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Collapsing spheres satisfying an “Euclidean condition”

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Abstract We study the general properties of fluid spheres satisfying the heuristic assumption that their areal and proper radius are equal (the Euclidean condition). Dissipative and non-dissipative models are considered. In the latter case, all models are necessarily geodesic and a subclass of the Lemaître–Tolman–Bondi solution is obtained. In the dissipative case solutions are non-geodesic and are characterized by the fact that all non-gravitational forces acting on any fluid element produces a radial three-acceleration independent on its inertial mass.

Keywords Relativistic stars, Gravitational collapse

1 Introduction

Analytical or numerical solutions to Einstein equations describing dissipative gravitational collapse are thought to be useful not only for describing specific astrophysical phenomena, but also as test-bed for probing cosmic censorship and hoop conjecture among other important issues.

In this work we consider a large family of solutions, derived from the heuristic assumption that the areal radius of any shell of fluid, which is the radius obtained from its area, equals the proper radial distance from the centre to the shell. Since these two quantities are always equal in the Euclidean geometry, systems described by solutions satisfying such a condition will be called “Euclidean stars”.

Some of the models are necessarily dissipative. This is appealing from a physical point of view, since gravitational collapse is a highly dissipative process (see

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[1; 2; 3] and references therein). This dissipation is required to account for the very large (negative) binding energy of the resulting compact object of the order of -10^{53} erg.

The resulting dissipative models have a distinct dynamical property, namely all non-gravitational forces acting on any fluid element, produce a radial three-acceleration being independent of the inertial mass density of the fluid element. This behaviour, which is characteristic of the gravitational force, is now shared, due to the Euclidean condition, by all forces. The specific case of the shear-free and conformally flat fluid, are considered in detail.

Non-dissipative models are necessarily geodesic, belonging to the Lemaître–Tolman–Bondi (LTB) solutions (more specifically to the parabolic subclass). They may describe collapsing dust or, more generally, anisotropic fluids [4].

2 The Euclidean condition and its consequences

We consider a spherically symmetric distribution of collapsing fluid, bounded by a spherical surface Σ . The fluid is assumed to be locally anisotropic with principal stresses unequal and undergoing dissipation in the form of heat flow. Choosing comoving coordinates inside Σ , the general interior metric can be written

$$ds_-^2 = -A^2 dt^2 + B^2 dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where A , B and R are functions of t and r and are assumed positive. We number the coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$.

The matter energy-momentum $T_{\alpha\beta}^-$ inside Σ has the form

$$T_{\alpha\beta}^- = (\mu + P_{\perp})V_{\alpha}V_{\beta} + P_{\perp}g_{\alpha\beta} + (P_r - P_{\perp})\chi_{\alpha}\chi_{\beta} + q_{\alpha}V_{\beta} + V_{\alpha}q_{\beta}, \quad (2)$$

where μ is the energy density, P_r the radial pressure, P_{\perp} the tangential pressure, q^{α} the heat flux, V^{α} the four-velocity of the fluid and χ^{α} a unit four-vector along the radial direction. These quantities satisfy

$$V^{\alpha}V_{\alpha} = -1, \quad V^{\alpha}q_{\alpha} = 0, \quad \chi^{\alpha}\chi_{\alpha} = 1, \quad \chi^{\alpha}V_{\alpha} = 0. \quad (3)$$

The four-acceleration a_{α} and the expansion Θ of the fluid are given by

$$a_{\alpha} = V_{\alpha;\beta}V^{\beta}, \quad \Theta = V^{\alpha}_{;\alpha}, \quad (4)$$

and its shear $\sigma_{\alpha\beta}$ by

$$\sigma_{\alpha\beta} = V_{(\alpha;\beta)} + a_{(\alpha}V_{\beta)} - \frac{1}{3}\Theta(g_{\alpha\beta} + V_{\alpha}V_{\beta}). \quad (5)$$

