GEOMETRIC QUANTIZATION: REGULAR REPRESENTATIONS AND MODULAR ALGEBRAS

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Abstract. A link is established between the geometric quantization programme and the decomposition theory of the regular representation of the Weyl group for homogeneous manifolds of constant curvature $K \leq 0$.

1. Motivation and results.

The purpose of this review is to look at geométric quantization in a novel light, namely the general contexts of the decomposition theory of regular representations and of the Tomita-Takesaki theory of modular algebras [30].

In this Section, we give our motivation and state the main results, the proof of which is sketched in Section 2. With this apparatus in hand, we present in Section 3 an application of these results to the passage from prequantization to quantization, as well as some general comments on the scheme outlined in Section 1 .

Recall first that, traditionally, three mathematical categories are involved in the discussion of quantization:

(a) unitary representations of Lie groups;
(b) symplectic geometry of classical mechanics;
(c) non-commutative operator algebras of quantum theories.

While it should be clear that the philosophical primacy of one of these categories over the others is largely a matter of personal taste, each of the possible choices of one of them, as the starting point of a quantum theory of dynamical systems, has its own history and disciples. The elusive passage from (b) to (c) is known as the Dirac problem; in its most restritive formulation, this problem was proven to have no solution (see e.g. [1] and references quoted there; a version of this proof can also be found in [8]). The formulation of (c) in terms of (a), but independently of (b), is examplified by the system of imprimitivity approach of Mackey [21]. The passage from (c) to (b) is known as the classical limit; for the homogeneous manifolds, discussed later in this section, this part of the programme was discussed in [8] using techniques from non-commutative harmonic analysis. Finally, the geometric quantization programme [1,13,16,26,27,28], at its most schematic level, is mostly motivated by (a); it starts by classifying the co-adjoint orbits of the group considered; as these are homogenous manifolds, naturally equipped with a symplectic form [16], a link with (b) is established; this structure is then used to construct a reducible (and hence partial) solution to the Dirac problem: this is known as the prequantization stage of the theory; the third step of the geometric quantization programme is the reduction of this representation into irreducible representations by the so-called polarization method.

The motivation for the approach taken in this lecture comes from the following remark [29], the origins of which may be traced back to [15,24].

While the prequantization map [1,13,16,26,27,28] is a solution of the Dirac problem that produces a *reducible* representation of the CCR, acting in the space $\mathcal{H}$ of square integrable functions on *phase-space*:

$$P = -i\hbar\partial_q \quad , \quad Q = +i\hbar\partial_p + q \quad , \tag{1-1}$$

the Hilbert space $\mathcal{H}$ also harbors an anti-representation of the CCR, namely

$$P' = +i\hbar\partial_q + p \quad , \quad Q' = -i\hbar\partial_p \quad , \tag{1-2}$$

which *commutes* with the representation (1-1).

The contact between this observation and the theory of modular algebras is established by the following result.

SCHOLIUM [7,10]. *The prequantization representation (1-1) generates a von Neumann algebra $\mathcal{N}$ which is a factor of type I, while the antirepresentation (1-2) generates the commutant*

$$\mathcal{N}' = \{ A \in B(\mathcal{H}) \mid [A,N] = 0 \ \forall\, N \in \mathcal{N} \} \tag{1-3}$$

of $\mathcal{N}$; moreover, there exists an involutive anti-unitary operator J that establishes an isomorphism between $\mathcal{N}$ and $\mathcal{N}'$, i.e.

