SPIN AS A KINEMATICAL EFFECT OF THE RELATIVITY

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Abstract

It is shown, that Pauli spin matrices are infinitesimal boost operators for irreducible representation of Lorentz group with first weight number equal 1/2, which do not contain space (P) inversion and time (T) reversal. Purely relativistic effect appears under orbital motion, which is the result of so-called Wigner rotation. The similar analysis was fulfilled for (P)-, (T)- and (PT)-conjugate irreducible representation of Lorentz group with respect to standard (i.e. proper, orthochronous, homogeneous) one. In every particular case explicit form of spin matrices are changed, but components of spin operators have the same commutative relations up to phase factor ±1 and they have the same physical sense.

Key-words: spin, groups, irreducible representations, discrete symmetry.

1. Introduction

The question on spin nature does not closed in spite of long history of spin conception [1] and it successful mathematical formalization [2] for the electron. The aim of this article is to draw attention to physical interpretation of mathematical formalism for one particular case namely spin 1/2. It is singled out, exclusive case due to successes, which are symbolized the scientific progress of the past century. For example, explanation of the Mendeleev periodical low, development of the nuclear and particle physics are connected with Pauli formalism.

At first [1] electron was called "spinning". Then in Pauli article [2] it is become "magnetic". Question on a pointlike particle rotation does not yield to reasonable understanding, therefore spin was refered as inner or proper characteristic of a particle . Evidently it does not become more clear. As it was shown by Pauli situation does not improve, if we shall imagine electron as a ball with classical radius $r_0 = e^2/m_e c^2$. In this case contradiction arises with relativity requirement. Instantaneous velocity at the radius end exceeds light velocity 70 times.

It should be noted, that practically at that time [3] natural understanding arises that circular motion of the electron leads to magnetic manifestations. But Thomas did not refuse superposition on "inner magnetic moment". Therefore he obtained the precession of the pointlike magnet instead of the electron spin. Comparison of his calculations with proposed here shows, that so called Thomas precession has the same nature as electron spin.

Another dogma of the theory promoting initially adopted representation is a belief that interacting electron is characterized by the same properties as free one. Usual argument looks as following. It is impossible to represent why electron has four degrees of freedom in an atom and three in free state [4]. Evidently it is not a prove that the electron has four degree of freedom in the free state . In opposition to this expression one can assume that all completeness of particle properties is become by interactions. Free Dirac equation describes both electron and positron. But it is possible to distinguish one from another by interaction with electromagnetic field.

As for our purpose main peculiarities of the 1/2 formalism are that $\sigma-$ matrices generate finite group, it has no superfluous components as in the case of high spin fields and it satisfies relativity requirements without any stipulations.

2. Pauli matrices and the spin nature

Let us start from the conventional definition, that the spin is the proper angular momentum, having the quantum nature and which does not connected with particle movement as the whole [5].It is usual to describe the proper angular momentum equal 1/2 by Pauli σ-matrices.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is easy to verify, that σ-matrices generate 16-order group. Let us denote this group as d_γ. The group has ten conjugate classes. Center of the group contains four elements. Group has eight one-dimensional and two inequivalent two-dimensional irreducible representations. Rank of the group is equal 3. This means that all three Pauli $\sigma-$matrices are necessary for generating the group.

Let us adopt the following notations [6]:

$$\sigma_z\sigma_y \equiv a_1, \qquad \sigma_x\sigma_z \equiv a_2, \qquad \sigma_y\sigma_x \equiv a_2$$

and

$$\sigma_x \equiv b_1, \qquad \sigma_y \equiv b_2, \qquad \sigma_z \equiv b_3$$

It is evidently, that

$$b_1 = a_1c, \quad b_2 = a_2c, \quad b_3 = a_3c, \tag{1}$$

where c is one of four $(I, -I, iI, -iI)$ elements of group centre and $c = \sigma_x\sigma_y\sigma_z = iI$. Here I is the unit 2×2 matrix. It means, that operators a_1, a_2, a_3 are connected with operators b_1, b_2, b_3 by simple relations for the given irreducible representation

$$b_1 = ia_1, \quad b_2 = ia_2, \quad b_3 = ia_3. \tag{2}$$

It should be noted also, that

$$a_2a_1a_2^{-1} = a_1^{-1} = a_1^3, \qquad a_1a_2 \equiv a_3, \qquad a_1^2 = a_2^2 = a_3^2. \tag{3}$$

