

RECEIVED: January 3, 2022

REVISED: April 7, 2022

ACCEPTED: April 12, 2022

PUBLISHED: May 9, 2022

Integrality structures in topological strings and quantum 2-functions

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ABSTRACT: In this article, we first prove the integrality of an explicit disc counting formula obtained by Panfil and Sulkowski for a class of toric Calabi-Yau manifolds named generalized conifolds. Then, motivated by the integrality structures in open topological string theory, we introduce a mathematical notion of “quantum 2-function” which can be viewed as the quantization of the notion of “2-function” introduced by Schwarz, Vologodsky and Walcher. Finally, we provide a basic example of quantum 2-function and discuss the quantization of 2-functions.

KEYWORDS: Differential and Algebraic Geometry, String Duality, Topological Field Theories, Topological Strings

ARXIV EPRINT: [2002.09813](https://arxiv.org/abs/2002.09813)

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1 Introduction

This paper concerns the integrality structures appearing naturally in topological string theory. The basic example of mirror symmetry constructed in [9] implies the integrality of instanton numbers $N_{0,d}$ which are defined through the genus zero Gromov-Witten invariants $K_{0,d}$ of the quintic. More precisely, the genus 0 Gromov-Witten potential takes the form

$$F_0 = \sum_{d \geq 1} K_{0,d} a^d = \sum_{d \geq 1} N_{0,d} \text{Li}_3(a^d) \quad (1.1)$$

where we used the notation of poly-logarithm $\text{Li}_r(x) = \sum_{k \geq 1} \frac{x^k}{k^r}$, and a is a parameter related to the Kähler class of the quintic. Formula (1.1) is usually referred as the multiple covering formula or Aspinwall-Morrison formula [2] in literatures.

In general, the Gromov-Witten invariants $K_{0,d}$ are rational numbers, which is obvious from both the definition in Gromov-Witten theory, as well as from the B-model computations. However, the integrality of $N_{0,d}$ is not clear from the formula (1.1). In [26], Kontsevich, Schwarz and Vologodsky proposed a mathematical proof of the integrality of $N_{0,d}$ by using the p -adic theory, see [41, 42] for further progresses. The physical explanation of the integrality was given in [17] by relating $N_{0,d}$ to the degeneracy of BPS states. More precisely, let X be a Calabi-Yau 3-fold and let $K_{g,Q}^X$ be the genus g Gromov-Witten invariant of X in the curve class $Q \in H_2(X, \mathbb{Z})$, Gopakumar and Vafa [17] expressed the Gromov-Witten invariants $K_{g,Q}^X$ in terms of integer invariants $N_{g,Q}^X$ obtained by BPS state counts

$$\begin{aligned} F^X &= \sum_{g \geq 0} g_s^{2g-2} \sum_{Q \neq 0} K_{g,Q}^X a^Q \\ &= \sum_{g \geq 0, d \geq 1} \sum_{Q \neq 0} \frac{1}{d} N_{g,Q}^X \left(2 \sin \frac{dg_s}{2} \right)^{2g-2} a^{dQ}. \end{aligned} \quad (1.2)$$

Usually, these predicted integer invariants $N_{g,Q}^X$ are referred as Gopakumar-Vafa invariants in literatures. It is clear that formula (1.1) is the genus 0 part of the above formula (1.2). For a compact Calabi-Yau 3-fold X , the mathematical proof of the integrality of $N_{g,Q}^X$ is still unknown. However, when X is a toric Calabi-Yau 3-fold, the integrality of $N_{g,Q}^X$ was first proved by P. Peng for the case of toric Del Pezzo surfaces [37]. The proof for general toric Calabi-Yau 3-folds was then given by Konishi in [22]. See also [18] for several explicit formula of the Gopakumar-Vafa invariants for local $\mathbb{P}^2$.

Now we consider the open topological strings theory on Calabi-Yau 3-fold X . Suppose $L \subset X$ is a Lagrangian submanifold which may be viewed as the support of a topological D-brane in the A-model. It is well-known that the classical deformation space of L modulo Hamiltonian isotopy is unobstructed and of dimension equal to $b_1(L)$. The superpotential W depending on the Kähler moduli of X and the choice of a flat bundle over L , is the generating function counting worldsheet instanton corrections from holomorphic disks ending on the Lagrangian L .

More precisely, the spacetime superpotential can be identified with the topological disk partition function and is conjectured to admit an expansion of the general form

$$W(a, x) = F_{\text{disk}}^{(X,L)}(a, x) = \sum_{Q,m} K_{0,Q,m} a^Q x^m = \sum_{Q,m} \sum_{k \geq 1} \frac{n_{0,Q,m}}{k^2} a^{kQ} x^{km}. \quad (1.3)$$

where the sum is over relative cohomology classes in $H_2(X, L)$, a denotes the closed string Kähler parameters of X and y is the open string deformation parameters. The final transformation in (1.3) is a resummation of the multi-cover contributions and it is conjectured in [36] that the resulting expansion coefficients $n_{0,Q,m}$ are integers which are interpreted as the counting of BPS states in class (Q, m) .

Remark 1.1. Sometimes, such as in [43], we use $\beta \in H_2(X, L)$ to denote the class in $H_2(X, L)$, then the above formula (1.3) can also be written as

$$W(q) = \sum_{\beta} K_{0,\beta} q^{\beta} = \sum_{\beta} n_{0,\beta} \text{Li}_2(q^{\beta}) \quad (1.4)$$

where q is the combination of moduli parameter of (X, L) .

When X is the quintic and L is the real locus, the superpotential W had been computed in [46]. See [1, 40, 47, 48] for more results about the superpotential W for the compact Calabi-Yau manifolds. However, the integrality of $n_{0,\beta}$ is not clear from the formula (1.4). A mathematical proof was proposed in [42] follows the work [26].

When X is a toric Calabi-Yau 3-fold and L is the special Lagrangian submanifold named Aganagic-Vafa A-brane [3], the mirror geometry information of (X, L) is encoded in a mirror curve. The superpotential (or the disc counting formula) of (X, L) can be derived from the mirror curve [3, 5]. Moreover, Aganagic and Vafa surprisingly found that the computation by using mirror symmetry and the result from Chern-Simons knot invariants are matched. In [5], Aganagic, Klemme and Vafa investigated the integer ambiguity appearing in the disc counting and discovered that the corresponding ambiguity in Chern-Simons theory was described by the framing of the knot. They checked that the

two ambiguities match for the case of the unknot, by comparing the disk amplitudes on both sides. Motivated by this, one can introduce an integer τ named framing to describe the ambiguity. Let $\hat{X}$ be the resolved conifold, and D_τ the Aganagic-Vafa A-brane which is the dual of the framed unknot U_τ , in [35], Mariño and Vafa carefully studied the open topological string theory on $(\hat{X}, D_\tau)$, they computed the disk counting amplitude $F_{\text{disk}}^{(\hat{X}, D_\tau)}$ for this model and obtained the explicit expression for the corresponding integer invariants $n_{0,Q,m}^{(\hat{X}, D_\tau)}$, see also [54] for this computations, where we use the notation $n_{m,0,Q}(\tau)$ to denote this integer invariant instead. The mathematical proof for the integrality of $n_{0,Q,m}^{(\hat{X}, D_\tau)}$ was given in [34]. Moreover, we find in [33, 53] an interesting explanations for the integrality of these numbers by quiver representation theory, this provides the first example of toric Calabi-Yau and quiver correspondence, see [52] for a review of these integrality results in topological strings. Then in [23, 24], a general knot-quiver correspondence was proposed. This correspondence for a large class of knot, and links was established in [13, 24, 38, 39].

Furthermore, with the help of the knot-quiver correspondence, M. Panfl and P. Sulkowski [38] obtained an explicit disc counting formula for the open topological string theory on a class of toric Calabi-Yau manifolds without compact four-cycles, also referred to as strip geometries or generalized conifold.

