derive new integrability aspects of the ABS equations, as well as hierarchies of integrable differential-difference equations.

Specifically, first we will introduce a new class of autonomous and affine linear difference equations which possess the Klein symmetry. Among the members of this class are Viallet's equation **Q5**, [32], and the ABS equations. Next, the consistency property will be used to derive an auto-Bäcklund transformation for each one of the ABS equations, in terms of which Lax pairs will also be constructed.

The second direction of our investigation will be the symmetries of the ABS equations. In particular, first we will present the symmetry analysis of the class of equations under consideration and apply it to **Q5**. Using the latter results, we will prove that the ABS equations admit infinite hierarchies of generalized symmetries, which are determined inductively using linear differential operators. Finally, these hierarchies will lead to corresponding hierarchies of integrable differential-difference equations, for which auto-Bäcklund transformations and Lax pairs will be explicitly constructed.

The paper is organized as follows. In Section 2 we introduce the notation used in sections that follow, and present the background material on symmetries of difference equations. In the next section, the class of autonomous and affine linear difference equations possessing the Klein symmetry is presented, along with **Q5** and the ABS equations.

Section 4 deals with the integrability aspects of the ABS equations, and, in particular, with their auto-Bäcklund transformations and Lax pairs. Section 5 contains the symmetry analysis of the equations belonging in the class under consideration, as well as the construction of infinite hierarchies of symmetries for all of the ABS equations.

2 Notation and preliminaries on symmetries of difference equations

We first introduce the notation that it will be used in what follows. Also, we present those definitions on symmetries of difference equations that will be used in the next sections. For details on the subject, we refer the reader to the clear and extended review by Levi and Winternitz [14].

A partial difference equation is a functional relation among the values of a function $u : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{C}$ at different points of the lattice, which may involve the

independent variables n, m and the lattice spacings α, β , as well, i.e. a relation of the form

$$\mathcal{E}(n, m, u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, \dots; \alpha, \beta) = 0. \quad (1)$$

In this relation, $u_{(i,j)}$ is the value of the function u at the lattice point $(n+i, m+j)$, e.g.

$$u_{(0,0)} = u(n, m), \quad u_{(1,0)} = u(n+1, m), \quad u_{(0,1)} = u(n, m+1),$$

and this is the notation that we will adopt for the values of the function u from now on.

The analysis of such equations is facilitated by the introduction of two translation operators acting on functions on \mathbb{Z}^2 , defined by

$$\left(\mathcal{S}_n^{(k)} u\right)_{(0,0)} = u_{(k,0)}, \quad \left(\mathcal{S}_m^{(k)} u\right)_{(0,0)} = u_{(0,k)}, \quad \text{where } k \in \mathbb{Z},$$

respectively.

Let G be a connected one-parameter group of transformations acting on the dependent variable $u_{(0,0)}$ of the lattice equation (1) as follows

$$G : u_{(0,0)} \longrightarrow \tilde{u}_{(0,0)} = \Phi(n, m, u_{(0,0)}; \varepsilon), \quad \varepsilon \in \mathbb{R}.$$

Then, the prolongation of the group action of G on the shifted values of u is defined by

$$G^{(k)} : (u_{(i,j)}) \longrightarrow (\tilde{u}_{(i,j)} = \Phi(n+i, m+j, u_{(i,j)}; \varepsilon)). \quad (2)$$

The transformation group G is a *Lie point symmetry* of the lattice equation (1) if it transforms any solution of (1) to another solution of the same equation. Equivalently, G is a symmetry of equation (1), if the latter is not affected by transformation (2), i.e.

$$\mathcal{E}(n, m, \tilde{u}_{(0,0)}, \tilde{u}_{(1,0)}, \tilde{u}_{(0,1)}, \dots; \alpha, \beta) = 0.$$

In essence, the action of the transformation group G is expressed by its *infinitesimal generator*, i.e. the vector field

$$\mathbf{x} = R(n, m, u_{(0,0)}) \partial_{u_{(0,0)}},$$

where $R(n, m, u_{(0,0)})$ is defined by

$$R(n, m, u_{(0,0)}) = \left. \frac{d}{d\varepsilon} \Phi(n, m, u_{(0,0)}; \varepsilon) \right|_{\varepsilon=0},$$

and is referred to as *the characteristic*.

Using the latter, one defines the k -th order forward prolongation of \mathbf{x} , namely the vector field

$$\mathbf{x}^{(k)} = \sum_{i=0}^k \sum_{j=0}^{k-i} \left(\mathcal{S}_n^{(i)} \circ \mathcal{S}_m^{(j)} R \right) (n, m, u_{(0,0)}) \partial_{u_{(i,j)}}.$$

This allows one to give an infinitesimal form of the criterion for G to be a symmetry of equation (1). It consists of the condition

$$\mathbf{x}^{(k)} (\mathcal{E} (n, m, u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, \dots; \alpha, \beta)) = 0, \quad (3)$$

which should hold for every solution of equation (1) and, therefore, the latter should be taken into account, when condition (3) is tested out explicitly.

Equation (3) delivers the most general infinitesimal Lie point symmetry of equation (1). The solutions of (3) determine the Lie algebra \mathfrak{g} of the corresponding symmetry group G and the latter can be constructed by exponentiation:

$$\Phi(n, m, u_{(0,0)}; \varepsilon) = \exp(\varepsilon \mathbf{x}) u_{(0,0)}.$$

On the other hand, one may consider a group of transformations Γ acting, not only on the dependent variable u , but on the lattice parameters α, β as well. This leads to the notion of *the extended Lie point symmetry*. The infinitesimal generator of the group action of Γ is a vector field of the form

$$\mathbf{v} = R(n, m, u_{(0,0)}) \partial_{u_{(0,0)}} + \xi(n, m, \alpha, \beta) \partial_\alpha + \zeta(n, m, \alpha, \beta) \partial_\beta,$$

and the infinitesimal criterion for a connected group Γ to be an extended Lie point symmetry of equation (1) is

$$\mathbf{v}^{(k)} (\mathcal{E} (n, m, u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, \dots; \alpha, \beta)) = 0,$$

where

$$\begin{aligned} \mathbf{v}^{(k)} = & \sum_{i=0}^k \sum_{j=0}^{k-i} \left(\mathcal{S}_n^{(i)} \circ \mathcal{S}_m^{(j)} R \right) (n, m, u_{(0,0)}) \partial_{u_{(i,j)}} \\ & + \xi(n, m, \alpha, \beta) \partial_\alpha + \zeta(n, m, \alpha, \beta) \partial_\beta \end{aligned}$$

is the k -th forward prolongation of the symmetry generator \mathbf{v} .

By extending the geometric transformations to the more general ones, which depend, not only on n, m and $u_{(0,0)}$, but also on the shifted values of u , we arrive naturally at the notion of the *generalized symmetry*. For example, a five point generalized symmetry may be given by the vector field

$$\mathbf{v} = R(n, m, u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(-1,0)}, u_{(0,-1)}) \partial_{u_{(0,0)}},$$

while an extended five point generalized symmetry is given by the vector field

$$\begin{aligned} \mathbf{v} = & R(n, m, u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(-1,0)}, u_{(0,-1)}) \partial_{u_{(0,0)}} \\ & + \xi(n, m, \alpha, \beta) \partial_\alpha + \zeta(n, m, \alpha, \beta) \partial_\beta, \end{aligned}$$

respectively.

3 A class of difference equations

In this section we introduce a class of lattice equations, which contains, among others, Viallet's equation **Q5**, [32], and the ABS equations **H1-H3** and **Q1-Q4**, [3].

