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Relativistic Scalar Particle Systems in a Spacetime with a Spiral-like Dislocation

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Abstract: We have analyzed solutions of bound states of a scalar particle in spacetime with torsion. In the first analysis, we investigate the confinement of a scalar particle in a cylindrical shell. In the second step, we investigate the Klein–Gordon oscillator. Then, we finish our analysis by searching for solutions of bound states of the Klein–Gordon oscillator by interacting with a hard-wall potential. In all these systems, we determine the relativistic energy profile in the background characterized by the presence of torsion in spacetime represented by a spiral-like dislocation.

Keywords: torsion; spiral-like dislocation; Klein–Gordon oscillator; solution of bound states

MSC: 81-10; 81Q05; 81T20; 81T45

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1. Introduction

In analogy to phase transitions in condensed matter systems, it is believed that the decoupling of the fundamental interactions in the early Universe gave rise to cosmological objects known as topological defects (TDs) [1]. The best known TDs in the literature are the domain wall [2], global monopole [3,4] and the cosmic string [5–7]. In particular, the cosmic string is an example of a linear TD associated with the curvature of spacetime [8]. In crystallography, this type of defect is known as disclination [9]. In addition to the association between disclination and curvature, there are also linear defects associated with the torsion in a continuous solid [9]. These types of defects are known as dislocations, which, in cylindrical symmetry, can be typified as screw-like dislocation and spiral-like dislocation [10]. Recently, these two dislocations have been studied in quantum mechanics systems. For example, the screw-like dislocation has been investigated on the Landau quantization [11,12], on the harmonic oscillator [13], on a harmonic oscillator subjected to a linear potential [14], on the doubly anharmonic oscillator [15] and in non-inertial effects on a non-relativistic Dirac particle [16]. The spiral-like dislocation has been studied on an electron subjected to an electric field and an uniform magnetic field [17] and on the harmonic oscillator [18]. Other topological structures have been investigated in the gravitational context. For example, studies have evaluated the Hawking effect for a massive Dirac spinor under the effects of $\mathbb{RP}^3$ geon [19], quantum-gravitational effects produced in Minkowski spacetime with a periodic boundary condition [20] and the structure of the density matrix for two Unruh–DeWitt detectors coupled with a massless scalar field in two locally flat topologically nontrivial spacetimes constructed from identifications of Minkowski spacetime.

The screw-like and spiral-like dislocations have been reformulated in $(1+3)$-dimensions in Einstein–Cartan geometry through the generalized concept of the Volterra process [21], also known as the "cut and paste" process, in distorted spacetimes [22]. Another process capable of producing these types of structures is through the periodic identification of one of the spatial coordinates, imposing a periodic boundary condition on the fields in the background [20]. In particular, in the relativistic quantum mechanic context, recently, the screw-like dislocation has been investigated in several quantum systems [23–31]. The spiral-like dislocation has been studied on a scalar field subjected to a hard-wall confining potential [32] and in rotating effects on a Dirac field [33]. The metric that describes a spacetime with a spiral-like dislocation is given by [32] ($c = \hbar = 1$)

$$ds^2 = -dt^2 + (d\rho + \beta d\varphi)^2 + \rho^2 d\varphi^2 + dz^2, \quad (1)$$

where $\rho = (x^2 + y^2)^{1/2}$ and $\beta > 0$ are the parameters associated with the TD of the spacetime.

In this manuscript, we analyzed the effects of torsion on the relativistic quantum dynamics of a scalar particle immersed in spacetime with a spiral-like dislocation described by the line element given in Equation (1). Our first step is to investigate a relativistic scalar particle confined into a cylindrical shell. In addition, we analyze the dynamics of the Klein–Gordon oscillator [34] in the spacetime described by Equation (1), and thus, extend this discussion to the confinement of a hard-wall confining potential by searching for analytical solutions to the Klein–Gordon equation.

