Study of gravitational theories through Hamiltonian methods and consistent deformations

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Als meus pares,
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Chapter 1

Introduction

General Relativity, motivated by Einstein's ideas and developed in a series of papers that culminated in the gravitational field equations [4], [5], is the most accurate classical theory of gravitation we have so far. It entailed a huge change of paradigm in the conception of the nature of the universe, by replacing the rigid Galilean space and time by a dynamical, (in general curved) spacetime. The newborn theory had a critical success in explaining the precession of the perihelion of Mercury, the deflection of light by the Sun and, later on, the gravitational redshift experienced by light (see, for example, [6]).

However, even the astonishing success of General Relativity had a chink: singularities [7]. They appeared, loosely, as points where spacetime breaks down or has an 'end'. The two most famous instances of singularities are, perhaps, the one in Schwarzschild black holes and the initial singularity of Friedman-Lemaître-Robertson-Walker cosmological models. Although the formalism allows such situations, singularities are largely regarded as undesirable and signal our lack of understanding of the more profound physics needed to completely understand these physical situations. In a sense, the theory itself strongly suggests that something is still missing and nicely tells us where to look. There was some hope of resolving this conundrum with the help of (also very recent) ideas in Quantum Mechanics. In particular, since General Relativity is a field theory, one would like to find a relativistic Quantum Field Theory for the gravitational field (or as usually said, to *quantize* gravity), along the lines successfully followed to understand other filed theories, in particular, in Quantum Electrodynamics, which eventually gave rise to the celebrated Standard Model of Particle Physics [8].

However, as it was conceived, Quantum Field Theory frontally clashes with General Relativity. Firstly, the privileged Minkowski spacetime is used as a background. Worse, the perturbative techniques that allowed to extract the relevant physical information from path integrals by employing Feynmann diagrams completely fails when trying to quantize gravity. The main obstacle for the development of the traditional quantization programme for General Relativity is its non-renormalizability.

This issue arises because the coupling constant of the theory, Newton's constant in this case, is dimensionful. This fact is made apparent in the one loop corrections, which diverge and can not be absorbed [9]–[11].

A possible explanation for this failure is that usual perturbative treatments in physics assume that the spacetime is a continuum at all scales. This works for particle physics, since the scales of particles are much larger than the Planck length, the scale in which quantum gravity effects should manifest themselves. However, for a theory of quantum gravity, this hypothesis may be completely wrong and there is no reason *a priori* to assume it. This fact motivated the development of non-perturbative theories for quantum gravity that can account for the micro structure of spacetime, with the benefit that this approach completely avoids the non-renormalizability issue. Among the plethora of proposals, two of them have had the largest impact and a number of results over the years; these competing approaches are String Theory and Loop Quantum Gravity. Although String Theory [12], [13] is not just a theory of quantum gravity (in fact, it is quite more ambitious), it has paid close attention to how gravity manifests itself in the whole of the theory [14], [15], and has produced numerous results on black holes [16] and cosmology [17]–[19] On the other hand, Loop Quantum Gravity is a very natural continuation of the ideas that arose in the development of General Relativity, that can be found in the extensive literature on the topic, for instance, [20]–[22]. This approach has also achieved some results on black holes [23]–[25] and in cosmology [26]–[28] under the name Loop Quantum Cosmology, a quantization of the cosmological sector of General Relativity.

However, after more than 30 years of intense scientific activity, no one has been able to achieve a full theory of quantum gravity yet. In view of this fact, it may be an interesting possibility to pursue perturbative approaches again. Evidently, since the traditional approach does not work, we need to come up with new methods to deal with the problem. In [29], Smolin sketched a new perturbative framework. One of its prominent features was that general covariance was not broken at any order; moreover, no background structure (such as the Minkowski metric) was needed. This is an important point since many of the obstacles seem to come from the breaking of the 4-diffeomorphism symmetry. Then, if the theory chosen as the unperturbed starting point is integrable, one can then use it as the zeroth order of a perturbative expansion. However, an additional requirement must be met in order for this approach to work in a reliable and systematic way, namely, the perturbation that recovers the full solution must be regular. If a perturbation is *singular* rather than regular, a perturbative scheme could still be built, however, the usual expansions do not work in this context, requiring an *ad hoc* scheme for the particular problem at hand. These type of ideas are connected to several currently relevant topics. In particular, the $U(1)^3$ model, which can be seen as a zeroth order perturbation model for General Relativity, has recently raised some interest [30]–[32]. One of the main goals of this work is precisely to study this approach.

This thesis is structured as follows. Chapter 2 is devoted to reviewing the Hamiltonian formalism, in particular, three different geometrical approaches to get the Hamiltonian description derived from an action are presented. Chapter 3 provides several examples of gravitational theories (apart from the interesting exception of the scalar field) as well as their Hamiltonian descriptions using the three methods described in Chapter 2. Chapter 4 studies the anti-self-dual Palatini action by means of the GNH method. Chapter 5 explores the ideas about consistent deformations and internal Abelianizations and applies them to the previously presented gravitational theories.

A few words on notation. The spacetime manifold (either in 3 or 4 dimensions) will be denoted by $\mathcal{M}$, will be assumed globally hyperbolic $\mathcal{M} \cong \mathbb{R} \times \Sigma$ and the leaves of foliations will be deonted by $\Sigma_t \cong \Sigma$. Differential forms in $\mathcal{M}$ will be written in boldface (e.g. $\mathbf{e}, \mathbf{A}$) and differential forms in Σ in standard face (e.g. e, A). Differential forms in $\mathcal{M}$ that are adapted to the foliation Σ_t will be written in boldface and marked with a t subscript or underlined (e.g. $\mathbf{e}_t, \underline{\mathbf{e}}$). The spacetime or foliation leaf exterior derivative will be written as $\mathbf{d}$, while the exterior derivative in the configuration space will be written as $\mathbb{d}$. All spacetimes $\mathcal{M}$ are assumed to have no boundary.

Chapter 2

Hamiltonian analysis

Hamiltonian analysis has become a staple method in the study of dynamical systems [33]. Using the tools of symplectic geometry in a phase space of positions and momenta and a *Hamiltonian* function, this approach is able to correctly predict the dynamics of a physical systems whose energy equals the Hamiltonian function. In particular, the integral curves of the Hamiltonian vector field are the curves that leave the Hamiltonian invariant along their trajectories, implementing the physical requirement that the energy be conserved. In mechanics, where the phase spaces are finite dimensional, there are no major technical difficulties in pursuing this approach. However, in field theories, infinite dimensional phase spaces enter into play making the (sometimes subtle but very important) functional analytic details much harder to deal with in comparison with the finite dimensional case [34]. In particular, Poisson brackets are hard to compute in some instances (for example, in field theories formulated in regions with boundaries), which is a big obstacle for the approaches to the Hamiltonian analysis that rely heavily on their computation. A more productive strategy is to think in a more geometrical way as in the work of Gotay [35], which besides achieving a better and clearer picture of the relevant structures, is also computationally more efficient. In this chapter we review three different geometric methods to obtain the Hamiltonian or Lagrangian formulation of a theory that will be used in the next chapters. The most basic concepts discussed in this chapter can be found in greater detail in [35], [36].

2.1 The Gotay, Nester and Hinds algorithm

A *(pre)symplectic system* is a triple $(\mathfrak{Q}, \omega, \alpha)$, where $\mathfrak{Q}$ is a manifold called *the phase space*, $\omega \in \Omega^2(\mathfrak{Q})$ a (pre)symplectic form and $\alpha \in \Omega^1(\mathfrak{Q})$ a closed 1-form. The dynamics of the system is given by the vector field $\mathbb{Z}$ determined by the *generalized Hamiltonian equation*

$$\imath_{\mathbb{Z}}\omega = \alpha \ , \tag{2.1}$$

whose integral curves give the evolution of the system. Note that the requirement of α being closed is equivalent to demanding that $\mathcal{L}_{\mathbb{Z}}\omega = 0$, since ω is itself closed.

If ω is a *symplectic* (i.e. closed and strongly non-degenerate) form, equation (2.1) has a unique (global) solution. In contrast, if ω is merely *presymplectic* (it has some degeneracy), the equation is globally inconsistent and has no solution. In this case, since one is still interested in the particular dynamics encoded by the presymplectic system, one can instead reinterpret equation (2.1) by seeking and restricting to a suitable submanifold of $\mathfrak{Q}$ where it can be properly solved globally. Hence, in the case of presymplectic systems we need a systematic method of finding the maximal submanifold $\mathfrak{N}$ of $\mathfrak{Q}$ where equation (2.1) can be consistently solved by a *vector field of* $\mathfrak{N}$. This is provided by the GNH algorithm [37], [38] (named after Gotay, Nester and Hinds), a procedure that outputs the so-called *final constraint manifold*. Note, however, that even when such a solution exists and is integrable, it will be non-unique in general.

A first necessary condition for equation (2.1) to be solvable is that α is in the image of the map

$$\begin{aligned} \flat_{\mathbf{p}} : T_{\mathbf{p}}\mathfrak{Q} &\longrightarrow T^*_{\mathbf{p}}\mathfrak{Q} \\ X &\mapsto \imath_X\omega \ . \end{aligned}$$

Then, one might try to solve (2.1) restricted to the submanifold defined as

$$\mathfrak{Q}_1 = \left\{ \mathbf{p} \in \mathfrak{Q} \mid \alpha_{\mathbf{p}} \in \operatorname{Im} \flat_{\mathbf{p}} \right\} .$$

Note that even though there exist solutions $\mathbb{Z}$ in $\mathfrak{Q}_1$, there might be some points $\mathbf{p} \in \mathfrak{Q}_1$ where the vector field is not tangent to $\mathfrak{Q}_1$, i.e. $\mathbb{X}_{\mathbf{p}} \notin T_{\mathbf{p}}\mathfrak{Q}_1$ but rather $\mathbb{X}_{\mathbf{p}} \in T_{\mathbf{p}}\mathfrak{Q} \setminus T_{\mathbf{p}}\mathfrak{Q}_1$, which is crucial in order to obtain a genuine vector field in $\mathfrak{Q}_1$ rather than a vector field of $\mathfrak{Q}$ along the inclusion of $\mathfrak{Q}_1$ in $\mathfrak{Q}$. Physically, the reason why one wants the solution to be a vector field in the constraint manifold is because, otherwise, the own dynamics of the system would make it evolve 'out of it'. This is known as the *consistency problem* and motivates further restrictions to the submanifold

$$\mathfrak{Q}_2 = \left\{ \mathbf{p} \in \mathfrak{Q}_1 \mid \alpha_{\mathbf{p}} \in \flat_{\mathbf{p}} \left(\overline{T}_{\mathbf{p}}\mathfrak{Q}_1 \right) \right\} ,$$

where we have used the notation $\overline{T}\mathfrak{Q}_1 := T\overline{\mathfrak{Q}_1}|_{\mathfrak{Q}_1} \subset T\mathfrak{Q}$ and $\overline{\mathfrak{Q}_1}$ is the closure of $\mathfrak{Q}_1$ in $\mathfrak{Q}$. However, after restricting ourselves to $\mathfrak{Q}_2$, the tangency of the solution to $\mathfrak{Q}_2$ may still not be guaranteed, similarly to what happened with $\mathfrak{Q}_1$. In order to ensure consistency one generates a sequence of submanifolds defined by

$$\mathfrak{Q}_{i+1} = \left\{ \mathbf{p} \in \mathfrak{Q}_i \mid \alpha_{\mathbf{p}} \in \flat_{\mathbf{p}} \left(\overline{T}_{\mathbf{p}}\mathfrak{Q}_i \right) \right\} . \tag{2.2}$$

Note that the actual restrictions imposed by the algorithm do not require tangency to the submanifold but rather to its closure. This relaxation of the desirable strict tangency condition is due to the fact that in field theories – with an infinite number of degrees of freedom – one can obtain a countably infinite chain of constraint submanifolds leading to a final constraint manifold defined by the intersection of all of them. This poses a problem in the functional-analytic sense, since the resulting submanifold might be very difficult to work with (for example, it can be a Fréchet manifold instead of a Banach one). The final solution proposed by Gotay was the relaxed condition (2.2) introduced in an effort to make the algorithm stop in a finite amount of steps. Note that this problem does now show up for systems with a finite number of degrees of freedom in which case the strict tangency condition can be used instead.

The behaviour of the sequence $(\mathfrak{Q}_k)_k$ determines the outcome of the algorithm, which is one of the four possibilities:

1. For some k, $\mathfrak{Q}_k = \varnothing$.
2. For some k, $\mathfrak{Q}_k \neq \varnothing$ and $\dim \mathfrak{Q}_k = 0$.
3. For some k, $\mathfrak{Q}_k = \mathfrak{Q}_{k+1} := \mathfrak{N}$.
4. There is an infinite number of constraints.

Let us make some comments on each case.

1. The system is inconsistent: there is no solution.
2. The system is consistent, but the final constraint manifold consists of isolated points, hence the evolution vector field must vanish, which yields no evolution.
3. The solution is consistent in the final constraint manifold $\mathfrak{N}$ and a solution to (2.1) tangent to $\overline{\mathfrak{N}}$ exists. The Hamiltonian vector field describes the dynamics of the system but is, in general, not unique, since one can add to it any element of $\ker\omega \cap \overline{T}\mathfrak{N}$ and will still satisfy (2.1). In the case $\ker\omega \cap \overline{T}\mathfrak{N} \neq 0$, this implies the existence of gauge symmetries. Also, an important and desirable property of the Hamiltonian vector field is its integrability. While the GNH algorithm guarantees the existence of a consistent Hamiltonian vector field, it does not say anything about its integrability, which is a separate and hard problem.
4. The algorithm never stops and produces an infinite chain of constraints. However, one still can take as final constraint manifold the intersection of all the submanifolds, $\mathfrak{N} := \cap_k \mathfrak{Q}_k$ with the topology induced by that of the full phase space. Note that since we have an infinite intersection, the resulting induced topology of $\mathfrak{N}$ may be much more complicated than that of each $\mathfrak{Q}_k$. Note that this can only happen for field theories. The resulting final constraint manifold $\mathfrak{N}$ can then be of the form of any of the cases 1, 2 or 3.

2.2 Physical systems

A popular way to determine the dynamics of field theories and mechanical systems is through action principles. Actions give rise to presymplectic systems in a canonical way through the Lagrangian used to define them. Given a configuration space Q (a differential manifold of finite or infinite dimension) of a physical system, the Lagrangian L is a real function on the tangent bundle TQ. Given the pair (Q, L) there are two canonical constructions producing presymplectic systems, one in the cotangent bundle T^*Q and the other in the tangent bundle TQ, usually called Hamiltonian and symplectic Lagrangian approaches, respectively.

Given a Lagrangian $L : TQ \longrightarrow \mathbb{R}$, the fiber derivative is the fiber preserving vector bundle map $FL : TQ \longrightarrow T^*Q$ defined by

$$FL(\mathrm{v})(\mathrm{w}) = \frac{d}{d\varepsilon} L(\mathrm{v} + \varepsilon \mathrm{w})\Big|_{\varepsilon=0} , \tag{2.3}$$

where $\mathrm{w} \in TQ$. With the help of this map we can identify and transport some objects between TQ and T^*Q. First, let us define the canonical momenta as

$$\mathbf{p}(\mathrm{w}) = FL(\mathrm{v})(\mathrm{w}) ,$$

and the energy function $E_L : TQ \longrightarrow \mathbb{R}$ as

$$E_L(\mathrm{v}) = \mathbf{p}(\mathrm{v}) - L(\mathrm{v}) , \tag{2.4}$$

The Hamiltonian is then defined as the function H in T^*Q (or a submanifold of it) such that

$$H \circ FL = E_L . \tag{2.5}$$

The properties of FL are very important and determine the character of the theory. If FL is not a diffeomorphism, then (2.5) only defines H in the image of FL. It is then possible to extend that function to the whole space T^*Q although such extensions are not unique. We say that a Lagrangian L is regular if its fiber derivative FL is a local diffeomorphism and L is singular if FL fails to be a local diffeomorphism. If L is a global diffeomorphism it is called hyperregular.

The main advantadge of working in the cotangent space T^*Q is that it has a canonical symplectic form. First define the symplectic potential $\Theta \in \Omega^1(T^*Q)$ at each point $\mathbf{p} \in T^*Q$ as

$$\Theta_{\mathbf{p}}(X) = \mathbf{p}(T\pi_Q X) ,$$

for vectors $X \in T_{\mathbf{p}}(T^*Q)$ (hence $T\pi_Q X \in T_{\pi_Q(\mathbf{p})}Q$). Then, the canonical symplectic form is the 2-form

$$\Omega = -\mathbb{d}\Theta .$$

If L is hyperregular, the image of the fiber derivative FL is the whole T^*Q. In particular, the Hamiltonian is defined in all of T^*Q and the dynamical Hamiltonian equation

$$\imath_{\mathbb{Z}}\Omega = \mathrm{d}H \ ,$$

is also defined in all T^*Q. Since Ω is symplectic, this equation has a unique solution for $\mathbb{Z}$, which is

$$Z_q = D_p H \ , \ Z_p = -D_q H \ .$$

Here D_qH, D_pH are the usual partial Fréchet derivatives (with respect to q and p) in a Banach space [34], where if

$$\begin{aligned} \imath_q : U &\longrightarrow T^*Q \ , \\ \imath_p : Q^* &\longrightarrow T^*Q \ , \end{aligned}$$

are the canonical inclusions to the first and second factor on a chart $U \times Q^*$ of T^*Q, respectively, then

$$\begin{aligned} D_q f &= \mathrm{d}\left(f \circ \imath_q\right) \ , \\ D_p f &= \mathrm{d}\left(f \circ \imath_p\right) \ , \end{aligned}$$

so that

$$\mathrm{d}f(\mathbb{X}) = D_q f \cdot X_q + D_p f \cdot X_p \ .$$

The situation is quite different in the singular case. Singular Lagrangians are important because the physically relevant field theories are typically of this type. In this case, the image of the fiber derivative is not all T^*Q but a submanifold of it and we refer to $\mathfrak{Q} = FL(TQ) \subsetneq T^*Q$ as the primary constraint submanifold. In this case, the Hamiltonian function is only defined in $\mathfrak{Q}$ (although it can often be extended in multiple ways to a function in the whole T^*Q). Moreover, since the presymplectic system can only be defined in the domain of H, we need to use the presymplectic form $\omega = \jmath^*\Omega$ induced by the symplectic form Ω via the pullback of the inclusion $\jmath : \mathfrak{Q} \hookrightarrow T^*Q$. Note that even if ω is the pullback of a strongly non-degenerate symplectic form, its degree of degeneracy depends on the character of FL; in general it will only be presymplectic.

Hence, the presymplectic system of interest in physical theories is then $\left(\mathfrak{Q}, \omega, \mathrm{d}H\right)$, and the dynamical equation (2.1) reads in this case

$$\imath_{\mathbb{Z}}\omega = \mathrm{d}H \ . \tag{2.6}$$

Then, one can use the GNH algorithm described in Section 2.1 to find a suitable solution to the system.

A second option is working in the tangent bundle, which unlike the cotangent bundle, does not have a canonical symplectic structure. We then need an alternative way to build a (pre)symplectic structure for a given Lagrangian L. Let us first see some canonical constructions in TQ. Given a curve γ in Q, its canonical lift is the curve in TQ (or alternatively, the vector field of Q *along* γ) given by

$$\dot{\gamma}(t) = \frac{d}{d\varepsilon}\gamma(t+\varepsilon)\Big|_{\varepsilon=0} ,$$

i.e., built with the tangent vectors to γ at each of its points.

The double tangent bundle $TTQ \xrightarrow{\pi_{TQ}} TQ$ has a canonical subbundle defined by $V(TQ) := \ker T\pi_Q$, called the vertical subspace. For each $\mathrm{v} \in T_xQ$ tangent vector, one can define a vertical lift $\xi_{\mathrm{v}} : T_xQ \longrightarrow V_{\mathrm{v}}(TQ)$ to the fiber over v as

$$\xi_{\mathrm{v}}(\mathrm{w}) := \frac{d}{d\varepsilon}(\mathrm{v}+\varepsilon\mathrm{w})\Big|_{\varepsilon=0} ,$$

where $\mathrm{w} \in T_xQ$.

The almost tangent structure is the vector-valued 1-form $\mathbb{J}_{\mathrm{v}} : T_{\mathrm{v}}(TQ) \longrightarrow T_{\mathrm{v}}(TQ)$ — which can also be regarded as a (vertical) linear endomorphism — defined by [38], [39]

$$\mathbb{J}(\mathbb{Z}) := \xi \circ T\pi_Q(\mathbb{Z}) ,$$

for $\mathbb{Z} \in \mathfrak{X}(TQ)$. It is clear by the definition that $V(TQ) = \mathrm{Im}\,\mathbb{J} = \ker \mathbb{J}$ and hence $\mathbb{J}^2 = 0$. Since $\mathbb{J}$ is an endomorphism, it induces a derivative $\imath_{\mathbb{J}}$ of degree 0 defined by:

$$\imath_{\mathbb{J}}\alpha\,(X_1, \cdots, X_p) = \sum_{i=1}^{p} \alpha(X_1, \cdots, \mathbb{J}X_i, \cdots, X_p) ,$$

and also defines the vertical derivative

$$\mathbb{d}_{\mathbb{J}} := \imath_{\mathbb{J}}\mathbb{d} - \mathbb{d}\imath_{\mathbb{J}} .$$

Note that up to this point, the objects discussed are canonical in TQ. In the same way as in the canonical approach in the cotangent bundle, we need to introduce a Lagrangian function $L : TQ \longrightarrow \mathbb{R}$ (or an action from which a Lagrangian can be derived) to define and study physical systems. From this, the Lagrangian canonical 1-form and the (pre)symplectic Lagrangian form are respectively defined as

$$\theta_L := \mathbb{d}_{\mathbb{J}}L ,$$
$$\omega_L := -\mathbb{d}\theta_L = -\mathbb{d}\mathbb{d}_{\mathbb{J}}L . \tag{2.7}$$

The (pre)symplectic Lagrangian form ω_L is clearly closed. Again, the properties of L determine the character of these structures. If the Lagrangian L is regular, then ω_L is *symplectic* and the symplectic-Lagrangian equation

$$\imath_{\mathbb{Z}}\omega_L = \mathbb{d}E_L , \tag{2.8}$$

has a unique solution $\mathbb{Z}$ which also satisfies the *second order condition*

$$\mathbb{J}\mathbb{Z} = \mathbb{V} \ , \tag{2.9}$$

where $\mathbb{V}$ is the Liouville vector field on TQ, defined at each point $\mathrm{v} \in TQ$ by

$$\mathbb{V}_{\mathrm{v}} := \xi_{\mathrm{v}}(\mathrm{v}) \ , \tag{2.10}$$

and we write the energy (2.4) as

$$E_L = \imath_{\mathbb{V}} \mathrm{d\!l} L - L \ .$$

The fact that the unique solution $\mathbb{Z}$ of (2.8) with L regular satisfies the second order condition (2.9) easily follows by acting with $\imath_{\mathbb{J}}$ on both sides of (2.8) and then using Lemma 2 and Lemma 3, stated below, to rewrite each side

$$-\imath_{\mathbb{V}}\omega_L \overset{\text{Lemma 3}}{=} \imath_{\mathbb{J}} \mathrm{d\!l} E_L = \imath_{\mathbb{J}}\imath_{\mathbb{Z}}\omega_L \overset{\text{Lemma 2}}{=} -\imath_{\mathbb{J}\mathbb{Z}}\omega_L \implies \imath_{\mathbb{J}\mathbb{Z}-\mathbb{V}}\omega_L = 0 \ ,$$

and since ω_L is *symplectic* it follows that $\mathbb{Z}$ satisfies $\mathbb{J}\mathbb{Z} = \mathbb{V}$.

The second order condition (2.9) for the vector field $\mathbb{Z}$ can equivalently be written as

$$T_{\pi(Q)}\mathbb{Z} = \pi_{TQ}\mathbb{Z} \ .$$

In local coordinates (q, v), the expressions of $\mathbb{V}$ and $\mathbb{J}$ are

$$\begin{aligned} \mathbb{V}_{(q,\mathrm{v})} &= (0, \mathrm{v}) \ , \\ \mathbb{J}_{(q,\mathrm{v})}(\mathbb{X}) &= (0, X_q) \ , \end{aligned}$$

which implies that the second order condition (2.9) for a vector $\mathbb{X}$ reads $X_q = \mathrm{v}$. This guarantees that $\mathbb{X}$ is actually the double lift of a curve γ in Q (i.e., X_v is the second derivative of γ), hence the name second order.

If L is singular, ω_L will only be presymplectic and the equation (2.8) will not have in general a unique solution over all TQ. In such a situation it is possible to resort to the GNH algorithm for the presymplectic system (TQ, ω_L, E_L).

Note that both constructions are canonical for the pair (Q, L) (in the sense that given those objects, their constructions are natural), though in a different way. The cotangent space T^*Q has a canonical symplectic form Ω, which naturally induces a (pre)symplectic form in every submanifold of T^*Q via the pullback of the inclusion. Then, the relevant subaminfold is determined by the Lagrangian L through its fiber derivative. On the other hand, the tangent space TQ has no canonical symplectic form. In this case, the Lagrangian L is used jointly with the canonical structures of TQ to build a (pre)symplectic form.

The presymplectic system (TQ, ω_L, E_L) is equivalent to the variational principle induced from L in the following sense.

Proposition 1. *[40] Let $S : \mathcal{P}(Q) \longrightarrow \mathbb{R}$ be the action*

$$S(\gamma) = \int_{t_1}^{t_2} L(\dot{\gamma}(t)) \ \mathrm{dt} \ ,$$

defined for paths $\gamma(t)$ with fixed endpoints. The action has a stationary point if, and only if, there exists a vector field $\mathbb{Z} \in \mathfrak{X}(TQ)$ satisfying the Euler-Lagrange equation

$$\imath_{\mathbb{Z}}\omega_L = \mathrm{d} E_L \ , \tag{2.11}$$

and the second-order condition

$$\mathbb{J}\mathbb{Z} = \mathbb{V} \ .$$

Before proving the theorem let us state two lemmas.

Lemma 2. *([41] p.3759) $\imath_{\mathbb{J}\mathbb{X}}\omega_L = -\imath_{\mathbb{J}}\imath_{\mathbb{X}}\omega_L$.*

Proof. Let us first prove that $\imath_{\mathbb{J}}\omega_L = 0$. Indeed,

$$\imath_{\mathbb{J}}\omega_L = -\imath_{\mathbb{J}}\mathrm{d}\mathrm{d}_{\mathbb{J}}L = -(\mathrm{d}_{\mathbb{J}} - \mathrm{d}\imath_{\mathbb{J}})\mathrm{d}_{\mathbb{J}}L = \mathrm{d}\imath_{\mathbb{J}}\mathrm{d}_{\mathbb{J}}L = \mathrm{d}\imath_{\mathbb{J}}(\imath_{\mathbb{J}}\mathrm{d} + \mathrm{d}\imath_{\mathbb{J}})L = \mathrm{d}\imath_{\mathbb{J}}\imath_{\mathbb{J}}\mathrm{d}L \ ,$$

but since $\mathrm{d}L$ is a 1-form,

$$\imath_{\mathbb{J}}\imath_{\mathbb{J}}\mathrm{d}L(\mathbb{X}) = \mathrm{d}L(\mathbb{J}^2 X) = 0 \ ,$$

since $\mathbb{J}^2 = 0$.

Let $\mathbb{X}, \mathbb{Y} \in \mathfrak{X}(TQ)$. Since $\imath_{\mathbb{J}}\omega_L = 0$, we have

$$0 = (\imath_{\mathbb{J}}\omega_L)(\mathbb{X}, \mathbb{Y}) = \omega_L(\mathbb{J}\mathbb{X}, \mathbb{Y}) + \omega_L(\mathbb{X}, \mathbb{J}\mathbb{Y}) = (\imath_{\mathbb{J}\mathbb{X}}\omega_L + \imath_{\mathbb{J}}\imath_{\mathbb{X}})(\mathbb{Y}) \ ,$$

which proves the result. □

Lemma 3. $\imath_{\mathbb{J}}\mathrm{d}E_L = -\imath_{\mathbb{V}}\omega_L$

Proof. Let us first prove that

$$\mathrm{d}_{\mathbb{J}}\imath_{\mathbb{V}} + \imath_{\mathbb{V}}\mathrm{d}_{\mathbb{J}} = \imath_{\mathbb{J}} \ .$$

It is enough to do it for 0-forms and exact 1-forms since any differential form can be written as $\alpha = f_{i_1,\dots,i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ in a coordinate chart. For a 0-form $f \in \mathcal{C}^\infty(\mathcal{M})$,

$$\left(\mathrm{d}_{\mathbb{J}}\imath_{\mathbb{V}} + \imath_{\mathbb{V}}\mathrm{d}_{\mathbb{J}}\right) f = \imath_{\mathbb{V}}\imath_{\mathbb{J}}\mathrm{d}f = 0 = \imath_{\mathbb{J}} f \ .$$

For an exact 1-form $\mathrm{d}f \in \Omega^1(\mathcal{M})$, keeping in mind that $\mathbb{J}\mathbb{V} = 0$,

$$\begin{aligned}
&\left[\left(\mathrm{d}_{\mathbb{J}}\imath_{\mathbb{V}} + \imath_{\mathbb{V}}\mathrm{d}_{\mathbb{J}}\right)\mathrm{d}f\right](\mathbb{Y}) = \imath_{\mathbb{Y}}\left(-\imath_{\mathbb{V}}\mathrm{d}\imath_{\mathbb{J}} + \imath_{\mathbb{J}}\mathrm{d}\imath_{\mathbb{V}}\right)\mathrm{d}f = \imath_{\mathbb{Y}}\left(-\mathcal{L}_{\mathbb{V}}\imath_{\mathbb{J}} + \imath_{\mathbb{J}}\mathcal{L}_{\mathbb{V}}\right)\mathrm{d}f \\
&= \imath_{\mathbb{J}\mathbb{Y}}\mathrm{d}\imath_{\mathbb{V}}\mathrm{d}f - V\left(\imath_{\mathbb{J}\mathbb{Y}}\mathrm{d}f\right) + \imath_{\mathbb{J}[\mathbb{V},\mathbb{Y}]}\mathrm{d}f = [\mathbb{J}\mathbb{Y}, \mathbb{V}]f + (\mathbb{J}[\mathbb{V}, \mathbb{Y}])\, f \\
&= (\mathbb{J}\mathbb{Y})\, f = \left(\imath_{\mathbb{J}}\mathrm{d}f\right)(\mathbb{Y}) \ ,
\end{aligned}$$

since

$$\mathbb{J}\mathbb{Y} = \mathbb{J}[\mathbb{V}, \mathbb{Y}] + [\mathbb{J}\mathbb{Y}, \mathbb{V}] \ .$$

Using this result and the fact that for functions f the operator $\mathbb{d}_{\mathbb{J}}$ is just $\mathbb{d}_{\mathbb{J}} f = \imath_{\mathbb{J}} \mathbb{d} f$ we obtain

$$\begin{aligned} \imath_{\mathbb{J}} \mathbb{d} E_L = \mathbb{d}_{\mathbb{J}} E_L = \mathbb{d}_{\mathbb{J}} \left(\imath_{\mathbb{V}} \mathbb{d} L - L \right) &= \imath_{\mathbb{J}} \mathbb{d} L - \imath_{\mathbb{V}} \mathbb{d}_{\mathbb{J}} \mathbb{d} L - \mathbb{d}_{\mathbb{J}} L \\ &= -\imath_{\mathbb{V}} \mathbb{d}_{\mathbb{J}} \mathbb{d} L = \imath_{\mathbb{V}} \mathbb{d} \mathbb{d}_{\mathbb{J}} L = -\imath_{\mathbb{V}} \omega_L \ . \end{aligned}$$

□

Proof of Proposition 1. Let

$$\mathcal{E} = \imath_{\mathbb{Z}} \omega_L - \mathbb{d} E_L \ . \tag{2.12}$$

Consider a 1-parameter family of curves γ_λ in Q with fixed endpoints such that $\gamma_0 = \gamma$. The variation of the action along the one-parameter family is

$$\begin{aligned} \frac{dS(\gamma_\lambda)}{d\lambda}\Big|_{\lambda=0} &= \int_{t_1}^{t_2} \frac{d}{d\lambda} \left(L \circ \dot{\gamma}_\lambda \right) \Big|_{\lambda=0} \mathrm{d}t = \int_{t_1}^{t_2} \mathbb{d} L_{\dot{\gamma}_\lambda} \left(\frac{d}{d\lambda} \dot{\gamma}_\lambda \right) \Big|_{\lambda=0} \mathrm{d}t \\ &= \int_{t_1}^{t_2} \left(\imath_{\mathbb{Y}} \mathbb{d} L \right) (\dot{\gamma}(t)) \ \mathrm{d}t \ , \end{aligned}$$

where $\mathbb{Y} = \frac{d}{d\lambda} \dot{\gamma}_\lambda \in \mathfrak{X}(TQ)$. Also, since the endpoints of γ_λ are fixed for every λ and $\mathbb{J}\mathbb{Y}_{(\gamma_\lambda(t), \dot{\gamma}_\lambda(t))} = \left(0, \frac{d}{d\lambda}|_{\lambda=0} \gamma_\lambda(t)\right)$, then $\mathbb{J}\mathbb{Y} = 0$ at $\gamma_\lambda(t_1), \gamma_\lambda(t_2)$. By construction, $\mathbb{Z} = \ddot{\gamma}_\lambda$ satisfies the second order condition $\mathbb{J}\mathbb{Z} = \mathbb{V}$. Moreover, since $\mathbb{Z}, \mathbb{Y}$ generate flows in the t and λ parameters, respectively, and the action of the flows commute, $[\mathbb{Z}, \mathbb{Y}] = 0$.

Using the definitions of $\mathcal{E}$ and ω_L, and then using the expresion for the exterior derivative of a 1-form we can write

$$\begin{aligned} \imath_{\mathbb{Y}} \mathbb{d} L &\overset{(2.4)}{=} \imath_{\mathbb{Y}} \mathbb{d} \imath_{\mathbb{V}} \mathbb{d} L - \imath_{\mathbb{Y}} \mathbb{d} E_L \overset{(2.12)}{=} \imath_{\mathbb{Y}} \mathbb{d} \imath_{\mathbb{V}} \mathbb{d} L + \imath_{\mathbb{Y}} \mathcal{E} - \imath_{\mathbb{Y}} \imath_{\mathbb{Z}} \omega_L \\ &\overset{(2.7)}{=} \mathbb{Y} \left(\imath_{\mathbb{V}} \mathbb{d} L \right) + \imath_{\mathbb{Y}} \mathcal{E} + \mathbb{Z} \left(\imath_{\mathbb{Y}} \mathbb{d}_{\mathbb{J}} L \right) - \mathbb{Y} \left(\imath_{\mathbb{Z}} \mathbb{d}_{\mathbb{J}} L \right) - \imath_{[\mathbb{Z}, \mathbb{Y}]} \mathbb{d}_{\mathbb{J}} L \\ &\overset{\text{Lemma 2}}{=} \imath_{\mathbb{Y}} \mathcal{E} + \mathbb{Y} \left(\imath_{\mathbb{V} - \mathbb{J}\mathbb{Z}} \mathbb{d} L \right) + \mathbb{Z} \left(\imath_{\mathbb{J}\mathbb{Y}} \mathbb{d} L \right) - \imath_{[\mathbb{Z}, \mathbb{Y}]} \mathbb{d}_{\mathbb{J}} L \ . \end{aligned} \tag{2.13}$$

By means of these expressions we can now prove the implications of the proposition.

($\Longleftarrow$) Let us first prove the reverse implication. Assume there is a vector field $\mathbb{Z}$ satisfying $\mathcal{E} = 0$ and $\mathbb{J}\mathbb{Z} = \mathbb{V}$. Then, there exists a curve γ such that $\ddot{\gamma} = \mathbb{Z}$.

Then, by (2.13) we can write, since (2.9) and $[\mathbb{Z}, \mathbb{Y}] = 0$ hold,

$$\begin{aligned} \frac{dS(\gamma_\lambda)}{d\lambda}\Big|_{\lambda=0} &= \int_{t_1}^{t_2} \imath_{\mathbb{Y}} \mathbb{d} L \ \mathbf{d}t = \int_{t_1}^{t_2} \mathbb{Z} \left(\imath_{\mathbb{J}\mathbb{Y}} \mathbb{d} L \right) \ \mathbf{d}t = \int_{t_1}^{t_2} \frac{d}{dt} \left(\imath_{\mathbb{J}\mathbb{Y}} \mathbb{d} L \right) \ \mathbf{d}t \\ &= \left(\imath_{\mathbb{J}\mathbb{Y}} \mathbb{d} L \right) (\gamma(t_2)) - \left(\imath_{\mathbb{J}\mathbb{Y}} \mathbb{d} L \right) (\gamma(t_1)) = 0 \ , \end{aligned}$$

since $\mathbb{JY}$ vanishes at the endpoints. Hence, γ is a stationary curve of the action S.

($\Longrightarrow$) Assume that there is a curve γ which is a stationary point of S. Hence, any 1-parameter family of curves γ_λ such that $\gamma_0 = \gamma$ must satisfy that $\frac{dS(\gamma_\lambda)}{d\lambda}|_{\lambda=0} = 0$. By (2.13), we can rewrite this as

$$0 = \frac{dS(\gamma_\lambda)}{d\lambda}\Big|_{\lambda=0} = \int_{t_1}^{t_2} \imath_{\mathbb{Y}}\mathcal{E} \; \mathbf{d}t \ ,$$

since $\mathbb{JZ} = \mathbb{V}$, $[\mathbb{Z}, \mathbb{Y}] = 0$. From this we cannot conclude that $\mathcal{E} = 0$ because there are conditions on the vector field $\mathbb{Y}$. Note that $\mathbb{Y}$ is not a vertical vector since $T\pi_Q \mathbb{Y} = \frac{d}{d\lambda}\gamma_\lambda \neq 0$ and lifting it again $\dot{\left(\frac{d}{d\lambda}\gamma_\lambda\right)} = \frac{d}{d\lambda}\dot{\gamma}_\lambda = \mathbb{Y}$ we recover the original vector field $\mathbb{Y}$, hence $\mathbb{Y}$ has no vertical component. This means that if we add an arbitrary vertical vector field $\mathbb{JW}$ for arbitrary $\mathbb{W} \in \mathfrak{X}(TQ)$, the linear combination $\mathbb{Y} + \mathbb{JW}$ is an arbitrary vector field in TQ.

Using Lemmas 2, 3 we can write

$$\imath_{\mathbb{J}}\mathcal{E} = \imath_{\mathbb{J}}\imath_{\mathbb{Z}}\omega_L - \imath_{\mathbb{J}}\mathbf{d}E_L = -\imath_{\mathbb{JZ}}\omega_L + \imath_{\mathbb{V}}\omega_L = \imath_{\mathbb{V}-\mathbb{JZ}}\omega_L \ ,$$

and the integrand as

$$\imath_{\mathbb{Y}}\mathcal{E} = \imath_{\mathbb{Y}+\mathbb{JW}}\mathcal{E} - \imath_{\mathbb{W}}\imath_{\mathbb{V}-\mathbb{JZ}}\omega_L = \imath_{\mathbb{Y}+\mathbb{JW}}\mathcal{E} \ .$$

Then,

$$0 = \frac{dS(\gamma_\lambda)}{d\lambda}\Big|_{\lambda=0} = \int_{t_1}^{t_2} \imath_{\mathbb{Y}+\mathbb{JW}}\mathcal{E}\mathbf{d}t \ ,$$

where now $\mathbb{Y} + \mathbb{JW}$ is an arbitrary vector field, which implies that $\mathcal{E} = 0$. □

Proposition 1 says that the Euler-Lagrange equations obtained from the variational principle and the symplectic Lagrangian equations are equivalent as long as the second order condition is satisfied. More precisely, the Euler-Lagrange equations of a variational principle written in an invariant form (i.e., without coordinates) are the symplectic Lagrangian equation (2.11), and if a curve γ is a stationary point of the action, then the vector field $\mathbb{Z}$ along the curve $\dot{\gamma}(t) \subset TQ$ given by $\mathbb{Z}_{\dot{\gamma}(t)} = \ddot{\gamma}_{\dot{\gamma}(t)} := \left(\dot{\gamma}_{\gamma(t)}, \ddot{\gamma}_{\dot{\gamma}(t)}\right)$ is a solution to (2.11). Of course, by construction, $\mathbb{Z}$ satisfies the second order condition (2.9) since at the point $\dot{\gamma}(t) \in TQ$ its value is $(Z_q)_{\dot{\gamma}(t)} = \dot{\gamma}(t)$. Conversely, obtaining a solution to the symplectic Lagrangian equation (2.11) is not enough to provide a solution to its related variational principle. The obstruction is that the solution vector field $\mathbb{Z}$ may not be the (double) lift of a curve in Q. Consider a general curve ψ in TQ given by $\psi(t) = (q(t), \mathrm{v}(t))$ and its tangent curve $\dot{\psi}$ in TTQ given by $\dot{\psi}(t) = (q(t), \mathrm{v}(t); \dot{q}(t), \dot{\mathrm{v}}(t))$. If ψ is not the lift of any curve in Q, then $\dot{q}(t) \neq \mathrm{v}(t)$ and $\dot{\psi}$ does not satisfy the second order condition. Hence, it is necessary that both (2.11) and (2.9) are simultaneously satisfied in order to recover a curve which is solution to the variational principle.