Since we assumed the metric (1) comoving then

$$V^{\alpha} = A^{-1}\delta_0^{\alpha}, \quad q^{\alpha} = qB^{-1}\delta_1^{\alpha}, \quad \chi^{\alpha} = B^{-1}\delta_1^{\alpha}, \quad (6)$$

where q is a function of t and r . From (4) with (6) we have for the four-acceleration and its scalar a ,

$$a_1 = \frac{A'}{A}, \quad a^2 = a^{\alpha}a_{\alpha} = \left(\frac{A'}{AB}\right)^2, \quad (7)$$

and for the expansion

$$\Theta = \frac{1}{A} \left(\frac{\dot{B}}{B} + 2 \frac{\dot{R}}{R} \right), \quad (8)$$

where the prime stands for r differentiation and the dot stands for differentiation with respect to t . With (6) we obtain for the shear (5) its non zero components

$$\sigma_{11} = \frac{2}{3} B^2 \sigma, \quad \sigma_{22} = \frac{\sigma_{33}}{\sin^2 \theta} = -\frac{1}{3} R^2 \sigma, \quad (9)$$

and its scalar

$$\sigma^{\alpha\beta} \sigma_{\alpha\beta} = \frac{2}{3} \sigma^2, \quad (10)$$

where

$$\sigma = \frac{1}{A} \left(\frac{\dot{B}}{B} - \frac{\dot{R}}{R} \right). \quad (11)$$

The mass function $m(t, r)$ introduced by Misner and Sharp [5] (see also [6]) reads

$$m = \frac{R^3}{2} R_{23}{}^{23} = \frac{R}{2} \left[\left(\frac{\dot{R}}{A} \right)^2 - \left(\frac{R'}{B} \right)^2 + 1 \right], \quad (12)$$

We can define the velocity U of the collapsing fluid as the variation of the areal radius with respect to proper time, i.e.

$$U = D_T R < 0 \quad (\text{in the case of collapse}), \quad (13)$$

where $D_T = (1/A)(\partial/\partial t)$ defines the derivative with respect to proper time. Then (12) can be rewritten as

$$E \equiv \frac{R'}{B} = \left(1 + U^2 - \frac{2m}{R}\right)^{1/2}. \quad (14)$$

The proper radial three-acceleration $D_T U$ of an infalling particle inside Σ can be calculated to obtain

$$D_T U = -\left(\frac{m}{R^2} + 4\pi P_r R\right) + E a, \quad (15)$$

feeding back this expression into the radial component of the Bianchi identities produces (see [7] for details)

$$\begin{aligned} (\mu + P_r) D_T U = & -(\mu + P_r) \left(\frac{m}{R^2} + 4\pi P_r R\right) - E^2 \left[D_R P_r + 2(P_r - P_\perp) \frac{1}{R} \right] \\ & - E \left[D_T q + 2q \left(2 \frac{U}{R} + \sigma\right) \right]. \end{aligned} \quad (16)$$

The physical meaning of different terms in (16) is discussed in detail in [1; 7; 8]. We would like just to recall that the first term on the right hand side describes the gravitational force term. As expected from the equivalence principle, its contribution to $D_T U$ is independent on the inertial mass density $\mu + P_r$. The two last terms describe non-gravitational force terms (i.e. their combination vanishes in a geodesic motion).

Two radii are determined for a collapsing spherical fluid distribution by the metric (1). The first is determined by $R(t, r)$ representing the radius as measured by its spherical surface, hence called its *areal radius*. The second is obtained out its radial integration $\int B(t, r) dr$, hence called *proper radius*. These two radii in general, in Einstein's theory, need not to be equal, unlike in Newton's theory. Here we assume those two radii to be equal. Hence with this condition we can write,

$$B = R', \quad (17)$$

implying from (14)

$$E = 1. \quad (18)$$

The field equations with this condition become

$$\kappa \mu = \frac{1}{A^2} \left(\frac{\dot{R}}{R} + 2 \frac{\dot{R}'}{R'} \right) \frac{\dot{R}}{R}, \quad (19)$$

$$\kappa q A R' = -2 \frac{\dot{R}}{R} \frac{A'}{A}, \quad (20)$$

$$\kappa P_r = -\frac{1}{A^2} \left[2 \frac{\ddot{R}}{R} - \left(2 \frac{\dot{A}}{A} - \frac{\dot{R}}{R} \right) \frac{\dot{R}}{R} \right] + 2 \frac{A'}{A} \frac{1}{R R'}, \quad (21)$$

