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PAPER

Generalizing Maxwell's equations to complex-valued electromagnetic fields

James A Reggia*

College of Computer, Mathematical and Natural Sciences, University of Maryland, College Park, MD United States of America
* Author to whom any correspondence should be addressed.

E-mail: reggia@umd.edu

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Abstract

There is a well-known asymmetry in classical electromagnetism, apparent in Maxwell's equations, that arises from the existence of electric but not magnetic charge. This has motivated numerous experimental searches for magnetic monopoles which have, to date, not been found. To address this asymmetry, the research reported here generalizes these equations to accommodate complex-valued electromagnetic fields, thereby making Maxwell's equations more symmetric. The resulting generalized equations remain consistent with the experimental predictions of the original Maxwell equations, and they are shown to continue to exhibit conservation of charge. The increased symmetry of the complex-valued equations is demonstrated via a duality transformation that is derived and verified here. Importantly, the generalized theory implies that a novel type of magnetic monopoles exists while simultaneously explaining why their detection has eluded previous experimental searches. Further study of the possibility that electromagnetic fields include imaginary-valued components is clearly merited because of the implications it could have for the foundations of classical electrodynamics.

1. Introduction

The core laws governing classical electrodynamics, Maxwell's equations, are given by

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho \tag{1.1a}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{1.1b}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{1.1c}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \tag{1.1d}$$

where $\boldsymbol{E}$ ($\boldsymbol{B}$) is the 3-component electric (magnetic) field, ρ is the electric charge density, $\boldsymbol{J}$ is the volume electric current density, ϵ_0 (μ_0) is the permittivity (permeability) of free space, and c is the speed of light.[1] There is a striking and widely recognized asymmetry in these equations: while there are electric charge and current densities ρ and $\boldsymbol{J}$, there are no corresponding magnetic charge and current densities ρ_{m} and $\boldsymbol{J}_m$, reflecting the fact that extensive experimental investigations over the years have consistently failed to detect the existence of magnetic monopoles (reviews in [1–4]). In this context, some theoreticians have argued that magnetic monopoles cannot exist. However, based on symmetry considerations, arguments such as Dirac's quantization condition [5], theoretical analysis in grand unified theories [6], and for other reasons [3, 7], many physicists today continue to believe that magnetic monopoles probably exist, and a rich variety of potential types of magnetic monopoles have been proposed (Dirac monopoles [5], 't Hooft-Polyakov monopoles [8, 9], etc;

[1] The SI system of units is used in this paper. Bold font indicates 3-component column vectors, real or complex.

reviewed in [2, 10]. The active interest in this topic has, if anything, been increasing, with a continuing stream of very creative experimental searches and theoretical ideas appearing in the recent literature [6, 11–18].

Contemporary articles and textbooks on electrodynamics often illustrate the idea of magnetic monopoles within classical electrodynamics by showing a version of Maxwell's equations where the hypothetically 'missing' magnetic charge and current terms ρ_m and $\boldsymbol{J}_m$ have been added, as in

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho \tag{1.2a}$$

$$\nabla \times \boldsymbol{E} = -\mu_0 \boldsymbol{J}_m - \frac{\partial \boldsymbol{B}}{\partial t} \tag{1.2b}$$

$$\nabla \cdot \boldsymbol{B} = \mu_0 \rho_m \tag{1.2c}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \tag{1.2d}$$

to demonstrate the elegant symmetry that would exist in this case and the invariance of these extended equations under an electromagnetic duality transformation, e.g., [15, 19–23]. In short, the dilemma is that there are very good theoretical reasons to expect magnetic monopoles to exist, but experimental search for them over many years has consistently failed to detect them in cosmic rays, accelerator experiments, etc. It is frequently suggested that our inability to detect magnetic monopoles in such experiments arises because they are either very rare and/or they are very massive.

In this paper we consider an alternative possible explanation for why existing monopoles have not been detected experimentally, cast solely within the scope of classical electrodynamics. The explanation given here differs fundamentally from past explanations based on monopole flux or mass in that it is based on the premise that our existing concept of electromagnetic fields in general is incomplete. The idea is that this incompleteness is responsible for the asymmetries that are manifest in Maxwell's equations. Specifically, the central hypothesis considered here is that the components of electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are complex valued rather than being restricted to having real values as has generally been assumed. Implicit in this hypothesis is the recognition that the imaginary portions of such field components are currently unobservable by us, accounting for why their existence has not been recognized.

In the following, we first consider a generalization of Maxwell's equations that accommodates complex-valued field vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ and thereby allows a more symmetric statement of the laws of classical electrodynamics. Creating this generalization is more challenging than one might anticipate at first because the generalized equations must remain consistent with the existing Maxwell's equations and existing experimental results, and because any predictions that they make about novel electromagnetic phenomena must not already be known to be non-existent. The implications of these generalized and more symmetric laws of electrodynamics and of duality transformations associated with them are derived, including the existence of electromagnetic waves that have imaginary-valued components. The generalized equations are found to remain consistent with the conservation of charge. Importantly, the more symmetric generalized equations predict that magnetic monopoles exist and are widespread, while providing a novel explanation for why they have not been detected by existing experimental methods.

2. Generalizing Maxwell's equations to complex-valued fields

In this section a set of more generalized equations for classical electrodynamics is described based on extending Maxwell's equations to C^3, 3D complex-valued space, rather than the customary 3D real-valued space R^3, while remaining consistent with Maxwell's original equations in R^3. We start by defining the electric and magnetic fields in C^3 to be

$$\boldsymbol{E} = \begin{bmatrix} E_{x1} + iE_{t1} \\ E_{x2} + iE_{t2} \\ E_{x3} + iE_{t3} \end{bmatrix} = \boldsymbol{E_x} + i\boldsymbol{E}_t \text{ and } \quad \boldsymbol{B} = \begin{bmatrix} B_{x1} + iB_{t1} \\ B_{x2} + iB_{t2} \\ B_{x3} + iB_{t3} \end{bmatrix} = \boldsymbol{B_x} + i\boldsymbol{B}_t, \tag{2.1}$$

respectively, where

$$\boldsymbol{E}_x = \begin{bmatrix} E_{x1} \\ E_{x2} \\ E_{x3} \end{bmatrix} \quad \boldsymbol{E}_t = \begin{bmatrix} E_{t1} \\ E_{t2} \\ E_{t3} \end{bmatrix} \quad \boldsymbol{B}_x = \begin{bmatrix} B_{x1} \\ B_{x2} \\ B_{x3} \end{bmatrix} \quad \text{and} \quad \boldsymbol{B}_t = \begin{bmatrix} B_{t1} \\ B_{t2} \\ B_{t3} \end{bmatrix}$$

all lie in R^3. Here $\boldsymbol{E}_\mathrm{x}$ and $\boldsymbol{B}_\mathrm{x}$ are the conventional electric and magnetic fields that correspond to $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively, as they are currently used in Maxwell's equations (equations (1.1)). In contrast, $\boldsymbol{E}_\mathrm{t}$ and $\boldsymbol{B}_\mathrm{t}$, both in R^3, represent the central hypothesis of this paper that electromagnetic fields exist in the larger space C^3 and have

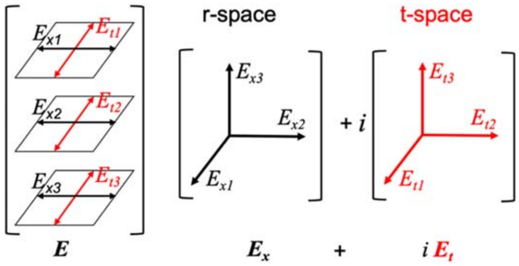

Figure 1. Two complementary geometric conceptions of a complex-valued vector $\boldsymbol{E}$ in C^3. **Left:** Sliced horizontally, we can view each of the three components of $\boldsymbol{E}$ as lying in a complex-valued (Argand) plane. **Right**: Sliced vertically, $\boldsymbol{E}$ can be viewed as the sum of a 3D real-valued vector $\boldsymbol{E_x}$ in r-space and a 3D purely imaginary-valued vector $i\,\boldsymbol{E_t}$ in t-space, where $\boldsymbol{E_t}$ itself has real-valued components. Similar comments apply to $\boldsymbol{B}$. Use of red font and arrows indicates imaginary-valued components and axes. ▢

unobservable imaginary-valued portions of their components. It is implicit in these definitions that the field vectors are functions of location in space $r = x + i\,t$, where x and t are both 3D real-valued vectors relating to the real and imaginary aspects of space, respectively. The field vectors are also functions of time t.