Let $\hat{X}$ be a generalized conifold with the Kähler parameters arising from two types $a_1, \dots, a_r$ and $A_1, \dots, A_s$ where $r, s \geq 0$, and let D_τ be the framed Aganagic-Vafa A-brane. Set $\mathbf{l} = (l_1, \dots, l_r)$, $\mathbf{k} = (k_1, \dots, k_s)$, and $|\mathbf{l}| = \sum_{j=1}^r l_j$, $|\mathbf{k}| = \sum_{j=1}^s k_j$. Given a positive integer m , we define

$$c_{m,\mathbf{l},\mathbf{k}}(\tau) = \frac{(-1)^{m(\tau+1)+|\mathbf{l}|}}{m^2} \binom{m\tau + |\mathbf{l}| + |\mathbf{k}| - 1}{m-1} \times \prod_{j=1}^r \binom{m}{l_j} \prod_{j=1}^s \frac{m}{m+k_j} \binom{m+k_j}{k_j}. \quad (1.5)$$

Then Panfl and Sulkowski obtained the following disk counting formula for $(\hat{X}, D_\tau)$ (cf. formula (4.19) in [38]):

$$\begin{aligned} F_{\text{disk}}^{(\hat{X}, D_\tau)} &= \sum_{m,\mathbf{l},\mathbf{k}} c_{m,\mathbf{l},\mathbf{k}}(\tau) a_1^{l_1} \dots a_r^{l_r} A_1^{k_1} \dots A_s^{k_s} x^m \\ &= \sum_{m,\mathbf{l},\mathbf{k}} \sum_{d \geq 1} n_{m,\mathbf{l},\mathbf{k}}(\tau) \frac{1}{d^2} a_1^{dl_1} \dots a_r^{dl_r} A_1^{dk_1} \dots A_s^{dk_s} x^{dm}. \end{aligned} \quad (1.6)$$

By Möbius inversion formula, we have the following explicit formula for the disc counting BPS invariants

$$n_{m,\mathbf{l},\mathbf{k}}(\tau) = \sum_{d|\gcd(m,\mathbf{l},\mathbf{k})} \frac{\mu(d)}{d^2} c_{m/d,\mathbf{l}/d,\mathbf{k}/d}(\tau) \quad (1.7)$$

In this article, we generalize the method used in [34] to prove that

Theorem 1.2. *For any m , $\mathbf{l}$ and $\mathbf{k}$ given above, we have*

$$n_{m,\mathbf{l},\mathbf{k}}(\tau) \in \mathbb{Z}. \quad (1.8)$$

Motivated by the multiple covering formulas (1.1) and (1.3), Schwarz, Vologodsky and Walcher [43] introduced the mathematical notion of *s-function* which is the integral linear combinations of poly-logarithms. We review the definition and properties of 2-functions in section 3, then it is easy to see that the proof of Theorem 1.2 immediately implies that

Corollary 1.3. *The disk counting formula $F_{\text{disk}}^{(\hat{X}, D_\tau)}$ given by formula (1.6) for the generalized conifold $(\hat{X}, D_\tau)$ is a 2-function.*

The disc counting formula (1.3) can be generalized to the higher genus case. Indeed, based on Ooguri and Vafa’s work [36], the generating function of all genus open Gromov-Witten invariants can also be expressed in terms of a series of new integers which were later refined by Labastida, Mariño and Vafa in [27–29]. Motivated by their results, we formulate a mathematical notation of quantum 2-function which can be viewed as the quantum version of the 2-function introduced in [43].

Definition 1.4. We call a formal power series

$$F(\lambda, \mathbf{z}, x) = \sum_{g \geq 0, m \geq 1} \sum_{\mathbf{d} > \mathbf{0}} \lambda^{2g} K_{g, \mathbf{d}, m} \mathbf{z}^{\mathbf{d}} x^m \in \mathbb{Q}[[\lambda^2, z_1, \dots, z_r, x]] \quad (1.9)$$

with rational coefficients $K_{g, \mathbf{d}, m}$ a *quantum 2-function* if it can be written in the following form

$$F(\lambda, \mathbf{z}, x) = \sum_{g \geq 0, m \geq 1} \sum_{\mathbf{d} > \mathbf{0}} \sum_{k \geq 1} n_{g, \mathbf{d}, m} \frac{2\lambda}{km} \sin\left(\frac{km\lambda}{2}\right) \left(2 \sin \frac{k\lambda}{2}\right)^{2g-2} \mathbf{z}^{\mathbf{d}} x^{km} \quad (1.10)$$

with $n_{g, \mathbf{d}, m} \in \mathbb{Z}$, where we used the multiple-index notations $\mathbf{z} = (z_1, \dots, z_r)$, $\mathbf{d} = (d_1, \dots, d_r)$ and $\mathbf{z}^{\mathbf{d}} = z_1^{d_1} \dots z_r^{d_r}$.

It is clear that when $\lambda = 0$, $F(0, \mathbf{z}, x)$ is just the 2-function in the sense of [43]. We hope that the quantum 2-functions have independent interests in mathematics.

Then we provide a basic example for quantum 2-function. We consider the open topological string model $(\hat{X}, D_\tau)$, where $\hat{X}$ a resolved conifold and D_τ is the Aganagic-Vafa A-brane which is the large N duality of the framed unknot U_τ with framing τ in Chern-Simons theory, we consider the generating function

$$F^{(\hat{X}, D_\tau)}(\lambda, a, x) = \sum_{g \geq 0, m \geq 1} \sum_{d > 0} \lambda^{2g} K_{g, d, m}^{(\hat{X}, D_\tau)} a^d x^m \quad (1.11)$$

where $K_{g, d, m}^{(\hat{X}, D_\tau)}$ are the one-hole open Gromov-Witten invariants of genus g with degree d and writhe number m , whose mathematical definition was given in [21]. We will show that the results obtained in our previous work [34] imply that

Theorem 1.5. *The function $F^{(\hat{X}, D_\tau)}(\lambda, a, x)$ given by formula (1.11) is a quantum 2-function.*

Finally, we discuss the question how to construct a quantum 2-function by quantizing a 2-function. Motivated by the method of topological recursion introduced in [11] and its applications in topological string theory [6, 12, 15], we briefly describe a natural method to construct an operator $\mathbf{Q}$ such that when apply it to a 2-function W , then $\mathbf{Q}(W)$ will be a quantum 2-function.

On the other hand, Schwarz, Vologodsky and Walcher [43] introduced a framing transformation operator $\mathbf{f}^\tau$ (with $\tau \in \mathbb{Z}$ and $\mathbf{f}^0 = id$) on the set of 2-functions. They claimed that $\mathbf{f}^\tau(W)$ is still a 2-function for any $\tau \in \mathbb{Z}$ if W is a 2-function. Therefore, we conjecture that $\mathbf{Q}(\mathbf{f}^\tau(W))$ will be a quantum 2-function for any $\tau \in \mathbb{Z}$ and any 2-function W .

Remark 1.6. Sometime, it is easy to see that $\mathbf{Q}(W)$ is a quantum 2-function, but it is very difficult to prove that $\mathbf{Q}(\mathbf{f}^\tau(W))$ is quantum 2-function for any $\tau \in \mathbb{Z}$. We leave the further discussions about the operator $\mathbf{Q}$ and quantum 2-functions to a separated paper.

