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Abstract. The study of gravitational collapse is a subject of great importance, both from an astrophysical and a holographic point of view. In this respect, exact solutions can be very helpful but known solutions are very scarce, especially when considering dynamical processes with rotation. We describe a setup in which gravitational collapse of rotating matter shells can be addressed with analytic tools, at the expense of going to higher dimensions and considering equal angular momenta spacetimes. The framework for an exact treatment of the dynamics, relying on a thin shell approximation, is developed. Our analysis allows the inclusion of a non-vanishing cosmological constant. Finally, we discuss applications of this machinery to the construction of stationary solutions describing matter around rotating black holes and to the cosmic censorship conjecture.

1. Introduction

The presence of rotation plays a major role in the formation and evolution of astrophysical systems and black holes are no exception. Half a century has passed since the discovery of the (arguably) most physically relevant black hole solution of the Einstein equations — the Kerr geometry [1]. We currently possess a good understanding of stationary, rotating black holes. However, we still have comparatively poor (analytic) control over highly dynamical scenarios involving rotation. In particular, very little is known about the associated non-spherical gravitational collapse [2]. Realistic collapses should include rotation and this issue becomes even more pressing since naked singularities are known to arise in fine-tuned non-rotating — and thus automatically non-generic — settings [3].

Clearly, the advantage of non-rotating setups is their large amount of symmetry. Imposing spherical symmetry reduces the study of the collapse to a problem in $1 + 1$ dimensions. Nevertheless, there exists a larger class of (rotating) black hole spacetimes that are stationary and whose metric depends on a single radial coordinate [4, 5]: these are the well-known Myers-Perry solutions [6] when restricted to equal angular momenta, which are blessed with the cohomogeneity-1 property. The price to pay for the convenience provided by cohomogeneity-1 spacetimes is the restriction to higher (odd) dimensions, namely $D = 2N+3$ with $N = 1, 2, 3, \ldots$.

2. Background geometries

Myers-Perry(-AdS) black holes in $2N+3$ dimensions possess isometry group $\mathbb{R} \times U(1)^N$. When all spin parameters are set equal, $a_i = a$, this symmetry is enhanced to $\mathbb{R} \times U(N)$, and coordinates can be found that reflect this large amount of symmetry, such that the metric depends on just one (radial) coordinate. Constant time and radial sections are squashed $(2N+1)$-spheres, which can be written as an S^1 fibre over the N-dimensional complex projective space CP^N.

The line element describing these cohomogeneity-1 black holes can be written as follows [5],

$$ds^2 = -f(r)^2 dt^2 + g(r)^2 dr^2 + r^2 \widehat{g}_{ab} dx^a dx^b + h(r)^2 \left[d\psi + A_a dx^a - \Omega(r) dt \right]^2 , \tag{1}$$

where the various metric functions are given by

$$g(r)^2 = \left(1 + \frac{r^2}{\ell^2} - \frac{2M}{r^{2N}} + \frac{2Ma^2}{r^{2N}\ell^2} + \frac{2Ma^2}{r^{2N+2}} \right)^{-1} , \tag{2}$$

$$h(r)^2 = r^2 \left(1 + \frac{2Ma^2}{r^{2N+2}} \right) , \qquad \Omega(r) = \frac{2Ma}{r^{2N}h(r)^2} , \qquad f(r) = \frac{r}{g(r)h(r)} . \tag{3}$$

The tensor $\widehat{g}_{ab}$ denotes the Fubini-Study metric on CP^N whereas $A = A_a dx^a$ is the associated Kähler potential, which for $N = 1$ are given respectively by

$$\widehat{g}_{ab} dx^a dx^b = \frac{1}{4} \left(d\theta^2 + \sin^2 \theta \, d\phi^2 \right) , \qquad A = \frac{1}{2} \cos \theta \, d\phi . \tag{4}$$

The angular coordinate ψ parametrizes the S^1 fibre and has 2π-periodicity.

These geometries accommodate a non-vanishing cosmological constant Λ. The parameter ℓ sets the (constant) curvature scale of the solutions and is related to the cosmological constant through $\Lambda = -(D-1)\ell^{-2}$ [5]. Thus, a (finite) real ℓ corresponds to asymptotically anti-de Sitter (AdS) black holes, and asymptotically flat spacetimes are recovered by taking the limit $\ell \to \infty$. In addition, the solutions (1) are fully determined by a mass parameter M and a spin parameter a, whose relation to the physical mass and angular momentum is explicitly given in Ref. [5].