Specifically, the members of this family are equations involving the values of a function u at the vertices of an elementary quadrilateral, as shown in Figure 1.

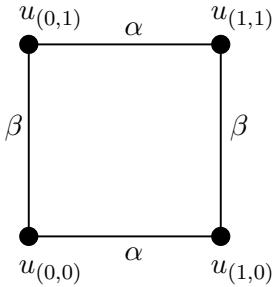


Figure 1.

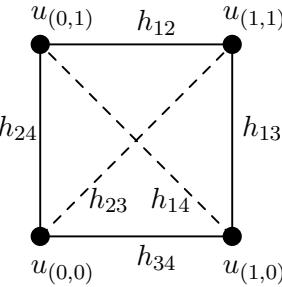


Figure 2.

They are autonomous

$$Q(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(1,1)}) = 0, \quad (4)$$

and the function Q satisfies two basic requirements: it is affine linear and depends explicitly on the four indicated values of u , i.e.

$$\partial_{u_{(i,j)}} Q(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(1,1)}) \neq 0$$

and

$$\partial_{u_{(i,j)}}^2 Q(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(1,1)}) = 0,$$

where $i, j = 0, 1$.

Moreover, we impose that, the function Q possesses the Klein symmetry:

$$\begin{aligned} Q(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(1,1)}) &= \tau Q(u_{(1,0)}, u_{(0,0)}, u_{(1,1)}, u_{(0,1)}) \\ &= \tau' Q(u_{(0,1)}, u_{(1,1)}, u_{(0,0)}, u_{(1,0)}), \end{aligned} \quad (5)$$

where $\tau = \pm 1$ and $\tau' = \pm 1$.

The affine linearity of Q implies that one can define six different polynomials in terms of Q and its derivatives [3, 4, 30], as illustrated in Figure 2. Specifically, a polynomial h_{ij} is defined by

$$h_{ij} = h_{j,i} := Q Q_{,ij} - Q_{,i} Q_{,j}, \quad i \neq j, \quad i, j = 1, \dots, 4,$$

where $Q_{,i}$ denotes the derivative of Q with respect to its i -th argument and $Q_{,ij}$ the second order derivative of Q with respect to its i -th and j -th argument. The

h_{ij} 's are bi-quadratic polynomials of the values of u assigned to the end-points of the edge or diagonal at which they correspond. Finally, the relations

$$h_{12} h_{34} = h_{13} h_{24} = h_{14} h_{23} \quad (6)$$

hold in view of the equation $Q = 0$.

The Klein symmetry of Q implies the following.

1. The polynomials assigned to the edges read as follows.

$$h_{ij}(x, y) = \begin{cases} h_1(x, y), & |i - j| = 1 \\ h_2(x, y), & |i - j| = 2 \end{cases}, \quad (7)$$

where $i \neq j$, $\{i, j\} \neq \{2, 3\}$, and h_1, h_2 are biquadratic and symmetric polynomials of their arguments.

2. The diagonal polynomials have the same form, i.e.

$$h_{14}(x, y) = h_{23}(x, y) = G(x, y), \quad (8)$$

where G is symmetric.

The most generic equation belonging in this class is Viallet's equation **Q5**, [32], for which the function Q in (4) has the form

$$\begin{aligned} Q(u, x, y, z) := & a_1 uxyz + a_2 (uyz + xyz + uxz + uxy) + a_3 (ux + yz) \\ & + a_4 (uz + xy) + a_5 (xz + uy) + a_6 (u + x + y + z) + a_7, \end{aligned} \quad (9)$$

where a_i are free parameters.

The ABS equations **H1–H3** and **Q1–Q4** follow from equation **Q5** by choosing appropriately the free parameters a_i . The specific choices are presented in the following list.

i) Equations **H1–H3** and **Q1–Q3** correspond to the choice $a_1 = a_2 = 0$, while the rest parameters are chosen according to the next table.

	a_3	a_4	a_5	a_6	a_7
H1	1	0	-1	0	$\beta - \alpha$
H2	1	0	-1	$\beta - \alpha$	$\beta^2 - \alpha^2$
H3	α	0	$-\beta$	0	$\delta(\alpha^2 - \beta^2)$
Q1	α	$\beta - \alpha$	$-\beta$	0	$\delta\alpha\beta(\alpha - \beta)$
Q2	α	$\beta - \alpha$	$-\beta$	$\alpha\beta(\alpha - \beta)$	$\alpha\beta(\beta - \alpha)(\alpha^2 - \alpha\beta + \beta^2)$
Q3	$\beta(\alpha^2 - 1)$	$\beta^2 - \alpha^2$	$\alpha(1 - \beta^2)$	0	$\frac{-\delta(\alpha^2 - \beta^2)(\alpha^2 - 1)(\beta^2 - 1)}{4\alpha\beta}$

ii) Equation **Q4** is the master equation of the members of the ABS list, [5], and corresponds to the following choices of the parameters:

$$\begin{aligned} a_1 &= a + b, \quad a_2 = -a\beta - b\alpha, \\ a_3 &= \frac{ab(a+b)}{2(\alpha-\beta)} + a\beta^2 - (2\alpha^2 - \frac{g_2}{4})b, \\ a_4 &= a\beta^2 + b\alpha^2 \\ a_5 &= \frac{ab(a+b)}{2(\beta-\alpha)} + b\alpha^2 - (2\beta^2 - \frac{g_2}{4})a, \\ a_6 &= \frac{g_3}{2}a_1 - \frac{g_2}{4}a_2, \quad a_7 = \frac{g_2^2}{16}a_1 - g_3a_2, \end{aligned}$$

where

$$a^2 = p(\alpha), \quad b^2 = p(\beta), \quad p(x) = 4x^3 - g_2x - g_3.$$

Remark 1. In the above list, α and β are the lattice parameters associated with the lattice directions, cf. Figure 1. Since the ABS equations depend explicitly on these parameters, in what follows we will denote them as

$$Q(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(1,1)}; \alpha, \beta) = 0.$$

It turns out [3, 30] that, the polynomials h_1, h_2 in equation (7) are related to a polynomial h as follows:

$$h_1(x, y) = h(x, y; \alpha, \beta), \quad h_2(x, y) = h(x, y; \beta, \alpha).$$

Remark 2. Equations **H1**, **H3** _{$\delta=0$} and **Q1** _{$\delta=0$} are also referred to as the discrete potential, modified and Schwarzian KdV equation, respectively, cf. review [17]. Equations **Q1** and **Q3** _{$\delta=0$} , are related to Nijhoff-Quispel-Capell (NQC) equation, [21], cf. also [3, 8]. Equation **Q4** was first presented by Adler in [2], as the superposition principle of the auto-Bäcklund transformation for the Krichever-Novikov equation.

4 Integrability aspects of the ABS equations: auto-Bäcklund transformations and Lax pairs

An important characteristic of the ABS equations is their multidimensional consistency [3], which can be regarded as the discrete analog of hierarchies of commuting flows of integrable differential equations, [22]. For the sake of self-containment, let us first explain the consistency property.