The structure of this paper is as follows: in Section 2, we investigate the topological effects of a spiral-like dislocation on the relativistic energy spectrum of a scalar particle into a cylindrical shell; in Section 3, we study the interaction between a scalar particle and the Klein–Gordon oscillator in spacetime with a spiral-like dislocation; in Section 4, we analytically determine the relativistic energy profile of the Klein–Gordon oscillator subjected to a hard-wall potential in this background with torsion; in Section 5, we present our conclusions.

2. On a Scalar Particle Confined to an Elastic Cylindrical Shell

A scalar particle associated with a field ϕ in a curved spacetime is described by the Klein–Gordon equation in the form [23]

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)\phi - m^2\phi = 0, \quad (2)$$

where $g = \det(g_{\mu\nu}) = -\rho^2$, by using Equation (1), $g^{\mu\nu} = (g_{\mu\nu})^{-1}$ and m is the parameter associated with the remaining mass of the field. From Equations (1) and (2), we obtain

$$-\frac{\partial^2\phi}{\partial t^2} + \left(1 + \frac{\beta^2}{\rho^2}\right)\frac{\partial^2\phi}{\partial\rho^2} + \left(\frac{1}{\rho} - \frac{\beta^2}{\rho^3}\right)\frac{\partial\phi}{\partial\rho} - \frac{2\beta}{\rho^2}\frac{\partial^2\phi}{\partial\rho\partial\varphi} + \frac{\beta}{\rho^3}\frac{\partial\phi}{\partial\varphi} + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2} - m^2\phi = 0. \quad (3)$$

Equation (3) describes the relativistic quantum motion of a spin-0 particle in spacetime with a spiral-like dislocation. The general solution to Equation (3) is

$$\phi(\rho, \varphi, z, t) = u(\rho)e^{il\varphi}e^{ikz}e^{-i\mathcal{E}t}, \quad (4)$$

where $l = 0, \pm1, \pm2 \ldots$ are the quantum numbers associated with the angular momentum operator $\hat{L}_z = -i\partial_\varphi$, $-\infty < k < \infty$ are the quantum numbers associated with the linear momentum operator $\hat{p}_z = -i\partial_z$, where $[\hat{L}_z, \hat{H}] = [\hat{H}, \hat{L}_z]$ and $[\hat{p}_z, \hat{H}] = [\hat{H}, \hat{p}_z]$, with $\hat{H}$ as the Hamiltonian operator, and $u(\rho)$ is a radial function. Then, by substituting Equation (4) into Equation (3), we obtain the axial wave equation

$$\left(1+\frac{\beta^2}{\rho^2}\right)\frac{d^2u}{d\rho^2}+\left(\frac{1}{\rho}-\frac{\beta^2}{\rho^3}-\frac{2il\beta}{\rho^2}\right)\frac{du}{d\rho}+\frac{il\beta}{\rho^3}u-\frac{l^2}{\rho^2}u+\alpha^2u=0, \quad (5)$$

where

$$\alpha^2=\mathcal{E}^2-m^2-k^2. \quad (6)$$

Now, in order to analytically solve Equation (5), let us consider the axial wave function [32]

$$u(\rho)=R(\rho)e^{il\arctan\left(\frac{\rho}{\beta}\right)}, \quad (7)$$

that is, by substituting Equation (7) into (5), we obtain

$$\left(1+\frac{\beta^2}{\rho^2}\right)\frac{d^2R}{d\rho^2}+\left(\frac{1}{\rho}-\frac{\beta^2}{\rho^3}\right)\frac{dR}{d\rho}-\frac{l^2}{(\rho^2+\beta^2)}R+\alpha^2R=0. \quad (8)$$

Let us define $s=\alpha(\rho^2+\beta^2)^{1/2}$; then, Equation (8) becomes

$$\frac{d^2R}{ds^2}+\frac{1}{s}\frac{dR}{ds}-\frac{l^2}{s^2}R+R=0. \quad (9)$$

Equation (9) is the Bessel equation [35] and its general solution is

$$R(s)=C_1J_{|l|}(s)+C_2N_{|l|}(s), \quad (10)$$

where C_1 and C_2 are constants, $J_l(s)$ is the first type of Bessel function and $N_l(s)$ is the Neumann function [35]. In the interval $0\le\rho<\infty$, we impose that $C_2=0$, since $N_{|l|}(s)\to\infty$ when $s\to0$ ($\rho\to0$). In this case, we obtain a simpler solution, $R(s)=C_1J_{|l|}(s)$; that is, the solution for a free scalar particle in spacetime with a spiral-like dislocation, as already discussed in Ref. [32].