Using these complex-valued fields, Maxwell's equations can now be revised to be both more general and more symmetric. The generalized electrodynamics equations proposed here are:

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho \tag{2.2a}$$

$$\nabla \times \boldsymbol{E} = -ic\mu_0\boldsymbol{J} - \frac{\partial \boldsymbol{B}}{\partial t} \tag{2.2b}$$

$$\nabla \cdot \boldsymbol{B} = ic\mu_0\rho \tag{2.2c}$$

$$\nabla \times \boldsymbol{B} = \mu_0\boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \tag{2.2d}$$

These generalized equations look superficially very similar to Maxwell's equations modified to include hypothesized magnetic charge terms as given earlier in equations (1.2). However, there are four fundamental differences. First, both of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ in these equations (2.2) have components in C^3, rather than in R^3. In other words, the fields involved are $\boldsymbol{E} = \boldsymbol{E_x} + i\boldsymbol{E}_t$ and $\boldsymbol{B} = \boldsymbol{B_x} + i\boldsymbol{B}_t$, unlike with classical electromagnetism. Second, the magnetic charge and current density terms on the right sides of equations (2.2b) and (2.2c) are imaginary valued. Third, the separate magnetic charge ρ_m and current $\boldsymbol{J}_m$ densities that appear in equations (1.2) are now taken instead to be associated with the usual electric charge and current densities ρ and $\boldsymbol{J}$, respectively, rather than introducing new physical entities to represent distinct magnetic charges. Fourth, the divergence and curl operators have been extended to C^3 in a way that remains consistent with their existing definitions in R^3. We elaborate on these differences in the rest of this section.

The complex fields $\boldsymbol{E}$ and $\boldsymbol{B}$ each have six components, raising the issue of how we can conceptually visualize these fields which effectively exist in a 6D space. To address this issue, equation (2.1) has been written to emphasize that we can think of fields in C^3 in two different but equivalent ways. First, conventionally, we can think of each field component as lying in the complex plane (Argand plane), as illustrated on the left in figure 1. Alternatively, we can view the real and imaginary portions of electromagnetic field vectors as lying in two 3D spaces, as shown on the right in figure 1. It is the latter viewpoint that will be emphasized in the following because it makes the observable/unobservable distinction more explicit. For clarity, in the following we will refer to the real portions $\boldsymbol{E_x}$ and $\boldsymbol{B}_x$ of the complex fields $\boldsymbol{E}$ and $\boldsymbol{B}$ as lying in *real-valued space*, or *r-space*, which corresponds to our current familiar notion of the 3D space in which experimental physics takes place and is what is assumed by the classical Maxwell equations. Physical quantities in familiar 3D r-space are observable (measurable experimentally). In contrast, we will refer to the imaginary portions $\boldsymbol{E}_t$ and $\boldsymbol{B}_t$ of the complex fields E and B as lying in a separate imaginary-valued *transcendent space*, or *t-space*. The term 'transcendent' here means solely that this aspect of space goes beyond (transcends) our routine experience of space. Based on this separated r-space/t-space viewpoint, we can clarify that the hypothesis considered here only states that electromagnetic fields extend into t-space. It does *not* state that other physical entities such as matter or charge extend into t-space.

Another difference between the equations (1.2) and the generalized equations introduced here is that the magnetic charge and current density terms on the right sides of equations (2.2b) and (2.2c) are imaginary valued, unlike in equation (1.2). They thus not only make these generalized equations more symmetric than Maxwell's original equations (like equation (1.2) does), but they also indicate explicitly that the electromagnetic fields E and B have imaginary components. The imaginary-valued quantities on the right side of these two equations are based on the usual electric charge and current densities ρ and J, respectively, rather than being novel magnetic charge ρ_m and current J_m densities that have not been observed experimentally in the past. Some implications of this latter difference between the generalized equations equations (2.2) and equations (1.2) are examined below in later sections.

Finally, we need to define the divergence and curl operations used in the generalized equations (2.2). For convenience and simplicity, we take a non-standard approach to defining these and related operations in C^3 whereby vector product operations in C^3 are 'reduced' to a linear sum of the standard corresponding R^3 operations in r-space and t-space. Hence, we refer to these C^3 vector products as *reduction vector products*, and their definitions can be viewed as just convenient notational abbreviations. First, we consider the general C^3 dot and cross product notation used in the rest of this paper.

Let $\boldsymbol{C} = \boldsymbol{C_x} + i\boldsymbol{C_t}$ and $\boldsymbol{C'} = \boldsymbol{C'_x} + i\boldsymbol{C'_t}$ be two arbitrary continuous vector fields in C^3 where, as above, $\boldsymbol{C_x}$, $\boldsymbol{C_t}$, $\boldsymbol{C'_x}$ and $\boldsymbol{C'_t}$ each lie in R^3. Define the reduction dot product $\cdot$ of $\boldsymbol{C}$ and $\boldsymbol{C'}$ in C^3 to be

$$\boldsymbol{C} \cdot \boldsymbol{C'} = \boldsymbol{C_x} \cdot \boldsymbol{C'_x} + i\, \boldsymbol{C_t} \cdot \boldsymbol{C'_t} \tag{2.3}$$

where the dot products on the right side of this equation are both the usual inner product in R^3. The dot product being defined on the left side of this equation acts on vectors in C^3 and in general returns a complex number. It should always be apparent in the following which meaning of the dot product is intended from the context, i.e., from whether the vectors being operated on are in C^3 or R^3. If $\boldsymbol{C}$ and $\boldsymbol{C'}$ are purely real-valued vectors in C^3 ($\boldsymbol{C_t} = \boldsymbol{C'_t} = \boldsymbol{0}$), then $\boldsymbol{C} \cdot \boldsymbol{C'}$ in C^3 corresponds to the standard dot product in R^3. While $\boldsymbol{C} \cdot \boldsymbol{C'}$ defined in equation (2.3) is commutative and distributive over addition, it does not meet the full definition of an inner product like the standard dot product in C^3 does (for example, $\boldsymbol{C} \cdot \boldsymbol{C'}$ in equation (2.3) is a complex value in general).

Similarly, we define a non-standard *reduction cross product* $\times$ of $\boldsymbol{C}$ and $\boldsymbol{C'}$ in C^3 to be

$$\boldsymbol{C} \times \boldsymbol{C'} = \boldsymbol{C_x} \times \boldsymbol{C'_x} + i\, \boldsymbol{C_t} \times \boldsymbol{C'_t} \tag{2.4}$$

where the symbols $\times$ on the right side of this equation are both the usual cross product in R^3. Defining the reduction cross product in C^3 in terms of conventional cross products in R^3 avoids the complexities confronted in extending the curl to spaces other than R^3 [24–27]. Again, if $\boldsymbol{C}$ and $\boldsymbol{C'}$ are purely real-valued vectors, then $\boldsymbol{C} \times \boldsymbol{C'}$ in C^3 corresponds to the standard cross product in R^3.

With the above notation, we can now characterize the divergence and curl operators that appear in equations (2.2) in C^3 in a similar fashion. Specifically, we define a reduction differential operator $\boldsymbol{\nabla} = \boldsymbol{\nabla_x} + i\,\boldsymbol{\nabla_t}$ in C^3, where

$$\boldsymbol{\nabla}_x = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\nabla}_t = \begin{bmatrix} \frac{\partial}{\partial t_1} \\ \frac{\partial}{\partial t_2} \\ \frac{\partial}{\partial t_3} \end{bmatrix} \tag{2.5}$$

are the usual R^3 del operators, here applied separately in r-space and t-space, respectively. With these definitions, if $\boldsymbol{C} = \boldsymbol{C_x} + i\boldsymbol{C_t}$ is an arbitrary continuous and differentiable vector field in C^3 (such as $\boldsymbol{E}$ or $\boldsymbol{B}$), and if $T = T_x + iT_t$ is an arbitrary continuous and differentiable scalar field in C^3, then the reduction gradient, divergence, and curl are defined as

$$\boldsymbol{\nabla} T = \boldsymbol{\nabla_x}\, T_x + i\, \boldsymbol{\nabla_t}\, T_t \tag{2.6a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{C} = \boldsymbol{\nabla_x} \cdot \boldsymbol{C_x} + i\, \boldsymbol{\nabla_t} \cdot \mathbf{C_t} \tag{2.6b}$$

$$\boldsymbol{\nabla} \times \mathbf{C} = \boldsymbol{\nabla_x} \times \boldsymbol{C_x} + i\, \boldsymbol{\nabla_t} \times \boldsymbol{C_t} \tag{2.6c}$$

respectively. It is straightforward but tedious to show that many of the usual relations for del in R^3 hold for $\boldsymbol{\nabla}$ in C^3 as the latter is defined here. For example, the above definitions and some simple algebra confirm that the familiar second derivative identities

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{C}) = 0 \tag{2.7a}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\, \mathrm{T}) = \boldsymbol{0} \tag{2.7b}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{C}) = \boldsymbol{\nabla}\, (\boldsymbol{\nabla} \cdot \mathbf{C}) - \boldsymbol{\nabla}^2 \boldsymbol{C} \tag{2.7c}$$

all hold in C^3, where $\boldsymbol{\nabla}^2 = \boldsymbol{\nabla}_x^2 + i\, \boldsymbol{\nabla}_t^2$ is the reduction Laplacian operator.