2 Proof of the theorem 1.2

We follow the notations used in [20]. Let p be any prime number, for any nonzero integer n , let the p -adic ordinal of n , denoted $\text{ord}_p n$, be the highest power of p which divides n , i.e. the greatest α such that $n = p^\alpha m$ for some integer m . If $n = 0$, we agree to write $\text{ord}_p 0 = \infty$. For any rational number $x = \frac{a}{b}$, we define

$$\text{ord}_p x = \text{ord}_p a - \text{ord}_p b. \quad (2.1)$$

Given any two rational numbers $x, y \in \mathbb{Q}$, it is obvious that

$$\text{ord}_p(x + y) \geq \min\{\text{ord}_p x, \text{ord}_p y\}. \quad (2.2)$$

For nonnegative integer n and prime number p , we introduce the following function

$$f_p(n) = \prod_{i=1, p \nmid i}^n i. \quad (2.3)$$

By its definition, $f_p(n)$ has no p -factor, i.e. $\text{ord}_p(f_p(n)) = 0$.

Lemma 2.1. Suppose $n \in \mathbb{Z}_+$, for odd prime numbers p and $\alpha \geq 1$ or for $p = 2$, $\alpha \geq 2$, we have

$$\text{ord}_p(f_p(p^\alpha n) - f_p(p^\alpha)^n) \geq 2\alpha. \quad (2.4)$$

For $p = 2, \alpha = 1$,

$$\text{ord}_2(f_2(2n) - (-1)^{[n/2]}) \geq 2. \quad (2.5)$$

Proof. We prove the Lemma 2.1 by induction. The case for $n = 1$ is obvious. Now suppose the Lemma 2.1 holds for $n - 1$. Since

$$\begin{aligned} & f_p(p^\alpha n) - f_p(p^\alpha)^n \\ &= f_p(p^\alpha n) - f_p(p^\alpha(n-1))f_p(p^\alpha) + f_p(p^\alpha(n-1))f_p(p^\alpha) - f_p(p^\alpha)^n \end{aligned} \quad (2.6)$$

Then

$$\begin{aligned} & \text{ord}_p(f_p(p^\alpha n) - f_p(p^\alpha)^n) \\ & \geq \min \left\{ \text{ord}_p(f_p(p^\alpha n) - f_p(p^\alpha(n-1))f_p(p^\alpha)), \right. \\ & \quad \left. \text{ord}_p((f_p(p^\alpha(n-1)) - f_p(p^\alpha)^{n-1})f_p(p^\alpha)) \right\}. \end{aligned} \quad (2.7)$$

By induction, $\text{ord}_p((f_p(p^\alpha(n-1)) - f_p(p^\alpha)^{n-1})) \geq 2\alpha$, hence we only need to show that

$$\text{ord}_p(f_p(p^\alpha n) - f_p(p^\alpha(n-1))f_p(p^\alpha)) \geq 2\alpha. \quad (2.8)$$

By a straightforward computation,

$$\begin{aligned} & f_p(p^\alpha n) - f_p(p^\alpha(n-1))f_p(p^\alpha) \\ & = f_p(p^\alpha(n-1)) \left(\prod_{j=1, p \nmid j}^{p^\alpha} (p^\alpha(n-1) + j) - \prod_{j=1, p \nmid j}^{p^\alpha} j \right) \\ & = f_p(p^\alpha(n-1))f_p(p^\alpha)p^\alpha(n-1) \sum_{j=1, p \nmid j}^{p^\alpha} \frac{1}{j}. \end{aligned} \quad (2.9)$$

For odd prime numbers p and $\alpha \geq 1$ or for $p = 2$, $\alpha \geq 2$, then $p^{\alpha-1}(p-1)$ is even, thus

$$\sum_{j=1, p \nmid j}^{p^\alpha} \frac{1}{j} = \sum_{j=1, p \nmid j}^{[p^\alpha/2]} \left(\frac{1}{j} + \frac{1}{p^\alpha - j} \right) = \sum_{j=1, p \nmid j}^{[p^\alpha/2]} \frac{p^\alpha}{j(p^\alpha - j)}. \quad (2.10)$$

Therefore, $\text{ord}_p(f_p(p^\alpha n) - f_p(p^\alpha(n-1))f_p(p^\alpha)) \geq 2\alpha$.

As to the case $p = 2$ and $\alpha = 1$, note that $f_2(2n) = (2n-1)!!$, then formula (2.5) is easy to check by induction. $\square$

In the following, suppose $r, s \geq 0$ are two given integers.

Lemma 2.2. *For odd prime number p such that $m = p^\alpha a, l_i = p^{\beta_i} b_i, k_j = p^{\gamma_j} c_j$, $p \nmid a, p \nmid b_i, p \nmid c_j$ for $1 \leq i \leq r, 1 \leq j \leq s$ and $\alpha \geq 1, \beta_i \geq 0, \gamma_j \geq 0$, we have*

$$\begin{aligned} & \text{ord}_p \left(\binom{m\tau + |\mathbf{l}| + |\mathbf{k}| - 1}{m-1} \prod_{i=1}^r \binom{m}{l_i} \prod_{j=1}^s \frac{m}{k_j} \binom{m+k_j-1}{k_j-1} \right. \\ & \quad \left. - \binom{\frac{m\tau + |\mathbf{l}| + |\mathbf{k}|}{p} - 1}{\frac{m}{p} - 1} \prod_{i=1}^r \binom{\frac{m}{p}}{\frac{l_i}{p}} \prod_{j=1}^s \frac{m}{k_j} \binom{\frac{m+k_j}{p} - 1}{\frac{k_j}{p} - 1} \right) \geq 2\alpha \end{aligned} \quad (2.11)$$

where the second term is defined to be zero if one of β_i or γ_j is zero.

Proof.

$$\begin{aligned}
 & \binom{m\tau + |\mathbf{l}| + |\mathbf{k}| - 1}{m - 1} \prod_{i=1}^r \binom{m}{l_i} \prod_{j=1}^s \frac{m}{k_j} \binom{m + k_j - 1}{k_j - 1} \\
 & - \binom{\frac{m\tau + |\mathbf{l}| + |\mathbf{k}|}{p} - 1}{\frac{m}{p} - 1} \prod_{i=1}^r \binom{\frac{m}{p}}{\frac{l_i}{p}} \prod_{j=1}^s \frac{m}{k_j} \binom{\frac{m + k_j}{p} - 1}{\frac{k_j}{p} - 1} \\
 & = \binom{\frac{m\tau + |\mathbf{l}| + |\mathbf{k}|}{p} - 1}{\frac{m}{p} - 1} \prod_{i=1}^r \binom{\frac{m}{p}}{\frac{l_i}{p}} \prod_{j=1}^s \frac{m}{k_j} \binom{\frac{m + k_j}{p} - 1}{\frac{k_j}{p} - 1} \\
 & \quad \times \left(\frac{f_p(m\tau + |\mathbf{l}| + |\mathbf{k}|)}{f_p(m)f_p(m(\tau - 1) + |\mathbf{l}| + |\mathbf{k}|)} \prod_{i=1}^r \frac{f_p(m)}{f_p(l_i)f_p(m - l_i)} \prod_{j=1}^s \frac{f_p(m)}{f_p(k_j)f_p(m - k_j)} - 1 \right)
 \end{aligned} \tag{2.12}$$

By Lemma 2.1, we have

$$\begin{aligned}
 & \text{ord}_p \left(\frac{f_p(m\tau + |\mathbf{l}| + |\mathbf{k}|)}{f_p(m)f_p(m(\tau - 1) + |\mathbf{l}| + |\mathbf{k}|)} \prod_{i=1}^r \frac{f_p(m)}{f_p(l_i)f_p(m - l_i)} \prod_{j=1}^s \frac{f_p(m)}{f_p(k_j)f_p(m - k_j)} - 1 \right) \\
 & \geq 2 \min(\alpha, \beta_1, \dots, \beta_i, \dots, \beta_r, \gamma_1, \dots, \gamma_j, \dots, \gamma_s).
 \end{aligned} \tag{2.13}$$

For brevity, we only compute $\text{ord}_p \left(\frac{f_p(m)}{f_p(l_i)f_p(m - l_i)} - 1 \right)$, the computation for (2.13) is the same. Indeed, by Lemma 2.1, we have

$$\begin{aligned}
 & \text{ord}_p \left(\frac{f_p(m)}{f_p(l_i)f_p(m - l_i)} - 1 \right) \\
 & = \text{ord}_p \left(\frac{f_p(m) - f_p(l_i)f_p(m - l_i)}{f_p(l_i)f_p(m - l_i)} \right) \\
 & = \text{ord}_p (f_p(m) - f_p(l_i)f_p(m - l_i)) \\
 & \geq \min \left(\text{ord}_p (f_p(m) - f_p(p^{\min(\alpha, \beta_i)} \frac{m}{\min(\alpha, \beta_i)})) \right. \\
 & \quad \left. \text{ord}_p (f_p(l_i)f_p(m - l_i) - f_p(p^{\min(\alpha, \beta_i)} \frac{m}{\min(\alpha, \beta_i)})) \right) \\
 & \geq 2 \min(\alpha, \beta_i).
 \end{aligned} \tag{2.14}$$

In order to compute the orders of the other parts of the righthand side of formula (2.12), we need to divide it into different cases to discuss.