$$\mathcal{N}' = J\mathcal{N}J \quad . \tag{1-4}$$

As usual, a technical remark should be made here, namely that all the operators appearing in (1-1) and (1-2) are evidently unbounded; they can nevertheless all be defined in such a manner that they are self-adjoint, and thus generate unitary groups,

$$\{ U(a) = \exp(-iaP) \mid a \in R \} \quad ; \quad \{ V(b) = \exp(-ibQ) \mid b \in R \} \tag{1-5}$$

$$\{ U'(a) = \exp(-iaP') \mid a \in R \} \quad ; \quad \{ V'(b) = \exp(-ibQ') \mid b \in R \} \tag{1-6}$$

satisfying

$$U(a)\, V(b) = \exp(-iab)\, V(b)\, U(a) \tag{1-7}$$

and

$$U'(a)V'(b) = \exp(+iab)\, V'(b)\, U'(a) \quad . \tag{1-8}$$

When we say that the von Neumann algebra $\mathcal{N}$ [resp. $\mathcal{N}'$] is "generated" by (1-1) [resp. (1-2)], we mean that $\mathcal{N}$ [resp. $\mathcal{N}'$] is the algebra of operators obtained as the weak-operator closure of all finite linear combinations of the unitary operators appearing in (1-5) [resp. (1-6)].

The commutation relations, written in the Weyl form (1-7), suggest the introduction of the "Weyl group "

$$\mathcal{W}_o = \{w = (\mathrm{a}, \mathrm{b}, \theta) \mid \mathrm{a} \in R^n, \mathrm{b} \in R^n, \theta \in R\}$$
$$(1\text{-}9) \qquad (\mathrm{a}, \mathrm{b}, \theta)(\mathrm{a}', \mathrm{b}', \theta') = (\mathrm{a} + \mathrm{a}', \mathrm{b} + \mathrm{b}', \theta + \theta' - \frac{1}{2}(\mathrm{a} \cdot \mathrm{b}' - \mathrm{a}' \cdot \mathrm{b})) \quad .$$

While in the physics literature, this group usually shows up in connection with quantum mechanics, it must be noted that it arises in classical mechanics as well when the momentum map is introduced [1]. It is important for our purpose to note that this group is a central extension

$$(1\text{-}10) \qquad 0 \to R \to \mathcal{W}_o \to R^2 \to 0$$

of the group of translations R^2 which acts transitively and freely on the phase space $T^* R$. The generalization to $T^* R^n$ is straightforward.

The new question we want to address here is whether the above Scholium is itself a consequence of a deeper result. In answering this question, we were led to an alternate derivation of the prequantization representation, summarized in the following statement.

THEOREM. *The prequantization algebra $\mathcal{N}$ appears as a factor in the central decomposition of the regular representation of the Weyl group.*

Together with the Scholium, this Theorem shows that: (i) the prequantization representation of the CCR can be obtained *directly* from the central decomposition of the regular representation of the Weyl group; and (ii) it generates a von Neumann algebra spatially anti-isomorphic to its commutant. This anti-isomorphism is at the heart of the Tomita-Takesaki theory of modular Hilbert algebras [30].

REMARK: This theorem is not a peculiarity of the flat configuration space R^n and it extends, in particular, to the Weyl group $\mathcal{W}_\kappa$ for simply-connected $n-$dimensional homogeneous manifolds $\mathcal{H}_\kappa^n$ of constant negative curvature $K = -\kappa^2 \leq 0$.

For illustrative purposes, in the first case of interest, namely $n = 2$, $\mathcal{W}_\kappa$ can be described explicitly as follows.

For $K = -1$, $\mathcal{H}_1^2$ is the Poincare half-plane, which is isometrically isomorphic to the unit space-like hyperboloid in the $(2+1)$- dimensional Minkowski space, where the Riemann metric on $\mathcal{H}_1^2$ is induced from the Minkowski metric. In order to be able to interpolate readily between the results relative to the curved configuration manifold $\mathcal{H}_1^2$ and those obtained for the corresponding flat manifold R^2, it is convenient to consider the Minkowski metric $ds^2 = dx^2 + dy^2 - c^2 dt^2$ for which the corresponding unit space-like hyperboloid $\mathcal{H}_\kappa^2$ has curvature $K = -\kappa^2$ with $\kappa = 1/c$. The considerations to be presented explicitly below for $\mathcal{H}_\kappa^2$ extend, for instance, to the three-dimensional unit space-like hyperboloid $\mathcal{H}_\kappa^3$ in $(3+1)-$Minkowski space.

