Tunnelling in a time dependent quartic potential: Possible implications for cosmology

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Abstract. The theory of a real scalar field with an arbitrary potential plays an important role in cosmology, particularly in the context of inflationary scenarios. However, in most applications, the potential is treated as independent of time, whereas in an evolving universe, for example, before the onset of inflation, the potential is actually likely to be changing with time. As pointed out by Berry in the context of single-particle quantum mechanics, the existence of multiple time scales can lead to results that are qualitatively different from those obtained with a static potential. The present paper reports on numerical investigations in a scalar field theory with a double-well potential that depends explicitly on time. The transition rate per unit volume for the decay of the false vacuum is found to depend strongly on time. Possible implications for old inflation are discussed.

1. Introduction

The quantum tunnelling in a time-dependent asymmetric double-well potential appears in a variety of physical systems. A typical asymmetric double-well potential will have local (energetically dis-favoured) minima in addition to the global minimum which defines the true vacuum. The purpose of this paper is to investigate how the tunnelling rate behaves depending on whether the relative depths of two minima of the potential change *slowly* or *rapidly*. Berry had studied quantum mechanics of a single particle in a time-varying potential. He had used the quantum mechanical wave function in Schrödinger's time dependent equation and concluded that results are significantly different from the static potential [1]. Berry's work was the main motivation for our work, and we have aimed to generalize his results for the field theoretic context. We have used the instanton method, which is by now a well-known tool for dealing with quantum tunnelling phenomena [2]. In a study of the decay of metastable states by quantum tunnelling, Coleman had introduced the concept of the bounce, and had shown that the bounce dominates the quantum tunnelling [3].

In the original version of cosmological inflation [4], the phase transition which ends the inflationary epoch is of first order, and proceeds through the quantum tunnelling of scalar fields from a false vacuum state to a true vacuum state. However, this model of inflation is plagued by the graceful exit problem [5]. We have suggested a possibility for solving the problem, using the time-dependent tunnelling rate.

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2. Basic formulation

In this section, we have reviewed the formalism used for calculating tunnelling probabilities. This discussion will be applicable to the single scalar field ϕ with static as well as the time dependent asymmetric double-well potentials, although with slight modifications in the latter case.

Let us consider a scalar field theory with a Lagrangian density, given by:

$$\mathcal{L}(x,t) = \frac{1}{2}\partial^{\mu}\partial_{\mu}\phi - V(\phi), \tag{1}$$

where the potential $V(\phi)$ has two minima at ϕ_f (false vacuum) and ϕ_t (true vacuum). A typical asymmetric double-well potential may look like:

$$V(\phi) = \frac{1}{2}A\phi^2 - \frac{1}{3}B\phi^3 + \frac{1}{4}C\phi^4, \tag{2}$$

where the coefficients A, B and C are constants and satisfy the inequality, $B^2 > 9/2AC$ [6].

Figure 1. The asymmetrical double-well potential for two different values of A. Note that as time evolves the depth of the true vacuum increases whereas the false vacuum remains fixed at the origin.

If the system is initially in the false vacuum phase, barrier tunnelling leads to the appearance of bubbles of a new phase with $\phi = \phi_t$. In the zero temperature limit, the rate, i.e., decay probability per unit time per unit volume, for this process is (upto leading order in $\hbar$):

$$\Gamma = K\ e^{-S_E[\phi]}, \tag{3}$$

where $S_E[\phi]$ is the action corresponding to the O(4)-symmetric solution of Euclidean equation of motion:

$$\frac{d^2\phi}{d\rho^2} + \frac{3}{\rho}\frac{d\phi}{d\rho} = \frac{dV(\phi)}{d\phi}, \tag{4}$$

where ρ is the four-dimensional radial coordinate, with boundary conditions:

$$\lim_{\rho\to\infty}\phi = \phi_f = constant, \quad and \quad \lim_{\rho\to 0}\frac{d\phi}{d\rho} = 0. \tag{5}$$

K is the determinantal factor of the order η^4, where η is the energy scale of phase transition. Our focus has been on the calculation of the action $S_E[\phi]$, which in the O(4)-symmetric case is given by:

$$S_E = 2\pi^2\int_0^{\infty} d\rho\rho^3\left[\frac{1}{2}\Big(\frac{d\phi}{d\rho}\Big)^2 + V(\phi)\right]. \tag{6}$$

2.1. Potentials with explicit time dependence

To get explicit time dependence, it is enough to make any one coefficient of the potential given in Eq. (2) a function of time. Although, there could be various other ways, we make the coefficient A, time dependent in the following way:

$$A(t) = a_2 + \frac{(a_1 - a_2)}{\cosh(t/\lambda)}, \tag{7}$$

where λ can be considered to be the time scale for the potential change. In this model, $A(t)$ starts from a_1 and goes to a_2 as $t \to \infty$. To solve the time-dependent case, we have taken $a_1 = 1.9$ (nearly degenerate) and $a_2 = 0.1$ (highly asymmetric) (see Figure 1). Further, we have fixed $B = 3.0$ and $C = 1.0$ for all calculations.

