Lectures on Dark Matter

U. Seljak*

Department of Physics, Princeton University, Princeton, NJ, USA

Lecture given at the
Summer School on Astroparticle Physics and Cosmology
Trieste, 12-30 June 2000

LNS014002

*useljak@princeton.edu

Abstract

These lectures concentrate on evolution and generation of dark matter perturbations. The purpose of the lectures is to present, in a systematic way, a comprehensive review of the cosmological parameters that can lead to observable effects in the dark matter clustering properties. We begin by reviewing the relativistic linear perturbation theory formalism. We discuss the gauge issue and derive Einstein's and continuity equations for several popular gauge choices. We continue by developing fluid equations for cold dark matter and baryons and Boltzmann equations for photons, massive and massless neutrinos. We then discuss the generation of initial perturbations by the process of inflation and the parameters of that process that can be extracted from the observations. Finally we discuss evolution of perturbations in various regimes and the imprint of the evolution on the dark matter power spectrum both in the linear and in the nonlinear regime.

Contents

1 Introduction

The study of the correlations in the dark matter and in the Cosmic Microwave Background (CMB) is revolutionizing cosmology. Fluctuations across a range of scales have been detected and the precision in the determination of the power spectra are improving steadily. Over the past few years, a coherent cosmological model appears to have emerged: CMB observations[1, 2, 3] suggest that the universe is flat; a host of astronomical observations[4] suggest that most of the matter in the universe is non-baryonic and that the energy density in this dark matter is not sufficient to explain the observed flatness; and supernova observations[5, 6] imply that the expansion of the universe is, surprisingly, accelerating. These observations suggest that most of the energy density of the universe is in some new form. This dark energy may be quintessence[7] associated with a light scalar field, or due to the energy of the vacuum. While this emerging standard model is an important intellectual triumph, the unknown nature of the dark matter and the dark energy suggests that there may be important missing physics in the model.

Among the questions that we still need to address are: What is the dark matter? What is the mass of the neutrino? What is the nature of the dark energy? Did structure form by gravitational instability? How did the universe begin? These questions were among those identified in the recent National Academy of Sciences' Committee on the Physics of the Universe study[8].

Determining the linear power spectrum of dark matter can provide answers to many of these questions. From the theoretical perspective the dark matter clustering can give information on the exact nature of dark matter. For example, the clustering of cold dark matter differs from that of hot or warm dark matter and this information can be extracted from the mass power spectrum. The latter is also sensitive to neutrino mass, since massive neutrinos strongly suppress the level of mass fluctuations on small scales because of the high neutrino momentum before they become nonrelativistic. In principle the sensitivity of upcoming surveys is such that it will be possible to test neutrino masses below 0.1-1eV [9], close to those suggested by recent Super-Kamiokande neutrino results [10].

Another family of models that recently received a lot of attention is that where a significant fraction of the dark matter is really dark energy that only weakly clusters on small scales. The family can be described by its equation of state, which in principle can evolve in time and by the contribution to the overall energy density. These models can also be studied through the effects on the dark matter power spectrum and its time evolution.

From the observational perspective there are several surveys, such as 2dF[1] and SDSS[2], which will measure redshifts of more than a million galaxies. 3-dimensional mass power spectrum is sensitive to a number of cosmological parameters, such as matter density, Hubble constant, primordial power spectrum slope and amplitude, massive neutrinos, baryon density etc. In addition, this sensitivity is further improved if information from higher redshifts is provided, such as from the CMB, Ly-α forest or weak lensing.

The purpose of these lectures is to present a comprehensive review of the theory behind the dark matter perturbations. These are determined both by the initial conditions and by their subsequent evolution. Because the perturbation wavelengths can be larger or smaller than the size of observable universe one must describe both the generation and the evolution of perturbations in the context of general relativity. We first review the relativistic linear perturbation theory formalism. We discuss the gauge issue and describe several popular gauge choices. We then derive Einstein's and continuity equations for these gauge choices. We continue by developing fluid equations for cold dark matter and baryons and Boltzmann equations for photons, massive and massless neutrinos. These equations describe a complete set which is sufficient to solve the perturbations starting from their initial conditions, as long as the evolution is linear (i.e., the perturbations are small).