Thus, while one can apply the GNH algorithm to the presymplectic system $\left(TQ, \omega_L, \mathrm{d}E_L\right)$, as a consequence of Proposition 1, there is an extra condition that we need to take into account, namely, the second order condition (2.9), that the GNH algorithm does not implement and that has to be separately taken into account. This can be simply done by applying the GNH algorithm but adding as an extra step condition (2.9):

$$\mathfrak{Q}_2 = \{\mathrm{v} \in \mathfrak{Q}_1 \mid \exists \mathbb{Z}_{\mathrm{v}} \text{ solving (2.11) and } \mathbb{J}\mathbb{Z}_{\mathrm{v}} = \mathbb{V}_{\mathrm{v}}\} \ .$$

Note that this additional constraint is particular from the tangent bundle and has no analogue in the cotangent bundle.

In mechanics, where configuration fields are finite dimensional, the construction proceeds as explained. However, most physical field theories are modelled on L^2 spaces. In particular, the presence of derivatives in the Lagrangian makes it necessary to use discontinuous functions. A solution to this problem is to restrict the domain of such functions to the so-called *manifold domains*, an approach introduced by Chernoff and Marsden [34]. A clear example where this is useful is given by the scalar field, further studied in Section 3.1 and in [35], [42]. In these cases, one should replace TQ by the corresponding manifold domain in the previous constructions, while keeping in mind that the appropriate manifold domain might not be reflexive, a property which is relevant for the constructions in the next section.

As a remark, it is interesting to compare the Hamiltonian and Lagrangian formalisms presented in this chapter. Since in physical theories both the Lagrangian and the Hamiltonian approaches are derived from the Lagrangian function, it is reasonable to ask whether they give rise to the same dynamics. This issue was first addressed in [35], [38] and later completely solved in [43]. The fiber derivative of the Lagrangian $FL : TQ \longrightarrow T^*Q$ is the map that connects both (pre)symplectic systems. When the Lagrangian is regular its fiber derivative is a local diffeomorphism, hence one can translate the symplectic structure from the cotangent to the tangent bundle and viceversa (via pullback or pushforward). However, this is not so clear in the singular case. The main result of [43] is that solutions to either (2.11) or (2.6) can be transformed to solutions to the other, and its constraints mapped into the others, making the dynamics equivalent.

2.3 The STL simplification

The procedure described in Sections 2.1 and 2.2 can be used to study physical systems. Nevertheless, in the form the constraints are written, it can be difficult to find the successive constraint submanifolds. Let us take the chance to describe the *geometric constraint algorithm* in tangent spaces TQ which also are STL manifolds (defined below), which simplifies the application of the GNH algorithm by characterizing the constraint submanifold as the zero set of some functions [44]. A pair

$(\mathfrak{Q},\omega)$ is said to be an *STL manifold* (named after Śniatycki, Tulczyjew and Lichnerowicz) if $\mathfrak{Q}$ is reflexive and, similarly to the map $\flat$ in T^*Q, the map $\flat$ induced by ω defined according to

$$\begin{aligned} \flat : & TTQ \longrightarrow T^*TQ \\ & \mathbb{X} \quad \mapsto \quad \imath_{\mathbb{X}}\omega_L \ , \end{aligned}$$

restricted to each fiber of $\mathfrak{Q}$ maps closed subspaces into closed subspaces. In a certain sense, the STL hypothesis ensures that some desirable properties of finite dimensional systems are still valid in infinite dimensions. The reason why these precise conditions are needed for field theories will become clear in the technical Proposition 4. Intuitively, we do not want the spaces to become too big in order for $\flat$ to be able to map to the adequate elements. Under these conditions the GNH algortihm in $\mathfrak{Q}$ can be simplified by characterising the submanifold $\mathfrak{Q}_1$ as the zero set of some functions. We will be denoting the successive constraint manifolds by $\mathcal{S}$.

As per the GNH algorithm, the first constraints to be taken into account are given by

$$\mathcal{S}_1 = \left\{ \mathrm{v} \in TQ \mid \left(\mathbb{d} E_L \right)_{\mathrm{v}} \in \operatorname{Im} \flat_{\mathrm{v}} \right\} .$$

In the STL case, one can actually simplify this condition into a form more suitable for practical computations.

Proposition 4. $\mathcal{S}_1 = \left\{ \mathrm{v} \in TQ \mid \left(\imath_{\mathbb{Z}} \mathbb{d} E_L \right)(\mathrm{v}) = 0 \ , \ \forall \mathbb{Z} \in \ker \omega_L \right\} .$

Proof.

- $(\subseteq)$ Let $\mathrm{v} \in \mathcal{S}_1$. Then, there exists a solution to (2.11), i.e., a vector $\mathbb{X} \in T_{\mathrm{v}}TQ$ such that $\imath_{\mathbb{X}}\omega_L = \mathbb{d} E_L$ is satisfied at v. But this implies that for $\mathbb{Z} \in \ker \omega$

$$0 = \imath_{\mathbb{Z}} \imath_{\mathbb{X}} \omega_L = \imath_{\mathbb{Z}} \mathbb{d} E_L \ .$$

- $(\supseteq)$ [37] Let $\mathrm{v} \in TQ$ such that $\left(\imath_{\mathbb{Z}} \mathbb{d} E_L \right)(\mathrm{v}) = 0 \ , \ \forall \mathbb{Z} \in \ker \omega_L$. If $V \subset TQ$ is a subspace, denote the annihilator of V by $V^{\vdash} \subset T^*Q$, the set of $\alpha \in T^*Q$ such that $\alpha(v) = 0$, $\forall v \in V$. Then, $\left(\mathbb{d} E_L \right)_{\mathrm{v}} \in (\ker \omega_{\mathrm{v}})^{\vdash}$.

  On the other hand, $\mathbb{Z} \in \ker \omega_L$ means that $\omega_L(\mathbb{Z}, \mathbb{X}) = 0$, $\forall \mathbb{X} \in TQ$, but naming $\alpha = \imath_{\mathbb{X}} \omega_L \in \operatorname{Im} \flat$ we can write $\alpha(\mathbb{Z}) = 0$, which means that $\mathbb{Z} \in (\operatorname{Im} \flat)^{\vdash}$, hence $\ker \omega_L = (\operatorname{Im} \flat)^{\vdash}$. Therefore $\left(\mathbb{d} E_L \right)_{\mathrm{v}} \in \left((\operatorname{Im} \flat_{\mathrm{v}})^{\vdash} \right)^{\vdash}$.

  Lemma 5. *If $\mathfrak{Q}$ is reflexive, $\left(\mathfrak{Q}^{\vdash} \right)^{\vdash} = \overline{\mathfrak{Q}}$.*

  Since TQ is an STL manifold, by Lemma 5, $\left(\mathbb{d} E_L \right)_{\mathrm{v}} \in \overline{\operatorname{Im} \flat_{\mathrm{v}}}$, and since the image of the fibers under $\flat$ are closed, then $\overline{\operatorname{Im} \flat_{\mathrm{v}}} = \operatorname{Im} \flat_{\mathrm{v}}$. Hence, $\mathrm{v} \in \mathcal{S}_1$.

□

Actually, the vertical vectors automatically satisfy the condition stated in Proposition 4. Let us introduce now the useful notation

$$\mathfrak{K} = \left\{ \mathbb{X} \in \mathfrak{X}(TQ) \mid \mathbb{J}\mathbb{X} \in \ker \omega_L \right\} .$$

Lemma 6. $\imath_{\mathbb{Z}} \mathrm{d} E_L = 0$, $\forall \mathbb{Z} \in \ker \omega_L \cap V(TQ)$

Proof. Since $\mathbb{Z}$ is vertical, it can be written as $\mathbb{Z} = \mathbb{J}\mathbb{X}$. Then, for $\mathbb{X} \in \mathfrak{K}$,

$$\mathrm{d} E_L(\mathbb{J}\mathbb{X}) = \imath_{\mathbb{X}} \imath_{\mathbb{J}} \mathrm{d} E_L = -\imath_{\mathbb{X}} \imath_{\mathbb{V}} \omega_L = -\imath_{\mathbb{X}} \imath_{\mathbb{J}\mathbb{Y}} \omega_L = -\imath_{\mathbb{X}} \imath_{\mathbb{J}} \imath_{\mathbb{Y}} \omega_L = (\imath_{\mathbb{J}\mathbb{X}} - \imath_{\mathbb{J}} \imath_{\mathbb{X}}) \, \imath_{\mathbb{Y}} \omega_L = 0 .$$

□

In the submanifold $\mathcal{S}_1$, equation (2.11) has solutions, but they do not satisfy the second order condition (2.9) in general. This is not addressed by the original GNH algorithm since it is an independent condition that must be specifically taken into consideration in the tangent bundle, hence next we impose this requirement. To this end we define the submanifold

$$\mathcal{S}_2 = \left\{ \mathrm{v} \in \mathcal{S}_1 \mid \exists \mathbb{Z} \in T_{\mathrm{v}} TQ \; : \; \imath_{\mathbb{Z}} \left(\omega_L \right)_{\mathrm{v}} = \left(\mathrm{d} E_L \right)_{\mathrm{v}} \text{ and } \mathbb{J}\mathbb{Z} = \mathbb{V}_{\mathrm{v}} \right\}$$

Proposition 7. *Let $\mathbb{Z}$ be a particular solution to* (2.11) *and $\mathbb{Y} \in \mathfrak{X}(TQ)$ such that $\mathbb{J}\left(\mathbb{Z} + \mathbb{Y}\right) = \mathbb{V}$. Then,*

$$\mathcal{S}_2 = \left\{ \mathrm{v} \in \mathcal{S}_1 \mid \left(\imath_{\mathbb{X}} \imath_{\mathbb{Y}} \omega_L \right)(\mathrm{v}) = 0 \; , \; \forall \mathbb{X} \in \mathfrak{K} \right\} \tag{2.14}$$

Proof.

- *The set $\mathcal{S}_2$ is independent of the choice of vector field $\mathbb{Y}$.* Choose two vector fields $\mathbb{Y}_1, \mathbb{Y}_2$, in such a way that $\mathbb{Z}+\mathbb{Y}_i$ satisfy the second order condition. Then,

  $$\mathbb{J}\left(\mathbb{Y}_2 - \mathbb{Y}_1\right) = \left(\mathbb{V} - \mathbb{J}\mathbb{Z}\right) - \left(\mathbb{V} - \mathbb{J}\mathbb{Z}\right) = 0 ,$$

  hence $\mathbb{Y}_2 - \mathbb{Y}_1$ is a vertical field, which in turn implies that there exists a vector field $\mathbb{U}$ such that $\mathbb{J}\mathbb{U} = \mathbb{Y}_2 - \mathbb{Y}_1$.

  $$\imath_{\mathbb{X}} \imath_{\mathbb{Y}_2 - \mathbb{Y}_1} \omega_L = \imath_{\mathbb{X}} \imath_{\mathbb{J}\mathbb{U}} \omega_L = -\imath_{\mathbb{X}} \imath_{\mathbb{J}} \imath_{\mathbb{U}} \omega_L = -\imath_{\mathbb{J}} \imath_{\mathbb{X}} \imath_{\mathbb{U}} \omega_L - \imath_{\mathbb{J}\mathbb{X}} \imath_{\mathbb{U}} \omega_L = -\imath_{\mathbb{J}\mathbb{X}} \imath_{\mathbb{U}} \omega_L = 0 ,$$

  since $\mathbb{X} \in \mathfrak{K}$, hence $\mathbb{J}\mathbb{X} \in \ker \omega_L$. Then, $\imath_{\mathbb{X}} \imath_{\mathbb{Y}_2} \omega_L = \imath_{\mathbb{X}} \imath_{\mathbb{Y}_1} \omega_L$ and $\mathcal{S}_2$ is independent of the choice of vector field $\mathbb{Y}$.

- ($\subseteq$) Let $\mathrm{v} \in \mathcal{S}_2$. Then, there are vectors $\mathbb{Z}$ and $\mathbb{Y}$ such that $\mathbb{J}\left(\mathbb{Z} + \mathbb{Y}\right) = \mathbb{V}_{\mathrm{v}}$ with $\mathbb{Y} \in \ker \omega$. Hence, $\imath_{\mathbb{Y}} \omega = 0$.

- ($\supseteq$) Let $\mathrm{v} \in \mathcal{S}_1$ such that $\left(\imath_{\mathbb{X}} \imath_{\mathbb{Y}} \omega \right)(\mathrm{v}) = 0$, $\forall \mathbb{X} \in \mathfrak{K}$ for $\mathbb{Y}$ satisfying the hypothesis.

□

The previous two conditions can be summarized in a single one.

Proposition 8. *For any* $\mathbb{U}$ *satisfying the second order condition* (2.9),

$$\mathcal{S}_2 = \left\{ \mathrm{v} \in TQ \mid \left(\imath_{\mathbb{X}} \left(\imath_{\mathbb{U}} \omega_L - \mathbb{d} E_L \right) \right) (\mathrm{v}) = 0 \ , \ \forall \mathbb{X} \in \mathfrak{K} \right\} \ .$$

Proof. We will see that this takes into account both conditions. Since $\ker \omega_L \subseteq \mathfrak{K}$, we distinguish two cases for $\mathbb{X} \in \mathfrak{K}$:

- If $\mathbb{X} \in \ker \omega_L$, then $\imath_{\mathbb{X}} \left(\imath_{\mathbb{U}} \omega_L - \mathbb{d} E_L \right) = -\imath_{\mathbb{X}} \mathbb{d} E_L$.
- If $\mathbb{X} \notin \ker \omega_L$, restrict to the submanifold $\mathcal{S}_1$ where there exist solutions to (2.11) and write $\mathbb{U} = \mathbb{Z} + \mathbb{Y}$, where $\mathbb{Z}$ is a particular solution to (2.11). Then, $\imath_{\mathbb{X}} \left(\imath_{\mathbb{U}} \omega_L - \mathbb{d} E_L \right) = \imath_{\mathbb{X}} \imath_{\mathbb{Y}} \omega_L$.

□

Finally, we have a characterization of the submanifold $\mathcal{S}_2$ where there exists a simultaneous solution $\mathbb{Z}$ to both equations (2.11) and (2.9). However, $\mathbb{Z}$ might not be tangent to $\mathcal{S}_2$ at all the points and, hence, not a vector field in $\mathcal{S}_2$. We now need to follow again the GNH algorithm. Let us then define the sequence of stabilization constraints

$$\mathcal{S}_{i+1} = \left\{ \mathrm{v} \in \mathcal{S}_i \mid \exists \mathbb{Z} \in T\mathcal{S}_i \text{ solution to (2.11) and (2.9)} \right\} \ .$$

These are the analogues of the constraint submanifolds (2.2) in the Hamiltonian case studied in Section 2.1. Hence, as explained there, the GNH algorithm reaches one of its possible outcomes depending on the behaviour of the generated chain of constraints.

2.4 Field equations method

Consider the family of actions $\tilde{S}_I : \tilde{Q} \to \mathbb{R}$ labelled by an interval I of $\mathbb{R}$, defined in some configuration space $\tilde{Q}$ by the expression

$$\tilde{S}_I(\tilde{q}) = \int_{I \times \Sigma} \mathcal{L} \, , \tag{2.15}$$

where $\mathcal{L} \in \Omega^n(\mathcal{M}) \otimes \mathcal{C}^\infty(\tilde{Q})$ is a top form in a spacetime $\mathcal{M}$ and also a function in $\tilde{Q}$. The space $\tilde{Q}$ induces a new configuration space Q of objects defined in Σ in the following way. Given a dynamical field $\tilde{q}$, denote by $\Pi_t(\tilde{q})$ and $\underline{\Pi}(\tilde{q})$ the two adapted components of $\tilde{q}$ as described in Appendix A. Recall that they are fields in $\mathcal{M}$. Define the curves $\gamma_t^{\tilde{q}}(s) = \jmath_s^* \Pi_t(\tilde{q})$, $\underline{\gamma}^{\tilde{q}}(s) = \jmath_s^* \underline{\Pi}(\tilde{q})$, where here $\jmath_s^*$ means the actual pullback for differential forms and the restriction for vector fields (since the adapted components are already tangent to the leaves). If $\tilde{Q} = \bigtimes_i \tilde{Q}_i$, define the configuration space

$$Q = \prod_i \left(\Pi_t(\tilde{Q}_i) \times \underline{\Pi}(\tilde{Q}_i) \right) \ ,$$

where the product is performed over all the dynamical fields $\tilde{q}$ of the configuration space $\tilde{Q}$. Then, one can rewrite $\tilde{S}_I(\tilde{q})$ as a function $S_I : \mathcal{P}(Q) \to \mathbb{R}$ of the space of curves γ in Q with fixed values in the extrema of I as

$$S_I(\gamma) = \int_I dt \int_\Sigma \left(\jmath_t^* i_{\partial_t} \mathcal{L} \right) (\dot{\gamma}) \ . \tag{2.16}$$

Note that the curve γ has components in each of the spaces $\Pi_t(\tilde{Q}_i)$ and $\underline{\Pi}(\tilde{Q}_i)$. Note that the Lagrangian function defined by the action (2.16) is $L = \int_\Sigma \jmath_t^* i_{\partial_t} \mathcal{L}$.

Since $\tilde{S}_I(\tilde{q}) = S_I(\gamma_t^{\tilde{q}}, \underline{\gamma}^{\tilde{q}})$, the two formulations are equivalent and the critical points of (2.15) are in one-to-one correspondence with stationary curves of (2.16).

The critical points of $\tilde{S}_I$ are characterized by

$$\mathbb{d}\tilde{S}_I = \int_{I \times \Sigma} E_{\tilde{q}} \wedge \mathbb{d}\tilde{q} = 0 \ ,$$

where $E_{\tilde{q}}$ are the Euler-Lagrange forms. Note that the Euler-Lagrange equations are $E_{\tilde{q}} = 0$.

As discussed after Proposition 1, a solution γ to the variational principle induces a vector field $\mathbb{Z} = \ddot{\gamma}$ that satisfies (2.11) and (2.9). Then, if Ψ is an Euler-Lagrange equation, the equations

$$\begin{aligned} \jmath_t^* \Psi &= 0 \ , \\ \jmath_t^* i_{\partial_t} \Psi &= 0 \ , \end{aligned} \tag{2.17}$$

give the solution to the symplectic Lagrangian equation (2.11).

In the particular case that all objects in the equations of motion are differential forms, this formulation is specially simple because the pullback in (2.17) interacts nicely with the natural operations of the exterior algebra. In the generic case, the first set of equations in (2.17) will produce the constraints. The second set will typically produce equations for the Hamiltonian vector field, since derivatives with respect to the curve parameter t might appear, and as discussed, derivatives of the fields correspond to their velocities and double derivatives of fields correspond to the evolution vector field $\mathbb{Z}$. Notice that, *a priori* the resulting vector field need not necessarily be tangent to the submanifold defined by the constraints, hence we still need to implement the chain of consistency constraints as in the GNH algorithm. Also note that the presymplectic form implicitly used in this procedure is the one described in Proposition 1. This method is useful because it gives a priori the constraints (and possibly in a simpler form than the obtained by using the other methods) and Hamiltonian equations in a much quicker and simpler way.

2.5 Lagrangians linear in velocities

Let us consider the special case of Lagrangians linear in velocities. These systems hvae been studied, for example, in [45]. Although this is a fairly restrictive condition, since most gravitational theories admit a first order formulation, they can be written in this form and as we will see, they have very interesting properties. Every 1-form $\mu \in \Omega^1(Q)$ in Q induces a linear function in the velocities in TQ given by $\hat{\mu}(\mathrm{v}) = \mu_{\pi_Q(\mathrm{v})}(\mathrm{v})$ by the linear nature of covectors. Then, an arbitrary function L in TQ linear in velocities can be written as

$$L = \hat{\mu} + \pi_Q^* h \ ,$$

with $\mu \in \Omega^1(Q)$ and $h \in \mathcal{C}^\infty(Q)$.

It is interesting to study both the Hamiltonian and Lagrangian symplectic settings for such a Lagrangian and to compare them. First, the energy is

$$\begin{aligned} \mathbb{d}L &= \mathbf{d}\mu \wedge \mathbb{d}q + \mu_q(\mathbb{d}\mathrm{v}) + \pi_Q^* \mathbb{d}h \ , \\ \imath_{\mathbb{V}} \mathbb{d}L &= \hat{\mu} \ , \\ E_L &= -\pi_Q^* h \ . \end{aligned}$$

In the Hamiltonian scheme one needs to compute the fiber derivative and find the Hamiltonian function

$$\begin{aligned} FL(\mathrm{v})(\mathrm{w}) &= \mu_q(\mathrm{w}) \ , \\ H &= -\pi_Q^* h \ . \end{aligned}$$

Then, the canonical symplectic form and its pullback are computed as

$$\begin{aligned} &\Omega(\mathbb{Z}, \mathbb{Y}) = \sum_i \big(\mathbf{Y}_i(Z^i) - \mathbf{Z}_i(Y^i) \big) \ , \\ &\mathbf{X}_i(\mathrm{w}) = \mathbf{d}\mu(X^i, \mathrm{w}) \ , \\ &\omega(\mathbb{Z}, \mathbb{Y}) = \sum_i \big(\mathbf{d}\mu(Y^i, Z^i) - \mathbf{d}\mu(Z^i, Y^i) \big) = -2 \sum_i \big(\mathbf{d}\mu(Z^i, Y^i) \big) \ , \\ &\omega = -\pi_Q^* \mathbf{d}\mu \ . \end{aligned}$$

Hence, the Hamiltonian equation (2.6) in this case reads

$$\imath_{\mathbb{Z}} \pi_Q^* \mathbf{d}\mu = \pi_Q^* \mathbb{d}h \ .$$

In the symplectic Lagrangian scheme one needs to compute the symplectic structure associated with the Lagrangian

$$\begin{aligned} \mathbb{d}_{\mathbb{J}} L &= \mu_q(\mathbb{d}q) \ , \\ \omega_L &= -\mathbf{d}\mu(\mathbb{d}q) = -\pi_Q^* \mathbf{d}\mu \ . \end{aligned}$$

Both ω_L and E_L are pullbacks by the projection π_Q to TQ of objects in Q.

The first step in the algorithm that we are going to use here is given by Proposition 4. In this case, the condition is

$$\imath_{\mathbb{Z}} \pi_Q^* \mathbb{d} h = 0 \ , \ \forall \mathbb{Z} \in \ker \pi_Q^* \mathbf{d}\mu \ .$$

However, due to Lemma 6, the vertical vectors do not impose any condition. Thus, we can only consider 'horizontal' vectors, in which case we can project the equation to

$$\imath_{\mathbb{Z}} \mathbb{d} h = 0 \ , \ \forall \mathbb{Z} \in \ker \mathbf{d}\mu \ ,$$

hence, the dynamical constraints are functions in Q and do not involve the velocities.

Next, we must implement the constraints given by (2.14), which in this case read

$$\imath_{\mathbb{X}} \imath_{\mathbb{Y}} \pi_Q^* \mathbf{d}\mu = 0 \ , \ \forall \mathbb{X} \colon \mathbb{J}\mathbb{X} \in \ker \pi_Q^* \mathbf{d}\mu \text{ and } \mathbb{J}\left(\mathbb{Z} + \mathbb{Y}\right) = \mathbb{V} \ ,$$

for $\mathbb{Z}$ a particular solution to (2.11). Since $\imath_{\mathbb{J}} \pi_Q^* \mathbf{d}\mu = 0$, the condition on $\mathbb{X}$ is always satisfied. Then, the second order condition translates to $Z_q + Y_q = \mathrm{v}$ for $Y_q \in \ker \mathbf{d}\mu$.

Finally, the Hamiltonian equation (2.11) in this case has the form

$$\begin{aligned} \left(D_q D_{\mathrm{v}} \hat{\mu} - D_{\mathrm{v}} D_q \hat{\mu}\right) Z_q &= D_q h \ , \\ Z_{\mathrm{v}} &\text{ arbitrary} \ , \end{aligned}$$

which should be solved together with the second order condition. Nevertheless this is still not the solution to the problem, since the issue of the tangency remains and as discussed so far, in general one will need to impose further constraints to ensure tangency of the Hamiltonian vector field to the constraint submanifold.

Chapter 3

Gravitational theories in tetrad and connection variables

This chapter is devoted to presenting some important examples of physical theories and their Hamiltonian analysis by means of the methods described in Chapter 2. In particular, the focus is set on a varied selection of gravitational theories of interest. Nevertheless, we start the list of examples with a very important but non-gravitational theory: the scalar field. Since it is of a very different nature than the rest of examples considered here, it is worth studying since it will provide further insight into the Hamiltonian methods.

Recall that we assume that the spacetimes $\mathcal{M}$ are manifolds without boundary.

3.1 Scalar fields

The action of the free, massless scalar field ϕ on the Minkowski spacetime $\mathcal{M}$ is

$$S_{\text{Scalar}}(\phi) = -\frac{1}{2}\int_{\mathcal{M}} \mathbf{d}\phi \wedge *\mathbf{d}\phi \ ,$$

where $*$ is the Hodge dual defined by a background (i.e., non-dynamical) Lorentzian metric g of $\mathcal{M}$.

Varying the action we obtain

$$\begin{aligned}\frac{d}{d\varepsilon}S_{\text{Scalar}}(\phi+\varepsilon\delta\phi)\Big|_{\varepsilon=0} &= -\frac{d}{d\varepsilon}\frac{1}{2}\int_{\mathcal{M}}(\mathbf{d}\phi+\varepsilon\mathbf{d}\delta\phi)\wedge *\,(\mathbf{d}\phi+\varepsilon\mathbf{d}\delta\phi)\Big|_{\varepsilon=0}\\ &= -\int_{\mathcal{M}}\mathbf{d}\delta\phi\wedge *\mathbf{d}\phi = \int_{\mathcal{M}}\delta\phi\ \mathbf{d}*\mathbf{d}\phi\ ,\end{aligned}$$

hence the equations of motion are

$$\mathbf{d}*\mathbf{d}\phi = 0\ . \tag{3.1}$$

We will now choose an inertial foliation Σ_t. A similar discussion can be done with a general foliation, however, that would give rise to a time-dependent Lagrangian, for which the methods discussed in the previous chapter are not appropriate. Since this example is meant to illustrate some points in the Hamiltonian analysis, we will restrict ourselves to this simpler case. The Lagrangian is

$$L_{\text{Scalar}}(\mathrm{v}) = \frac{1}{2}\int_\Sigma (\mathrm{v}_\phi \wedge \#\mathrm{v}_\phi - \mathrm{d}\phi \wedge \#\mathrm{d}\phi) \ ,$$

where $\#$ is the Hodge dual associated to the pullback of g to Σ. Note that this depends on the foliation chosen, in particular, on the choice of lapse and shift functions, which ultimately is a choice on how to decompose the metric of $\mathcal{M}$. In this case a lapse $N = 1$ and a shift $\mathbf{N} = 0$ were chosen. The configuration space of this Lagrangian is not straightforward. Note that it is not $L^2(\Sigma)$ as one might initially think, since the field ϕ appears differentiated, operation that is not defined for generic elements of $L^2(\Sigma)$. It seems then reasonable to require $Q = H^1(\Sigma)$ (the first Sobolev space where it makes sense to talk about first derivatives), however, note that the velocities need not be differentiable, hence we would unreasonably restrict the domain of our Lagrangian if we used $TQ = H^1(\Sigma) \times H^1(\Sigma)$. An appropriate balance is reached by using the so called *manifold domains*, introduced by Chernoff and Marsden in [34], which are fiber bundles with a certain restriction on the base while keeping the fibers untouched. In our case, the appropriate manifold domain to use is $T_{H^1}L^2(\Sigma) := H^1(\Sigma) \times L^2(\Sigma)$, with $H^1(\Sigma)$ in the base to ensure ϕ is differentiable but $L^2(\Sigma)$ in the fibers since there is no need to restrict it further.

3.1.1 Hamiltonian analysis

The Hamiltonian analysis of the scalar field is extensively discussed in [35], [42]. The energy (2.4) is

$$E_{L_{\text{Scalar}}} = \frac{1}{2}\int_\Sigma (\mathrm{v}_\phi \wedge \#\mathrm{v}_\phi + \mathrm{d}\phi \wedge \#\mathrm{d}\phi) \ ,$$

and the fiber derivative (2.3) is $FL : H^1(\Sigma) \times L^2(\Sigma) \to H^1(\Sigma) \times L^2(\Sigma)^* \cong H^1(\Sigma) \times L^2(\Sigma)$ defined by

$$\mathbf{p}_\phi(\mathrm{w}) := FL_{\text{Scalar}}(\mathrm{v})(\mathrm{w}) = \int_\Sigma \mathrm{v} \wedge \#\mathrm{w} \ ,$$

hence the primary constraint submanifold is $\mathfrak{Q} = H^1(\Sigma) \times L^2(\Sigma)^* \cong H^1(\Sigma) \times L^2(\Sigma) = T_{H^1(\Sigma)}L^2(\Sigma)$. Note that the Riesz-Fréchet representative (in $L^2(\Sigma)$) of $\mathbf{p}_\phi$ is v. For a generic point $\mathbf{p} \in T^*Q$ let us denote its Riesz-Fréchet representative as $p \in L^2(\Sigma)$. Then, the Hamiltonian (2.5) is

$$H(\mathbf{p}) = \frac{1}{2}\int_\Sigma p \wedge \#p + \mathrm{d}\phi \wedge \#\mathrm{d}\phi \ ,$$

whose differential is

$$\mathbb{d}H(\mathbb{Y}) = \int_\Sigma Y_p \, \# p - Y_\phi \, \mathrm{d}\#\mathrm{d}\phi \; . \tag{3.2}$$

The canonical symplectic form of T^*Q is

$$\Omega(\mathbb{X}, \mathbb{Y}) = \mathbf{Y}_\phi(X_\phi) - \mathbf{X}_\phi(Y_\phi) \; ,$$

and its pullback to the primary constraint submanifold is

$$\omega(\mathbb{Z}, \mathbb{Y}) = \int_\Sigma Y_p \, \# Z_\phi - Z_p \, \# Y_\phi \; , \tag{3.3}$$

with $Z_\phi, Y_\phi \in H^1(\Sigma)$ while $Z_p, Y_p \in L^2(\Sigma)$.

Equating (3.2) and (3.3) we see that the Hamiltonian vector field is given by

$$Z_\phi = p \; , \tag{3.4}$$

$$Z_p = -\#\mathrm{d}\#\mathrm{d}\phi =: \Delta\phi \; , \tag{3.5}$$

where Δ stands for the Laplacian of Σ. It is important to notice that the form of the Hamiltonian vector field introduces a new subtle constraint. Recall that $\phi \in H^1(\Sigma)$, however, (3.5) requires that the Laplacian of ϕ exists, which is only possible in the submanifold $H^2(\Sigma) \subset H^1(\Sigma)$. In turn, because of (3.4), then we must have $p \in H^1(\Sigma)$, giving rise to the constraint submanifold

$$\mathfrak{Q}_1 = \{(\phi, p) \in H^2(\Sigma) \times H^1(\Sigma)\} \; .$$

It remains to check the tangency conditions. Since the closure of $\mathfrak{Q}_1$ in $\mathfrak{Q}$ is $\overline{\mathfrak{Q}_1} = H^1(\Sigma) \times L^2(\Sigma)$, it is clear that $\mathbb{Z} \in \overline{T}\mathfrak{Q}_1$ and the GNH algorithm stops. It is interesting to see in this example why this modified tangency condition is useful. Were the condition of strict tangency, the vector field $\mathbb{Z}_{\mathbf{p}} = (p, \Delta\phi) \notin T_{\mathbf{p}}(H^2(\Sigma) \times H^1(\Sigma)) = H^2(\Sigma) \times H^1(\Sigma)$ because $\Delta\phi \in L^2(\Sigma)$ but in general not in $H^1(\Sigma)$. Hence, we should further restrict to

$$\mathfrak{Q}_2 = \{(\phi, p) \in H^3(\Sigma) \times H^2(\Sigma)\} \; ,$$

but this causes the same problem again, since now $\Delta\phi \in H^1(\Sigma)$ but by strict tangency it should be in $H^2(\Sigma)$. This generates an infinite chain of constraint submanifolds yielding as final constraint submanifold $\mathcal{C}^\infty(\Sigma) \times \mathcal{C}^\infty(\Sigma)$ with the induced topology, which makes it a Fréchet (rather than a Banach) manifold [35]. This is inconvenient because many useful properties and theorems available in Banach spaces are not valid in Fréchet spaces.

3.1.2 Equations of motion analysis

By decomposing the equations of motion (3.1) one obtains

$$\mathbf{d} * \mathbf{d}\phi = \mathbf{d}t \wedge \left(-\mathcal{L}_{\partial_t}\left(\#\dot{\phi}\right) + \underline{\mathbf{d}}\#\underline{\mathbf{d}}\phi\right) - \underline{\mathbf{d}}\#\dot{\phi} = \mathbf{d}t \wedge \left(-\ddot{\phi}\ \mathsf{vol}_\Sigma + \underline{\mathbf{d}}\#\underline{\mathbf{d}}\phi\right) \ .$$

Since this decomposition only has a $\mathbf{d}t$ component, we obtain no constraints and the equation that we get for the evolution vector field by projecting to Σ is

$$Z_{\mathrm{v}}\ \mathsf{vol}_\Sigma = \dot{Z}_\phi\ \mathsf{vol}_\Sigma = \mathrm{d}\#\mathrm{d}\phi \ .$$

By taking the Hodge dual $\#$, we obtain

$$-Z_{\mathrm{v}} = \#\mathrm{d}\#\mathrm{d}\phi \ ,$$

which together with the equation

$$Z_\phi = \mathrm{v} \ ,$$

is equivalent to the equation (3.5) found in the previous section. The tangency analysis proceeds essentially as before.

3.2 Chern-Simons

In 1974, Chern and Simons studied invariant polynomials of the curvature in a principal bundle [46], obtaining as a particular result invariants of 3-dimensional manifolds. The constructions and the result are purely geometrical and topological in nature, however it was quickly realized that they could be leveraged in physics. The invariant quantity would be later known as the Chern-Simons action and attracted attention in the context of gravitational field theories and specially in string theory, where topological invariants are relevant. In 3-dimensional manifolds, it turns out that adequately written, the Chern-Simons action describes any 3-dimensional gravitational theory, making it play a very important role in this area.

In order to present the necessary elements for the Chern-Simons theory, we first review some ideas about Lie groups. A good presentation of these topics is made in [47]. Let $\mathcal{G}$ be a Lie group, $\mathfrak{g}$ its Lie algebra with Lie bracket $[\cdot,\cdot]$ and $\langle\cdot,\cdot\rangle$ be a $\mathcal{G}$-invariant non-degenerate metric on $\mathfrak{g}$. The invariance of the metric means

$$\langle \mathrm{Ad}(g)v, \mathrm{Ad}(g)w\rangle = \langle v, w\rangle \ , \ \forall g \in \mathcal{G} \ , \ v, w \in \mathfrak{g} \ ,$$

where the map $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ is the differential at the identity of the map $AD_g : \mathcal{G} \to \mathcal{G}$ defined by $AD_g(h) = hgh^{-1}$. By computing this differential one obtains that $\mathcal{G}$-invariance of the metric implies

$$\langle [T, v], w\rangle + \langle v, [T, w]\rangle = 0 \ , \quad \forall T, v, w \in \mathfrak{g} \ . \tag{3.6}$$

The invariant bilinear forms of $\mathfrak{g}$ are just the quadratic Casimir operators of $\mathfrak{g}$.

The Lie bracket of $\mathfrak{g}$ induces a multiplication operation on $\mathfrak{g}$-valued forms. Let $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$ be $\mathfrak{g}$-valued p and r-forms respectively. Define the $\mathfrak{g}$-valued $(p+r)$-form $[\![\boldsymbol{\omega}_1, \boldsymbol{\omega}_2]\!]$ as

$$[\![\boldsymbol{\omega}_1, \boldsymbol{\omega}_2]\!](X_1, ..., X_{p+r}) = \sum_{\sigma \in S_{p+r}} [\boldsymbol{\omega}_1(X_{\sigma(1)}, ..., X_{\sigma(p)}), \boldsymbol{\omega}_2(X_{\sigma(p+1), ..., X_{\sigma(p+r)}})] \ .$$

Note that this operation is bigraded

$$[\![\boldsymbol{\omega}_1, \boldsymbol{\omega}_2]\!] = (-1)^{pr+1} [\![\boldsymbol{\omega}_2, \boldsymbol{\omega}_1]\!] \ .$$

In particular, for a 1-form $\boldsymbol{\omega}$, we have

$$[\![\boldsymbol{\omega}, \boldsymbol{\omega}]\!](X, Y) = [\boldsymbol{\omega}(X), \boldsymbol{\omega}(Y)] - [\boldsymbol{\omega}(Y), \boldsymbol{\omega}(X)] = 2[\boldsymbol{\omega}(X), \boldsymbol{\omega}(Y)] \ .$$

If we choose a basis T_i of $\mathfrak{g}$, one can write $\mathfrak{g}$-valued differential forms as the sum $\boldsymbol{\omega} = \boldsymbol{\omega}^i T_i$, where now each $\boldsymbol{\omega}^i$ is a differential form. As a consequence,

$$\begin{aligned} [\![\boldsymbol{\omega}_1, \boldsymbol{\omega}_2]\!] &= \boldsymbol{\omega}_1^i \wedge \boldsymbol{\omega}_2^j \ [T_i, T_j] \ , \\ \langle \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \rangle &= \boldsymbol{\omega}_1^i \wedge \boldsymbol{\omega}_2^j \ \langle T_i, T_j \rangle \ . \end{aligned}$$

Also, we denote the components of the metric $\langle \cdot, \cdot \rangle$ in the basis T_i as

$$g_{ij} = \langle T_i, T_j \rangle \ .$$

A clarification is in order here. In some texts on this topic where $\mathcal{G}$ is a matrix group one sees expressions such as $\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2$ with $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$ two Lie algebra valued forms. This actually means

$$\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2 = \boldsymbol{\omega}_1^i \wedge \boldsymbol{\omega}_2^j \ T_i T_j \ ,$$

where now $T_i T_j$ is the matrix product. Note that since the matrix product was used instead of the Lie bracket, $\boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2$ is not, in general, a Lie algebra valued form (at least in $\mathfrak{g}$). Then, if $\boldsymbol{\omega}$ is a Lie algebra valued 1-form, by the antisymmetry of the wedge product, one has the relation

$$\boldsymbol{\omega} \wedge \boldsymbol{\omega} = \frac{1}{2} [\![\boldsymbol{\omega}, \boldsymbol{\omega}]\!] \ .$$

The *Maurer-Cartan form* of a group $\mathcal{G}$ is the $\mathfrak{g}$-valued 1-form θ defined by

$$\begin{aligned} \theta_g : T_g\mathcal{G} &\to T_e\mathcal{G} \cong \mathfrak{g} \\ v &\mapsto (L_{g^{-1}})_* v \ , \end{aligned}$$

where L_g is the left and R_g the right multiplication of $\mathcal{G}$. It satisfies the properties

$$\begin{aligned} L_g^* \theta_h &= \theta_{gh} \ , \\ R_g^* \theta_h &= \mathrm{Ad}_{g^{-1}} \theta_{hg} \ , \end{aligned}$$

and the structural equation

$$\mathrm{d}\theta + \frac{1}{2}[\![\theta, \theta]\!] = 0 \ .$$

A $\mathcal{G}$-connection $\mathbf{A}$ in a principal $\mathcal{G}$-bundle $P \xrightarrow{\pi} \mathcal{M}$ can be understood as having a Maurer-Cartan form θ_x in each fiber $P_x = \pi^{-1}(x)$ for each $x \in \mathcal{M}$. Then, if $\imath_x : P_x \hookrightarrow P$ is the inclusion of the fiber, $\theta_x = \imath_x^* \mathbf{A}$. Clearly,

$$R_g^* \mathbf{A} = \mathrm{Ad}_{g^{-1}} \mathbf{A} \ .$$

The curvature of the connection $\mathbf{A}$ is

$$\mathbf{F} = \mathbf{dA} + \frac{1}{2}[\![\mathbf{A}, \mathbf{A}]\!] \ .$$

Note that due to the structural equation of the Maurer-Cartan form, the pullback of the curvature to each fiber $\imath_x^* \mathbf{F} = 0$ vanishes. Then, $\mathbf{F}$ is actually horizontal.

We are interested in $\mathcal{G}$-invariant forms $f(\mathbf{F}, \cdots, \mathbf{F})$ on $\mathfrak{g}$ of the curvature of a connection. Such forms are closed, hence, they define characteristic cohomology classes in P. The relevant property for field theory is that they are actually exact and their antiderivatives are also invariants, in some sense that will be made explicit later on, and which are obtained by integrating a certain function in the space of connections of P. These are the contents of the following theorem.

Theorem 9 (Chern-Simons [46]). *Let f be an invariant symmetric multilinear map $f : \mathfrak{g}^k \to \mathbb{R}$ with $k \in \mathbb{N}$ and $\mathbf{A}$ a connection on a principal $\mathcal{G}$-bundle P over $\mathcal{M}$. Write $\mathbf{A}_t = t\mathbf{A}$ and*

$$\mathbf{F}_t = \mathbf{dA}_t + \frac{1}{2}[\![\mathbf{A}_t, \mathbf{A}_t]\!] \ .$$

Then, the $(2k-1)$-form $\omega_k(\mathbf{A}) \in \Omega^{2k-1}(P)$ defined by

$$\omega_k(\mathbf{A}) := k \int_0^1 f(\mathbf{A}, \mathbf{F}_t, \cdots, \mathbf{F}_t)\mathrm{d}t \ ,$$

satisfies

$$f(\mathbf{F}, \cdots, \mathbf{F}) = \mathbf{d}\omega_k(\mathbf{A}) \ .$$

The proof and further considerations can be found in [46], [48].