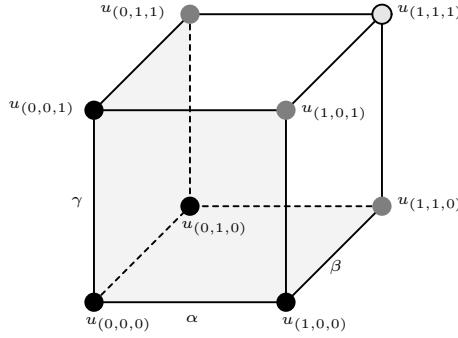


Figure 3. Three dimensional consistency

Suppose that, the function u depends also on a third lattice variable k , with which the parameter γ is associated. Moreover, each face of the cube carries a copy of the equation, cf. Figure 3, which involves the values of u assigned to the vertices of the face, as well as the corresponding parameters assigned to the edges. Now, starting with the values of u at the black vertices and using the equations on the gray faces, one can uniquely evaluate the values of u at the gray vertices of the cube. Next, the equations on the white faces provide three different ways to calculate the value $u_{(1,1,1)}$. The equation possesses the consistency property if these different ways lead to the same result. Moreover, if the resulting value $u_{(1,1,1)}$ is independent of $u_{(0,0,0)}$, then the equation fulfills the tetrahedron property.

The ABS equations possess both of the above properties, [3]. The consistency and the tetrahedron properties imply that, the polynomial h related to the edges is factorized as

$$h(x, y; \alpha, \beta) = k(\alpha, \beta) f(x, y, \alpha),$$

where the function $k(\alpha, \beta)$ is antisymmetric, i.e.

$$k(\beta, \alpha) = -k(\alpha, \beta).$$

Furthermore, the discriminant

$$d(x) = f_{,y}^2 - 2 f f_{,yy}$$

is independent of the lattice parameters. The functions f , k and G corresponding to each one of the ABS equations are presented in Appendix A.

Identifying the shifts of u with respect to the third lattice variable k with a new function \tilde{u} , the consistency property leads to Bäcklund transformations for the equations under consideration, [6, 33].

Proposition 1. *The system*

$$\mathbb{B}_d(u, \tilde{u}, \lambda) := \begin{cases} Q(u_{(0,0)}, u_{(1,0)}, \tilde{u}_{(0,0)}, \tilde{u}_{(1,0)}; \alpha, \lambda) = 0 \\ Q(u_{(0,0)}, u_{(0,1)}, \tilde{u}_{(0,0)}, \tilde{u}_{(0,1)}; \beta, \lambda) = 0 \end{cases} \quad (10)$$

defines an auto-Bäcklund transformation for the ABS equation

$$Q(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(1,1)}; \alpha, \beta) = 0. \quad (11)$$

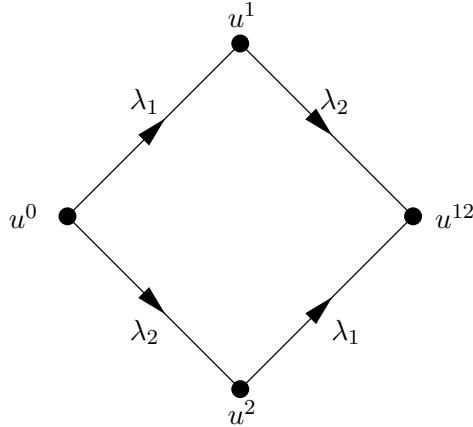


Figure 4. Bianchi commuting diagramm

A direct consequence of the previous result is

Proposition 2 (Superposition principle). *Let u^1, u^2 be solutions of the ABS equation (11), generated by means of the auto-Bäcklund transformation \mathbb{B}_d from a known solution u^0 via the Bäcklund parameters λ_1 and λ_2 , respectively. Then, there is a third solution u^{12} , which is given algebraically by*

$$Q(u^0, u^1, u^2, u^{12}; \lambda_1, \lambda_2) = 0,$$

and is constructed according to Bianchi commuting diagramm, Figure 4.

Remark 3. Auto-Bäcklund transformations different from the ones presented here, as well as some hetero-Bäcklund transformations, were given recently by Atkinson in [6].

The consistency property also serves in the construction of a Lax pair, through an algorithmic procedure [9, 13, 16]. Alternatively, using the fact that, a Bäcklund transformation may be regarded as a gauge transformation for the Lax pair, [10], one may construct a Lax pair for the ABS equations using the equations constituting \mathbb{B}_d [33]. The two approaches lead essentially to the same result, which can be stated as

Proposition 3. *The equation of the ABS class*

$$Q(u_{(0,0)}, u_{(1,0)}, u_{(0,1)}, u_{(1,1)}; \alpha, \beta) = 0$$

arises in the compatibility condition of the linear system

$$\Psi_{(1,0)} = L(u_{(0,0)}, u_{(1,0)}; \alpha, \lambda) \Psi_{(0,0)}, \quad \Psi_{(0,1)} = L(u_{(0,0)}, u_{(0,1)}; \beta, \lambda) \Psi_{(0,0)}, \quad (12)$$

where

$$L(x^1, x^2; a, \lambda) = \frac{1}{\sqrt{k(a, \lambda) f(x^1, x^2, a)}} \begin{pmatrix} Q_{,4} & -Q_{,34} \\ Q & -Q_{,3} \end{pmatrix}, \quad (13)$$

with $Q = Q(x^1, x^2, x^3, x^4; a, \lambda)$ and its derivatives ($Q_{,i} = \partial_{x^i} Q$) being evaluated at $x^3 = x^4 = 0$.

5 Infinite hierarchies of generalized symmetries

In this section first we deal with the symmetry analysis of the equations belonging in the class presented in Section 3, which generalizes the corresponding results of [30]. Next, we apply this analysis to equation **Q5** and determine its symmetries. Using the latter results along with the ones of [30], we explicitly construct infinite hierarchies of generalized symmetries for all of the ABS equations. Finally, using the correspondence of symmetry generators and group of transformations, hierarchies of integrable differential-difference equations are presented, along with their Lax pairs.

The symmetry analysis of the equations presented in Section 3 is contained in the following two propositions.

Proposition 4. *Every equation in this class admits two three point generalized symmetries with generators the vector fields*

$$\mathbf{v}_n = \left(\frac{h_1(u_{(0,0)}, u_{(1,0)})}{u_{(1,0)} - u_{(-1,0)}} - \frac{1}{2} h_{1,u_{(1,0)}}(u_{(0,0)}, u_{(1,0)}) \right) \partial_{u_{(0,0)}},$$

and

$$\mathbf{v}_m = \left(\frac{h_2(u_{(0,0)}, u_{(0,1)})}{u_{(0,1)} - u_{(0,-1)}} - \frac{1}{2} h_{2,u_{(0,1)}}(u_{(0,0)}, u_{(0,1)}) \right) \partial_{u_{(0,0)}},$$

respectively.

Proposition 5. *Let an equation in this class be such that, the matrices*

$$\mathcal{G}_i = \left. \begin{pmatrix} h_i(x, y) & G(x, z) & G(x, w) \\ h_{i,x}(x, y) & G_{,x}(x, z) & G_{,x}(x, w) \\ h_{i,xx}(x, y) & G_{,xx}(x, z) & G_{,xx}(x, w) \end{pmatrix} \right|_{x=0}, \quad i = 1, 2, \quad (14)$$

are invertible. Then, the generator of any five point generalized symmetry of this equation will be necessarily a vector field of the form

$$\mathbf{v} = a(n)\mathbf{v}_n + b(m)\mathbf{v}_m + \frac{1}{2} \psi(n, m, u_{(0,0)}) \partial_{u_{(0,0)}},$$

where the functions $a(n)$, $b(m)$ and $\psi(n, m, u_{(0,0)})$ satisfy the determining equation

$$\begin{aligned}
 & (a(n) - a(n+1)) h_1(u_{(0,0)}, u_{(1,0)})^2 \partial_{u_{(1,0)}} \left(\frac{G(u_{(1,0)}, u_{(0,1)})}{h_1(u_{(0,0)}, u_{(1,0)})} \right) \\
 & + (b(m) - b(m+1)) h_2(u_{(0,0)}, u_{(0,1)})^2 \partial_{u_{(0,1)}} \left(\frac{G(u_{(1,0)}, u_{(0,1)})}{h_2(u_{(0,0)}, u_{(0,1)})} \right) \\
 & + G(u_{(1,0)}, u_{(0,1)}) \psi(n, m, u_{(0,0)}) + h_2(u_{(0,0)}, u_{(0,1)}) \psi(n+1, m, u_{(1,0)}) \\
 & + h_1(u_{(0,0)}, u_{(1,0)}) \psi(n, m+1, u_{(0,1)}) - Q_{,u_{(1,1)}}^2 \psi(n+1, m+1, u_{(1,1)}) = 0.
 \end{aligned}$$