From now on we shall restrict to the case $N = 1$ for concreteness, corresponding to five spacetime dimensions. However, the whole analysis can be generalised to higher odd dimensions.

3. Junction conditions and the stress-energy tensor

The cohomogeneity-1 property makes an exact (thin-shell) calculation possible, 'gluing' an interior to an exterior geometry [7]. We take advantage of the high degree of symmetry by considering shells that respect the full set of spatial isometries. Indeed, for test shells the dynamics on the $CP^1 \simeq S^2$ and on the S^1 separate [8] and the geometry of the shells only changes by a mere scaling of the 'radial' coordinate r. However, since their geometry is squashed — as opposed to a round sphere — one can anticipate that pressure anisotropies emerge during gravitational collapse of matter-filled rotating thin shells.

We take both the exterior $(+)$ and interior $(-)$ metrics to be of the form (1) and make use of the Darmois-Israel junction conditions [9] along a timelike hypersurface Σ, defined parametrically by $t = \mathcal{T}(\tau)$ and $r = \mathcal{R}(\tau)$, where τ is for the proper time for an observer comoving with the hypersurface.[1] The junction conditions determine the (dis)continuity of the first and second fundamental forms characterising the hypersurface. If $[C_{ij...}] \equiv C_{ij...}^{(+)} - C_{ij...}^{(-)}$ denotes the jump in a quantity $C_{ij...}$ across the shell, then the junction conditions read

$$[\mathfrak{g}_{ij}] = 0 \qquad \Longrightarrow \qquad \mathfrak{g}_{ij}^{(+)} = \mathfrak{g}_{ij}^{(-)} \equiv \mathfrak{g}_{ij} , \tag{5}$$

$$[k_{ij}] - \mathfrak{g}_{ij}[k] = -8\pi G \mathcal{S}_{ij} , \tag{6}$$

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[1] We take the coordinates $y^i = \{\tau, \psi, \theta, \phi\}$, with $i, j = 0, \ldots, 3$, to parametrize the surface Σ, but we shall use indices a, b to refer to the coordinates on CP^1, i.e., they run over $\{2, 3\}$.

where $\mathfrak{g}_{ij}^{(\pm)}$ is the induced metric on Σ, as inferred from either the exterior or the interior. Similarly, $k_{ij}^{(\pm)}$ are the extrinsic curvatures and $k^{(\pm)}$ their traces. $\mathcal{S}_{ij}$ stands for the surface stress-energy tensor. This formalism has been applied to rotating spacetimes in $(2+1)$ dimensions by Mann *et al.* [10]. This lower dimensional case is however very special, since general relativity possesses no propagating degrees of freedom in spacetime dimensions lower than four. In what follows we generalize to $D = 5$ following Ref. [11].

Compared to the $D = 3$ case, we get (for any $N \in \mathbb{N}$) one additional constraint from the first junction condition,[2]

$$h_+(\mathcal{R}) = h_-(\mathcal{R}) \equiv h(\mathcal{R}) \qquad \Longrightarrow \qquad M_+ a_+^2 = M_- a_-^2 \,. \tag{7}$$

This can be understood as the requirement that the areal radius of the shell, which is proportional to $r^{2N}h(r)$, must be the same when seen from the inside or outside. Note that Eq. (7) forbids the matching of an exterior black hole spacetime with a flat interior. However, working in the slow rotation limit (i.e., up to order $O(a^2)$) this constraint is avoided.

The second junction condition requires the (corotating) shell to possess the stress-energy tensor of an imperfect fluid,

$$\mathcal{S}_{ij} = (\rho + P)u_i u_j + P\,\mathfrak{g}_{ij} + 2\varphi\,u_{(i}\xi_{j)} + \Delta P\,\mathcal{R}^2\widehat{g}_{ij} \,, \tag{8}$$

where $u = \partial_\tau$ is the fluid velocity, $\xi = h(\mathcal{R})^{-1}\partial_\psi$ is the normalised Killing angular vector field, and $\widehat{g}_{ij}dy^i dy^j = \widehat{g}_{ab}dx^a dx^b$. The quantities ΔP and φ are pressure anisotropy and intrinsic momentum terms, respectively, and we recover a perfect fluid when $\Delta P = \varphi = 0$.