When $r = s = 0$, formula (2.13) implies that formula (2.11) holds.

When $r = 1$ and $s = 0$ (or $r = 0$ and $s = 1$), if $\alpha > \beta$, then

$$\begin{aligned}
 & \text{ord}_p \left(\left(\frac{\frac{m\tau + l}{p} - 1}{\frac{m}{p} - 1} \right) \binom{\frac{m}{p}}{\frac{l}{p}} \right) \\
 & = \text{ord}_p \left(\frac{m}{m\tau + l} \frac{m}{l} \binom{\frac{m\tau + l}{p}}{\frac{m}{p}} \binom{\frac{m}{p} - 1}{\frac{l}{p} - 1} \right) \\
 & = 2(\alpha - \beta)
 \end{aligned} \tag{2.15}$$

if $\alpha \leq \beta$, then $\text{ord}_p \left(\left(\frac{m\tau+l}{\frac{m}{p}-1} - 1 \right) \left(\frac{\frac{m}{p}}{\frac{l}{p}} \right) \right) \geq 0$. We obtain the formula (2.11) by using formula (2.13).

Now, we discuss the case when $r, s \geq 1$.

Case 1: $\min(\alpha, \beta_1, \dots, \beta_i, \dots, \beta_r, \gamma_1, \dots, \gamma_j, \dots, \gamma_s) = \alpha$, since

$$\text{ord}_p \left(\left(\frac{m\tau + |\mathbf{l}| + |\mathbf{k}| - 1}{m-1} \right) \prod_{i=1}^r \binom{m}{l_i} \prod_{j=1}^s \frac{m}{k_j} \binom{m+k_j-1}{k_j-1} \right) \geq 0, \quad (2.16)$$

then together with formula (2.13) gives (2.11).

Case 2: $\min(\alpha, \beta_1, \dots, \beta_i, \dots, \beta_r, \gamma_1, \dots, \gamma_j, \dots, \gamma_s) \neq \alpha$, and

$$\min(\alpha, \beta_1, \dots, \beta_i, \dots, \beta_r, \gamma_1, \dots, \gamma_j, \dots, \gamma_s) = \beta_1$$

without loss of generality.

Case 2a: all the other terms are bigger than β_1 ,

$$\begin{aligned} & \text{ord}_p \left(\left(\frac{m\tau + |\mathbf{l}| + |\mathbf{k}| - 1}{m-1} \right) \prod_{i=1}^r \binom{m}{l_i} \prod_{j=1}^s \frac{m}{k_j} \binom{m+k_j-1}{k_j-1} \right) \\ &= \text{ord}_p \left(\frac{m}{m\tau + |\mathbf{l}| + |\mathbf{k}|} \binom{m\tau + |\mathbf{l}| + |\mathbf{k}|}{m} \frac{m}{l_1} \binom{m-1}{l_1-1} \prod_{i=2}^r \binom{m}{l_i} \prod_{j=1}^s \frac{m}{k_j} \binom{m+k_j-1}{k_j-1} \right) \\ &\geq 2(\alpha - \beta_1), \end{aligned} \quad (2.17)$$

together with formula (2.13) imply (2.11).

Case 2b: at least one of other β_i or γ_j equal to β_1 . Without loss of generality, suppose $\gamma_1 = \beta_1$. Then

$$\begin{aligned} & \text{ord}_p \left(\left(\frac{m\tau + |\mathbf{l}| + |\mathbf{k}| - 1}{m-1} \right) \prod_{i=1}^r \binom{m}{l_i} \prod_{j=1}^s \frac{m}{k_j} \binom{m+k_j-1}{k_j-1} \right) \\ &= \text{ord}_p \left(\left(\frac{m\tau + |\mathbf{l}| + |\mathbf{k}| - 1}{m-1} \right) \frac{m}{l_1} \binom{m-1}{l_1-1} \prod_{i=2}^r \binom{m}{l_i} \frac{m}{k_1} \binom{m+k_1-1}{k_1-1} \prod_{j=2}^s \frac{m}{k_j} \binom{m+k_j-1}{k_j-1} \right) \\ &\geq 2(\alpha - \beta_1), \end{aligned} \quad (2.18)$$

together with formula (2.13) also imply (2.11). $\square$

Lemma 2.3. For $m = 2^\alpha a, l_i = 2^{\beta_i} b_i, k_j = 2^{\gamma_j} c_j, 2 \nmid a, 2 \nmid b_i, 2 \nmid c_j$ for $1 \leq i \leq r, 1 \leq j \leq s$ and $\alpha \geq 1, \beta_i \geq 0, \gamma_j \geq 0$, we have

$$\begin{aligned} & \text{ord}_2 \left((-1)^{m(\tau+1)+|\mathbf{l}|} \cdot \left(\frac{m\tau + |\mathbf{l}| + |\mathbf{k}| - 1}{m-1} \right) \prod_{i=1}^r \binom{m}{l_i} \prod_{j=1}^s \frac{m}{k_j} \binom{m+k_j-1}{k_j-1} \right) \\ & - (-1)^{\frac{m(\tau+1)+|\mathbf{l}|}{2}} \left(\frac{\frac{m\tau+|\mathbf{l}|+|\mathbf{k}|}{2} - 1}{\frac{m}{2} - 1} \right) \prod_{i=1}^r \binom{\frac{m}{2}}{\frac{l_i}{2}} \prod_{j=1}^s \frac{m}{k_j} \binom{\frac{m+k_j}{2} - 1}{\frac{k_j}{2} - 1} \geq 2\alpha \end{aligned} \quad (2.19)$$

where the second term is defined to be zero if one of β_i or γ_j is zero.

Proof. Case 1: all the $\alpha, \beta_i, \gamma_j \geq 2$, then $(-1)^{m(\tau+1)+|\mathbf{l}|} = (-1)^{\frac{m(\tau+1)+|\mathbf{l}|}{2}} = 1$, in this case the proof is the same as in Lemma 2.2.

Case 2: only one of β_i (or γ_j) is equal to zero, then

$$\begin{aligned} & \text{ord}_2 \left(\frac{m(\tau+1) + |\mathbf{l}| + |\mathbf{k}| - 1}{m-1} \right) \\ &= \text{ord}_2 \left(\frac{m}{m(\tau+1) + |\mathbf{l}| + |\mathbf{k}|} \binom{m(\tau+1) + |\mathbf{l}| + |\mathbf{k}|}{m} \right) \geq \alpha \end{aligned} \quad (2.20)$$

together with

$$\text{ord}_2 \binom{m}{l_i} = \text{ord}_2 \left(\frac{m}{l_i} \binom{m-1}{l_i-1} \right) \geq \alpha \quad (2.21)$$

imply formula (2.19).