We showed elsewhere [11] (see also [8,25,31]) that the natural generalization of the Weyl group $\mathcal{W}_o$ to the case where the configuration space is $\mathcal{H}_\kappa^2$ is a $5-$dimensional, simply-connected, exponential and tame Lie group $\mathcal{W}_\kappa$ on which a coordinate system can be chosen in such a manner that the only non-vanishing brackets in its Lie algebra are:

$$(1\text{-}11) \qquad [\xi_{a^1}, \xi_{a^2}] = \kappa \xi_{a^1} \quad , \quad [\xi_{b^1}, \xi_{a^1}] = \kappa \xi_{b^2} + \xi_\theta \quad , \quad [\xi_{b^2}, \xi_{a^2}] = \kappa \xi_{b^2} + \xi_\theta \, .$$

Clearly this group contracts, in the flat case limit $\kappa \to 0$ to the Weyl group $\mathcal{W}_o$ for the configuration manifold R^2. It is a central extension

$$(1\text{-}12) \qquad 0 \to R \to \mathcal{W}_\kappa \to G_\kappa \to 0$$

of a 4 – dimensional group G_κ that acts transitively and freely on phase space, i.e. on the cotangent bundle

$$(1\text{-}13) \qquad T^* \mathcal{H}_\kappa^2 \simeq \mathcal{W}_\kappa / Z_o$$

where $Z_o \simeq R$ is the center of $\mathcal{W}_\kappa$. In (1-12), the group G_κ is itself an extension

$$(1\text{-}14) \qquad 0 \to R^2 \to G_\kappa \to H_\kappa \to 0$$

where H_κ is the group of lower-diagonal 2×2 matrices, that occur in the Iwasawa decomposition (see e.g. [14]) $K \cdot H_\kappa$ of $SL(2, R)$ (where K is the compact isotropy group of rotations).

2. Sketch of the proof for the main theorem.

For the sake of simplicity, the proof of the main theorem is conducted in this Section for the flat case R^n; see the concluding comments in Section 3 for an indication of some among the technical precautions that have to be taken when dealing with the non-flat manifolds $\mathcal{H}_\kappa^n$. The proof proceeds in two steps: the first is an explicit application of the central decomposition theory of the regular representation for the Weyl group $\mathcal{W}_o$; the second step establishes the link between the primary representations so obtained and those obtained by prequantization.

Let $\mathcal{N}^\#$ (with $\#$ standing for either R or L) be the von Neumann algebra generated by the (right- or left-) regular representation $U^\#$ of the Weyl group $\mathcal{W}_o$, i.e.

$$(2\text{-}1) \qquad [U^R(w)\Psi](w_o) = \Psi(w_o w) \quad , \quad [U^L(w)\Psi](w_o) = \Psi(w^{-1} w_o) \quad ;$$

and let J be the involutive anti-unitary operator

$$(2\text{-}2) \qquad [J\Psi](w_o) = \Psi(w_o^{-1})^* \quad .$$

Then (see for instance [5,6,16]) :

$$(2\text{-}3) \qquad J U^R(w) J = U^L(w) \quad , \quad J \mathcal{N}^R J = \mathcal{N}^L = [\mathcal{N}^R]' \quad .$$

Because of (1-10), we have immediately that the central decomposition ([5,6])

$$U^{\#}(w) = \int_R^{\oplus} d\lambda\, U_\lambda^{\#}(w) \tag{2-4}$$

is implemented, in the sense of Gelfand triplets [12], by the Fourier transform

$$\Psi_\lambda(\mathbf{a}_o, \mathbf{b}_o) = \frac{1}{\sqrt{2\pi}} \int_R d\theta e^{-i\lambda\theta_o} \Psi(\mathbf{a}_o, \mathbf{b}_o, \theta_o) \;, \tag{2-5}$$

namely (with # standing again for either R or L, and $\epsilon^R = +1\,,\ \epsilon^L = -1$):