Example: Here we consider an example both to illustrate the use of the reduction vector operations defined above as well as to illustrate the derivation of one of the second derivative identities, equation (2.7c), that is based on these definitions and well known identities in R^3, as follows.

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{C})$$

$$= (\nabla_x + i\ \nabla_t) \times (\nabla_x + i\ \nabla_t) \times (C_x + iC_t)$$
$$= (\nabla_x + i\ \nabla_t) \times (\nabla_x \times C_x + i\ \nabla_t\ \times C_t)$$
$$= \nabla_x\ \times (\nabla_x \times C_x) + i\ \nabla_t\ \times (\nabla_t \times C_t)$$
$$= \nabla_x\ (\nabla_x \cdot C_x) - \nabla_x^2\ C_x + i\,(\nabla_t(\nabla_t \cdot C_t) - \nabla_t^2\ C_t)$$
$$= \nabla_x\ (\nabla_x \cdot C_x) + i\ \nabla_t\ (\nabla_t \cdot C_t) - (\nabla_x^2 C_x + i\ \nabla_t^2\ C_t)$$
$$= (\nabla_x + i\ \nabla_t)(\nabla_x \cdot C_x + i\ \nabla_t\ \cdot C_t) - (\nabla_x^2 C_x + i\nabla_t^2\ C_t)$$
$$= (\nabla_x + i\ \nabla_t)((\nabla_x + i\ \nabla_t) \cdot (C_x + iC_t)) - (\nabla_x^2 + i\ \nabla_t^2)(C_x + iC_t)$$
$$= \nabla\ (\nabla \cdot \mathrm{C}) - \nabla^2\ C$$

With the above descriptions, the generalization of Maxwell's equations to C^3 as given in equations (2.2) is now fully defined. We next turn to addressing several questions. Do the charge ρ and current J densities in these equations also extend into t-space? Are the generalized equations consistent with the original Maxwell equations? Do they obey a duality transform, and if so, what does that transform look like? What does the wave equation look like? Is the expansion of electromagnetic fields to have imaginary valued components still consistent with the conservation of charge? The answers to these questions are derived from equations (2.2) in the following sections.

3. Basic properties of the generalized equations

Allowing the fields E and B to extend into imaginary valued t-space in the generalized equations (2.2) raises the issue of whether charge ρ and current J densities might also do so. While it may not be apparent on first glance, the equations (2.2) imply that, unlike electromagnetic fields, charge and current in equations (2.2) cannot have imaginary valued components. To see this, suppose that $\rho = \rho_x + i\rho_t$ where $\rho_t \neq 0$; in other words, suppose that charge has an imaginary component. Then according to equation (2.2c),

$$\nabla_x \cdot B_x + i\ \nabla_t\ \cdot B_t = ic\mu_0(\rho_x + i\rho_t) = -c\mu_0\rho_t + ic\mu_0\rho_x \tag{3.1}$$

and $\nabla_x \cdot B_x = -c\mu_0\rho_t$ would follow from equating the real parts of this, something that is inconsistent with past experimental results showing that always $\nabla_x \cdot B_x = \mathbf{0}$. It follows that ρ_t must be zero. Similarly, if $J = J_x + i J_t$ where $J_t \neq \mathbf{0}$, then it would follow from equation (2.2b) that

$$\nabla_x \times E_x = -\frac{\partial B_x}{\partial t} + c\ \mu_0\ J_t \tag{3.2}$$

which is inconsistent with experimental results showing that always $\nabla_x \times E_x = -\frac{\partial B_x}{\partial t}$, so it must be that $J_t = \mathbf{0}$. To summarize, while the theory presented here postulates that the fields E and B extend into imaginary valued t-space, this hypothesized extension does not include material charge/current.[2] Accordingly, in the following we will continue to use the symbol ρ to represent real-valued charge density, and to use J for current density, where J can be thought of as representing either $J_x + i\,\mathbf{0}$ in C^3 (as in equations (2.2)) or just J_x in R^3 unambiguously as determined by context.

Each of the four generalized electrodynamics equations, equations (2.2), involve complex-valued fields in C^3, so they each represent two equations, one in r-space and one in t-space. For example, writing out equation (2.2c) using the definitions above gives

$$\nabla \cdot B = (\nabla_x + i\,\nabla_t) \cdot (B_x + iB_t) = \nabla_x \cdot B_x + i\,\nabla_t \cdot B_t = ic\mu_0\rho. \tag{3.3}$$

Equating the real and imaginary parts of the rightmost equality of equation (3.3) to each other gives two separate equations, each in R^3,

$$\nabla_x \cdot B_x = 0 \tag{3.4a}$$
$$\nabla_t \cdot B_t = c\mu_0\rho \tag{3.4b}$$

the first involving r-space, and the second involving t-space (in other words, there is an implicit i on both sides of equation (3.4b)). Similarly, writing out equation (2.2b) gives

$$\nabla \times E = (\nabla_x + i\,\nabla_t) \times (E_x + iE_t) = \nabla_x\ \times E_x + i\,\nabla_t\ \times E_t \tag{3.5}$$

[2] Since electric charge is always associated with matter (particles) having mass, it seems reasonable to conjecture that all matter having mass does not extend into t-space. Since the photons composing electromagnetic fields are massless and do not carry a charge, they would be an exception to this conjectured rule and can extend into t-space as is hypothesized to occur here.

for the left side, and

$$-ic\mu_0 \boldsymbol{J} - \frac{\partial \boldsymbol{B}}{\partial t} = -ic\mu_0 \boldsymbol{J} - \frac{\partial \boldsymbol{B}_x}{\partial t} - i\frac{\partial \boldsymbol{B}_t}{\partial t} = -\frac{\partial \boldsymbol{B}_x}{\partial t} + i\left(-c\mu_0 \boldsymbol{J} - \frac{\partial \boldsymbol{B}_t}{\partial t}\right) \tag{3.6}$$

for the right side. Equating the real parts of the rightmost portions of equations (3.5) and (3.6) to each other, and the corresponding imaginary portions to each other, gives two separate equations

$$\boldsymbol{\nabla_x} \times \boldsymbol{E_x} = -\frac{\partial \boldsymbol{B}_x}{\partial t} \tag{3.7a}$$

$$\boldsymbol{\nabla_t} \times \boldsymbol{E_t} = -c\mu_0 \boldsymbol{J} - \frac{\partial \boldsymbol{B}_t}{\partial t} \tag{3.7b}$$

the first in r-space and the second in t-space.

Extending this process to all four of the generalized equations (2.2) indicates that they can be partitioned into two separate sets of equations corresponding to r-space and imaginary-valued t-space. The four *r-space electrodynamics equations* are

$$\nabla_x \cdot \boldsymbol{E_x} = \frac{1}{\epsilon_0}\rho \tag{3.8a}$$

$$\nabla_x \times \boldsymbol{E_x} = -\frac{\partial \boldsymbol{B_x}}{\partial t} \tag{3.8b}$$

$$\nabla_x \cdot \boldsymbol{B_x} = 0 \tag{3.8c}$$

$$\nabla_x \times \boldsymbol{B_x} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E_x}}{\partial t} \tag{3.8d}$$

These are exactly the familiar Maxwell's equations, equations (1.1), without magnetic charge (with the symbols $\boldsymbol{E}$ and $\boldsymbol{B}$ there replaced by the equivalent $\boldsymbol{E}_x$ and $\boldsymbol{B}_x$ here, respectively). It is important to recognize this equivalence with equations (1.1) because it shows that the equations (2.2) are a true generalization of Maxwell's equations and thus consistent with classical electrodynamics. The generalized equations encompass all of the experimental results that the original Maxwell's equations encompass in observable physical space (r-space), including prediction of the absence of magnetic charge that is detectable in r-space, and they do not predict additional electromagnetic phenomena that have not been observed in r-space.

