Physics Letters B 789 (2019) 160–166

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Curvature couplings of massless fermions in analog gravity

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ARTICLE INFO

Article history:
Received 28 October 2018
Received in revised form 1 December 2018
Accepted 11 December 2018
Available online 13 December 2018
Editor: M. Cvetič

ABSTRACT

Bogoliubov quasiparticles moving in the background of superfluid He^3-A see an apparently curved space–time metric when the background superfluid vacuum is in motion. We study how this curvature couples with the spins of the effectively massless quasiparticles. First, we set up the problem in null Fermi coordinates for radial as well as circular geodesics and then use it in the context of the analog metric seen by the quasiparticles. We obtain an effective magnetic interaction due to curvature coupling, and provide numerical estimates. Some possible implications of these results are then pointed out.

1. Introduction

It is well known that in general relativity, gravitational effects on fermions provide an effective magnetic field like interaction, that might have important experimental signatures [1], [2], [3], [4]. The purpose of this paper is to study the equivalent effect in an analog gravity black hole setup in the context of superfluid He^3-A. Analog gravity setups have seen tremendous interest in the past few decades and a large volume of literature is available on the subject, both in the context of theory and experiment.[1] On the experimental front, exciting progress has been made recently by Steinhauer and his group. Following their important works that appeared in [13], [14], [15] and [16], the recent work of [17] reported the observation of Hawking radiation from black holes in analog gravity, thus opening up the exciting possibility of further testing black hole physics in the laboratory.

In a similar spirit, in this paper, we will consider an analog black hole in a generalized draining bathtub geometry advocated in [18], and focus on such relativistic effects on fermionic quasiparticles (Weyl fermions), which are effectively massless in the background of an analog gravity metric but move with finite speed. The motivation for this work comes from the fact that such analysis (in real black hole scenarios) involving curved space–times are difficult to envisage, especially in the context of massless fermions, which would then move with the speed of light. The analog gravity picture on the other hand offers a somewhat simpler situation to consider, from which useful physical insights can be gleaned.

The equation for massless Weyl fermions in a curved analog background was written down by Volovik [19]. Here, following the usual procedure, we analyze the Lagrangian corresponding to this equation and in particular study the interaction Lagrangian for massless fermions in the background of the analog metric. This is done in null Fermi coordinates, and we obtain an interaction term (the curvature coupling) via an *effective* magnetic field for these fermions that arise out of analog gravity effects (this is distinct from any real magnetic effects present in the system and arise purely due to the background curvature). Standard analysis in quantum mechanics then implies interesting effects that should arise due to this coupling. In order to obtain numerical estimates of our results, we use the uncertainty relation and an energy condition as applicable to low energy quasiparticles in He^3-A.

It is known [20] that in an analog gravity setup of superfluids, there is a special class of observers, the "inner" observers, who view the inhomogeneous fluid as a space–time in which free quasiparticles should move in (roughly) geodesic trajectories. We will mainly focus on such observers, and the relativistic computations here are carried out in coordinates that are locally flat along a geodesic trajectory. These are the Fermi normal coordinates. In general, spinning particles do not follow geodesic trajectories, but the deviation from the latter are known to be small. Since our computations are relevant for effectively massless fermions, we use null Fermi coordinates, and interpret the results as those that will be seen by a null observer who moves along a (null) geodesic trajectory.

We should point out here that in visualizing a classical trajectory for the quasiparticles, we are in a sense assuming that the latter are localised near a geodesic path. We note that this is an

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[1] For excellent reviews and more recent works, see [5], [6], [7], [8], [9], [10], [11], [12].

https://doi.org/10.1016/j.physletb.2018.12.027

assumption which has been abundantly used in previous literature. We will proceed with this assumption here, and our results are valid only near such geodesic trajectories of the quasiparticles.

In the rest of this paper, we will analyse analog gravity coupling to massless fermionic quasiparticles in the context of an acoustic black hole. After providing the explicit analog gravity metric in the next section following [18], in section 3 we provide the general formalism for treating Weyl fermions in null Fermi coordinates. Sections 4 and 5 deal with radial and circular geodesics in these coordinates, and expressions for the effective magnetic field are obtained. In section 6, we provide numerical estimates of our results and conclude the paper with discussions in section 7.

2. Analog black hole in He^3-A

We will be interested in an analog black hole in the context of He^3-A. Following [18], we consider a 2-D thin film of He^3-A, which forms the background of the system, and is moving towards the orifice at the center where it goes into the third (vertical) dimension. In this case, it can be shown [18] that the energy spectrum for the low-energy Bogoliubov fermionic quasiparticles yields

$$(E - \mathbf{p}\cdot\mathbf{v})^2 = c^2(p_x^2 + p_y^2) + v_F^2(p_z - ep_F)^2 \tag{1}$$

where $e = \pm 1$. The velocity of the quasiparticles along the film ($c \sim 3$ cm/s) is much smaller than the velocity normal to the film ($v_F \sim 55$ m/s). So the degree of anisotropy of the velocity is large. This energy spectrum of the Bogoliubov quasiparticles in Eq. (1) can be recast into an effective motion of a charged, massless relativistic particle in a $(3+1)$-dimensional curved space–time with the following form of the metric [18] that is a generalization of the draining bathtub geometry described in [21]

$$ds^2 = -\left(c^2 - v^2(r)\right)dt^2 + 2v(r)drdt + dr^2 + r^2 d\phi^2 + \frac{c^2}{v_F^2}dz^2 \tag{2}$$

where (r, θ) are polar coordinates on the 2-D film, z is the vertical direction, and we have denoted $\vec{v} = (v_x, v_y)$. The above line element also shows that the g_{00} component of the metric changes sign as $v(r)$ becomes greater than c inside $r = r_h$ confirming the formation of a black hole.