Case 3: at least two β_i or γ_j (suppose they are β_i and γ_j) are equal to zero, then

$$\text{ord}_2 \binom{m}{k_j} = \text{ord}_2 \left(\frac{m}{k_j} \binom{m-1}{k_j-1} \right) \geq \alpha. \quad (2.22)$$

together with formula (2.21) imply formula (2.19).

For the remain cases, we compute similarly as in (2.12)

$$\begin{aligned} & (-1)^{m(\tau+1)+|\mathbf{l}|} \binom{m\tau + |\mathbf{l}| + |\mathbf{k}| - 1}{m-1} \prod_{i=1}^r \binom{m}{l_i} \prod_{j=1}^s \frac{m}{k_j} \binom{m+k_j-1}{k_j-1} \\ & - (-1)^{\frac{m(\tau+1)+|\mathbf{l}|}{2}} \binom{\frac{m\tau+|\mathbf{l}|+|\mathbf{k}|}{2} - 1}{\frac{m}{2}-1} \prod_{i=1}^r \binom{\frac{m}{2}}{l_i} \prod_{j=1}^s \frac{m}{k_j} \binom{\frac{m+k_j}{2}-1}{\frac{k_j}{2}-1} \\ & = (-1)^{m(\tau+1)+|\mathbf{l}|} \binom{\frac{m\tau+|\mathbf{l}|+|\mathbf{k}|}{2} - 1}{\frac{m}{2}-1} \prod_{i=1}^r \binom{\frac{m}{2}}{l_i} \prod_{j=1}^s \frac{m}{k_j} \binom{\frac{m+k_j}{2}-1}{\frac{k_j}{2}-1} \\ & \quad \times \left(\frac{f_2(m\tau + |\mathbf{l}| + |\mathbf{k}|)}{f_2(m)f_2(m(\tau-1) + |\mathbf{l}| + |\mathbf{k}|)} \prod_{i=1}^r \frac{f_2(m)}{f_2(l_i)f_2(m-l_i)} \prod_{j=1}^s \frac{f_2(m)}{f_2(k_j)f_2(m-k_j)} \right. \\ & \quad \left. - (-1)^{\frac{m(\tau+1)+|\mathbf{l}|}{2}} \right). \end{aligned} \quad (2.23)$$

For the case $\alpha = 1$, it remains to show that

$$\begin{aligned} & \text{ord}_2 \left(\frac{f_2(m\tau + |\mathbf{l}| + |\mathbf{k}|)}{f_2(m)f_2(m(\tau-1) + |\mathbf{l}| + |\mathbf{k}|)} \prod_{i=1}^r \frac{f_2(m)}{f_2(l_i)f_2(m-l_i)} \right. \\ & \quad \left. \times \prod_{j=1}^s \frac{f_2(m)}{f_2(k_j)f_2(m-k_j)} - (-1)^{\frac{m(\tau+1)+|\mathbf{l}|}{2}} \right) \geq 2. \end{aligned} \quad (2.24)$$

For the case $\alpha \geq 2$, if only one of β_i (or γ_j) is equal to 1, then

$$\begin{aligned} & \text{ord}_2 \left(\left(\frac{\frac{m\tau+|\mathbf{l}|+|\mathbf{k}|}{2} - 1}{\frac{m}{2} - 1} \right) \left(\frac{\frac{m}{2}}{\frac{l_i}{2}} \right) \right) \\ &= \text{ord}_2 \left(\frac{m}{m\tau + |\mathbf{l}| + |\mathbf{k}|} \left(\frac{\frac{m\tau+|\mathbf{l}|+|\mathbf{k}|}{2} - 1}{\frac{m}{2} - 1} \right) \frac{m}{l_i} \left(\frac{\frac{m}{2} - 1}{\frac{l_i}{2} - 1} \right) \right) \geq 2(\alpha - 1). \end{aligned} \quad (2.25)$$

If at least two β_i or γ_j (suppose they are β_i and γ_j) are equal to 1, then

$$\text{ord}_2 \left(\left(\frac{\frac{m}{2}}{\frac{l_i}{2}} \right) \frac{m}{k_j} \right) = \text{ord}_2 \left(\frac{m}{k_j} \frac{m}{l_i} \left(\frac{\frac{m}{2} - 1}{\frac{l_i}{2} - 1} \right) \right) \geq 2(\alpha - 1). \quad (2.26)$$

Therefore, it also remains to show the inequality (2.24) which can be obtained by applying the Lemma 2.1. We leave the details to the reader. $\square$

Now, we can finish the proof of Theorem 1.2.

Proof.

$$\begin{aligned} n_{m,\mathbf{l},\mathbf{k}}(\tau) &= \sum_{d|\text{gcd}(m,\mathbf{l},\mathbf{k})} \frac{\mu(d)}{d^2} c_{m/d,\mathbf{l}/d,\mathbf{k}/d}(\tau) \\ &= \frac{1}{m^2} \sum_{d|\text{gcd}(m,\mathbf{l},\mathbf{k})} \mu(d) \cdot (-1)^{\frac{m(\tau+1)+|\mathbf{l}|}{d}} \left(\frac{\frac{m\tau+|\mathbf{l}|+|\mathbf{k}|}{d} - 1}{\frac{m}{d} - 1} \right) \\ &\quad \times \prod_{j=1}^r \left(\frac{\frac{m}{d}}{\frac{l_j}{d}} \right) \prod_{j=1}^s \frac{m}{m+k_j} \left(\frac{\frac{m+k_j}{d}}{\frac{k_j}{d}} \right). \end{aligned} \quad (2.27)$$

Suppose we have the prime factorization $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, we only need to show that the summation term is divisible by $p_t^{2\alpha_t}$ for every $1 \leq t \leq n$.

Given any such p_t , if $p_t \nmid \text{gcd}(m, \mathbf{l}, \mathbf{k})$, then by Lemma 2.2 and Lemma 2.3, every terms in the above summation is divisible by $p_t^{2\alpha_t}$.

If $p_t | \text{gcd}(m, \mathbf{l}, \mathbf{k})$, then

$$\begin{aligned} & \sum_{d|\text{gcd}(m,\mathbf{l},\mathbf{k})} \mu(d) \cdot (-1)^{\frac{m(\tau+1)+|\mathbf{l}|}{d}} \left(\frac{\frac{m\tau+|\mathbf{l}|+|\mathbf{k}|}{d} - 1}{\frac{m}{d} - 1} \right) \prod_{j=1}^r \left(\frac{\frac{m}{d}}{\frac{l_j}{d}} \right) \prod_{j=1}^s \frac{m}{m+k_j} \left(\frac{\frac{m+k_j}{d}}{\frac{k_j}{d}} \right) \\ &= \pm \sum_{d_t | \frac{\text{gcd}(m,\mathbf{l},\mathbf{k})}{p_t}} \left((-1)^{\frac{m(\tau+1)+|\mathbf{l}|}{d_t}} \left(\frac{\frac{m\tau+|\mathbf{l}|+|\mathbf{k}|}{d_t} - 1}{\frac{m}{d_t} - 1} \right) \prod_{j=1}^r \left(\frac{\frac{m}{d_t}}{\frac{l_j}{d_t}} \right) \prod_{j=1}^s \frac{m}{m+k_j} \left(\frac{\frac{m+k_j}{d_t}}{\frac{k_j}{d_t}} \right) \right. \\ &\quad \left. - (-1)^{\frac{m(\tau+1)+|\mathbf{l}|}{d_t p_t}} \left(\frac{\frac{m\tau+|\mathbf{l}|+|\mathbf{k}|}{d_t p_t} - 1}{\frac{m}{d_t p_t} - 1} \right) \prod_{j=1}^r \left(\frac{\frac{m}{d_t p_t}}{\frac{l_j}{d_t p_t}} \right) \prod_{j=1}^s \frac{m}{m+k_j} \left(\frac{\frac{m+k_j}{d_t p_t}}{\frac{k_j}{d_t p_t}} \right) \right). \end{aligned} \quad (2.28)$$

By Lemma 2.2 and Lemma 2.3, the above terms is divisible by $p_t^{2\alpha_t}$. $\square$

3 Quantum 2-functions

3.1 2-functions

Motivated by the multiple covering formulas (1.1) and (1.3), Schwarz, Vologodsky and Walcher [43], introduced the notion of *s-function* as integral linear combinations of poly-logarithms. Here we review the definition of 2-function.