The stress-energy tensor components are dictated by the metric components according to [11]

$$\rho = -\frac{(\beta_+ - \beta_-)(\mathcal{R}^2 h)'}{8\pi\mathcal{R}^3} \,, \qquad P = \frac{h}{8\pi\mathcal{R}^3}\left[\mathcal{R}^2(\beta_+ - \beta_-)\right]' \,, \tag{9}$$

$$\varphi = -\frac{(\Omega'_+ - \Omega'_-)h^2}{16\pi\mathcal{R}} \,, \qquad \Delta P = \frac{(\beta_+ - \beta_-)}{8\pi}\left[\frac{h}{\mathcal{R}}\right]' \,, \tag{10}$$

where we have introduced the quantities

$$\beta_\pm \equiv f_\pm(\mathcal{R})\sqrt{1 + g_\pm(\mathcal{R})^2\dot{\mathcal{R}}^2} \,, \tag{11}$$

and where primes and overdots stand for $d/d\mathcal{R}$ and $d/d\tau$, respectively. Observe that the intrinsic momentum φ and the anisotropic pressure ΔP necessarily vanish in the non-rotating limit.

4. The shell's equation of motion

For simplicity we now assume a linear equation of state, $P = w\rho$, as in Ref [11]. Note, however, that other equations of state, e.g., polytropic, can be considered while still allowing the analytical integration of the equations of motion. Employing expressions (9) and (11), one obtains an effective equation governing the radial motion of the shell,

$$\dot{\mathcal{R}}^2 + V_{\text{eff}}(\mathcal{R}) = 0 \,, \tag{12}$$

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[2] We could have taken different r coordinates for the interior and exterior spacetimes but the first junction condition would impose their equality.

where the effective potential V_{eff} is given by

$$V_{\text{eff}}(\mathcal{R}) = 1 + \frac{\mathcal{R}^2}{\ell^2} + \frac{2Ma^2}{\mathcal{R}^4} + \frac{2Ma^2}{\ell^2\mathcal{R}^2} - \frac{M_+ + M_-}{\mathcal{R}^2} - \frac{1}{4}\left(\frac{m_0}{\mathcal{R}^2}\right)^{2+3w}\left(1 + \frac{2Ma^2}{\mathcal{R}^4}\right)^{1-w}$$
$$- \left(\frac{M_+ - M_-}{m_0}\right)^2\left(\frac{\mathcal{R}^2}{m_0}\right)^{3w}\left(1 + \frac{2Ma^2}{\mathcal{R}^4}\right)^{w-1}. \tag{13}$$

The parameter m_0 is a (positive) integration constant, which corresponds to the rest mass of the shell in the case of non-rotating dust. The second term in the potential is nothing but the usual infinite barrier (for positive ℓ^{-2}) responsible for the confining nature of AdS spacetime. The third term behaving like $\sim \mathcal{R}^{-4}$ represents the centrifugal repulsion due to rotation.

Solutions describing the radial trajectory of thin shells are found by simply integrating (12). If desired, the acceleration of the shell can be determined by

$$\ddot{\mathcal{R}} = -\frac{1}{2}V'_{\text{eff}}(\mathcal{R}). \tag{14}$$

5. Stationary solution in AdS

When considering a negative cosmological constant, the confining nature of the effective potential (when $-1 < w < 1/3$), in conjunction with the centrifugal barrier, suggests the possibility of stationary solutions consisting of a rotating shell hovering around an AdS black hole.

Indeed, these solutions exist and can be constructed as follows. First, the radial location of the shell must be constant, i.e., $\mathcal{R}(\tau) = \mathcal{R}_*$, and therefore Eqs. (12) and (14) imply

$$V_{\text{eff}}(\mathcal{R}_*) = V'_{\text{eff}}(\mathcal{R}_*) = 0. \tag{15}$$

Second, for the shell to be stable against radial oscillations, $\mathcal{R}_*$ should correspond to a local minimum of the potential,

$$V''_{\text{eff}}(\mathcal{R}_*) > 0. \tag{16}$$

For the case of dust ($w = 0$) the conditions (15) reduce to high order polynomial equations in $\mathcal{R}_*^2$, but generically they are non-polynomial equations. However, the conditions are simple quadratic equations in M_+ and M_-, and so in practice it is more convenient to fix these mass parameters in terms of $\{w, m_0, \mathcal{R}_*, Ma^2\}$.