More interestingly, there is a second, new set of four *t-space electrodynamics equations* derived from the generalized equations (2.2) that emerges from equating their imaginary-valued portions during the above procedure:

$$\nabla_t \cdot \boldsymbol{E_t} = 0 \tag{3.9a}$$

$$\nabla_t \times \boldsymbol{E_t} = -c\mu_0 \boldsymbol{J} - \frac{\partial \boldsymbol{B_t}}{\partial t} \tag{3.9b}$$

$$\nabla_t \cdot \boldsymbol{B_t} = c\mu_0 \rho \tag{3.9c}$$

$$\nabla_t \times \boldsymbol{B_t} = \frac{1}{c^2}\frac{\partial \boldsymbol{E_t}}{\partial t} \tag{3.9d}$$

Equations (3.9) in t-space are analogous to the original Maxwell equations (3.8) in r-space, but they now characterize electromagnetic fields $\boldsymbol{E}_t$ and $\boldsymbol{B}_t$ that both exist in the purely imaginary-valued t-space. The similarity to Maxwell's equations (3.8) becomes more evident if one notes that the roles of the electric and magnetic fields are effectively reversed in equations (3.9) relative to (3.8). For example, in t-space there are no electric monopoles (equation (3.9*a*)), but instead there are magnetic monopoles (equation (3.9*c*)), as will be discussed in the next section.

The generalized electrodynamics equations (2.2) have a number of additional basic properties that are reminiscent of those of the original Maxwell's equations, such as a continuity equation. To see this, apply the divergence $\nabla\cdot$ to both sides of equation (2.2*d*). This gives zero on the left side (by equation (2.7*a*)), and

$$\begin{aligned}\nabla \cdot \left(\mu_0 \boldsymbol{J} + \tfrac{1}{c^2}\tfrac{\partial \boldsymbol{E}}{\partial t}\right) &= (\nabla_x + i\nabla_t) \cdot \left(\left(\mu_0 \boldsymbol{J} + \tfrac{1}{c^2}\tfrac{\partial \boldsymbol{E}_x}{\partial t}\right) + i\tfrac{1}{c^2}\tfrac{\partial \boldsymbol{E}_t}{\partial t}\right)\\ &= \mu_0 \nabla_x \cdot \boldsymbol{J} + \tfrac{1}{c^2}\tfrac{\partial}{\partial t}\nabla_x \cdot \boldsymbol{E}_x + i\tfrac{1}{c^2}\tfrac{\partial}{\partial t}\nabla_t \cdot \boldsymbol{E}_t\\ &= \mu_0 \nabla_x \cdot \boldsymbol{J} + \tfrac{1}{c^2}\tfrac{\partial}{\partial t}\left(\tfrac{\rho}{\epsilon_o}\right) + i0 = \mu_0 \nabla_x \cdot \boldsymbol{J} + \tfrac{1}{c^2\epsilon_o}\tfrac{\partial \rho}{\partial t}\end{aligned}$$

on the right side, since $\nabla_x \cdot \boldsymbol{E}_x = \frac{\rho}{\epsilon_o}$ by equation (3.8*a*) and $\nabla_t \cdot \boldsymbol{E}_t = 0$ by equation (3.9*a*). Because the last line of this result equals zero, and using $c^2 = (\epsilon_0\mu_0)^{-1}$, this produces a continuity equation in C^3 of

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}. \tag{3.10}$$

Equation (3.10) shows that the generalized equations (2.2) continue to be consistent with experimentally observed conservation of electric charge.

Further, the generalized equations (2.2), like the more symmetrical extended Maxwell's equations (1.2) that include hypothesized magnetic charge in r-space, are invariant under a duality transformation between the electric and magnetic fields, although now these fields are complex-valued rather than real-valued. Specifically, the duality transformation

$$\boldsymbol{E}' = c\boldsymbol{B}, \qquad \boldsymbol{B}' = -\frac{1}{c}\boldsymbol{E}, \qquad \rho' = i\rho, \qquad \boldsymbol{J}' = i\boldsymbol{J} \tag{3.11}$$

when applied to equations (2.2), leaves these equations unchanged. For example, for equation (2.2*a*)

$$\nabla \cdot \boldsymbol{E}' = c\nabla \cdot \boldsymbol{B} = ic^2\mu_0\rho = ic^2\mu_0(-i\rho') = \frac{1}{\epsilon_0}\rho'$$

since $c^2 = (\epsilon_0\mu_0)^{-1}$ and since $\rho = -i\rho'$ by equations (3.11). Similarly, for equation 2.2d we have

$$\nabla \times \boldsymbol{B}' = -\frac{1}{c}\nabla \times \boldsymbol{E} = -\frac{1}{c}\left[-ic\mu_0\boldsymbol{J} - \frac{\partial \boldsymbol{B}}{\partial t}\right] = \frac{1}{c}\left[ic\mu_0(-i\boldsymbol{J}') + \frac{1}{c}\frac{\partial \boldsymbol{E}'}{\partial t}\right] = \mu_0\boldsymbol{J}' + \frac{1}{c^2}\frac{\partial \boldsymbol{E}'}{\partial t}$$

since $\boldsymbol{J} = -i\boldsymbol{J}'$ by the transformation equations (3.11). Equations (2.2b) and (2.2c) are analogous.

There is another type of duality transformation of interest that does not involve invariance and that, unlike equation (3.11), does not have an analogous transformation involving the original Maxwell equations. This additional *cross-domain duality transformation* maps the 3D r-space equations (3.8) into the 3D t-space equations (3.9), and vice versa, both sets of equations involving real-valued vectors having three components. This second novel type of duality transformation is given by

$$\nabla_x = \nabla_t \quad \boldsymbol{E}_x = c\,\boldsymbol{B}_t \quad \boldsymbol{B}_x = -\frac{1}{c}\boldsymbol{E}_t \tag{3.12}$$

Applied to the set of equations equations (3.8) as a whole, these rules produce the set equations (3.9); applied to equations (3.9), they produce equations (3.8). For example, for equation (3.8*a*) we have

$$\nabla_x \cdot \boldsymbol{E}_x = \frac{1}{\epsilon_0}\rho \implies \nabla_t \cdot (c\boldsymbol{B}_t) = \frac{1}{\epsilon_0}\rho \implies \nabla_t \cdot \boldsymbol{B}_t = \frac{1}{c\epsilon_0}\rho = c\mu_0\rho$$

which is equation (3.9*c*), and analogous substitutions in equation (3.8*c*) give equation (3.9*a*). Similarly, for equation (3.8d)

$$\nabla_x \times \boldsymbol{B}_x = \mu_0\boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}_x}{\partial t} \implies -\frac{1}{c}\nabla_t \times \boldsymbol{E}_t = \mu_0\boldsymbol{J} + \frac{1}{c^2}c\frac{\partial \boldsymbol{B}_t}{\partial t}$$
$$\implies \nabla_t \times \boldsymbol{E}_t = -c\mu_0\boldsymbol{J} - \frac{\partial \boldsymbol{B}_t}{\partial t}$$

which is equation (3.9*b*), and analogous substitutions in equation (3.8*b*) give equation (3.9*d*). Thus, the cross-domain duality transformation given by equation (3.12) maps the set of equations (3.8), taken as a whole, into the set of equations (3.9). The same cross-domain transformation with minor algebraic manipulations ($\nabla_t = \nabla_x$, $\boldsymbol{B}_t = \frac{1}{c}\boldsymbol{E}_x$, and $\boldsymbol{E}_t = -c\boldsymbol{B}_x$), when applied to equations (3.9) in a similar fashion, gives equations (3.8). Note that this cross-domain transformation equation (3.12) necessarily involves switching between ∇_x and ∇_t. Equation (3.12) indicate an additional type of symmetry in the theory developed here that is most evident by comparing equations (3.9) to equations (3.8) where the roles of the electric and magnetic fields are reversed, as mentioned earlier in this section.