From now on, let us consider a scalar particle confined to a cylindrical shell; that is, the relativistic quantum particle is restricted to move in interval $s_a\le s\le s_b$, where $s_a=\alpha\sqrt{\rho_a^2+\beta^2}$ and $s_b=\alpha\sqrt{\rho_b^2+\beta^2}$, with $\rho_a=a$ and $\rho_b=b$ fixed and $b>a$. In addition, let us consider the boundaries of this region as impenetrable walls, such that the axial wave function satisfies

$$R(s_a)=R(s_b)=0. \quad (11)$$

This type of confinement has been studied in spacetime with curvature and torsion [23], in Safka–Witten spacetime [36], in curved spacetime [37], on a non-relativistic particle in an environment with a magnetic dislocation [38], and on a neutral particle interacting with a dipole moment [39]. Equation (11) yields the following relation:

$$J_{|l|}(s_a)N_{|l|}(s_b)-J_{|l|}(s_b)N_{|l|}(s_a)=0. \quad (12)$$

Let us consider the case where $s_a\gg1$, $s_b\gg1$ and l are fixed numbers. In this specific case, $J_l(s)$ and $N_l(s)$ functions are rewritten, respectively, in the form [40]

$$J_{|l|}(s_i)\sim\sqrt{\frac{2}{\pi s_i}}\left[\cos\left(s_i-\frac{l\pi}{2}-\frac{\pi}{4}\right)-\frac{(4l^2-1)}{8s_i}\sin\left(s_i-\frac{l\pi}{2}-\frac{\pi}{4}\right)\right], \quad (13)$$

and

$$N_{|l|}(s_i) \sim \sqrt{\frac{2}{\pi s_i}}\left[\sin\left(s_i - \frac{l\pi}{2} - \frac{\pi}{4}\right) + \frac{(4l^2-1)}{8s_i}\cos\left(s_i - \frac{l\pi}{2} - \frac{\pi}{4}\right)\right], \tag{14}$$

where $i = a, b$. By substituting Equations (13) and (14) into Equation (12), we obtain

$$\alpha^2 \sim \frac{\pi^2 n^2}{(\sqrt{b^2+\beta^2} - \sqrt{a^2+\beta^2})^2} + \frac{4l^2-1}{4\sqrt{(a^2+\beta^2)(b^2+\beta^2)}}, \tag{15}$$

with $n = 0, 1, 2, \ldots$. By substituting Equation (6) into Equation (15), we obtain

$$\mathcal{E}_{k,l,n} \approx \sqrt{m^2 + k^2 + \frac{\pi^2 n^2}{(\sqrt{b^2+\beta^2} - \sqrt{a^2+\beta^2})^2} + \frac{4l^2-1}{4\sqrt{(a^2+\beta^2)(b^2+\beta^2)}}}. \tag{16}$$

Equation (16) gives us the relativistic energy spectrum of a scalar particle into a cylindrical shell in an environment with a spiral-like dislocation. We can see that the relativistic energy profile of this system is influenced by the spacetime topology. This influence is the correction made on the fixed axial radius $\rho_{\text{eff}} = \sqrt{\rho_i^2 + \beta^2}$, with $i = a, b$. In contrast to ref. [23], there is no gravitational effect analogous to the Aharonov–Bohm effect for bound states on the allowed energy values of the relativistic quantum system [26–28,30]. By making $\beta = 0$ into Equation (16) we recover the result discussed in refs. [23,36]; that is, the relativistic energy spectrum of a scalar particle into a cylindrical shell in Minkowski spacetime.