It means, that elements a_1, a_2 generate quaternion subgroup [7]. Let us denote it as - $Q_2[a_1, a_2]$.

Assuming that element of the group d_γ are the generators of an algebra, we obtain the following commutative relations for the algebra based on the d_γ group (standard representation)

$$\begin{array}{lll} [a_1, a_2] = 2a_3, & [a_2, a_3] = 2a_1, & [a_3, a_1] = 2a_2, \\ [b_1, b_2] = -2a_3, & [b_2, b_3] = -2a_1, & [b_3, b_1] = -2a_2, \\ [a_1, b_1] = 0, & [a_2, b_2] = 0, & [a_3, b_3] = 0, \\ [a_1, b_2] = 2b_3, & [a_1, b_3] = -2b_2, & \\ [a_2, b_3] = 2b_1, & [a_2, b_1] = -2b_3, & \\ [a_3, b_1] = 2b_2, & [a_3, b_2] = -2b_1. & \end{array} \tag{4}$$

Up to the factor 2, these commutative relations coincide with commutative relations of the proper homogeneous orthochronous Lorentz group [8].

It follows from the first row of commutative relations (CR) (4) , that elements a_1, a_2, a_3 and all theirs products form the subgroup of 3-dimensional rotations. As it follows from derivation of commutative relations [8] $b_1 = \sigma_x, \quad b_2 = \sigma_y, \quad b_3 = \sigma_z$ have the sense of infinitesimal operators of Lorentz transformations.

Taking into account anticommutation of the operators b_1, b_2, b_3, the second upper row of commutative relations (6) takes the form:

$$b_1 b_2 = -a_3, \quad b_2 b_3 = -a_1, \quad b_3 b_1 = -a_2, \tag{5}$$

All three equalities express in infinitesimal form well known fact - turn through some fixed angle one inertial system with respect to another under relativistic motion [9]. Further transition to repeating motion, for example orbital, leads to rotation. Then the total rotation and total momentum connected with total rotation will be sum of the orbital and so called spin momentum.

Thus the analysis of σ-matrix group on the base of CR (4), which are direct corollary of the Lorentz transformations, demonstrates, that so-called proper momentum of 1/2 spin particle is the result of a definite character of motion. It must be not free, nonlinear. This conclusion is in agreement with familiar fact. It is impossible to measure magnetic moment of the electron, connected with proper momentum, if it moves freely [4].

Explicit form of the operators a_1, a_2, a_3 and b_1, b_2, b_3 for irreducible representations allows to evaluate weight numbers, which specifies uniquely irreducible representations of the Lorentz group. It is necessary for this purpose to construct operators:

$$\begin{array}{ll} H_+ = ia_1 - a_2, & F_+ = ib_1 - b_2, \\ H_- = ia_1 + a_2, & F_- = ib_1 + b_2, \\ H_3 = ia_3, & F_3 = ib_3. \end{array}$$

Weight numbers is eigenvalue of the operators $H_3 = ia_3$ and $F_3 = ib_3$ [8]. Calculation of the eigenvalue gives $l_0 = 1/2$ for the standard σ- matrices. We see, that first weight number (l_0) coincides with spin value. One can show [11], that operators a_1, a_2, a_3 as the elements of d_γ-group have the same fourth order. Therefore any permutations of the elements a_1, a_2, a_3 inside the operators H_+, H_-, H_3 do not change value l_0 and any of space axis may be selected as quantization axis.