Let us continue the construction of the action in the particular case of a 3-dimensional manifold $\mathcal{M}$ (hence $k = 2$), where $\omega_{\mathrm{CS}} := \omega_2$ is called the (3-dimensional) *Chern-Simons form.* Since $\omega_{\mathrm{CS}} \in \Omega^3(P)$ is a form in the principal bundle P, in order to proceed with the variational treatment we would like to pull it back and integrate it on $\mathcal{M}$ first. For simplicity, let us restrict the discussion to trivializable bundles

P, for which global sections exist (a similar result exists for general bundles). Let $\sigma : \mathcal{M} \to P$ be a section and define

$$S_{\text{CS}}(\sigma, \mathbf{A}) = \int_{\mathcal{M}} \sigma^* \omega_{\text{CS}}(\mathbf{A}) \ .$$

We need to understand how this quantity depends on the section chosen.

A gauge transformation is a fiber preserving map $\varphi : P \to P$, which can be characterized by a map $g : P \to \mathcal{G}$ defined by $\varphi(p) = p \lhd g(p)$, where $\lhd$ denotes the right action of the bundle. Any two sections $\sigma_{1,2} : \mathcal{M} \to P$ are related by a gauge transformation φ_{21} according to $\sigma_2 = \varphi_{21} \circ \sigma_1$, hence the action changes as

$$S_{\text{CS}}(\sigma_2, \mathbf{A}) = S_{\text{CS}}(\varphi_{21} \circ \sigma_1, \mathbf{A}) = S_{\text{CS}}(\sigma_1, \varphi_{21}^* \mathbf{A}) \ ,$$

so we need to compute the pullback of the connection by a gauge transformation. Since

$$\begin{aligned} \varphi^* \mathbf{A} &= \text{Ad}_{g^{-1}} \mathbf{A} + g^* \theta \ , \\ \varphi^* \mathbf{F} &= \text{Ad}_{g^{-1}} \mathbf{F} \ , \end{aligned}$$

we get

$$\varphi^* \omega_{\text{CS}}(\mathbf{A}) = \omega_{\text{CS}}(\mathbf{A}) + \mathbf{d} \langle \text{Ad}_{g^{-1}} \mathbf{A}, g^* \theta \rangle - \frac{1}{6} \langle g^* \theta, [\![g^* \theta, g^* \theta]\!] \rangle \ .$$

Then,

$$S_{\text{CS}}(\varphi \circ \sigma, \mathbf{A}) = S_{\text{CS}}(\sigma, \mathbf{A}) - \frac{1}{6} \int_{\mathcal{M}} \langle g^* \theta, [\![g^* \theta, g^* \theta]\!] \rangle \ ,$$

so that the difference between the actions induced by two different sections is the integral of $g^* \omega_{\text{CS}}(\theta) \in \Omega^3(\mathcal{M})$. At this point, the usual requirement for the metric $\langle \cdot, \cdot \rangle$ is that for any closed 3-submanifold $X \subset \mathcal{G}$ of $\mathcal{G}$, the integral [49], [50]

$$\int_X \omega_{\text{CS}}(\theta) \in \mathbb{Z} \ ,$$

is an integer. In the end this means that

$$S_{\text{CS}}(\mathbf{A}) = S_{\text{CS}}(\sigma, \mathbf{A}) \pmod 1 \ .$$

Hence, $S_{\text{CS}}(\mathbf{A})$ is an invariant in $\mathbb{R}/\mathbb{Z}$.

The construction of the invariant action can be extended to any odd dimension $2k-1$ by choosing an appropriate $f : \mathfrak{g}^k \to \mathbb{R}$. Then, since $f(\mathbf{F}, \cdots, \mathbf{F})$ is a $2k$-form, its antiderivative will be the $(2k-1)$-dimensional Chern-Simons form.

Let us now consider the particular case of the function $f(A, B) = \langle A, B \rangle$ (which is indeed invariant since $\langle \cdot, \cdot \rangle$ is) in a 3-dimensional $\mathcal{M}$. By Theorem 9, $f(\mathbf{F}, \mathbf{F}) = \langle \mathbf{F}, \mathbf{F} \rangle$ is exact and its antiderivative is

$$\omega_{\text{CS}}(\mathbf{A}) = 2 \int_0^1 f(\mathbf{A}, \mathbf{F}_t) \text{d}t = 2 \int_0^1 \langle \mathbf{A}, t \mathbf{d} \mathbf{A} + \frac{t^2}{2} [\![\mathbf{A}, \mathbf{A}]\!] \rangle \text{d}t = \langle \mathbf{A}, \mathbf{d} \mathbf{A} + \frac{1}{3} [\![\mathbf{A}, \mathbf{A}]\!] \rangle \ .$$

From now on, denote by $\mathbf{A} \in \Omega(\mathcal{M}) \otimes \mathfrak{g}$ a $\mathfrak{g}$-valued connection 1-form on $\mathcal{M}$ (rather than a connection on P). The (3-dimensional) Chern-Simons action is

$$S_{\text{CS}}(\mathbf{A}) = \frac{1}{2} \int_{\mathcal{M}} \langle \mathbf{A}, \mathbf{dA} + \frac{1}{3} [\![\mathbf{A}, \mathbf{A}]\!] \rangle \ . \tag{3.7}$$

Note that, although the global factor is determined by the integrality requirement for the metric, in the variational approach it is unimportant since it does not affect the field equations. Nevertheless, we follow the convention used in, for instance [51], with a global factor of $\frac{1}{2}$. In components, the action reads

$$S_{\text{CS}}(\mathbf{A}) = \frac{1}{2} \int_{\mathcal{M}} g_{ij} \left(\mathbf{A}^i \wedge \mathbf{dA}^j + \frac{1}{3} f^j{}_{kl} \mathbf{A}^i \wedge \mathbf{A}^k \wedge \mathbf{A}^l \right) \ ,$$

where $f^j{}_{kl}$ are the structure constants of $\mathfrak{g}$ in the chosen basis T_i and $[T_i, T_j] = f^k{}_{ij} T_k$.

Varying the action with respect to the field $\mathbf{A}$ we obtain

$$\frac{d}{d\epsilon} S_{\text{CS}}(\mathbf{A} + \epsilon \delta \mathbf{A}) \Big|_{\epsilon=0} = \int_{\mathcal{M}} g_{ij} \delta \mathbf{A}^i \wedge \mathbf{F}^j \ ,$$

hence the equations of motion are

$$\mathbf{F}^i = 0 \ , \tag{3.8}$$

where $\mathbf{F}^i = \mathbf{dA}^i + \frac{1}{2} f^i{}_{jk} \mathbf{A}^j \wedge \mathbf{A}^k$ is the curvature of $\mathbf{A}^i$.

The configuration space is $Q = \mathcal{C}^\infty(\Sigma)^3 \times \Omega^1(\Sigma)^3$ and the Chern-Simons Lagrangian is

$$L_{\text{CS}}(\text{v}) = \int_\Sigma g_{ij} \left(2 A^i_t F^j - A^i \wedge \text{v}^j_A \right) \tag{3.9}$$

Note that the action depends on the choice of the Lie algebra $\mathfrak{g}$ (or equivalently, Lie group $\mathcal{G}$) and the choice of an invariant non-degenerate metric. The requirement that the metric is non-degenerate yields an action with kinetic terms for all the components A^i of the field, while requiring invariance of the metric yields a gauge invariant action.

3.2.1 Variations on the Chern-Simons action

It turns out that a wide variety of theories can be written in the simple Chern-Simons form (3.7). The trick is to write the $\mathfrak{g}$-valued 1-form A in a certain basis T_i of $\mathfrak{g}$ with the components in that basis being 1-forms [51], [52].

We say that a group $\mathcal{G}$ is *inhomogeneous* if we can decompose it as $\mathcal{G} = \mathcal{H} \times \mathcal{N}$, where $\mathcal{N}$ is an Abelian normal subgroup and $\mathcal{H}$ a closed subgroup such that $\mathcal{N} \cap \mathcal{H} = \{\text{Id}\}$, in that case we write $\mathcal{G} = \mathcal{H} \ltimes \mathcal{N}$ and call it a *semidirect product*. Consider the particular case of the canonical inhomogeneous group $\mathbf{I}\mathcal{G} = \mathcal{G} \ltimes \mathfrak{g}^* \cong$

$T^*\mathcal{G}$ associated with any Lie group $\mathcal{G}$, where the dual of the Lie algebra $\mathfrak{g}^*$ is to be thought as an Abelian group in the addition operation. The inhomogeneous algebra is $\mathfrak{ig} = \mathfrak{g} \oplus \mathfrak{g}^*$ and its Lie bracket is given by

$$[(v,\alpha),(w,\beta)]_{\mathfrak{g}\oplus\mathfrak{g}^*} = ([v,w],[w,\alpha]_{\mathfrak{g}^*} - [v,\beta]_{\mathfrak{g}^*}) \ , \quad v,w \in \mathfrak{g} \ , \ \alpha,\beta \in \mathfrak{g}^* \ ,$$

where

$$[v,\alpha]_{\mathfrak{g}^*} = \alpha([v,\cdot]) \in \mathfrak{g}^* \ .$$

From now on, the subscript of the brackets indicating the algebra in which they are defined will not be indicated since there will be no room for confusion. Choosing the basis J_i of $\mathfrak{g}$ and the basis P^i of the dual $\mathfrak{g}^*$, with the convention that by J_i, P^i we mean $(J_i,0),(0,P^i) \in \mathfrak{ig}$ respectively, we have

$$[P^i,P^j] = 0 \ , \ [P^i,J_j] = f^i_{\ jk}P^k \ , \ [J_i,J_j] = f_{ij}^{\ \ k}J_k \ ,$$

where $f^i_{\ jk}$ are the structure constants of $\mathfrak{g}$. In $\mathbf{I}\mathcal{G}$ there is a canonical invariant metric determined by

$$\langle P^i,P^j\rangle = 0 \ , \ \langle J_i,J_j\rangle = 0 \ , \ \langle P^i,J_j\rangle = \eta^i_{\ j} \ . \tag{3.10}$$

Since the P^i form an Abelian ideal, the elements of $\mathfrak{g}^*$ can be interpreted as the generators of 'translations', while the elements of $\mathfrak{g}$ can be interpreted as the generators of 'rotations'. Note that by construction, there is the same number of generators for the translations than rotations. This is not usually the physical case, where one uses the Poincaré group (or inhomogeneous Lorentz group) in $d+1$ dimensions defined by

$$ISO(1,d) := SO(1,d) \ltimes \mathbb{R}^d \ .$$

Only in the particular case of $d = 3$, where $\dim SO(1,2) = \dim \mathbb{R}^3 = 3$, the canonical inhomogeneous Lorentz group and the Poincaré group coincide $\mathbf{I}SO(1,2) = ISO(1,2)$.

We can write a $\mathfrak{ig}$-valued 1-form $\mathbf{A}$ in the mentioned basis $\mathbf{A} = \mathbf{e}_i P^i + \boldsymbol{\omega}^i J_i$, with $\mathbf{e}_i, \boldsymbol{\omega}^i \in \Omega^1(\mathcal{M})$ for $i = 1,\ldots,\dim\mathfrak{g}$. Then,

$$\begin{aligned}
[\![\mathbf{A},\mathbf{A}]\!] =& 2\left(\mathbf{e}_i\wedge\mathbf{e}_j[P^i,P^j] + 2\mathbf{e}_i\wedge\boldsymbol{\omega}^j[P^i,J_j] + \boldsymbol{\omega}^i\wedge\boldsymbol{\omega}^j[J_i,J_j]\right)\\
=& 2\left(2\mathbf{e}_i\wedge\boldsymbol{\omega}^j f^i_{\ jk}P^k + \boldsymbol{\omega}^i\wedge\boldsymbol{\omega}^j f_{ij}^{\ \ k}J_k\right) \ ,\\
\frac{1}{3}\langle\mathbf{A},[\![\mathbf{A},\mathbf{A}]\!]\rangle =& \frac{2}{3}\left(2\mathbf{e}_l\wedge\mathbf{e}_i\wedge\boldsymbol{\omega}^j f^i_{\ jk}\langle P^l,P^k\rangle + \mathbf{e}_l\wedge\boldsymbol{\omega}^i\wedge\boldsymbol{\omega}^j f^k_{\ ij}\langle P^l,J_k\rangle\right.\\
&\left. +\ 2\boldsymbol{\omega}^l\wedge\mathbf{e}_i\wedge\boldsymbol{\omega}^j f^i_{\ jk}\langle J_l,P^k\rangle + \boldsymbol{\omega}^l\wedge\boldsymbol{\omega}^i\wedge\boldsymbol{\omega}^j f^k_{\ ij}\langle J_l,J_k\rangle\right) \ ,\\
\langle\mathbf{A},\mathbf{dA}\rangle =& \mathbf{e}_i\wedge\mathbf{d}\mathbf{e}_j\langle P^i,P^j\rangle + e_i\wedge\mathbf{d}\boldsymbol{\omega}^j\langle P^i,J_j\rangle\\
&+\boldsymbol{\omega}^i\wedge\mathbf{d}\mathbf{e}_j\langle J_i,P^j\rangle + \boldsymbol{\omega}^i\wedge\mathbf{d}\boldsymbol{\omega}^j\langle J_i,J_j\rangle \ .
\end{aligned}$$

By using the metric (3.10) then the $\mathbf{I}\mathcal{G}$-Chern-Simons action (3.7) becomes

$$
\begin{aligned}
S_{\text{CS}}(\mathbf{e},\boldsymbol{\omega}) &= \frac{1}{2}\int_{\mathcal{M}} \mathbf{e}_i \wedge \mathbf{d}\boldsymbol{\omega}^i + \boldsymbol{\omega}^i \wedge \mathbf{d}\mathbf{e}_i + \frac{2}{3} 3 f^i_{\ jk}\, e_i \wedge \boldsymbol{\omega}^j \wedge \boldsymbol{\omega}^k \\
&= \int_{\mathcal{M}} \mathbf{e}_i \wedge \left(\mathbf{d}\boldsymbol{\omega}^i + f^i_{\ jk}\boldsymbol{\omega}^j \wedge \boldsymbol{\omega}^k\right) = \int_{\mathcal{M}} \mathbf{e}_i \wedge \mathbf{F}^i \ .
\end{aligned}
$$

The term

$$
S_{\text{3-P}}(\mathbf{e},\boldsymbol{\omega}) = \int_{\mathcal{M}} \mathbf{e}_i \wedge \mathbf{F}^i \ , \tag{3.11}
$$

is the $\mathcal{G}$-Cartan-Palatini action in 3 dimensions. Hence, the $\mathcal{G}$-Cartan-Palatini theory is equivalent to the $\mathbf{I}\mathcal{G}$-Chern-Simons theory.

There is another interesting construction from a Lie group $\mathcal{G}$. Denote $\lambda\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$ equipped with the Lie bracket

$$
[P^i, P^j] = \lambda f^{ijk} J_k \ , \ [P^i, J_j] = f^i_{\ jk} P^k \ , \ [J_i, J_j] = f_{ij}^{\ \ k} J_k \ . \tag{3.12}
$$

Note that this is no longer an inhomogeneous algebra, but the previous $\mathfrak{ig}$ is recovered for $\lambda = 0$. Then, the group $\lambda\mathcal{G}$ is the image under the exponential map of $\lambda\mathfrak{g}$. A shortcoming of this construction is that for $\lambda = 0$, if the exponential map is not surjective, one does not recover the original group.

In addition to the metric (3.10), for $\lambda \neq 0$, in this algebra there also exists the (non-degenerate) metric

$$
\langle P^i, P^j \rangle = \lambda\ \eta^{ij} \ , \ \langle J_i, J_j \rangle = \eta^{ij} \ , \ \langle P^i, J_j \rangle = 0 \ . \tag{3.13}
$$

Note that this is not a metric on $\mathfrak{ig}$ because, if $\lambda = 0$, then the bilinear form becomes denegerate.

We then have two choices of metric when writing the $\lambda\mathcal{G}-$Chern-Simons action. If we use the metric (3.10), we obtain

$$
S_{\Lambda}(\mathbf{e},\boldsymbol{\omega}) = \int_{\mathcal{M}} \left(\mathbf{e}_i \wedge \mathbf{F}^i + \frac{\lambda}{6} f_{ijk}\, \mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k \right) \ ,
$$

which is the $\mathcal{G}$-Cartan-Palatini action with the addition of a cosmological constant term. On the other hand, using the metric (3.13) one obtains the so-called Witten's exotic action [52]

$$
S_{\text{Exotic}}(\mathbf{e},\boldsymbol{\omega}) = \int_{\mathcal{M}} \left(\lambda \mathbf{e}_i \wedge \left(\mathbf{d}\mathbf{e}^i + 2 f^i_{\ jk}\boldsymbol{\omega}^j \wedge \mathbf{e}^k\right) + \boldsymbol{\omega}_i \wedge \left(\mathbf{d}\boldsymbol{\omega}^i + \frac{2}{3} f^i_{\ jk}\boldsymbol{\omega}^j \wedge \boldsymbol{\omega}^k \right) \right) \ .
$$

Note that since both (3.10) and (3.13) are metrics, one can consider a more general metric given by

$$
\langle P^i, P^j \rangle = \mu_1 \eta^{ij} \ , \ \langle J_i, J_j \rangle = \mu_2 \eta^{ij} \ , \ \langle P^i, J_j \rangle = \mu_3 \eta^{ij} \ ,
$$

where $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ are constants. By changing from the basis (J_i, P_i) to the new basis (J_i, T_i) where $T_i = P_i + \frac{q}{2} J_i$ with q constant, the algebra and metric read

$$\begin{aligned} &\langle T^i, T^j \rangle = \sigma_1 \eta^{ij} \ , && \langle J_i, J_j \rangle = \sigma_2 \eta^{ij} \ , \quad \langle T^i, J_j \rangle = \sigma_3 \eta^{ij} \ , \\ &[T^i, T^j] = f^{ij}{}_k \left(q T^k + p J^k \right) \ , && [T^i, J_j] = f^i{}_{jk} T^k \ , \ [J_i, J_j] = f_{ijk} J^k \ , \end{aligned}$$

where the new constants are related to the old ones by

$$\begin{aligned} \sigma_1 &= \mu_1 + \frac{q^2}{4} \mu_2 + q \mu_3 \ , \\ \sigma_2 &= \mu_2 \ , \\ \sigma_3 &= \mu_3 + \frac{q}{2} \mu_2 \ , \\ p &= \lambda - \frac{q^2}{4} \ , \end{aligned} \tag{3.14}$$

with the additional condition required by the invariance of the metric (3.6) given by

$$\langle [J_i, T^j], T^k \rangle + \langle J_i, [T^j, T^k] \rangle = 0 \ ,$$

which is equivalent to

$$\sigma_1 = p \sigma_2 + q \sigma_3 \ .$$

It is straightforward to check that by using the relations (3.14), this condition translates to

$$\mu_1 = \lambda \mu_2 \ .$$

Using this basis to write $\mathbf{A} = \boldsymbol{\omega}^i J_i + \mathbf{e}_i T^i$ in the Chern-Simons action (3.7) leads to the most general gravitational action in 3 dimensions, first introduced by Mielke and Baekler [53], [54]

$$\begin{aligned} S_{\mathrm{MB}}(\mathbf{e}, \boldsymbol{\omega}) = \int_{\mathcal{M}} \Big(\sigma_1 \mathbf{e}_i \wedge \mathbf{D} \mathbf{e}^i + \sigma_2 \boldsymbol{\omega}_i \wedge \left(\mathbf{d} \boldsymbol{\omega}^i + \frac{1}{3} [\boldsymbol{\omega}, \boldsymbol{\omega}]^i \right) \\ + 2 \sigma_3 \mathbf{e}_i \wedge \mathbf{F}^i + \frac{\sigma_4}{3} f_{ijk} \mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k \Big) \ . \end{aligned}$$

Similarly, one can obtain the Husain action [55] by using a deformation (different from the λ deformation presented before although we use the same symbol) of the group $ISO(2)$, whose algebra has generators P_i, J, $i = 1, 2$. The deformation consists in making the generators of the transations P^i non-commuting

$$[P^i, P^j] = \lambda \epsilon^{ij} J \ , \quad [J, P^i] = \epsilon^i{}_j P^j \ ,$$

with the metric

$$\langle P^i, P^j \rangle = \lambda \ \delta^{ij} \ , \ \langle J, J \rangle = 1 \ , \ \langle P^i, J \rangle = 0 \ .$$

Note that for $\lambda = 0$ we recover the algebra $\mathfrak{iso}(2)$, while for $\lambda > 0$ and $\lambda < 0$ it is isomorphic to $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$, respectively. If we write $\mathbf{A} = \boldsymbol{\omega} J + \mathbf{e}_i P^i$ we get the action

$$S_{\text{Husain}}(\mathbf{e}, \boldsymbol{\omega}) = \int_{\mathcal{M}} \boldsymbol{\omega} \wedge \mathbf{d}\boldsymbol{\omega} + \lambda \mathbf{e}_i \wedge \left(\mathbf{d}\mathbf{e}^i - \epsilon^i{}_j \boldsymbol{\omega} \wedge \mathbf{e}^j \right) \ .$$

This action is interesting on its own and we study it in detail in Appendix C. The reason why this is a relevant model is that it describes 2-geometries (surfaces modulo diffeomorphisms) of constant curvature. Moreover, in some sense, it is a $2+1$ dimensional version of the Husain-Kuchař action (that will be discussed in Section 3.3), which is a theory of 3-geometries.

3.2.2 Hamiltonian analysis

We start by computing the fiber derivative (2.3) of the Lagrangian (3.9)

$$FL_{\text{CS}}(\mathrm{v})(\mathrm{w}) = -\int_{\Sigma} g_{ij} A^i \wedge \mathrm{w}_A^j \ .$$

This defines the canonical momenta

$$\begin{aligned} \mathbf{p}_{A_t}(\mathrm{w}) &= 0 \ , \\ \mathbf{p}_A(\mathrm{w}) &= -\int_{\Sigma} g_{ij} A^i \wedge \mathrm{w}^j \ . \end{aligned}$$

Clearly, $FL_{\text{CS}} : TQ \longrightarrow T^*Q$ is not an diffeomorphism, hence the system is singular.

The energy (2.4) is

$$E_{L_{\text{CS}}}(\mathrm{v}) = -2\int_{\Sigma} g_{ij} A_t^i F^j \ , \tag{3.15}$$

and, hence, the Hamiltonian (2.5) is

$$H(\mathbf{p}) = -2\int_{\Sigma} g_{ij} A_t^i F^j \ ,$$

and its differential

$$\mathbb{d} H(\mathbb{Y}) = -2\int_{\Sigma} g_{ij} \left(Y_{A_t}^i F^j - DA_t^i \wedge Y_A^j \right) \ . \tag{3.16}$$

Notice that the energy is independent of the velocities and the Hamiltonian is independent of the momenta.

The canonical symplectic form of T^*Q is

$$\Omega\left(\mathbb{Z}, \mathbb{Y}\right) = \mathbf{Y}_{A_t}(Z_{A_t}) - \mathbf{Z}_{A_t}(Y_{A_t}) + \mathbf{Y}_A(Z_A) - \mathbf{Z}_A(Y_A) \ ,$$

and its pullback to the primary constraint submanifold $\mathfrak{Q}$ is

$$\omega\left(\mathbb{Z}, \mathbb{Y}\right) = 2\int_{\Sigma} g_{ij} Z^i_A \wedge Y^j_A \ . \tag{3.17}$$

Equating (3.16) and (3.17) tells us that the submanifold where the Hamiltonian equation (2.6) has solutions is characterized by

$$F^i = 0 \ .$$

and the Hamiltonian vector field of H is given by

$$Z^i_A = DA^i_t \ ,$$

while $Z^i_{A_t}$ remains undetermined. We still need to check tangency to the constraint submanifold. This is easily done

$$\imath_{\mathbb{Z}} \mathbb{d} F^i = DZ^i_A = D^2 A^i_t = f^i_{\ jk} F^j A^k_t = 0 \ ,$$

which are automatically satisfied, hence we do not need any further constraints.

3.2.3 Lagrangian analysis

Since the configuration space for the Chern-Simons Lagrangian (3.9) is

$$Q = \mathcal{C}^{\infty}\left(\Sigma\right)^3 \times \Omega^1\left(\Sigma\right)^3 \ ,$$

we can use the STL simplification described in Section 2.3. Then, a typical vector field at a point $\mathrm{v} \in TQ$ is of the form

$$\mathbb{X}_{\mathrm{v}} = \left(X_{A_t}, X_A, X_{\mathrm{v}_{A_t}}, X_{\mathrm{v}_A}\right) \in T_{\mathrm{v}}TQ \ .$$

The presymplectic form (2.7) is

$$\omega_{L_{\mathrm{CS}}}\left(\mathbb{Z}, \mathbb{Y}\right) = 2\int_{\Sigma} g_{ij} Z^i_A \wedge Y^j_A \ ,$$

hence, its kernel is obviously given by

$$\ker \omega_{L_{\mathrm{CS}}} = \left\{\mathbb{X} \in \mathfrak{X}(TQ) \mid X_A = 0\right\} \ .$$

Now we find the constraints as per the STL version of the GNH algorithm. By Proposition 4, and since

$$\imath_{\mathbb{X}}\left(\mathbb{d} E_{L_{\mathrm{CS}}}\right) = -2\int_{\Sigma} g_{ij}\left(X^i_{A_t} F^j + A^i_t D X^j_A\right) = -2\int_{\Sigma} g_{ij} X^i_{A_t} F^j \ ,$$

one concludes that the submanifold $\mathcal{S}_1$ is given by the constraints

$$F^i = 0 \ .$$

Next, we apply Proposition 7 by computing the constraint (2.14). The condition in $\mathbb{X}$ is empty and since $Y_A = 0$, $\imath_{\mathbb{X}}\imath_{\mathbb{Y}}\left(\omega_{L_{\mathrm{CS}}}\right)_{\mathrm{v}} = 0$ is automatically satisfied. Then, solving the Hamiltonian equation (2.11) together with the second order condition gives

$$
\begin{aligned}
DA^i_t &= \mathrm{v}^i_A \ , \\
Z^i_{A_t} + Y^i_{A_t} &= \mathrm{v}^i_{A_t} \ .
\end{aligned}
$$

One still needs to check the tangency conditions, which is done in the same way as in Subsection 3.2.2 since the expressions involved are identical.

3.2.4 Equations of motion analysis

We are now going to use the equations of motion method described in Section 2.4. We need to compute (2.17) for the Chern-Simons equations of motion (3.8). Decomposing it as described in Appendix A one obtains

$$
\begin{aligned}
\mathbf{F}^i &= \mathbf{d}t \wedge \left(\mathcal{L}_{\partial_t}\underline{\mathbf{A}}^i - \underline{\mathbf{d}}\mathbf{A}^i_t\right) + \underline{\mathbf{d}}\underline{\mathbf{A}}^i + \frac{1}{2}\epsilon^i{}_{jk}\left(\mathbf{d}t\mathbf{A}^j_t + \underline{\mathbf{A}}^j\right) \wedge \left(\mathbf{d}t\mathbf{A}^k_t + \underline{\mathbf{A}}^k\right) \\
&= \underline{\mathbf{d}}\underline{\mathbf{A}}^i + \frac{1}{2}\epsilon^i{}_{jk}\underline{\mathbf{A}}^j \wedge \underline{\mathbf{A}}^k + \mathbf{d}t \wedge \left(\mathcal{L}_{\partial_t}\underline{\mathbf{A}} - \underline{\mathbf{d}}\mathbf{A}_t - \epsilon^i{}_{jk}\underline{\mathbf{A}}^j \wedge \mathbf{A}^k_t\right) \\
&= \underline{\mathbf{D}}\underline{\mathbf{A}}^i + \mathbf{d}t \wedge \left(\mathcal{L}_{\partial_t}\underline{\mathbf{A}}^i - \underline{\mathbf{D}}\mathbf{A}^i_t\right) \\
&= \underline{\mathbf{F}}^i + \mathbf{d}t \wedge \left(\mathcal{L}_{\partial_t}\underline{\mathbf{A}}^i - \underline{\mathbf{D}}\mathbf{A}^i_t\right) \ .
\end{aligned}
$$

Then, extracting the adapted components and projecting them to Σ, we obtain the constraints

$$
F^i = 0 \ ,
$$

and the equations for the Hamiltonian vector field

$$
Z^i_A = DA^i_t \ .
$$

The constraints and the expression of the Hamiltonian vector field are identical to the ones obtained previously. Hence, the tangency analysis is identical to the one discussed in Subsection 3.2.2 and the implementation of the second order condition follows as in Subsection 3.2.3.

3.3 Husain-Kuchař

Let $\mathcal{M}$ be an orientable 4-dimensional manifold. Let $\mathbf{e}^i$ be 3 linearly independent 1-forms and $\mathbf{F}^i = \mathbf{d}\boldsymbol{\omega}^i + \frac{1}{2}\epsilon^i{}_{jk}\boldsymbol{\omega}^j \wedge \boldsymbol{\omega}^k$ the curvature of an $SO(3)$ connection $\boldsymbol{\omega}^i$. The Husain-Kuchař action proposed in [56] is

$$
S_{\mathrm{HK}}\left(\mathbf{e}, \boldsymbol{\omega}\right) = \int_{\mathcal{M}} \epsilon_{ijk}\mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{F}^k \ , \tag{3.18}
$$

where ϵ_{ijk} is an $SO(3)$ volume form. This action is invariant under diffeomorphisms of $\mathcal{M}$ and under $SO(3)$ gauge transformations.

Taking variations of the action with respect to the fields gives

$$\begin{aligned}\frac{d}{d\epsilon}S_{\text{HK}}(\mathbf{e}+\epsilon\delta\mathbf{e},\boldsymbol{\omega}+\epsilon\delta\boldsymbol{\omega})\Big|_{\epsilon=0} &= \int_{\mathcal{M}}\epsilon_{ijk}\Big(2\delta\mathbf{e}^i\wedge\mathbf{e}^j\wedge\mathbf{F}^k+\mathbf{e}^i\wedge\mathbf{e}^j\wedge\mathbf{D}(\delta\boldsymbol{\omega}^k)\Big)\\ &= \int_{\mathcal{M}}\epsilon_{ijk}\Big(2\delta\mathbf{e}^i\wedge\mathbf{e}^j\wedge\mathbf{F}^k-\mathbf{D}\left(\mathbf{e}^i\wedge\mathbf{e}^j\right)\wedge\delta\boldsymbol{\omega}^k\Big)\ ,\end{aligned}$$

hence the equations of motion are

$$\begin{aligned}\epsilon_{ijk}\mathbf{e}^j\wedge\mathbf{D}\mathbf{e}^k&=0\ ,\\ \epsilon_{ijk}\mathbf{e}^j\wedge\mathbf{F}^k&=0\ .\end{aligned}\tag{3.19}$$

The action (3.18) is structurally very similar to the Cartan-Palatini action (3.32) which will be studied in Section 3.4, however, the subtle but big difference between the two is the internal group, which in the Cartan-Palatini case is $SO(1,3)$ rather than $SO(3)$. A direct consequence of this is the fact that we only have 3 linearly independent 1-forms $\mathbf{e}^i$ in a 4-dimensional manifold, that is, we do not have a coframe. We can still build a 3-metric γ by taking $\gamma=\delta_{ij}\mathbf{e}^i\otimes\mathbf{e}^j$, which is degenerate. Another possible construction form the $\mathbf{e}^i$ is the vector density (actually defined as an element of the double dual)

$$U(\cdot)=\left(\frac{\cdot\wedge\epsilon_{ijk}\mathbf{e}^i\wedge\mathbf{e}^j\wedge\mathbf{e}^k}{\mathsf{vol}}\right)\ ,$$

which characterises the degenerate directions of γ, since it satisfies $U(\mathbf{e}^i)=\imath_U\mathbf{e}^i=0$ and hence

$$\gamma(U,\cdot)=\delta_{ij}\ \imath_U\mathbf{e}^i\ \mathbf{e}^j=0\ .$$

If we take the interior product of Equations (3.19) with the distinguished vector field U, we get the equations

$$\begin{aligned}\epsilon_{ijk}\mathbf{e}^j\wedge\imath_U\mathbf{D}\mathbf{e}^k&=0\ ,\\ \epsilon_{ijk}\mathbf{e}^j\wedge\imath_U\mathbf{F}^k&=0\ ,\end{aligned}$$

which by using Lemma 21 of Appendix B, are equivalent to

$$\begin{aligned}\imath_U\mathbf{D}\mathbf{e}^k&=0\ ,\\ \imath_U\mathbf{F}^k&=0\ .\end{aligned}$$

By making use of $\imath_U\mathbf{e}^i=0$, these are equivalent to

$$\begin{aligned}0=\imath_U\mathbf{D}\mathbf{e}^i=\imath_U\left(\mathbf{d}\boldsymbol{\omega}^i+\frac{1}{2}\epsilon^i{}_{jk}\boldsymbol{\omega}^j\wedge\boldsymbol{\omega}^k\right)&=\mathcal{L}_U\boldsymbol{\omega}^i-\mathbf{d}\imath_U\boldsymbol{\omega}^i+\epsilon^i{}_{jk}\imath_U\boldsymbol{\omega}^j\boldsymbol{\omega}^k\\ =\mathcal{L}_U\boldsymbol{\omega}^i-\mathbf{D}\left(\imath_U\boldsymbol{\omega}^i\right)\ ,&\\ 0=\imath_U\mathbf{F}^i=\imath_U\left(\mathbf{d}\mathbf{e}^i+\epsilon^i{}_{jk}\boldsymbol{\omega}^j\wedge\mathbf{e}^k\right)=\mathcal{L}_U\mathbf{e}^i+\epsilon^i{}_{jk}\imath_U\boldsymbol{\omega}^j\wedge\mathbf{e}^k\ ,&\end{aligned}$$

and by identifying $\tau^i := \imath_U \boldsymbol{\omega}^i$, we can write

$$\begin{aligned}\mathcal{L}_U \boldsymbol{\omega}^i &= \mathbf{D}\tau^i \ , \\ \mathcal{L}_U \mathbf{e}^i &= -\epsilon^i{}_{jk}\tau^j \mathbf{e}^k \ .\end{aligned}$$

Thus, Lie-dragging the fields $\mathbf{e}^i, \boldsymbol{\omega}^i$ along the direction of U is equivalent to performing an $SO(3)$ gauge transformation on them with parameter τ^i. Since the 3-metric is just $\gamma = \mathbf{e}_i \otimes \mathbf{e}^i$, it will remain invariant under Lie-dragging along the integral curves of U, since

$$\mathcal{L}_U \gamma = \mathcal{L}_U \mathbf{e}_i \otimes \mathbf{e}^i + \mathbf{e}_i \otimes \mathcal{L}_U \mathbf{e}^i = -\epsilon_{ijk}\tau^j \mathbf{e}^k \otimes \mathbf{e}^i - \mathbf{e}_i \otimes \epsilon^i{}_{jk}\tau^j \mathbf{e}^k = 0 \ .$$

Hence, this model describes 3-geometries, i.e. metrics on a 3-manifold, modulo diffeomorphisms.

The Lagrangian defined in TQ is

$$L_{\mathrm{HK}}(\mathrm{v}) = \int_\Sigma \left(2\epsilon_{ijk} e^i_t e^j \wedge F^k + \epsilon_{ijk} e^i \wedge e^j \wedge \left(\mathrm{v}^k_A - DA^k_t\right)\right) \ , \tag{3.20}$$

with the configuration space being $Q = \mathcal{C}^\infty(\Sigma)^3 \times \Omega^1(\Sigma)^3 \times \mathcal{C}^\infty(\Sigma)^3 \times \Omega^1(\Sigma)^3$.

3.3.1 Hamiltonian analysis

The fiber derivative (2.3) of the Lagrangian (3.20) is

$$FL_{\mathrm{HK}}(\mathrm{v})(\mathrm{w}) = \int_\Sigma \epsilon_{ijk} e^i \wedge e^j \wedge \mathrm{w}^k_A \ ,$$

then the canonical momenta are given by

$$\begin{aligned}\mathbf{p}_{e_t} &= 0 \ , \\ \mathbf{p}_{e} &= 0 \ , \\ \mathbf{p}_{A_t} &= 0 \ , \\ \mathbf{p}_A(\mathrm{w}) &= \int_\Sigma \epsilon_{ijk} e^i \wedge e^j \wedge \mathrm{w}^k_A \ ,\end{aligned}$$

and define the primary constraint submanifold $\mathfrak{Q}$ of T^*Q. Clearly, $FL_{\mathrm{HK}} : TQ \longrightarrow T^*Q$ is not a diffeomorphism, hence the system is singular.