Figure 1 provides the relativistic energy levels for the first five radial modes of the system, which shows us how the permitted energy values of the system vary for each radial mode, both for the positive part and for the negative part (antiparticle). Figures 2 and 3 show the variation in the relativistic energy levels of the system through the variation in the TD. Figure 2 gives us the ground state $(n = 0)$ of the system as a function of the torsion in spacetime, while Figure 3 gives us the first four excited states of the system as a function of the dislocation, both for $l = 1$. By observing Figures 2 and 3, we can note that the higher the value of n, the more abrupt the increase in energy (analyzing the positive side), while, in the fundamental state, the variation is minimal and occurs very tenuously when $m\beta \to 0$.

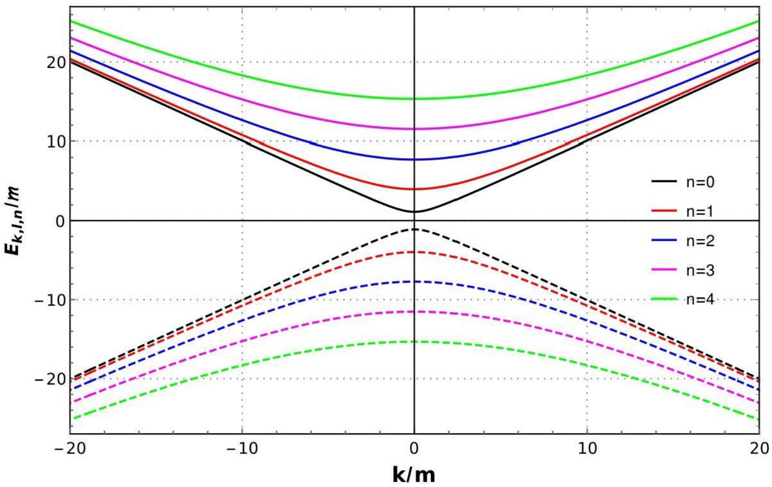

Figure 1. The positive and negative relativistic energy levels given in Equation (16) for $n = 0, 1, 2, 3$ and 4, with $a = 1$, $b = 2$, $l = 1$ and $\beta m = 1$.

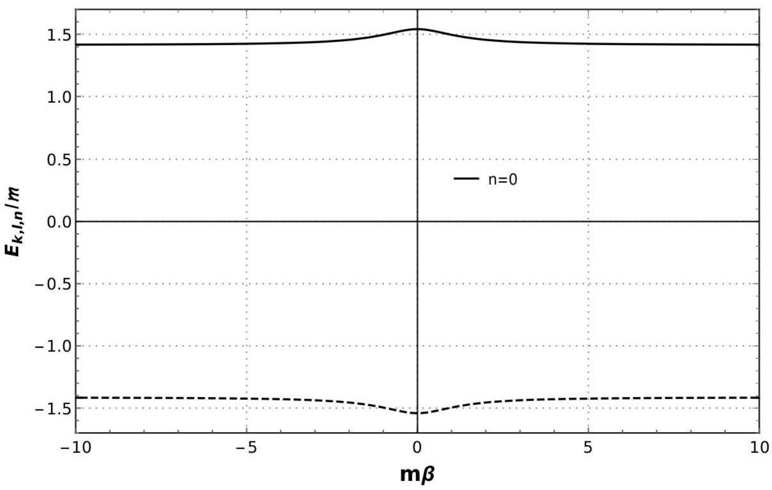

Figure 2. Relativistic energy level given in Equation (16) for the radial mode $n = 0$, with $a = 1$, $b = 2$, $l = 1$ and $k/m = 1$.