We mention in passing that one can, in this case, make a coordinate transformation

$$t = \tau + \int \frac{v(r)dr}{c^2 - v(r)^2} \tag{3}$$

This equation is integrable, and in terms of the coordinate τ, yields the metric

$$ds^2 = -\left(c^2 - v^2\right)d\tau^2 + \frac{c^2}{c^2 - v^2}dr^2 + r^2 d\phi^2 + \frac{c^2}{v_F^2}dz^2 \tag{4}$$

The metrics in Eq. (2) or (4) give equivalent results. We will use the form in Eq. (2) in what follows.

In the remainder of this paper, we study massless fermionic quasiparticles in the background of the geometry of Eq. (4) (equivalently Eq. (2)).

3. Curvature coupling of quasiparticles

Let us imagine that the superfluid excitations, known as Bogoliubov quasiparticles, move in geodesics in an effectively curved space–time. Since these quasiparticles are excitations of superfluid He^3-A vacuum, they are just the dressed He^3 atoms having Bogoliubov spin, and are fermionic in nature. As a result, they will exhibit the characteristic signatures of their spin while moving in (nearly) geodesic trajectories by getting coupled with the intrinsic curvature of the space–time metric that they see. As mentioned before, we have in mind an observer who makes a measurement on the fermions, in coordinates that are locally flat all along a given geodesic. These are the Fermi normal coordinates [22].

Consider a set of four orthogonal vectors which satisfies the following two relations along a timelike geodesic of a massive particle [22]

$$\hat{e}_\alpha \cdot \hat{e}_\beta = \eta_{\alpha\beta}\ , \quad \nabla_{\nu'}(\hat{e}_\alpha^{\mu'})\hat{e}_0^{\nu'} = 0 \tag{5}$$

where ∇ denotes covariant derivative, $\eta_{\alpha\beta}$ is the usual Minkowski metric with signature $(-,+,+,+)$ and $\hat{e}_0$ represents the tangent vector to the timelike geodesic. The primed indices refer to the components of the vectors in the original coordinate system of the metric, and the unprimed indices refer to the corresponding components in Fermi normal coordinates. The structures $\hat{e}_\alpha^{\mu'}, \hat{e}_\beta^{\nu'} \ldots$ define the different elements of the coordinate transformation matrix from general coordinates to Fermi normal coordinates. Therefore, once the above tetrad is set as the basis of Fermi normal coordinate system, we can in principle compute the components of every tensor in this locally flat system. For Riemann curvature tensor, these components are

$$R_{\alpha\beta\gamma\delta} = \hat{e}_\alpha^{\mu'}\hat{e}_\beta^{\nu'}\hat{e}_\gamma^{\lambda'}\hat{e}_\delta^{\sigma'} R_{\mu'\nu'\lambda'\sigma'} \tag{6}$$

The metric close to the geodesic (G), now, looks like, up to second order in coordinates [22,23]

$$g_{00} = -1 - R_{0l0m}|_G\, x^l x^m, \quad g_{0i} = -\frac{2}{3}R_{0lim}|_G\, x^l x^m,$$
$$g_{ij} = \delta_{ij} - \frac{1}{3}R_{iljm}|_G\, x^l x^m \tag{7}$$

where the Latin indices $i, j, k, \ldots$ take the values 1,2 and 3. Here, the observer's time dependence enters the metric only through the curvature tensor components as they are evaluated at a particular proper time along the geodesic G. After obtaining such a coordinate system, we can study the covariant Dirac Lagrangian given by

$$\mathcal{L} = \sqrt{-g}(i\bar{\psi}\gamma^\alpha\mathcal{D}_\alpha\psi - m\bar{\psi}\psi) \tag{8}$$

where γ^α are the flat space Dirac matrices in the Weyl basis as appropriate for massless fermions.[2] Here, we have defined

$$\mathcal{D}_\alpha \equiv (\partial_\alpha - \frac{i}{4}\omega_{\beta\gamma\alpha}\sigma^{\beta\gamma}) \tag{9}$$

where the spin connection $(\omega_{\beta\gamma\alpha})$ and $\sigma^{\beta\gamma}$ are given, respectively, by

$$\omega_{\beta\gamma\alpha} = \hat{e}_{\beta\mu'}(\partial_\alpha\hat{e}_\gamma^{\mu'} + \Gamma^{\mu'}_{\nu'\rho'}\hat{e}_\gamma^{\nu'}\hat{e}_\alpha^{\rho'})\ , \quad \sigma^{\beta\gamma} = \frac{i}{2}[\gamma^\beta, \gamma^\gamma] \tag{10}$$

In the above expressions, $\Gamma^{\mu'}_{\nu'\rho'}$ are the Christoffel connections, and $e_\alpha^{\mu'}$ denotes, as stated before, the coefficient of the transformation matrix connecting the curved and flat space–times. If the expression of $\mathcal{D}_\alpha$ is put in the Lagrangian equation, i.e., Eq. (8), the corresponding term coming from the spin connection involves an interaction Lagrangian of the form $\bar{\psi}\gamma^\alpha\gamma^5 b_\alpha\psi$ [3,4].

[2] Here α being a flat space index, the effects of curvature enter via the spin connection in eq. (10).

In analog gravity of He^3-A, the quasiparticles are in principle massless fermions moving in a curved space–time as described earlier. Therefore, it is more useful to consider a null observer, and we need to reformulate the above analysis for null geodesics, and find the modified expressions of curvature couplings. The first question that arises in this regard is how to define the notion of Fermi normal coordinates for null geodesics. The construction here is somewhat subtle, and has been recently addressed in [24]. The technical subtlety here is that since the tangent to a null geodesic is a null vector, the corresponding set of four vectors which act as the basis of null Fermi coordinates cannot be orthonormal.