The canonical symplectic form of T^*Q is

$$\Omega(\mathbb{Z}, \mathbb{Y}) = \mathbf{Y}_{A_{ti}}(Z^i_{A_t}) - \mathbf{Z}_{A_{ti}}(Y^i_{A_t}) + \mathbf{Y}_{A_i}(Z^i_A) - \mathbf{Z}_{A_i}(Y^i_A) \tag{3.21}$$

$$+ \mathbf{Y}_{e_{ti}}(Z^i_{e_t}) - \mathbf{Z}_{e_{ti}}(Y^i_{e_t}) + \mathbf{Y}_{e_i}(Z^i_e) - \mathbf{Z}_{e_i}(Y^i_e) \ , \tag{3.22}$$

and its pullback to the primary constraint manifold $\mathfrak{Q}$ is

$$\omega(\mathbb{Z}, \mathbb{Y}) = 2\int_\Sigma \epsilon_{ijk} \left(Y^i_e \wedge Z^j_A - Y^i_A \wedge Z^j_e\right) \wedge e^k \ . \tag{3.23}$$

The energy (2.4) is

$$E_{L_{\mathrm{HK}}} = \int_\Sigma \left(\epsilon_{ijk} e^i \wedge e^j \wedge DA_t^k - 2\epsilon_{ijk} e_t^i e^j \wedge F^k\right) \ , \tag{3.24}$$

and since it does not depend on the momenta, as in the Chern-Simons case in equation (3.15), the Hamiltonian (2.5) can be written with the same expression as (3.24):

$$H = \int_\Sigma \left(\epsilon_{ijk} e^i \wedge e^j \wedge DA_t^k - 2\epsilon_{ijk} e_t^i e^j \wedge F^k\right) \ .$$

Then,

$$\begin{aligned} \mathbb{d} H(\mathbb{Y}) = &\int_\Sigma \Big(2\epsilon_{ijk} Y_e^i \wedge e^j \wedge DA_t^k + \epsilon_{ijk} e^i \wedge e^j \wedge \left(DY_{A_t}^k + \epsilon^k{}_{lm} Y_A^l A_t^m\right) \\ &- 2\epsilon_{ijk} Y_{e_t}^i e^j \wedge F^k - 2\epsilon_{ijk} e_t^i Y_e^j \wedge F^k - 2\epsilon_{ijk} e_t^i e^j DY_A^k\Big) \\ = 2 &\int_\Sigma \Big(Y_{e_t}^i \epsilon_{ijk} e^j \wedge F^k - Y_{A_t}^i \epsilon_{ijk} De^j \wedge e^k \\ &+ Y_e^i \wedge \epsilon_{ijk} \left(e^j \wedge DA_t^k + e_t^j F^k\right) \\ &- Y_A^i \wedge \epsilon_{ijk} \left(D(e_t^i e^j) - e_i \wedge e^j A_{t_j}\right)\Big) \ . \end{aligned} \tag{3.25}$$

Comparing (3.23) and (3.25) for all $\mathbb{Y}$, leads to the set of contraints

$$\begin{aligned} C_{Fi} &:= \epsilon_{ijk} e^j \wedge F^k = 0 \ , \\ C_{ei} &:= \epsilon_{ijk} e^j \wedge De^k = 0 \ , \end{aligned} \tag{3.26}$$

and equations for the Hamiltonian vector field

$$\epsilon_{ijk} Z_A^j \wedge e^k = \epsilon_{ijk} \left(e^j \wedge DA_t^k + e_t^j F^k\right) \ , \tag{3.27a}$$

$$\epsilon_{ijk} Z_e^j \wedge e^k = \epsilon_{ijk} D(e_t^j e^k) - e_i \wedge e^j A_{t_j} \ . \tag{3.27b}$$

The solution to (3.27a) is directly given by Lemma 13

$$\begin{aligned} Z_A^i = &\left(\frac{e^j \wedge \epsilon_{jkl} \left(e^k \wedge DA_t^l + e_t^k F^l\right)}{2\ \mathsf{vol}_e}\right) e^i - \left(\frac{e^i \wedge \epsilon_{jkl} \left(e^k \wedge DA_t^l + e_t^k F^l\right)}{\mathsf{vol}_e}\right) e^j \\ = &\epsilon_{jkl} \left(\frac{e^j \wedge e^k \wedge DA_t^l}{2\ \mathsf{vol}_e}\right) e^i + \epsilon_{jkl} e_t^k \left(\frac{e^j \wedge F^l}{2\ \mathsf{vol}_e}\right) e^i \\ &- \epsilon_{jkl} \left(\frac{e^i \wedge e^k \wedge DA_t^l}{\mathsf{vol}_e}\right) e^j - \epsilon_{jkl} e_t^k \left(\frac{e^i \wedge F^l}{\mathsf{vol}_e}\right) e^j \\ = &\frac{1}{2} \epsilon_{jkl} \epsilon^{jkm} \left(DA_t^l\right)_m e^i - \epsilon_{jkl} \epsilon^{ikm} \left(DA_t^l\right)_m e^j - \epsilon_{jkl} e_t^k \left(\frac{e^i \wedge F^l}{\mathsf{vol}_e}\right) e^j \\ = &\frac{1}{2} 2 \left(DA_t^l\right)_l e^i - \left(\left(DA_t^l\right)_l e^i - \left(DA_t^i\right)_j e^j\right) - \epsilon_{jkl} e_t^k \left(\frac{e^i \wedge F^l}{\mathsf{vol}_e}\right) e^j \\ = &DA_t^i + \epsilon_{jkl} \left(\frac{e^i \wedge F^j}{\mathsf{vol}_e}\right) e_t^k e^l \ . \end{aligned}$$

Likewise, the solution to (3.27b) is

$$
\begin{aligned}
Z_e^i =& \left(\frac{e^j \wedge \left(\epsilon_{jkl} D(e_t^k e^l) - e_j \wedge e^k A_{t_k}\right)}{2 \,\mathsf{vol}_e}\right) e^i - \left(\frac{e^i \wedge \left(\epsilon_{jkl} D(e_t^k e^l) - e_j \wedge e^k A_{t_k}\right)}{\mathsf{vol}_e}\right) e^j \\
=& \epsilon_{jkl} \left(\frac{e^j \wedge De_t^k \wedge e^l}{2 \,\mathsf{vol}_e}\right) e^i + \epsilon_{jkl} e_t^k \left(\frac{e^j \wedge De^l}{2 \,\mathsf{vol}_e}\right) e^i - A_{t_k} \left(\frac{e^j \wedge e_j \wedge e^k}{2 \,\mathsf{vol}_e}\right) e^i \\
&- \epsilon_{jkl} \left(\frac{e^i \wedge De_t^k \wedge e^l}{\mathsf{vol}_e}\right) e^j - \epsilon_{jkl} e_t^k \left(\frac{e^i \wedge De^l}{\mathsf{vol}_e}\right) e^j + A_{t_k} \left(\frac{e^i \wedge e_j \wedge e^k}{\mathsf{vol}_e}\right) e^j \\
=& \frac{1}{2} \epsilon_{jkl} \epsilon^{jml} \left(De_t^k\right)_m e^i - \epsilon_{jkl} \epsilon^{iml} \left(De_t^k\right)_m e^j - \epsilon_{jkl} e_t^k \left(\frac{e^i \wedge De^l}{\mathsf{vol}_e}\right) e^j + \epsilon^i{}_{jk} A_t^k e^j \\
=& \frac{1}{2} 2 \left(De_t^k\right)_k e^i - \left(De_t^k\right)_k e^i + \left(De_t^i\right)_j e^j - \epsilon^i{}_{jk} A_t^j e^k + \epsilon_{jkl} \left(\frac{e^i \wedge De^j}{\mathsf{vol}_e}\right) e_t^k e^l \\
=& De_t^i - \epsilon^i{}_{jk} A_t^j e^k + \epsilon_{jkl} \left(\frac{e^i \wedge De^j}{\mathsf{vol}_e}\right) e_t^k e^l \ .
\end{aligned}
$$

The components that do not appear in the equations are not fixed so far. In summary, the Hamiltonian vector field is

$$
\begin{aligned}
Z_A^i &= DA_t^i + \epsilon_{jkl} \left(\frac{e^i \wedge F^j}{\mathsf{vol}_e}\right) e_t^k e^l \ , \\
Z_e^i &= De_t^i - \epsilon^i{}_{jk} A_t^j e^k + \epsilon_{jkl} \left(\frac{e^i \wedge De^j}{\mathsf{vol}_e}\right) e_t^k e^l \ , \\
& Z_{e_t}, Z_{A_t} \text{ arbitrary} \ .
\end{aligned} \tag{3.28}
$$

One still needs to make sure that the resulting Hamiltonian vector field is tangent to the solution submanifold by imposing the tangency conditions

$$
\begin{aligned}
0 &= \imath_{\mathbb{Z}} \mathbf{d} C_F = \epsilon_{ijk} Z_e^j \wedge F^k + \epsilon_{ijk} e^j \wedge DZ_A^k \ , \\
0 &= \imath_{\mathbb{Z}} \mathbf{d} C_e = \epsilon_{ijk} Z_e^j \wedge De^k + \epsilon_{ijk} e^j \wedge DZ_e^k + \epsilon_{ijk} \epsilon^k{}_{lm} e^j \wedge Z_A^l \wedge e^m \ .
\end{aligned}
$$

It is very useful to get rid of the terms containing covariant derivatives of the Hamiltonian vector field by taking the covariant derivative of expressions (3.27), which yields

$$
\begin{aligned}
&\epsilon_{ijk} \left(e^j \wedge DZ_A^k + Z_A^j \wedge De^k\right) = -\epsilon_{ijk} \left(De^j \wedge DA_t^k - \epsilon^k{}_{lm} e^j \wedge F^l A_t^m + De_t^j \wedge F^k\right) \ , \\
&\epsilon_{ijk} \left(e^j \wedge DZ_e^k + Z_e^j \wedge De^k\right) \\
&\qquad = -\epsilon_{ijk} \epsilon^k{}_{lm} e_t^l F^j \wedge e^m + De^i \wedge e^j A_{tj} - e^i \wedge De^j A_{tj} + e^i \wedge e^j \wedge DA_{tj} \ .
\end{aligned} \tag{3.29}
$$

By using (3.29), the expression for the Hamiltonian vector field (3.28) and the alternative form of the constraints (3.26)

$$
\begin{aligned}
e^i \wedge F^j - e^j \wedge F^i = 0 \ , \\
e^i \wedge De^j - e^j \wedge De^i = 0 \ ,
\end{aligned}
$$

one finds that

$$
\imath_{\mathbb{Z}} \mathbf{d} C_F = \epsilon_{ijk} Z_e^j \wedge F^k - \epsilon_{ijk} Z_A^j \wedge De^k
$$

$$
\begin{aligned}
&- \epsilon_{ijk} \left(De^j \wedge DA_t^k - \epsilon^k{}_{lm} e^j \wedge F^l A_t^m + De_t^j \wedge F^k \right) \\
&= \epsilon_{ijk} \left(De_t^j - \epsilon^j{}_{lm} A_t^l e^m + \epsilon_{lmp} \left(\frac{e^i \wedge De^l}{\mathsf{vol}_e} \right) e_t^m e^p \right) \wedge F^k \\
&\quad - \epsilon_{ijk} \left(DA_t^j + \epsilon_{lmp} \left(\frac{e^i \wedge F^l}{\mathsf{vol}_e} \right) e_t^m e^p \right) \wedge De^k \\
&\quad - \epsilon_{ijk} \left(De^j \wedge DA_t^k - \epsilon^k{}_{lm} e^j \wedge F^l A_t^m + De_t^j \wedge F^k \right) \\
&= \epsilon_{ijk} \left(\epsilon^k{}_{lm} A_t^m e^j \wedge F^l - \epsilon^j{}_{lm} A_t^l e^m \wedge F^k \right) \\
&\quad + \epsilon_{ijk} \epsilon_{lmp} e_t^m \left(\left(\frac{e^j \wedge De^l}{\mathsf{vol}_e} \right) F^k \wedge e^p + \left(\frac{e^k \wedge F^l}{\mathsf{vol}_e} \right) De^j \wedge e^p \right) \\
&= \epsilon_{ijk} \epsilon^j{}_{lm} A_t^l \left(e^k \wedge F^m - e^m \wedge F^k \right) \\
&\quad + \epsilon_{ijk} \epsilon_{lmp} e_t^m \left(\left(\frac{e^j \wedge De^l}{\mathsf{vol}_e} \right) F^k \wedge e^p + \left(\frac{e^p \wedge De^j}{\mathsf{vol}_e} \right) F^l \wedge e^k \right) \\
&= \epsilon_{ijk} \epsilon_{lmp} e_t^m \left(\frac{e^j \wedge De^l}{\mathsf{vol}_e} \right) \left(F^k \wedge e^p - F^p \wedge e^k \right) \\
&= 0 \; ,
\end{aligned}
$$

$$
\begin{aligned}
\imath_{\mathbb{Z}} \mathsf{d} C_e &= -\epsilon_{ijk} \epsilon^k{}_{lm} e_t^l F^j \wedge e^m + De^i \wedge e^j A_{tj} - e^i \wedge De^j A_{tj} + e^i \wedge e^j \wedge DA_{tj} \\
&\quad + \epsilon_{ijk} \epsilon^k{}_{lm} e^j \wedge Z_A^l \wedge e^m \\
&= \left(De^i \wedge e^j - e^i \wedge De^j \right) A_{tj} + e^i \wedge e^j \wedge DA_{tj} - \epsilon_{ijk} \epsilon^k{}_{lm} e_t^l F^j \wedge e^m \\
&\quad + \epsilon_{ijk} \epsilon^k{}_{lm} e^j \wedge \left(DA_t^l + \epsilon_{pqr} \left(\frac{e^l \wedge F^p}{\mathsf{vol}_e} \right) e_t^q e^r \right) \wedge e^m \\
&= e_i \wedge e^j \wedge DA_{tj} + e^j \wedge DA_{ti} \wedge e_j - e^j \wedge DA_{tj} \wedge e_i \\
&\quad + \epsilon_{ijk} \epsilon^k{}_{lm} \left(\epsilon_{pqr} \left(\frac{e^l \wedge F^p}{\mathsf{vol}_e} \right) e_t^q e^j \wedge e^r \wedge e^m - e_t^l F^j \wedge e^m \right) \\
&= \epsilon_{ijk} \epsilon^k{}_{lm} \left(\epsilon_{pqr} \epsilon^{jrm} e_t^q e^l \wedge F^p - e_t^l F^j \wedge e^m \right) \\
&= \epsilon_{ijk} \epsilon^k{}_{lm} \left(e_t^j e^l \wedge F^m - e_t^m e^l \wedge F^j - e_t^l F^j \wedge e^m \right) \\
&= \epsilon_{ijk} \epsilon^k{}_{lm} e_t^l \left(e^m \wedge F^j - F^j \wedge e^m \right) \\
&= 0 \; ,
\end{aligned}
$$

hence, $\mathbb{Z}$ is already tangent to the constraint submanifold and there are no further secondary constraints.

3.3.2 Lagrangian analysis

Since the configuration space for the Husain-Kuchař Lagrangian (3.20) is $Q = \mathcal{C}^\infty(\Sigma)^3 \times \Omega^1(\Sigma)^3 \times \mathcal{C}^\infty(\Sigma)^3 \times \Omega^1(\Sigma)^3$, we can use the STL simplification described in Section 2.3. Then, a typical vector field at a point $\mathrm{v} \in TQ$ is of the form

$$
\mathbb{X}_{\mathrm{v}} = \left(X_{A_t}, X_A, X_{e_t}, X_e, X_{\mathrm{v}_{A_t}}, X_{\mathrm{v}_A}, X_{\mathrm{v}_{e_t}}, X_{\mathrm{v}_e} \right) \in T_{\mathrm{v}} TQ \; .
$$

The symplectic form (2.7) is

$$\omega_{L_{\rm HK}}(\mathbb{Z},\mathbb{Y}) = 2\int_\Sigma \epsilon_{ijk}\left(Y_e^i \wedge Z_A^j - Y_A^i \wedge Z_e^j\right)\wedge e^k \ . \tag{3.30}$$

Clearly, its kernel is given by the vectors satisfying

$$\begin{aligned}\epsilon_{ijk}e^j \wedge Z_e^k = 0 \ ,\\ \epsilon_{ijk}e^j \wedge Z_A^k = 0 \ ,\end{aligned}$$

which by Lemma 13 of Appendix B is equivalent to

$$\ker \omega_{L_{\rm HK}} = \{\mathbb{X} \in \mathfrak{X}(TQ) \mid X_A = X_e = 0\} \ .$$

Then, for vectors $\mathbb{X} \in \ker\omega_L$ contracted with the differential of the energy (3.24) gives

$$\begin{aligned}\imath_{\mathbb{X}}\mathrm{d}E_{L_{\rm HK}} &= \int_\Sigma \left(\epsilon_{ijk}e^i \wedge e^j \wedge DX^k_{A_t} - 2\epsilon_{ijk}X^i_{e_t}e^j \wedge F^k\right)\\ &= -\,2\int_\Sigma \left(X^i_{A_t}\epsilon_{ijk}De^j \wedge e^k + X^i_{e_t}\epsilon_{ijk}e^j \wedge F^k\right) \ .\end{aligned} \tag{3.31}$$

According to Proposition 4, the submanifold $\mathcal{S}_1$ where (3.31) vanishes for all $\mathbb{X} \in \ker\omega_L$ is then given by the constraints

$$\begin{aligned}\epsilon_{ijk}De^j \wedge e^k = 0 \ ,\\ \epsilon_{ijk}e^j \wedge F^k = 0 \ .\end{aligned}$$

The condition (2.14) on $\mathbb{X}$ in Proposition 7 is satisfied by all $\mathbb{X} \in \mathfrak{X}(TQ)$, hence we just need to ensure that $\imath_{\mathbb{Y}}\omega_{L_{\rm HK}} = 0$ for a vector field $\mathbb{Y}$ such that $\mathbb{J}(\mathbb{Z}+\mathbb{Y}) = \mathbb{V}$ with $\mathbb{Z}$ being a solution to (2.11). But, by the explicit form of (3.30) and Lemma 13 again, one must conclude that given a solution $\mathbb{Z}$, the constraint submanifold $\mathcal{S}_2$ is determined by

$$\begin{aligned}\mathrm{v}_e = Z_e \ ,\\ \mathrm{v}_A = Z_A \ .\end{aligned}$$

The Hamiltonian equations (2.11) are

$$\begin{aligned}\epsilon_{ijk}Z_A^j \wedge e^k &= \epsilon_{ijk}\left(e^j \wedge DA_t^k + e_t^j F^k\right) \ ,\\ \epsilon_{ijk}Z_e^j \wedge e^k &= \epsilon_{ijk}D(e_t^j e^k) - e_i \wedge e^j A_{t_j} \ ,\end{aligned}$$

which are identical to the ones found in Subsection 3.3.1, hence their resolution and tangency analysis are the same.

3.3.3 Equations of motion analysis

We are now going to use the equations of motion method described in Section 2.4. We need to compute (2.17) for the equations of motion (3.19). Decomposing the equations of motion with the help of (A.3) and (A.4), we get

$$\begin{aligned}\epsilon_{ijk}\mathbf{e}^j \wedge \mathbf{D}\mathbf{e}^k &= \epsilon_{ijk}\underline{\mathbf{e}}^j \wedge \underline{\mathbf{D}}\underline{\mathbf{e}}^k \\ &\quad + \mathbf{d}t \wedge \epsilon_{ijk}\left(\mathbf{e}^j_t \mathbf{D}\mathbf{e}^k - \underline{\mathbf{e}}^j \wedge \mathcal{L}_{\partial_t}\underline{\mathbf{e}}^k + \underline{\mathbf{e}}^j \wedge \underline{\mathbf{D}}\mathbf{e}^k_t - \epsilon^k{}_{lm}\mathbf{A}^l_t \underline{\mathbf{e}}^j \wedge \underline{\mathbf{e}}^m\right) \ , \\ \epsilon_{ijk}\mathbf{e}^j \wedge \mathbf{F}^k &= \epsilon_{ijk}\underline{\mathbf{e}}^j \wedge \underline{\mathbf{F}}^k + \mathbf{d}t \wedge \epsilon_{ijk}\left(\mathbf{e}^j_t \mathbf{F}^k - \underline{\mathbf{e}}^j \wedge \mathcal{L}_{\partial_t}\underline{\mathbf{A}}^k + \underline{\mathbf{e}}^j \wedge \underline{\mathbf{D}}\mathbf{A}^k_t\right) \ .\end{aligned}$$

Then, extracting the adapted components and projecting them onto Σ, we obtain the constraints

$$\begin{aligned}\epsilon_{ijk}e^j \wedge De^k &= 0 \ , \\ \epsilon_{ijk}e^j \wedge F^k &= 0 \ ,\end{aligned}$$

and the equations for the Hamiltonian vector field

$$\begin{aligned}\epsilon_{ijk}e^j \wedge Z^k_e &= \epsilon_{ijk}e^j_t De^k + \epsilon_{ijk}e^j \wedge \left(De^k_t - \epsilon^k{}_{lm}A^l_t e^m\right) \ , \\ \epsilon_{ijk}e^j \wedge Z^k_A &= \epsilon_{ijk}e^j_t F^k + \epsilon_{ijk}e^j \wedge DA^k_t \ .\end{aligned}$$

The resolution to the equations and the consistency conditions are identical to those discussed in Subsection 3.3.1.

3.4 Cartan-Palatini action

The Einstein-Hilbert action [4], [57]

$$S_{\mathrm{EH}}(g) = \int_{\mathcal{M}} R \ \mathsf{vol}_g \ ,$$

which depends on a metric g and where R is the Ricci curvature scalar (computed using the Levi-Civita connection), whose equations of motion are Einstein's field equations was proposed in an effort to be able to obtain General Relativity from a variational principle. Although it was successful, the equations obtained through this method are second-order partial differential equations. This could be solved by promoting the connection to an independent field instead of forcing it to be the Levi-Civita connection of the metric. In this way, the equations for both the metric and the connection become first order. This achieves an effect similar to the standard technique of splitting a second order differential equation into two first-order differential equations. Both actions are equivalent because the equation for the connection forces it to be precisely the Levi-Civita one and the other becomes Einstein's equations. A further version of the action was developed by Einstein and Weyl [58], [59], who introduced Cartan's idea of moving frames into the theory by replacing the metric by a tetrad as independent dynamical variables.

By using a tetrad and considering the connection as an independent variable, one obtains the Cartan-Palatini action. Besides yielding first order equations, this action for General Relativity shares some similarities with the very important Yang-Mills actions for gauge theories. In some sense, General Relativity can then be understood as a $SO(1,3)$ gauge theory. The tetradic modern form of the action is

$$S_{\mathrm{CP}}(\mathbf{e},\boldsymbol{\omega}) = \int_{\mathcal{M}} \epsilon_{IJKL}\mathbf{e}^I \wedge \mathbf{e}^J \wedge \mathbf{F}^{KL} \ , \tag{3.32}$$

where $\mathbf{F}^{IJ} = \mathbf{d}\boldsymbol{\omega}^{IJ} + \boldsymbol{\omega}^I{}_K \wedge \boldsymbol{\omega}^{KJ}$ is the curvature of an $SO(1,3)$ connection $\boldsymbol{\omega}^{IJ}$ and $\mathbf{e}^I$ is a tetrad.

The variations of the action (3.32) give the equations

$$\begin{aligned} \epsilon_{IJKL}\mathbf{e}^I \wedge \mathbf{D}\mathbf{e}^J = 0 \ , \\ \epsilon_{IJKL}\mathbf{e}^J \wedge \mathbf{F}^{KL} = 0 \ . \end{aligned} \tag{3.33}$$

For nondegenerate tetrads $\mathbf{e}^I$, the previous equations can be reduced [60], [61], by using Lemma 20 of Appendix B, to

$$\mathbf{D}\mathbf{e}^I = 0 \ , \tag{3.34}$$

$$\epsilon_{IJKL}\mathbf{e}^J \wedge \mathbf{F}^{KL} = 0 \ . \tag{3.35}$$

Equation (3.34) implies that the affine connection defined by $\boldsymbol{\omega}^{IJ}$ and $\mathbf{e}^I$ is torsionless and equations (3.35) are equivalent to Einstein's field equations. In this way we recover General Relativity.

The configuration space is then

$$Q = \mathcal{C}^\infty(\Sigma)^4 \times \Omega^1(\Sigma)^4 \times \mathcal{C}^\infty(\Sigma)^6 \times \Omega^1(\Sigma)^6 \ ,$$

and the Lagrangian is defined by

$$L_{\mathrm{CP}}(\mathrm{v}) = \int_\Sigma \epsilon_{IJKL}\left(2e_t^I e^J \wedge F^{KL} + e^I \wedge e^J \wedge \left(\mathrm{v}_\omega^{KL} - D\omega_t^{KL}\right)\right) \ . \tag{3.36}$$

3.4.1 Hamiltonian analysis

The computations done in this subsection are generalized by the analysis of the parametrized unimodular Holst action in [62]. In order to recover the expressions for the Palatini action from that paper one must take the limit $\gamma \to \infty$ and $\Lambda = 0$. We start our analysis by computing the fiber derivative (2.3) of the Lagrangian (3.36)

$$FL_{\mathrm{CP}}(\mathrm{v})(\mathrm{w}) = \int_\Sigma \epsilon_{IJKL}\left(e^I \wedge e^J \wedge \mathrm{w}_\omega^{KL}\right) \ ,$$

which defines the canonical momenta

$$\begin{aligned}
\mathbf{p}_\omega(\mathrm{w}) &= \int_\Sigma \epsilon_{IJKL}\left(e^I \wedge e^J \wedge \mathrm{w}_\omega^{KL}\right) \ , \\
\mathbf{p}_e &= 0 \ , \\
\mathbf{p}_{e_t} &= 0 \ , \\
\mathbf{p}_{\omega_t} &= 0 \ .
\end{aligned}$$

Clearly, $FL_{\mathrm{CP}} : TQ \longrightarrow T^*Q$ is not a diffeomorphism, hence the system is singular.

The Hamiltonian (2.5) is

$$H(\mathbf{p}) = \int_\Sigma \epsilon_{IJKL}\left(e^I \wedge e^J \wedge D\omega_t^{KL} - 2e_t^I e^J \wedge F^{KL}\right) \ . \tag{3.37}$$

The canonical symplectic form of T^*Q is

$$\begin{aligned}
\Omega(\mathbb{Z},\mathbb{Y}) = &\mathbf{Y}_e(X_e) - \mathbf{X}_e(Y_e) + \mathbf{Y}_\omega(X_\omega) - \mathbf{X}_\omega(Y_\omega) \\
&+ \mathbf{Y}_{e_t}(X_{e_t}) - \mathbf{X}_{e_t}(Y_{e_t}) + \mathbf{Y}_{\omega_t}(X_{\omega_t}) - \mathbf{X}_{\omega_t}(Y_{\omega_t}) \ ,
\end{aligned}$$

and its pullback to the primary constraint manifold

$$\omega(\mathbb{Z},\mathbb{Y}) = 2\int_\Sigma \epsilon_{IJKL} e^I \wedge \left(Y_e^J \wedge Z_\omega^{KL} - Z_e^J \wedge Y_\omega^{KL}\right) \ . \tag{3.38}$$

We compute the differential of the Hamiltonian (3.37)

$$\mathrm{d}H(\mathbb{Y}) = \int_\Sigma \Big(- 2Y_{e_t}^I \epsilon_{IJKL} e^J \wedge F^{KL} \tag{3.39}$$

$$- Y_{\omega_t}^{IJ} \epsilon_{IJKL} \wedge D(e^K \wedge e^L) \tag{3.40}$$

$$+ 2Y_e^I \wedge \epsilon_{IJKL}\left(e^J \wedge D\omega_t^{KL} + e_t^J \wedge F^{KL}\right) \tag{3.41}$$

$$+ 2Y_\omega^{IJ} \wedge \left(\epsilon_{IKLM}\omega_{tJ}{}^K e^L \wedge e^M - \epsilon_{IJKL} D(e_t^K e^L)\right)\Big) \ . \tag{3.42}$$

By equating (3.38) and (3.39) one sees that the Hamiltonian equation (2.6) can only be solved in the submanifold defined by the constraints

$$\begin{aligned}
\epsilon_{IJKL} e^J \wedge F^{KL} &= 0 \ , \\
\epsilon_{IJKL} D(e^K \wedge e^L) &= 0 \ ,
\end{aligned} \tag{3.43}$$

and the Hamiltonian vector field is determined by the equations

$$\begin{aligned}
\epsilon_{IJKL} e^J \wedge Z_\omega^{KL} &= \epsilon_{IJKL}\left(e^J \wedge D\omega_t^{KL} + e_t^J F^{KL}\right) \ , \\
\epsilon_{IJKL} e^K \wedge Z_e^L &= \epsilon_{IJKL} D(e_t^K e^L) - \epsilon_{IKLM}\omega_{tJ}{}^K e^L \wedge e^M \ .
\end{aligned} \tag{3.44}$$

Note that the components Z_{ω_t} and Z_{e_t} are undetermined. A particularity that the equations (3.44) present is that the internal indices I run from 0 to 3, while the manifold where they are posed is only 3-dimensional. An effective way to study these equations is to separately consider the cases $I = 0$ and $I = i$. When doing this, we make use of $\epsilon_{ijk} := \epsilon_{0ijk}$, where $i, j, k = 1, 2, 3$. We then get

$$\begin{aligned}
\epsilon_{jkl} e^j \wedge Z_\omega^{kl} &= \epsilon_{jkl} \left(e^j \wedge D\omega_t^{kl} + e_t^j F^{kl} \right) \ , \\
\epsilon_{ijk} \left(2e^j \wedge Z_\omega^{0k} - e^0 \wedge Z_\omega^{jk} \right) &= \epsilon_{ijk} \left(2e^j \wedge D\omega_t^{0k} + 2e_t^j F^{0k} - e^0 \wedge D\omega_t^{jk} - e_t^0 F^{jk} \right) \ , \\
\epsilon_{ijk} e^j \wedge Z_e^k &= \epsilon_{ijk} D(e_t^j e^k) - \epsilon_{jkl} \omega_{ti}{}^j e^k \wedge e^l \ , \\
\epsilon_{ijk} \left(e^0 \wedge Z_e^k - e^k \wedge Z_e^0 \right) &= \epsilon_{ijk} D \left(e_t^0 e^k - e_t^k e^0 \right) - \epsilon_{ilm} \left(-\omega_{tj}{}^0 e^l \wedge e^m + 2\omega_{tj}{}^l e^0 \wedge e^m \right) \ .
\end{aligned}$$

Note that in the previous expressions, although only spatial indices are visible, the covariant derivatives (including the curvature) contain sums over all indices, for instance, $F^{kl} = d\omega^{kl} + \omega^k{}_m \wedge \omega^{ml} + \omega^k{}_0 \wedge \omega^{0l}$. By making use of the constraints (3.43) and using Lemmas 13, 14 and 15 of Appendix B one is able to solve these equations to obtain after a lot of work

$$\begin{aligned}
Z_e^I &= De_t^I - \omega_t{}^I{}_K e^K - \frac{1}{2} e_t^I \epsilon_{jkl} \left(\frac{e^k \wedge De^l}{\mathsf{vol}_e} \right) e^j - \epsilon_{jkl} e_t^j \left(\frac{e^I \wedge De^k}{\mathsf{vol}_e} \right) e^l \ , \\
Z_\omega^{ij} &= D\omega_t^{ij} - \epsilon_{ij}{}^k \left(\tau_{kl} e^l + \frac{1}{4} \epsilon_{klm} \epsilon^{pqr} e_{tp} \left(\frac{e^l \wedge F_{qr}}{\mathsf{vol}_e} \right) e^m \right) \ , \\
Z_\omega^{i0} &= D\omega_t^{i0} - \tau_{jk} \left(\frac{e^k \wedge e^i \wedge e^0}{\mathsf{vol}_e} \right) e^j \\
&\quad + \frac{1}{4} \epsilon_{jkl} e_t^0 \left(\frac{e^j \wedge F^{kl}}{\mathsf{vol}_e} \right) e^i + \frac{1}{2} \epsilon_{jkl} e_t^j \left(\frac{e^k \wedge F^l{}_0}{\mathsf{vol}_e} \right) e^i \\
&\quad - \frac{1}{2} \epsilon_{jkl} e_t^0 \left(\frac{e^i \wedge F^{kl}}{\mathsf{vol}_e} \right) e^j + \epsilon_{jkl} e_t^k \left(\frac{e^i \wedge F^l{}_0}{\mathsf{vol}_e} \right) e^j \\
&\quad - \frac{1}{2} e_t^j \left(\frac{e^k \wedge F_{kl}}{\mathsf{vol}_e} \right) \left(\frac{e^j \wedge e^l \wedge e^0}{\mathsf{vol}_e} \right) e^i + \frac{1}{4} e_t^j \left(\frac{e^j \wedge F_{kl}}{\mathsf{vol}_e} \right) \left(\frac{e^k \wedge e^l \wedge e^0}{\mathsf{vol}_e} \right) e^i \\
&\quad - \frac{1}{2} e_{tj} \left(\frac{e^k \wedge F_{kl}}{\mathsf{vol}_e} \right) \left(\frac{e^l \wedge e^i \wedge e^0}{\mathsf{vol}_e} \right) e^j + \frac{1}{2} e_{tj} \left(\frac{e^j \wedge F_{kl}}{\mathsf{vol}_e} \right) \left(\frac{e^l \wedge e^i \wedge e^0}{\mathsf{vol}_e} \right) e^k \ ,
\end{aligned}$$

where the $\tau_{ij} \in \mathcal{C}^\infty(\Sigma)$ are arbitrary functions symmetric in ij.

Having obtained these expressions, it is still necessary to check the tangency of the obtained solution to the constraint submanifold. This can be done as explained in [62] by taking the appropriate limits. In that process some additional secondary constraints appear

$$e^{(i} \wedge De^{j)} = 0 \ .$$

Note that these constraints can be easily obtained in a straightforward way by using the equations of motion method explained in Subsection 3.4.3.

3.4.2 Lagrangian analysis

Since the configuration space of the Lagrangian (3.36) is

$$Q = \mathcal{C}^\infty(\Sigma)^4 \times \Omega^1(\Sigma)^4 \times \mathcal{C}^\infty(\Sigma)^6 \times \Omega^1(\Sigma)^6 \ ,$$

we can use the STL simplification. We must first compute the submanifold given by Proposition 4. The presymplectic form is

$$\omega_{L_{\rm CP}}(\mathbb{Z},\mathbb{Y}) = 2\int_\Sigma \epsilon_{IJKL} e^I \wedge \left(Y_e^J \wedge Z_\omega^{KL} - Z_e^J \wedge Y_\omega^{KL}\right) \ ,$$

which has the same expression as (3.38), as expected by the discussion in Section 2.5, and its kernel is given by

$$\ker \omega_{L_{\rm CP}} = \{\mathbb{X} \in \mathfrak{X}(TQ) \mid \epsilon_{IJKL} e^I \wedge X_e^J = 0 \ , \ \epsilon_{IJKL} e^J \wedge X_\omega^{KL} = 0 \ \} \ .$$

By splitting indices, the expressions become

$$\begin{aligned} \epsilon_{ijk} e^0 \wedge X_e^i - \epsilon_{ijk} e^i \wedge X_e^0 &= 0 \ , \\ \epsilon_{ijk} e^i \wedge X_e^j &= 0 \ , \\ \epsilon_{ijk} e^i \wedge X_\omega^{jk} &= 0 \ , \\ 2\epsilon_{ijk} e^j \wedge X_\omega^{0k} - \epsilon_{ijk} e^0 \wedge X_\omega^{jk} &= 0 \ , \end{aligned}$$

and by using Lemmas 14 and 15, we find that these conditions are actually equivalent to

$$\ker \omega_{L_{\rm CP}} = \{\mathbb{X} \in \mathfrak{X}(TQ) \mid X_e^J = 0 \ , \ X_\omega^{KL} = 0 \ \} \ . \tag{3.45}$$

Likewise, the energy is

$$E_{L_{\rm CP}}(\mathrm{v}) = \int_\Sigma \epsilon_{IJKL} \left(e^I \wedge e^J \wedge D\omega_t^{KL} - 2e_t^I e^J \wedge F^{KL}\right) \ ,$$

which has the same expression as (3.37).

Now we use Proposition 4. For any $\mathbb{X} \in \ker \omega_{L_{\rm CP}}$, compute

$$\begin{aligned} \imath_{\mathbb{X}} \mathrm{d} E_{L_{\rm CP}} &= \int_\Sigma \epsilon_{IJKL} \left(e^I \wedge e^J \wedge D X_{\omega_t}^{KL} - 2X_{e_t}^I e^J \wedge F^{KL}\right) \\ &= -\int_\Sigma 2\epsilon_{IJKL} \left(e^I \wedge De^J X_{\omega_t}^{KL} + X_{e_t}^I e^J \wedge F^{KL}\right) \ . \end{aligned}$$

The points where this vanishes define the submanifold $\mathcal{S}_1$, determined by the constraints

$$\begin{aligned} \epsilon_{IJKL} e^I \wedge De^J &= 0 \ , \\ \epsilon_{IJKL} e^J \wedge F^{KL} &= 0 \ . \end{aligned}$$

Because of (3.45), one has $\mathfrak{K} = \mathfrak{X}(TQ)$, hence (2.14) of Proposition 7 requires

$$0 = \int_\Sigma \epsilon_{IJKL} e^I \wedge \left(\left(\mathrm{v}_e^J - Z_e^J\right) \wedge X_\omega^{KL} - X_e^J \wedge \left(\mathrm{v}_\omega^{KL} - Z_\omega^{KL}\right)\right) \ ,$$

for any $\mathbb{X} \in \mathfrak{X}(TQ)$ and $\mathbb{Z}$ any solution to (2.11) in $\mathcal{S}_1$. Splitting the indices of these expressions we obtain

$$\begin{aligned}\epsilon_{IJKL} e^I \wedge \left(\mathrm{v}_e^J - Z_e^J\right) \wedge X_\omega^{KL} &= \epsilon_{ijk}\left(e^0 \wedge \left(\mathrm{v}_e^j - Z_e^j\right) - e^i \wedge \left(\mathrm{v}_e^0 - Z_e^0\right)\right) \wedge X_\omega^{kl} \\ &\quad + 2\epsilon_{ijk} e^i \wedge \left(\mathrm{v}_e^j - Z_e^j\right) \wedge X_\omega^{0k} \ , \\ \epsilon_{IJKL} e^I \wedge X_e^J \wedge \left(\mathrm{v}_\omega^{KL} - Z_\omega^{KL}\right) &= \epsilon_{ijk}\left(2e^j \wedge \left(\mathrm{v}_\omega^{0k} - Z_\omega^{0k}\right) - e^0 \wedge \left(\mathrm{v}_\omega^{jk} - Z_\omega^{jk}\right)\right) \wedge X_e^i \\ &\quad + \epsilon_{ijk} e^i \wedge \left(\mathrm{v}_\omega^{jk} - Z_\omega^{jk}\right) \wedge X_e^0 \ ,\end{aligned}$$

and since this must vanish for every $\mathbb{X}$, by Lemmas 14 and 15 this implies that the additional constraints $\mathcal{S}_2$ are given by

$$\begin{aligned} Z_e^I &= \mathrm{v}_e^I \ , \\ Z_\omega^I &= \mathrm{v}_\omega^I \ ,\end{aligned}$$

which implement the second order condition. Finally, computing (2.11) yields the equations for the Hamiltonian vector field

$$\begin{aligned}\epsilon_{IJKL} e^J \wedge Z_\omega^{KL} &= \epsilon_{IJKL}\left(e^J \wedge D\omega_t^{KL} + e_t^J F^{KL}\right) \ , \\ \epsilon_{IJKL} e^K \wedge Z_e^L &= \epsilon_{IKLM} \omega_{tJ}{}^K e^L \wedge e^M - \epsilon_{IJKL} D(e_t^K e^L) \ .\end{aligned}$$

Since the constraints and equations for the Hamiltonian vector field are identical to those found in Subsection 3.4.1, their solution and tangency analysis proceed in the same way.

3.4.3 Equations of motion analysis

We are now going to use the equations of motion method described in Section 2.4. We need to compute (2.17) for the equations of motion (3.34) and (3.35). As shown in [61], the equations of motion decompose into

$$\begin{aligned}\mathbf{D}\mathbf{e}^I &= \mathbf{d}t \wedge \left(\mathrm{v}_{\mathbf{e}}^I - \underline{\mathbf{D}}\mathbf{e}_t^I + \boldsymbol{\omega}_t{}^I{}_J \underline{\mathbf{e}}^J\right) + \underline{\mathbf{D}}\underline{\mathbf{e}}^I \ , \\ \epsilon_{IJKL}\mathbf{e}^J \wedge \mathbf{F}^{KL} &= \epsilon_{IJKL}\underline{\mathbf{e}}^J \wedge \underline{\mathbf{F}}^{KL} + \mathbf{d}t \wedge \epsilon_{IJKL}\left(\mathbf{e}_t^J \underline{\mathbf{F}}^{KL} - \underline{\mathbf{e}}^J \wedge \left(\mathrm{v}_{\boldsymbol{\omega}}^{KL} - \underline{\mathbf{D}}\boldsymbol{\omega}_t^{KL}\right)\right) \ ,\end{aligned}$$

hence by extracting the adapted components of these equations we obtain the constraints

$$\begin{aligned} De^I &= 0 \ , \\ \epsilon_{IJKL} e^J \wedge F^{KL} &= 0 \ ,\end{aligned}$$

and the equations for $\mathbb{Z}$

$$\begin{aligned} Z_e^I &= De_T^I - \omega_t{}^I{}_J e^J \ , \\ \epsilon_{IJKL} e^J \wedge \left(Z_\omega^{KL} - D\omega_t^{KL}\right) &= \epsilon_{IJKL} e_t^J F^{KL} \ ,\end{aligned}$$

while Z_{e_t}, Z_{ω_t} are arbitrary. The expression of the Hamiltonian vector field and the tangency analysis proceed in the same way as in Subsection 3.4.1.

It is remarkable how using this method we directly get the constraints $De^I = 0$ rather than the constraints $\epsilon_{IJKL} D(e^K \wedge e^L) = 0$ obtained in Subsection 3.4.1. This happened because we were able to find the alternative expression (or partially solve) (3.34), (3.35) for the equations of motion (3.33). In particular, notice that the equations $\epsilon_{IJKL}\mathbf{e}^I \wedge \mathbf{D}\mathbf{e}^J = 0$ are 3-forms in a 4-dimensional manifold with 2 antisymmetric indices, which means that they amount to 24 equations. The equations $\mathbf{D}\mathbf{e}^I = 0$ are 2-forms with a free index, which amount to 24 equations again. However, when taking the pullback of both sets, the number of independent constraints change. In the former case we now have 3-forms in a *3-dimensional manifold* with 2 antisymmetric indices, that is, 6 equations. In the latter case we have 2-forms with 1 free index, i.e. 12 equations. Of course, since in the first approach there appear less constraints a priori, when properly performing the GNH analysis, six more constraints must appear to match the twelve constraints found by the second approach. This shows that the possibility of rewriting the field equations in different forms may help find the constraints in a simple and direct way, much simpler, in fact, that the standard implementation of the GNH procedure (or even the same field equations method without rewriting the equations).

Chapter 4

Anti-self-dual action

In the 80s and early 90s there was a particular interest in gravitational theories of self-dual connections [63]–[65]. Since the self-dual part of the spin connection equals the connection resulting from the equations of motion of the self-dual action, one can see that, although self-dual connections have half as many components as their general counterparts, the self-dual action also yields General Relativity. Hence, the idea was to use such actions with a reduced number of fields with the hope of dealing with a simpler theory. However, an unintended consequence of that approach is the necessity to work with complex fields in order to recover Lorentzian General Relativity. This gives rise to some hard technical difficulties, in particular, in order to have real metrics one has to introduce by hand some reality conditions whose consistency must me checked. Moreover, the Hamiltonian formulation of field theories defined by complex Lagrangians presents some subtleties and differs in a significant way from the real case. It is in this sense that real fields are preferred in these treatments, since they automatically clarify or bypass many difficulties caused by the presence of complex fields. Because of this, when the real Ashtekar variables appeared [66], self-dual theories with complex fields were quickly forgone.

The interest for self-dual formulations of General Relativity originated a long time ago. Plebański introduced in [67] a formulation for General Relativity where self-dual 2-forms played a central role, which was later seen to be equivalent to Ashtekar's formulation [64]. Also, the vacuum Einstein equations were found to be the condition that the curvature of a self-dual connection is self-dual in [68]. Moreover, Petrov's classification of spaces [69] heavily relies on self-duality. It is clear that self-dual connections have a privileged spot in the variational setting for General Relativity. ith that in mind, it is worth paying some attention to the Euclidean version of self-dual formulations of General Relativity, which are interesting for a few reasons. The path integral formulation for the quantization of this type of theories yields a Gaussian-like probability distribution, which can be computed through a perturbative expansion [70]. Then, one can attempt to perform a Wick rotation hoping to recover the Lorentzian version. Also, Donaldson invariants in 4-manifolds

[71] are interesting from a mathematical point of view and are connected to Yang-Mills theories. In this chapter, we will retake the study of (anti)-self-dual Euclidean General Relativity, which is defined in terms of real fields as opposed to its Lorentzian version, and carefully perform its Hamiltonian analysis.