Next we discuss the generation of initial perturbations by the process of inflation and the parameters of that process that can be observationally accessible. Finally we discuss evolution of perturbations in various regimes and the imprint of the evolution on the dark matter power spectrum both in the linear and in the nonlinear regime.

2 Relativistic perturbation theory

The goal of studying cosmological perturbations is to understand the evolution of the structure in the universe. Relativistic perturbation theory describes this evolution in a general relativistic context. In cosmology such an approach is required, because we want to describe perturbations not just on small scales where Newtonian laws suffice, but also on scales comparable

[1] http://www.mso.anu.edu.au/2dFGRS/
[2] http://www.astro.princeton.edu/BBOOK

or larger to the Hubble length, which can be used as a typical scale of observable universe. Before we introduce the perturbations we must however describe the unperturbed (homogeneous) evolution of the universe, to which we turn next.

2.1 Homogeneous universe

The fundamental assumption of cosmology is that the universe is homogeneous and isotropic on average. To be more precise, our paradigm is that we live in a perturbed Robertson-Walker universe, in which the metric perturbations are small, so the averaging procedure is well defined and the backreaction of the metric fluctuations on the homogeneous equations is negligible. One can write the line element in a homogeneous and isotropic universe using conformal time τ and comoving coordinates x^i as

$$ds^2 = \gamma_{\mu\nu} dx^\mu dx^\nu = a^2(\tau)\left\{-d\tau^2 + \gamma_{ij} dx^i dx^j\right\}. \tag{1}$$

This is the Robertson-Walker metric. We will often use greek indices to denote 4-tensors and latin to denote spatial 3-tensors. Here $a(\tau)$ is the scale factor expressed in terms of conformal time τ, which is related to the proper time t via $dt = ad\tau$. Similarly, proper coordinates r_i are related to the comoving coordinates x_i via $r_i = ax_i$. We adopt units such that $c = 1$. The space part of the background metric can be written as

$$\gamma_{ij} dx^i dx^j = d\chi^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

$$r = \sin_K \chi \equiv \begin{cases} K^{-1/2} \sin K^{1/2}\chi, \ K > 0 \\ \chi, \ K = 0 \\ (-K)^{-1/2} \sinh(-K)^{1/2}\chi, \ K < 0 \end{cases} \tag{2}$$

where K is the curvature term which can be expressed using the present density parameter in all components Ω_0 and the present Hubble parameter H_0 as $K = (\Omega_0 - 1)H_0^2$. The density parameter Ω_0 can have contributions from mass density Ω_m which can consist of baryons, cold dark matter (CDM), massive neutrinos (mixed dark matter MDM, warm dark matter WDM etc.), cosmological constant Ω_λ or some more general dark energy (or quintessence) Ω_ϕ, $\Omega_0 = \Omega_m + \Omega_\lambda + \Omega_\phi$. The advantage of using the conformal time τ is that the metric becomes conformally Euclidean ($K = 0$), 3-sphere ($K > 0$) or 3-hyperboloid ($K < 0$) and leads to a simple geometrical description of light propagation and other processes.

Einstein's equation is

$$G^{\nu}_{\mu} = 8\pi G T^{\nu}_{\mu}, \tag{3}$$

where G^{ν}_{μ} is the Einstein tensor and $T^{\mu}{}_{\nu}$ is the stress-energy tensor. Einstein's tensor is related to the spacetime Ricci tensor $R_{\mu\nu}$ by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} \ , \quad R \equiv R^{\mu}{}_{\mu} \ , \quad R_{\mu\nu} \equiv R^{\kappa}{}_{\mu\kappa\nu} \ . \tag{4}$$