It is then just a matter of scanning the parameter space to obtain solutions such that:

(i) $\mathcal{R} = \mathcal{R}_*$ is a stable point, i.e., inequality (16) is satisfied;

(ii) The interior geometry has an event horizon located at $r = r_h^- < \mathcal{R}_*$;

(iii) The exterior geometry features no horizons, i.e., $r_h^+ < \mathcal{R}_*$;[3]

(iv) Energy conditions for the matter on the shell are obeyed.

In Ref. [11] the weak energy conditions were imposed and it was found that it is easy to satisfy simultaneously all the conditions above. An example is presented in Fig. 1.

While the confining potential is absent in the asymptotically flat case ($\ell \to \infty$), this does not exclude the existence of analogous stationary solutions describing thin shells around rotating black holes. In fact, we have been able to obtain such configurations, but in all cases found the shell sits at a local maximum of the potential and is therefore unstable against radial oscillations.

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[3] The alternative would be to consider the exterior spacetime to be over-rotating, thus corresponding to a naked singularity. However, it can be shown using condition (7) that this would imply also a nakedly singular interior.

$m_0/\ell^2 = 0.324,\ w = 0.285,\ Ma^2/\ell^4 = 0.02,\ \mathcal{R}_*/\ell = 1.8$
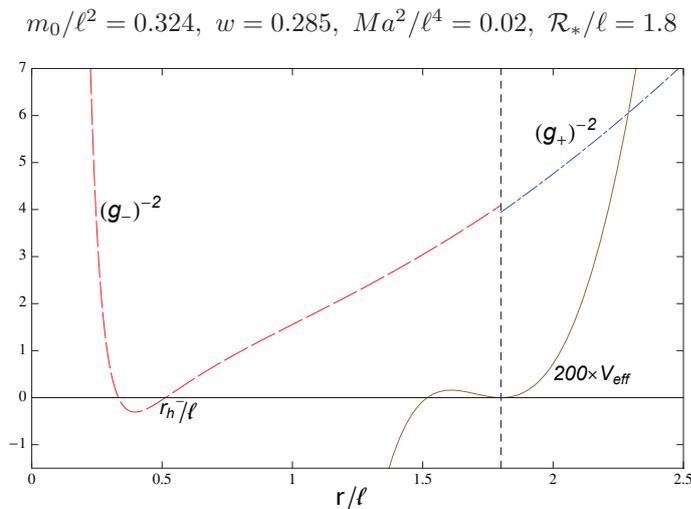
Figure 1. Stable configuration of a stationary rotating shell surrounding an asymptotically AdS black hole. The dashed vertical line identifies the location $\mathcal{R}_*$ of the shell. The metric function $g(r)^{-2}$ is shown in both the exterior domain, $r > \mathcal{R}_*$ (blue), and the interior domain, $r < \mathcal{R}_*$ (red). The largest real root of $g_-{}^{-2}$ marks the event horizon radius, r_h^-. The brown curve represents the effective radial potential (amplified to ease interpretation) and $r = \mathcal{R}_*$ corresponds to a local minimum of the potential.

6. Full collapse onto a black hole and cosmic censorship

Within the framework developed it is also possible to obtain collapse of a rotating thin shell onto a (previously existing) black hole. An interesting scenario is that of a shell collapsing from infinity, initially at rest, in an asymptotically flat spacetime ($\ell \to \infty$). The effective potential must then asymptote to zero as $\mathcal{R} \to \infty$. Using Eq. (13) it is easy to show that this can only occur for dust ($w = 0$) and only if the condition $m_0 = M_+ - M_-$ is satisfied. Having fixed these quantities it is not hard to find conditions on the remaining parameters, $\{Ma^2, M_+, M_-\}$, so that the effective potential is strictly negative — having no roots, and therefore no turning points. Simultaneously, the weak energy conditions can be obeyed (see Ref. [11]).