Definition 3.1. Given t variables $z_1, \dots, z_t$, we call a formal power series

$$W(z_1, \dots, z_t) = \sum_{d_1, \dots, d_t \geq 1} m_{d_1, \dots, d_t} z_1^{d_1} \cdots z_t^{d_t} \in \mathbb{Q}[[z_1, \dots, z_t]] \quad (3.1)$$

with rational coefficients $m_{d_1, \dots, d_t}$ a 2-function if it can be written as an integral linear combination of di-logarithms

$$W(z_1, \dots, z_t) = \sum_{d_1, \dots, d_t \geq 1} n_{d_1, \dots, d_t} \text{Li}_2(z_1^{d_1} \cdots z_t^{d_t}). \quad (3.2)$$

Lemma 3.2. $W(z_1, \dots, z_t) \in \mathbb{Q}[[z_1, \dots, z_t]]$ is a 2-function if and only if

$$m_{d_1, \dots, d_t} - \frac{1}{p^2} m_{\frac{d_1}{p}, \frac{d_2}{p}, \dots, \frac{d_t}{p}}$$

is p -integral for all $p, d_1, \dots, d_t$, where $m_{\frac{d_1}{p}, \frac{d_2}{p}, \dots, \frac{d_t}{p}} = 0$ if $p \nmid \gcd(d_1, \dots, d_t)$.

Proof. The proof is essentially given in [26, 43]. By using the formula

$$\begin{aligned} W(z_1, \dots, z_t) &= \sum_{d_1, \dots, d_t \geq 1} m_{d_1, \dots, d_t} z_1^{d_1} \cdots z_t^{d_t} \\ &= \sum_{d_1, d_2, \dots, d_t, k \geq 1} \frac{n_{d_1, d_2, \dots, d_t}}{k^2} (z_1^{d_1} \cdots z_t^{d_t})^k \end{aligned} \quad (3.3)$$

we obtain

$$m_{d_1, \dots, d_t} = \sum_{k | \gcd(d_1, \dots, d_t)} \frac{1}{k^2} n_{\frac{d_1}{k}, \dots, \frac{d_t}{k}}. \quad (3.4)$$

Applying the Möbius inversion formula, we find

$$n_{d_1, \dots, d_t} = \sum_{k | \gcd(d_1, \dots, d_t)} \frac{\mu(k)}{k^2} m_{\frac{d_1}{k}, \dots, \frac{d_t}{k}}. \quad (3.5)$$

Indeed, by the definition of Möbius function $\mu(k) = 0$ if k is not squarefree, $\mu(k) = (-1)^r$ if $k = p_1 \cdots p_r$ is the product of r distinct primes. It follows that

$$\sum_{k|d} \mu(k) = \delta_{1d}. \quad (3.6)$$

Applying formula (3.6), we obtain

$$\begin{aligned}
 \sum_{k|\gcd(d_1, \dots, d_t)} \frac{\mu(k)}{k^2} m_{\frac{d_1}{k}, \dots, \frac{d_t}{k}} &= \sum_{k|\gcd(d_1, \dots, d_t)} \frac{\mu(k)}{k^2} \sum_{l|\gcd(\frac{d_1}{k}, \dots, \frac{d_t}{k})} \frac{1}{l^2} n_{\frac{d_1}{kl}, \dots, \frac{d_t}{kl}} \\
 &= \sum_{p|\gcd(d_1, \dots, d_t)} \sum_{k|p} \frac{\mu(k)}{p^2} n_{\frac{d_1}{p}, \dots, \frac{d_t}{p}} \\
 &= n_{d_1, \dots, d_t}
 \end{aligned} \tag{3.7}$$

which is just the formula (3.5).

“ $\Rightarrow$ ”: by the formula (3.4),

$$\begin{aligned}
 m_{d_1, \dots, d_t} - \frac{1}{p^2} m_{\frac{d_1}{p}, \dots, \frac{d_t}{p}} &= \sum_{k|\gcd(d_1, \dots, d_t)} \frac{1}{k^2} n_{\frac{d_1}{k}, \dots, \frac{d_t}{k}} - \frac{1}{p^2} \sum_{l|\frac{\gcd(d_1, \dots, d_t)}{p}} \frac{1}{l^2} n_{\frac{d_1}{pl}, \dots, \frac{d_t}{pl}} \\
 &= \sum_{\substack{k|\gcd(d_1, \dots, d_t) \\ p \nmid k}} \frac{1}{k^2} n_{\frac{d_1}{k}, \dots, \frac{d_t}{k}}
 \end{aligned} \tag{3.8}$$

Note the sum is restricted to those k has no prime factor p , and therefore the righthand side is p -integral if for any $n_{d_1, \dots, d_t} \in \mathbb{Z}$.

“ $\Leftarrow$ ”: since $\mu(k) = 0$ if k is divisible by p^2 , and $\mu(pk) = -\mu(k)$ if $p \nmid k$, by formula (3.5), we get

$$\begin{aligned}
 n_{d_1, \dots, d_t} &= \sum_{k|\gcd(d_1, \dots, d_t)} \frac{\mu(k)}{k^2} m_{\frac{d_1}{k}, \dots, \frac{d_t}{k}} \\
 &= \sum_{\substack{k|\gcd(d_1, \dots, d_t) \\ p \nmid k}} \frac{\mu(k)}{k^2} \left(m_{\frac{d_1}{k}, \dots, \frac{d_t}{k}} - \frac{1}{p^2} m_{\frac{d_1}{pk}, \dots, \frac{d_t}{pk}} \right)
 \end{aligned} \tag{3.9}$$

with the same understanding that $m_{\frac{d_1}{pk}, \dots, \frac{d_t}{pk}} = 0$ if $p \nmid d$. We see that if $m_{d_1, \dots, d_t} - \frac{1}{p^2} m_{\frac{d_1}{p}, \dots, \frac{d_t}{p}}$ are p -integral for all $p, d_1, \dots, d_t$, then $n_{d_1, \dots, d_t}$ are p -integral for any p , hence integral. $\square$

Indeed, the proof Theorem 1.2 implies that

Theorem 3.3. *The disc counting formula for generalized conifold given by formula (1.6)*

$$F_{\text{disk}}^{(\widehat{X}, D_\tau)} = \sum_{m, \mathbf{l}, \mathbf{k}} c_{m, \mathbf{l}, \mathbf{k}}(\tau) x^m a_1^{l_1} \cdots a_r^{l_r} A_1^{k_1} \cdots A_s^{l_s} \tag{3.10}$$

is a 2-function.

Proof. By Lemma 3.2, we only need to show that

$$c_{m, \mathbf{l}, \mathbf{k}}(\tau) - \frac{1}{p^2} c_{\frac{m}{p}, \frac{\mathbf{l}}{p}, \frac{\mathbf{k}}{p}}(\tau) \tag{3.11}$$

is p -integral for all prime p and positive integers $m, \mathbf{l}, \mathbf{k}$.