Proof. The proofs of the above propositions follow from the corresponding ones given in [30] by making the following changes

$$h(u, x, \alpha, \beta) \longrightarrow h_1(u, x), \quad h(u, x, \beta, \alpha) \longrightarrow h_2(u, x).$$

■

Applying the above symmetry analysis to equation **Q5**, one concludes that the latter admits only the pair of three point generalized symmetries generated by \mathbf{v}_n and \mathbf{v}_m given in Proposition 4 with

$$\begin{aligned}
 h_1(x, y) &= (a_3 + a_1xy + a_2(x+y))(a_7 + a_3xy + a_6(x+y)) - \\
 &\quad (a_6 + a_5x + (a_4 + a_2x)y)(a_6 + a_4x + (a_5 + a_2x)y), \\
 h_2(x, y) &= (a_5 + a_1xy + a_2(x+y))(a_7 + a_5xy + a_6(x+y)) - \\
 &\quad (a_6 + a_4x + (a_3 + a_2x)y)(a_6 + a_3x + (a_4 + a_2x)y).
 \end{aligned}$$

Furthermore, it can be shown by a direct computation that, the commutator of these two symmetry generators is a trivial symmetry, [23], i.e. the characteristic of the resulting vector field vanishes on solutions of equation **Q5**. In view of this observation, we will write

$$[\mathbf{v}_n, \mathbf{v}_m] = 0. \quad (15)$$

Since **Q5** contains all of the ABS equations, the above result is also valid for the corresponding symmetry generators of the latter equations.

On the other hand, [30], all of the ABS equations admit a pair of three point generalized symmetries, as well as a pair of extended generalized symmetries, the generators of which are the vector fields

$$\mathbf{v}_n \equiv \mathbf{v}_n^{[0]} = R_n^{[0]} \partial_{u_{(0,0)}}, \quad \mathbf{v}_m \equiv \mathbf{v}_m^{[0]} = R_m^{[0]} \partial_{u_{(0,0)}}, \quad (16)$$

and

$$\mathbf{V}_n = n R_n^{[0]} \partial_{u_{(0,0)}} - r(\alpha) \partial_\alpha, \quad \mathbf{V}_m = m R_m^{[0]} \partial_{u_{(0,0)}} - r(\beta) \partial_\beta, \quad (17)$$

respectively, with

$$\begin{aligned} R_n^{[0]} &= \frac{f(u_{(0,0)}, u_{(1,0)}, \alpha)}{u_{(1,0)} - u_{(-1,0)}} - \frac{1}{2} f_{,u_{(1,0)}}(u_{(0,0)}, u_{(1,0)}, \alpha), \\ R_m^{[0]} &= \frac{f(u_{(0,0)}, u_{(0,1)}, \beta)}{u_{(0,1)} - u_{(0,-1)}} - \frac{1}{2} f_{,u_{(0,1)}}(u_{(0,0)}, u_{(0,1)}, \beta). \end{aligned}$$

The form of the function r appearing in (17) depends on the particular equation and is given in the following table.

Equation	H1	H2	H3	Q1	Q2	Q3	Q4
$r(x)$	1	1	$-\frac{x}{2}$	1	1	$-\frac{x}{2}$	$-\frac{1}{2} \sqrt{4x^3 - g_2x - g_3}$

The generalized symmetries have been used effectively to derive reductions of partial difference equations to discrete analogues of the Painlevé equations. The first result in this direction was presented by Nijhoff and Papageorgiou in [19], where the reduction of **H1** to a discrete analogue of Painlevé II was given.

The extended symmetries can be used effectively in two different ways. The first way is the construction of solutions through symmetry reductions, [31, 33]. Assuming that the function u depends continuously on the lattice parameters α, β , one may derive solutions of the given difference equation which remain invariant under the action of both of the extended symmetries generated by the vector fields $\mathbf{V}_n, \mathbf{V}_m$. In this fashion, one is led to a system of differential-difference equations, which can be equivalently written as an integrable system of differential equations. On the one hand, this differential system is related to the so called generating partial differential equations, introduced by Nijhoff, Hone and Joshi in [18], cf. also [28, 29]. On the other hand, some of its similarity solutions reveal new connections between discrete and continuous Painlevé equations, [31].

The other way to use the extended symmetries was suggested by Rasin and Hydon in [24]. Specifically, it was pointed out that, the above extended generalized symmetries can be regarded as master symmetries of the corresponding generalized ones. To prove this, one has to show that the following commutation relations hold

$$[\mathbf{V}_n, \mathbf{v}_n^{[0]}] \neq 0, \quad [[\mathbf{V}_n, \mathbf{v}_n^{[0]}], \mathbf{v}_n^{[0]}] = 0,$$

as well as similar relations for the generators $\mathbf{V}_m, \mathbf{v}_m^{[0]}$. It should be noted that, the commutators $[\mathbf{V}_n, \mathbf{v}_m^{[0]}], [\mathbf{V}_m, \mathbf{v}_n^{[0]}]$ lead to trivial symmetries because of the relation (15), e.g. $[\mathbf{V}_n, \mathbf{v}_m^{[0]}] = n[\mathbf{v}_n^{[0]}, \mathbf{v}_m^{[0]}]$.

Now, writing out explicitly the commutators $[\mathbf{V}_n, \mathbf{v}_n^{[0]}]$ and $[\mathbf{V}_m, \mathbf{v}_m^{[0]}]$, one arrives at

$$\mathbf{v}_n^{[1]} := [\mathbf{V}_n, \mathbf{v}_n^{[0]}] = R_n^{[1]} \partial_{u_{(0,0)}}, \quad (18)$$

$$\mathbf{v}_m^{[1]} := [\mathbf{V}_m, \mathbf{v}_m^{[0]}] = R_m^{[1]} \partial_{u_{(0,0)}}, \quad (19)$$

where

$$\begin{aligned} R_n^{[1]} &= \left(\left(S_n R_n^{[0]} \right) \partial_{u_{(1,0)}} - \left(S_n^{(-1)} R_n^{[0]} \right) \partial_{u_{(-1,0)}} - r(\alpha) \partial_\alpha \right) R_n^{[0]} \\ &= \frac{f(u_{(0,0)}, u_{(1,0)}, \alpha) f(u_{(0,0)}, u_{(-1,0)}, \alpha) (u_{(2,0)} - u_{(-2,0)})}{(u_{(1,0)} - u_{(-1,0)})^2 (u_{(2,0)} - u_{(0,0)}) (u_{(-2,0)} - u_{(0,0)})}, \end{aligned} \quad (20)$$

$$\begin{aligned} R_m^{[1]} &= \left(\left(S_m R_m^{[0]} \right) \partial_{u_{(0,1)}} - \left(S_m^{(-1)} R_m^{[0]} \right) \partial_{u_{(0,-1)}} - r(\beta) \partial_\beta \right) R_m^{[0]} \\ &= \frac{f(u_{(0,0)}, u_{(0,1)}, \beta) f(u_{(0,0)}, u_{(0,-1)}, \beta) (u_{(0,2)} - u_{(0,-2)})}{(u_{(0,1)} - u_{(0,-1)})^2 (u_{(0,2)} - u_{(0,0)}) (u_{(0,-2)} - u_{(0,0)})}. \end{aligned} \quad (21)$$

Using definitions (18-19) and the Jacobi identity, it can be verified straightforwardly that:

$$[v_n^{[1]}, v_m^{[0]}] = [v_n^{[1]}, \mathbf{V}_m] = [v_m^{[1]}, v_n^{[0]}] = [v_m^{[1]}, \mathbf{V}_n] = [v_n^{[1]}, v_m^{[1]}] = 0.$$

On the other hand, the commutators

$$[v_n^{[1]}, v_n^{[0]}] = [v_m^{[1]}, v_m^{[0]}] = 0$$

can be verified by using (16), (18-21) and the properties of polynomial f , i.e. it is biquadratic and symmetric.