The solution thus obtained describes the collapse of a shell, initially located at $\mathcal{R} \to \infty$ with vanishing radial velocity. The spacetime in the interior of the shell consists of a rotating black hole with a horizon at $r = r_h^-$ and spin parameter $a_- = \sqrt{Ma^2/M_-}$. At larger radii, the spacetime is the same as that of a black hole with horizon radius $r_h^+ > r_h^-$ and spin parameter $a_+ = \sqrt{Ma^2/M_+} < a_-$. As the shell falls past the radius r_h^+, a new apparent horizon appears at that location. At this point nothing can prevent the shell from meeting its doom, as it will eventually fall onto the curvature singularity at $r = 0$. The end state is a black hole with larger mass and spin. Such an example (satisfying the weak energy conditions) is presented in Fig. 2. The centrifugal barrier at smaller radii is apparent, but not strong enough to prevent full collapse from occurring. However, the shell's radial trajectory turns into a bounce if one increases sufficiently the spin parameter a.

An analogous construction can be performed in AdS, i.e., with a finite ℓ, even though it is more laborious because it involves more parameters. The main difference is that it is no longer possible to have a shell at rest at infinity: either the AdS potential barrier dominates at large radii or the (negative) w-dependent terms in Eq. (13) make the potential diverge negatively as $\mathcal{R} \to \infty$. Fig. 3 shows a case in which a shell starting from rest at a finite radius fully collapses onto a black hole.

Finally, we point out that none of the collapsing solutions we have described can violate the cosmic censorship conjecture [12]. Indeed, it is easy to show, using identity (7) and $M_+ > M_-$, that if the interior solution is sub-extremal ($g_-(r)^{-2} = 0$ has a real and positive root) the global solution after full collapse cannot be over-extremal. Thus, the curvature singularity at $r = 0$ remains covered by a horizon at all times.
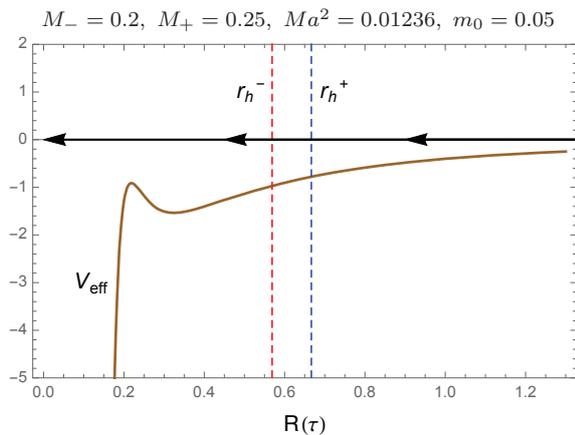
Figure 2. Full collapse of a rotating thin shell onto an asymptotically flat black hole. The vertical dashed lines mark the horizons of the interior and exterior geometries.
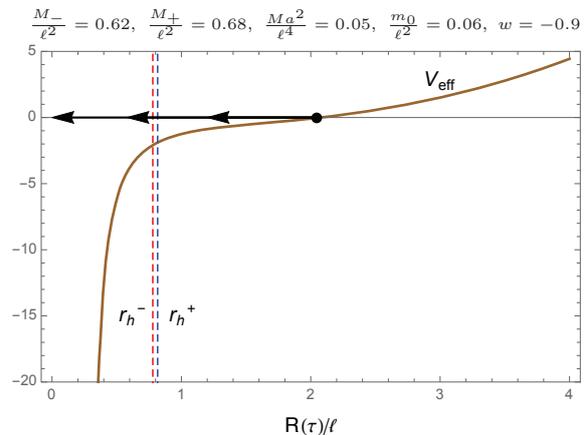
Figure 3. Full collapse of a rotating thin shell onto an asymptotically AdS black hole. The vertical dashed lines mark the horizons of the interior and exterior geometries.

7. Conclusion and outlook

We have described a simple framework to study the dynamics of thin matter shells around rotating black holes in five dimensions. In the presence of rotation the matter must take the form of an imperfect fluid. We constructed stationary solutions consisting of rotating shells surrounding spinning black holes in AdS and applied the formalism to address gravitational collapse with rotation, finding that cosmic censorship is not violated.

The extension of this work to higher (odd) dimensions and the exhaustive exploration of the parameter space is left for the future. Also, it would be interesting to consider the collapse of *null* shells in AdS. This would allow a confrontation with Ref. [13], which studied the (perturbative) black hole spin-up process with light-like test particles in AdS.

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