Gravitational waves and cosmological back reaction

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Abstract. Cosmological back reaction of gravitational waves and matter density fluctuations is revisited through a new general mean field theory of relativistic gravitation. Computations are done for all wave-lengths and Isaacson's results are confirmed for high-frequency waves. It is found that back reaction is typically much stronger for gravitational waves than for matter density fluctuations. In particular, a bath of gravitational waves of relative amplitude 10^{-5} and frequency 10^{-12} Hz (compatible with today's constraints) would generate an effective large-scale radiation of amplitude comparable to the unperturbed matter density of the universe.

1. Introduction

It is now well established [7] that the universe is, on large scale, expanding in an homogeneous and isotropic manner. Several authors [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] have recently argued that small-scale fluctuations around this large-scale expansion may, by non-linearity, contribute substantially to the large-scale energy repartition generating the expansion. This effect is termed cosmological back reaction.

This article focuses on the back reaction generated by gravitational waves (GWs) and evaluates it within the framework of the recently developed, general mean field theory of relativistic gravitation [8, 9, 5]. We first compute the back reaction due to a background of incoherent GWs of arbitrary frequency; in particular, we confirm, in the high frequency limit, the seminal result originally derived by Isaacson [23]. Our computation, combined with the current direct and indirect observational evidence, leaves only the so-called very long wave-length GWs (10^{-12} Hz) as possible sources for non negligible back reaction. We also compute

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the back reaction due to matter density fluctuations and find it is typically much smaller than the back reaction produced by GWs.

2. Mean field theory

2.1. Notation

In this article, the metric has signature $(+,-,-,-)$. We shall use mixed components T_μ^ν of the stress-energy tensor; with this signature, for a perfect fluid at rest with density ρ and pressure p, we have $T_0^0 = \rho$ and $T_i^i = -p$.

2.2. General framework

Averaging classical gravitational fields necessitates a mean field theory of general relativity. Such a theory has been introduced in [8, 9]; we include a brief overview here for completeness. Some applications to black hole physics are presented in [19, 20, 21].

Let $\mathcal{M}$ be a fixed manifold and $g(\omega)$ be a Lorentzian metric on $\mathcal{M}$ indexed by a random parameter ω. For example, $g(\omega)$ may represent a gravitational wave of random phase and wave vector around a given reference space-time. With each space-time $\mathcal{S}(\omega) = (\mathcal{M}, g(\omega))$ are associated the Einstein tensor $G(\omega)$ of the metric $g(\omega)$, and a stress-energy tensor $\mathcal{T}(\omega)$, satisfying the Einstein equation

$$G(\omega) = 8\pi\mathcal{T}(\omega). \tag{1}$$

Suppose one wants to extract from such a collection of space-times an average, mean gravitational field. There is a unique way to do so [8, 9] provided one imposes that (i) motions in the $\mathcal{S}(\omega)$'s average at least locally into motions in the mean field (ii) the mean gravitational field is described by General Relativity. The mean field has then to be represented by a mean metric $\bar{g} = \langle g(\omega)\rangle$ and the associated stress-energy tensor $\bar{\mathcal{T}}$ is linked by Einstein equation to the Einstein tensor $\bar{G}$ of $\bar{g}$: $\bar{\mathcal{T}} = (1/8\pi)\bar{G}$. Since the Einstein tensor depends non-linearly on the metric, $\bar{\mathcal{T}}$ does *not* coincide with $\langle\mathcal{T}(\omega)\rangle$ and the difference

$$\mathcal{T}^{\mathrm{app}}{}_\mu^\nu = \bar{G}_\mu^\nu/8\pi - \left\langle \mathcal{T}_\mu^\nu(\omega)\right\rangle, \tag{2}$$

can be interpreted as the stress-energy of an 'apparent matter' which traces the net back reaction of the fluctuations on the average field. In particular, the vanishing of $\mathcal{T}(\omega)$ for all ω does not necessarily imply the vanishing of $\bar{\mathcal{T}}$. The mean stress-energy tensor $\bar{\mathcal{T}}$ can therefore be non-vanishing in regions where the unaveraged stress-energy tensor actually vanishes. We will see that this happens, for instance, with gravitational waves.

Note also that once the statistical ensemble of metrics has been chosen, there is no gauge choice involved above since the definition of apparent matter is manifestly covariant; however, the choice of the ensemble may, in some cases, reflect the point of view of some particular observer, akin to fixing a preferred referential for the observations.