Following the construction of [24], let us define four pseudo-orthonormal vectors satisfying the same two relations as given in Eq. (5) along a null geodesic $\mathcal{N}$

$$\hat{E}_A \cdot \hat{E}_B = \eta_{AB} \ , \quad \nabla_{\nu'}(\hat{E}^{\mu'}_A)\hat{E}^{\nu'}_+ = 0 \tag{11}$$

where $\hat{E}_+$ is tangent to the null geodesic and η_{AB} is still the flat Minkowski metric but expressed in a new E^A-basis. The matrix form of η_{AB} in this new basis and the corresponding line element along $\mathcal{N}$ are given by

$$\eta_{AB} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \ ds^2|_{\mathcal{N}} = 2E^+E^- + \delta_{ab}E^aE^b \tag{12}$$

where each $A, B, \ldots$ takes the values $(+, -, 2, 3)$ and each $a, b, \ldots$ takes $(2, 3)$. The corresponding Fermi coordinates of a point x are denoted as $(x^A) = (x^+, x^-, x^a)$ and their definition is given in [24]. Once again, the quantities $\hat{E}^{\mu'}_A$ represent the different elements of the basis transformation matrix from the actual $x^{\mu'}$ coordinate system to the Fermi coordinate system x^A. Along $\mathcal{N}$, the two coordinate systems are related by

$$\left.\frac{\partial x^A}{\partial x^{\mu'}}\right|_{\mathcal{N}} = \hat{E}^A_{\mu'} \ , \quad \left.\frac{\partial x^{\mu'}}{\partial x^A}\right|_{\mathcal{N}} = \hat{E}^{\mu'}_A \tag{13}$$

The components of Riemann curvature tensor in Fermi coordinates are evaluated from the equation which is similar to Eq. (6), and are given by

$$R_{ABCD} = \hat{E}^{\mu'}_A \hat{E}^{\nu'}_B \hat{E}^{\lambda'}_C \hat{E}^{\sigma'}_D R_{\mu'\nu'\lambda'\sigma'} \tag{14}$$

The components of the metric tensor in the vicinity of the geodesic $\mathcal{N}$, up to second order, can be shown to be given by

$$\begin{aligned} g_{++} &= -R_{+\bar{c}+\bar{d}}\,|_{\mathcal{N}}\ x^{\bar{c}}x^{\bar{d}} \ , \quad g_{--} = -\frac{1}{3}R_{-\bar{c}-\bar{d}}\ |_{\mathcal{N}}\, x^{\bar{c}}x^{\bar{d}} \ , \\ g_{+-} &= 1 - \frac{2}{3}R_{+\bar{c}-\bar{d}}\,|_{\mathcal{N}}\ x^{\bar{c}}x^{\bar{d}} \ , \quad g_{ab} = \delta_{ab} - \frac{1}{3}R_{a\bar{c}b\bar{d}}\,|_{\mathcal{N}}\ x^{\bar{c}}x^{\bar{d}} \ , \\ g_{+a} &= -\frac{2}{3}R_{+\bar{c}a\bar{d}}\,|_{\mathcal{N}}\ x^{\bar{c}}x^{\bar{d}} \ , \quad g_{-a} = -\frac{1}{3}R_{-\bar{c}a\bar{d}}\,|_{\mathcal{N}}\ x^{\bar{c}}x^{\bar{d}} \end{aligned} \tag{15}$$

where $(\bar{a}) = (-, a)$.

We will now analyse the covariant Dirac Lagrangian for massless fermions appropriate to the Weyl equations obtained in [19]. This is given by

$$\mathcal{L} = i\sqrt{-g}\bar{\psi}\gamma^A\mathcal{D}_A\psi \tag{16}$$

where (see, e.g. [19]) $\mathcal{D}_A$ is given by Eq. (9) (apart from a term involving an effective gauge field), with α's replaced by A's, and the corresponding expressions of spin connection and σ^{AB} are also similar to Eq. (10):

$$\omega_{BCA} = \hat{E}_{B\mu'}(\partial_A\hat{E}^{\mu'}_C + \Gamma^{\mu'}_{\nu'\rho'}\hat{E}^{\nu'}_C\hat{E}^{\rho'}_A) \ , \ \ \sigma^{BA} = \frac{i}{2}[\gamma^B, \gamma^A] \tag{17}$$

We note here that inclusion of the additional term involving the effective gauge field [19] makes the expressions cumbersome, and for the moment we will work with the terms of Eq. (9) purely for ease of presentation, and the term involving the gauge field will be introduced later, following Eq. (28).

Here, we will have to be careful in defining γ^A. Unlike the previous case where each γ^α represents one of the standard Dirac matrices, the forms of γ^A's, in this case are different. Note that the Lagrangian in flat space–time for massless fermions can be decomposed into two parts

$$\mathcal{L}' = i\bar{\psi}\gamma^\mu\partial_\mu\psi = iu^\dagger_-\sigma^\mu\partial_\mu u_- + iu^\dagger_+\bar{\sigma}^\mu\partial_\mu u_+ \tag{18}$$

where $\sigma^\mu = (1, \sigma^i)$, $\bar{\sigma}^\mu = (1, -\sigma^i)$ with σ^i's being the Pauli matrices and $\psi = (u_+, u_-)^T$. In case of a massive fermion, u_+ and u_- cannot be separated completely, but we can describe a massless fermion by u_+ or u_- alone with the respective equation of motion given by

$$i\bar{\sigma}^\mu\partial_\mu u_+ = 0, \ \text{ or } \ i\sigma^\mu\partial_\mu u_- = 0 \tag{19}$$

These equations are the well known Weyl equations for massless fermions, and involve Pauli matrices. Now, let us apply this analysis to the covariant Dirac Lagrangian for massless fermions expressed in null Fermi coordinates, Eq. (16)

$$\mathcal{L} = i\sqrt{-g}\ u^\dagger_-\tilde{\sigma}^A\mathcal{D}_Au_- + i\sqrt{-g}\ u^\dagger_+\bar{\tilde{\sigma}}^A\mathcal{D}_Au_+ \ (\equiv \mathcal{L}_1 + \mathcal{L}_2) \tag{20}$$

The corresponding Weyl equations for u_+ or u_- will be respectively

$$i\bar{\tilde{\sigma}}^A\mathcal{D}_Au_+ = 0, \ \text{ or } \ i\tilde{\sigma}^A\mathcal{D}_Au_- = 0 \tag{21}$$

Let us compare the second expressions of equations (19) and (21). These expressions are similar with ∂_μ replaced by $\mathcal{D}_A$ and σ^μ replaced by $\tilde{\sigma}^A$. The difference between σ^μ and $\tilde{\sigma}^A$ is easy to understand. σ^μ in Eq. (19) is just the Pauli matrices with the background flat metric given by Diag$(-1, 1, 1, 1)$. The forms of the $\tilde{\sigma}^A$ in Eq. (21) are different from Pauli matrices. The reason for this is that $\tilde{\sigma}^A$ is expressed in terms of null Fermi coordinates and if we follow the definition (11) of pseudo-orthonormal Fermi frames with $\hat{E}^+$ and $\hat{E}^-$ being null vectors, the corresponding background locally flat metric along a null geodesic takes the form given in the first expression of Eq. (12).