The spacetime Riemann tensor is defined according to the convention

$$R^{\mu}{}_{\nu\kappa\lambda} \equiv \partial_{\kappa}\Gamma^{\mu}{}_{\nu\lambda} - \partial_{\lambda}\Gamma^{\mu}{}_{\nu\kappa} + \Gamma^{\mu}{}_{\alpha\kappa}\Gamma^{\alpha}{}_{\nu\lambda} - \Gamma^{\mu}{}_{\alpha\lambda}\Gamma^{\alpha}{}_{\nu\kappa} \ , \tag{5}$$

where the affine connection coefficients are

$$\Gamma^{\mu}{}_{\nu\lambda} \equiv \frac{1}{2} g^{\mu\kappa} \left(\partial_{\nu} g_{\kappa\lambda} + \partial_{\lambda} g_{\kappa\nu} - \partial_{\kappa} g_{\nu\lambda}\right) \ . \tag{6}$$

The Einstein field equations 3 show that the stress-energy tensor provides the source for the metric variables. The stress-energy tensor takes the well-known form

$$T^{\mu\nu} = (\rho + p) u^{\mu} u^{\nu} + p g^{\mu\nu} + p \Pi^{\mu\nu} \ , \tag{7}$$

where ρ and p are the energy density and pressure, $u^{\mu} = dx^{\mu}/d\lambda$ (where $d\lambda^2 \equiv -ds^2$) is the fluid 4-velocity and $p\Pi^{\mu\nu}$ is the shear stress absent for a perfect fluid. In locally flat coordinates in the fluid frame, $T^{00} = \rho$, $T^{0i} = 0$, and $T^{ij} = p\delta^{ij}$ for a perfect fluid.

Einstein's equations applied to the background metric gives the evolution of the expansion factor $a(\tau)$,

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv \eta^2 = \frac{8\pi}{3} G\bar{\rho} a^2 - K, \quad \dot{\eta} = -\frac{4\pi}{3} G a^2 (\bar{\rho} + 3\bar{p}). \tag{8}$$

Overdots denote derivatives with respect to the conformal time τ. For convenience we introduced comoving Hubble parameter $\eta = \dot{a}/a$, which will appear often in the equations below. Its value today is H_0. These are the Friedmann equations applied to the Robertson-Walker metric. The density $\bar{\rho}$ is related to the density parameter Ω_0 via $8\pi G\bar{\rho}_0/3 = H_0^2\Omega_0$.

The mean density of the universe $\bar{\rho}$ (and similarly the mean pressure $\bar{p}$) can be written as a sum of matter, radiation, cosmological constant or any other dark energy contributions,

$$\bar{\rho} = \bar{\rho}_m a^{-3} + \bar{\rho}_r a^{-4} + \bar{\rho}_{\lambda} + \bar{\rho}_{\phi} a^{-3(1+w)}, \tag{9}$$

where w is the equation of state of the dark energy.

The energy-momentum tensor is required to obey local conservation law $T^{\mu\nu}_{;\mu} = 0$, which gives

$$\dot{\rho} + 3\eta(\rho + p) = 0. \tag{10}$$

One can also derive the evolution equation for a homogeneous scalar field ϕ evolving in a potential $V(\phi)$. Its Lagrangian is

$$L = \sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\phi + V(\phi)\right], \tag{11}$$

where $g^{\mu\nu}$ is the metric. Stress energy tensor in this case is

$$\rho_\phi = \frac{\dot{\phi}^2}{2a^2} + V(\phi) \qquad p_\phi = \frac{\dot{\phi}^2}{2a^2} - V(\phi). \tag{12}$$

Equation of state $w = p/\rho$ is in general a function of time. Continuity equation (10) gives

$$\ddot{\phi} + 2\eta\dot{\phi} + a^2V' = 0. \tag{13}$$

The scalar field source has to be added to the Friedmann equations above and modifies the expansion of the universe. It obeys the same energy-momentum conservation as the other fluids and can be easily integrated to find energy density as a function of time (or expansion factor as in equation 9). In the limit where kinetic term is negligible compared to the potential scalar field reduces to the cosmological constant with $w = -1$. This case is relevant both for inflation and for the possible late time contribution from the dark energy.