2.3. Small amplitude fluctuations

We now investigate the case when the metrics $g_{\mu\nu}(\omega)$ are all close to a reference metric $g^{\mathrm{ref}}{}_{\mu\nu}$. More precisely, we assume that there is a small parameter ε such that, for any value of the random parameter ω,

$$g_{\mu\nu}(\omega) = g^{\mathrm{ref}}{}_{\mu\nu} + \varepsilon g^{(1)}{}_{\mu\nu}(\omega) + \varepsilon^2 g^{(2)}{}_{\mu\nu}(\omega) + O(\varepsilon^3) \tag{3}$$

and we will expand the theory above at second order in ε.

Consider 'centered' fluctuations *i.e.* fluctuations whose averages vanish at first order in ε: $\langle g^{(1)}(\omega)\rangle = 0$. A direct computation then delivers:

$$\mathcal{T}^{\mathrm{app}}{}_\mu^\nu = -\frac{\varepsilon^2}{16\pi}\left\langle (\mathcal{D}^2 G_\mu^\nu)(g^{(1)}(\omega), g^{(1)}(\omega))\right\rangle, \tag{4}$$

where $\mathcal{D}^2 G_\mu^\nu$ is the (functional) second derivative of the Einstein tensor with respect to the metric g^{ref}.

It is to be noted that the effect is at second order in ε, which was to be expected since at first order, gravitation is by definition linear. What is more interesting is that $g^{(2)}$ does not appear in the result. This reflects the fact that non-linearities acting on the second-order term $g^{(2)}$ will only produce higher-order terms.

Often, it makes more physical sense to define the fluctuations in terms of the sources rather than the metric, *i.e.* to first prescribe physically meaningful fluctuations $\mathcal{T}^{(1)}$ and $\mathcal{T}^{(2)}$ of the stress-energy tensor and then look for $g^{(1)}$ and $g^{(2)}$ solving the Einstein equation. Since $g^{(2)}$ vanishes from the above result, to compute the desired effect it is actually enough to solve the *linearized* Einstein equation around g^{ref}, for a given equation of state. In the sequel we will focus on choices of $g^{(1)}$ arising from physically interesting stress-energy tensors, such as gravitational waves or fluctuations of the density of matter.

3. Fluctuations around dust cosmologies

3.1. Basics

We now apply the above to the case of either gravitational waves or density fluctuations around a homogeneous and isotropic, spatially flat dust universe (flat Friedmann–Lemaï¿½tre–Robertson–Walker metric). The reference metric and stress-energy tensor of such a space-time are, in conformal coordinates [22]:

$$g^{\text{ref}} = a(\eta)^2(d\eta^2 - dx^2 - dy^2 - dz^2) \qquad T_0^0 = \rho(\eta) \quad T_i^0 = T_i^j = 0 \tag{5}$$

where a is the so-called expansion factor and ρ is the energy density. The Einstein equation delivers $a(\eta) = C\eta^2$ and $8\pi\rho(\eta) = 3\dot{a}^2/a^4 = 12/C^2\eta^6$, where C is an arbitrary (positive) constant. Proper time is $\tau = C\eta^3/3$ and the Hubble 'constant' is $H = \dfrac{1}{a}\dfrac{da}{d\tau} = \dfrac{\dot{a}}{a^2} = \dfrac{2C}{\eta^3}$.

The perturbations $g^{(1)}$ considered in this article will be of two types: gravitational waves and matter density fluctuations. They can be written as sums or integrals of spatial Fourier modes (this makes sense since g^{ref} is spatially flat). Each term in such a series is of the form $F(\eta)\exp\left(i(\mathbf{q}.\mathbf{r} + \omega_{\mathbf{q}})\right)$ where $F(\eta)$ is some tensor, $\mathbf{q}$ is a three-dimensional wave vector, and $\omega_{\mathbf{q}}$ is a phase associated with mode $\mathbf{q}$.

Averaging a given mode $\mathbf{q}$ on spatial scales much larger than the wave-length $1/|\mathbf{q}|$ is equivalent to averaging this mode over the phase $\omega_{\mathbf{q}} \in [0; 2\pi]$. We therefore choose the set of all phases $(\omega_{\mathbf{q}})$ as our random parameter, and perform all averagings over these phases. By a simple superposition argument, which we omit, one can easily check that if several Fourier modes are present but statistically independent, then the averaging can be performed separately for each mode (at least at second order). Hence, in the sequel, we will use a single Fourier mode. Note finally that averaging over $\omega_{\mathbf{q}}$ can also be interpreted as statistically averaging over a superposition of interfering GWs of the same wave-number $\mathbf{q}$ but different phases.