Therefore, the transformation relations from $\eta_{\mu\nu} \longrightarrow \eta_{AB}$ have to be applied on σ^μ to obtain the corresponding expressions of $\tilde{\sigma}^A$ in the new coordinate system. The forms of $\tilde{\sigma}^A$, after this transformation, are given as $\tilde{\sigma}^A = (\tilde{\sigma}^+, \tilde{\sigma}^-, \tilde{\sigma}^2, \tilde{\sigma}^3)$, where we have defined

$$\begin{aligned} \tilde{\sigma}^+ &= -\frac{1}{\sqrt{2}}\sigma^0 + \frac{1}{\sqrt{2}}\sigma^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} , \\ \tilde{\sigma}^- &= \frac{1}{\sqrt{2}}\sigma^0 + \frac{1}{\sqrt{2}}\sigma^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} , \\ \tilde{\sigma}^2 &= \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \tilde{\sigma}^3 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{22}$$

Since the forms of $\tilde{\sigma}^A$ are changed from the usual Pauli matrices, so do those of the corresponding (4×4) γ^A matrices, and as a result, they do not exactly resemble the Dirac matrices. In particular, we will use the following forms of the γ^A matrices:

$$\gamma^+ = \begin{pmatrix} 0 & \tilde{\sigma}^+ \\ -\tilde{\sigma}^- & 0 \end{pmatrix}, \quad \gamma^- = \begin{pmatrix} 0 & \tilde{\sigma}^- \\ -\tilde{\sigma}^+ & 0 \end{pmatrix},$$
$$\gamma^a = \begin{pmatrix} 0 & \tilde{\sigma}^a \\ -\tilde{\sigma}^a & 0 \end{pmatrix} \quad (a = 2, 3) \tag{23}$$

With the new definitions and expressions of $\tilde{\sigma}^A$, we are now in a position to define the curvature coupling of massless fermions expressed in null Fermi coordinates. The expressions of vierbeins can be shown to be

$$\hat{E}^A_+ = \delta^A_+ - \frac{1}{2} R^A{}_{\bar{c}+\bar{d}}\,|_{\mathcal{N}}\, x^{\bar{c}} x^{\bar{d}}, \quad \hat{E}^A_- = \delta^A_- - \frac{1}{6} R^A{}_{\bar{c}-\bar{d}}\,|_{\mathcal{N}}\, x^{\bar{c}} x^{\bar{d}},$$
$$\hat{E}^A_a = \delta^A_a - \frac{1}{6} R^A{}_{\bar{c}a\bar{d}}\,|_{\mathcal{N}}\, x^{\bar{c}} x^{\bar{d}} \tag{24}$$

The corresponding expressions of affine connections, in Fermi coordinates, are

$$\Gamma^A{}_{B+}\,|_{\mathcal{N}} = R^A{}_{B\bar{a}+}|_{\mathcal{N}}\, x^{\bar{a}}, \quad \Gamma^A{}_{\bar{b}\bar{c}}\,|_{\mathcal{N}} = -\frac{1}{3}\left(R^A{}_{\bar{b}\bar{c}\bar{d}} + R^A{}_{\bar{c}\bar{b}\bar{d}}\right)|_{\mathcal{N}}\, x^{\bar{d}} \tag{25}$$

Now, if we expand the first term of the Lagrangian (20) in Fermi coordinates by using the expressions of Eq. (15), Eq. (24) and Eq. (25), it takes the following form

$$\mathcal{L}_1 = \sqrt{-g}\, u_-^\dagger \left(i\tilde{\sigma}^A \partial_A + b^A \tilde{\sigma}_A + i a^A \tilde{\sigma}_A\right) u_- \tag{26}$$

The third term which is anti-hermitian vanishes when its conjugate part is added to the Lagrangian. Therefore, the only interaction term that survives is the second one which is hermitian. The expressions of the components of this gravitational coupling term (b^A) come out to be

$$b^+ = \frac{1}{4}\epsilon^{1ab}\left[\frac{1}{6}\left(R_{+\bar{m}ab} - R_{-ab\bar{m}}\right) + \frac{1}{3}\left(R_{+ab\bar{m}} + R_{-ba\bar{m}}\right) + \frac{1}{2}\left(R_{-\bar{m}ab} + R_{+ba\bar{m}}\right)\right] x^{\bar{m}}$$
$$b^- = -\frac{1}{4}\epsilon^{1ab}\left[\frac{1}{6}R_{-\bar{m}ab} + \frac{7}{6}R_{+\bar{m}ab} + \frac{1}{3}\left(R_{+ab\bar{m}} + R_{-ab\bar{m}}\right) + \frac{1}{2}\left(R_{-ba\bar{m}} + R_{+ba\bar{m}}\right)\right] x^{\bar{m}}$$
$$b^c = \frac{1}{4}\epsilon^{1ac}\left[\frac{7}{6}R_{-a\bar{m}+} - R_{+a+\bar{m}} + \frac{1}{6}R_{-a-\bar{m}} + \frac{1}{3}R_{+a-\bar{m}} + \frac{1}{2}\left(R_{+-a\bar{m}} + R_{--a\bar{m}}\right)\right] x^{\bar{m}} \tag{27}$$

where again $(\bar{a}) = (-, a)$ and $a, b, c, \ldots$ take values $(2, 3)$.