2.2 Perturbations in the metric and in the energy-momentum tensor

Our universe is not homogeneous: we see inhomogeneities caused by gravity present on all scales, from planets to clusters, superclusters and beyond. We want to describe the deviations from the isotropy and homogeneity of the universe using general relativity. Small perturbations $h_{\mu\nu}$ around the Robertson-Walker metric are

$$g_{\mu\nu} = a^2(\gamma_{\mu\nu} + h_{\mu\nu}). \tag{14}$$

In the most general form one can write the perturbed line element using conformal time τ and comoving coordinates x^i as

$$ds^2 = a^2(\tau)\left\{-(1+2A)d\tau^2 - 2B_i d\tau dx^i + [(1+2H_L)\gamma_{ij} + 2h_{ij}]\,dx^i dx^j\right\}. \tag{15}$$

The perturbations have been decomposed into time-time component $-2A$, time-space component $-2B_i$, trace of the space-space component $2H_L$ and traceless space-space component $2h_{ij}$. Vector field B_i can be further decomposed into a scalar component, which arises from a gradient of a scalar field, and a pure vector component, which is the remainder of what is left. Similarly we can decompose tensor h_{ij} into a scalar, vector and tensor components. As we show below these perturbations can be decomposed into scalar ($m = 0$, compressional), vector ($m = \pm 1$, vortical) and tensor ($m = \pm 2$, gravitational wave) eigenmode components, which differ in their transformation properties under spatial rotations. The advantage of this decomposition is that the linearized equations decouple into separate scalar, vector and tensor components, with no cross-coupling between them.

In linear theory, each eigenmode of the Laplacian for the perturbation evolves independently, and so it is useful to decompose the perturbations via the eigentensor $\mathbf{Q}^{(m)}$, where

$$\nabla^2 \mathbf{Q}^{(m)} \equiv \gamma^{ij} \mathbf{Q}^{(m)}_{|ij} = -k^2 \mathbf{Q}^{(m)}, \tag{16}$$

with "|" representing covariant differentiation with respect to the three metric γ_{ij}. Note that the eigentensor $\mathbf{Q}^{(m)}$ has $|m|$ indices (suppressed in the above). To obtain a pure vector component we must subtract out a component of a vector field that can be obtained from a scalar field. In real space this is a gradient of a scalar field. This mean that vector modes satisfy the auxiliary condition

$$Q_i^{(\pm 1)|i} = 0 \tag{17}$$

which represents the divergenceless condition for vorticity waves.

Similarly, to obtain a pure tensor mode we must subtract out components that can be formed from a scalar and vector field. The auxiliary conditions are

$$\gamma^{ij} Q_{ij}^{(\pm 2)} = Q_{ij}^{(\pm 2)|i} = 0\,, \tag{18}$$

which represent the divergenceless and transverse-traceless conditions respectively, as appropriate for gravity waves.

We will often focus on perturbations in flat space, both because they lead to simplified expressions and because they seem to be observationally favored. In this case the eigenmodes are particularly simple. If we assume the direction of the wavevector $\vec{k}$ in $\hat{e}_3$ then

$$Q^{(\pm m)}_{i_1 \ldots i_m} \propto (\hat{e}_1 \pm i\hat{e}_2)_{i_1} \ldots (\hat{e}_1 \pm i\hat{e}_2)_{i_m} \exp(i\vec{k}\cdot\vec{x})\,, \qquad (K = 0, m \geq 0)\,, \tag{19}$$

where the presence of $\hat{e}_i$, which forms a local orthonormal basis with $\hat{e}_3 = \hat{k}$, ensures the divergenceless and transverse-traceless conditions. One can see now the transformation properties of eigenmodes under rotation in the plane perpendicular to $\hat{e}_3$ is such that

$$\tilde{Q}^{(\pm m)} = Q^{(\pm m)} e^{\mp i m \psi}, \tag{20}$$

where ψ is the rotation angle.