3.2. Gravitational waves

Consider a single gravitational wave propagating along the above background [23]. This wave admits two polarizations [22]; since the background is isotropic, there is no loss of generality in assuming the wave propagates along, say the x-axis. The first-order metric perturbation then reads, for the first polarization:

$$g^{(1)}{}_{22}(\omega) = -a(\eta)^2 e^{i(q(x-\eta)+\omega)}(1 - i/q\eta)/\eta^2 \qquad g^{(1)}{}_{33} = -g^{(1)}{}_{22} \tag{6}$$

with the other components equal to 0. Here q is the wave number in conformal coordinates, and $\omega \in [0; 2\pi]$.

The statistical averaging corresponds to a uniform averaging over $\omega \in [0; 2\pi]$; this models situations in which the system is observed at a resolution much larger than the perturbation wave-length $1/q$.

The quantity $n_{\rm osc} = q\eta$ measures the typical number of oscillations (periods) in that part of the universe accessible to an observer situated at time η. The relative amplitude of the perturbation $\varepsilon g^{(1)}$ at time η, compared to $g^{\rm ref}$, is $\tilde{\varepsilon}(\eta) = \varepsilon/\eta^2$. We will express the results in terms of those quantities.

The apparent stress-energy corresponding to a superposition of statistically independent gravitational waves sharing a common frequency and amplitude, but propagating along random spatial directions reads:

$$\mathcal{T}^{\rm app}{}^0_0 = \tilde{\varepsilon}(\eta)^2 n_{\rm osc}^2 \, \frac{1 - 14/n_{\rm osc}^2 - 39/2n_{\rm osc}^4}{48} \, \rho(\eta) \tag{7}$$

$$\mathcal{T}^{\rm app}{}^1_1 = \mathcal{T}^{\rm app}{}^2_2 = \mathcal{T}^{\rm app}{}^3_3 = -\tilde{\varepsilon}(\eta)^2 n_{\rm osc}^2 \, \frac{1/3 - 4/3n_{\rm osc}^2 - 45/6n_{\rm osc}^4}{48} \, \rho(\eta). \tag{8}$$

In the particular case $n_{\rm osc} \gg 1$ (high frequency), these expressions confirm Isaacson's result [23] that a background of gravitational waves of high frequency behaves like radiation, with positive pressure equal to a third of its energy density (at this order in $\tilde{\varepsilon}(\eta)^2 n_{\rm osc}^2$). Since high frequency waves do not see the curvature of the background, this result is actually background independent and is also valid, *e.g.* in the presence of vacuum energy or standard electromagnetic radiation.

The first-order metric perturbation for the other polarization is given by

$$g^{(1)}{}_{23}(\omega) = (C^2\eta^4) e^{i(q(x-\eta)+\omega)} (1 - i/q\eta)/\eta^2. \tag{9}$$

The stress-energy tensor of the apparent matter associated with this polarization is identical to (7) and (8) and does not warrant separate discussion.

Orders of magnitude. The important factor in (7) and (8) is $\tilde{\varepsilon}(\eta)^2 n_{\rm osc}^2$. The energy density and pressure of apparent matter are (at this order) quadratic, not only in the amplitude $\tilde{\varepsilon}(\eta)$ of the perturbation, but also in its 'frequency' $n_{\rm osc}$. Thus, $\tilde{\varepsilon}(\eta) \ll 1$ does not necessarily translate into negligible energy density and pressure of apparent matter: the smallness of $\tilde{\varepsilon}(\eta)$ can be compensated by a sufficiently high frequency $n_{\rm osc}$. Naturally, very stringent, frequency-dependent constraints exist on $\Omega_{\rm GW}$ [24, 29]. Very low frequency waves ($10^{-17} \sim 10^{-15}$ Hz) are constrained by studying the power spectrum of the CMB B-polarizations; high frequency waves ($10^{-4} \sim 10^{4}$ Hz) are tested by laser interferometers, while the amplitude of intermediate frequency waves ($10^{-9} \sim 10^{-7}$ Hz) is indirectly bounded by studying millisecond pulsars. Current constraints definitely rule out any non negligible back reaction in all the above ranges. But a gravitational wave of frequency of order 10^{-12} Hz (corresponding to a wave-length of $1/n_{\rm osc} = 10^{-5}$ times the radius of the observable Universe) and relative amplitude $\tilde{\varepsilon}(\eta) \approx 10^{-5}$ would, by (7) and (8), generate a $\mathcal{T}^{\rm app}$ of order unity, which is quite compatible with the current constraints [26, 27] $\Omega_{\rm GW}\, h_{100}^2 \leq 0.1$ and $\Omega_{\rm GW}\, h_{100}^2 \leq 0.5$ around this frequency.