The internal Hodge dual of an $SO(n)$ or $SO(1,n-1)$ connection $\boldsymbol{\omega}^I{}_J$, depending on the signature $\varepsilon = \pm 1$, satisfies

$$\left(*^2\boldsymbol{\omega}\right)^I{}_J = \varepsilon \boldsymbol{\omega}^I{}_J \ .$$

Note that in the Lorentzian case, $*$ is diagonalizable in the complex numbers and its eigenvalues are $\pm i$, while in the Euclidean case it is diagonalizable in the real numbers with eigenvalues ± 1. This implies that a Lorentzian theory of (anti)-self-dual fields forces them to be complex, with

$$\left(*\boldsymbol{\omega}^{\pm}\right)^I{}_J = \pm i \left(\boldsymbol{\omega}^{\pm}\right)^I{}_J \ ,$$

for the self-dual sector $\boldsymbol{\omega}^+$ and the anti-self-dual sector $\boldsymbol{\omega}^-$, while in the Euclidean case we have

$$\left(*\boldsymbol{\omega}^{\pm}\right)^I{}_J = \pm \left(\boldsymbol{\omega}^{\pm}\right)^I{}_J \ .$$

Hence the anti-self-dual Euclidean Cartan-Palatini action is

$$S_{\text{ASD-CP}}(\mathbf{e},\boldsymbol{\omega}) = \int_{\mathcal{M}} \epsilon_{IJKL} \mathbf{e}^I \wedge \mathbf{e}^J \wedge \left(\mathbf{F}^-\right)^{KL} \ , \tag{4.1}$$

where $\left(\mathbf{F}^-\right)^{KL} = \mathbf{d}\left(\boldsymbol{\omega}^-\right)^{KL} + \left(\boldsymbol{\omega}^-\right)^K{}_J \wedge \left(\boldsymbol{\omega}^-\right)^{JL}$, which is very similar to (3.32) but with the difference that $\left(\boldsymbol{\omega}^-\right)^{KL}$ is now an anti-self-dual $SO(4)$ connection $\left(*\boldsymbol{\omega}^-\right)^{KL} = -\left(\boldsymbol{\omega}^-\right)^{KL}$, this is,

$$\left(\boldsymbol{\omega}^-\right)^{IJ} = -\frac{1}{2}\epsilon^{IJ}{}_{KL}\left(\boldsymbol{\omega}^-\right)^{KL} \ .$$

Notice that in this form, the variations of $\left(\boldsymbol{\omega}^-\right)^{KL}$ only run through the anti-self-dual connections, which can be an unpleasant restriction to deal with. An alternative way to write the action (4.1) is

$$S_{\text{ASD-CP}}(\mathbf{e},\boldsymbol{\omega}) = \int_{\mathcal{M}} \epsilon_{IJKL} \mathbf{e}^I \wedge \mathbf{e}^J \wedge P_-{}^{KL}{}_{MN} \mathbf{F}^{MN} \ ,$$

where $\mathbf{F}^{IJ} = \mathbf{d}\boldsymbol{\omega}^{IJ} + \boldsymbol{\omega}^I{}_K \wedge \boldsymbol{\omega}^{KJ}$ and $P_-{}^{KL}{}_{MN}$ is the projector onto the anti-self-dual sector of $\mathfrak{so}(4)$. By doing this, the variations of $\boldsymbol{\omega}^{IJ}$ can be taken now to be arbitrary. The full expression of $P_\pm{}^{IJ}{}_{KL}$ is

$$P_\pm{}^{IJ}{}_{KL} = \frac{1}{2}\left(\delta^I{}_K \delta^J{}_L - \delta^J{}_K \delta^I{}_L \pm \epsilon^{IJ}{}_{KL}\right) \ .$$

Since (anti)-self-dual connections only have three independent components, this relation allows us to introduce three 1-forms $\boldsymbol{\omega}^i$, $i = 1,2,3$

$$\boldsymbol{\omega}^i = 2\left(\boldsymbol{\omega}^-\right)^{0i} \ ,$$

related to the other components of $\boldsymbol{\omega}^{IJ}$ by

$$\left(\boldsymbol{\omega}^{-}\right)^{ij} = -\frac{1}{2}\epsilon^{ij}{}_{k}\boldsymbol{\omega}^{k} \ .$$

Hence, one can write $\left(\boldsymbol{\omega}^{-}\right)^{IJ}$ — and in general elements of the anti-self-dual sector of $\mathfrak{so}(4)$ — as

$$\left(\boldsymbol{\omega}^{-}\right)^{IJ} = \frac{1}{2}\begin{pmatrix} 0 & \boldsymbol{\omega}^1 & \boldsymbol{\omega}^2 & \boldsymbol{\omega}^3 \\ -\boldsymbol{\omega}^1 & 0 & -\boldsymbol{\omega}^3 & \boldsymbol{\omega}^2 \\ -\boldsymbol{\omega}^2 & \boldsymbol{\omega}^3 & 0 & -\boldsymbol{\omega}^1 \\ -\boldsymbol{\omega}^3 & -\boldsymbol{\omega}^2 & \boldsymbol{\omega}^1 & 0 \end{pmatrix} \ .$$

This yields the relations

$$\left(\mathbf{F}^{-}\right)^{0i} = \frac{1}{2}\mathbf{F}^{i} \ , \ \left(\mathbf{F}^{-}\right)^{ij} = -\frac{1}{2}\epsilon^{ij}{}_{k}\mathbf{F}^{k} \ ,$$

where F^i is a new curvature defined as $\mathbf{F}^i = \mathbf{d}\boldsymbol{\omega}^i + \frac{1}{2}\epsilon^{i}{}_{jk}\boldsymbol{\omega}^j \wedge \boldsymbol{\omega}^k$. Similarly, elements of the self-dual sector of $\mathfrak{so}(4)$ defined by

$$B^{IJ} = \frac{1}{2}\epsilon^{IJ}{}_{KL}B^{KL} \ .$$

can be written as

$$B^{IJ} = \begin{pmatrix} 0 & B^1 & B^2 & B^3 \\ -B^1 & 0 & B^3 & -B^2 \\ -B^2 & -B^3 & 0 & B^1 \\ -B^3 & B^2 & -B^1 & 0 \end{pmatrix} \ .$$

Let us also rename $\boldsymbol{\alpha} := \mathbf{e}^0$. Now rewriting the action (4.1) with the new variables $(\boldsymbol{\alpha}, \mathbf{e}^i, \boldsymbol{\omega}^i)$ by splitting the indices in 0 and $i = 1, 2, 3$ we obtain [72]

$$S_{\text{ASD}}(\mathbf{e}, \boldsymbol{\omega}, \boldsymbol{\alpha}) = \int_{\mathcal{M}} \left(\epsilon_{ijk}\mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{F}^k - 2\boldsymbol{\alpha} \wedge \mathbf{e}_i \wedge \mathbf{F}^i\right) \ . \tag{4.2}$$

Note that the curvature $\mathbf{F}^i$ is the same as in the Husain-Kuchař model studied in Section (3.3).

Note that we can treat i, j, k as $SO(3)$ indices since the gauge transformation

$$\begin{aligned} \delta_{\boldsymbol{\Lambda}}\boldsymbol{\omega}^i &:= (Z_{\boldsymbol{\Lambda}})^i_{\boldsymbol{\omega}} = -\mathbf{D}\Lambda^i \ , \\ \delta_{\boldsymbol{\Lambda}}\boldsymbol{\alpha} &:= (Z_{\boldsymbol{\Lambda}})_{\alpha} = 0 \ , \\ \delta_{\boldsymbol{\Lambda}}\mathbf{e}^i &:= (Z_{\boldsymbol{\Lambda}})^i_{\mathbf{e}} = -\epsilon^{i}{}_{jk}\mathbf{e}^j\Lambda^k \ , \end{aligned} \tag{4.3}$$

which is a linear combination of a self-dual and anti-self-dual $SO(4)$ transformations, acts as an $SO(3)$ transformation on the fields. The gauge transformations corresponding to the other $SO(3)$ factor (recall that $SO(4) = SO(3) \otimes SO(3)$) are [72]

$$\begin{aligned}
\delta_{\boldsymbol{\tau}} \boldsymbol{\omega}^i &:= (Z_{\boldsymbol{\tau}})^i_{\boldsymbol{\omega}} = 0 \ , \\
\delta_{\boldsymbol{\tau}} \boldsymbol{\alpha} &:= (Z_{\boldsymbol{\tau}})_{\alpha} = \tau_i \mathbf{e}^i \ , \\
\delta_{\boldsymbol{\tau}} \mathbf{e}^i &:= (Z_{\boldsymbol{\tau}})^i_{\mathbf{e}} = -\tau^i \boldsymbol{\alpha} + \epsilon^i{}_{jk} e^j \tau^k \ .
\end{aligned} \tag{4.4}$$

We use the Z notation for the gauge transformations since we can interpret them as vectors in the kernel of the exterior differential of the action, as explained in Chapter 5. It is remarkable that setting $\boldsymbol{\alpha} = 0$ in (4.2) we obtain exactly the Husain-Kuchař action (3.18), and that by adding an additional term we arrive at Euclidean General Relativity.

The equations of motions given by the action (4.2) are

$$\mathbf{D}\left(\boldsymbol{\alpha} \wedge \mathbf{e}_k\right) + \epsilon_{ijk} \mathbf{e}^i \wedge \mathbf{D}\mathbf{e}^j = 0 \ , \tag{4.5}$$

$$\mathbf{e}_i \wedge \mathbf{F}^i = 0 \ , \tag{4.6}$$

$$\epsilon_{ijk} \mathbf{e}^j \wedge \mathbf{F}^k + \boldsymbol{\alpha} \wedge \mathbf{F}_i = 0 \ . \tag{4.7}$$

It is actually possible to solve equation (4.5) for $\boldsymbol{\omega}^i$, yielding the solution

$$\begin{aligned}
\boldsymbol{\omega}^i =& \frac{1}{2}\left(\frac{\left(\boldsymbol{\alpha} \wedge \mathbf{d}\boldsymbol{\alpha} - \mathbf{e}_k \wedge \mathbf{d}\mathbf{e}^k\right) \wedge \mathbf{e}^i - \epsilon^i{}_{jk} \mathbf{e}^j \wedge \mathbf{e}^k \wedge \mathbf{d}\boldsymbol{\alpha}}{\mathsf{vol}_{\mathbf{e}}}\right) \boldsymbol{\alpha} \\
&- \frac{1}{2}\left(\frac{\boldsymbol{\alpha} \wedge \mathbf{e}_k \wedge \mathbf{d}\mathbf{e}^k + \epsilon_{jkl} \mathbf{e}^j \wedge \mathbf{e}^k \wedge \mathbf{d}\mathbf{e}^l}{\mathsf{vol}_{\mathbf{e}}}\right) \mathbf{e}^i \\
&+ \left(\frac{\boldsymbol{\alpha} \wedge \mathbf{e}^i \wedge \mathbf{d}\mathbf{e}_j}{\mathsf{vol}_{\mathbf{e}}}\right) \mathbf{e}^j + \epsilon_{kl(j} \left(\frac{\mathbf{e}^{i)} \wedge \mathbf{e}^k \wedge \mathbf{d}\mathbf{e}^l}{\mathsf{vol}_{\mathbf{e}}}\right) \mathbf{e}^j \\
&+ \frac{1}{2} \epsilon^i{}_{jk} \left(\frac{\boldsymbol{\alpha} \wedge \mathbf{d}\boldsymbol{\alpha} \wedge \mathbf{e}^k}{\mathsf{vol}_{\mathbf{e}}}\right) \mathbf{e}^j \ .
\end{aligned}$$

Then, by using this expression in equations (4.6), (4.7), one obtains Einstein's field equations.

The Lagrangian obtained from the action (4.2) in the configuration space

$$Q = \mathcal{C}^\infty(\Sigma)^3 \times \Omega^1(\Sigma)^3 \times \mathcal{C}^\infty(\Sigma)^3 \times \Omega^1(\Sigma)^3 \times \mathcal{C}^\infty(\Sigma) \times \Omega^1(\Sigma) \ ,$$

is

$$\begin{aligned}
L_{\mathrm{ASD}}(\mathrm{v}) = \int_\Sigma \Bigg(&\left(\epsilon_{ijk} e^i \wedge e^j - 2\alpha \wedge e_k\right) \wedge \mathrm{v}^k_\omega + D\left(\epsilon_{ijk} e^i \wedge e^j - 2\alpha \wedge e_k\right) \omega^k_t \\
&- 2\alpha_t e_i \wedge F^i + 2 e^i_t \left(\epsilon_{ijk} e^j \wedge F^k + \alpha \wedge F^i\right) \Bigg) \ ,
\end{aligned}$$

and its fiber derivative (2.3) is

$$FL_{\mathrm{ASD}}(\mathrm{v})(\mathrm{w}) = \int_\Sigma \left(\epsilon_{ijk} e^i \wedge e^j - 2\alpha \wedge e_k\right) \wedge \mathrm{w}^k_\omega \ .$$

Then, the canonical momenta, which define the primary constraint manifold, are

$$
\begin{aligned}
\mathbf{p}_\omega(\mathrm{w}) &= \int_\Sigma \left(\epsilon_{ijk}e^i \wedge e^j - 2\alpha \wedge e_k\right) \wedge \mathrm{w}^k_\omega \ , \\
\mathbf{p}_e &= 0 \ , \\
\mathbf{p}_\alpha &= 0 \ , \\
\mathbf{p}_{e_t} &= 0 \ , \\
\mathbf{p}_{\omega_t} &= 0 \ , \\
\mathbf{p}_{\alpha_t} &= 0 \ .
\end{aligned}
$$

The Hamiltonian (2.5) is

$$
H(\mathbf{p}) = -\int_\Sigma \left(D\left(\epsilon_{ijk}e^i \wedge e^j - 2\alpha \wedge e_k\right)\omega^k_t - 2\alpha_t e_i \wedge F^i + 2e^i_t\left(\epsilon_{ijk}e^j \wedge F^k + \alpha \wedge F_i\right)\right) \ , \tag{4.8}
$$

and its exterior derivative

$$
\begin{aligned}
&\mathbb{d}H(\mathbb{Y}) = \\
&\quad -2\int_\Sigma \Big(Y_\alpha \wedge \left(e_{ti}F^i + \omega_{ti}De^i - d(\omega_{ti}e^i)\right) \\
&\qquad + Y^i_e \wedge \left(\epsilon_{ijk}\left(\omega^j_t De^k + D(e^j\omega^k_t + F^j e^k_t)\right) - \omega_{ti}d\alpha + D(\omega_{ti}\alpha) - \alpha_t F_i\right) \\
&\qquad + Y^i_\omega \wedge \left(D(e_{ti}\alpha - \alpha_t e_i + \epsilon_{ijk}e^j_t e^k) + e^j \wedge e_i\omega_{tj} - \epsilon_{ijk}\alpha \wedge e^j\omega^k_t\right) \\
&\qquad - Y_{\alpha_t} e_i \wedge F^i \\
&\qquad + Y^i_{\omega_t}\left(\epsilon_{ijk}De^j \wedge e^k - d\alpha \wedge e_i + \alpha \wedge De_i\right) \\
&\qquad + Y^i_{e_t}\left(\epsilon_{ijk}e^j \wedge F^k + \alpha \wedge F_i\right)\Big) \ .
\end{aligned}
$$

The canonical symplectic form of T^*Q is

$$
\begin{aligned}
\Omega(\mathbb{Z},\mathbb{Y}) = {}& \mathbf{Y}_e(X_e) - \mathbf{Z}_e(Y_e) + \mathbf{Y}_\omega(Z_\omega) - \mathbf{Z}_\omega(Y_\omega) + \mathbf{Y}_\alpha(Z_\alpha) - \mathbf{Z}_\alpha(Y_\alpha) \\
& + \mathbf{Y}_{e_t}(Z_{e_t}) - \mathbf{Z}_{e_t}(Y_{e_t}) + \mathbf{Y}_{\omega_t}(Z_{\omega_t}) - \mathbf{Z}_{\omega_t}(Y_{\omega_t}) + \mathbf{Y}_{\alpha_t}(Z_{\alpha_t}) - \mathbf{Z}_{\alpha_t}(Y_{\alpha_t}) \ ,
\end{aligned}
$$

and its pullback to the primary constraint submanifold

$$
\begin{aligned}
\omega(\mathbb{Z},\mathbb{Y}) = 2\int_\Sigma \Big(& Y^i_e \wedge \left(\epsilon_{ijk}e^j + \delta_{ik}\alpha\right) \wedge Z^k_\omega - Y_\alpha \wedge e_k \wedge Z^k_\omega \\
& - Y^k_\omega \wedge \left(Z^i_e \wedge \left(\epsilon_{ijk}e^j + \delta_{ik}\alpha\right) - Z_\alpha \wedge e_k\right)\Big) \ .
\end{aligned}
$$

The equations for the components of the Hamiltonian vector field are

$$
\begin{aligned}
e_{ti}F^i - D\omega_{ti} \wedge e^i &= e_k \wedge Z^k_\omega \ , \\
\epsilon_{ijk}\left(F^j e^k_t - D\omega^j_t \wedge e^k\right) + D\omega_{ti} \wedge \alpha - \alpha_t F_i &= -\left(\epsilon_{ijk}e^j + \delta_{ik}\alpha\right) \wedge Z^k_\omega \ , \\
D(e_{ti}\alpha - \alpha_t e_i + \epsilon_{ijk}e^j_t e^k) + e^j \wedge e_i\omega_{tj} - \epsilon_{ijk}\alpha \wedge e^j\omega^k_t & \\
&= Z^j_e \wedge \left(\epsilon_{ijk}e^k + \delta_{ij}\alpha\right) - Z_\alpha \wedge e_i \ ,
\end{aligned}
$$

which are better written as

$$e_{ti}F^i = e_k \wedge \left(Z_\omega^k - D\omega_t{}^i\right) \ , \tag{4.9a}$$

$$\left(\epsilon_{ijk}e_t^j + \delta_{ik}\alpha_t\right) F^k = \left(\epsilon_{ijk}e^j + \delta_{ik}\alpha\right) \wedge \left(Z_\omega^k - D\omega_t^k\right) \ , \tag{4.9b}$$

$$\begin{aligned}(Z_\alpha - D\alpha_t) \wedge e_i + \left(\epsilon_{ijk}e_t^j - \delta_{ik}\alpha_t\right) De^k + e_{ti}D\alpha \\ = \left(\epsilon_{ijk}e^j - \delta_{ik}\alpha\right) \wedge \left(Z_e^k - De_t^k - \epsilon^k{}_{lm}e^l\omega_t^m\right) \ .\end{aligned} \tag{4.9c}$$

Finally, the constraints are

$$\begin{aligned} e_i \wedge F^i &= 0 \ , \\ D\left(\epsilon_{ijk}e^j \wedge e^k - 2\alpha \wedge e_i\right) &= 0 \ , \\ \left(\epsilon_{ijk}e^j + \delta_{ik}\alpha\right) \wedge F^k &= 0 \ . \end{aligned} \tag{4.10}$$

Other useful alternative expressions for the constraints are

$$\begin{aligned} e_i \wedge F^i &= 0 \ , \\ D\left((e_k\delta_{ij} - \epsilon_{ijk}\alpha) \wedge e^j\right) &= 0 \ , \\ e^j \wedge F^k - e^k \wedge F^j &= -\epsilon^{ijk}\alpha \wedge F_i \ . \end{aligned}$$

Since the system of equations (4.9) is linear, in order to find the general solution we need to both find a particular solution and the solution to the homogeneous system, which is equivalent to finding the kernel of a linear operator. Let us first find the solutions of the homogeneous system defined by (4.9), which is

$$e_k \wedge Z_\omega^k = 0 \ , \tag{4.11a}$$

$$\left(\epsilon_{ijk}e^j + \delta_{ik}\alpha\right) \wedge Z_\omega^k = 0 \ , \tag{4.11b}$$

$$Z_e^j \wedge \left(\epsilon_{ijk}e^k + \delta_{ij}\alpha\right) - Z_\alpha \wedge e_i = 0 \ . \tag{4.11c}$$

Since we are requiring that e^i is a non-degenerate frame (recall that Σ is a 3-dimensional manifold), we can write

$$\begin{aligned} Z_e^i &= \mu^i{}_j e^j \ , \\ Z_\omega^i &= \rho^i{}_j e^j \ , \\ Z_\alpha &= \lambda_i e^i \ . \end{aligned} \tag{4.12}$$

By using (4.12), equation (4.11a) directly gives the condition

$$\rho^k{}_l \epsilon_k{}^{lm} = 0 \ , \tag{4.13}$$

while equation (4.11b) transforms into

$$\left(\delta^l{}_k \delta^m{}_i - \delta^l{}_i \delta^m{}_k + \epsilon^{jlm}\alpha_j \delta_{ik}\right) \rho^k{}_l = 0 \ ,$$

where α_i is defined by $\alpha = \alpha_i e^i$, or alternatively, by Lemma 11 of Appendix B, in terms of densities,

$$\alpha_i = \frac{1}{2}\epsilon_{ijk}\left(\frac{\alpha \wedge e^j \wedge e^k}{\mathsf{vol}_e}\right) \ .$$

The tensors

$$(Q_\pm)^m{}_{i;k}{}^l = \delta^m{}_i\delta^l{}_k - \delta^m{}_k\delta^l{}_i \pm \delta_{ik}\epsilon^{jml}\alpha_j \ , \tag{4.14}$$

can be thought of as matrices in the multi-indices mi and kl, with the former upstairs and the latter downstairs. Note that $(Q_\pm)^m{}_{i;k}{}^l$ is neither symmetric nor antisymmetric in those multiindices. Since we will be interested in solving equations involving these tensors, but with images in no particular subspace, we need to compute the inverse of the matrix in the sense

$$\left(Q_\pm^{-1}\right)^p{}_{q;m}{}^i (Q_\pm)^m{}_{i;k}{}^l = \delta^p_k\delta^l_q \ .$$

This is given by

$$\begin{aligned}\left(Q_\pm^{-1}\right)^p{}_{q;m}{}^i =& \frac{1}{2\left(1+\alpha^2\right)}\Bigg(\delta^p_q\delta^i_m - 2\delta^p_m\delta^i_q + \alpha^2\left(\delta^{pi}\delta_{qm} - \delta^p_m\delta^i_q\right)\\ &+ \delta^p_q\alpha^i\alpha_m - \delta^p_m\alpha^i\alpha_q + \delta^i_m\alpha^p\alpha_q - \delta^i_q\alpha^p\alpha_m - \delta^{pi}\alpha_q\alpha_m - \delta_{qm}\alpha^p\alpha^i\\ &\pm \alpha^r\left(\epsilon_r{}^p{}_q\left(\delta^i_m + \alpha^i\alpha_m\right) + \epsilon_{rm}{}^i\left(\delta^p_q + \alpha^p\alpha_q\right) - 2\epsilon_r{}^{pi}\left(\delta_{qm} + \alpha_q\alpha_m\right)\right)\\ &- \alpha^p\alpha^i\alpha_q\alpha_m\Bigg) \ ,\end{aligned} \tag{4.15}$$

where $\alpha^2 = \alpha_i\alpha^i$.

Interesting properties of the inverse of Q are

$$\left(Q_-^{-1}\right)^p{}_{q;m}{}^i = \left(Q_+^{-1}\right)^i{}_{m;q}{}^p \ , \tag{4.16a}$$

$$\left(Q_\pm^{-1}\right)^p{}_{p;m}{}^i = \frac{1}{2}\left(\delta^i_m - \alpha_m\alpha^i \pm \alpha^r\epsilon_{rm}{}^i\right) \ , \tag{4.16b}$$

$$\epsilon_{ap}{}^q\left(Q_\pm^{-1}\right)^p{}_{q;m}{}^i = \epsilon^i{}_{ma} \pm \delta^i_a\alpha_m \ , \tag{4.16c}$$

$$\alpha_p\left(Q_\pm^{-1}\right)^p{}_{q;m}{}^i = \frac{1}{2}\left(\alpha_q\delta^i_m - 2\alpha_m\delta^i_q - \alpha^i\alpha_q\alpha_m \pm \epsilon_{rm}{}^i\alpha^r\alpha_q\right) \ . \tag{4.16d}$$

Hence, the equation for $\rho^k{}_l$ becomes

$$(Q_-)^m{}_{i;k}{}^l\rho^k{}_l = 0 \ ,$$

and since Q_- is invertible, this means that $\rho^k{}_l = 0$.

By introducing the expressions (4.12) in (4.11c), one obtains

$$0 = \left(\epsilon_{ijk}\mu^j{}_l + \mu^j{}_l\delta_{ij}\alpha_k - \lambda_l\delta_{ik}\right)e^l\wedge e^k \ ,$$

hence the kernel of the linear operator is determined by the solutions to

$$\left(\epsilon_{ij[k} + \delta_{ij}\alpha_{[k}\right)\mu^j{}_{l]} = \lambda_{[l}\delta_{k]i} \ . \tag{4.17}$$

These are 9 relations (since in (4.17) i is free and is antisymmetric in lk) between the 12 functions ${\mu^i}_j, \lambda_i$, revealing that this sector of the kernel can be parametrized by only 3 functions. It is thus possibly more sensible then to try to write the 9 functions ${\mu^i}_j$ in terms of the 3 functions λ^i. In order to do so, one must, in a sense, invert the relation (4.17). By multiplying by δ^{ik} in the previous equation, one obtains

$$\lambda_l = \frac{1}{2}{\mu^j}_k \left({\delta^k}_l \alpha_j - {\delta^k}_j \alpha_l - {\epsilon^k}_{jl}\right) \ .$$

Multiply both sides by the tensor

$${B^j}_{kr} = {\epsilon^j}_{kr} - {\delta^j}_r \alpha_k \ ,$$

which is built in such a way that

$$\frac{1}{2}\left({\delta^k}_l \alpha_j - {\delta^k}_j \alpha_l - {\epsilon^k}_{jl}\right) {B^j}_{kr} = \delta_{rl} \ .$$

This implies that the expression of ${\mu^i}_j$ in terms of λ^i is

$${\mu^i}_j = {\epsilon^i}_{jk}\lambda^k - {\delta^i}_k \alpha_j \lambda^k \ .$$

Then, the kernel of the homogeneous system (4.11) is given by

$$\begin{aligned} Z_\omega^{0\,i} &= 0 \ , \\ Z_e^{0\,i} &= {\epsilon^i}_{jk} e^j \lambda^k - \alpha\lambda^i \ , \\ Z_\alpha^0 &= \lambda_i e^i \ , \end{aligned} \tag{4.18}$$

parametrized by the 3 functions λ^i. This gives the $SO(3)$ symmetry generated by (4.4).

Let us now find a particular solution to the system (4.9). Write $\Xi^i = Z_\omega^i - D\omega_t^i$ and $\Xi^i = {\Xi^i}_j e^j$. Then, by right-multiplying equation (4.9b) by $\cdot \wedge e^m$, it can be conveniently written as

$$\left(\epsilon_{ijk} + \delta_{ik}\alpha_j\right) {\Xi^k}_l e^j \wedge e^l \wedge e^m = \left(\epsilon_{ijk} e_t^j + \delta_{ik}\alpha_t\right) F^k \wedge e^m \ ,$$

which written in terms of densities and by expanding and making use of (4.14) yields

$${(Q_-)^m}_{i;k}{}^l \ {\Xi^k}_l = \left({\delta^l}_k {\delta^m}_i - {\delta^l}_i {\delta^m}_k + \delta_{ik}\epsilon^{jlm}\alpha_j\right) {\Xi^k}_l = \left(\epsilon_{ijk} e_t^j + \delta_{ik}\alpha_t\right) \left(\frac{F^k \wedge e^m}{\mathsf{vol}_e}\right) \ .$$

By using the inverse of Q_- given by (4.15), we find

$${\Xi^p}_q = {\left(Q_-^{-1}\right)^p}_{q;m}{}^i \left(\epsilon_{ijk} e_t^j + \delta_{ik}\alpha_t\right) \left(\frac{F^k \wedge e^m}{\mathsf{vol}_e}\right) \ ,$$

and hence, the expression for the ω component of the Hamiltonian vector field is

$$Z_\omega^i = D\omega_t^i + {\left(Q_-^{-1}\right)^i}_{p;m}{}^q \left(\epsilon_{qjk} e_t^j + \delta_{qk}\alpha_t\right) \left(\frac{F^k \wedge e^m}{\mathsf{vol}_e}\right) e^p \ . \tag{4.19}$$

This must also be a solution to (4.9a). By plugging it into the equation, we will either find that it is automatically satisfied or that it generates a secondary constraint. The expression is

$$e_{ti}F^i = \left(Q_-^{-1}\right)^i{}_{p;m}{}^q \left(\epsilon_{qjk}e_t^j + \delta_{qk}\alpha_t\right)\left(\frac{F^k \wedge e^m}{\mathsf{vol}_e}\right) e_i \wedge e^p \ . \tag{4.20}$$

Since

$$\begin{aligned}\left(Q_-^{-1}\right)^i{}_{p;m}{}^q e_i \wedge e^p &= e^i \wedge e_m \\ &- \frac{1}{2(1+\alpha^2)}\alpha^r \Big(\epsilon_{rpq}e^p \wedge e^q \left(\delta^i_m + \alpha^i\alpha_m\right) - 2\epsilon_{rp}{}^i \left(e^p \wedge e_m + \alpha_m e^p \wedge \alpha\right)\Big) \ ,\end{aligned}$$

the right hand side of (4.20) becomes, by grouping terms by e_t^i and α_t,

$$\begin{aligned}& e_t^j\Bigg(\epsilon_{pjk}\delta_{lm} - \frac{1}{2(1+\alpha^2)}\alpha^r\Big(\epsilon_{mjk}\epsilon_{rpl} + \alpha_m\alpha^q\epsilon_{qjk}\epsilon_{rpl} \\ &\qquad - 2\left(\delta_{rj}\delta_{pk} - \delta_{rk}\delta_{pj}\right)\left(\delta_{lm} + \alpha_l\alpha_m\right)\Big)\Bigg)\left(\frac{F^k \wedge e^m}{\mathsf{vol}_e}\right) e^p \wedge e^l \\ &+ \alpha_t\Bigg(\delta_{pk}\delta_{lm} - \frac{1}{2(1+\alpha^2)}\alpha^r\Big(\epsilon_{rpl} - 2\epsilon_{rpk}\left(\delta_{lm} + \alpha_l\alpha_m\right)\Big)\Bigg)\left(\frac{F^k \wedge e^m}{\mathsf{vol}_e}\right) e^p \wedge e^l \ .\end{aligned}$$

By using the usual relations between δ^i_j and ϵ_{ijk}, the constraints

$$\begin{aligned}\epsilon_{ijk}e^j \wedge F^k &= -\alpha \wedge F_i \ , \\ e_i \wedge F^i &= 0 \ ,\end{aligned}$$

and the relation

$$e^i \wedge e^j = \frac{1}{2}\epsilon_{klm}\epsilon^{ijk}e^l \wedge e^m \ ,$$

given by Lemma 12 of Appendix B, one finally obtains that the right hand side of (4.20) becomes just

$$e_t^j F_j \ ,$$

which satisfies the equation (4.20), hence no new secondary constraints are generated at this point.

Next, using Lemma 16, equation (4.9c) can be solved for Z_α with

$$\begin{aligned}Z_\alpha = D\alpha_t &+ \frac{1}{2}\epsilon_{ijk}\left(\frac{\left(\epsilon^i{}_{lm}e^l - \delta^i{}_m\alpha\right) \wedge \left(Z_e^m - De_t^m - \epsilon^m{}_{pq}e^p\omega_t^q\right) \wedge e^j}{\mathsf{vol}_e}\right) e^k \\ &- \frac{1}{2}\epsilon_{ijk}\left(\frac{\left(\epsilon^i{}_{lm}e_t^l - \delta^i{}_m\alpha_t\right) De^m \wedge e^j + e_t{}^i D\alpha \wedge e^j}{\mathsf{vol}_e}\right) e^k \ ,\end{aligned} \tag{4.21}$$

with the restriction for Z_e^i given by

$$\left(Z_e^m - De_t^m - \epsilon^m{}_{pq}e^p\omega_t^q\right) \wedge \left(e^l\epsilon_{lm}{}^{(i} - \alpha\delta_m^{(i}\right) \wedge e^{j)}$$
$$+ \left(e_t^l\epsilon_{lm}{}^{(i} - \alpha_t\delta_m^{(i}\right) e^{j)} \wedge De^m + e_t{}^{(i}e^{j)} \wedge D\alpha = 0 \ .$$

By naming $\Psi^i = Z_e^i - De_t^i - \epsilon^i{}_{jk}e^j\omega_t^k$ and defining $\Psi^i{}_j$ by $\Psi^i = \Psi^i{}_j e^j$, one can rewrite the equation as

$$T^{ij}{}_{mq}\Psi^{mq} = \left(\frac{-e_t{}^{(i}e^{j)} \wedge D\alpha - e_t^l\epsilon_{lm}{}^{(i}e^{j)} \wedge De^m + \alpha_t e^{(i} \wedge De^{j)}}{\mathsf{vol}_e}\right) \ , \tag{4.22}$$

where $T^{ij}{}_{mq}$ is though of as the matrix in the multiindices ij and mq given by

$$T^{ij}{}_{mq} = \delta_m^{(i}\delta_q^{j)} - \delta^{ij}\delta_{mq} - \alpha_l\epsilon_q{}^{l(i}\delta_m^{j)} \ .$$

The matrix T is not invertible: its kernel is 3-dimensional and is spanned by

$$(K_l)^{mq} = \delta_l^m\alpha^q - \epsilon_l{}^{mq} \ .$$

The kernel of the transpose of T, $T_{mq}{}^{ij}$ is also 3-dimensional and is spanned by

$$(K_l^T)_{ij} = \epsilon_{lij} \ ,$$

which tells us that the image of T are the tensors symmetric in ij. Hence, the right hand side of (4.22) indeed is in the image of $T^{ij}{}_{mq}$ and a solution to the equation exists without the need of introducing additional constraints. Hence,

$$Z_e^i = De_t^i + \epsilon^i{}_{jk}e^j\omega_t^k + \sigma^i\alpha - \sigma_p\epsilon^{pi}{}_j e^j + \Psi^i \ ,$$

where σ^p are arbitrary and $\Psi^i := \Psi^i{}_j e^j$ is a particular solution to (4.22). It is hard but straightforward to check (by using the constraints (4.10) and properties (4.16b), (4.16c), (4.16d)) that

$$\Psi^i{}_j = -\left(Q_-^{-1}\right)^r{}_{p;j}{}^k\left(\epsilon_{kl}{}^i e_t^l + \delta_k^i\alpha_t\right)\left(\frac{De_r \wedge e^p}{\mathsf{vol}_e}\right) \ ,$$

indeed satisfies (4.22). Hence, the e^i component of a particular solution of the Hamiltonian vector field is given by

$$Z_e^i = De_t^i + \epsilon^i{}_{jk}e^j\omega_t^k + \sigma^i\alpha - \sigma_p\epsilon^{pi}{}_j e^j - \left(Q_-^{-1}\right)^r{}_{p;j}{}^k\left(\epsilon_{kl}{}^i e_t^l + \delta_k^i\alpha_t\right)\left(\frac{De_r \wedge e^p}{\mathsf{vol}_e}\right)e^j \ . \tag{4.23}$$

With this expression we can rewrite (4.21) as

$$Z_\alpha = D\alpha_t - \sigma_k e^k + e_{tl}\left(Q_-^{-1}\right)^r{}_{p;k}{}^l\left(\frac{De_r \wedge e^p}{\mathsf{vol}_e}\right)e^k \ . \tag{4.24}$$

In summary, by joining the particular solution given by (4.19), (4.23) and (4.24) with the homogeneous solution (4.18), the general solution for $\mathbb{Z}$ is given by

$$
\begin{aligned}
Z_\omega^i =& D\omega_t^i + \left(Q_-^{-1}\right)^i{}_{p;m}{}^q \left(\epsilon_{qlk} e_t^l + \delta_{qk}\alpha_t\right) \left(\frac{F^k \wedge e^m}{\mathsf{vol}_e}\right) e^p \ , \\
Z_e^i =& De_t^i + \epsilon^i{}_{jk} e^j \omega_t^k + \epsilon^i{}_{jk} e^j \lambda^k - \lambda^i \alpha + \sigma^i \alpha - \epsilon^i{}_{jk} e^j \sigma^k \\
& - \left(Q_-^{-1}\right)^r{}_{p;j}{}^k \left(\epsilon_{kl}{}^i e_t^l + \delta_k^i \alpha_t\right) \left(\frac{De_r \wedge e^p}{\mathsf{vol}_e}\right) e^j \ , \\
Z_\alpha =& D\alpha_t + \lambda_i e^i - \sigma_i e^i + e_{tl} \left(Q_-^{-1}\right)^r{}_{p;k}{}^l \left(\frac{De_r \wedge e^p}{\mathsf{vol}_e}\right) e^k \ .
\end{aligned}
$$

Note that in this solution there appear 13 arbitrary functions $(e_t^i, \omega_t^i, \alpha_t, \lambda^i,\ \sigma^i)$, however due to the particular form in which λ^i and σ^i show up, they can be combined together by defining a new arbitrary parameter $\tau^i := \lambda^i - \sigma^i$, hence the number of independent arbitrary functions is reduced to 10.

However, there still are more equations that the Hamiltonian vector field must satisfy as per the GNH algorithm, namely, the tangency conditions derived from (4.10), which are

$$
\begin{aligned}
Z_{ei} \wedge F^i + e_i \wedge DZ_\omega^i = 0 \ , \\
D\left(\epsilon_{ijk} e^j \wedge Z_e^k - \alpha \wedge Z_{ei} - e_i \wedge Z_\alpha\right) + Z_\omega^j \wedge \left(e_i \wedge e_j - \epsilon_{ijk}\alpha \wedge e^k\right) = 0 \ , \\
\epsilon_{ijk} Z_e^j \wedge F^k + Z_\alpha \wedge F_i + \left(\epsilon_{ijk} e^j + \delta_{ik}\alpha\right) \wedge DZ_\omega^k = 0 \ .
\end{aligned}
$$

In order to work with these equations it is more convenient to rewrite them and eliminate the covariant derivatives of the Hamiltonian vector field, since their components are written in terms of densities. It is a better strategy to take the covariant derivative of equations (4.9) and substitute the matching terms with covariant derivatives of $\mathbb{Z}$. Doing this and using the constraints when necessary yields

$$
\left(Z_{ei} - De_{ti} - \epsilon_{ijk} e^j \omega_t^k\right) \wedge F^i + De_i \wedge \left(Z_\omega^i - D\omega_t{}^i\right) = 0 \ , \tag{4.25a}
$$

$$
\begin{aligned}
& D\omega_t^k \wedge \left(e_k \delta_{ij} - \epsilon_{ijk}\alpha\right) \wedge e^j - e_t^j \left(\epsilon_{ijk}\alpha + \delta_{jk} e_i\right) \wedge F^k - \alpha_t \epsilon_{ijk} F^j \wedge e^k \\
& \qquad - \epsilon^k{}_{lm} \left(\epsilon_{ijk} e^j - \delta_{ik}\alpha\right) \wedge Z_\omega^l \wedge e^m = 0 \ ,
\end{aligned} \tag{4.25b}
$$

$$
\begin{aligned}
& \epsilon_{ijk} \left(Z_e^j - De_t^j - \epsilon^j{}_{lm} e^l \omega_t^m\right) \wedge F^k + \left(Z_\alpha - D\alpha_t\right) \wedge F_i \\
& \qquad + D\left(\epsilon_{ijk} e^j + \delta_{ik}\alpha\right) \wedge \left(Z_\omega^k - D\omega_t^k\right) = 0 \ .
\end{aligned} \tag{4.25c}
$$

By plugging the values of Z_e^i and Z_ω^i given by (4.23) and (4.19) in condition (4.25a) we immediately find that it is already satisfied. Equation (4.25b) can also be seen to hold by plugging the value of Z_ω^i (4.19) in it and then using relations (4.16b), (4.16c), (4.16d). One would also need to check that condition (4.25c), when using the expressions (4.8), is satisfied.

Then, since none of the arbitrary functions have been fixed in the tangency conditions, the general form of the Hamiltonian vector field is

$$
\begin{aligned}
Z_\omega^i =& D\omega_t^i + \left(Q_-^{-1}\right)^i{}_{p;m}{}^q \left(\epsilon_{qlr} e_t^l + \delta_{qr}\alpha_t\right) \left(\frac{F^r \wedge e^m}{\mathsf{vol}_e}\right) e^p \ , \\
Z_e^i =& De_t^i + \epsilon^i{}_{jk} e^j \omega_t^k + \epsilon^i{}_{jk} e^j \tau^k - \alpha\tau^i \\
& - \left(Q_-^{-1}\right)^r{}_{p;j}{}^k \left(\epsilon_{kl}{}^i e_t^l + \delta_k^i \alpha_t\right) \left(\frac{De_r \wedge e^p}{\mathsf{vol}_e}\right) e^j \ , \\
Z_\alpha =& D\alpha_t + \tau_i e^i + e_{tl} \left(Q_-^{-1}\right)^r{}_{p;k}{}^l \left(\frac{De_r \wedge e^p}{\mathsf{vol}_e}\right) e^k \ ,
\end{aligned}
\tag{4.26}
$$

which depend on the 10 arbitrary functions $e_t^i, \omega_t^i, \alpha_t, \tau^i$. Notice that this coincides with the number of expected symmetries (6 coming from the $SO(4)$ group and 4 from the diffeomorphisms). Moreover, it can be read off from their expressions that ω_t^i and τ^i are the parameters of the two $SO(3)$ gauge symmetries that arise from the decomposition $SO(4) = SO(3) \otimes SO(3)$ (whose representations are given by (4.3) and (4.4)), and that e_t^i, α_t parameterize the diffeomorphism symmetry. Also notice that the solution depends linearly in the parameters, making all the evolution of the system gauge. This is consistent with the fact that the Hamiltonian (4.8) vanishes on the final constraint submanifold.