Thus, we have proven that the symmetry generators \mathbf{V}_n , \mathbf{V}_m are master symmetries of $v_n^{[0]}$ and $v_m^{[0]}$, respectively. Consequently, infinite hierarchies of generalized symmetries can be constructed in this fashion, the members of which are defined inductively:

$$v_i^{[k+1]} = R_i^{[k+1]} \partial_{u_{(0,0)}} := [\mathbf{V}_i, v_i^{[k]}], \quad k = 0, 1, \dots \quad \text{and} \quad i = n, m. \quad (22)$$

The characteristics $R_i^{[k]}$ involve the values of u at $(2k+3)$ points in the i direction of the lattice and are determined by applying successively the linear differential operators

$$\mathcal{R}_n = \sum_{\ell=-\infty}^{\infty} \ell \left(S_n^{(\ell)} R_n^{[0]} \right) \partial_{u_{(\ell,0)}} - r(\alpha) \partial_\alpha, \quad (23)$$

$$\mathcal{R}_m = \sum_{\ell=-\infty}^{\infty} \ell \left(S_m^{(\ell)} R_m^{[0]} \right) \partial_{u_{(0,\ell)}} - r(\beta) \partial_\beta \quad (24)$$

on $R_i^{[0]}$, i.e.

$$R_i^{[k]} = \mathcal{R}_i^k R_i^{[0]}, \quad k = 0, 1, \dots \quad \text{and} \quad i = n, m. \quad (25)$$

Remark 4. The linear operators $\mathcal{R}_n, \mathcal{R}_m$ may be regarded as recursion operators for equations **Q3** and **Q4**. This follows from the fact [30] that, these equations admit only the symmetries generated by $\mathbf{v}_n^{[0]}$ and $\mathbf{v}_m^{[0]}$.

Remark 5. It is worth mentioning some previous results on higher order generalized symmetries for the ABS equations. Specifically, a procedure for the construction of hierarchies of generalized symmetries was presented in [13], where the relation of the ABS equations to Yamilov's discrete Krichever–Novikov equation (YdKN), [34, 35], was explored. On the other hand, particular results about **H1** and **Q1** _{$\delta=0$} were presented in [11] and [12], respectively, the derivation of which based on the associated spectral problem. The advantage of our approach compared to the above outcomes is that, it leads straightforwardly to explicit expressions for the higher order generalized symmetries for all of the ABS equations.

The correspondence of the generators $\mathbf{v}_i^{[k]}$ to group of transformations, [23], leads to hierarchies of integrable differential-difference equations, the first members of which are special cases of YdKN equation, [13], cf. also [5].

Dropping the one of the two indices corresponding to shifts with respect to n or m , and denoting by a the corresponding lattice parameter, these hierarchies have the form

$$\frac{du}{d\epsilon_k} = \mathcal{R}^k \left(\frac{f(u, u_1, a)}{u_1 - u_{-1}} - \frac{1}{2} f_{,u_1}(u, u_1, a) \right), \quad k = 0, 1, \dots, \quad (26)$$

where ϵ_k is the corresponding group parameter,

$$\mathcal{R} = \sum_{\ell=-\infty}^{\infty} \ell \left(\mathcal{S}^{(\ell)} \left(\frac{f(u, u_1, a)}{u_1 - u_{-1}} - \frac{1}{2} f_{,u_1}(u, u_1, a) \right) \right) \partial_{u_\ell} - r(a) \partial_a \quad (27)$$

and \mathcal{S} is the shift operator in the corresponding direction.

The corresponding ABS equation, written now in the form

$$Q(u, u_1, \tilde{u}, \tilde{u}_1; a, \lambda) = 0,$$

is an auto-Bäcklund transformation of equations (26) [13], while the latter equations admit the following Lax pair

$$\Psi_1 = \mathbf{L}(u, u_1; a, \lambda) \Psi, \quad \frac{d\Psi}{d\epsilon_k} = \mathbf{N}_k \Psi, \quad k = 0, 1, \dots. \quad (28)$$

In the above relations, the matrix $\mathbf{L}(u, u_1; a, \lambda)$ is given by (13) and

$$\mathbf{N}_k := \mathcal{R}^k \left(\frac{1}{u_1 - u_{-1}} \mathbf{X} - \frac{1}{2} \mathbf{X}_{,u_1} \right), \quad k = 0, 1, \dots, \quad (29)$$

where

$$\mathbf{X} := -f(u, u_1, a) (\mathbf{L}(u, u_1; a, \lambda))^{-1} \partial_u \mathbf{L}(u, u_1; a, \lambda). \quad (30)$$

6 Conclusions and perspectives

We have presented some recent and new results about the integrability and the symmetries of difference equations. We have used as a representative example the equations of the ABS classification [3]. These equations possess the consistency property which provides the means to explore Bäcklund transformations and Lax pairs for all of them. In the same fashion, Bäcklund transformations and Lax pairs have been derived for systems of difference equations possessing the consistency property, e.g. the discrete Boussinesq system [26, 27].

The extended generalized symmetries play the key role of master symmetries leading to the explicit construction of infinite hierarchies of symmetries for all of the ABS equations. Moreover, the study of solutions of the latter equations, which remain invariant under the action of these symmetries, reveals a new link among the ABS and integrable differential equations, as well as among discrete and continuous Painlevé equations, [30]. It would be interesting to study systems of difference equations, such as the Boussinesq system and the generalization of the discrete KdV equation presented in [20], from this point of view.

Appendix. The characteristic polynomials of the ABS equations

H1 $f(u, x, \alpha) = 1, \quad G(x, y) = (x - y)^2, \quad k(\alpha, \beta) = \beta - \alpha.$

H2 $f(u, x, \alpha) = 2(u + x + \alpha), \quad G(x, y) = (x - y)^2 - (\alpha - \beta)^2, \quad k(\alpha, \beta) = \beta - \alpha.$

H3 $f(u, x, \alpha) = ux + \alpha\delta, \quad G(x, y) = (y\alpha - x\beta)(y\beta - x\alpha), \quad k(\alpha, \beta) = \alpha^2 - \beta^2.$

Q1 $f(u, x, \alpha) = ((u - x)^2 - \alpha^2\delta^2)/\alpha, \quad G(x, y) = \alpha\beta((x - y)^2 - (\alpha - \beta)^2\delta^2),$
 $k(\alpha, \beta) = -\alpha\beta(\alpha - \beta).$

Q2 $f(u, x, \alpha) = ((u - x)^2 - 2\alpha^2(u + x) + \alpha^4)/\alpha,$
 $G(x, y) = \alpha\beta((x - y)^2 - 2(\alpha - \beta)^2(x + y) + (\alpha - \beta)^4),$
 $k(\alpha, \beta) = -\alpha\beta(\alpha - \beta).$