It is also useful to construct (auxiliary) vector and tensor objects out of the fundamental scalar and vector modes through covariant differentiation

$$Q_i^{(0)} = -k^{-1} Q_{|i}^{(0)}, \qquad Q_{ij}^{(0)} = k^{-2} Q_{|ij}^{(0)} - \frac{1}{3}\gamma_{ij} Q^{(0)}, \tag{21}$$

$$Q_{ij}^{(\pm 1)} = -(2k)^{-1}(Q_{i|j}^{(\pm 1)} + Q_{j|i}^{(\pm 1)}). \tag{22}$$

For $K = 0$ this becomes

$$Q_i^{(0)} = -ik^{-1} k_i Q^{(0)}, \qquad Q_{ij}^{(0)} = -k^{-2} k_i k_j Q^{(0)} - \frac{1}{3}\gamma_{ij} Q^{(0)}, \tag{23}$$

$$Q_{ij}^{(\pm 1)} = -i(2k)^{-1}(k_j Q_i^{(\pm 1)} + k_i Q_j^{(\pm 1)}). \tag{24}$$

The eigenmodes form a complete set, so that any perturbation can be expanded in terms of these.

The metric perturbations can be broken up into the normal modes of scalar ($m = 0$), vector ($m = \pm 1$) and tensor ($m = \pm 2$) type,

$$\begin{aligned} A &= A^{(0)} Q^{(0)}, H_L = H_L^{(0)} Q^{(0)} \\ B_i &= -\sum_{m=-1}^{1} B^{(m)} Q_i^{(m)}, \\ h_{ij} &= \sum_{m=-2}^{2} H_T^{(m)} Q_{ij}^{(m)}. \end{aligned} \tag{25}$$

The stress energy tensor can likewise be broken up into scalar, vector, and tensor contributions. The fluctuations can be decomposed into the normal modes as

$$\begin{aligned} \delta T^0_{\ 0} &= -\delta\rho^{(0)}\, Q^{(0)}, \\ \delta T^0_{\ i} &= \textstyle\sum_{m=-1}^{1} [(\rho + p)(v^{(m)} - B^{(m)})]\, Q_i^{(m)}, \\ \delta T_0^{\ i} &= \textstyle\sum_{m=-1}^{1} [(\rho + p) v^{(m)}\, Q^{(m)i}, \\ \delta T^i_{\ j} &= \delta p^{(0)} \delta^i_{\ j} Q^{(0)} + \textstyle\sum_{m=-2}^{2} p \Pi^{(m)} Q^{(m)i}_{\ \ \ \ j}. \end{aligned} \tag{26}$$

Note that the two mixed time-space components are not equal. We introduced above the density perturbation $\delta\rho^{(0)}$, pressure perturbation $\delta p^{(0)}$, velocity perturbation $v^{(m)}$, where $m = 0, \pm 1$ and anistropic stress perturbation $p\Pi^{(m)}$, which can be scalar ($m = 0$), vector ($m = \pm 1$) or tensor ($m = \pm 2$) type. These are in general a sum from all of the species present,

$$\delta\rho^{(0)} = \sum_i \delta\rho_i^{(0)}, \tag{27}$$

where the index i stands for baryons, CDM, photons, neutrinos (massive and massless), dark energy etc.

A minimally coupled scalar field φ also has perturbations, $\varphi = \phi + \delta\phi$, which are related to the fluid quantities as [11, 12]

$$\begin{aligned}
\delta\rho_\phi^{(0)} &= a^{-2}(\dot{\phi}\dot{\delta\phi}^{(0)} - A^{(0)}\dot{\phi}^2) + V'\delta\phi^{(0)}\,, \\
\delta p_\phi^{(0)} &= a^{-2}(\dot{\phi}\dot{\delta\phi}^{(0)} - A^{(0)}\dot{\phi}^2) - V'\delta\phi^{(0)}\,, \\
(\rho_\phi + p_\phi)(v_\phi^{(0)} - B^{(0)}) &= a^{-2}k\dot{\phi}\delta\phi\,, \\
p_\phi\Pi_\phi^{(0)} &= 0\,,
\end{aligned} \tag{28}$$

where V' denotes derivative with respect to ϕ. This shows that there are no vector or tensor modes associated with the scalar field, as expected in the linear order. Applying energy-momentum conservation one finds

$$\delta\ddot{\phi} + 2\eta\delta\dot{\phi} + (k^2 + a^2V'')\delta\phi = (\dot{A}^{(0)} - 3\dot{H_L}^{(0)} - kB^{(0)})\dot{\phi} - 2a^2V'A^{(0)}. \tag{29}$$