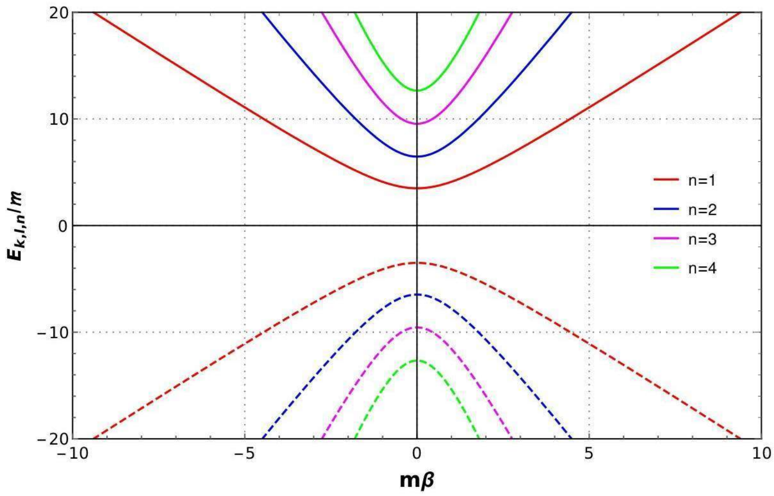

Figure 3. Positive and negative relativistic energy levels given in Equation (16) for $n = 1, 2, 3$ and 4, with $a = 1$, $b = 2$, $l = 1$ and $k/m = 1$.

By taking $b \to a$, we have a relativistic quantum particle restricted to move in a circle of radius of a. In Ref. [23], this confinement configuration has been investigated on a scalar field immersed in a spacetime with curvature and (time-like and space-like) torsion, while in Ref. [36], this confinement configuration is analyzed in a background characterized by a tubular matter source with axial interior magnetic field and a vanishing exterior magnetic field. However, for $b \to a$ into Equation (16), we have $\mathcal{E} \to \infty$, which is not a physically acceptable result. Next, let us introduce a potential of attractive nature in the interval $a < \rho < b$ to compete with the growth of the allowed energy values for the radial modes of the quantum system. In this case, Equation (16) is rewritten as follows

$$\mathcal{E}_{k,l} \approx \sqrt{m^2 + k^2 + \frac{l^2}{a^2 + \beta^2} - \frac{1}{4(a^2 + \beta^2)}}, \tag{17}$$

which represents the relativistic energy spectrum of a scalar particle confined into a quantum ring in the spacetime with spiral-like dislocation. We can note that the permitted values of relativistic energy of this system are influenced by the torsion present in the spacetime. This influence is the contribution that gives rise to an effective radius $\rho_{\text{eff}} = \sqrt{a^2 + \beta^2}$. By making $\beta = 0$ into Equation (17), we obtain a relativistic energy profile of a scalar particle confined into a quantum ring in Minkowski spacetime.

3. Klein–Gordon Oscillator Under Effect of a Spiral-like Dislocation

Inspired by the Dirac oscillator [41,42], Bruce and Minning [34] proposed a relativistic quantum oscillator model for scalar particles which became known as the Klein–Gordon oscillator (KGO) in the literature. This relativistic quantum oscillator model, in addition to providing an analytical solution at the non-relativistic limit, falls on the quantum harmonic oscillator described by the Schröndiger equation [43]. KGO has been investigated in a noncommutative space [44], on anti-de Sitter space [45], on a generalized uncertainty principle framework [46], on a topologically nontrivial spacetime [47], in position-dependent mass systems [48–50], in a curved spacetime [51], in the Som–Raychaudhuri spacetime [52], in possible Lorentz symmetry violation scenarios [53,54], and in the global monopole spacetime [55]. However, KGO has not yet been analyzed in a spacetime with a spiral-like dislocation. Here, we investigate the topological effects of a spiral-type dislocation on KGO. In this case, Equation (2) in the spacetime considered in this study can be rewritten as [29]

$$\frac{1}{\sqrt{-g}}(\partial_\mu + m\omega X_\mu)(\sqrt{-g}g^{\mu\nu})(\partial_\nu + m\omega X_\nu)\phi - m^2\phi = 0, \tag{18}$$