Indeed, by formula (1.5), we have

$$\begin{aligned}
 & c_{m,\mathbf{l},\mathbf{k}}(\tau) - \frac{1}{p^2} c_{\frac{m}{p}, \frac{\mathbf{l}}{p}, \frac{\mathbf{k}}{p}}(\tau) \\
 &= \frac{1}{m^2} \left((-1)^{m(\tau+1)+|\mathbf{l}|} \binom{m\tau+|\mathbf{l}|+|\mathbf{k}|-1}{m-1} \prod_{j=1}^r \binom{m}{l_j} \prod_{j=1}^s \frac{m}{m+k_j} \binom{m+k_j}{k_j} \right. \\
 &\quad \left. - (-1)^{\frac{m(\tau+1)+|\mathbf{l}|}{p}} \binom{\frac{m\tau+|\mathbf{l}|+|\mathbf{k}|}{p}-1}{\frac{m}{p}-1} \prod_{j=1}^r \binom{m}{l_j} \prod_{j=1}^s \frac{m}{m+k_j} \binom{\frac{m+k_j}{p}}{\frac{k_j}{p}} \right).
 \end{aligned} \tag{3.12}$$

Then Lemma 2.2 and Lemma 2.3 implies that $c_{m,\mathbf{l},\mathbf{k}}(\tau) - \frac{1}{p^2} c_{\frac{m}{p}, \frac{\mathbf{l}}{p}, \frac{\mathbf{k}}{p}}(\tau)$ is p -integral for all $p, m, \mathbf{l}, \mathbf{k}$. Hence, we obtain Theorem 3.3 by Lemma 3.2. $\square$

3.2 Quantum 2-functions

Motivated by Ooguri-Vafa's work [36] which generalized the disc counting formula (1.3) to the higher genus case, we introduce the notion of quantum 2-function, that means there exists a deformation parameter λ , such that when $\lambda \rightarrow 0$, the quantum 2-function reduced to the 2-function in the sense of Schwarz-Vologodsky-Walcher [43]. The notion of quantum 2-function may have independent interests.

We introduce some notations first. Set $\mathbf{z} = (z_1, \dots, z_r)$ for r variables $z_1, \dots, z_r$, and $\mathbf{d} = (d_1, \dots, d_r)$ for r nonnegative integers $d_1, \dots, d_r$, in particular $\mathbf{0} = (0, \dots, 0)$, we denote $\mathbf{z}^{\mathbf{d}} = \prod_{i=1}^r z_i^{d_i}$.

Definition 3.4. We call a formal power series

$$F(\lambda, \mathbf{z}, x) = \sum_{g \geq 0} \sum_{m \geq 1} \sum_{\mathbf{d} > \mathbf{0}} \lambda^{2g} K_{g,\mathbf{d},m} \mathbf{z}^{\mathbf{d}} x^m \in \mathbb{Q}[[\lambda^2, z_1, \dots, z_r, x]] \tag{3.13}$$

with rational coefficients $K_{g,\mathbf{d},m}$ a *quantum 2-function* if it can be written in the following form

$$F(\lambda, \mathbf{z}, x) = \sum_{g \geq 0} \sum_{m \geq 1} \sum_{\mathbf{d} > \mathbf{0}} \sum_{k \geq 1} n_{g,\mathbf{d},m} \frac{2\lambda}{km} \sin\left(\frac{km\lambda}{2}\right) \left(2 \sin \frac{k\lambda}{2}\right)^{2g-2} \mathbf{z}^{\mathbf{d}} x^{km} \tag{3.14}$$

with $n_{g,\mathbf{d},m} \in \mathbb{Z}$.

It is clear that when the parameter $\lambda = 0$, $F(0, \mathbf{z}, x)$ is just the 2-function in the sense of Definition 3.1.

For convenience, let $\hat{F}(\lambda, \mathbf{z}, x) = \sqrt{-1} \lambda^{-1} F(\lambda, \mathbf{z}, x)$. We set $q = e^{\sqrt{-1}\lambda}$ and $\{m\} = q^{\frac{m}{2}} - q^{-\frac{m}{2}}$. Let $\hat{n}_{g,\mathbf{d},m} = (-1)^{g-1} n_{g,\mathbf{d},m}$, and we introduce the function

$$\hat{f}_m(q, \mathbf{z}) = \sum_{g \geq 0} \sum_{\mathbf{d} > \mathbf{0}} \hat{n}_{g,\mathbf{d},m} (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{2g-2} \mathbf{z}^{\mathbf{d}}. \tag{3.15}$$

For $k \in \mathbb{Z}_+$, we define the k -th Adams operator Ψ_k as the $\mathbb{Q}$ -algebra map on $\mathbb{Q}(q^{\frac{1}{2}})[[x, \mathbf{z}]]$ by

$$\Psi_k(g(x, q, \mathbf{z})) = g(x^k, q^k, \mathbf{z}^k). \tag{3.16}$$

Then, the formula (3.14) can be rewritten as

$$\hat{F}(q, \mathbf{z}, x) = \sum_{k \geq 1} \frac{1}{k} \Psi_k \left(\sum_{m \geq 1} \frac{\{m\}}{m} \hat{f}_m(q, \mathbf{z}) x^m \right). \quad (3.17)$$

The Möbius inversion formula leads to

$$\sum_{m \geq 1} \hat{f}_m(q, \mathbf{z}) x^m = \sum_{k \geq 1} \frac{\mu(k)}{k} \Psi_k \left(\hat{F}(q, \mathbf{z}, x) \right) = \sum_{k \geq 1} \frac{\mu(k)}{k} \hat{F}(q^k, \mathbf{z}^k, x^k). \quad (3.18)$$

Therefore, for $m \geq 1$, we have

$$\hat{f}_m(q, \mathbf{z}) = \frac{\sqrt{-1}}{\{m\}} \sum_{k|m} \frac{\mu(k)}{k^{2-2g}} \sum_{g \geq 0, \mathbf{d} > 0} \lambda^{2g-1} K_{g, \mathbf{d}, \frac{m}{k}} \mathbf{z}^{k\mathbf{d}}. \quad (3.19)$$

In conclusion, we obtain

Proposition 3.5. *The function $F(\lambda, \mathbf{z}, x)$ given by formula (3.13) is a quantum 2-function if and only the function $\hat{f}_m(q, \mathbf{z})$ given by formula (3.19) belongs to the ring $z^{-2} \mathbb{Z}[z^2, \mathbf{z}]$, where $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$.*

However, in general, it is difficult to prove the above statement for a function given by formula (3.13). Based on the works [27–29, 36], it is expected that the generating functions of certain type open Gromov-Witten invariants in topological string theory provide many examples of quantum 2-functions.

Let us study the basic model $(\hat{X}, D_\tau)$ with $\hat{X}$ the resolved conifold and D_τ the Aganagic-Vafa A-brane which is the dual of framed unknot U_τ with framing τ in Chern-Simons theory. We consider the generating function

$$F^{(\hat{X}, D_\tau)}(\lambda, a, x) = \sum_{g \geq 0, m \geq 1} \sum_{d > 0} \lambda^{2g} K_{g, d, m}^{(\hat{X}, D_\tau)} a^d x^m \quad (3.20)$$

where $K_{g, d, m}^{(\hat{X}, D_\tau)}$ are the one-hole genus g open Gromov-Witten invariants with degree d and writhe number m , whose mathematical definition was given in [21].

According to the Mariño-Vafa's formula proposed in [35], and proved by [31, 50], one can show that the corresponding formula (3.19) in this case, denoted by

$$\hat{f}_m^{(\hat{X}, D_\tau)}(q, a) \quad (3.21)$$

can be obtained as follow.

Let $n \in \mathbb{Z}$ and μ, ν denote the partitions. We introduce the following notations

$$\{n\}_x = x^{\frac{n}{2}} - x^{-\frac{n}{2}}, \quad \{\mu\}_x = \prod_{i=1}^{l(\mu)} \{\mu_i\}_x.$$

In particular, let $\{n\} = \{n\}_q$ and $\{\mu\} = \{\mu\}_q$.