Q3 $f(u, x, \alpha) = \frac{-1}{4\alpha(\alpha^2 - 1)}(4\alpha(x - \alpha u)(\alpha x - u) + (\alpha^2 - 1)^2\delta^2),$
 $G(x, y) = \frac{(\alpha^2 - 1)(\beta^2 - 1)}{4\alpha\beta}(4\alpha\beta(\alpha y - \beta x)(\beta y - \alpha x) + (\alpha^2 - \beta^2)\delta^2),$
 $k(\alpha, \beta) = (\alpha^2 - \beta^2)(\alpha^2 - 1)(\beta^2 - 1).$

Q4 $f(u, x, \alpha) = ((ux + \alpha(u + x) + g_2/4)^2 - (u + x + \alpha)(4\alpha ux - g_3)) / a,$
 $G(x, y) = (a_1xy + a_2(x + y) + a_4)(a_4xy + a_6(x + y) + a_7)$
 $\quad - (a_2xy + a_3y + a_5x + a_6)(a_2xy + a_3x + a_5y + a_6),$
 $k(\alpha, \beta) = \frac{ab(a^2b + ab^2 + [12\alpha\beta^2 - g_2(\alpha + 2\beta) - 3g_3]a + [12\beta\alpha^2 - g_2(\beta + 2\alpha) - 3g_3]b)}{4(\alpha - \beta)}.$

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Reduction of multidimensional non-linear d'Alembert equations to two-dimensional equations and classes of the reduced equations

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We study conditions of reduction of the multidimensional wave equation $\square u = F(u)$ — a system of the d'Alembert and Hamilton equations: $y_\mu y_\mu = r(y, z)$; $y_\mu z_\mu = q(y, z)$; $z_\mu z_\mu = s(y, z)$; $\square y = R(y, z)$; $\square z = S(y, z)$. We prove necessary conditions for compatibility of such system of the reduction conditions. Possible types of the reduced equations represent interesting classes of two-dimensional parabolic, hyperbolic and elliptic equations. Ansatzes and methods used for reduction of the d'Alembert (n -dimensional wave) equation can be also used for arbitrary Poincaré-invariant equations. This seemingly simple and partial problem involves many important aspects in the studies of the PDE.

1 Introduction

We study conditions of reduction of the multidimensional wave equation

$$\begin{aligned} \square u &= F(u), \\ \square &\equiv \partial_{x_0}^2 - \partial_{x_1}^2 - \cdots - \partial_{x_n}^2, \quad u = u(x_0, x_1, \dots, x_n) \end{aligned} \tag{1}$$

by means of the ansatz with two new independent variables [1, 2]

$$u = \varphi(y, z), \tag{2}$$

where y, z are new variables. Henceforth n is the number of independent space variables in the initial d'Alembert equation.

These conditions are a system of the d'Alembert and Hamilton equations:

$$\begin{aligned} y_\mu y_\mu &= r(y, z), \quad y_\mu z_\mu = q(y, z), \quad z_\mu z_\mu = s(y, z), \\ \square y &= R(y, z), \quad \square z = S(y, z). \end{aligned} \tag{3}$$

We prove necessary conditions for compatibility of such system of the reduction conditions. This paper is a development of research started jointly with W.I. Fushchych in 1990s [3], and we present some new results and ideas.

Possible types of the reduced equations represent interesting classes of two-dimensional parabolic, hyperbolic and elliptic equations. Ansatzes and methods used for reduction of the d'Alembert (n -dimensional wave) equation can be also used for arbitrary Poincaré-invariant equations. This seemingly simple and partial problem involves many important aspects in the studies of the PDE.

Classes of exact solutions of non-linear equations having respective symmetry properties can be constructed by means of symmetry reduction of these equations to equations with smaller number of independent variables or to ordinary differential equations (for the algorithms see the books [4–7]).

Reductions and solutions of equation (1) by means of symmetry reduction or ansatzes were considered in numerous papers [8–13]. See [14] for a review of results related to reduction of a number of wave equations. In the paper [15] an alternative was proposed for the method of application of ansatzes for equation (1) with a degree nonlinearity.

The method of symmetry reduction does not give exhaustive description of all exact solutions for an equation, so other methods for construction of exact solutions may be expedient.

A so-called “direct method” for search of exact solutions of nonlinear partial differential equations (giving wider classes of solutions than the symmetry reduction) was proposed by P. Clarkson and M. Kruskal [16] (see also [17, 18] and the papers cited therein). It is easy to see that this method for majority of equations results in considerable difficulties as it requires investigation of compatibility and solution of cumbersome reduction conditions of the initial equation. These reduction conditions are much more difficult for investigation and solution in the case of equations containing second and/or higher derivatives for all independent variables, and for multidimensional equations — e.g. in the situation of the nonlinear wave equations.

The direct method, if applied “completely” (with full solution of compatibility conditions), is exhaustive to some extent — it allows obtaining all reductions of the original equations that may be obtained from Q -conditional symmetry (see more comments on symmetry in Section 4).

In this paper we were not able to achieve such complete application of the direct reduction to equation (1) — the presented results are only a step to such application. To do that it is necessary to find a general solution of the reduction

Direct reduction with utilisation of ansatzes or exhaustive description of conditional symmetries (even Q -conditional symmetries) cannot be regarded as algorithmic to the same extent as the standard symmetry reduction. Majority of papers on application of the direct method are devoted to evolution equations or other equations that contain variables of the order not higher than one for at least one of the independent variables, with not more than three independent variables. In such cases solution of the reduction conditions is relatively simple.

We consider general reduction conditions of equation (1) by means of a general ansatz with two new independent variables. We found necessary compatibility conditions for the respective reduction conditions — we developed the conditions

found in [3]. We also describe respective possible forms of the reduced equations. Thus we proved that the reduced equations may have only a particular form. conditions.

A similar problem was considered by previous authors for an ansatz with one independent variable

$$u = \varphi(y), \quad (4)$$

where y is a new independent variable.

Compatibility analysis of the d'Alembert–Hamilton system

$$\square u = F(u), \quad u_\mu u_\mu = f(u) \quad (5)$$

in the three-dimensional space was done in [19]. For more detailed review of investigation and solutions of this system see [20, 21]).

The compatibility condition of the system (5) for $f(u) = 0$ was found in the paper [22].

Complete investigation of compatibility of overdetermined systems of differential equations with fixed number of independent variables may be done by means of Cartan's algorithm [23], however, it is very difficult for practical application even in the case of three independent variables, and not applicable for arbitrary number of independent variables. For this reason some ad hoc techniques for such cases should be used even for search of necessary compatibility conditions.

It is evident that the d'Alembert–Hamilton system (5) may be reduced by local transformations to the form

$$\square u = F(u), \quad u_\mu u_\mu = \lambda, \quad \lambda = 0, \pm 1. \quad (6)$$

Necessary compatibility conditions of the system (6) for four independent variables were studied in [24] (see also [21]). The necessary compatibility conditions for the system (6) for arbitrary number of independent variables were found in [25]:

Proposition 1. *For the system (6) (n is arbitrary) to be compatible it is necessary that the function F has the following form:*

$$F = \frac{\lambda \partial_u \Phi}{\Phi}, \quad \partial_u^{n+1} \Phi = 0.$$

W.I. Fushchych, R.Z. Zhdanov and I.V. Revenko [20, 26, 27] found a general solution of the system (6) for three space variables (that is four independent variables), as well as necessary and sufficient compatibility conditions for this system [26]:

Proposition 2. *For the system (6) ($u = u(x_0, x_1, x_2, x_3)$) to be compatible it is necessary and sufficient that the function F has the following form:*

$$F = \frac{\lambda}{N(u + C)}, \quad N = 0, 1, 2, 3.$$

The results presented in this papers may be regarded as a development of the above Propositions.