$$\kappa P_\perp = -\frac{1}{A^2} \left[\frac{\ddot{R}}{R} + \frac{\ddot{R}'}{R'} - \frac{\dot{A}}{A} \frac{\dot{R}}{R} - \left(\frac{\dot{A}}{A} - \frac{\dot{R}}{R} \right) \frac{\dot{R}'}{R'} \right]$$

$$+ \left[\frac{A''}{A} - \left(\frac{R''}{R'} - \frac{R'}{R} \right) \frac{A'}{A} \right] \frac{1}{R'^2}; \quad (22)$$

while the mass function (12) now reads,

$$m = \frac{R}{2} \left(\frac{\dot{R}}{A} \right)^2. \quad (23)$$

It is clear from (23) that if $\dot{R} = 0$ then $m = 0$ and spacetime becomes Minkowskian. Therefore *all Euclidean stars are necessarily non-static*. Furthermore, using (13), (23) can be rewritten as

$$\frac{m}{R} = \frac{U^2}{2}. \quad (24)$$

Hence, (24) can be interpreted as the Newtonian kinetic energy (per unit mass) of the collapsing particles being equal to their Newtonian potential energy.

From (20), we observe that if the system is dissipating in the form of heat flow, the collapsing source needs $A' \neq 0$, implying because of (7) $a^\alpha \neq 0$. This means that dissipation does not allow collapsing particles to follow geodesics. Inversely, of course, *non-dissipative Euclidean models are necessarily geodesic*, since $q = 0$ implies because of (7) and (20) that $a^\alpha = 0$.

It is interesting to observe that due to the Euclidean condition, the dynamical equation (15) or (16) becomes,

$$D_T U = - \left(\frac{m}{R^2} + 4\pi P_r R \right) - \frac{\kappa q R}{2U}, \quad (25)$$

implying that the non-gravitational force term (the last on the right hand side) contributes to $D_T U$, for any fluid element, independently on its inertial mass density. In other words, the Euclidean condition produces a “gravitational-like” behaviour in non-gravitational forces (which are controlled by q). Thus, the effect of non-gravitational forces amounts to modify the gravitational force term, leaving a “gravitational-like” force term producing a radial three-acceleration independent on the inertial mass density of the fluid element.

The Weyl tensor $C_{\alpha\beta\gamma\delta}$ for metric (1) with (17) has the following non zero components,

$$\begin{aligned} C_{0101} = \frac{A^2}{3} \left\{ \left[\frac{\ddot{R}}{R} - \frac{\ddot{R}'}{R'} + \left(\frac{\dot{A}}{A} + \frac{\dot{R}}{R} \right) \left(\frac{\dot{R}'}{R'} - \frac{\dot{R}}{R} \right) \right] \left(\frac{R'}{A} \right)^2 \right. \\ \left. + \frac{A''}{A} - \left(\frac{R''}{R'} + \frac{R'}{R} \right) \frac{A'}{A} \right\}, \end{aligned} \quad (26)$$

and all the other non zero components are proportional to (26),

$$\begin{aligned} \frac{R^2}{2} C_{0101} &= -B^2 C_{0202} = - \left(\frac{B}{\sin \theta} \right)^2 C_{0303} \\ &= A^2 C_{1212} = \left(\frac{A}{\sin \theta} \right)^2 C_{1313} = -\frac{1}{2} \left(\frac{AB}{R \sin \theta} \right)^2 C_{2323}. \end{aligned} \quad (27)$$

With (11), (21) and (22) we can rewrite (26) like

$$C_{0101} = \frac{AR'^2}{3} \left[\kappa(P_\perp - P_r)A + 2\frac{\dot{R}}{r}\sigma \right], \quad (28)$$

showing that for isotropic systems the shear-free conditions implies a conformally flat source.

We consider next the non-dissipative case.

3 Collapse with $q = 0$

As mentioned before, for this case we have from (20) that $A' = 0$ which means $A = A(t)$ and by rescaling t we can have

$$A = 1. \quad (29)$$

Of course such models are members of the Lemaître–Tolman–Bondi (LTB) spacetimes [9; 10; 11], furthermore they correspond to the parabolic case. Indeed, the general metric for LTB spacetimes read,

$$ds^2 = -dt^2 + \frac{R'^2}{1-K(r)}dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (30)$$

where $K(r)$ is an arbitrary function of r .