The value of the first weight number is determined formally by operators a_1, a_2, a_3, i.e. by the subgroup of three-dimensional rotations. But generation of the spin rotation impossible without relativity, as it follows from above mentioned. It is undoubtedly truly to connect spin with quite definite quantum number, if the quantum numbers are interpreted as indexes of groups [10]. But it is unprovable assumption to endow spin notion with physical value property, which exists in separation from motion as a whole.

Thus starting from conventional 1/2 spin formalism and taking no any additional assumption, we obtain one particular irreducible representation of the Lorentz group and as consequence physical interpretation of Pauli σ-matrices. Rigorously speaking it is applicable for description of the electrons or objects, which structures does not take into account.

3. T-conjugate representation

Exhaustive analysis of the Dirac γ-matrix group and consequent development [11], [12], [13] indicated necessity to supplement standard representation of Lorentz group by P-,T- and (PT)-conjugate representations. It was obtained in explicit form three groups $f_\gamma, b_\gamma, c_\gamma$ in addition to d_γ group. All they realize different irreducible representations of the Lorenz group with first weight number $l_0 = 1/2$. These four groups are connected with four classes of transformations of the Lorentz group. It is usual to distinguish classes by determinant value (± 1) and by sign change ($\pm$) of the time component of the four-vector under transformations.

Four groups $d_\gamma, f_\gamma, b_\gamma, c_\gamma$ are connected between themselves by discrete transformations or by T-, P- and (PT)-reflections. Such kind of transformations are known as an analytical continuations by group parameters.

Two groups namely b_γ and f_γ are the substructures of Dirac γ-matrix group besides of d_γ one. The group b_γ contains quaternion group as subgroup, but it is not isomorphic to d_γ. As a result commutative relations have the following form on the base of b_γ:

$$\begin{array}{lll}
[a_1, a_2] = 2a_3, & [a_2, a_3] = 2a_1, & [a_3, a_1] = 2a_2, \\
[b'_1, b'_2] = 2a_3, & [b'_2, b'_3] = 2a_1, & [b'_3, b'_1] = 2a_2, \\
[a_1, b'_1] = 0, & [a_2, b'_2] = 0, & [a_3, b'_3] = 0, \\
[a_1, b'_2] = 2b'_3, & [a_1, b'_3] = -2b'_2, & \\
[a_2, b'_3] = 2b'_1, & [a_2, b'_1] = -2b'_3, & \\
[a_3, b'_1] = 2b'_2, & [a_3, b'_2] = -2b'_1. &
\end{array} \tag{6}$$

This set of CR is differed from (4) by sign of three commutators in the second upper row. It was named as (T)-conjugate with respect to standard one. The time as the parameter appears under Lorentz transformations just at this row. All remaining commutators are corollary of the first six commutators disposed at the first two lines.

Analog of σ-matrices for this representation, i.e. matrix form of b'_1, b'_2, b'_3 has the view:

$$b'_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}; \quad b'_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}; \quad b'_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Transition from CR (4) to CR (6) is realized by the substitution $b_k \to b'_k = ib_k$, $(k = 1, 2, 3)$. In this case we have transition between the groups $d_\gamma \to b_\gamma$. (T)-conjugation does not touch operators a_1, a_2, a_3, therefore it does not changes $l_0 = 1/2$ and type of spin, i.e. all tree space axes are remained equivalent. It is the reason why particle and antiparticle have the same spin properties [13].