where ω is the angular frequency of KGO and $X_\mu = (0, \rho, 0, 0)$. Then, by following the steps from Equation (3) to Equation (5), we obtain the axial wave equation

$$\left(1 + \frac{\beta^2}{\rho^2}\right)\frac{d^2u}{d\rho^2} + \left(\frac{1}{\rho} - \frac{\beta^2}{\rho^3} - \frac{2il\beta}{\rho^2}\right)\frac{du}{d\rho} + \frac{il\beta}{\rho^3}u - \frac{l^2}{\rho^2}u - m^2\omega^2\rho^2 u + \gamma u = 0, \tag{19}$$

with

$$\gamma = \mathcal{E}^2 - m^2 - k^2 - m^2\omega^2\beta^2 - 2m\omega. \tag{20}$$

By substituting Equation (7) into Equation (19), we obtain the differential equation

$$\left(1 + \frac{\beta^2}{\rho^2}\right)\frac{d^2R}{d\rho^2} + \left(\frac{1}{\rho} - \frac{\beta^2}{\rho^3}\right)\frac{dR}{d\rho} - \frac{l^2}{(\rho^2 + \beta^2)}R - m^2\omega^2\rho^2 R + \gamma R = 0. \tag{21}$$

Let us define the new variable $r = m\omega(\rho^2 + \beta^2)$, such that we obtain

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{l^2}{4r^2}R + \frac{\delta}{r}R - \frac{1}{4}R = 0, \tag{22}$$

with

$$\delta = \frac{\gamma + m^2\omega^2\beta^2}{4m\omega} = \frac{\mathcal{E}^2 - m^2 - k^2 - 2m\omega}{4m\omega}. \tag{23}$$

We are interested in a well-behaved solution to Equation (22) in the limits $r \to 0$ and $r \to \infty$, and we have the following general solution:

$$R(r) = r^{\frac{|l|}{2}} e^{-\frac{r}{2}} f(r), \tag{24}$$

where $f(r)$ is a function to be determined. By substituting Equation (24) into Equation (22), we obtain

$$r\frac{d^2f}{dr^2} + (|l| + 1 - r)\frac{df}{dr} + \left(\gamma - \frac{|l|}{2} - \frac{1}{2}\right)f = 0, \tag{25}$$

which is the confluent hypergeometric differential equation [35] and $f(r)$ is the confluent hypergeometric series: $f(r) = {}_1F_1(A, B; r)$, with

$$A = \frac{1}{2} + \frac{|l|}{2} - \gamma; \quad B = |l| + 1. \tag{26}$$

The confluent hypergeometric series becomes a polynomial of degree n by imposing that $A = -n = 0, 1, 2, \ldots$, of which we obtain

$$\mathcal{E}_{k,l,n} = \pm\sqrt{m^2 + k^2 + 4m\omega\left(n + \frac{|l|}{2} + 1\right)}. \tag{27}$$

By observing Equation (27), we can see that the relativistic energy profile of KGO in the spiral-like dislocation spacetime is equal to the relativistic energy profile of KGO in the Minkowski spacetime [27,29,56]; that is, the relativistic energy levels of KGO are not influenced by the spacetime topology. This is due to Equation (23); note that the parameter δ is defined in terms of the parameter γ, where the latter is added by the term $m^2\omega^2\beta^2$, which cancels out with the term $-m^2\omega^2\beta^2$ that exists in the definition of the parameter γ. However, the eigenfunctions of KGO depend on the TD, since they are defined in terms of the confluent hypergeometric polynomials, which in turn depend on the parameter $A = A[\gamma(\beta^2)]$ given in Equation (26). In addition, there is no gravitational effect analogous to the Aharonov–Bohm effect for bound states on relativistic energy levels of the quantum system [26–28,30].