The above expressions were evaluated for the form of the covariant derivative given in Eq. (9). Including the vector potential term, the full covariant derivative is given by [19]:

$$\mathcal{D}_A \equiv \partial_A - \frac{i}{4}\omega_{BDA}\sigma^{BD} - i\tilde{A}_A \tag{28}$$

where, $\tilde{A}_A = A_A + \chi_A$, with the expressions of χ_A and A_A being

$$\chi_A = \frac{1}{8}\epsilon^{\lambda'\gamma'\mu'\nu'} E_{A\lambda'}\, E^B_{\gamma'}\left(\partial_{\mu'}E_{B\nu'} - \partial_{\nu'}E_{B\mu'}\right),$$
$$A_A = (0, 0, 0, p_F) \tag{29}$$

The corresponding expressions of χ^A in Fermi coordinates are evaluated to be

$$\chi^+ = 0, \quad \chi^- = \frac{1}{4}\left(\frac{1}{3}R_{+32\bar{m}} + \frac{2}{3}R_{+\bar{m}23} - \frac{1}{3}R_{+23\bar{m}}\right) x^{\bar{m}},$$
$$\chi^2 = \frac{1}{4}\left(\frac{2}{3}R_{+\bar{m}3-} - \frac{1}{3}R_{+3-\bar{m}} + \frac{1}{3}R_{+-3\bar{m}}\right) x^{\bar{m}},$$
$$\chi^3 = \frac{1}{4}\left(-\frac{2}{3}R_{+\bar{m}2-} + \frac{1}{3}R_{+2-\bar{m}} - \frac{1}{3}R_{+-2\bar{m}}\right) x^{\bar{m}} \tag{30}$$

The total magnetic field including the gauge field term is now given by

$$B^A = b^A + \chi^A + A^A \tag{31}$$

with the form of b^A given in Eq. (27). Note that the last term in Eq. (31) is a constant term, and we will ignore this in our analysis. In what follows, we will focus on the first two terms of Eq. (31) and in the next section, we proceed to evaluate the components of B^A for both radial and circular null geodesics in the background of analog gravity and study its characteristic features in some details.

4. Massless fermionic quasiparticles in radial null geodesics

For null geodesics in the space–time of Eq. (2), the normalization of the four-velocity yields

$$\dot{r}^2 + 2v(r)\dot{t}\dot{r} + r^2\dot{\phi}^2 - \left(c^2 - v^2(r)\right)\dot{t}^2 + \frac{c^2}{v_F^2}\dot{z}^2 = 0 \tag{32}$$

where over-dots represent derivative with respect to an affine parameter along the null geodesic. For timelike geodesics, a standard choice of this affine parameter is the proper time. But in case of a null geodesic, the affine parameter cannot be the proper time. Instead, normal coordinate time or radial distance may be considered as the affine parameter, if they satisfy the geodesic equation of the form

$$\nabla_{\nu'}(u^{\mu'})u^{\nu'} = 0 \tag{33}$$

where $u^{\mu'}$ is the tangent vector to the null geodesic under consideration. Here, by radial null geodesics, we mean the set of null geodesics for which $\dot{\phi} = \frac{d\phi}{d\lambda} = 0$, with λ being the affine parameter. Therefore, for radial null geodesics outside the orifice, Eq. (32) becomes

$$\dot{r}^2 + 2v(r)\dot{t}\dot{r} - \left(c^2 - v^2(r)\right)\dot{t}^2 = 0 \tag{34}$$

With condition (34) in mind, we can find out the pseudo-orthonormal Fermi tetrad basis for the analog metric along a null radial geodesic as:

$$\hat{E}^{\mu'}_- = \left(\frac{v(r) - c}{2c^2} - \frac{kc + kv(r)}{2\left(c^2 - v^2(r)\right)}, \frac{c^2 - v(r)^2}{2c^2} - \frac{k}{2}, \frac{k}{r}, 0\right),$$
$$\hat{E}^{\mu'}_3 = \left(0, 0, 0, \frac{v_z}{c}\right), \quad \hat{E}^{\mu'}_+ = \left(\frac{c + v(r)}{c^2 - v(r)^2}, 1, 0, 0\right),$$
$$\hat{E}^{\mu'}_2 = \left(-\frac{kc + kv(r)}{c^2 - v^2(r)}, -k, \frac{1}{r}, 0\right), \tag{35}$$

where k is a constant. The tangent vector to the geodesic, $u^{\mu'}$ or $\hat{E}^{\mu'}_+$ takes the form $\hat{E}^{\mu'}_+ = (\dot{t}, \dot{r}, 0, 0)$, for a general affine parameter λ. In the present case, we have set $\dot{r} = 1$, i.e., we have explicitly chosen r as the affine parameter along the geodesic $\mathcal{N}$. This choice simplifies the calculation as well as it satisfies the required geodesic equation condition.

Now the above choice of tetrad has to satisfy the required conditions of Eq. (11). Let us rewrite the first condition of Eq. (11)

and analyze it using the tetrad defined above. From the condition $\hat{E}_A \cdot \hat{E}_B = \eta_{AB}$, we obtain

$$\hat{E}_- \cdot \hat{E}_- = \eta_{--} = 0$$
$$(\text{for } A = B = -) \Rightarrow g_{\mu'\nu'}\hat{E}_-^{\mu'}\hat{E}_-^{\nu'} = 0 \Rightarrow k(k-1) = 0 \tag{36}$$

So only two values of the constant k satisfy the required conditions for $\hat{E}_-$, i.e. k takes two values, 0 and 1. All other components of the tetrad automatically satisfy the required conditions of (11). Therefore, we have two different set of tetrads with $k = 0$ and $k = 1$ which can be chosen as the basis of null Fermi frame.