Reduction of equation (1) by means of the ansatz (2) was considered in [27] for a special case (when the second independent variable enters the reduced equation only as a parameter), described all respective ansatzes for the case of four independent variables, and found the respective solutions. Some solutions of such type for arbitrary n were also considered in [28].

In [29] reduction of the nonlinear d'Alembert equation by means of ansatz $u = \phi(\omega_1, \omega_2, \omega_3)$ was considered for the case $\square\omega_1 = 0, \omega_{1\mu}\omega_{1\mu} = 0$ (that is ω_1 entered the reduced equation only as a parameter). The respective compatibility conditions were studied and new (non-Lie) exact solutions were found. Note that this case does not include completely the case considered here — the case of the ansatz with two new independent variables.

2 Necessary compatibility conditions of the system of the d'Alembert–Hamilton equations for two functions or for a complex-valued function.

Reduction of multidimensional equations to two-dimensional ones may be interesting as solutions of two-dimensional partial differential equations, including nonlinear ones, may be investigated more comprehensively than solutions of multidimensional equations, and such two-dimensional equations may have more interesting properties than ordinary differential equations. Two-dimensional reduced equations also may have interesting properties with respect to conditional symmetries.

Substitution of ansatz (2) equation (1) leads to the following equation:

$$\begin{aligned} \varphi_{yy}y_\mu y_\mu + 2\varphi_{yz}z_\mu y_\mu + \varphi_{zz}z_\mu z_\mu + \varphi_y \square y + \varphi_z \square z &= F(\varphi) \\ \left(y_\mu = \frac{\partial y}{\partial x_\mu}, \quad \varphi_y = \frac{\partial \varphi}{\partial y} \right), \end{aligned} \quad (7)$$

whence we get a system of equations:

$$\begin{aligned} y_\mu y_\mu &= r(y, z), \quad y_\mu z_\mu = q(y, z), \quad z_\mu z_\mu = s(y, z), \\ \square y &= R(y, z), \quad \square z = S(y, z). \end{aligned} \quad (8)$$

System (8) is a reduction condition for the multidimensional wave equation (1) to the two-dimensional equation (7) by means of ansatz (2).

The system of equations (8), depending on the sign of the expression $rs - q^2$, may be reduced by local transformations to one of the following types:

1) elliptic case: $rs - q^2 > 0$, $v = v(y, z)$ is a complex-valued function,

$$\begin{aligned} \square v &= V(v, v^*), \quad \square v^* = V^*(v, v^*), \\ v_\mu^* v_\mu &= h(v, v^*), \quad v_\mu v_\mu = 0, \quad v_\mu^* v_\mu^* = 0 \end{aligned} \quad (9)$$

(the reduced equation is of the elliptic type);

2) hyperbolic case: $rs - q^2 < 0$, $v = v(y, z)$, $w = w(y, z)$ are real functions,

$$\begin{aligned} \square v &= V(v, w), & \square w &= W(v, w), \\ w_\mu w_\mu &= h(v, w), & v_\mu v_\mu &= 0, & w_\mu w_\mu &= 0 \end{aligned} \quad (10)$$

(the reduced equation is of the hyperbolic type);

3) parabolic case: $rs - q^2 = 0$, $r^2 + s^2 + q^2 \neq 0$, $v(y, z)$, $w(y, z)$ are real functions,

$$\begin{aligned} \square v &= V(v, w), & \square w &= W(v, w), \\ v_\mu w_\mu &= 0, & v_\mu v_\mu &= \lambda \ (\lambda = \pm 1), & w_\mu w_\mu &= 0 \end{aligned} \quad (11)$$

(if $W \neq 0$, then the reduced equation is of the parabolic type);

4) first-order equations: $(r = s = q = 0)$, $y \rightarrow v$, $z \rightarrow w$

$$\begin{aligned} v_\mu v_\mu &= w_\mu w_\mu = v_\mu w_\mu = 0, \\ \square v &= V(v, w), & \square w &= W(v, w). \end{aligned} \quad (12)$$

Let us formulate necessary compatibility conditions for the systems (9)–(12).

Theorem 1. *System (9) is compatible if and only if*

$$V = \frac{h(v, v^*) \partial_{v^*} \Phi}{\Phi}, \quad \partial_{v^*} \equiv \frac{\partial}{\partial v^*},$$

where Φ is an arbitrary function for which the following condition is satisfied

$$(h \partial_{v^*})^{n+1} \Phi = 0.$$

The function h may be represented in the form $h = \frac{1}{R_{vv^*}}$, where R is an arbitrary sufficiently smooth function, R_v , R_{v^*} are partial derivatives by the respective variables.

Then the function Φ may be represented in the form $\Phi = \sum_{k=0}^{n+1} f_k(v) R_v^k$, where $f_k(v)$ are arbitrary functions, and

$$V = \frac{\sum_{k=1}^{n+1} k f_k(v) R_v^k}{\sum_{k=0}^{n+1} f_k(v) R_v^k}.$$

The respective reduced equation will have the form

$$h(v, v^*) \left(\phi_{vv^*} + \phi_v \frac{\partial_{v^*} \Phi}{\Phi} + \phi_{v^*} \frac{\partial_v \Phi^*}{\Phi^*} \right) = F(\phi). \quad (13)$$

The equation (13) may also be rewritten as an equation with two real independent variables ($v = \omega + \theta$, $v^* = \omega - \theta$):

$$2\tilde{h}(\omega, \theta)(\phi_{\omega\omega} + \phi_{\theta\theta}) + \Omega(\omega, \theta)\phi_\omega + \Theta(\omega, \theta)\phi_\theta = F(\phi). \quad (14)$$

We will not adduce here cumbersome expressions for Ω , Θ that may be found from (13).

Theorem 2. *System (10) is compatible if and only if*

$$V = \frac{h(v, w)\partial_w\Phi}{\Phi}, \quad W = \frac{h(v, w)\partial_v\Psi}{\Psi},$$

where the functions Φ, Ψ for which the following conditions are satisfied

$$(h\partial_v)^{n+1}\Psi = 0, \quad (h\partial_w)^{n+1}\Phi = 0.$$

The function h may be presented in the form $h = \frac{1}{R_{vw}}$, where R is an arbitrary sufficiently smooth function, R_v, R_w are partial derivatives by the respective variables. Then the functions Φ, Ψ may be represented in the form

$$\Phi = \sum_{k=0}^{n+1} f_k(v)R_v^k, \quad \Psi = \sum_{k=0}^{n+1} g_k(w)R_w^k,$$

where $f_k(v), g_k(w)$ are arbitrary functions,

$$V = \frac{\sum_{k=1}^{n+1} kf_k(v)R_v^k}{\sum_{k=0}^{n+1} f_k(v)R_v^k}, \quad W = \frac{\sum_{k=1}^{n+1} kg_k(w)R_w^k}{\sum_{k=0}^{n+1} g_k(w)R_w^k}.$$

The respective reduced equation will have the form

$$h(v, w) \left(\phi_{vw} + \phi_v \frac{\partial_w\Phi}{\Phi} + \phi_w \frac{\partial_v\Psi}{\Psi} \right) = F(\phi). \quad (15)$$

Theorem 3. *System (11) is compatible if and only if*

$$V = \frac{\lambda\partial_v\Phi}{\Phi}, \quad \partial_v^{n+1}\Phi = 0, \quad W \equiv 0.$$

Equation (1) by means of ansatz (2) cannot be reduced to a parabolic equation — in this case one of the variables will enter the reduced ordinary differential equation of the first order as a parameter.