$$\begin{aligned} &[U_\lambda^{\#}(\mathbf{a}, \mathbf{b}, \theta)\Psi_\lambda](\mathbf{a}_o, \mathbf{b}_o) \equiv [U^{\#}(\mathbf{a}, \mathbf{b}, \theta)\Psi]_\lambda(\mathbf{a}_o, \mathbf{b}_o) \\ &\quad = \exp\{-i\lambda[-\epsilon^{\#}\theta + \frac{1}{2}(\mathbf{a}_o \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b}_o)]\}\, \Psi_\lambda(\mathbf{a}_o + \epsilon^{\#}\mathbf{a}, \mathbf{b}_o + \epsilon^{\#}\mathbf{b}) \\ &[J_\lambda\Psi_\lambda](\mathbf{a}_o, \mathbf{b}_o) \equiv [J\Psi]_\lambda(\mathbf{a}_o, \mathbf{b}_o) = \Psi_\lambda(-\mathbf{a}_o, -\mathbf{b}_o)^* \end{aligned} \tag{2-6}$$

We read directly from these expressions:

$$J_\lambda U_\lambda^R(w) J_\lambda = U_\lambda^L(w) \tag{2-7a}$$

and, with

$$\mathcal{N}_\lambda^{\#} \equiv \{U_\lambda^{\#}(\mathbf{a}, \mathbf{b}) \mid (\mathbf{a}, \mathbf{b}) \in R^{2n}\}'' \,, \tag{2-7b}$$

$$J_\lambda \mathcal{N}_\lambda^R J_\lambda = \mathcal{N}_\lambda^L = [\mathcal{N}_\lambda^R]' \tag{2-7c}$$

$$\mathcal{N}_\lambda^R \cap \mathcal{N}_\lambda^L = C\, I\,. \tag{2-7d}$$

Since the R^{2n} part of $\mathcal{W}_o$ acts transitively and freely on the cotangent bundle $T^*R^n \simeq \mathcal{W}_o/Z_o$ (where $Z_o \simeq R$ is the center of $\mathcal{W}_o$), we introduce the coordinate identification

$$(\mathbf{a}, \mathbf{b}) \in R^{2n} \leftrightarrow (-\mathbf{q}, \mathbf{p}) \in T^*R^n \tag{2-8}$$

corresponding to the classical action

$$\exp\{-(X_{\mathbf{p}} \cdot \mathbf{a} \,+\, X_{\mathbf{q}} \cdot \mathbf{b})\} : (\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{q} - \mathbf{a}, \mathbf{p} + \mathbf{b}) \tag{2-9}$$

where X_H is the Hamiltonian vector field for the Hamiltonian function $H(\mathbf{p}, \mathbf{q})$ (i.e. $X_H \rfloor \omega = -dH$), and ω is the symplectic form $d\mathbf{p} \wedge d\mathbf{q}$. Upon defining the generators of the representation $U^{\#}(\mathcal{W}_o)$ by

$$\begin{gathered} U^{\#}(\mathbf{a}, 0, 0) \equiv \exp(-i\mathbf{a} \cdot \mathbf{P}_\lambda^{\#}/\hbar) \quad , \quad U^{\#}(0, \mathbf{b}, 0) \equiv \exp(-i\mathbf{b} \cdot \mathbf{Q}_\lambda^{\#}/\hbar) \\ U^{\#}(0, 0, \theta) \equiv \exp(-i\theta\, \Theta_\lambda^{\#}/\hbar)\,, \end{gathered} \tag{2-10}$$

we obtain

$$
\text{(2-11)} \quad \begin{aligned} \mathbf{P}^{\#}_{\lambda} = -i\hbar\epsilon^{\#}\partial_{\mathrm{q}} - \frac{1}{2}(\lambda\hbar)\mathbf{p} \quad , \quad \mathbf{Q}^{\#}_{\lambda} = +i\hbar\epsilon^{\#}\partial_{\mathrm{p}} - \frac{1}{2}(\lambda\hbar)\mathbf{q} \\ \Theta^{\#}_{\lambda} = -(\lambda\hbar)\epsilon^{\#} I \\ [\mathbf{P}^{\#}_{\lambda}, \mathbf{Q}^{\#}_{\lambda}] = +i\hbar(\lambda\hbar)\epsilon^{\#} I \quad . \end{aligned}
$$