4. Magnetic monopoles

The full set of generalized electrodynamics equations (equations (2.2)) represent a novel solution to the dilemma discussed in the Introduction of having good theoretical reasons to expect that magnetic monopoles exist but of not being able to detect them experimentally. Specifically, the ubiquitous existence of magnetic monopoles is implied by $\nabla_t \cdot \boldsymbol{B}_t = c\mu_0\rho$ (equation (3.9*c*)) which indicates that charge serves as a source/sink for radially directed magnetic fields $\boldsymbol{B}_t$ that lie solely in the imaginary-valued t-space of equations (2.2). This is illustrated in figure 2. In classical electrodynamics, electric fields $\boldsymbol{E}_x$ can of course originate at positive electric charges and terminate on negative electric charges in familiar r-space only, as sketched in figures 2(a) and (c). The absence of magnetic charge means that magnetic field lines for $\boldsymbol{B}_x$ are either loops (closing on themselves) or that they extend to infinity—there are no magnetic monopoles. In contrast to this, with the theory presented here charges serve as both electric and magnetic monopoles, as caricatured in figures 2(b) and (d). These individual

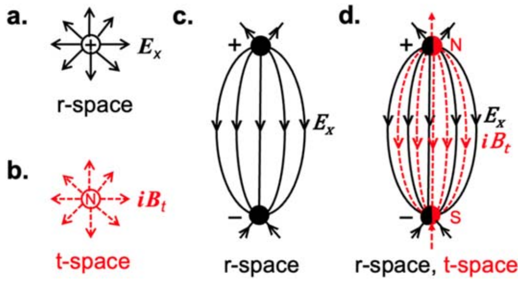

Figure 2. (**a**). In classical electrodynamics, electric fields $\boldsymbol{E_x}$ (solid arrows) can originate at positive electric charges and terminate on negative electric charges in r-space only, as with the positive electric charge sketched here. No magnetic fields B_x originate/terminate on magnetic charges. (**b**). In the theory presented in this paper, magnetic field lines (red font and dashed arrows) can originate at north poles N (positive electric charges) and terminate at south poles S (negative electric charges), producing magnetic fields $\boldsymbol{B_t}$ in t-space only. (**c**). In existing theory, the electric fields associated with electric dipoles can be caricatured as sketched here in r-space. (**d**). In contrast, the theory presented here predicts that electric dipoles also represent magnetic dipoles, and have both real-valued electric fields in r-space and imaginary-valued (red) magnetic fields in t-space. □

monopoles serve as sources/sinks for magnetic fields that extend radially in t-space, and at rest they do not produce electric fields in t-space (equation (3.9*a*)). Magnetic field lines originate at north poles N (associated with positive electric charges) and terminate at south poles S (associated with negative electric charges), representing magnetic fields $\boldsymbol{B_t}$ in t-space. In other words, charges are conceptually sources and sinks for both electric and magnetic fields, making magnetic monopoles common rather than rare (every proton, every electron, etc). In figure 2(d) the +/N pole can be viewed as a field source, and the −/S pole can be viewed as a field sink.

Equation (2.2*b*), $\nabla \times \boldsymbol{E} = -ic\mu_0\boldsymbol{J} - \frac{\partial \boldsymbol{B}}{\partial t}$, implies also that moving charge produces an imaginary-valued electric field $i\boldsymbol{E_t}$ in t-space, as indicated by the imaginary portion of this equation, $\nabla_t \times \boldsymbol{E_t} = -c\mu_0\boldsymbol{J} - \frac{\partial \boldsymbol{B_t}}{\partial t}$. This is consistent with the cross-domain duality transform which specifies that E_t in t-space behaves like B_x in r-space, as noted earlier.

As pointed out previously [3], the existence of magnetic monopoles would imply a magnetostatic analog to the electrostatic equation $\boldsymbol{E_x} = \frac{1}{4\pi\epsilon_o}\frac{q}{r^2}\hat{\boldsymbol{r}}$ for a stationary isolated charge q located at the origin in r-space (here $\hat{\boldsymbol{r}}$ is a unit vector in the direction of point $\boldsymbol{r}$). Applying the cross-domain duality transformation to this equation for $\boldsymbol{E_x}$ results in an analogous purely imaginary valued

$$\boldsymbol{B_t} = \frac{c\mu_o}{4\pi}\frac{q}{|\boldsymbol{t}|^2}\hat{\boldsymbol{t}} \tag{4.1}$$

for the associated magnetic field at location $\boldsymbol{t}$ in t-space ($\hat{\boldsymbol{t}}$ is a unit vector in the direction of $\boldsymbol{t}$). The point is that, according to the theory developed here, such magnetic monopoles are experimentally unobservable in our conventional R^3 space (r-space) as their fields do not extend into r-space.

Since particles that carry magnetic charge are the same as those that carry electric charge in the generalized equations (2.2), according to the theory presented here we know *a priori* that magnetic monopoles are widespread (every proton, electron, etc) and that they are stable particles. Further, the continuity equation for charge derived above (equation (3.10)), which indicates that the generalized electrodynamics equations (2.2) continue to respect the conservation of electric charge, also implies that magnetic charge is conserved. Just as with electric charge, two types of magnetic charge exist, with positive electric charges corresponding to magnetic north poles N, and negative electric charges corresponding to magnetic south poles S. Put otherwise, the theory presented here predicts that *all electrically charged particles in r-space are also magnetically charged*, but that this is not normally evident to us because the imaginary-valued magnetic fields involved are solely in t-space and unobservable. This accounts for why magnetic monopoles have not been detected in the many careful experiments searching for them that have been carried out previously.

The idea that charge serves as a source/sink for both electric and magnetic radially directed fields has occasionally been proposed previously in various contexts. However, to the author's knowledge, such theoretical proposals for magnetic monopoles have always involved magnetic fields that lie solely in r-space rather than existing in t-space as is proposed here. While such past theories may be consistent with equations (1.2), they are not consistent with the new equations (2.2) described in this paper. Examples of such past proposals include hypothetical particles called dyons [28] and particles that oscillate over time between carrying electric and magnetic charge [29]. Like magnetic monopoles, particles such as dyons have not yet been observed experimentally in spite of some impressive recent efforts to search for them [30].

5. Generalized wave equations

One of the most important implications of Maxwell's equations is the prediction that electromagnetic waves exist. It is straightforward to show that the generalized electrodynamics equations (2.2) continue to imply that electromagnetic waves exist, but that they would be expected to propagate into t-space as well as r-space. Specifically, one can derive wave equations in C^3 in the absence of charge from equations (2.2) as

$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \tag{5.1a}$$

and

$$\nabla^2 \boldsymbol{B} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2} \tag{5.1b}$$

that are analogous to those for the original Maxwell equations. For example, to derive equation (5.1*a*), first apply $\nabla\times$ to the left side of equation (2.2*b*) to obtain

$$(i)\ \nabla \times (\nabla \times \boldsymbol{E}) = \nabla_x (\nabla_x \cdot \boldsymbol{E_x}) - \nabla_x^2 \boldsymbol{E_x} + i(\boldsymbol{\nabla_t} (\boldsymbol{\nabla_t} \cdot \mathbf{E_t}) - \boldsymbol{\nabla_t^2} \mathbf{E_t}) = -(\boldsymbol{\nabla_x^2} \mathbf{E_x} + \mathbf{i}\,\boldsymbol{\nabla_t^2} \mathbf{E_t}) = -\nabla^2 \boldsymbol{E}$$

because both $\boldsymbol{\nabla_x} \cdot \boldsymbol{E_x}$ and $\boldsymbol{\nabla_t} \cdot \boldsymbol{E_t}$ are zero in the absence of charge (equations (3.8*a*) and (3.9*a*)).