In practice, one might not want to find the general solution to (4.9), which are complicated equations. An alternative that allows us to greatly simplify them is fixing the gauge. In particular, one can use the so-called time gauge which forces the timelike covector in the coframe to be aligned with the normal to the leaves Σ_t of the foliation. This amounts to setting $\alpha = 0$ while letting α_t be arbitrary. Note that by doing this we are removing a whole $SO(3)$ factor from the symmetry group, so that only the diffeomorphisms and a single $SO(3)$ symmetry group remain. By using this fixing, the Hamiltonian equations (4.9) become

$$
e_{ti} F^i = e_k \wedge \left(Z_\omega^k - D\omega_t{}^i\right) \ , \tag{4.27a}
$$

$$
\left(\epsilon_{ijk} e_t^j + \delta_{ik}\alpha_t\right) F^k = \epsilon_{ijk} e^j \wedge \left(Z_\omega^k - D\omega_t^k\right) \ , \tag{4.27b}
$$

$$
-D\alpha_t \wedge e_i + \left(\epsilon_{ijk} e_t^j - \delta_{ik}\alpha_t\right) De^k = \epsilon_{ijk} e^j \wedge \left(Z_e^k - De_t^k - \epsilon^k{}_{lm} e^l \omega_t^m\right) \ , \tag{4.27c}
$$

and the constraints (4.10) are

$$
e_i \wedge F^i = 0 \ , \tag{4.28a}
$$

$$
\epsilon_{ijk} De^j \wedge e^k = 0 \ , \tag{4.28b}
$$

$$
\epsilon_{ijk} e^j \wedge F^k = 0 \ . \tag{4.28c}
$$

From equations (4.27b) and (4.27c), using Lemma 13 from Appendix B we directly obtain the solutions

$$
\begin{aligned}
Z_e^i &= De_t^i + \epsilon^i{}_{jk} e^j \omega_t^k + \epsilon_{jkl} e_t^l \left(\frac{e^i \wedge De^k}{\mathsf{vol}_e} \right) e^j \\
&\quad + \alpha_t \left(\frac{e_j \wedge De^i}{\mathsf{vol}_e} \right) e^j - \frac{1}{2} \alpha_t \left(\frac{e_k \wedge De^k}{\mathsf{vol}_e} \right) e^j - \left(\frac{e^i \wedge e_j \wedge D\alpha_t}{\mathsf{vol}_e} \right) e^j \ , \\
Z_\omega^i &= D\omega_t^i + \epsilon_{jkl} e_t^l \left(\frac{e^i \wedge F^k}{\mathsf{vol}_e} \right) e^j - \alpha_t \left(\frac{e^i \wedge F_j}{\mathsf{vol}_e} \right) e^j \ .
\end{aligned} \tag{4.29}
$$

It remains to check that solution (4.29) also satisfies equation (4.27a), otherwise, the system would be incompatible. By plugging the solution one obtains

$$
\begin{aligned}
e_i \wedge Z_\omega^i &= e_i \wedge \left(D\omega_t^i + \epsilon_{jkl} e_t^l \left(\frac{e^i \wedge F^k}{\mathsf{vol}_e} \right) e^j - \alpha_t \left(\frac{e^i \wedge F^j}{\mathsf{vol}_e} \right) e^j \right) \\
&= -D\omega_t^i \wedge e_i + \epsilon_{jkl} e_t^l \left(\frac{e^i \wedge F^k}{\mathsf{vol}_e} \right) e_i \wedge e^j - \alpha_t \left(\frac{e^i \wedge F^j}{\mathsf{vol}_e} \right) e^i \wedge e^j \ .
\end{aligned} \tag{4.30}
$$

The second term of (4.30) can be rewritten by using Lemma 12 on the 2-form $e_i \wedge e^j$:

$$
\begin{aligned}
\epsilon_{jkl} e_t^l \left(\frac{e^i \wedge F^k}{\mathsf{vol}_e} \right) e_i \wedge e^j &= \epsilon_{jkl} e_t^l \left(\frac{e^i \wedge F^k}{\mathsf{vol}_e} \right) \frac{1}{2} \epsilon_{pqr} \left(\frac{e_i \wedge e^j \wedge e^p}{\mathsf{vol}_e} \right) e^q \wedge e^r \\
&= \frac{1}{2} e_t^l \epsilon_{jkl} \epsilon_{pqr} \epsilon_i{}^{jp} \left(\frac{e^i \wedge F^k}{\mathsf{vol}_e} \right) e^q \wedge e^r \\
&= \frac{1}{2} e_t^l \epsilon_{pqr} \left(\left(\frac{e_l \wedge F^p}{\mathsf{vol}_e} \right) e^q \wedge e^r - \left(\frac{e^i \wedge F_i}{\mathsf{vol}_e} \right) \delta^p{}_l e^q \wedge e^r \right) \\
&= \frac{1}{2} e_t^l \epsilon_{pqr} \left(\frac{e_l \wedge F^p}{\mathsf{vol}_e} \right) e^q \wedge e^r = \frac{1}{2} e_t^l \epsilon_{pqr} \left(\frac{e^p \wedge F_l}{\mathsf{vol}_e} \right) e^q \wedge e^r = e_t^l F_l \ ,
\end{aligned}
$$

where we have used the constraints (4.28a) and (4.28c) respectively in the last two lines. The third term of (4.30) vanishes since $e^i \wedge F^j$ is symmetric while $e^i \wedge e^j$ is antisymmetric in ij. Hence, equation (4.27a) is also satisfied by the solution (4.29).

We still need to check tangency of the Hamiltonian vector field (4.29) to the constraint submanifold defined by (4.28). The conditions are

$$
\begin{aligned}
\epsilon_{ijk} Z_e^j \wedge F^k + \epsilon_{ijk} e^j \wedge DZ_\omega^k &= 0 \ , \\
Z_e^i \wedge F_i + e_i \wedge DZ_\omega^i &= 0 \ , \\
\epsilon_{ijk} \left(DZ_e^j + \epsilon^j{}_{lm} Z_\omega^l \wedge e^m \right) \wedge e^k + \epsilon_{ijk} De^j \wedge Z_e^k &= 0 \ .
\end{aligned}
$$

As in the full case, dealing with covariant derivatives of the components of the Hamiltonian vector field is inconvenient, so a better strategy is to take the covariant derivative of equations (4.27) and use them to rewrite the tangency conditions. This yields

$$
\begin{aligned}
\epsilon_{ijk} \left(Z_e^j - De_t^j - \epsilon^j{}_{lm} e^l \omega_t^m \right) \wedge F^k + \epsilon_{ijk} De^j \wedge \left(Z_\omega^k - D\omega_t^k \right) - D\alpha_t \wedge F_i &= 0 \ , \\
\left(Z_e^i - De_t^i - \epsilon^i{}_{jk} e^j \omega_t^k \right) \wedge F_i + De_i \wedge \left(Z_\omega^i - D\omega_t^i \right) &= 0 \ , \\
\left(Z_\omega^k - D\omega_t^k \right) \wedge e^i \wedge e_k - e_t^k e_k \wedge F^i &= 0 \ .
\end{aligned}
$$

By plugging the expressions found (4.29) for the Hamiltonian vector field, and by using the constraints (4.28), it is easy to check that the tangency conditions are automatically satisfied, hence there are no new secondary constraints.

One can actually recover the result (4.29) from the more general solution (4.26). In order to implement the time gauge, one must not only set $\alpha = 0$, but in addition, some of the arbitrary functions in (4.26) need to be fixed (since we are actually reducing the gauge group), namely the three τ_i which correspond to the $SO(3)$ symmetry that acts on α. The right choice of these parameters is

$$\tau_i = -\frac{1}{2}\epsilon_{ijk}\left(\frac{D\alpha_t \wedge e^j \wedge e^k}{\mathsf{vol}_e}\right) - \frac{1}{2}e^i_t\left(\frac{De_k \wedge e^k}{\mathsf{vol}_e}\right) + e^l_t\left(\frac{De^i \wedge e_l}{\mathsf{vol}_e}\right) ,$$

which yields $Z_\alpha = 0$ and recovers the expressions for Z^i_e, Z^i_ω given in (4.29) as expected. This is easily seen by using the constraint (4.28b) and the identity

$$0 = 8\epsilon_{[ijk}\delta_{l]r} = \epsilon_{ijk}\delta_{lr} - \epsilon_{jkl}\delta_{ir} + \epsilon_{kli}\delta_{jr} - \epsilon_{lij}\delta_{kr} .$$

Chapter 5

New perturbative approaches to gravity

Perturbative approaches to field theories have been extremely fruitful. However, due to the non-renormalizability of gravity, standard perturbative methods do not work in this context. In this chapter we consider a totally covariant approach to perturbative treatments of the Einstein-Hilbert action, namely, *perturbative methods based on consistent deformations of internal Abelianizations.* This procedure does not depend on auxiliary background structures such as the Minkowski metric, but rather relies on the exact treatment of the unperturbed theory. This chapter is devoted to exploring this technique. In Section 5.1 we discuss the gauge structure of an action and the structure tensors and equations that arise. In Section 5.2 we outline the construction of the BRST antifield formalism, which is the basis for the techniques based on consistent deformations. In Section 5.3 we apply the approach to several important actions related to gravitational theories. Detailed and in-depth accounts of the BRST formalism and its implications for gauge theories are given in [73]–[77].

First, a comment on notation. The *Euler forms* E_A are defined by

$$\mathbb{d}S = \int_{\mathcal{M}} E_A \wedge \mathbb{d}\phi^A \ , \tag{5.1}$$

where the ϕ^A are the dynamical fields and A is an index running over all fields. Then, the equations of motion are given by

$$E_A = 0 \ .$$

In this context, the gauge symmetries are determined by $\mathbb{d}S$: the vector fields $\mathbb{Z}$ in the kernel of $\mathbb{d}S$ are called *gauge generators.* Henceforth these gauge transformations will be denoted as $\mathbb{Z}_\alpha$ and their components as Z^A_α, where α is an index running from $1, \ldots, m$ where $m = \dim \mathcal{G}$ is the dimension of the gauge symmetry group.

We will use a very convenient (and traditional) notation. Indices on tensors are regarded as collective indices, which contain both discrete and continuous variables (spacetime coordinates), which are summed or integrated over, respectively. More explicitly, what we mean is

$$K^a{}_b K^b{}_c = \sum_b \int_{\mathcal{M}} \mathrm{d}y \; K^a{}_b(x,y) K^b{}_c(y,z) \; ,$$

where a, b, c represent any type of index. Although a bit cumbersome, the conciseness of this notation is useful to avoid cluttered expressions.

In this chapter we rely on the most widely used approach, which is to introduce derivatives of the fields via Dirac delta distributions and their derivatives, which then by integration by parts yield the wanted result of differentiating the original field. An alternative approach is to use the space of jets of the fields, which directly account for derivatives. However, since *a priori* arbitrarily high-order derivatives could appear, one needs to work in an infinite jet space in order to include all the possible terms. This is explored in [78].

5.1 Gauge structure of an action

A theory with a finite-dimensional gauge symmetry Lie group $\mathcal{G}$ has directions in the configuration space that leave the action unchanged. These directions correspond to the vector fields in the kernel of $\mathbb{d}S$ that we parametrize with the so-called *gauge generators*. They will be denoted by $\mathbb{Z}_\alpha$ and their components by $Z^A{}_\alpha$, where α is an index running from $1\dots, m$ and $m = \dim \mathcal{G}$ is the dimension of the gauge symmetry group. Note that since the gauge group is a finite-dimensional Lie group, the distribution generated by $\mathbb{Z}_\alpha$ must be integrable, which means that the commutators $[\mathbb{Z}_\alpha, \mathbb{Z}_\beta]$ must also be in the kernel of $\mathbb{d}S$ and hence be themselves gauge generators. Since the algebra generated by the gauge generators — the *gauge algebra* — is not necessarily closed in gauge field theories, one might need to impose the condition that nested commutators also are gauge generators. A gauge transformation of the field ϕ^A is of the form

$$\mathbb{d}\phi^A = Z^A{}_\alpha \epsilon^\alpha \; ,$$

where ϵ^α are the gauge parameters (that is, functions in the Lie algebra $\mathfrak{g}$) and $Z^A{}_\alpha$ is a linear distributional operator whose elements of the image are gauge transformations. In our case of interest, since we work with the exterior algebra, the gauge transformations will be of the form

$$\mathbb{d}\phi^A = z^{0A}{}_\alpha \epsilon^\alpha + z^{1A}{}_\alpha \mathbf{d}\epsilon^\alpha + \dots \; ,$$

plus other terms that preserve the exterior algebra built with, for instance, interior products, Hodge duals (if there is a metric) and in general other terms written with the help of background structures. For convenience, we will follow the usual approach in the literature, which is to consider the objects ${Z^A}_\alpha$ to be distributions of the form

$$ {Z^A}_\alpha(x,y) = {z^{0A}}_\alpha \delta(x,y) + {{z^{1A}}_\alpha}^\mu \partial_\mu \delta(x,y) + \dots \, . $$

Note that the derivatives of Dirac deltas are meant to charge their derivatives to the fields via integration by parts. A more rigorous approach would be to work in the jet space $J^\infty(\mathfrak{g})$ to account for derivatives directly as coordinates. A better look into this approach is given in [78].

A particularly simple but interesting case to look at are field theories involving k-forms and their exterior derivatives. A particular example of such an action is

$$ S(\mathbf{B}) = \int_{\mathcal{M}} *\mathbf{d}\mathbf{B}_A \wedge \mathbf{d}\mathbf{B}^A \; , \tag{5.2} $$

(with each $\mathbf{B}^A \in \Omega^k(\mathcal{M})$ a k-form) which will be used throughout the section to illustrate several concepts. The gauge transformations of $\mathbf{B}$ are $\mathbb{d}\mathbf{B}^A = \mathbf{d}\epsilon^A$, with ϵ^A any $(k-1)$-form $\epsilon^A \in \Omega^{k-1}(\mathcal{M})$, since they leave the action invariant. Hence, the gauge generators are maps

$$ \mathbb{Z} : \Omega^{k-1}(\mathcal{M}) \longrightarrow \ker \mathbb{d} S \; , $$

given by $Z^A(x,y)\cdot\epsilon = \epsilon^A \mathbf{d}\delta(x,y)$, or equivalently, ${Z^A}_\alpha(x,y) = {\delta^A}_\alpha \mathbf{d}\delta(x,y)$.

A gauge transformation is called trivial if it can be written as

$$ \mathbb{d}\phi^A = E_B \wedge T^{BA} \; , $$

with $T^{AB} = (-1)^{(n-p)(n-q)+1} T^{BA}$, but otherwise, arbitrary bi-forms in space-time of degree $p = \deg \phi^A$ in the first variable, $q = \deg \phi^B$ in the second and degree 1 in configuration space.

The *regularity condition* requires that in the submanifold where the equations of motion hold, the Hessian of the action (which must be understood as the matrix built with the coordinates corresponding to the second derivatives in the space of jets) has rank equal to the number of fields that enter dynamically into the action. Equivalently, this means that the only null vectors of the Hessian of the action are the gauge generators. This condition is desirable because, when it holds, the box of the Hessian corresponding to the dynamical degrees of freedom has an inverse and there exist propagators for the dynamical fields. We will henceforth assume that the regularity condition holds.

The general form of the gauge generators $\mathbb{X}$ is given by

$$ E_A \wedge X^A = 0 \implies X^A = {Z^A}_\alpha C^\alpha + E_B \wedge T^{BA} \; , \tag{5.3} $$

this is, a linear combination of gauge generators plus a trivial gauge transformation, since

$$ \mathbb{d}S(\mathbb{X}) = E_A \wedge X^A = E_A \wedge {Z^A}_\alpha C^\alpha + E_A \wedge E_B \wedge T^{BA} = 0 \; , $$

where the first terms vanishes because the $\mathbb{Z}_\alpha$ are gauge generators and the second term vanishes because of its symmetries.

One can regard $\mathbb{Z}$ as an operator from the space of gauge parameters to the space of gauge generators. If this map is a bijection on-shell, then one can parametrize the space of gauge generators with the $m_0 := m$ gauge parametrs ϵ^α. In this case, the theory is called *irreducible*. On the other hand, if $\mathbb{Z}$ is not a bijection (which is equivalent to saying that it is not injective) the theory is called *reducible*. In this case, $\ker \mathbb{Z}$ is not 0 and we need to characterize it. Let R_1 be a linear operator with components ${R_1}^\alpha{}_{\alpha_1}$ parametrizing $\ker \mathbb{Z}$ with m_1 parameters ϵ^{α_1}. Note that the composition $Z^A{}_\alpha {R_1}^\alpha{}_{\alpha_1} \epsilon^{\alpha_1}$ must vanish since ${R_1}^\alpha{}_{\alpha_1}\epsilon^{\alpha_1} \in \ker Z$ by construction. However, since we only require it to vanish on-shell, and not in the whole configuration space, we instead impose

$$\boxed{Z^A{}_\alpha {R_1}^\alpha{}_{\alpha_1} = E_B \wedge K^{BA}{}_{\alpha_1}} \tag{5.4}$$

where $K^{BA}{}_{\alpha_1}$ are some functionals. Hence, instead of m_0, the amount of independent gauge generators is $m_0 - \text{rank } R_1$.

Now, if the map R_1 is not bijective itself, higher orders of reducibility appear. In this case, $\ker R_1 \neq 0$, hence we need to parametrize $\ker R_1$ again. To this end we introduce the map R_2 with components ${R_2}^{\alpha_1}{}_{\alpha_2}$ parametrizing $\ker R_1$ with m_2 parameters ϵ^{α_2}. Again, by construction,

$$\boxed{{R_1}^\alpha{}_{\alpha_1} {R_2}^{\alpha_1}{}_{\alpha_2} = E_B \wedge K^{B\alpha}{}_{\alpha_2}}$$

Thus, the rank of ${R_1}^\alpha{}_{\alpha_1}$ equals $m_1 - \text{rank } R_2$, hence the amount of independent gauge generators is $m_0 - m_1 + \text{rank } R_2$. Still, the kernel of this map might not be 0 so we still do not have a proper parametrization of $\ker \mathbb{d}S$. This generates a cascade of parametrizations R_{k+1} with components ${R_{k+1}}^{\alpha_k}{}_{\alpha_{k+1}}$ of $\ker R_k$ with m_k parameters ϵ^{α_k}. This also generates a cascade of *reducibility conditions*

$$\boxed{{R_{k-1}}^{\alpha_{k-2}}{}_{\alpha_{k-1}} {R_k}^{\alpha_{k-1}}{}_{\alpha_k} = E_B \wedge K^{B\alpha_{k-2}}{}_{\alpha_k}} \tag{5.5}$$

with $\alpha_k = 1, \cdots, m_k$ and where we use as a convention $\alpha_0 := \alpha$, $\alpha_{-1} := A$ and ${R_0}^{\alpha_{-1}}{}_{\alpha_0} := Z^A{}_\alpha$ to include condition (5.4). A condition similar to (5.3) also applies to coefficients in higher orders of reducibility

$${R_k}^{\alpha_{k-1}}{}_{\alpha_k} X^{\alpha_k} = E_A \wedge M^{A\alpha_{k-1}} \implies X^{\alpha_k} = {R_{k+1}}^{\alpha_k}{}_{\alpha_{k+1}} J^{\alpha_{k+1}} + E_A \wedge S^{A\alpha_k} \ . \tag{5.6}$$

If for some L the map R_{k+1} is bijective (rank $R_L = m_L$), then no more conditions are generated and we say that it is a gauge theory of L-th reducibility order. In this case, the number of independent gauge generators is

$$\sum_{i=0}^{L} (-1)^i m_i \ .$$

Let us continue with our example of a k-form field $\mathbf{B}$ with action (5.2). As stated before, its gauge transformations are of the form $\mathbb{d}\phi^A = \mathbf{d}\epsilon^A$, with $\epsilon^A \in \Omega^{k-1}(\mathcal{M})$ and the gauge generators are

$$\mathbb{Z} : \Omega^{k-1}(\mathcal{M}) \longrightarrow \ker \mathbb{d}S \ ,$$

given by $Z^A{}_\alpha(x,y) = \delta^A{}_\alpha \mathbf{d}\delta(x,y)$. If $k > 1$, clearly, not every $\epsilon^A \in \Omega^k(\mathcal{M})$ yields a non-trivial transformation, since exact $(k-1)-$forms make the transformations vanish. Hence, $\ker Z \cong \Omega^{k-2}(\mathcal{M})$. We may now try to parametrize it in terms of

$$R_1 : \Omega^{k-2}(\mathcal{M}) \longrightarrow \ker \mathbb{Z} \ ,$$

with $R_1{}^\alpha{}_{\alpha_1}(x,y) = \delta^\alpha{}_{\alpha_1}\mathbf{d}\delta(x,y)$ and parameters $\epsilon^{\alpha_1} \in \Omega^{k-2}(\mathcal{M})$. However, if $k > 2$, $R_1{}^\alpha{}_{\alpha_1}$ has as kernel the exact $(k-2)$-forms, which means $\ker R_1 \cong \Omega^{k-3}(\mathcal{M})$. Hence, we need to introduce a further parametrization

$$R_2 : \Omega^{k-3}(\mathcal{M}) \longrightarrow \ker R_1 \ ,$$

$R_2{}^{\alpha_1}{}_{\alpha_2}(x,y) = \delta^{\alpha_1}{}_{\alpha_2}\mathbf{d}\delta(x,y)$. This process continues until we reach the 0-forms, which cannot be exact. Thus, this is a theory with $(k-1)$-th reducibility order.

Now we turn to the restrictions that the gauge generators must satisfy in order to constitute an integrable distribution. Since $[\mathbb{Z}_\alpha, \mathbb{Z}_\beta]$ must also be a gauge generator, by (5.3) we can write

$$\boxed{[\mathbb{Z}_\alpha, \mathbb{Z}_\beta]^A = Z^A{}_\gamma C^\gamma{}_{\alpha\beta} + E_B \wedge T^{BA}{}_{\alpha\beta}} \tag{5.7}$$

Note that by the antisymmetry of the vector bracket, both $C^\gamma{}_{\alpha\beta}$ and $T^{BA}{}_{\alpha\beta}$ must be antisymmetric in $\alpha\beta$. In general, $C^\gamma{}_{\alpha\beta}$ and $T^{BA}{}_{\alpha\beta}$ may be functions of the fields and are called *structure functions* (or *tensors*). Note that if $C^\gamma{}_{\alpha\beta}$ are actually constant and all $T^{BA}{}_{\alpha\beta} = 0$, then the algebra of gauge generators is a genuine Lie algebra. If $T^{BA}{}_{\alpha\beta} = 0$, the algebra is called *closed*, and otherwise, *open*.

Multiplying (5.7) by $R_1{}^\beta{}_{\alpha_1}$ yields

$$\begin{aligned} Z^A{}_\gamma &\left(\mathbb{Z}_\alpha\left(R_1{}^\gamma{}_{\alpha_1}\right) - C^\gamma{}_{\alpha\beta} R_1{}^\beta{}_{\alpha_1}\right) \\ &= E_B \wedge \left(\mathbb{Z}_\alpha\left(K^{BA}{}_{\alpha_1}\right) - T^{BA}{}_{\alpha\beta} R_1{}^\beta{}_{\alpha_1} - K^{BC}{}_{\alpha_1}\partial_C Z^A{}_\alpha\right) \ . \end{aligned}$$

By (5.6), we can write

$$\boxed{\mathbb{Z}_\alpha\left(R_1{}^\gamma{}_{\alpha_1}\right) - C^\gamma{}_{\alpha\beta} R_1{}^\beta{}_{\alpha_1} = R^\gamma{}_{\delta_1} C^{\delta_1}{}_{\alpha\alpha_1} + E_C \wedge T^{C\gamma}{}_{\alpha\alpha_1}}$$

which are new structural equations for the tensors $C^{\delta_1}{}_{\beta\alpha_1}$ and $T^{C\alpha}{}_{\beta\alpha_1}$.

The Jacobi identity of the bracket imposes restrictions on the structure functions C and T. In general, they do not need to be automatically satisfied if the algebra is not closed, in which case one needs to introduce new structure tensors and relations. A direct computation gives

$$[\mathbb{Z}_\alpha, [\mathbb{Z}_\beta, \mathbb{Z}_\gamma]]^A = Z^A{}_\lambda \left(\mathbb{Z}_\alpha \left(C^\lambda{}_{\beta\gamma}\right) + C^\lambda{}_{\alpha\delta} C^\delta{}_{\beta\gamma}\right) + E_B \wedge B^{BA}{}_{\alpha\beta\gamma} \ ,$$

where B is a function of T and Z and the Jacobi identity is equivalent to the vanishing of the antisymmetrization of this expression in the indices $\alpha\beta\gamma$. Then it can be written as

$$Z^A{}_\lambda J^\lambda{}_{\alpha\beta\gamma} = E_B \wedge B^{BA}{}_{[\alpha\beta\gamma]} \ , \tag{5.8}$$

where

$$J^\lambda{}_{\alpha\beta\gamma} = \mathbb{Z}_{[\alpha} \left(C^\lambda{}_{\beta\gamma]}\right) + C^\lambda{}_{[\alpha|\delta} C^\delta{}_{\beta\gamma]} \ ,$$

and which, by (5.6), implies

$$\boxed{J^\lambda{}_{\alpha\beta\gamma} = R_1{}^\lambda{}_{\alpha_1} G^{\alpha_1}{}_{\alpha\beta\gamma} + E_B \wedge T^{B\lambda}{}_{\alpha\beta\gamma}} \tag{5.9}$$

where $G^{\alpha_1}{}_{\alpha\beta\gamma}$ and $T^{B\lambda}{}_{\alpha\beta\gamma}$ are new structure tensors. By using the general form of (5.9) in the original equation (5.8), we find the sufficient conditions:

$$\begin{aligned} 0 &= Z^A{}_\lambda \left(R^\lambda{}_{\alpha_1} G^{\alpha_1}{}_{\alpha\beta\gamma} + E_B \wedge D^{B\lambda}{}_{\alpha\beta\gamma}\right) - E_B \wedge B^{BA}{}_{[\alpha\beta\gamma]} \\ &= E_B \wedge \left(K^{BA}{}_{\alpha\beta\gamma} G^{\alpha_1}{}_{\alpha\beta\gamma} + Z^A{}_\lambda D^{B\lambda}{}_{\alpha\beta\gamma} - E_B \wedge B^{BA}{}_{[\alpha\beta\gamma]}\right) \ , \end{aligned}$$

which, as a consequence of (5.3), imply

$$K^{BA}{}_{\alpha\beta\gamma} G^{\alpha_1}{}_{\alpha\beta\gamma} + Z^A{}_\lambda D^{B\lambda}{}_{\alpha\beta\gamma} - E_B \wedge B^{BA}{}_{[\alpha\beta\gamma]} = Z^B{}_\lambda C^{A\lambda}{}_{\alpha\beta\gamma} + E_C \wedge T^{CAB}{}_{\alpha\beta\gamma} \ ,$$

and gives new structure tensors $C^{A\lambda}{}_{\alpha\beta\gamma}$ and $T^{CAB}{}_{\alpha\beta\gamma}$.

In a completely similar fashion, we need to impose that further nested commutators of the gauge generators are themselves gauge generators. Also, by multiplying the obtained relations by the appropriate reducibility coefficients, one finds new necessary equations that may give rise to new structure tensors. By doing this, one gets at each step more new structure tensors with more indices and their corresponding structural equations. If the theory is reasonable (finite dimensional gauge group, finite reducibility order), this process must stop at some point. Then, the structure tensors encode and determine the gauge structure of the action. Since the number of structure tensors grows very quickly with the reducibility order of the theory, studying the gauge structure is quite complicated. A better suited approach is the one provided by the BRST formalism, which we will review in the next section.

Turning again to our example of the p-forms $\mathbf{B}$, we have (5.7) that the gauge generators commute

$$[Z^A \cdot \epsilon, Z^B \cdot \epsilon] = 0 \ ,$$

yielding $C^{\gamma}{}_{\alpha\beta} = 0, T^{BA}{}_{\alpha\beta}$ and making the algebra of gauge generators a Lie algebra. Then, clearly the Jacobi identity is automatically satisfied and expression (5.9) tells us that the corresponding structure tensors are

$$J^{\lambda}{}_{\alpha\beta\gamma} = 0,\ G^{\alpha_1}{}_{\alpha\beta\gamma} = 0,\ T^{B\lambda}{}_{\alpha\beta\gamma} = 0\ .$$

Since the algebra is commutative, further conditions are automatically satisfied and their structure tensors are 0.

5.2 BRST antifield formalism

Ghost fields were first introduced in order to retain unitarity when performing gauge fixings. Even after fixing the gauge, the action possesses a symmetry whose transformations involve both fields and ghosts. This is the Becchi-Rouet-Stora-Tyutin symmetry. In order to incorporate these heuristically found ideas into a systematic study of field theories, the BRST formalism was developed. Later, Batalin and Vilkovisky introduced antifields [76], which acted as sources for the BRST transformation and a symplectic structure: these are the main elements of the the antifield formalism. This turned out to be a very powerful method to quantize gauge theories. Nevertheless, it is very much worth using the tools it provides for the study of field theories at the classical level, since it encodes all the information about the gauge structure of the action.

The original formalism was developed by taking into account both *bosonic* and *fermionic* fields (commuting and anticommuting, respectively) in order to write the usual actions of the Standard Model. Since for our purposes we only need bosonic fields, we will omit fermionic fields from the discussion. By doing this, some signs depending on the Grassmann parity of the fields involved are removed from most expressions involving operations in graded spaces. The original expressions can be found in [73].

The construction of the BRST configuration space is as follows [73], [77], [79], [80]. For each gauge invariance at k-th order, introduce a ghost field $\mathcal{C}^{\alpha_k}$. Next, for each field $\Phi^A = \{\phi^A, \mathcal{C}^{\alpha_k}\}$, introduce an antifield $\Phi^*{}_A$. The set of fields and antifields $\{\Phi^A, \Phi^*_A\}$ constitute the extended configuration space in which the BRST (sometimes even called BV) formalism takes place. In this space, several gradings are induced by ghosts and antifields. In particular, we have a grading whose degree is called *pureghost number* and a gradation whose degree is called *antighost number* defined by

$$\begin{aligned}
\mathsf{puregh}\ (\mathcal{C}^{\alpha_k}) &= k+1\ ,\\
\mathsf{puregh}\ (\mathcal{C}^*{}_{\alpha_k}) &= 0\ ,\\
\mathsf{antigh}\ (\mathcal{C}^{\alpha_k}) &= 0\ ,\\
\mathsf{antigh}\ (\mathcal{C}^*{}_{\alpha_k}) &= k+2\ ,
\end{aligned}$$

for $k \geq -1$, where we use as a convention $\alpha_0 := \alpha$, $\alpha_{-1} := A$, $\mathcal{C}^*{}_{\alpha_{-1}} := \phi^*_A$ and $\mathcal{C}^{\alpha_{-1}} = \phi^A$. A further grading is defined whose degree is called *ghost number* given by $\mathsf{gh} = \mathsf{puregh} - \mathsf{antigh}$.

There exists a canonical symplectic structure in the extended configuration space

$$\Omega = \mathbb{d}\Phi^A \wedge \mathbb{d}\Phi^*_A$$

whose Poisson bracket is called the *antibracket* (this name derives from the fact that it is a bracket for fields and antifields) defined by the basic relations

$$(\Phi^A, \Phi^*_B) = \delta^A{}_B \ ,$$

so one can think of Φ^*_A as the conjugate variable to Φ^A. In particular, the antibracket is antisymmetric and satisfies the Jacobi identity.

One can then look for a functional $\mathcal{S}$ called the *BRST action* such that $\mathsf{gh}\ \mathcal{S} = 0$ and

$$(\mathcal{S}, \mathcal{S}) = 0 \ , \tag{5.10}$$

which is known as the *master equation*. In general, solutions to (5.10) are not unique, however, if we additionally require that the rank of the Hessian of $\mathcal{S}$ equals the number of fields, then the solution is unique up to canonical transformations.

As an additional condition we require that

$$\mathcal{S}|_{\Phi^*=0} = S \ .$$

This is needed in order for the BRST action to be able to give back the original action.

Notice that with these ingredients we can define an operator s, that raises the ghost number by 1 according to,

$$sF = (F, \mathcal{S}) \ . \tag{5.11}$$

As a consequence of the master equation (5.10) and the Jacobi identity of the antibracket, $s^2 = 0$, so s can be used to define a differential complex in the space of fields and antifields. This operator is the generator of the BRST symmetry. The functionals $\mathcal{O}$ such that $s\mathcal{O} = 0$ but $\mathcal{O} \neq sX$ for any X are called *classical observables* of the theory. By the properties of s, observables form an algebra. Equivalently, observables are elements $\mathcal{O} \in H_0(s)$ in the zeroth cohomology group of s. This is a powerful identification, since it allows us to transform the problem of finding $\mathcal{S}$ into a cohomological one, for which many tools are available [75], [77].

Since the ghost number of $\mathcal{S}$ is 0, the pureghost and antighost numbers of each of its terms must be equal. Then, one can expand $\mathcal{S}$ as a series in the antighost number. The first few terms of the expansion are of the form

$$\begin{aligned}\mathcal{S} &= S \\ &+ \phi^*_A Z^A{}_\alpha \mathcal{C}^\alpha \\ &+ \mathcal{C}^*_\alpha \left(R_1{}^\alpha{}_{\alpha_1} \mathcal{C}^{\alpha_1} + C^\alpha{}_{\beta\gamma} \mathcal{C}^\beta \mathcal{C}^\gamma \right) + \phi^*_A \phi^*_B \left(K^{AB}{}_{\alpha_1} \mathcal{C}^{\alpha_1} + K^{AB}{}_{\alpha\beta} \mathcal{C}^\alpha \mathcal{C}^\beta \right) \\ &+ \mathcal{C}^*_{\alpha_1} \left(R_2{}^{\alpha_1}{}_{\alpha_2} \mathcal{C}^{\alpha_2} + K^{\alpha_1}{}_{\beta_1\gamma} \mathcal{C}^{\beta_1} \mathcal{C}^\gamma + K^{\alpha_1}{}_{\beta\gamma\delta} \mathcal{C}^\beta \mathcal{C}^\gamma \mathcal{C}^\delta \right) \\ &\quad + \mathcal{C}^*_\alpha \phi^*_A \left(K^{\alpha A}{}_{\beta_2} \mathcal{C}^{\beta_2} + K^{\alpha A}{}_{\beta_1\gamma} \mathcal{C}^{\beta_1} \mathcal{C}^\gamma + K^{\alpha A}{}_{\beta\gamma\delta} \mathcal{C}^\beta \mathcal{C}^\gamma \mathcal{C}^\delta \right) \\ &\quad + \phi^*_A \phi^*_B \phi^*_C \left(K^{ABC}{}_{\beta_2} \mathcal{C}^{\beta_2} + K^{ABC}{}_{\beta_1\gamma} \mathcal{C}^{\beta_1} \mathcal{C}^\gamma + K^{ABC}{}_{\beta\gamma\delta} \mathcal{C}^\beta \mathcal{C}^\gamma \mathcal{C}^\delta \right) \\ &+ (\text{terms with antighost number} > 3) \ .\end{aligned}$$

Now, using this expansion in the master equation (5.10) gives a set of conditions on the tensors $Z^A{}_\alpha, R_{k+1}{}^{\alpha_k}{}_{\alpha_{k+1}}$ and all the K's with different amounts of indexes. Note that *a priori*, these quantities have nothing to do with the gauge structure of the action. The master equation should be solved order by order, i.e., the coefficient of each ghost and antifield term must vanish separately. Notice that as we go up in the reducibility order, the amount of terms in $\mathcal{S}$, and hence the amount of conditions, grows very fast. The first few terms give the equations

$$E_A \wedge Z^A{}_\alpha \mathcal{C}^\alpha = 0 \ , \tag{5.12}$$

$$\phi^*_A \left(Z^A{}_\alpha R_1{}^\alpha{}_{\alpha_1} - E_B \wedge K^{BA}{}_{\alpha_1} \right) \mathcal{C}^{\alpha_1} = 0 \ , \tag{5.13}$$

$$\phi^*_A \left(\mathbb{Z}_\beta \left(Z^A{}_\alpha \right) + Z^A{}_\gamma K^\gamma{}_{\alpha\beta} - E_B \wedge K^{BA}{}_{\alpha\beta} \right) \mathcal{C}^\alpha \mathcal{C}^\beta = 0 \ , \tag{5.14}$$

$$\mathcal{C}^*_\delta \left(\mathbb{Z}_\alpha \left(C^\delta{}_{\beta\gamma} \right) + C^\delta{}_{\lambda\alpha} C^\lambda{}_{\beta\gamma} - R_1{}^\delta{}_{\alpha_1} K^{\alpha_1}{}_{\alpha\beta\gamma} - E_B \wedge K^{B\delta}{}_{\alpha\beta\gamma} \right) \mathcal{C}^\alpha \mathcal{C}^\beta \mathcal{C}^\gamma = 0 \ . \tag{5.15}$$

Notice that the ghosts (and also the antifields in higher terms) have the effect of symmetrizing the expressions inside the parentheses. Equations (5.12) tells us that $Z^A{}_\alpha$ actually are the gauge generators of the action. Equation (5.13) are the reducibility conditions (5.4). Equations (5.14) are the equations (5.7) requiring that the commutator of two gauge generators is a gauge generator. Equations (5.15) are the structure equations (5.9) coming from the Jacobi identity. By continuing this, one obtains a set of equations that match all the structure equations and reducibility conditions of the action S. Hence, the coefficients of the expansion of $\mathcal{S}$ are precisely (up to a number, which appears due to the symmetries of the terms involved and that is unimportant since it can be compensated by modifying the factors in the expasion) the structure constants and the reducibility coefficients. Hence, a functional $\mathcal{S}$ solution to the master equation encodes all the information about the gauge generators, their structure and their algebra. In particular, the gauge generators can be extracted from the term containing $\phi^*_A \mathcal{C}^\alpha$, the coefficients satisfying all the reducibility identities (5.5) can be extracted from the terms containing $\mathcal{C}^*_{\alpha_k} \mathcal{C}^{\alpha_{k+1}}$, the structure constants can be extracted from the term containing $\mathcal{C}^*_\alpha \mathcal{C}^\beta \mathcal{C}^\gamma$, etc. Moreover, an important consequence is that $S = \mathcal{S}|_{\Phi^*=0}$ is gauge invariant under the gauge transformations generated by $\mathbb{Z}_\alpha$.

5.3 Consistent deformations from internally Abelianized theories

In the early 90s, Smolin proposed in [29] a potential perturbative scheme for the quantization of gravity. His idea was based on considering Newton's constant G as a coupling constant appearing in the definition of the curvature and the covariant differential as

$$\begin{aligned}\mathbf{F}^{IJ} &= \mathbf{d}\boldsymbol{\omega}^{IJ} + G\boldsymbol{\omega}^{I}{}_{K} \wedge \boldsymbol{\omega}^{KJ} \ , \\ \mathbf{D}\mathbf{e}^{I} &= \mathbf{d}\mathbf{e}^{I} + G\boldsymbol{\omega}^{I}{}_{J} \wedge \mathbf{e}^{J} \ .\end{aligned}$$

Then, he used similar expressions with a self-dual connection and found that the self-dual Cartan-Palatini action can be written as the sum of a lower order term, independent of G, and a coupling term proportional to G. This idea can be readily compared to the traditional perturbative approach in which the metric is written as the Minkowski metric plus a perturbation, leading to an action with a 'free' part (i.e. quadratic in the fields) describing massless, spin 2 particles, known as gravitons, propagating in a Minkowski background geometry and interaction terms proportional to powers of G. The new approach championed by Smolin is more desirable in the sense that the perturbed action only has two terms (the perturbation is only up to linear terms in G), each of the terms retains diffeomorphism invariance on its own and the procedure does not depend on any background objects. The version of this model with the time gauge fixing is called the $U(1)^3$ model (since it yields a $U(1)^3$ connection) and is studied in, for instance, [30].

However, Smolin's proposal has some drawbacks. First, it is not clear why this way of introducing the coupling G is the right one. More importantly, it is not physically clear what taking G to 0 means or even how to use it as a perturbative parameter since it is a dimensionful quantity. Even if these points were clarified, it might still happen that the perturbative scheme is not feasible. First, the unperturbed theory must be integrable in order to produce suitable 'free' solutions. Furthermore, in order to write the full solutions as an asymptotic expansion, the perturbation must be regular. Neither of these properties are guaranteed and should be checked.

A possible systematic approach that can be used to find the kind of splitting put forward by Smolin is described in the following. Given an action S^0, a *deformation* of it is a new action S that can be written as a (possibly infinite) series in the deformation parameter g and such that $S|_{g=0} = S^0$; this is

$$S = \sum_{k=0} g^k S^k \ . \tag{5.16}$$

A deformation is *consistent* if it preserves the number of gauge symmetries, even if their form changes. Note that this also implies that the number of degrees of freedom in the theory is mantained. If the gauge transformations of the original action S^0 are denoted by ${\mathbb{Z}_\alpha}^0$, then the transformations of the deformed action are also of the form

$$\mathbb{Z}_\alpha = \sum_{k=0} g^k {\mathbb{Z}_\alpha}^k .$$

In a similar way, the equations of motion (and hence the Euler forms) are also deformed

$$E_A = \sum_{k=0} g^k {E_A}^k .$$

The deformed action will induce a deformed BRST action

$$\mathcal{S} = \sum_{k=0} g^k \mathcal{S}^k \ ,$$

which must solve the master equation to guarantee that the deformation from S^0 to S is consistent.

Given a theory, we build its *internally Abelianized* version by replacing its internal symmetry group $\mathcal{G}$ by a (dim $\mathcal{G}$)-dimensional Abelian group. At the level of its Lie algebra $\mathfrak{g}$, this is equivalent to requiring the Lie bracket to vanish. By doing this, the interaction terms that arise from the higher order terms in the curvature and covariant derivatives disappear. Then, if the theory given by the resulting action is integrable, one can add as an interaction the term that was eliminated from the original action by the process of internal Abelianization. If the deformation is consistent, then a regular perturbative scheme can be set up as long as the solutions of the Abelianized theory can be found in a sufficiently explicit form. This is precisely the case in Yang-Mills actions, whose internal Abelianizations are multiple independent copies of the Maxwell action.