In particular, for the choice

$$
\text{(2-12)} \qquad (\lambda\hbar) = -1
$$

$\{P^R_\lambda, \mathbf{Q}^R_\lambda\}$ is a representation of the CCR, while $\{\mathbf{P}'_\lambda, \mathbf{Q}'_\lambda\} \equiv \{\mathbf{P}^L_\lambda, \mathbf{Q}^L_\lambda\}$ is an anti-representation of the CCR. Moreover (2-11/12) generate two von Neumann algebras $\mathcal{N}_\lambda = \mathcal{N}^R_\lambda$ and $[\mathcal{N}_\lambda]' = \mathcal{N}^L_\lambda$ that are the commutant of one another, and satisfy

$$
\text{(2-13)} \qquad J_\lambda \mathcal{N}_\lambda J_\lambda = [\mathcal{N}_\lambda]'.
$$

Finally, the apparent discrepency between (2-11) and (1-1/2) is only due to a residual ambiguity [16,18] in the usual geometric quantization proceedure, which can be "gauged away rq'by the unitary multiplication operator $U_\chi = \exp i\chi/\hbar$ with $\chi \in C^\infty(T^*R^n)$, corresponding to the fact that, in the definition of the prequantization map, the one-form η appearing in the connection

$$
\text{(2-14)} \qquad \nabla_{\eta;X} = X - i\hbar^{-1}\eta(X)
$$

is uniquely determined only up to an additive term $d\chi$. This completes the proof, for the flat configuration space R^n, of the results announced in Section 1.

3. Application and Comments.

The results of Section 1 (namely that there exists an anti-unitary isomorphism between: (i) the von Neumann algebra $\mathcal{N}_\lambda$ generated by the primary representation of the CCR usually obtained by prequantization; and (ii) its commutant $[\mathcal{N}_\lambda]'$) have several consequences [2,7,10]. The most important one is that (1-4) [i.e. (2-7c)] allows one to identify in terms of observables quantities (i.e. in terms of elements in $\mathcal{N}_\lambda$) the maximal abelian algebra $\mathcal{A}_\lambda$ in $[\mathcal{N}_\lambda]'$ corresponding [5,6,16] to a decomposition of the primary representation $U_\lambda(\mathcal{W}_\kappa)$ as a direct integral of irreducible representations. For instance, if the decomposition is made with respect to

$$
\text{(3-1)} \qquad \mathcal{A}_\lambda = J_\lambda \mathcal{Q}_\lambda J_\lambda \subset [\mathcal{N}_\lambda]'
$$

where $\mathcal{Q}_\lambda$ is the maximal abelian subalgebra of $\mathcal{N}_\lambda$ generated by the position operators on $\mathcal{H}^n_\kappa$, the decomposition of $U^R_\lambda(\mathcal{W}_\kappa)$ precisely produces a direct integral of irreducible systems of imprimitivity [8,21] for the group H^n_κ acting on the configuration space $\mathcal{H}^n_\kappa$. In general, the choice of a decomposition with respect to a maximal abelian subalgebra (i.e. a complete set of commuting observables) $\mathcal{A}_\lambda \subset [\mathcal{N}_\lambda]' = J_\lambda \mathcal{N}_\lambda J_\lambda$ is the algebraic equivalent of the geometric reduction procedure [19] associated with the choice of polarization when one carries out explicitly the geometric quantization programme for the group $\mathcal{W}_{\kappa}$.