Applying $\nabla\times$ to the right side of equation (2.2*b*) in a vacuum (no charge/current present) gives

$$(ii) -\nabla \times \frac{\partial \boldsymbol{B}}{\partial t} = -(\nabla_x + i\nabla_t)\times\left(\frac{\partial \boldsymbol{B_x}}{\partial t} + i\frac{\partial \boldsymbol{B_t}}{\partial t}\right) = -\left(\frac{\partial}{\partial t}(\nabla_x \times \boldsymbol{B_x}) + i\frac{\partial}{\partial t}(\nabla_t \times \boldsymbol{B_t})\right)$$
$$= -\left(\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \boldsymbol{E_x}}{\partial t}\right) + i\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \boldsymbol{E_t}}{\partial t}\right)\right) = -\frac{1}{c^2}\left(\frac{\partial^2 \boldsymbol{E_x}}{\partial t^2} + i\frac{\partial^2 \boldsymbol{E_t}}{\partial t^2}\right) = -\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

where the values of $\nabla_x \times \boldsymbol{B_x}$ and $\nabla_t \times \boldsymbol{B_t}$ were taken from equations (3.8*d*) and (3.9*d*), respectively. Equating (i) and (ii) gives the wave equation (5.1*a*) in C^3 for $\boldsymbol{E}$. The derivation of the wave equation (5.1*b*) for $\boldsymbol{B}$ is similar, starting instead from equation (2.2*d*).

As with the generalized equations (2.2), Equations (5.1) each represent a set of two wave equations, the usual wave equations for $\boldsymbol{E_x}$ and $\boldsymbol{B_x}$ in r-space

$$\nabla_x^2 \boldsymbol{E_x} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E_x}}{\partial t^2} \text{ and } \nabla_x^2 \boldsymbol{B_x} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{B_x}}{\partial t^2} \tag{5.2}$$

and

$$\nabla_t^2 \boldsymbol{E_t} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E_t}}{\partial t^2} \text{ and } \nabla_t^2 \boldsymbol{B_t} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{B_t}}{\partial t^2} \tag{5.3}$$

in imaginary valued t-space. According to these results, electromagnetic waves extend into t-space, not just r-space, and they propagate at the same speed c in empty t-space as they do in empty r-space.

What do solutions to the generalized wave equations (5.1) look like? To give an example, consider the monochromatic sinusoidal plane wave that is a simple but very important solution (linear combinations, Fourier series, etc) to the standard wave equation (5.2) in r-space that are the same as the wave equations derived from Maxwell's original equations. Even though such waves involving $\boldsymbol{E_x}$ and $\boldsymbol{B_x}$ are considered in classical electrodynamics to exist solely in r-space, they are often represented as complex-valued functions in C^3 by using the exponential $e^{i\phi}$, where ϕ is the wave phase. This is motivated by the computational conveniences that such a complex-valued formulation entails, and importantly, such representations are almost always accompanied by an explicit statement that only the real parts of such equations represent actual physical fields. However, in the context of the generalized formulation of Maxwell's equations considered here, it is reasonable to ask whether the full complex-valued plane waves $e^{i\phi}$ used in the past should instead be taken at face value. In other words, does the theory introduced here in equations (2.2) mean that the imaginary parts of such complex-valued waves should *not* be discarded as is currently done, and if so, what does that imply?

Consider a simple situation where a single source emits a brief electromagnetic wave pulse (e.g., a flash of light) that travels as an expanding spherical wave through both r-space and t-space. Far from the source the spherical wavefront can be closely approximated by a complex-valued

$$\boldsymbol{E} = \boldsymbol{E_o} e^{i\phi} \tag{5.4}$$

for a monochromatic sinusoidal plane wave in C^3. Here $\boldsymbol{E}_o$ is a constant real-valued vector in R^3. Quantity $\phi = \boldsymbol{k} \cdot \boldsymbol{x} + \boldsymbol{k} \cdot \boldsymbol{t} - \omega t + \delta$ is the wave phase where $\boldsymbol{k}$ is the wave propagation vector having magnitude $k = |\boldsymbol{k}|$ as the wave number, $\boldsymbol{x} + i\boldsymbol{t}$ is a point in C^3 space, ω is the wave's angular frequency, and real scalar δ is a phase constant. The *full* equation (5.4) can be shown to be a solution to the generalized equation (5.1*a*), as demonstrated immediately below. This is consistent with the notion that electromagnetic waves in the theory studied here propagate both in r-space and in t-space. Equation (5.4) represents a solution where the wave fronts in t-space are $\pi/2$ out of phase with those in r-space.

That equation (5.4) is a solution to equation (5.1*a*) is shown by substituting $\boldsymbol{E}_o e^{i\phi}$ into both sides of the wave equation (equation (5.1*a*)) and obtaining an identity. Substitution into the left side gives

$$\begin{aligned}(i)\ \nabla^2 \boldsymbol{E} &= (\nabla_x^2 + i\,\nabla_t^2)\boldsymbol{E}_o e^{i\phi}\\ &= \boldsymbol{E}_o(\nabla_x^2 \cos\phi + i\,\nabla_t^2 sin\phi)\\ &= \boldsymbol{E}_o(\nabla_x \cdot \nabla_x \cos\phi + i\,\nabla_t \cdot \nabla_t \sin\phi)\\ &= \boldsymbol{E}_o(-\nabla_x \cdot k \sin\phi + i\,\nabla_t \cdot k \cos\phi)\\ &= -k^2 \boldsymbol{E}_o e^{i\phi}.\end{aligned}$$

Similarly, substituting $\boldsymbol{E}_o e^{i\phi}$ into the right side of equation (5.1*a*) gives

$$\begin{aligned}(ii)\quad \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} &= \frac{1}{c^2}\boldsymbol{E}_o\frac{\partial^2}{\partial t^2}(\cos\phi + i\,\sin\phi)\\ &= \frac{1}{c^2}\boldsymbol{E}_o\frac{\partial}{\partial t}(\omega\sin\phi - i\,\omega\cos\phi)\\ &= \frac{\omega}{c^2}\boldsymbol{E}_o(-\omega\cos\phi - i\,\omega\sin\phi)\\ &= -\frac{\omega^2}{c^2}\boldsymbol{E}_o(\cos\phi + isin\phi)\\ &= -k^2\boldsymbol{E}_o e^{i\phi}\end{aligned}$$

where the relation $k = \omega/c$ was used on the last step. Since (i) = (ii), $\boldsymbol{E}_o e^{i\phi}$ is a solution to equation (5.1*a*). Analogous results can be obtained for equation (5.1*b*) involving $\boldsymbol{B}$.

6. Energy considerations

Extending electromagnetic fields to include imaginary-valued components $\boldsymbol{E}_t$ and $\boldsymbol{B}_t$ raises the issue of how that might affect the energy of charged particles as we measure it in r-space. Specifically, a key question is this: Is the generalization of Maxwell's equations (2.2) to complex-valued fields consistent with the conservation of energy that we can observe experimentally for systems of charges and current in r-space? To examine this question, we first need to consider how charged particles in r-space would be affected by electromagnetic fields that are now complex valued.

Maxwell's equations are classically complemented by the Lorentz force law, which in the notation used here is given by

$$\boldsymbol{F_x} = q\ [\boldsymbol{E_x} + (\boldsymbol{v} \times \boldsymbol{B_x})]. \tag{6.1}$$

This law describes the force $\boldsymbol{F_x}$ in r-space acting on a particle having an electrical charge $q = q_e$ and moving with velocity $\boldsymbol{v}$ through r-space in the presence of electromagnetic fields. When hypothetical magnetic charge is discussed in the literature, this law is often expanded to

$$\boldsymbol{F_x} = q_e\ [\boldsymbol{E_x} + (\boldsymbol{v} \times \boldsymbol{B_x})] + q_m\left[\boldsymbol{B_x} - \tfrac{1}{c^2}(\boldsymbol{v} \times \boldsymbol{E_x})\right] \tag{6.2}$$

which includes forces that would be attributable to magnetic charge q_m [3, 19, 20]; .[3] Unlike the experimentally derived original law equation (6.1), this addition involving hypothetical q_m is derived based on theoretical considerations such as an electromagnetic duality transformation.

In the theory presented here, we proceed in a similar fashion to what has been done in the past, but in this case applying the cross-domain duality transformation equation (3.12) to the Lorentz force law equation (6.1), and find that

$$\boldsymbol{F_t} = q\left[c\ \boldsymbol{B_t} - \frac{1}{c}(\boldsymbol{v} \times \boldsymbol{E_t})\right] \tag{6.3}$$

would describe the additional forces occurring in t-space, so that

$$\boldsymbol{F} = \boldsymbol{F_x} + i\ \boldsymbol{F_t} = q\ [\boldsymbol{E_x} + (\boldsymbol{v} \times \boldsymbol{B_x})] + iq\left[c\ \boldsymbol{B_t} - \frac{1}{c}(\boldsymbol{v} \times \boldsymbol{E_t})\right] \tag{6.4}$$

would be the full extended force equation. While this appears similar to equation (6.2), which is solely in r-space, note that $\boldsymbol{F}$ in equation (6.4) exists in C^3, that q serves to represent both q_e and q_m, and that the forces associated

[3] Sometimes this equation is expressed as a force density rather than as a force, e.g., [23, page 49].

with fields $\boldsymbol{B_t}$ and $\boldsymbol{E_t}$ are taken to be purely imaginary valued and thus extend only through t-space. There is a magnetic analog to Coulomb's law in electrostatics (based on equation (4.1)), with particles having opposite magnetic charges attracting one another in t-space, and those with like magnetic charges repelling one another in t-space.