4. On the Klein–Gordon Oscillator Subjected to a Hard-Wall Potential in Spacetime with a Spiral-like Dislocation

In this section, we investigate the effects of the spacetime topology on KGO subjected to a hard-wall. This confinement type has been investigated in several quantum systems, for example, on a Dirac oscillator [57,58], in quantum systems under non-inertial effects [59,60], in a Landau-type quantization plus a Dirac field [61], in quantum system of geometric phase [62], in an environment with a pointlike defect [63] and on a massive scalar field under effects of the aether-like Lorentz symmetry violation [64]. This confining potential is important due its similarity to a box of certain dimensions, which is a very good approximation to consider when discussing the quantum properties of a gas molecule system and other particles, which are necessarily confined in a box.

Then, let us restrict a scalar particle in the following form:

$$u(\rho_0) = 0, \tag{28}$$

where $\rho_0 = \text{const.}$. The boundary condition given in Equation (28) indicates that the radial wave function vanishes at a fixed radius ρ_0; that is, the quantum particle is under the effects of the hard-wall potential. To obtain the relativistic energy levels of this quantum system, let us consider the specific case $\gamma \gg 1$ and with l fixed. This means that parameters A and B is larger and fixed, respectively. Under these mathematical constraints, the confluent hypergeometric series is rewritten as follows

$${}_1F_1(A, B; r_0) \propto \cos\left(\frac{\pi}{4} - \frac{B\pi}{2} + \sqrt{2Br_0 - 4Ar_0}\right). \tag{29}$$

By substituting Equations (7) and (24) into Equation (28), we have

$$\mathcal{E}_{k,l,n} \approx \pm\sqrt{m^2 + k^2 + \frac{\pi^2}{(\rho_0^2 + \beta^2)}\left(n + \frac{|l|}{2} + \frac{3}{4}\right)^2 + 2m\omega}. \tag{30}$$

Equation (30) gives us the relativistic energy spectrum of KGO plus a hard-wall potential in the spacetime with a spiral-like dislocation, influenced by the TD present in spacetime. We can note this influence by the contribution that gives rise to an effective radius $\rho_{\text{eff}} = \sqrt{\rho_0^2 + \beta^2}$. We can also that there is no gravitational effect analogous to the Aharonov–Bohm effect for bound states on the relativistic energy profile of the quantum system [26–28,30]. In addition, by taking $\omega \to 0$ and $\beta \neq 0$ into Equation (30) we recover the result obtained in Ref. [32]. By making $\omega \neq 0$ and $\beta = 0$ into Equation (30) we obtain the relativistic energy spectrum of KGO interacting with a hard-wall potential in the Minkowski spacetime discussed in Ref. [65].

5. Conclusions

We have investigated some possible scenarios of a scalar particle interacting with confining potentials in spacetime with a spiral-like dislocation. We started our analysis with a scalar particle restricted in a cylindrical shell where we obtained the relativistic energy levels, which are influenced by the TD present in the spacetime. This influence is described explicitly through an effective radius defined by the parameter associated with the torsion. Next, we analyzed KGO in this background and analytically defined its relativistic energy levels, which are not influenced by the spiral-like dislocation; that is, the energy profile of KGO remains the same, despite being immersed in a background with torsion. In addition, we obtain the relativistic energy levels of KGO plus a hard-wall confining potential, which, again, are influenced by the spiral-like dislocation through an effective radius defined by the parameter associated with the TD. Finally, unlike the effects of the cosmic string and screw dislocation on confined quantum systems, we have observed that in all cases analyzed, there is no gravitational effect analogous to the Aharonov–Bohm effect for bound states on relativistic energy levels of the quantum system [26–28,30].

It is worth mentioning that the above results open up paths for extensions or generalizations. For example, the scalar particle subjected to relativistic Landau quantization [66] has already been studied in several distorted backgrounds [27,29], for different types of confinements [27,31,67]; however, this system has not been studied under the effects of spiral-like dislocation. Furthermore, confined quantum systems have recently been the object of study in the thermodynamic context [68–79], in which the effects of curvature and torsion on thermodynamic quantities, such as internal energy, entropy, specific heat, etc., are analyzed. In this sense, as a future perspective, the results obtained here can be used for this type of investigation.

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