Physically, this choice can be interpreted in terms of the choice of a measuring process, which in turn is reflected in the correspondance one obtains between classical and quantum observables, i.e. the choice of an ordering. For instance, in the quantization of a one-dimensional harmonic oscillator, we can choose for $\mathcal{A}$ the maximal abelian algebra generated by the Hamiltonian $H = \frac{1}{2}P^2 + \frac{1}{2}Q^2$; the correspondance one obtains from this choice is then given by the so-called anti-normal ordering. This result, discussed in [2], follows from fact that the decomposition of the prequantization representation into irreducible representations can be implemented by a reproducing kernel K that is easily interpreted in terms of coherent states. In fact, the prequantization representation itself, namely

$$\text{(3-2)} \qquad \mathbf{P}^R_\lambda = -i\hbar\partial_{\mathbf{q}} + \frac{1}{2}\mathbf{p} \quad , \quad \mathbf{Q}^R_\lambda = +i\hbar\partial_{\mathbf{p}} + \frac{1}{2}\mathbf{q}$$

obtained in Section 2 as a factor in the central decomposition of the right-regular representation of the Weyl group $\mathcal{W}_o$, coincides with a representation of the CCR that also shows up in [29], where Streater points out its relation with the Bargmann formalism [4] (and related works in flat-space quantum field theory [24]).

To be more explicit about the way the quantum ordering (or correspondance "principle ") enters into our theory, we recall that the Hilbert space $\mathcal{H}$ (of square integrable functions on phase-space), recovered through (2-5), not only carries the representation (1-1) and the antirepresentation (1-2), recovered in (2-11), but can also be viewed as the Hilbert space on which the Koopman formalism [17] for classical mechanics operates: every classical observable f, being a real valued function on phase-space, can be viewed as the multiplication operator M_f, acting in $\mathcal{H}$, defined by

$$\text{(3-3)} \qquad [M_f \Psi](p,q) = f(p,q)\, \Psi(p,q)$$

(we ignore here the domain questions as these can be taken care of by the usual techniques). The correspondance between classical and quantum observables is then given by

$$\text{(3-4)} \qquad f \mapsto K\, M_f\, K$$

where K is the reproducing kernel introduced above; this gives directly the specific ordering corresponding to K and thus to the maximal abelian alegebra $\mathcal{A}$. An explicit example of the construction of such an ordering is given in [2].

From the point of view of the mathematical structure of the theory presented here as compared to the usual derivation of the prequantization representation, it should be remarked that the choice (2-12) that selects one particular primary representation in the central decomposition (2-4) matches exactly the choice one makes in the geometric quantization programme for the value of the constant $\hbar \neq 0$ that enters in the definition of the prequantization map when one imposes that the curvature $\mathcal{R}_\eta$ of the connection (2-14) is proportional to a specific multiple of the symplectic form ω, namely $-i\hbar^{-1}\omega$.

Finally, we elected to present the proofs in Section 2 for the particular case where the configuration space is R^n. The analysis of the central decomposition of the right- and left-regular representations of $\mathcal{W}_\kappa$, with $\kappa \neq 0$, can essentially be conducted as in section 2, with however two technical differences:

(i) $\mathcal{W}_\kappa$ is not unimodular, and neither is G_κ nor even $\mathcal{H}_\kappa$;

(ii) the non-abelianness of the extension G_κ of H^n_κ also implies that the first-order differential operators appearing in $\{\mathbf{P}^R_{\lambda,\kappa}, Q^R_{\lambda,\kappa}\}$ are different from those appearing in $\{\mathbf{P}^L_{\lambda,\kappa}, Q^L_{\lambda,\kappa}\}$) (a distinction that is blurred in the flat case); they are now the left-(resp. right-) invariant Hamiltonian vector fields of the classical description.

With these precautions, one can then indeed generalize the argument presented in section 2, and prove that the results stated in section 1 remain true in the case where the flat configuration space R^n is replaced by the curved homogeneous manifold $\mathcal{H}^n_\kappa$.

As for the short list of open questions, with which a review should end, we would like to mention two directions of possible extension for the approach to quantization presented here. The first line of investigation would be concerned with non-tame Lie groups, and with group actions that appear in classical dynamical systems and lead to non-type I representations [20,23,32]. The second line would be to consider the special infinite-dimensional Lie groups that occur when one considers systems with infinitely many degrees of freedom [3,9].

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