4. P-conjugate representation

Next kind of CR was obtained on the base of group f_γ [13]:

$$\begin{array}{lll}
[a_1, a_2'] = 2a_3', & [a_2', a_3'] = -2a_1, & [a_3', a_1] = 2a_2', \\
[b_1', b_2'] = -2a_3', & [b_2', b_3'] = 2a_1, & [b_3', b_1'] = -2a_2', \\
[a_1, b_1'] = 0, & [a_2', b_2'] = 0, & [a_3', b_3'] = 0, \\
[a_1, b_2'] = 2b_3', & [a_1, b_3'] = -2b_2', & \\
[a_2', b_3'] = -2b_1', & [a_2', b_1'] = -2b_3', & \\
[a_3', b_1'] = 2b_2', & [a_3', b_2'] = 2b_1'. &
\end{array} \tag{7}$$

This representation was called (P)-conjugate, because distinctions are arisen at the level of 3-dimensional rotation subgroup with respect to standard group d_γ, i.e. at the first row. The transition from (4) to (7) is equivalent to following change $a_2 \to ia_2'$. By definition we obtain $a_3 \to ia_3'$. All changes of the signs in more lower rows are consequence of this primary change. In this case quaternion subgroup $Q_2[a_1, a_2']$ transforms in $q_2[a_1, a_2]$ with the same determining relations between generators (see equalities (3)). One can show [13], that elements a_1, a_2, a_3 of d_γ group have fourth order whereas elements a_2', a_3' of f_γ have second order and element a_1 has fourth order as previously. So that we have in the similar cases non-equivalence of space directions, or so-called asymmetry between left and right. One can show [13], that first weight number is obtained equal $l_0 = 1/2$ for f_γ, if $H_3 = ia_1$. The number l_0 is obtained pure imaginary for two other operators ($H_3' = a_2', H_3'' = a_3'$), i.e. they have no physical sense for three-dimensional rotation subgroup.

Similar non-equivalence is observed for b_1', b_2', b_3' values. They explicit form is such for f_γ group:

$$b_1' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad b_2' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad b_3' = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Nevertheless the structure of commutative relations is the same on the whole as previously.

5. (PT)-conjugate representation

The fourth type of CR is connected with group c_γ:

$$\begin{array}{lll}
[a_1, a_2'] = 2a_3', & [a_2', a_3'] = -2a_1, & [a_3', a_1] = 2a_2', \\
[b_1'', b_2''] = 2a_3', & [b_2'', b_3''] = -2a_1, & [b_3'', b_1''] = 2a_2', \\
[a_1, b_1''] = 0, & [a_2', b_2''] = 0, & [a_3', b_3''] = 0, \\
[a_1, b_2''] = 2b_3'', & [a_1, b_3''] = -2b_2'', & \\
[a_2', b_3''] = -2b_1'', & [a_2', b_1''] = -2b_3'', & \\
[a_3', b_1''] = 2b_2'', & [a_3', b_2''] = 2b_1''. &
\end{array} \tag{8}$$

This kind of CR was called (TP)-conjugate with respect to standard one (4), because it is consecutive action two conjugations. They commute between themselves therefore one can write (TP)=(PT). Comparison of CR (7) and (8) shows, that f_γ and c_γ are (T)-conjugate in the same way as d_γ and b_γ.

The first rows of CR (7) and (8) coincide, therefore all said about the first weight number l_0 for f_γ is correctly for c_γ. It is confirmed a rule that T-conjugation does not change spin properties. Matrix form of operators b_1'', b_2'', b_3'' , i.e. analog of σ-matrices for c_γ has the view:

$$b_1'' = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \quad b_2'' = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}; \quad b_3'' = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The study of possible types of lepton equations [12], [13],[14] exhibits, that presence in they structures of c_γ group leads to the particle description having longitudinal polarization only.

6. Conclusion

Main results of this article my be classified as a theoretical observations, because no assumptions was made. One particular irreducible representation of the Lorentz group was obtained and as a consequence we got physical interpretation of the σ-matrices having used well known formalism of Pauli matrices . Identical coincidence of Pauli σ-matrices with generators of irreducible representation Lorentz group makes words such as "spin is internal property (or proper angular momentum) of particle" not necessary or inappropriate.

Proposed interpretation of Pauli matrices does not cancel anything connected with spin electron, when it is placed in atom or in some other binding system. Difference arises from a conventional notion, when we think the free electron. The point is that particle properties are become apparent by interactions in all completeness. The spin is not exclusion, as it is shown.

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