Having obtained the Fermi tetrad basis, we can readily find out the components of the Riemann curvature tensor in null Fermi coordinates using Eqs. (14) and (35). Then we use Eqs. (27) and (30) to calculate the components of the effective magnetic field due to curvature coupling and find that the corresponding expressions are given by

$$B^+ = B^- = B^2 = 0$$
$$B^3 = \left[kr(4h - ky)v'(r)^2 + v(r)\left(kr(4h - ky)v''(r) + \left(k(-4h + 2ky - y) + y\right)v'(r)\right)\right]\left[24c^2 r\right]^{-1} \tag{37}$$

where h, y, z represent observer's coordinates or Fermi coordinates.

The above expression of B^3 has been evaluated for a general $v(r)$. But in case of the draining bathtub type geometry, the specific form of $v(r)$ happens to be $v(r) = -\frac{cr_h}{r}$. So if we put this form of $v(r)$ in Eq. (37), we obtain the following expression of B^3:

$$B^3 = \frac{r_h^2\left[16hk + \left(-1 + k - 5k^2\right)y\right]}{24r^4} \tag{38}$$

The expressions of B^3 for $k = 0$ and $k = 1$ are given by

$$B^3 = -\frac{r_h^2 y}{24r^4} \ (k = 0)\ ,\ B^3 = \frac{r_h^2(16h - 5y)}{24r^4} \ (k = 1) \tag{39}$$

We need to analyze this result in more detail. Eq. (38) tells us that the effective magnetic field component (b^3) diverges at $r = 0$. Since $r = r_h$ represents the position of event horizon of the analog black hole and we are particularly interested in the phenomena occurring outside r_h, the effective magnetic field is always finite in this region. From Eq. (38) we see that the effective magnetic field falls off as r^{-4} as a function of the radial distance.

5. Massless fermionic quasiparticles in circular null geodesic

We will now compute the curvature coupling of fermionic quasiparticles in circular null geodesics. This is of course a special case, as we discuss. For such geodesics, it can be checked from Eq. (2) that the only allowed value of the radial coordinate is $r = \sqrt{2}r_h$, for $v(r) = -cr_h/r$ as in the previous subsection. This is the analog of the photon sphere in GR [25], and a null observer in a circular geodesic is uniquely located at this value of r. This is to be kept in mind in the analysis that follows.

Similar to the radial case, first we need to set up the pseudo-orthonormal Fermi frame for a circular null geodesic $\mathcal{G}$. Then we find out components of Riemann tensor in Fermi coordinates and use it to calculate the effective magnetic field. By "circular null geodesic" we mean the family of null geodesics for which r is constant, i.e., $\dot{r} = \ddot{r} = 0$. The corresponding Fermi frame for a circular null geodesic $\mathcal{G}$ are found to be

$$\hat{E}_+^{\mu'} = \left(\frac{1}{\sqrt{c^2 - v(r)^2}}, 0, \frac{1}{r}, 0\right)$$
$$\hat{E}_-^{\mu'} = \left(\frac{r\phi^2 v(r)v'(r)}{2c^2\sqrt{c^2 - v(r)^2}} + \frac{\phi v(r)}{c^2} - \frac{1}{2\sqrt{c^2 - v(r)^2}}, -\frac{r\phi v(r)v'(r)}{c^2}, \frac{\phi^2 v(r)v'(r)}{2c^2} + \frac{1}{2r}, 0\right)$$
$$\hat{E}_2^{\mu'} = \left(-\frac{\phi}{c} + \frac{v(r)}{c\sqrt{c^2 - v(r)^2}}, \frac{\sqrt{c^2 - v(r)^2}}{c}, -\frac{\phi\sqrt{c^2 - v(r)^2}}{cr}, 0\right)$$
$$\hat{E}_3^{\mu'} = \left(0, 0, 0, \frac{v_z}{c}\right) \tag{40}$$

with the condition $rv(r)v'(r) + c^2 - v(r)^2 = 0$. It has to be remembered that the above expressions in Eq. (40) have to be evaluated at $r = \sqrt{2}r_h$ and we have denoted $v'(r) = \frac{\partial v(r)}{\partial r}$.

It is to be noted that the tetrad components depend explicitly on ϕ. This might seem surprising, given that the analog metric that we start with is spherically symmetric, but is due to the fact that the tetrad must satisfy the pseudo-orthonormal and parallel transport conditions given in Eq. (11), along the null geodesic. As the first two vectors of the tetrad basis $\hat{E}^+$ and $\hat{E}^-$ are null, for circular geodesic they demand its components to depend explicitly on the affine parameter which in this case is chosen to be the arc-length of the circular geodesic $\sqrt{2}r_h\phi$. Similar dependence can also be seen for the timelike circular geodesic where ϕ dependence comes into the phases of harmonic functions [26] but not explicitly, the reason being that the tetrad basis is made of the timelike tangent vector and three spacelike vectors.

The corresponding expressions of the components of effective magnetic field in this case are

$$B^+ = B^- = B^2 = 0\ ,\ B^3 = \frac{8\sqrt{2}h\phi\left(\phi^2 + 4\right) - \left(\phi^4 - 96\right)y}{384r_h^2} \tag{41}$$

6. Numerical estimates

It now remains to provide numerical estimates of the B^A that we have evaluated. In order to do this, we will take recourse to various approximations that we now discuss. We note that the dimension of B^A for both radial and circular geodesics is an inverse length (contrary to usual magnetic fields that come in dimensions of $1/L^2$). In order to convert B^A into a quantity having dimensions of a magnetic field (Gauss or Tesla), we will need to divide it by the Bohr magneton, expressed in GeV per Tesla (or GeV per Gauss) [27]. Doing this, it can be checked that a magnetic field of 10^{-12} Gauss translates to a value of $B^A \sim 10^{-29}$ GeV. This is the limit of measurability of the magntic field, as of now. In the analysis that follows, we will use energy units only, for convenience.