Imposing the Euclidean condition (17) in (30), one obtains $K = 0$, which defines parabolic LTB spacetimes. Further, assuming that the source consists of pure dust ($P_r = P_\perp = 0$) then it follows from the field equations that

$$R(t, r) = [c_1(r)t + c_2(r)]^{2/3}, \quad (31)$$

and

$$\kappa\mu = \frac{4c_1c'_1}{3(c_1t + c_2)(c'_1t + c'_2)}, \quad (32)$$

where $c_1(r)$ and $c_2(r)$ are integration functions. Hence the solution reduces to parabolic LTB collapsing dust [9; 10; 11].

From (29) and (31) we have for (26),

$$C_{0101} = \left(\frac{2}{3}\right)^4 \frac{c_1(c'_1t + c'_2)^2}{(c_1t + c_2)^{5/3}} \sigma, \quad (33)$$

where σ , from (11), is

$$\sigma = \frac{c_2c'_1 - c_1c'_2}{(c_1t + c_2)(c'_1t + c'_2)}. \quad (34)$$

In the shear-free case, $c_1 = c_2$, the system becomes conformally flat too, and with the freedom for choosing the r coordinate we can assume $c_1 = r^{3/2}$ recovering the Friedmann critical dust sphere.

Of course more general models can be obtained by relaxing the condition of vanishing pressure, we recall that LTB spacetime is compatible with an anisotropic fluid [12; 13].

4 Collapse with $q \neq 0$

We consider now the dissipative case. For simplicity we assume the fluid to be shear-free. In this latter case the line element can be written as [14]

$$ds_-^2 = -A^2 dt^2 + B^2 [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)], \quad (35)$$

then the Euclidean condition becomes

$$B = (Br)' \rightarrow B = f(t) \quad (36)$$

implying

$$R = f(t)r, \quad (37)$$

where f is an arbitrary function of t .

The field equations (19–22) now read,

$$\kappa\mu = \frac{3}{A^2} \left(\frac{\dot{f}}{f} \right)^2, \quad (38)$$

$$\kappa q = -2 \frac{\dot{f}}{f^2} \frac{A'}{A^2}, \quad (39)$$

$$\kappa P_r = -\frac{1}{A^2} \left[2 \frac{\ddot{f}}{f} - \left(2 \frac{\dot{A}}{A} - \frac{\dot{f}}{f} \right) \frac{\dot{f}}{f} \right] + \frac{2}{f^2 r} \frac{A'}{A}, \quad (40)$$

$$\kappa P_\perp = -\frac{1}{A^2} \left[2 \frac{\ddot{f}}{f} - \left(2 \frac{\dot{A}}{A} - \frac{\dot{f}}{f} \right) \frac{\dot{f}}{f} \right] + \frac{1}{f^2} \left(\frac{A''}{A} + \frac{1}{r} \frac{A'}{A} \right). \quad (41)$$

From (40) and (41) we have

$$\kappa(P_\perp - P_r) = \frac{1}{f^2 A} \left(A'' - \frac{A'}{r} \right). \quad (42)$$

From (28) it follows that (in the shear-free case) if the collapsing source is conformally flat, it must be isotropic in its pressures, and vice-versa.

The general form of all conformally flat and shear-free metrics is known [15], it reads

$$A = [e_1(t)r^2 + 1]B, \quad (43)$$

where e_1 is an arbitrary function of t , and

$$B = \frac{1}{e_2(t)r^2 + e_3(t)}, \quad (44)$$

where e_2 and e_3 are arbitrary functions of t .

The Euclidean condition then implies

$$e_2 = 0, \quad e_3 = \frac{1}{f}. \quad (45)$$

An approximate solution of this kind has been presented and discussed in [15]. Furthermore, an exact solution is also known [16], which in turn is a particular case of a family of solutions found in [17]. It reads (see Case III in [16])

$$f(t) = (\beta_1 + \beta_2)^2 e^{-2\alpha r_\Sigma t} \quad (46)$$

and

$$A = (\alpha r^2 + 1)f, \quad (47)$$

where α , β_1 and β_2 are constants. The above solution satisfies junction conditions and its physical properties have been discussed in [16]. Thus we shall not elaborate any further on it. Suffice to say at this point that its physical properties are reasonable and a thermodynamic analysis brings out the relevance of relaxational effects on the evolution of the system.

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