The deformations (5.16) that we will obtain using this procedure are just of the form

$$S = S^0 + gS^1 \ ,$$

where g is the perturbative parameter and S^1 collects all the terms lost in the process of internal Abelianization. The perturbed gauge transformations and Euler forms are expanded in the same way. This yields a BRST generator $\mathcal{S}$ that must satisfy the master equation (5.10), and in particular, (5.12) (which must always hold for both reducible and irreducible theories). Among the theories that we will consider, only one of them is reducible and coincidentally, the deformation that it defines is not consistent. For the rest of them, one still must take into account the closure of the gauge algebra (5.14) and the Jacobi identity (5.15). Hence, in the described particular case, this imposes, up to first order, that

$$
\begin{aligned}
0 = \text{d}\!\!\text{I} S(\mathbb{Z}_\alpha) &= \int_{\mathcal{M}} E_A \wedge Z^A{}_\alpha \\
&= \int_{\mathcal{M}} \left(E^0_A \wedge Z^{0A}{}_\alpha + g\left(E^1_A \wedge Z^{0A}{}_\alpha + E^0_A \wedge Z^{1A}{}_\alpha \right) + O(g^2) \right) \ . \qquad (5.17)
\end{aligned}
$$

To zeroth order in g, (5.17) is automatically satisfied since this is the condition that the internally Abelianized action is gauge invariant under its own gauge transformations $Z^{0A}{}_\alpha$. However, to first order (5.17) is only true when the deformation is consistent. In this case, $\text{d}\!\!\text{I} S^1(\mathbb{Z}^0_\alpha)$ must vanish when the unperturbed equations of motion E^0_A hold. Furthermore, we can write

$$
g\int_{\mathcal{M}} E^1_A \wedge Z^{0A}{}_\alpha = -g\int_{\mathcal{M}} E^0_A \wedge Z^{1A}{}_\alpha \ . \qquad (5.18)
$$

From expression (5.18) one can read off the deformation of the symmetries $Z^{1A}{}_\alpha$. Notice that, if the deformation *preserves* the gauge symmetries, i.e. if $Z^{1A}{}_\alpha = 0$, then $\text{d}\!\!\text{I} S^1(\mathbb{Z}^0_\alpha)$ vanishes identically.

It should be noted that it is relevant which action we take as the unperturbed theory and which one as the deformation. First, the chosen unperturbed theory should be integrable, but even if both are, the order may change the results. As an example, consider the 3-dimensional Cartan-Palatini and Yang-Mills actions

$$
S_{\text{3-P}}(\mathbf{e},\boldsymbol{\omega}) = \int_{\mathcal{M}} \mathbf{e}_i \wedge \mathbf{F}^i \ , \quad S_{YM}(\boldsymbol{\omega}) = \int_{\mathcal{M}} \mathbf{F}_i \wedge *\mathbf{F}^i \ .
$$

In 3 dimensions, General Relativity has no local degrees of freedom, while Yang-Mills has $1 \times \dim SO(3) = 1 \times 3 = 3$. The sum of both actions

$$
S(\mathbf{e},\boldsymbol{\omega}) = \int_{\mathcal{M}} (\mathbf{e}_i + *\mathbf{F}_i) \wedge \mathbf{F}^i \ ,
$$

yields the equations of motion

$$
\begin{aligned}
2\ \mathbf{D} \star \mathbf{F}^i + \mathbf{D}\mathbf{e}^i &= 0 \ , \\
\mathbf{F}^i &= 0 \ ,
\end{aligned}
$$

which are trivially equivalent to

$$
\begin{aligned}
\mathbf{D}\mathbf{e}^i &= 0 \ , \\
\mathbf{F}^i &= 0 \ .
\end{aligned}
$$

These are equivalent to the equations of General Relativity in 3 dimensions. Hence, if we take as starting point the Cartan-Palatini action and treat the Yang-Mills term as an interaction, the resulting theory has the same equations of motion and degrees of freedom than the original one, namely, zero. On the other hand, if the starting point is the Yang-Mills action and we add the Cartan-Palatini term as an interaction, the original action has 3 degrees of freedom wheras the deformed one has none, hence, we do not have a consistent deformation.

We are now going to discuss the internal Abelianization strategy for some actions that are relevant to study gravitational theories. Note that we are going to treat them in a fully covariant way without turning to 3+1 decompositions or canonical methods, since we will be able to give explicit solutions for the fields in the whole spacetime manifold.

5.3.1 Chern-Simons

The internally Abelianized version of the Chern-Simons action (3.7) is

$$S_{\mathrm{CS}}(\mathbf{A}) = \frac{1}{2}\int_{\mathcal{M}} g_{ij}\mathbf{A}^i \wedge \mathbf{d}\mathbf{A}^j \ ,$$

and its equations of motion are

$$\mathbf{d}\mathbf{A}^i = 0 \ .$$

The solutions to these equations are easily obtained. They are

$$\mathbf{A}^i = \mathbf{d}f^i + \lambda^i{}_a \Phi^a \ ,$$

with $f^i \in \mathcal{C}^\infty(M)$ an arbitrary function, $\lambda^i{}_a \in \mathbb{R}$ and $\Phi^a \in H^1_{dR}(\mathcal{M})$ representatives of the first de Rham cohomology group. One would then like to obtain all the possible interaction terms that can be added to this theory. This has already been studied and the main results are summarised in the following theorem:

Theorem 10. *[75, page 560, eq. (14.13)] The consistent deformations of Abelian Chern-Simons actions are Chern-Simons actions based on arbitrary groups of the same dimension.*

It is perhaps more illustrative to particularize the discussion to the 3-dimensional Cartan-Palatini action (3.11) which arises by choosing the group $SO(1,2)$ in the Chern-Simons action. Then, internally Abelianizing would mean replacing $SO(1,2)$ by the Abelian group $U(1)^3$. In this case, the curvature $\mathbf{F}^i$ reduces to $\mathbf{d}\boldsymbol{\omega}^i$. The resulting action reads

$$S^0_{\text{3-P}}(\mathbf{e}, \boldsymbol{\omega}) = 2\int_{\mathcal{M}} \mathbf{e}_i \wedge \mathbf{d}\boldsymbol{\omega}^i \ , \tag{5.19}$$

whose Euler forms and field equations are simply

$$E^{0i}_{\mathbf{e}} := 2\mathbf{d}\boldsymbol{\omega}^i = 0 \ ,$$
$$E^{0i}_{\boldsymbol{\omega}} := 2\mathbf{d}\mathbf{e}^i = 0 \ .$$

As with the general Chern-Simons case, the equations have explicit solutions on an arbitrary manifold $\mathcal{M}$ characterized by the representatives Φ^a of the equivalence classes of the first de Rham cohomology group $H^1_{dR}(\mathcal{M})$ of $\mathcal{M}$:

$$\begin{aligned} \mathbf{e}^i &= \mathbf{d}f^i + \lambda^i{}_a\Phi^a \ , \\ \boldsymbol{\omega}^i &= \mathbf{d}g^i + \mu^i{}_a\Phi^a \ , \end{aligned} \tag{5.21}$$

where $f^i, g^i \in \mathcal{C}^\infty(M)$ are arbitrary functions and $\lambda^i{}_a, \mu^i{}_a \in \mathbb{R}$. Note that this fully parametrizes the space of solutions by 6 functions and $2n_1$ real numbers where $n_k = \dim H^k_{dR}(\mathcal{M})$. To separate the gauge degrees of freedom from the physical ones, we must look at the presymplectic structure in the space of solutions of the field equations [81]. The symplectic form in the covariant space is

$$\boldsymbol{\Omega} = -\int_\Sigma \mathbb{d}\mathbf{e}_i \wedge \mathbb{d}\boldsymbol{\omega}^i \ .$$

By pulling it back to the solution space defined by (5.21) one gets

$$\begin{aligned} \boldsymbol{\Omega}_{\mathbb{S}} = &-\int_\Sigma \Big(\mathbf{d}\mathbb{d} f_i \wedge \mathbf{d}\mathbb{d} g^i + \mathbf{d}\mathbb{d} f_i \wedge \mathbb{d}\mu^i{}_b \wedge \Phi^b \\ &+ \mathbb{d}\lambda_{ia} \wedge \Phi^a \wedge \mathbf{d}\mathbb{d} g^i + \mathbb{d}\lambda_{ia} \wedge \Phi^a \wedge \mathbb{d}\mu^i{}_b \wedge \Phi^b\Big) \\ = &-\left(\int_\Sigma \Phi^a \wedge \Phi^b\right) \mathbb{d}\lambda^i{}_a \wedge \mathbb{d}\mu_{ib} \ , \end{aligned}$$

since the rest of the terms are exact (the Φ^a are closed) and do not contribute to the bulk of the integral. Note that the arbitrary functions f^i, g^i do not play any role in the presymplectic form, hence, they are gauge degrees of freedom of the $U(1)^6$ gauge group. In fact, $\boldsymbol{\Omega}$ does not depend on any field, hence the theory has no local degrees of freedom. It has, nevertheless, $2n_1$ global (or topological) degrees of freedom described by $\lambda^i{}_a, \mu^i{}_a$.

The symmetries of the action (5.19) are

$$\begin{aligned} (Z^0_{\text{Diff}})^i_{\mathbf{e}} \cdot \boldsymbol{\rho} &= \mathcal{L}_{\boldsymbol{\rho}}\mathbf{e}^i \ , & (Z^0_{\text{Diff}})^i_{\boldsymbol{\omega}} \cdot \boldsymbol{\rho} &= \mathcal{L}_{\boldsymbol{\rho}}\boldsymbol{\omega}^i \ , \\ (Z^0_{\text{T}})^i_{\mathbf{e}} \cdot \boldsymbol{\tau} &= 0 \ , & (Z^0_{\text{T}})^i_{\boldsymbol{\omega}} \cdot \boldsymbol{\tau} &= \mathbf{d}\tau^i \ , \\ (Z^0_{\text{R}})^i_{\mathbf{e}} \cdot \boldsymbol{\lambda} &= \mathbf{d}\lambda^i \ , & (Z^0_{\text{R}})^i_{\boldsymbol{\omega}} \cdot \boldsymbol{\lambda} &= 0 \ , \end{aligned}$$

where $\mathbb{Z}^0_{\text{Diff}}$ implements the diffeomorphisms and $\mathbb{Z}^0_{\text{T}}, \mathbb{Z}^0_{\text{R}}$ correspond to the $U(1)^6$ gauge group. Note that the latter can be easily read off from the expressions of the solutions (5.21). The theory is irreducible, since λ^i, τ^i are 0-forms. The generator algebra is given by

$$\begin{aligned} [\mathbb{Z}_{\text{T}} \cdot \boldsymbol{\tau}, \mathbb{Z}_{\text{R}} \cdot \boldsymbol{\lambda}] &= 0 \ , \\ [\mathbb{Z}_{\text{R}} \cdot \boldsymbol{\lambda}, \mathbb{Z}_{\text{Diff}} \cdot \boldsymbol{\rho}] &= \mathbb{Z}_{\text{R}} \cdot \imath_{\boldsymbol{\rho}}\mathbf{d}\boldsymbol{\lambda} \ , \\ [\mathbb{Z}_{\text{T}} \cdot \boldsymbol{\tau}, \mathbb{Z}_{\text{Diff}} \cdot \boldsymbol{\rho}] &= \mathbb{Z}_{\text{T}} \cdot \imath_{\boldsymbol{\rho}}\mathbf{d}\boldsymbol{\tau} \ . \end{aligned}$$

Let us turn to its possible consistent deformations. According to Theorem 10, they are all of the form

$$S^1_{\text{3-P}} = \int_{\mathcal{M}} \frac{1}{3}\langle \mathbf{A}, [\![\mathbf{A},\mathbf{A}]\!]\rangle \ , \tag{5.22}$$

where now $\langle\cdot,\cdot\rangle$ and $[\![\cdot,\cdot]\!]$ are an invariant metric and the Lie brackets of an arbitrary 6-dimensional Lie algebra. Not surprisingly, of particular relevance is the family of algebras $\lambda\mathfrak{iso}(2,1)$ defined in (3.12) with the invariant metric (3.10). If $\lambda = 0$, then this algebra is just $\mathfrak{iso}(2,1)$. For $\lambda > 0$ it is isomorphic to $\mathfrak{so}(3,1)$ and for $\lambda < 0$ it is isomorphic $\mathfrak{so}(2,2)$. By using the change of basis $\mathbf{A} = \mathbf{e}_i P^i + \boldsymbol{\omega}^i J_i$ (as discussed in Section 3.2), it is possible to write (5.22) as

$$S^1_{\text{3-P}} = \int_{\mathcal{M}} \left(\epsilon_{ijk}\mathbf{e}^i \wedge \boldsymbol{\omega}^j \wedge \boldsymbol{\omega}^k + \frac{\lambda}{3}\epsilon_{ijk}\mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k\right) \ .$$

Together with the original action the deformed action reads

$$S_{\text{3-P}} = 2\int_{\mathcal{M}} \left(\mathbf{e}_i \wedge \mathbf{F}^i + \Lambda \mathsf{vol}_{\mathbf{e}}\right) \ , \tag{5.23}$$

which is the Cartan-Palatini action for 3-dimensional gravity with a cosmological constant Λ term. Note that Λ can be made zero, positive or negative by choosing the appropriate group.

A straightforward computation gives

$$\begin{aligned}
&\mathbb{d}S^1_{\text{CS}}(\mathbb{Z}^0_\alpha \cdot \epsilon^\alpha)\\
&= \int_{\mathcal{M}} \Bigg(\mathbf{d}\imath_{\boldsymbol{\rho}} \left(\epsilon_{ijk}\mathbf{e}^i \wedge \boldsymbol{\omega}^j \wedge \boldsymbol{\omega}^k + \frac{\lambda}{3}\epsilon_{ijk}\mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k\right)\\
&\quad + 2\mathbf{d}\left(\epsilon_{ijk}\mathbf{e}^i \wedge \boldsymbol{\omega}^j \tau^k\right) - \epsilon_{ijk} E^{0i}_{\boldsymbol{\omega}} \wedge \boldsymbol{\omega}^j \tau^k + \epsilon_{ijk} E^{0i}_{\mathbf{e}} \wedge \mathbf{e}^j \tau^k\\
&\quad + \mathbf{d}\left(\epsilon_{ijk}\lambda^i\left(\boldsymbol{\omega}^j \wedge \boldsymbol{\omega}^k + \lambda \mathbf{e}^j \wedge \mathbf{e}^k\right)\right) - \lambda^i \epsilon_{ijk}\left(E^{0j}_{\mathbf{e}} \wedge \boldsymbol{\omega}^k + \lambda E^{0j}_{\boldsymbol{\omega}} \wedge \mathbf{e}^k\right)\Bigg) \approx 0 \ .
\end{aligned}$$

Since all the bulk terms are proportional to the Euler forms, the deformation is consistent. This is not surprising in hindsight, because the deformed action (5.23) has no local degrees of freedom as in the case for its internally Abelianized version. Hence, the deformation process did not change their number, as expected per the features of a consistent deformation. According to (5.18), the symmetries of the deformed action are given by

$$\begin{aligned}
(Z_{\text{Diff}})^i_{\mathbf{e}} \cdot \boldsymbol{\rho} &= \mathcal{L}_{\boldsymbol{\rho}}\mathbf{e}^i &&, \quad (Z_{\text{Diff}})^i_{\boldsymbol{\omega}} \cdot \boldsymbol{\rho} = \mathcal{L}_{\boldsymbol{\rho}}\boldsymbol{\omega}^i \ ,\\
(Z_{\text{T}})^i_{\mathbf{e}} \cdot \boldsymbol{\tau} &= -g\epsilon^i{}_{jk}\mathbf{e}^j\tau^k &&, \quad (Z_{\text{T}})^i_{\boldsymbol{\omega}} \cdot \boldsymbol{\tau} = \mathbf{d}\tau^i + g\epsilon^i{}_{jk}\boldsymbol{\omega}^j\tau^k \ ,\\
(Z_{\text{R}})^i_{\mathbf{e}} \cdot \boldsymbol{\lambda} &= \mathbf{d}\lambda^i + g\epsilon^i{}_{jk}\boldsymbol{\omega}^j\lambda^k &&, \quad (Z_{\text{R}})^i_{\boldsymbol{\omega}} \cdot \boldsymbol{\lambda} = g\epsilon^i{}_{jk}\mathbf{e}^j\lambda^k \ .
\end{aligned}$$

5.3.2 Husain-Kuchař

Since the internal group of the Husain-Kuchař action (3.18) is $SO(3)$, its internally Abelianized version is

$$S^0_{\mathrm{HK}}(\mathbf{e},\boldsymbol{\omega}) = \int_{\mathcal{M}} \epsilon_{ijk}\mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{d}\boldsymbol{\omega}^k \ , \tag{5.24}$$

where $\boldsymbol{\omega}^i$ is now a $U(1)^3$ connection.

The symmetries of S^0_{HK} can be read directly from it. They are

$$(Z^0_{\mathrm{Diff}})^i_{\mathbf{e}} \cdot \boldsymbol{\rho} = \mathcal{L}_{\boldsymbol{\rho}}\mathbf{e}^i \ , \quad (Z^0_{\mathrm{Diff}})^i_{\boldsymbol{\omega}} \cdot \boldsymbol{\rho} = \mathcal{L}_{\boldsymbol{\rho}}\boldsymbol{\omega}^i \ , \tag{5.25a}$$

$$(Z^0_{\mathrm{R}})^i_{\mathbf{e}} \cdot \boldsymbol{\tau} \ = 0 \qquad , \quad (Z^0_{\mathrm{R}})^i_{\boldsymbol{\omega}} \cdot \boldsymbol{\tau} \ = \mathbf{d}\tau^i \ , \tag{5.25b}$$

where $\boldsymbol{\rho} \in \mathfrak{X}(\mathcal{M})$ is an arbitrary vector field in $\mathcal{M}$ and $\tau^i \in \mathcal{C}^\infty(\mathcal{M})$. The symmetries involving $\boldsymbol{\rho}$ are the 4-diffeomorphisms of $\mathcal{M}$, while those involving $\boldsymbol{\tau}$ generate internal $U(1)^3$ rotations. The theory is irreducible. The algebra is

$$\begin{aligned}
[\mathbb{Z}_{\mathrm{R}} \cdot \boldsymbol{\tau}, \mathbb{Z}_{\mathrm{R}} \cdot \boldsymbol{\tau}] \quad &= 0 \ , \\
[\mathbb{Z}_{\mathrm{Diff}} \cdot \boldsymbol{\rho}, \mathbb{Z}_{\mathrm{Diff}} \cdot \boldsymbol{\rho}] &= 0 \ , \\
[\mathbb{Z}_{\mathrm{R}} \cdot \boldsymbol{\tau}, \mathbb{Z}_{\mathrm{Diff}} \cdot \boldsymbol{\rho}] \quad &= \mathbb{Z}_{\mathrm{R}} \cdot \imath_{\boldsymbol{\rho}}\mathbf{d}\boldsymbol{\tau} \ .
\end{aligned}$$

The Euler forms obtained by varying (5.24) are

$$E^{0i}_{\mathbf{e}} := -2\epsilon^i{}_{jk}\mathbf{e}^j \wedge \mathbf{d}\boldsymbol{\omega}^k \ , \tag{5.26a}$$

$$E^{0i}_{\boldsymbol{\omega}} := -\epsilon^i{}_{jk}\mathbf{d}(\mathbf{e}^j \wedge \mathbf{e}^k) \ . \tag{5.26b}$$

The presymplectic form in the covariant space is

$$\boldsymbol{\Omega} = 2\int_\Sigma \epsilon_{ijk} \mathbb{d}\mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbb{d}\boldsymbol{\omega}^k \ .$$

Although it is not necessary for the upcoming discussion, one would like to have a parametrization of the space of solutions of the internally Abelianized model. Even though the equations are not as involved as the ones appearing in General Relativity, to this date this solution is still unknown.

The deformation that one needs to add in order to recover the full action (3.18) is

$$S^1_{\mathrm{HK}}(\mathbf{e},\boldsymbol{\omega}) = \int_{\mathcal{M}} \mathbf{e}_i \wedge \mathbf{e}_j \wedge \boldsymbol{\omega}^i \wedge \boldsymbol{\omega}^j \ .$$

We now need to compute (5.18) in order to check the consistency of the deformation. A direct computation gives

$$\begin{aligned}
\mathbb{d}S^1_{\mathrm{HK}}(\mathbb{Z}^0_\alpha \cdot \epsilon^\alpha) = \int_{\mathcal{M}} \bigg(&\mathbf{d}\left(2\tau^i \mathbf{e}_i \wedge \mathbf{e}_j \wedge \boldsymbol{\omega}^j + \imath_\rho \left(\mathbf{e}_i \wedge \mathbf{e}_j \wedge \boldsymbol{\omega}^i \wedge \boldsymbol{\omega}^j\right)\right) \\
&- \tau^i \varepsilon_{ijk}\left(\boldsymbol{\omega}^j \wedge E^{0k}_{\boldsymbol{\omega}} + \mathbf{e}^j \wedge E^{0k}_{\mathbf{e}}\right)\bigg) \approx 0 \ .
\end{aligned}$$

Hence, this deformation is consistent. We can now get the deformed symmetries by using (5.18); they are

$$
\begin{aligned}
(Z_{\text{Diff}})^i_{\mathbf{e}} \cdot \boldsymbol{\rho} &= \mathcal{L}_{\boldsymbol{\rho}} \mathbf{e}^i &, \quad (Z_{\text{Diff}})^i_{\boldsymbol{\omega}} \cdot \boldsymbol{\rho} &= \mathcal{L}_{\boldsymbol{\rho}} \boldsymbol{\omega}^i \ , \\
(Z_{\text{R}})^i_{\mathbf{e}} \cdot \boldsymbol{\tau} &= -g\varepsilon^i{}_{jk}\tau^j \mathbf{e}^k &, \quad (Z_{\text{R}})^i_{\boldsymbol{\omega}} \cdot \boldsymbol{\tau} &= \mathbf{d}\tau^i + g\epsilon^i{}_{jk}\boldsymbol{\omega}^j \tau^k .
\end{aligned}
$$

5.3.3 Cartan-Palatini

The internal group of the (4-dimensional) Cartan-Palatini action (3.32) is $SO(1,3)$, hence its internal Abelianization must be built with a $U(1)^6$ connection $\boldsymbol{\omega}^{IJ}$. The Abelianized action is

$$
S^0_{\text{CP}}(\mathbf{e}, \boldsymbol{\omega}) = \int_{\mathcal{M}} \epsilon_{IJKL} \mathbf{e}^I \wedge \mathbf{e}^J \wedge \mathbf{d}\boldsymbol{\omega}^{KL} \ , \tag{5.27}
$$

and the Euler forms (5.1) are

$$
\begin{aligned}
E^{0I}_{\mathbf{e}} &= -2\epsilon^I{}_{JKL} \mathbf{e}^J \wedge \mathbf{d}\boldsymbol{\omega}^{KL} \ , \\
E^{0IJ}_{\boldsymbol{\omega}} &= -\epsilon^{IJ}{}_{KL} \mathbf{d}\left(\mathbf{e}^K \wedge \mathbf{e}^L\right) \ .
\end{aligned}
$$

Requiring that they vanish yields the equations of motion, which according to Lemma 20 of Appendix B are equivalent to

$$
\mathbf{d}\mathbf{e}^I = 0 \ , \tag{5.28a}
$$

$$
\epsilon_{IJKL} \mathbf{d}\left(\mathbf{e}^J \wedge \boldsymbol{\omega}^{KL}\right) = 0 \ . \tag{5.28b}
$$

Equation (5.28a) is readily solved by

$$
\mathbf{e}^I = \mathbf{d}h^I + \eta^I{}_a \Phi^{(1)^a} \ , \tag{5.29}
$$

with $h^I \in \mathcal{C}^\infty(\mathcal{M})$, $\eta^I{}_a \in \mathbb{R}$ and $\Phi^{(1)^a} \in \Omega^1(\mathcal{M})$ representatives of the first de Rham cohomology group.

Equation (5.28b) implies

$$
\epsilon_{IJKL} \mathbf{e}^J \wedge \boldsymbol{\omega}^{KL} = \mathbf{d}g_I + \gamma_{Ia} \Phi^{(2)^a} =: \mathbf{B}_I \ ,
$$

with $g_I \in \mathcal{C}^\infty(\mathcal{M})$, $\gamma_{Ia} \in \mathbb{R}$ and $\Phi^{(2)^a} \in \Omega^2(\mathcal{M})$ representatives of the second de Rham cohomology group. The solution is given by Lemma 18 and is

$$
\boldsymbol{\omega}^{KL} = -\frac{1}{2}\left(\frac{\mathbf{B}_I \wedge \mathbf{e}^K \wedge \mathbf{e}^L}{\mathsf{vol}_{\mathbf{e}}}\right)\mathbf{e}^I - \frac{1}{2}\left(\frac{\mathbf{B}_I \wedge \mathbf{e}^I \wedge \mathbf{e}^{[K}}{\mathsf{vol}_{\mathbf{e}}}\right)\mathbf{e}^{L]} \ . \tag{5.30}
$$

In the simply connected case, the solution for the metric is

$$
g = \eta_{IJ} \mathbf{d}h^I \otimes \mathbf{d}h^J \ ,
$$

which is, locally, the Minkowski metric. As in other more traditional perturbative approaches, the leading term in the metric is Minkowski.

The presymplectic form in the covariant space is

$$\mathbf{\Omega}_{\mathbb{S}} = \int_{\Sigma} \epsilon_{IJKL} \mathbb{d}\mathbf{e}^I \wedge \mathbf{e}^J \wedge \mathbb{d}\boldsymbol{\omega}^{KL} \ ,$$

and its pullback to the solution space defined by (5.29) and (5.30) is

$$\mathbf{\Omega} = \Big(\int_{\Sigma} \Phi^{(1)^a} \wedge \Phi^{(2)^b} \Big) \, \mathbb{d}\eta_a^I \wedge \mathbb{d}\gamma_{Ib} \ .$$

As $\mathbf{\Omega}$ does not depend on any fields, the theory has no local degrees of freedom. At this point, one should be suspicious that the deformation leading to the full action might not be consistent since the local degrees of freedom change abruptly. In a moment we will check this explicitly.

Since we already know the solutions for $\mathbf{e}^I, \boldsymbol{\omega}^{KL}$ in terms of $h^I, \mathbf{B}^I$, it is easy to find the gauge generators of the action by varying $h^I, \mathbf{B}^I$

$$\begin{aligned}
(Z^0_{\mathrm{T}})^I_{\mathbf{e}} \cdot \boldsymbol{\tau} &= \mathbf{d}\tau^I \ , \\
(Z^0_{\mathrm{T}})^I_{\boldsymbol{\omega}} \cdot \boldsymbol{\tau} &= \frac{1}{2}\epsilon_{KLPQ} \left(\frac{\mathbf{d}\tau^L \wedge \boldsymbol{\omega}^{PQ} \wedge \mathbf{e}^I \wedge \mathbf{e}^J}{\mathsf{vol}_{\mathbf{e}}} \right) \mathbf{e}^K \\
&\quad + \frac{1}{2}\epsilon_{KLPQ} \left(\frac{\mathbf{d}\tau^L \wedge \boldsymbol{\omega}^{PQ} \wedge \mathbf{e}^K \wedge \mathbf{e}^{[I}}{\mathsf{vol}_{\mathbf{e}}} \right) \mathbf{e}^{J]} \ , \\
(Z^0_{\mathrm{R}})^I_{\mathbf{e}} \cdot \boldsymbol{\chi} &= 0 \ , \\
(Z^0_{\mathrm{R}})^I_{\boldsymbol{\omega}} \cdot \boldsymbol{\chi} &= -\frac{1}{2} \left(\frac{\mathbf{d}\chi_K \wedge \mathbf{e}^I \wedge \mathbf{e}^J}{\mathsf{vol}_{\mathbf{e}}} \right) \mathbf{e}^K - \frac{1}{2} \left(\frac{\mathbf{d}\chi_K \wedge \mathbf{e}^K \wedge \mathbf{e}^{[I}}{\mathsf{vol}_{\mathbf{e}}} \right) \mathbf{e}^{J]} \ .
\end{aligned}$$

Note that the theory is first-order reducible, since the parameter $\chi_K = \mathbf{d}\gamma_K$, for $\gamma_K \in \mathcal{C}^\infty(\mathcal{M})$, makes the gauge generator $\mathbb{Z}_{\mathrm{R}}$ vanish.

The interaction term that we have to add to (5.28) to recover the full Cartan-Palatini action (3.32) is

$$S^1_{\mathrm{CP}} = \int_{\mathcal{M}} \epsilon_{IJKL} \mathbf{e}^I \wedge \mathbf{e}^J \wedge \boldsymbol{\omega}^K{}_M \wedge \boldsymbol{\omega}^{ML} \ , \tag{5.31}$$

A direct computation gives

$$\begin{aligned}
&\mathbb{d}S^1_{\mathrm{CP}}(\mathbb{Z}^0_\alpha \cdot \epsilon^\alpha) \qquad\qquad (5.32)\\
&= \int_{\mathcal{M}} 2\mathbf{d} \left(\epsilon_{IJKL}\tau^I \left(\mathbf{e}^J \wedge \boldsymbol{\omega}^K{}_M \wedge \boldsymbol{\omega}^{ML} + \mathbf{e}^M \wedge \boldsymbol{\omega}^J{}_M \wedge \boldsymbol{\omega}^{KL} \right) + \chi_I \wedge \mathbf{e}^J \wedge \boldsymbol{\omega}^I{}_J \right) \\
&\quad - 2\epsilon_{IJKL}\tau^I \mathbf{d} \left(\mathbf{e}^J \wedge \boldsymbol{\omega}^K{}_M \wedge \boldsymbol{\omega}^{ML} + \mathbf{e}^M \wedge \boldsymbol{\omega}^J{}_M \wedge \boldsymbol{\omega}^{KL} \right) + 2\chi_I \wedge \mathbf{d} \left(\mathbf{e}^J \wedge \boldsymbol{\omega}^I{}_J \right) \ .
\end{aligned}$$

We wish to know whether (5.32) vanishes on-shell or not. This amounts to checking whether the term $\mathbf{d}\left(\mathbf{e}^J \wedge \boldsymbol{\omega}^I{}_J\right)$ vanishes for all the solutions of the equations of motion (5.28). Let us show that this is not the case by exhibiting a solution not satisfying this condition in the simply connected case. Consider an arbitrary function $h^I \in \mathcal{C}^\infty(\mathcal{M})$ and set $\mathbf{e}^I = \mathbf{d}h^I$, $\boldsymbol{\omega}^{0i} = 0$ and $\boldsymbol{\omega}^{ij} = \epsilon^{ij}{}_k h^0 \mathbf{d}h^k$. Indeed, this a solution to the equations of motion: equation (5.28a) is identically satisfied, and for equation (5.28b), by separating the cases of $I = 0$ and $I = i$, we have

$$\begin{aligned}
\epsilon_{0jkl}\mathbf{d}h^j \wedge \mathbf{d}\boldsymbol{\omega}^{kl} &= \epsilon_{jkl}\mathbf{d}h^j \wedge \epsilon^{kl}{}_m \mathbf{d}h^0 \wedge \mathbf{d}h^m = 2\mathbf{d}h_m \wedge \mathbf{d}h^0 \wedge \mathbf{d}h^m = 0 \ , \\
\epsilon_{iJKL}\mathbf{d}h^J \wedge \mathbf{d}\boldsymbol{\omega}^{KL} &= \epsilon_{i0kl}\mathbf{d}h^0 \wedge \epsilon^{kl}{}_m \mathbf{d}h^0 \wedge \mathbf{d}h^m + 2\epsilon_{ij0l}\mathbf{d}h^j \wedge \mathbf{d}\boldsymbol{\omega}^{0l} = 0 \ .
\end{aligned}$$

On the other hand, we evaluate $\mathbf{d}\left(\mathbf{e}^J \wedge \boldsymbol{\omega}^I{}_J\right)$ and find

$$\begin{aligned}
&\mathbf{d}\left(\mathbf{e}^J \wedge \boldsymbol{\omega}^0{}_J\right) = 0 \ , \\
&\mathbf{d}\left(\mathbf{e}^J \wedge \boldsymbol{\omega}^i{}_J\right) = \mathbf{d}h^j \wedge \mathbf{d}\boldsymbol{\omega}^i{}_j = -\mathbf{d}h^j \wedge \epsilon^i{}_{jk}\mathbf{d}h^0 \wedge \mathbf{d}h^k \neq 0 \ .
\end{aligned}$$

Since it does not vanish, the deformation (5.31) is *not consistent*. In hindsight, this was expected, since the internally Abelianized theory has no local degrees of freedom, while it is well known that General Relativity has 2. As a consequence, although it is an integrable system, it is not possible to set a regular perturbative scheme leading to General Relativity.

Although the most interesting possibility of recovering full GR as a consistent deformation ultimately fails, it is worth exploring other consistent deformations. In particular, let us deform the action by introducing a cosmological constant term. The interaction in this case is

$$S^1_\Lambda = \frac{\Lambda}{4!}\int_{\mathcal{M}} \epsilon_{IJKL}\mathbf{e}^I \wedge \mathbf{e}^J \wedge \mathbf{e}^K \wedge \mathbf{e}^L \ .$$

It is straightforward to see that

$$\mathbb{d} S^1_\Lambda(\mathbb{Z}^0_\alpha \cdot \epsilon^\alpha) = \int_{\mathcal{M}} \left(\mathbf{d}\left(\frac{\Lambda}{6}\epsilon_{IJKL}\tau^I \mathbf{e}^J \wedge \mathbf{e}^K \wedge \mathbf{e}^L \right) - \frac{\Lambda}{4}\tau^I \mathbf{e}^J \wedge E^0_{\boldsymbol{\omega}_{IJ}} \right) \approx 0 \ ,$$

hence, it is indeed a consistent deformation. Notice that the last term implies a deformation of the τ symmetry by $-g\frac{\Lambda}{8}\left(\tau^I \mathbf{e}^J - \tau^J \mathbf{e}^I\right)$.

5.3.4 Anti-self-dual action

The internal symmetry group of the anti-self-dual action (4.2) is $SO(4)$, hence its internal Abelianization is built with the group $U(1)^6$, yielding

$$S^0_{\text{ASD}}(\mathbf{e}, \boldsymbol{\alpha}, \boldsymbol{\omega}) = \int_{\mathcal{M}} \left(\epsilon_{ijk}\mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{d}\boldsymbol{\omega}^k - 2\boldsymbol{\alpha} \wedge \mathbf{e}_i \wedge \mathbf{d}\boldsymbol{\omega}^i \right) \ , \qquad (5.33)$$

whose Euler forms (5.1) are

$$\begin{aligned}
E^0_{\boldsymbol{\alpha}} &= 2\mathbf{e}_i \wedge \mathbf{d}\boldsymbol{\omega}^i \ , \\
E^0_{\mathbf{e}i} &= -2\left(\epsilon_{ijk}\mathbf{e}^j \wedge \mathbf{d}\boldsymbol{\omega}^k + \boldsymbol{\alpha} \wedge \mathbf{d}\boldsymbol{\omega}_i\right) \ , \\
E^0_{\boldsymbol{\omega}k} &= -\mathbf{d}\left(\epsilon_{ijk}\mathbf{e}^i \wedge \mathbf{e}^j - 2\boldsymbol{\alpha} \wedge \mathbf{e}_k\right) \ .
\end{aligned}$$

The action (5.33) has the following gauge generators

$$
\begin{aligned}
&(Z^0_{\mathrm{R}})_{\boldsymbol{\omega}}{}^{i}\cdot\boldsymbol{\tau}=\mathbf{d}\tau^i\ , && (Z^0_{\mathrm{T}})_{\boldsymbol{\omega}}{}^{i}\cdot\boldsymbol{\lambda}=0 &&, && (Z^0_{\mathrm{Diff}})_{\boldsymbol{\omega}}{}^{i}\cdot\boldsymbol{\rho}=\mathcal{L}_{\boldsymbol{\rho}}\boldsymbol{\omega}^i\ ,\\
&(Z^0_{\mathrm{R}})_{\mathbf{e}}{}^{i}\cdot\boldsymbol{\tau}\ =0 \quad , && (Z^0_{\mathrm{T}})_{\mathbf{e}}{}^{i}\cdot\boldsymbol{\lambda}\ =-\lambda^i\boldsymbol{\alpha}+\epsilon^i{}_{jk}\mathbf{e}^j\lambda^k\ , && && (Z^0_{\mathrm{Diff}})_{\mathbf{e}}{}^{i}\cdot\boldsymbol{\rho}\ =\mathcal{L}_{\boldsymbol{\rho}}\mathbf{e}^i\ ,\\
&(Z^0_{\mathrm{R}})_{\boldsymbol{\alpha}}\cdot\boldsymbol{\tau}\ =0 \quad , && (Z^0_{\mathrm{T}})_{\boldsymbol{\alpha}}\cdot\boldsymbol{\lambda}\ =\lambda_i\mathbf{e}^i &&, && (Z^0_{\mathrm{Diff}})_{\boldsymbol{\alpha}}\cdot\boldsymbol{\rho}\ =\mathcal{L}_{\boldsymbol{\rho}}\boldsymbol{\alpha}\ ,
\end{aligned}
$$

where $\boldsymbol{\rho}\in\mathfrak{X}(\mathcal{M})$ is an arbitrary vector field in $\mathcal{M}$ and $\tau^i,\Lambda^i\in\mathcal{C}^\infty(\mathcal{M})$. Then, $\mathbb{Z}^0_{\mathrm{Diff}}$ generates the diffeomorphisms and $\mathbb{Z}^0_{\mathrm{T}}$ and $\mathbb{Z}^0_{\mathrm{R}}$ generate the internal group $U(1)^6$. The theory is irreducible. The algebra they form is

$$
\begin{aligned}
&[\mathbb{Z}^0_{\mathrm{T}}\cdot\boldsymbol{\tau},\mathbb{Z}^0_{\mathrm{R}}\cdot\boldsymbol{\lambda}]=0\ ,\\
&[\mathbb{Z}^0_{\mathrm{R}}\cdot\boldsymbol{\lambda},\mathbb{Z}^0_{\mathrm{Diff}}\cdot\boldsymbol{\rho}]=0\ ,\\
&[\mathbb{Z}^0_{\mathrm{T}}\cdot\boldsymbol{\tau},\mathbb{Z}^0_{\mathrm{Diff}}\cdot\boldsymbol{\rho}]=\mathbb{Z}^0_{\mathrm{T}}\cdot\imath_{\boldsymbol{\rho}}\mathbf{d}\boldsymbol{\tau}\ ,
\end{aligned}
$$

while the rest of commutators are 0.

The deformation that we need to add to recover the whole action is

$$
S^1_{\mathrm{ASD}}=\int_{\mathcal{M}}\left(\mathbf{e}_i\wedge\mathbf{e}_j\wedge\boldsymbol{\omega}^i\wedge\boldsymbol{\omega}^j-\epsilon_{ijk}\boldsymbol{\alpha}\wedge\mathbf{e}^i\wedge\boldsymbol{\omega}^j\wedge\boldsymbol{\omega}^k\right)\ .
$$

Then, computing (5.18) directly gives

$$
\begin{aligned}
\mathbb{d}S^1_{\mathrm{ASD}}\left(\mathbb{Z}^0_{\alpha}\cdot\epsilon^\alpha\right)=\int_{\mathcal{M}}&\Big(2\mathbf{d}\left(\tau^i\left(\mathbf{e}_i\wedge\mathbf{e}_j\wedge\boldsymbol{\omega}^j+\epsilon_{ijk}\boldsymbol{\alpha}\wedge\mathbf{e}^j\wedge\boldsymbol{\omega}^k\right)\right)\\
&+\mathbf{d}\imath_{\boldsymbol{\xi}}\left(\mathbf{e}_i\wedge\mathbf{e}_j\wedge\boldsymbol{\omega}^i\wedge\boldsymbol{\omega}^j-\epsilon_{ijk}\boldsymbol{\alpha}\wedge\mathbf{e}^i\wedge\boldsymbol{\omega}^j\wedge\boldsymbol{\omega}^k\right)\\
&+\tau^i\left(-\epsilon_{ijk}\boldsymbol{\omega}^j\wedge E^{0k}_{\boldsymbol{\omega}}-\mathbf{e}_i\wedge E^0_{\boldsymbol{\alpha}}+\boldsymbol{\alpha}\wedge E^0_{\mathbf{e}i}\right)\Big)\approx 0\ ,
\end{aligned}
$$

hence the deformation is consistent. Also, there is only one deformed symmetry given by

$$
\begin{aligned}
&(Z_{\mathrm{T}})_{\boldsymbol{\omega}}{}^{i}\cdot\boldsymbol{\tau}=\mathbf{d}\tau^i+g\epsilon^i{}_{jk}\boldsymbol{\omega}^j\tau^k=\mathbf{D}\tau^i\ ,\\
&(Z_{\mathrm{T}})_{\mathbf{e}}{}^{i}\cdot\boldsymbol{\tau}=g\tau^i\boldsymbol{\alpha}\ ,\\
&(Z_{\mathrm{T}})_{\boldsymbol{\alpha}}\cdot\boldsymbol{\tau}=-g\tau^i\mathbf{e}_i\ ,
\end{aligned}
$$

while the $\mathbb{Z}_{\mathrm{Diff}}$ and $\mathbb{Z}_{\mathrm{R}}$ transformations remain the same.

It is remarkable that the anti-self-dual action (4.2) can be obtained as the deformation of its internally Abelianized version, in contrast to the Cartan-Palatini action (3.32), since they both yield General Relativity (though in the anti-self-dual case it is its Euclildean version). This suggests the possibility of an action similar to (4.2) that can be obtained as a deformation of its internally Abelianized version and gives Lorentzian General Relativity. Notice that the same approach used to derive the action (4.2) would not work in the Lorentzian case, since (anti)-self-dual connections would be complex. Nevertheless, it is interesting to explore the possibility of a complexified variational principle based on the Lorentizan anti-self-dual action, which potentially could lead to a suitable perturbative scheme using the approach followed in this section. The main obstacle for this strategy is that the treatment of complex variational principles is not well understood, as exemplified by the fact that it is not clear how to consistently implement the so called reality conditions [72], [82].