3.3. Fluctuations in the density of matter

The first-order expressions for the metric and stress-energy tensors corresponding to a matter density fluctuation around a spatially flat, homogeneous and isotropic universe are well-known and given in [22]. These expressions can be used to compute the stress-energy of apparent matter from the formula (4).

We discuss here the simplest such perturbation, which reads:

$$\begin{aligned} &{g^{(1)}}_{00} = 0 \qquad {g^{(1)}}_{11}(\omega) = a(\eta)^2\eta^2\cos(qx+\omega) \\ &{g^{(1)}}_{22}(\omega) = {g^{(1)}}_{33}(\omega) = -a(\eta)^2\frac{10}{q^2}\cos(qx+\omega). \end{aligned} \tag{10}$$

It corresponds to the following first-order stress-energy tensor perturbation

$$\mathcal{T}^{(1)}{}^0_0(\omega) = \rho\,\frac{\eta^2}{2}\cos(qx+\omega) \qquad \mathcal{T}^{(1)}{}^j_i = \mathcal{T}^{(1)}{}^j_0 = 0. \tag{11}$$

This stress-energy tensor describes a co-moving, shear-free spatial density fluctuation. Note that the relative amplitude of the perturbation increases with time, which traces the aggregating effect of gravitation.

The quantity $\tilde{\varepsilon}(\eta) = \varepsilon\eta^2$ measures the effective relative magnitude of the perturbation $\varepsilon g^{(1)}$ with respect to g^{ref}. The quantity $n_{\mathrm{osc}} = q\eta$ represents the number of oscillations (periods) in that part of the universe accessible to an observer situated at time η; typically $n_{\mathrm{osc}} \gg 1$.

As above, the averaging is over $\omega \in [0; 2\pi]$, and the stress-energy tensor of apparent matter can be obtained from (4) by direct computation. This gives

$$\mathcal{T}^{\mathrm{app}}{}^0_0 = -\tilde{\varepsilon}(\eta)^2\frac{1-75/n_{\mathrm{osc}}^2}{16\pi}\rho(\eta) \tag{12}$$

$$\mathcal{T}^{\mathrm{app}}{}^1_1 = \tilde{\varepsilon}(\eta)^2\frac{25}{16\pi n_{\mathrm{osc}}^2}\rho(\eta) \tag{13}$$

$$\mathcal{T}^{\mathrm{app}}{}^2_2 = \mathcal{T}^{\mathrm{app}}{}^3_3 = \tilde{\varepsilon}(\eta)^2\frac{7+50/n_{\mathrm{osc}}^2}{32\pi}\rho(\eta) \tag{14}$$

and all the other terms are 0 at this order in ε.

The apparent matter associated with these fluctuations is thus characterized, at this order, by a negative energy density and a negative pressure. Loosely speaking, the negative energy could be interpreted in a semi-Newtonian setting as the gravitational energy of the fluctuations and the negative pressure represents the collapsing effects of gravitation.

There is an important difference with respect to the gravitational wave case above, namely that the effect simply scales like the square of the effective amplitude of the perturbation, with no n_{osc}^2 factor (compare (7–8)). Thus, the net large-scale effect of high-frequency gravitational waves is much more important than the net large-scale effect of matter density fluctuations of comparable wave-length, at least at this order in ε. Note however that matter density fluctuations 'may become quite large' [22] with time, and, thus, escape the perturbative regime in which our results are obtained.

4. Conclusion

We have investigated perturbatively how gravitational waves and matter density fluctuations influence the large-scale homogeneous an isotropic expansion of the universe. Our results indicate that the so-called back reaction effect is dominated by gravitational waves, rather than matter density fluctuations. The relative importance of the effective large-scale stress-energy generated by gravitational waves scales as the squared product of their amplitude by their frequency. Thus, even small amplitude waves can generate an important effect provided their frequencies are high enough. For example, it is found that waves of current amplitude $\sim 10^{-5}$ and current physical frequency 10^{-12} Hz (thus compatible with currently known constraints in this part of the spectrum) would generate a large-scale stress-energy comparable to the dust energy.

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