A particle having charge q would be expected to react to these forces. It follows from equation (6.4) that such a particle would be accelerated in both r-space and in t-space. According to equation (6.4), the forces due to fields in r-space cause acceleration affecting $\boldsymbol{v}$ in r-space exactly as we observe. In general, two isolated resting charges of opposite sign that are near one another would be affected by each other's $\boldsymbol{E_x}$ and $\boldsymbol{B_t}$ fields, causing them to move towards each other in both r-space and t-space. We would not expect to observe any direct impact on a particle's movement in r-space due to the $\boldsymbol{B_t}$ and $\boldsymbol{E_t}$ fields of other charges, and our observations in r-space would remain precisely as we observe them.

Given the force law equation (6.4), we can now ask whether complex valued electromagnetic fields governed by the generalized electrodynamics of equations (2.2), are consistent with the conservation of energy that we can observe experimentally for systems of charges and current in r-space. This provides a check that the theory pursued here agrees with existing findings, verifying that the extended fields do not imply *measurable* phenomena that have not been observed. To address this issue, we use the same approach that is commonly taken to assessing the energy associated with a system of charged particles restricted to a volume V in r-space (no charge leaves/enters V from outside) having charge density ρ and current density $\boldsymbol{J}$, but now modified by using the full extended force equation (6.4) above. We only consider the behavior of charge in r-space because that is what past studies are based on.

It is customary to define the total electromagnetic energy associated with isolated charge ρ and current $\boldsymbol{J}$ densities in r-space as the total reversible work required to create ρ and $\boldsymbol{J}$ and their associated fields by bringing charge in from infinity, e.g., [23]. Accordingly, we adopt this definition where the work is assumed to be done by the complex fields $\boldsymbol{E} = \boldsymbol{E_x} + i\boldsymbol{E}_t$ and $\boldsymbol{B} = \boldsymbol{B_x} + i\boldsymbol{B}_t$ in assembling that system of charges in r-space from an initial situation where all of the charge is located spatially at infinity. Then by equation (6.4), the rate at which the complex fields $\boldsymbol{E}$ and $\boldsymbol{B}$ do mechanical work on this charge distribution is given by

$$\begin{aligned}\frac{dW}{dt} &= \int_V q[\boldsymbol{E_x} + (\boldsymbol{v} \times \boldsymbol{B_x})] \cdot \boldsymbol{v} d^3x + i\int_V q\left[c\,\boldsymbol{B_t} - \frac{1}{c}(\boldsymbol{v} \times \boldsymbol{E_t})\right] \cdot \boldsymbol{v} d^3x \\ &= \int_V \boldsymbol{E_x} \cdot q\boldsymbol{v} d^3x + i\int_V c\,\boldsymbol{B_t} \cdot q\boldsymbol{v} d^3x \\ &= \int_V \boldsymbol{E_x} \cdot \boldsymbol{J} d^3x + i\int_V c\,\boldsymbol{B_t} \cdot \boldsymbol{J}\, d^3x \end{aligned} \tag{6.5}$$

where d^3x is a volume element in r-space, and we are integrating over V in r-space because that is what is done in classical electrodynamics. Interestingly, work and energy in the theory given here, even though considered solely in r-space, involve an unobservable imaginary-valued second term arising from equation (6.4) that is not present in classical electrodynamics. This second term in equation (6.5) is always guaranteed to be purely imaginary-valued due to the i preceding the integral, and because $\boldsymbol{B}_t$ and $\boldsymbol{J}$ are both 3D real-valued vectors, so $\boldsymbol{B_t} \cdot \boldsymbol{J}$ is always real (see equation (2.1) and the first paragraph of Section 3). It follows that the *observable* mechanical work *dW/dt* only involves the first real-valued integral over $\boldsymbol{E_x}$, consistent with what is measured experimentally. It is also interesting to note that, just as the magnetic field $\boldsymbol{B}_x$ in r-space does no work, the electric field $\boldsymbol{E_t}$ in t-space does no work on charges.

Replacing $\boldsymbol{J}$ in each of the two integrals in equation (6.5) using equations (3.8*d*) and (3.9*b*) respectively, applying an identity for the divergence of the cross-product of two vectors in R^3, and then replacing $\nabla_x \times \boldsymbol{E}_x$ and $\nabla_t \times \boldsymbol{B}_t$ using equations (3.8*b*) and (3.9*d*), respectively, gives

$$\begin{aligned}\frac{dW}{dt} &= \int_V \left[-\epsilon_o \boldsymbol{E}_x \cdot \frac{\partial \boldsymbol{E}_x}{\partial t} - \frac{1}{\mu_o}\boldsymbol{B}_x \cdot \frac{\partial \boldsymbol{B}_x}{\partial t} - \nabla_x \cdot \frac{1}{\mu_o}(\boldsymbol{E}_x \times \boldsymbol{B}_x)\right] d^3x \\ &\quad + i\int_V \left[-\epsilon_o \boldsymbol{E}_t \cdot \frac{\partial \boldsymbol{E}_t}{\partial t} - \frac{1}{\mu_o}\boldsymbol{B}_t \cdot \frac{\partial \boldsymbol{B}_t}{\partial t} - \nabla_t \cdot \frac{1}{\mu_o}(\boldsymbol{E}_t \times \boldsymbol{B}_t)\right] d^3x. \end{aligned} \tag{6.6}$$

where we again made use of the identity $c^2 = \frac{1}{\mu_o \epsilon_o}$. For an arbitrary real-valued vector $\boldsymbol{C}$ in R^3, such as $\boldsymbol{E_x}$, $\boldsymbol{E}_t$, $\boldsymbol{B_x}$ or $\boldsymbol{B}_t$, it holds that $\boldsymbol{C} \cdot \frac{\partial \boldsymbol{C}}{\partial t} = \frac{1}{2}\frac{\partial |\boldsymbol{C}|^2}{\partial t}$. Further, the quantity $\boldsymbol{S}_x = \frac{1}{\mu_o}(\boldsymbol{E}_x \times \boldsymbol{B}_x)$ in r-space on the first line of equation (6.6) is Poynting's vector, and on the second line we define an analogous quantity $\boldsymbol{S}_t = \frac{1}{\mu_o}(\boldsymbol{E}_t \times \boldsymbol{B}_t)$. Making these substitutions into equation (6.6) gives

$$\begin{aligned}\frac{dW}{dt} &= \int_V \left[-\frac{\partial}{\partial t}\frac{1}{2}\left(\epsilon_o E_x^2 + \frac{1}{\mu_o}B_x^2\right) - \nabla_x \cdot \boldsymbol{S}_x\right] d^3x + i\int_V \left[-\frac{\partial}{\partial t}\frac{1}{2}\left(\epsilon_o E_t^2 + \frac{1}{\mu_o}B_t^2\right) - \nabla_t \cdot \boldsymbol{S}_t\right] d^3x \\ &= \int_V \left[-\frac{\partial}{\partial t}u_x - \nabla_x \cdot \boldsymbol{S}_x\right] d^3x + i\int_V \left[-\frac{\partial}{\partial t}u_t - \nabla_t \cdot \boldsymbol{S}_t\right] d^3x \end{aligned} \tag{6.7}$$

where $u_x = \frac{1}{2}\left(\epsilon_o E_x^2 + \frac{1}{\mu_o}B_x^2\right)$ is the familiar energy density for r-space electromagnetic fields, and we define an analogous imaginary-valued energy density $u_t = \frac{1}{2}\left(\epsilon_o E_t^2 + \frac{1}{\mu_o}B_t^2\right)$ for the t-space fields. The last line of equation (6.7) indicates two things. First, the left-most integral shows that what we currently observe for conservation of energy in r-space is consistent with the theory presented here. Second, the right-most integral indicates that there a purely imaginary-valued aspect of energy u_t predicted by the current theory that has not been previously recognized.