Let us first consider the case of radial geodesics and consider the case $k = 0$, i.e. the first expression given in Eq. (39). Since this has dimension L^{-1}, we convert this to energy units by multiplying with $\hbar c_L$, where c_L is the speed of light ($= 3 \times 10^8$ m/s). Thus we have $B^3(k = 0) = -r_h^2 y\hbar c_L/(24r^4)$. In order to get an estimate for the coordinate y, we use the uncertainty relation. Remembering that the quasiparticles are dressed Helium-3 atoms of mass m^*, moving with speed $c = 3$ cm/s, we have $y \sim \hbar/p = \hbar/(m^* c)$. Plugging this in, we have in electron-volts,

$$|B^3(k = 0)| = \frac{1}{24}\frac{r_h^2\hbar^2 c_L}{r^4 m^* c \times q_e}\ \text{eV} \tag{42}$$

where q_e is the electron charge. We now use the fact that m^* is 3 times the mass of Helium-3 atoms, which is given by 3.016

atomic mass units. A numerical estimate of $|B^3(k=0)|$ is obtained by putting in this value of m^*.

Now we note that the theory of low energy quasiparticles is valid for

$$E \ll \frac{\Delta_v^2}{p_F v_F}\ \text{, i.e.}\ \ E \ll 10^{-10}\ \text{eV} \tag{43}$$

Since B^A appears in the interaction energy term in the Lagrangian of Eq. (26), as an estimate we equate $B^3(k=0) \sim 10^{-10}$, to obtain $r = 0.067\sqrt{r_h}$ (in metres). For $E \ll 10^{-10}$eV, we therefore require $r \gg 0.067\sqrt{r_h}$. A typical value of $r_h \sim 1$ μm will thus ensure that our results are valid for $r \gg 0.0067$ cm. Hence on a radial geodesic, at say $r \sim 1$ cm, our results should be robust, and at this radial distance, we have $B^3(k=0) \sim 10^{-28}$ GeV. Smaller values of the radial distance may push up this value to $\sim 10^{-27} - 10^{-26}$ GeV, while respecting the energy condition. This discussion was for $k = 0$. For $k = 1$, the analysis of the second expression of Eq. (39) is qualitatively similar, and yields similar numerical estimates.

We will now turn to circular geodesics. An analysis similar to the one outlined above shows that in this case, setting $y = 0$ in Eq. (41) implies that $B^3 = (1/r_h^2)(5.6 \times 10^{-15}\phi + 1.4 \times 10^{-15}\phi^3)$ eV. If we now set a typical value of $\phi = \pi$, then in order to satisfy $E \ll 10^{-10}$eV, we require $r_h \gg 2.4$ cm. As an estimate, if we set $r_h = 10$ cm, we obtain $B^3 \sim 10^{-21}$ GeV.

Our numerical analysis above establishes the fact that the effective magnetic field that is seen by an inner observer in superfluid He^3-A are withing bounds reachable by present experiments, i.e. these are not vanishingly small. Hence, such an observer should measure effects that are known in quantum mechanics regarding the interaction of spins with such magnetic fields. In this case, however, the effective magnetic field is non-uniform. For radial null geodesics, this falls off as the fourth power of the radial distance, while for circular null geodesics, it is explicitly dependent on the angular variable. From Eq. (39), using the condition $r \gg 0.067\sqrt{r_h}$, it is seen that for relatively large values of r (compared to r_h), B^3 varies slowly. Hence, if we approximate B^3 by a uniform (average) value, for such large r, one should expect the inner observer to see oscillations between a spin up and a spin down state of the massless fermionic quasiparticles when the system evolves from a general spin state. Similar analysis holds for null circular geodesics, for small values of the angular coordinate. The external observer (moving at 3 cm per sec) along a radial coordinate or moving at a fixed radius, however, perceives these quasiparticles as dressed Helium-3 atoms. It would therefore seem that such an observer is likely to see the spin of the quasiparticles to also oscillate between an up-spin state and a down-spin state.

We note here that the time difference between two events for the external observer Δt is related to that for the inner observer $\Delta\tau$ by [20] $\Delta t = \Delta\tau/\sqrt{1 - r_h^2/r^2}$. Hence, the characteristic frequency of oscillation (assuming a uniform magnetic field) for the external observer is dilated by a factor of $\sqrt{1 - r_h^2/r^2}$, as compared to the inner observer. For null radial geodesics, for small values of r_h/r, $\Delta t \sim \Delta\tau$. This is relevant for us, as we have already seen that the energy condition demands that here, $r \gg 0.067\sqrt{r_h}$, and that the effective magnetic field can be approximated to a constant for small values of r_h/r. For circular geodesics, since $r = \sqrt{2}r_h$, the dilation factor is $\sqrt{2}$.

7. Discussions and conclusions

In this paper, we have presented an analog gravity equivalent of fermion curvature couplings to gravity in general relativity. To the best of our knowledge, this analysis, which applies to massless fermions, is novel. In particular, we have used the recent construction of [24] of null Fermi coordinates to understand such curvature couplings. We started with a He^3-superfluid system where the vacuum excitations are Bogoliubov fermions, which are dressed He^3 atoms and see an effective curved space–time with moving superfluid vacuum in the background. We first established the Fermi coordinates along a null radial as well as a null circular geodesics, and calculated the components of the Riemann curvature tensor in these coordinates. Having obtained the curvature tensor, we determined an effective magnetic field due to curvature coupling. The whole analysis was done in the analog black hole draining bath-tub geometry discussed by Volovik in [18], and numerical estimates of the *effective* magnetic field have been provided.

As is known, spinning particles do not follow exactly geodesic trajectories, but the difference of the latter from their actual paths is small. We can then envisage an inner null observer of section 2 (feeling the analog metric), on such a geodesic trajectory, who makes a measurement on the fermionic system. This inner null observer moves with a finite speed (~ 3 cm/s) and sees the non-trivial effect of curvature coupling to the fermionic quasiparticles. It is however somewhat unclear how a precise experimental signature of this should be realisable. In principle, this would imply that one introduces a model interaction between the two sets of observers in our theoretical computations. This should be interesting to probe further.

We should also point out that our analysis in this paper concerns massless fermions in analog gravity. A natural extension should be to analyse massive fermions in such models. Indeed, massive bosons were introduced in analog models via a two-component Bose–Einstein condensate in [28]. We are not aware of a corresponding analysis for fermions, although we expect that this should be more complicated than the bosonic case. This would be an interesting direction for further study.