Chapter 6

Conclusions

This thesis has focused on two main problems. On the one hand, in Chapter 4, an alternative action for the Euclidean version of anti-self-dual General Relativity has been studied in detail, in particular, the GNH algorithm has been used to find its Hamiltonian formulation. The studied action is the result of plugging the concrete expression of anti-self-dual connections into the Cartan-Palatini action. In this way, one obtains an action composed by two terms: the first describes the Husain-Kuchař model while the second is linear in the 0-th element of the coframe $\mathbf{e}$. A problem we have faced is that the gauge group $SO(4)$ is naturally split in two $SO(3)$ components, one of which is not so explicitly manifest in the sense that it is not directly reflected by the internal indices of the fields used to write the action. It appears then, that although one manages to split the symmetry in smaller groups, which are easier to handle, their complexity still shows up somewhere else. The solution for the Hamiltonian vector field (from which all the information can be extracted) has been seen to depend on 10 arbitrary functions: these correspond to the $SO(4)$ and diffeomorphism symmetries.

On the other hand, in Chapter 5 a new perturbative scheme for field theories has been proposed and its viability has been analysed in a variety of interesting gravitational theories. The idea is to use internally Abelianized versions of the theories as the starting point of a perturbative expansion, i.e., replacing the internal symmetry group of the action by an Abelian group of the same dimension. In terms of the Lie algebra, this has the effect of setting all the Lie brackets to 0, which removes the quadratic terms in the connection from the action. This greatly simplifies the resulting equations of motion and has allowed us to solve them directly in the cases at hand without the need to resort to Hamiltonian methods.

A first requirement for the perturbative scheme to work is that the unperturbed theory be integrable and the solution is expressible in a reasonable form in order to serve as the starting point of the perturbative expansion. However, this is not enough. An equally important condition is that the perturbation is a consistent deformation of the base action (i.e., the deformation introduced to the action preserves the number of gauge symmetries, although their expressions may change). This is equivalent to saying that the perturbation is regular rather than singular. Indeed, for regular perturbations one can write an expansion in power series, while for singular perturbations this is not possible. Note that in the singular case, perturbative schemes are still available, however they require different approaches that may be difficult to find. Only when these two conditions hold it is reasonable to pursue a perturbative theory along these lines. A particularly nice feature of the method discussed in the thesis is that it does not need any auxiliary background objects, which are fundamental in the traditional approach to perturbative gravity, namely, the Minkowski metric. Also, each term of the perturbative expannsion of the action is covariant, which may be beneficial since there is no breaking of the diffeomorphism symmetry (which as mentioned before, might lead to unexpected unpleasant effects). These two facts combined make this approach very reasonable for a perturbative scheme for gravitational theories.

The consistent deformations method can be applied in a wide range of gravitational theories as shown in Chapter 5. Unfortunately, in the actions relevant for General Relativity, namely Cartan-Palatini (and also the Holst action [2]) the scheme does not work, since the resulting deformation is not consistent. This is due to the fact that General Relativity has two local degrees of freedom, while its internally Abelianized version (also called the $U(1)^3$ model) is topological, i.e., it has no local degrees of freedom. In this sense, the model for (anti-self-dual) Euclidean General Relativity presented in Chapter 4 is interesting, since it actually is a consistent deformation of its internally Abelianized version, while Euclidean Cartan-Palatini is not. This motivates the search of an action that yields Lorentzian General Relativity and to which the consistent deformation procedure can be applied.

It is somewhat surprising that, given two different actions for the same physical theory, one can be consistently deformed from its internal Abelianization while the other can not. It is also very surprising that precisely the actions that yield the physically relevant gravitational theory are not consistent deformations of their internal Abelianizations, while other simpler actions are. It would be worth investigating if there is some physical property of a theory that determines whether this process works or not. Obtaining such a characterization could open the possibility for new actions admitting consistent deformations.

An also very interesting (but also very difficult) problem would be to find *all the possible* consistent deformations of a given action. The appropriate context to deal with this problem is cohomology and the use of the BRST symmetry. Then, the task is equivalent to computing some cohomology groups (as in [75] for the Chern-Simons case), which is in general hard to do.

Another possible future project could be to implement the perturbation scheme for quantization. For a given theory, first one would need to quantize of its internally Abelianized version and then find a way to implement the quantization of the deformation perturbatively. Of particular interest would be the Chern-Simons action (which might be the simplest of the studied in this thesis because of the low dimensionality), since its quantization has already been done by Witten [83] in another way. Similarly, the Husain-Kuchař model is also interesting because it yields 3-geometries and it is quantizable in the Loop Quantum Gravity scheme since the difficult-to-quantize Hamiltonian constraint is not present. It could also be interesting to analyze the quantum version of the $U(1)^3$ model, equipped with a well defined set of observables, and study the implications it has for the full quantum General Relativity.

Appendix A

Some useful expressions for 3+1 decompositions

We can decompose any tensor field with the help of the $(1,1)$-tensor field

$$\Pi = \mathrm{Id} - \mathbf{d}t \otimes \partial_t \ ,$$

by contraction. Also, by extension, it is possible to decompose other types of dynamical objects that act via pullbacks on tensors. As an example, consider a one-form $\boldsymbol{\alpha}$ pulled-back by a diffeomorphism ϕ. Then, one can use the decomposition for vector fields to write

$$\phi^*\boldsymbol{\alpha}(X) = \boldsymbol{\alpha}(\phi_* X) = \boldsymbol{\alpha}\left(\phi_*(\mathbf{d}t(X)\partial_t + \Pi(X))\right) = \mathbf{d}t(X)\boldsymbol{\alpha}(\phi_*\partial_t) + \boldsymbol{\alpha}(\phi_*\Pi(X)) \ ,$$

hence

$$\phi^*\boldsymbol{\alpha} = \boldsymbol{\alpha}(\phi_*\partial_t)\mathbf{d}t + \boldsymbol{\alpha}(\phi_*\Pi\cdot) \ .$$

This allows us to decompose a wide variety of actions.

In the particular case (that will be the one of interest throughout the thesis) of differential forms, one can adapt a $p-$form $\boldsymbol{\alpha} \in \Omega^p(\mathcal{M})$ to the foliation by defining

$$\boldsymbol{\alpha}_t := \imath_{\partial_t}\boldsymbol{\alpha} \in \Omega^{p-1}(\mathcal{M}) \ , \tag{A.1}$$

$$\underline{\boldsymbol{\alpha}} := \Pi(\alpha) = \imath_{\partial_t}(\mathbf{d}t \wedge \boldsymbol{\alpha}) \in \Omega^p(\mathcal{M}) \ , \tag{A.2}$$

We call $\boldsymbol{\alpha}_t$ and $\underline{\boldsymbol{\alpha}}$ the *adapted components* of $\boldsymbol{\alpha}$ to the foliation. Note that although (A.1) are fields on $\mathcal{M}$, they have no component in $\mathbf{d}t$, making them adapted to the foliation Σ_t. Equivalently, if $\boldsymbol{\beta}$ is a form adapted to the foliation, $\imath_{\partial_t}\boldsymbol{\beta} = 0$. This also means that when they act on vector fields $\mathbb{X} \in \mathfrak{X}(\mathcal{M})$ on $\mathcal{M}$, only the projection of $\mathbb{X}$ onto the leaves of the foliation will survive.

We denote their pullbacks to Σ as

$$\alpha_t := \jmath_t^*\boldsymbol{\alpha}_t \in \Omega^{p-1}(\Sigma) \ ,$$

$$\alpha := \jmath_t^*\underline{\boldsymbol{\alpha}} \in \Omega^p(\Sigma) \ ,$$

where $\jmath_t : \Sigma \longrightarrow \mathcal{M}$ is the inclusion of each leaf Σ_t.

It is also useful to introduce the operation $\underline{d}$ defined by

$$\underline{\mathbf{d}}\boldsymbol{\alpha} = \imath_{\partial_t}\left(\mathbf{d}t \wedge \mathbf{d}\boldsymbol{\alpha}\right) \ ,$$

whose application always yields an adapted form. This allows us to decompose the exterior derivative as

$$\mathbf{d}\boldsymbol{\alpha} = \mathbf{d}t \wedge \left(\mathcal{L}_{\partial_t}\underline{\boldsymbol{\alpha}} - \underline{\mathbf{d}}\boldsymbol{\alpha}_t\right) + \underline{\mathbf{d}}\underline{\boldsymbol{\alpha}} \ .$$

Some useful decompositions of other common objects are

$$\mathbf{D}\mathbf{e}^k = \underline{\mathbf{D}}\underline{\mathbf{e}}^k + \mathbf{d}t \wedge \left(\mathcal{L}_{\partial_t}\underline{\mathbf{e}}^k - \underline{\mathbf{D}}\mathbf{e}^k_t + \epsilon^k{}_{lm}\mathbf{A}^l_t\underline{\mathbf{e}}^m\right) \ , \tag{A.3}$$

$$\mathbf{F}^i = \underline{\mathbf{D}}\underline{\mathbf{A}}^i + \mathbf{d}t \wedge \left(\mathcal{L}_{\partial_t}\underline{\mathbf{A}}^i - \underline{\mathbf{D}}\mathbf{A}^i_t\right) \ , \tag{A.4}$$

$$\mathbf{F}^{IJ} = \underline{\mathbf{F}}^{IJ} + \mathbf{d}t \wedge \left(\mathcal{L}_{\partial_t}\underline{\boldsymbol{\omega}}^{IJ} - \underline{\mathbf{D}}\boldsymbol{\omega}^{IJ}_t\right) \ . \tag{A.5}$$

Appendix B

Some algebraic equations involving differential forms

B.1 Equations in 3-dimensional manifolds

Lemma 11. *Let Σ be an orientable 3-dimensional manifold, $\alpha \in \Omega^1(\Sigma)$ and $e^i \in \Omega^1(\Sigma)$ a nondegenerate coframe. Then,*

$$\alpha = \frac{1}{2}\epsilon_{ijk}\left(\frac{\alpha \wedge e^i \wedge e^j}{\mathsf{vol}_e}\right) e^k \ . \tag{B.1}$$

Proof. Write $\alpha = \alpha_i e^i$. By multiplying both sides by $\wedge e^j \wedge e^k$ we get

$$\alpha \wedge e^j \wedge e^k = \alpha_i \epsilon^{ijk} \mathsf{vol}_e \ .$$

Now, multiplying by ϵ_{ljk} one obtains

$$\epsilon_{ljk}\alpha \wedge e^j \wedge e^k = 2\alpha_l \mathsf{vol}_e \ .$$

Finally, relabelling, manipulating terms and multiplying by e^l we obtain

$$\alpha = \frac{1}{2}\epsilon_{ijk}\left(\frac{\alpha \wedge e^i \wedge e^j}{\mathsf{vol}_e}\right) e^k \ .$$

□

Lemma 12. *Let Σ be an orientable 3-dimensional manifold, $\alpha \in \Omega^2(\Sigma)$ and $e^i \in \Omega^1(\Sigma)$ a nondegenerate coframe. Then,*

$$\alpha = \frac{1}{2}\epsilon_{ijk}\left(\frac{\alpha \wedge e^i}{\mathsf{vol}_e}\right) e^j \wedge e^k \ . \tag{B.2}$$

Proof. Write the 2-form α in components in the triad basis $\alpha = \alpha_{ij} e^i \wedge e^j$, where $\alpha_{ij} \in \mathcal{C}^\infty(\Sigma)$ is antisymmetric in ij. Then, by multiplying by $\cdot \wedge e^k$, we get

$$\alpha \wedge e^k = \alpha_{ij} e^i \wedge e^j \wedge e^k = \alpha_{ij}\epsilon^{ijk}\mathsf{vol}_e \ .$$

By multiplying by $\epsilon_{klm}\cdot$, this implies in terms of densities

$$\epsilon_{klm}\left(\frac{\alpha\wedge e^k}{\mathsf{vol}_e}\right) = 2\alpha_{lm}\ ,$$

which directly implies the desired result. □

Lemma 13. *Let Σ be an orientable 3-dimensional manifold. For given $\beta_i \in \Omega^2(\Sigma)$ and $e^i \in \Omega^1(\Sigma)$ a nondegenerate coframe, consider the following system of equations in the unknowns $\alpha^i \in \Omega^1(\Sigma)$*

$$\epsilon_{ijk}e^j \wedge \alpha^k = \beta_i\ . \tag{B.3}$$

Then the solution is

$$\alpha^i = \left(\frac{e^j\wedge\beta_j}{2\ \mathsf{vol}_e}\right)e^i - \left(\frac{e^i\wedge\beta_j}{\mathsf{vol}_e}\right)e^j\ . \tag{B.4}$$

Proof. Writing the 1-form $\alpha^k = \alpha^k{}_m e^m$ in components, where $\alpha^k{}_m \in \mathcal{C}^\infty(\Sigma)$, and multiplying the initial equation (B.3) by $e^l\wedge$ one obtains

$$\epsilon_{ijk}\alpha^k{}_m e^l \wedge e^j \wedge e^m = e^l \wedge \beta_i\ .$$

The top form $e^l\wedge e^j\wedge e^m$ can be written as $\epsilon^{ljm}\mathsf{vol}_e$, hence

$$\left(\frac{e^l\wedge\beta_i}{\mathsf{vol}_e}\right) = \epsilon_{ijk}\epsilon^{ljm}\alpha^k{}_m = \delta^l{}_i\alpha^k{}_k - \alpha^l{}_i\ .$$

In particular, for $i=l$

$$2\alpha^k{}_k = \left(\frac{e^k\wedge\beta_k}{\mathsf{vol}_e}\right)\ ,$$

which implies that

$$\alpha^l{}_i = \left(\frac{e^k\wedge\beta_k}{2\ \mathsf{vol}_e}\right)\delta^l{}_i - \left(\frac{e^l\wedge\beta_i}{\mathsf{vol}_e}\right)\ .$$

Multiplying by e^i and relabelling the indices, one finally obtains the desired formula (B.4). This proves that a solution to (B.3) has to be of the form (B.4), but there might be additional constraints. To check this, one must insert the solution in the original equation and see whether it is automatically satisfied.

$$\epsilon_{ijk}e^j\wedge\alpha^k = \epsilon_{ijk}e^j\wedge\left(\left(\frac{e^l\wedge\beta_l}{2\ \mathsf{vol}_e}\right)e^k - \left(\frac{e^k\wedge\beta_l}{\mathsf{vol}_e}\right)e^l\right)$$

Using (12), one can rewrite the second term as

$$\begin{aligned}\epsilon_{ijk}\left(\frac{e^k\wedge\beta_l}{\mathsf{vol}_e}\right)e^j\wedge e^l &= \epsilon_{ijk}\left(\frac{e^k\wedge\beta_l}{\mathsf{vol}_e}\right)\frac{1}{2}\epsilon_{pqr}\epsilon^{pjl}e^q\wedge e^r\\ &= \frac{1}{2}\left(\epsilon_{iqr}\left(\frac{e^k\wedge\beta_k}{\mathsf{vol}_e}\right) - \epsilon_{kqr}\left(\frac{e^k\wedge\beta_i}{\mathsf{vol}_e}\right)\right)e^q\wedge e^r\ .\end{aligned}$$

Hence,

$$\begin{aligned}\epsilon_{ijk}e^j \wedge \alpha^k &= \epsilon_{ijk}e^j \wedge \left(\frac{e^l \wedge \beta_l}{2\,\mathsf{vol}_e}\right) e^k - \frac{1}{2}\left(\epsilon_{iqr}\left(\frac{e^k \wedge \beta_k}{\mathsf{vol}_e}\right) - \epsilon_{kqr}\left(\frac{e^k \wedge \beta_i}{\mathsf{vol}_e}\right)\right) e^q \wedge e^r \\ &= \frac{1}{2}\epsilon_{kqr}\left(\frac{e^k \wedge \beta_i}{\mathsf{vol}_e}\right) e^q \wedge e^r = \beta_i \ ,\end{aligned}$$

by use of (12) again. Since equation (B.3) is already recovered without requiring additional conditions, the result is proved. □

Lemma 14. *Let Σ be an orientable 3-dimensional manifold, $e^i \in \Omega^1(\Sigma)$ a non-degenerate coframe, $\Phi \in \Omega^2(\Sigma)$ and $\alpha^{jk} \in \Omega^1(\Sigma)$ antisymmetric in jk. Then, the solution to the equation for α^{jk}*

$$\epsilon_{ijk}e^i \wedge \alpha^{jk} = \Phi \ ,$$

is given by

$$\alpha^{jk} = -\frac{1}{2}\left(\frac{\Phi \wedge e^{[j}}{\mathsf{vol}_e}\right) e^{k]} + \epsilon^{jkl}\zeta_{lm}e^m \ ,$$

where $\zeta_{lm} \in \mathcal{C}^\infty$ are arbitrary functions such that $\zeta_{lm} = \zeta_{ml}$.

Proof. Write $\alpha^{jk} = \epsilon^{jkl}\left(\xi_{lm} + \zeta_{lm}\right) e^m$ with ξ_{lm} antisymmetric and ζ_{lm} symmetric in lm. By using this expression in the equation and multiplying by $\wedge e^p$ one obtains

$$\begin{aligned}\Phi \wedge e^p &= \epsilon_{ijk}e^i \wedge \epsilon^{jkl}\left(\xi_{lm} + \zeta_{lm}\right) e^m \wedge e^p = \epsilon_{ijk}\epsilon^{jkl}\epsilon^{imp}\left(\xi_{lm} + \zeta_{lm}\right)\mathsf{vol}_e \\ &= \left(\delta^m{}_j\delta^p{}_k - \delta^m{}_k\delta^p{}_j\right)\epsilon^{jkl}\left(\xi_{lm} + \zeta_{lm}\right)\mathsf{vol}_e = 2\epsilon^{plm}\left(\xi_{lm} + \zeta_{lm}\right)\mathsf{vol}_e \\ &= 2\epsilon^{plm}\xi_{lm}\mathsf{vol}_e \ .\end{aligned}$$

By multiplying by ϵ_{pab}, this implies in terms of densities

$$\xi_{ab} = \frac{1}{4}\epsilon_{pab}\left(\frac{\Phi \wedge e^p}{\mathsf{vol}_e}\right) \ ,$$

hence

$$\begin{aligned}\alpha^{jk} &= \epsilon^{jkl}\left(\frac{1}{4}\epsilon_{plm}\left(\frac{\Phi \wedge e^p}{\mathsf{vol}_e}\right) + \zeta_{lm}\right) e^m \\ &= \frac{1}{4}\left(\delta^j{}_m\delta^k{}_p - \delta^j{}_p\delta^k{}_m\right)\left(\frac{\Phi \wedge e^p}{\mathsf{vol}_e}\right) e^m + \epsilon^{jkl}\zeta_{lm}e^m \\ &= \frac{1}{4}\left(\frac{\Phi \wedge e^k}{\mathsf{vol}_e}\right) e^j - \frac{1}{4}\left(\frac{\Phi \wedge e^j}{\mathsf{vol}_e}\right) e^k + \epsilon^{jkl}\zeta_{lm}e^m \\ &= -\frac{1}{2}\left(\frac{\Phi \wedge e^{[j}}{\mathsf{vol}_e}\right) e^{k]} + \epsilon^{jkl}\zeta_{lm}e^m \ .\end{aligned}$$

Since there is no restriction on ζ_{lm}, it remains arbitrary. Plugging this expression in the original equation one sees that it is automatically satisfied and, hence, no other necessary conditions arise. □

Lemma 15. *Let Σ be an orientable 3-dimensional manifold, $e^i \in \Omega^1(\Sigma)$ a non-degenerate coframe, $\alpha \in \Omega^1(\Sigma)$, $\Phi_{ij} \in \Omega^2(\Sigma)$ antisymmetric in ij. Then, the solution to the equation for α*

$$\epsilon_{ijk} e^k \wedge \alpha = \Phi_{ij} \ ,$$

is given by

$$\alpha = \frac{1}{2} \left(\frac{\Phi_{ij} \wedge e^j}{\mathsf{vol}_e} \right) e^i \ .$$

Proof. Write $\alpha = \alpha_l e^l$ and multiply the equation by $\wedge e^m$. This yields

$$\Phi_{ij} \wedge e^m = \epsilon_{ijk} \alpha_l \epsilon^{klm} \mathsf{vol}_e = \left(\alpha_i \delta^m_j - \alpha_j \delta^m_i \right) \mathsf{vol}_e \ .$$

By multiplying by δ^j_m we obtain that

$$2\alpha_i \mathsf{vol}_e = \Phi_{ij} \wedge e^j \ ,$$

and by rearranging and multiplying by e^i we obtain

$$\alpha = \frac{1}{2} \left(\frac{\Phi_{ij} \wedge e^j}{\mathsf{vol}_e} \right) e^i \ .$$

□

Lemma 16. *Let Σ be an orientable 3-dimensional manifold, $e^i \in \Omega^1(\Sigma)$ a non-degenerate coframe, $\alpha \in \Omega^1(\Sigma)$, $\Phi^i \in \Omega^2(\Sigma)$. Consider the equation for α*

$$\alpha \wedge e^i = \Phi^i \ .$$

The solution to the equation exists if, and only if, the condition

$$\Phi^{(i} \wedge e^{j)} = 0 \ ,$$

is satisfied, in which case it is given by

$$\alpha = \frac{1}{2} \epsilon_{ijk} \left(\frac{\Phi^i \wedge e^j}{\mathsf{vol}_e} \right) e^k \ .$$

Proof. Write $\alpha = \alpha_l e^l$ and multiply the equation by $\wedge e^k$. This yields

$$\Phi^i \wedge e^k = \alpha_j \epsilon^{jik} \mathsf{vol}_e \ .$$

Now multiplying by ϵ_{lik} we get

$$2\alpha_l \mathsf{vol}_e = \epsilon_{lik} \Phi^i \wedge e^k \ ,$$

and by rearranging, multiplying by e^l and relabelling we get

$$\alpha = \frac{1}{2} \epsilon_{ijk} \left(\frac{\Phi^i \wedge e^j}{\mathsf{vol}_e} \right) e^k \ .$$

To check whether there are additional necessary conditions, we plug this expression into the original equation, to obtain

$$\frac{1}{2}\epsilon_{ijk}\left(\frac{\Phi^i \wedge e^j}{\mathsf{vol}_e}\right) e^k \wedge e^p = \Phi^p \ .$$

The left hand side can be expanded into

$$\begin{aligned}\frac{1}{2}\epsilon_{ijk}\left(\frac{\Phi^i \wedge e^j}{\mathsf{vol}_e}\right)\frac{1}{2}\epsilon_{abc}\left(\frac{e^k \wedge e^p \wedge e^a}{\mathsf{vol}_e}\right) e^b \wedge e^c &= \frac{1}{4}\epsilon_{ijk}\epsilon_{abc}\epsilon^{kpa}\left(\frac{\Phi^i \wedge e^j}{\mathsf{vol}_e}\right) e^b \wedge e^c \\ &= \frac{1}{4}\epsilon_{abc}\left(\frac{\Phi^p \wedge e^a - \Phi^a \wedge e^p}{\mathsf{vol}_e}\right) e^b \wedge e^c \ ,\end{aligned}$$

while the right hand side can be written as

$$\frac{1}{2}\epsilon_{abc}\left(\frac{\Phi^p \wedge e^a}{\mathsf{vol}_e}\right) e^b \wedge e^c \ .$$

Equating both sides leads to the necessary condition

$$\frac{1}{2}\left(\Phi^p \wedge e^a - \Phi^a \wedge e^p\right) = \Phi^p \wedge e^a \ ,$$

which can be written as

$$\Phi^{(p} \wedge e^{a)} = 0 \ .$$

□

Lemma 17. *Let Σ be an orientable 3-dimensional manifold, $e^i \in \Omega^1(\Sigma)$ a non-degenerate coframe, $\alpha_i \in \Omega^1(\Sigma)$, $\Phi \in \Omega^2(\Sigma)$. Consider the equation for α_i*

$$\alpha_i \wedge e^i = \Phi \ .$$

Then, its solution is

$$\alpha_i = -\frac{1}{2}\epsilon_{ijk}\left(\frac{e^j \wedge \Phi}{\mathsf{vol}_e}\right) e^k + \zeta_{ij} e^j \ ,$$

with arbitrary $\zeta_{ij} \in \mathcal{C}^\infty(\Sigma)$ symmetric in ij.

Proof. Write $\alpha_i = (\alpha_{ij} + \zeta_{ij})\, e^j$, where α_{ij} is antisymmetric and ζ_{ij} symmetric in ij. Then, the equation reads

$$\Phi = (\alpha_{ij} + \zeta_{ij})\, e^j \wedge e^i = \alpha_{ij} e^j \wedge e^i \ ,$$

and multiplying by $\wedge e^k$, this implies in terms of densities

$$\alpha_{ij}\epsilon^{jik} = \left(\frac{\Phi \wedge e^k}{\mathsf{vol}_e}\right) \ ,$$

and by multiplying by ϵ_{kpq} and expanding one obtains

$$2\alpha_{pq} = \epsilon_{kpq}\left(\frac{\Phi \wedge e^k}{\mathsf{vol}_e}\right) \ .$$

Since there is no equation for ζ_{ij}, it remains arbitrary, and joining the pieces gives the expected result. Plugging the solution in the original equation automatically satisfies it, hence there are no additional necessary conditions. □

B.2 Equations in 4-dimensional manifolds

Lemma 18. *Let $\mathcal{M}$ be a 4-dimensional parallelizable manifold and $\mathbf{e}^I \in \Omega^1(\mathcal{M})$ a nondegenerate coframe. The solution to*

$$\epsilon_{IJKL}\mathbf{e}^J \wedge \boldsymbol{\alpha}^{KL} = \Phi_I \ ,$$

for $\boldsymbol{\alpha}^{KL} \in \Omega^1(\mathcal{M})$ antisymmetric in KL and $\Phi_I \in \Omega^2(\mathcal{M})$ is

$$\boldsymbol{\alpha}^{KL} = -\frac{1}{2}\left(\frac{\Phi_I \wedge \mathbf{e}^K \wedge \mathbf{e}^L}{\mathsf{vol}_\mathbf{e}}\right)\mathbf{e}^I - \frac{1}{2}\left(\frac{\Phi_I \wedge \mathbf{e}^I \wedge \mathbf{e}^{[K}}{\mathsf{vol}_\mathbf{e}}\right)\mathbf{e}^{L]} \ .$$

Proof. Write $\boldsymbol{\alpha}^{KL} = \boldsymbol{\alpha}^{KL}{}_M\mathbf{e}^M$ and multiply the equation by $\wedge\mathbf{e}^P \wedge \mathbf{e}^Q$ to obtain

$$\Phi_I \wedge \mathbf{e}^P \wedge \mathbf{e}^Q = \epsilon_{IJKL}\boldsymbol{\alpha}^{KL}{}_M\epsilon^{JMPQ}\mathsf{vol}_\mathbf{e} \ .$$

Now, the contraction of two ϵ yields

$$\epsilon_{IJKL}\epsilon^{JMPQ} = -\delta^{MPQ}_{IKL} \ ,$$

where δ^{MPQ}_{IKL} is the generalised delta, which has the property that for a completely antisymmetric tensor T_{MPQ}

$$\delta^{MPQ}_{IKL}T_{MPQ} = T_{IKL} \ .$$

Then,

$$\Phi_I \wedge \mathbf{e}^P \wedge \mathbf{e}^Q = -2\left(\boldsymbol{\alpha}^{PQ}{}_I + \boldsymbol{\alpha}^{QM}{}_M\delta^P_I + \boldsymbol{\alpha}^{MP}{}_M\delta^Q_I\right)\mathsf{vol}_\mathbf{e} \ . \tag{B.5}$$

By multiplying by δ^I_P, we obtain the relation

$$\frac{1}{4}\left(\frac{\Phi_I \wedge \mathbf{e}^I \wedge \mathbf{e}^Q}{\mathsf{vol}_\mathbf{e}}\right) = \boldsymbol{\alpha}^{IQ}{}_I \ ,$$

which we can introduce back in (B.5). By doing so we obtain

$$-\frac{1}{2}\left(\frac{\Phi_I \wedge \mathbf{e}^P \wedge \mathbf{e}^Q}{\mathsf{vol}_\mathbf{e}}\right) = \boldsymbol{\alpha}^{PQ}{}_I + \frac{1}{2}\left(\frac{\Phi_M \wedge \mathbf{e}^M \wedge \mathbf{e}^{[P}}{\mathsf{vol}_\mathbf{e}}\right)\delta^{Q]}_I \ .$$

Finally, reordering, multiplying by e^I and relabelling we obtain the claimed result. □

Lemma 19. *Let $\mathcal{M}$ be a 4-dimensional parallelizable manifold and $\mathbf{e}^I \in \Omega^1(\mathcal{M})$ a nondegenerate coframe. The solution to*

$$\epsilon_{IJKL}\mathbf{e}^K \wedge \boldsymbol{\alpha}^L = \Phi_{IJ} \ ,$$

for $\boldsymbol{\alpha}^L \in \Omega^1(\mathcal{M})$ and $\Phi_{IJ} \in \Omega^2(\mathcal{M})$ is

$$\boldsymbol{\alpha}^I = \frac{1}{2}\left(\frac{\Phi_{KJ} \wedge \mathbf{e}^I \wedge \mathbf{e}^J}{\mathsf{vol}_\mathbf{e}}\right)\mathbf{e}^K - \frac{1}{6}\left(\frac{\Phi_{KL} \wedge \mathbf{e}^K \wedge \mathbf{e}^L}{\mathsf{vol}_\mathbf{e}}\right)\mathbf{e}^I \ .$$

Proof. By multiplying the equation by $\cdot \wedge \mathbf{e}^A \wedge \mathbf{e}^B$ and writing $Z^I = Z^I{}_J \mathbf{e}^J$, we have

$$\epsilon_{IJKL}\epsilon^{KMAB} Z^L{}_M \mathsf{vol}_{\mathbf{e}} = \Phi_{IJ} \wedge \mathbf{e}^A \wedge \mathbf{e}^B \ .$$

$$-2\left(Z^B{}_{[I}\delta^A{}_{J]} - Z^A{}_{[I}\delta^B{}_{J]} + Z^K{}_K \delta^A{}_{[I}\delta^B{}_{J]}\right) = \left(\frac{\Phi_{IJ} \wedge \mathbf{e}^A \wedge \mathbf{e}^B}{\mathsf{vol}_{\mathbf{e}}}\right) \ . \tag{B.6}$$

By multiplying by $\delta^I_A \delta^J_B$, equation (B.6) yields $Z^K{}_K = -\frac{1}{6}\left(\frac{\Phi_{IJ}\wedge \mathbf{e}^I \wedge \mathbf{e}^J}{\mathsf{vol}_{\mathbf{e}}}\right)$, while by multiplying by δ^J_B, equation (B.6) yields

$$Z^A{}_B = \frac{1}{2}\left(\frac{\Phi_{BJ} \wedge \mathbf{e}^A \wedge \mathbf{e}^J}{\mathsf{vol}_{\mathbf{e}}}\right) + Z^K{}_K \delta^A{}_B \ .$$

Combining both results and multiplying by $\cdot \wedge \mathbf{e}^B$, one arrives at the desired result. $\square$

Lemma 20. *Let $\mathcal{M}$ be a 4-dimensional parallelizable manifold and $\mathbf{e}^I \in \Omega^1(\mathcal{M})$ a nondegenerate coframe. The solution to*

$$\epsilon_{IJKL} \mathbf{e}^I \wedge \boldsymbol{\alpha}^J = \Phi_{KL} \ ,$$

for $\boldsymbol{\alpha}^J \in \Omega^2(\mathcal{M})$ and $\Phi_{KL} \in \Omega^3(\mathcal{M})$ is

$$\boldsymbol{\alpha}^J = \frac{1}{2}\left(\frac{\Phi_{KL} \wedge \mathbf{e}^J}{\mathsf{vol}_{\mathbf{e}}}\right) \mathbf{e}^K \wedge \mathbf{e}^L + \frac{1}{2}\left(\frac{\Phi_{LM} \wedge \mathbf{e}^M}{\mathsf{vol}_{\mathbf{e}}}\right) \mathbf{e}^J \wedge \mathbf{e}^L \ .$$

Proof. Write $\boldsymbol{\alpha}^J = \alpha^J{}_{MN} \mathbf{e}^M \wedge \mathbf{e}^N$. By using this in the equation and multiplying by $\wedge \mathbf{e}^P$ we get

$$\epsilon_{IJKL} \alpha^J{}_{MN} \mathbf{e}^I \wedge \mathbf{e}^M \wedge \mathbf{e}^N \wedge \mathbf{e}^P = \Phi_{KL} \wedge \mathbf{e}^P \ ,$$

which implies in terms of densities

$$\left(\frac{\Phi_{KL} \wedge \mathbf{e}^P}{\mathsf{vol}_{\mathbf{e}}}\right) = \epsilon_{IJKL}\epsilon^{IMNP} \alpha^J{}_{MN} = 2\left(\alpha^J{}_{JK}\delta^P_L + \alpha^J{}_{LJ}\delta^P_K + \alpha^P{}_{KL}\right) \ . \tag{B.7}$$

Now, by multiplying by δ^L_P we obtain

$$\frac{1}{2}\left(\frac{\Phi_{KL} \wedge \mathbf{e}^L}{\mathsf{vol}_{\mathbf{e}}}\right) = 4\alpha^J{}_{JK} + \alpha^J{}_{KJ} + \alpha^L{}_{KL} = 2\alpha^J{}_{JK} \ .$$

Using this expression in Equation (B.7) we get

$$\alpha^P{}_{KL} = \frac{1}{2}\left(\frac{\Phi_{KL} \wedge \mathbf{e}^P}{\mathsf{vol}_{\mathbf{e}}}\right) + \frac{1}{4}\left(\frac{\Phi_{LM} \wedge \mathbf{e}^M}{\mathsf{vol}_{\mathbf{e}}}\right)\delta^P_K - \frac{1}{4}\left(\frac{\Phi_{LM} \wedge \mathbf{e}^M}{\mathsf{vol}_{\mathbf{e}}}\right)\delta^P_L \ ,$$

from which we recover

$$\boldsymbol{\alpha}^P = \alpha^P{}_{KL}\mathbf{e}^K \wedge \mathbf{e}^L = \frac{1}{2}\left(\frac{\Phi_{KL} \wedge \mathbf{e}^P}{\mathsf{vol}_{\mathbf{e}}}\right) \mathbf{e}^K \wedge \mathbf{e}^L + \frac{1}{2}\left(\frac{\Phi_{LM} \wedge \mathbf{e}^M}{\mathsf{vol}_{\mathbf{e}}}\right) \mathbf{e}^P \wedge \mathbf{e}^L \ .$$

$\square$

Lemma 21. *Let $\mathcal{M}$ be a 4-dimensional parallelizable manifold and $\mathbf{e}^i \in \Omega^1(\mathcal{M})$, with $i = 1, 2, 3$, be three linearly independent 1-forms and $\boldsymbol{\alpha}^i \in \Omega^1(M)$ be another three 1-forms. Then, the solution for $\boldsymbol{\alpha}^i$ of the equation*

$$\epsilon_{ijk}\mathbf{e}^j \wedge \boldsymbol{\alpha}^k = 0 \ ,$$

is

$$\boldsymbol{\alpha}^i = 0 \ .$$

Proof. Let us complete the $\mathbf{e}^i$ with a fourth 1-form $\mathbf{e}^0$ linearly independent with $\mathbf{e}^i$ such that $\{\mathbf{e}^0, \mathbf{e}^i\}$ is a coframe (and $\mathsf{vol}_\mathbf{e} = \epsilon_{ijk}\mathbf{e}^0 \wedge \mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k$ is a volume form). Let $U \in \mathfrak{X}(\mathcal{M})$ be the vector field such that $\mathbf{e}^0(U) = 1$ and $\mathbf{e}^i(U) = 0$.

Write $\boldsymbol{\alpha}^i = \boldsymbol{\alpha}^i{}_j\mathbf{e}^j + \boldsymbol{\alpha}^i{}_0\mathbf{e}^0$. Then, by plugging this expression into the equation we get

$$\epsilon_{ijk}\mathbf{e}^j \wedge \left(\boldsymbol{\alpha}^k{}_l\mathbf{e}^l + \boldsymbol{\alpha}^k{}_0\mathbf{e}^0\right) = 0 \ ,$$

which by taking the interior product with U implies

$$\epsilon_{ijk}\mathbf{e}^j\boldsymbol{\alpha}^k{}_0 = 0 \ ,$$

which leads to $\boldsymbol{\alpha}^k{}_0 = 0$.

On the other hand, multiply the equation by $\mathbf{e}^0 \wedge \mathbf{e}^m \wedge \cdot$ to obtain

$$0 = \epsilon_{ijk}\mathbf{e}^0 \wedge \mathbf{e}^m \wedge \mathbf{e}^j \wedge \boldsymbol{\alpha}^k{}_l\mathbf{e}^l = \boldsymbol{\alpha}^k{}_l\epsilon_{ijk}\epsilon^{mjl}\mathsf{vol}_\mathbf{e} = \left(\boldsymbol{\alpha}^k{}_k\delta^m{}_i - \boldsymbol{\alpha}^m{}_i\right)\mathsf{vol}_\mathbf{e} \ .$$

By taking the trace by multiplying by δ^i_m, we find that $\boldsymbol{\alpha}^k{}_k = 0$, which in turn means that $\boldsymbol{\alpha}^m{}_i = 0$. Hence, $\boldsymbol{\alpha}^i = 0$. □

Appendix C

Analysis of the Husain action

The Husain action in a globally hyperbolic 3-dimensional manifold $\mathcal{M} \cong \mathbb{R} \times \Sigma$ is

$$S_{\text{Husain}}(\mathbf{e}, \boldsymbol{\omega}) = \int_{\mathcal{M}} \boldsymbol{\omega} \wedge \mathbf{d}\boldsymbol{\omega} + \lambda \mathbf{e}_i \wedge \mathbf{D}e^i \ , \tag{C.1}$$

where $\mathbf{e}^i \in \Omega^1(\mathcal{M})$ is a non-degenerate dyad, $\boldsymbol{\omega} \in \Omega^1(\mathcal{M})$ defines an $SO(2)$ connection 1-form $\boldsymbol{\omega}^i{}_j := -\epsilon^i{}_j \boldsymbol{\omega} \in \Omega^1(\mathcal{M})$, where ϵ_{ij} is the volume form in $SO(2)$ and the covariant derivative is $\mathbf{D}\mathbf{e}^i = \mathbf{d}\mathbf{e}^i + \boldsymbol{\omega}^i{}_j \wedge e^j$ and $\lambda \in \mathbb{R}$ a real number. The action (C.1) yields the equations of motion

$$\begin{aligned} \mathbf{D}\mathbf{e}^i &= 0 \ , \\ \mathbf{d}\boldsymbol{\omega} + \frac{\lambda}{2} \epsilon_{ij} \mathbf{e}^i \wedge \mathbf{e}^j &= 0 \ . \end{aligned} \tag{C.2}$$

Given a fixed volume form vol, define the vector density as an element of the double dual

$$U(\cdot) = \left(\frac{\cdot \wedge \epsilon_{ij} \mathbf{e}^i \wedge \mathbf{e}^j}{\mathsf{vol}} \right) \ .$$

Then,

$$\imath_U \mathbf{e}^i = \left(\frac{e^i \wedge \epsilon_{jk} \mathbf{e}^j \wedge \mathbf{e}^k}{\mathsf{vol}} \right) = 0 \ . \tag{C.3}$$

This vector field satisfies the particular relations, which can be found by using (C.2) and (C.3),

$$\begin{aligned} \mathcal{L}_U \mathbf{e}^i &= \imath_U \mathbf{d}\mathbf{e}^i + \mathbf{d}\imath_U \mathbf{e}^i = \imath_U \mathbf{d}\mathbf{e}^i = \imath_U \left(\epsilon^i{}_j \boldsymbol{\omega} \wedge \mathbf{e}^j \right) = \epsilon^i{}_j \left(\imath_U \boldsymbol{\omega} \right) \mathbf{e}^j \ , \\ \mathcal{L}_U \boldsymbol{\omega} &= \imath_U \mathbf{d}\boldsymbol{\omega} + \mathbf{d}\imath_U \boldsymbol{\omega} = -\frac{\lambda}{2} \imath_U \left(\epsilon_{ij} \mathbf{e}^i \wedge \mathbf{e}^j \right) + \mathbf{d}\imath_U \boldsymbol{\omega} = \mathbf{d} \left(\imath_U \boldsymbol{\omega} \right) \ . \end{aligned}$$

If we think of the quantity $\Lambda^i{}_j = -\epsilon^i{}_j \imath_U \boldsymbol{\omega}$ as a gauge parameter, then, Lie-dragging the fields along the integral curves of U just has the effect of an $SO(2)$ gauge transformation

$$\begin{aligned} \mathcal{L}_U \mathbf{e}^i &= -\Lambda^i{}_j \mathbf{e}^j \ , \\ \mathcal{L}_U \boldsymbol{\omega}^i{}_j &= \mathbf{d}\Lambda^i{}_j \ . \end{aligned}$$

Choosing a foliation $\mathcal{M} = \mathbb{R} \times \Sigma$ such that the pullback of $\mathbf{e}^i$ is non-degenerate in Σ and U is transverse to Σ, and by taking the pullback of the equations of motion to Σ we obtain

$$\begin{aligned} \mathrm{d}e^1 &= \omega \wedge e^2 \ , \\ \mathrm{d}e^2 &= -\omega \wedge e^1 \ , \\ \mathrm{d}\omega &= -\lambda e^1 \wedge e^2 \ . \end{aligned}$$

Those are Cartan's structural equations for the surface Σ [84]. In particular, the last equation tells us that the Gaussian curvature of Σ is constant $K = \lambda$ (hence, constant scalar curvature $R = 2\lambda$). As a consequence, since the direction along the vector field U only has the effect of an internal rotation (gauge transformation), we can interpret this model as describing surfaces modulo diffeomorphisms (known as 2-geometries) with constant Gaussian curvature.

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