3. Numerical simulations

The computation of the Euclidean action for explicit time dependent potential requires us to modify Eq. (6). We have used the following for $S_E[\phi]$ [7] as:

$$S_E = \int d\tau S_3, \quad where \quad S_3 = 4\pi \int_0^\infty dr r^2 \left[\frac{1}{2}\left(\frac{d\phi}{dr}\right)^2 + V(\phi, \tau)\right]. \tag{8}$$

The procedure to calculate S_E in the quasi-static approximation is as follows: For a fixed λ (say 8.0), we have divided the allowed time period into N (say 100) steps. At the midpoint of each step, we have calculated the value of A using Eq. (7), and subsequently solve the static equation of motion using O(3)-symmetry (see Figure 2). Then, we have calculated the action S_3 corresponding to this instantaneous static solution. We have repeated this for the midpoints of subsequent steps. Finally, we have summed (integrated) up the contributions, thus, obtained to get the so called quasi-static action S_E for a single λ. Although, this procedure is not strictly justifiable, it is a good approximation for large enough λ (in practice $\lambda > 5$). Table 1 shows actions S_E for different values of λ.

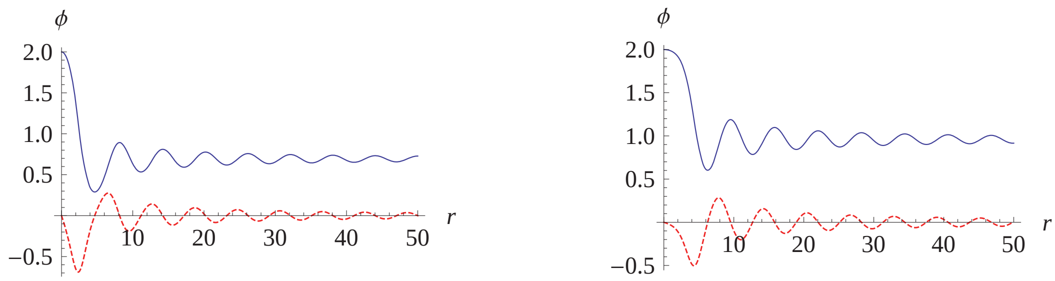

Figure 2. Numerically obtained bounce solutions as functions of r for two different time scales. The solid line represents ϕ, while dashed one represents $d\phi/dr$.

3.1. Inferences from the numerical study

The observation, that as we vary the parameter (here time scale), the tunnelling rate (negative exponential of the action) changes by orders of magnitude, leads to the following conclusion: If the potential is explicitly time dependent, then in particular, the tunnelling probability is quite different from the case of a static potential; and in general, physics can be quite different [1, 8]. The key result of this paper can be stated qualitatively as: *A slowly varying potential renders*

Table 1. Action as a function of time scale.

Time Scale (λ)	Action (S_E)
6.0	1504.87
7.0	1561.08
8.0	2009.40
10.0	2513.93
12.0	3018.46
14.0	3523.00

the tunnelling rate small, a rapidly varying potential increases it. Here, 'rapidly varying' is a relative term, since all values of λ are chosen to lie within the domain of validity of the quasi-static approximation. Apart from cosmology, this result may be relevant to any study dealing with tunnelling in time-dependent potentials, for example, Bose-Einstein condensates [9], and condensed matter physics [10, 11].

4. Implications for the old inflationary models

The quantity of interest for cosmology is $\epsilon = \Gamma/H^4$, where ϵ is the number of bubbles nucleated per Hubble volume per Hubble time. In theory, inflation ends when true vacuum bubbles nucleate, expand and collide. However, in Guth's model, percolation never occurs as bubbles inflate away faster than they form and grow. As bubbles do not merge, no reheating takes place. In terms of ϵ, the horizon and flatness problems are solved if $\epsilon < 0.001$, whereas percolation occurs only if $\epsilon \approx 0.1$ [12].

In Guth's model, both Γ and H are constant, and the requirements of sufficient inflation and percolation are incompatible (graceful exit problem). Obviously, if ϵ is allowed to vary with time, the graceful exit problem can be solved. As we have observed in the numerical study, the tunnelling rate Γ is strongly dependent on the time scales of the time dependent potentials. Hence, if somehow a time dependence can be introduced that causes the potential to change slowly in the beginning and rapidly later, ϵ will be small initially, allowing sufficient inflation to occur, and will become large later on for the percolation to occur; this might solve incompatibility problem.

Although, we have used a toy model for the time dependence, in a cosmological setting the effective potential automatically acquires time dependence. Perhaps there is a case for revisiting old inflationary models with explicitly time dependent potentials. Of course, one needs to carry out a self-consistent calculation. Work on this is in progress.

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