Compatibility and solutions of such system for $n = 3$ were considered in [27]; for this case necessary and sufficient compatibility conditions, as well as a general solution, were found.

System (12) is compatible only in the case if $V = W \equiv 0$, that is the reduced equation may be only an algebraic equation $F(u)=0$. Thus we cannot reduce equation (1) by means of ansatz (2) to a first-order equation.

Proof of these theorems is done by means of utilisation of lemmas similar to those adduced in [24, 25], and of the well-known Hamilton–Cayley theorem, in accordance to which a matrix is a root of its characteristic polynomial.

We will present an outline of proof of Theorem 2 for the hyperbolic case. For other cases the proof is similar.

We will operate with matrices of dimension $(n + 1) \times (n + 1)$ of the second variable of the functions v and w :

$$\hat{V} = \{v_{\mu\nu}\}, \quad \hat{W} = \{w_{\mu\nu}\}.$$

With respect to operations with these matrices we utilise summation arrangements customary for the Minkowsky space: $v_0 = i\partial_{x_0}$, $v_a = -i\partial_{x_a}$ ($a = 1, \dots, n$), $v_\mu v_\mu = v_0^2 - v_1^2 - \dots - v_n^2$.

Lemma 1. *If the functions v and w are solutions of the system (10), then the following relations are satisfied for them for any k :*

$$\text{tr} \hat{V} = \frac{(-1)^k}{(k-1)!} (h(v, w) \partial_w)^{k+1} V(v, w),$$

$$\text{tr} \hat{W} = \frac{(-1)^k}{(k-1)!} (h(v, w) \partial_v)^{k+1} W(v, w).$$

Lemma 2. *If the functions v and w are solutions of the system (10), then $\det \hat{V} = 0$, $\det \hat{W} = 0$.*

Lemma 3. *Let $M_k(\hat{V})$ be the sum of principal minors of the order k for the matrix \hat{V} . If the functions v and w are solutions of the system (10), then the following relations are satisfied for them for any k :*

$$M_k(\hat{V}) = \frac{(h(v, w) \partial_w)^k \Phi}{k! \Phi}, \quad M_k(\hat{W}) = \frac{(h(v, w) \partial_v)^k \Psi}{k! \Psi},$$

where the functions Φ, Ψ satisfy the following conditions

$$(h \partial_v)^{n+1} \Psi = 0, \quad (h \partial_w)^{n+1} \Phi = 0.$$

These lemmas may be proved with the method of mathematical induction similarly to [25] with utilisation of the Hamilton–Cayley theorem (E is a unit matrix of the dimension $(n + 1) \times (n + 1)$).

$$\sum_{k=0}^{n-1} (-1)^k M_k \hat{V}^{n-k} + (-1)^n E \det \hat{V} = 0.$$

It is evident that the statement of Theorem 2 is a direct consequence of Lemma 3 for $k = 1$.

Note 1. Equation (9) may be rewritten for a pair of real functions $\omega = \text{Re } v$, $\theta = \text{Im } v$. Though in this case necessary the respective compatibility conditions would have extremely cumbersome form.

Note 2. Transition from (8) to (9)–(12) is convenient only from the point of view of investigation of compatibility. The sign of the expression $rs - q^2$ may change for various y, z , and the transition is being considered only within the region where this sign is constant.

3 Examples of solutions of the system of d'Alembert–Hamilton equations

Let us adduce explicit solutions of systems of the type (8) and the respective reduced equations. Parameters $a_\mu, b_\mu, c_\mu, d_\mu$ ($\mu = 0, 1, 2, 3$) satisfy the conditions:

$$\begin{aligned} -a^2 = b^2 = c^2 = d^2 = -1 \quad (a^2 \equiv a_0^2 - a_1^2 - \cdots - a_3^2), \\ ab = ac = ad = bc = bd = cd = 0; \end{aligned}$$

y, z are functions of x_0, x_1, x_2, x_3 .

- 1) $y = ax, \quad z = dx, \quad \varphi_{yy} - \varphi_{zz} = F(\varphi);$
- 2) $y = ax, \quad z = ((bx)^2 + (cx)^2 + (dx)^2)^{1/2}, \quad \varphi_{yy} - \varphi_{zz} - \frac{2}{z}\varphi_z = F(\varphi);$

In this case the reduced equation is a so-called radial wave equation, the symmetry and solutions of which were investigated in [38, 39].

- 3) $y = bx + \Phi(ax + dx), \quad z = cx, \quad -\varphi_{zz} - \varphi_{yy} = F(\varphi);$
- 4) $y = ((bx)^2 + (cx)^2)^{1/2}, \quad z = ax + dx, \quad -\varphi_{yy} - \frac{1}{y}\varphi_y = F(\varphi).$

4 Symmetry aspects

Solutions obtained by the direct reduction are related to symmetry properties of the equation — Q -conditional symmetry of this equation [6, 30, 31] (symmetries of such type are also called non-classical or non-Lie symmetries [16, 32, 33]). For more theoretical background of conditional symmetry and examples see also [14, 34, 35].

Conditional symmetry and solutions of various non-linear two-dimensional wave equations that may be regarded as reduced equations for equation (1) were considered in [36–40]. It is also possible to see from these papers that symmetry of the two-dimensional reduced equations is often wider than symmetry of the initial equation, that is the reduction to two-dimensional equations allows to find new non-Lie solutions and hidden symmetries of the initial equation (see e.g. [41]).

The Hamilton equation may also be considered, irrespective of the reduction problem, as an additional condition for the d'Alembert equation that allows extending the symmetry of this equation. The symmetry of the system

$$\square u = F(u), \quad u_\mu u_\mu = 0$$

was described in [42]. In [25] a conformal symmetry of the system (5) was found that is was a new conditional symmetry for the d'Alembert equation. Conditional symmetries of this system were also described in [27, 29].

5 Conclusions

The results of investigation of compatibility and solutions of the systems (9)–(12) may be utilised for investigation and search of solutions also of other Poincaré–invariant wave equations, beside the d'Alembert equation, e.g. Dirac equation, Maxwell equations and equations for the vector potential.

Thus, in the present paper we found

- 1) necessary compatibility conditions for the system of the d'Alembert–Hamilton equations for two dependent functions, that is reduction conditions of the non-linear multidimensional d'Alembert equation by means of ansatz (2) to a two-dimensional equation; such compatibility conditions for equations of arbitrary dimensions cannot be found by means of the standard procedure.
- 2) possible types of the two-dimensional reduced equations that may be obtained from equation (1) by means of ansatz (2).

The found reduction conditions and types of ansatzes may be also used for arbitrary Poincaré–invariant multidimensional equation. In [43] the general form of the scalar Poincaré–invariant multidimensional equations were described; it is easy to prove that by means of ansatz (2) it is possible to reduce all these equations to PDE in two independent variables.

6 Further Research

1. Study of Lie and conditional symmetry of the system of the reduction conditions (symmetry of the system of the d'Alembert equations for the complex function was investigated in [44]).
2. Investigation of Lie and conditional symmetry of the reduced equations. Finding exact solutions of the reduced equations.
3. Relation of the equivalence group of the class of the reduced equations with symmetry of the initial equation.
4. Group classification of the reduced equations.
5. Finding of sufficient compatibility conditions and of a general solution of the compatibility conditions for lower dimensions ($n = 2, 3$).
6. Finding and investigation of compatibility conditions and classes of the reduced equations for other types of equations, in particular, for Poincaré–invariant scalar equations.

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