Let

$$\mathcal{Z}_m(q, a) = (-1)^{m\tau} \sum_{|\nu|=m} \frac{1}{\mathfrak{z}_\nu} \frac{\{m\nu\tau\}}{\{m\}\{m\tau\}} \frac{\{\nu\}_a}{\{\nu\}}$$

where $\mathfrak{z}_\nu = |Aut(\nu)| \prod_{i=1}^{l(\nu)} \nu_i$, see section 2 in [34] for these notations. Then we have the following formula for the expression (3.21)

$$\hat{f}_m^{(\hat{X}, D_\tau)}(q, a) = \sum_{d|m} \mu(d) \mathcal{Z}_{m/d}(q^d, a^d). \quad (3.22)$$

In [34], we have proved that, for any $m \geq 1$,

$$\hat{f}_m^{(\hat{X}, D_\tau)}(q, a) \in z^{-2} \mathbb{Z}[z^2, a^{\pm 1}], \quad (3.23)$$

where $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. Therefore, we have

Theorem 3.6. *The function $F^{(\hat{X}, D_\tau)}(\lambda, a, x)$ given by formula (3.20) is a quantum 2-function.*

3.3 Quantization and framing transformation

We have shown that the 2-function can be viewed as the classical limit of quantum 2-function, that's why we use the terminology “quantum” here. Now we consider the converse question, given a 2-function, how to construct its quantization?

Let $\mathbf{T}$ and $\mathbf{qT}$ denote the set of 2-functions and the set of quantum 2-functions respectively, we need to construct a quantized operator $\mathbf{Q}$ from $\mathbf{T}$ to $\mathbf{qT}$.

Motivated by the method of topological recursion introduced in [11] and its applications in topological string theory [6, 12, 15], we briefly describe a natural way to construct this operator $\mathbf{Q}$.

First, comparing to the relationship between the superpotential and mirror curve in topological string, one can construct a spectral curve $\mathcal{C}_W$ for a given 2-function $W \in \mathbf{T}$. Next, we apply the method of topological recursion [11] to this spectral curve $\mathcal{C}_W$, and we will obtain a series of symplectic invariants $\{F_{g,n}(\mathcal{C}_W)\}$. Finally, we collect all the $n = 1$ terms $F_{g,1}(\mathcal{C}_W)$ to construct a generating function $F(\mathcal{C}_W)$. Then $F(\mathcal{C}_W)$ is the expected quantum 2-function, in other words, we have

Conjecture 3.7. *$F(\mathcal{C}_W)$ is a quantum 2-function for any $W \in \mathbf{T}$.*

The Conjecture 3.7 allows us to introduce a formal operator $\mathbf{Q} : \mathbf{T} \rightarrow \mathbf{qT}$ by defining $\mathbf{Q}(W) = F(\mathcal{C}_W)$ for any 2-function $W \in \mathbf{T}$.

On the other hand side, motivated by the notion of the framing introduced in [3, 5] which describes the ambiguity in toric computations, Schwarz, Vologodsky and Walcher [43] considered the framing transformation on 2-function. For any $\tau \in \mathbb{Z}$, there is a framing transformation operator $\mathbf{f}^\tau$. The main result stated in [43] is that, for any 2-function W , $\mathbf{f}^\tau(W)$ is also a 2-function.

Hence one can lift the framing transformation operator $\mathbf{f}^\tau$ from $\mathbf{T}$ to $\mathbf{Q}(\mathbf{T})$, denoting the resulting operator by $\hat{\mathbf{f}}^\tau$, then

$$\hat{\mathbf{f}}^\tau(\mathbf{Q}(W)) = \mathbf{Q}(\mathbf{f}^\tau(W)). \quad (3.24)$$

Therefore, by Conjecture 3.7, we obtain a lot of quantum 2-functions by quantization and framing transformations.

Example 3.8. Here we provide a basic example to explain the above constructions. We consider the basic model $(\hat{X}, D_\tau)$ with $\hat{X}$ the resolved conifold and D_τ the Aganagic-Vafa A-brane which is the duality of the framed unknot U_τ with integer framing τ in Chern-Simons theory. The disc counting formula for this model is given by (cf. formula (22) in [54]).

$$F_{disk}^{(\hat{X}, D_\tau)} = \sum_{m \geq 1} \frac{1}{m^2} \sum_{l \geq 0} (-1)^{m\tau + m + l} \binom{m}{l} \binom{m\tau + l - 1}{m - 1} a^l x^m, \quad (3.25)$$

where the variable x is equal to $a^{-\frac{1}{2}}x$ in [54]. According to the works [34, 54] or as the special case of Theorem 3.3, we conclude that $F_{disk}^{(\hat{X}, D_\tau)}$ given by the formula (3.25) is a 2-function.

In general, the spectral curve for the 2-function $F_{disk}^{(\hat{X}, D_\tau)}$ is the algebraic curve $A(x, y) = 0$ determined by the relation

$$\log y = x \frac{dF_{disk}^{(\hat{X}, D_\tau)}}{dx}. \quad (3.26)$$

In the above model, such algebraic curve $A(x, y) = 0$ is given by the following formula (cf. formula (19) in [54])

$$y - 1 - (-1)^\tau xy^\tau (ay - 1) = 0. \quad (3.27)$$

Then, we can run the topological recursion [11] on it to obtain a series of symplectic invariants, and the $n = 1$ part gives rise to the quantum 2-function which is the one-hole part of the generating function of all genus Gromov-Witten invariants. Such calculations have been performed in literatures, such as [7]. Furthermore, the large N duality of Chern-Simons and topological string in this case was proved in [31, 50], so the one-hole part of the generating function of all genus Gromov-Witten invariants, i.e. quantum 2-function can also be given by the formula (3.17) with the expression $\hat{f}_m(q, \mathbf{z})$ given by the formula (3.21).

Remark 3.9. The definition of 2-function is inspired from the disc counting function in topological A-model of toric Calabi-Yau 3-fold. Based on the mirror symmetry [19], the mirror curve always exists, which provides the spectral curve that we can perform the topological recursion. Although this method is effective to compute the quantum 2-function for terms in low order, it is difficult to derive a closed formula for quantum 2-function. We leave the further study of the quantized procedure for 2-function to a separated paper, where more examples of quantum 2-functions will be provided.

4 Discussions and further questions

In this final section, we give some related questions which deserve to be studied further.

1. Finding more examples of 2-functions and quantum 2-functions. The existed examples of 2-functions given in [43] are the superpotentials or disc counting formulas in open topological string theory. Motivated by the large N duality of Chern-Simons and topological string theory, the Chern-Simons partition function of a knot which is a generating function of colored HOMFLYPT invariants of the knot [51], carries the natural integrality structure inherited from topological string theory. This statement is referred as to be the Labastida-Mariño-Ooguri-Vafa (LMOV) conjecture in [8, 30]. Therefore, one can define the (quantum) 2-function for any knot/link via the LMOV conjecture. If we consider the framed knot $\mathcal{K}_\tau$ with an integer framing τ , the corresponding framed LMOV conjecture was studied in [8, 10]. It is expected that the quantum 2-function of the framed knot $\mathcal{K}_\tau$ can be written as $\mathbf{Q}(\mathbf{f}^\tau(W))$, where W is the 2-function of the knot $\mathcal{K}_0$ with zero framing.
2. Studying the open topological string model beyond the Aganagic-Vafa A-brane. For example, Zaslow et al's works [45, 49] proposed the wavefunction for some Lagrangian brane which are asymptotic to Legendrian surface of genus g , they conjectured the wavefunction encodes all-genus open Gromov-Witten invariants. Therefore, one can derive a quantum 2-function from this wavefunction. The basic number theory method used in section 2 can be applied to prove the integrality of some formulas appearing in [45, 49].
3. In [44], the concept of 2-function was generalized to the situation of algebraic number field by replacing the rational number field $\mathbb{Q}$ with algebraic number field K in its definition, this generalization was motivated by the work in topological string [48]. So it is also interesting to consider the quantum 2-function in the situation of algebraic number field.

Acknowledgments

This work was supported by National Natural Science Foundation of China (Grant No. 12061014).

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