Before ending this paper, we should point out that the analysis that we have presented here is limited by the fact that it is applicable only to two special class of geodesics, i.e. radial or circular. A generic geodesic path may be neither of these. However, this last case is difficult to analyse analytically, and we leave a study of such a situation for a future publication.

Acknowledgements

It is a pleasure to thank K. Bhattacharya and V. Subhrahmanyam for useful discussions.

References

[1] R. Bluhm, V.A. Kostelecký, Lorentz and *CPT* tests with spin-polarized solids, Phys. Rev. Lett. 84 (Feb. 2000) 1381–1384.
[2] R. Bluhm, V.A. Kostelecký, C.D. Lane, *CPT* and Lorentz tests with muons, Phys. Rev. Lett. 84 (Feb. 2000) 1098–1101.
[3] S. Mohanty, B. Mukhopadhyay, A.R. Prasanna, Experimental tests of curvature couplings of fermions in general relativity, Phys. Rev. D 65 (May 2002) 122001.
[4] U. Debnath, B. Mukhopadhyay, N. Dadhich, Spacetime curvature coupling of spinors in early universe: neutrino asymmetry and a possible source of baryogenesis, Mod. Phys. Lett. A 21 (05) (2006) 399–408.
[5] C. Barcelo, S. Liberati, M. Visser, Analogue gravity, Living Rev. Relativ. 8 (2005) 12, arXiv:gr-qc/0505065 [gr-qc].
[6] S. Weinfurtner, S. Liberati, M. Visser, Analogue model for quantum gravity phenomenology, J. Phys. A 39 (2006) 6807–6814, arXiv:gr-qc/0511105 [gr-qc].
[7] D.V. Fursaev, Entanglement entropy in critical phenomena and analogue models of quantum gravity, Phys. Rev. D 73 (2006) 124025, arXiv:hep-th/0602134 [hep-th].
[8] T.K. Das, N. Bilic, S. Dasgupta, Black-hole accretion disc as an analogue gravity model, J. Cosmol. Astropart. Phys. 0706 (2007) 009, arXiv:astro-ph/0604477 [astro-ph].
[9] G. Krein, G. Menezes, N.F. Svaiter, Analog model for quantum gravity effects: phonons in random fluids, Phys. Rev. Lett. 105 (2010) 131301, arXiv:1006.3350 [hep-th].

[10] S. Hossenfelder, Analog systems for gravity duals, Phys. Rev. D 91 (12) (2015) 124064, arXiv:1412.4220 [gr-qc].
[11] B. Cropp, S. Liberati, R. Turcati, Vorticity in analog gravity, Class. Quantum Gravity 33 (12) (2016) 125009, arXiv:1512.08198 [gr-qc].
[12] C. Chakraborty, O. Ganguly, P. Majumdar, Intertial frame dragging in an acoustic analogue spacetime, arXiv:1510.01436 [gr-qc].
[13] O. Lahav, A. Itah, A. Blumkin, C. Gordon, S. Rinott, A. Zayats, J. Steinhauer, Realization of a sonic black hole analog in a Bose–Einstein condensate, Phys. Rev. Lett. 105 (24) (2010) 240401.
[14] I. Shammass, S. Rinott, A. Berkovitz, R. Schley, J. Steinhauer, Phonon dispersion relation of an atomic Bose–Einstein condensate, Phys. Rev. Lett. 109 (19) (2012) 195301.
[15] R. Schley, A. Berkovitz, S. Rinott, I. Shammass, A. Blumkin, J. Steinhauer, Planck distribution of phonons in a Bose–Einstein condensate, Phys. Rev. Lett. 111 (5) (2013) 055301.
[16] J. Steinhauer, Observation of self-amplifying Hawking radiation in an analogue black-hole laser, Nat. Phys. 10 (11) (2014) 864.
[17] J. Steinhauer, Observation of quantum Hawking radiation and its entanglement in an analogue black hole, Nat. Phys. 12 (10) (2016) 959.
[18] G. Volovik, Simulation of a Panlevé–Gullstrand black hole in a thin $^3\mathrm{He} - a$ film, JETP Lett. 69 (9) (1999) 705–713.
[19] G. Volovik, Analog of gravity in superfluid ^{3}He-A, JETP Lett. 44 (8) (1986).
[20] G. Volovik, The Universe in a Helium Droplet, Int. Ser. Monogr. Phys., vol. 117, Clarendon Press, Oxford, 2003.
[21] M. Visser, Acoustic black holes: horizons, ergospheres, and Hawking radiation, Class. Quantum Gravity 15 (1998) 1767–1791, arXiv:gr-qc/9712010 [gr-qc].
[22] F.K. Manasse, C.W. Misner, Fermi normal coordinates and some basic concepts in differential geometry, J. Math. Phys. 4 (6) (1963) 735–745.
[23] E. Poisson, A Relativist's Toolkit, the Mathematics of Black-Hole Mechanics, Cambridge University Press, 2004.
[24] M. Blau, D. Frank, S. Weiss, Fermi coordinates and Penrose limits, Class. Quantum Gravity 23 (2006) 3993–4010, arXiv:hep-th/0603109 [hep-th].
[25] C.-M. Claudel, K.S. Virbhadra, G.F.R. Ellis, The geometry of photon surfaces, J. Math. Phys. 42 (2001) 818–838, arXiv:gr-qc/0005050 [gr-qc].
[26] A. Dey, A. Samanta, T. Sarkar, Fermi normal coordinates and fermion curvature couplings in general relativity, Phys. Rev. D 89 (May 2014) 104008.
[27] R. Bluhm, V.A. Kostelecky, Lorentz and CPT tests with spin polarized solids, Phys. Rev. Lett. 84 (2000) 1381–1384, arXiv:hep-ph/9912542 [hep-ph].
[28] M. Visser, S. Weinfurtner, Massive Klein–Gordon equation from a BEC-based analogue spacetime, Phys. Rev. D 72 (2005) 044020, arXiv:gr-qc/0506029 [gr-qc].