Equation (6.7) is essentially a generalization of Poynting's theorem. The real portion corresponds to the classical Poynting's theorem, while the imaginary part is analogous but derived from the imaginary-valued fields $\boldsymbol{E}_t$ and $\boldsymbol{B}_t$. Additional analogies can be noted, as follows. In the case where no mechanical work is being done on the particles in V (static situation, empty space, etc), then both the real and imaginary parts of the right side of equation (6.7) are individually zero. Since V is arbitrary (i.e., since these integrals hold over any V), both

$$\frac{\partial}{\partial t}u_x = -\nabla_x \cdot \boldsymbol{S}_x \tag{6.8a}$$

and

$$\frac{\partial}{\partial t}u_t = -\nabla_t \cdot \boldsymbol{S}_t \tag{6.8b}$$

follow. The first of these is the well-known 'continuity equation' for energy in classical electrodynamics that indicates the local conservation of energy in r-space; the second is an analogous statement of local conservation of electromagnetic energy involving the imaginary valued t-space fields. Re-grouping the terms in equation (6.7) gives

$$\begin{aligned}\frac{dW}{dt} &= \int_V \left[-\frac{\partial}{\partial t}u_x - i\frac{\partial}{\partial t}u_t\right] d^3x + \int_V \left[-\nabla_x \cdot \boldsymbol{S}_x - i\ \nabla_t \cdot \boldsymbol{S}_t\right] d^3x \\ &= \int_V \left[-\frac{\partial}{\partial t}(u_x + i\ u_t) - \nabla \cdot (\boldsymbol{S}_x + i\boldsymbol{S}_t)\right] d^3x \\ &= \int_V \left[-\frac{\partial u}{\partial t} - \nabla \cdot \boldsymbol{S}\right] d^3x \end{aligned} \tag{6.9}$$

where we define $u = u_x + iu_t$ and $\boldsymbol{S} = \boldsymbol{S}_x + i\boldsymbol{S}_t$ as the overall energy density and generalized Poynting vector, respectively. This provides a concise statement of the rate at which the complex-valued fields $\boldsymbol{E}$ and $\boldsymbol{B}$ do mechanical work on a charge distribution as we would observe it in r-space. The key point is that the real portion of this does not contradict what we observe experimentally in r-space.

7. Discussion

The laws of classical electromagnetics incorporate a number of asymmetries [31], of which the most well-known is the existence of electric but not magnetic charge. Motivated by this, the central issue considered in this paper is determining, solely within the scope of classical electromagnetism, the implications of hypothesizing complex-valued electromagnetic fields that can accommodate a new type of magnetic monopoles.

While the work described in this paper has been developed in the context of classical electrodynamics, it is important to note that there has been significant past theoretical consideration of complex-valued electromagnetic fields in quantum electrodynamics. For example, past work exploring the development of a wave function for the photon—based in part on analogies between Maxwell's equations and Dirac's equation for the electron—introduced imaginary components into the electromagnetic fields [32, 33]. While this work is intriguing in that it bridges the gap between classical and quantum electrodynamics, it differs substantially from what is done in the current paper in being based on fields having the form $\boldsymbol{E} \pm ic\boldsymbol{B}$ and using six-component vectors of the form $\begin{pmatrix}\boldsymbol{E}\\ ic\boldsymbol{B}\end{pmatrix}$. This earlier work was also motivated by different considerations than those involved here, and did not explore its potential implications for hypothetical magnetic monopoles. Several subsequent studies have also proposed models involving complex fields [34–37]. These latter studies differ from the work done here in that when magnetic charge has been considered, it has taken the forms $\rho = \rho_e + i\rho_m$ and $\boldsymbol{J} = \boldsymbol{J}_e + i\boldsymbol{J}_m$, where ρ_m and $\boldsymbol{J}_m$ are distinct entities from ρ_e and $\boldsymbol{J}_e$ and related to previously proposed magnetic monopoles (e.g., Dirac's), and/or in that the differential operators used are much more complicated (for example, a curl operator having 48 terms).

In the current article , Maxwell's equations were generalized to accommodate complex fields $\boldsymbol{E}$ and $\boldsymbol{B}$ while remaining consistent with the original equations. Allowing these fields to have imaginary components produced increased symmetry in the Maxwell equations. As a consequence, it was possible to derive and verify a duality transform for the generalized equations in a way that is not possible with the original Maxwell equations. It was found that the complex fields $\boldsymbol{E}$ and $\boldsymbol{B}$ associated with the generalized equations can usefully be represented as

$\boldsymbol{E} = \boldsymbol{E_x} + i\boldsymbol{E}_t$ and $\boldsymbol{B} = \boldsymbol{B_x} + i\boldsymbol{B}_t$, where $\boldsymbol{E}_x$ and $\boldsymbol{B}_x$ are the conventional electric and magnetic fields that exist in r-space (real-valued observable 3D physical space), and where $\boldsymbol{E}_t$ and $\boldsymbol{B}_t$ are novel aspects of electromagnetic fields that exist in t-space (a purely imaginary-valued 3D space that is unobservable). Just as $\boldsymbol{E}_x$ and $\boldsymbol{B}_x$ are governed by the original Maxwell equations, the analysis done here showed that $\boldsymbol{E}_t$ and $\boldsymbol{B}_t$ are governed by a complementary set of equations that characterize their behaviors in t-space, and where the roles of electric and magnetic fields are reversed. A second cross-domain duality transform for converting bidirectionally between these complementary r-space and t-space equations, and having no analog in existing classical electrodynamics, was also derived and verified.

The complex-valued electrodynamics equations proved to be consistent with the original Maxwell equations and so they do not contradict past experimental observations. The complex fields they describe continue to exhibit well known features of classical electrodynamics, such as the existence of a continuity equation that indicates the conservation of charge. They predict the existence of electromagnetic waves that propagate through not only r-space (what we currently observe) but also through t-space (imaginary valued field components that are not observable). In spite of the imaginary-valued extensions to the electromagnetic fields, the complex-valued Maxwell equations continue to be consistent with our current concepts of energy conservation limited to r-space, although they suggest that there are unobservable imaginary-valued aspects of energy that have not been considered in the past.

Perhaps the most intriguing prediction of the extended, complex-valued Maxwell's equations is that magnetic monopoles exist. The monopoles predicted here are qualitatively different than those usually discussed in contemporary physics. They are simpler than those predicted in past quantum theoretical work (Dirac monopoles, 't Hooft-Polyakov monopoles, etc), having been derived within the scope of an extended classical, non-quantum physics. To the author's knowledge, the explanation given here for how magnetic monopoles can exist in the face of the existing contrary experimental evidence is the simplest theoretical explanation for this that has been offered to date (Occam's razor). The prediction is that every particle carrying an electric charge, such as a proton or an electron, is also carrying magnetic charge, and is therefore also a magnetic monopole. The reason that these magnetic monopoles have not been detected in past experimental searches for magnetic charge is that their magnetic fields $\boldsymbol{B}_t$ in t-space do not extend into observable r-space. Thus, past experimental searches, which have generally assumed that magnetic fields are limited to r-space, would never recognize that these particles are magnetic monopoles. According to the theory presented here, we therefore already know a great deal about magnetic monopoles: they are common, they have relatively low mass, they come in two types (north and south poles), they are long-lived stable particles, they are not dyons, and they are consistent with energy conservation as it is measured in r-space.

The work presented here raises many questions meriting further study, such as a deeper analysis of energy considerations and issues concerning the meaning of the imaginary-valued components of electromagnetic fields. However, its most critical limitation is that it is unclear at present how one can refute or confirm the existence of the imaginary components of electromagnetic fields. In other words, while the model presented here is supported by symmetry arguments, its ability to explain why potentially existing magnetic monopoles have not been detected experimentally, and so forth, the model can be viewed as speculative in that so far there is no experimental proof that it is correct. To the author's knowledge, past work raising the possibility that electromagnetic fields might have imaginary-valued components is not widely known (e.g., not discussed in many textbooks), and thus theoretical or experimental analysis of the challenging question of how to detect such components has not been pursued. It is thus hoped that the basic theoretical results presented here will encourage future work in this direction. Surely such work is merited as it could ultimately have an enormous impact on our fundamental understanding of electrodynamics.

Data availability statement

No new data were created or analysed in this study.

ORCID iDs

James A Reggia https://orcid.org/0009-0003-8517-8405

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