2.3 Perturbed Einstein's equations

The general covariant form of linearized perturbed Einstein's equations follows from the definitions above. The algebra is straightforward (although lengthy) and leads to the following set of equations

2.3.1 Scalar perturbations

$$\begin{aligned}
&(k^2 - 3K)\left[H_L + \tfrac{1}{3}H_T + \eta k^{-2}(kB - \dot{H}_T)\right] = 4\pi Ga^2\left[\delta\rho + 3\eta(\bar{\rho} + \bar{p})(v - B)/k\right] \\
&\eta A - \dot{H}_L - \tfrac{1}{3}\dot{H}_T - \tfrac{K}{k^2}(kB - \dot{H}_T) = 4\pi Ga^2(\bar{\rho} + \bar{p})(v - B)/k \\
&\left(2\tfrac{\ddot{a}}{a} - 2\eta^2 + \eta\tfrac{\partial}{\partial\tau} - \tfrac{1}{3}k^2\right)A - \left(\tfrac{\partial}{\partial\tau} + \eta\right)(\dot{H}_L + \tfrac{k}{3}B) = 4\pi Ga^2(\delta p + \tfrac{1}{3}\delta\rho) \\
&k^2(A + H_L + \tfrac{1}{3}H_T) + \left(\tfrac{\partial}{\partial\tau} + 2\eta\right)\left(kB - \dot{H}_T\right) = 8\pi Ga^2 p\Pi.
\end{aligned} \tag{30}$$

These are density (Poisson), momentum, pressure and anisotropic stress equations, respectively. The superscripts (0) have been dropped.

Corresponding energy-momentum tensor conservation equations are continuity and Euler's equations, which in the general covariant form are

$$\begin{aligned}&\left(\tfrac{\partial}{\partial\tau}+3\eta\right)\delta\rho+3\eta\delta p=-(\bar\rho+\bar p)(kv+3\dot H_L)\\&\left(\tfrac{\partial}{\partial\tau}+4\eta\right)[\bar\rho+\bar p)(v-B)/k]=\delta p-\tfrac{2}{3}(1-3\tfrac{K}{k^2})p\Pi+(\bar\rho+\bar p)A.\end{aligned}\tag{31}$$

Energy-momentum tensor conservation equations are not independent of Einstein's equations, since they follow from Bianchi identities. They are nevertheless useful, since they involve only first derivatives instead of second as in the case of Einstein's equations. In total we have 4 equations for 4 metric perturbations, which are sourced by 4 components of energy-momentum tensor. As we will see below the number of equations can be further reduced by the gauge freedom.

2.3.2 Vector perturbations

Since density and pressure are scalar quantities there are only two Einstein's equations for vectors, momentum and anisotropic stress,

$$\begin{aligned}&\left(1-\tfrac{2K}{k^2}\right)(kB-\dot H_T)=16\pi Ga^2(\bar\rho+\bar p)(v-B)/k\\&\left(\tfrac{\partial}{\partial\tau}+2\eta\right)(kB-\dot H_T)=-8\pi Ga^2p\Pi.\end{aligned}\tag{32}$$

Energy-momentum conservation consists of Euler's equation only,

$$2\left(\frac{\partial}{\partial\tau}+4\eta\right)[(\bar\rho+\bar p)(v-B)/k]=\left(\frac{2K}{k^2}-1\right)p\Pi\tag{33}$$

and we suppressed ±1 superscript on all the variables. Here again we have 2 Einstein's equations for 2 metric variables, but fixing the gauge freedom reduces this to a single equation.

2.3.3 Tensor perturbations

There is only one equation in this case, corresponding to the tensor part of the anisotropic spatial component in Einstein's equations,

$$\ddot H_T+2\eta\dot H_T+(k^2+2K)H_T=4\pi Ga^2p\Pi,\tag{34}$$

where again superscript ±2 was suppressed. There is no gauge freedom in the tensor case, as we discuss next. Equation above is a wave equation and describes gravity wave propagation and sourcing in an expanding universe.

2.4 Gauge transformations

So far we have described the perturbations in a given coordinate system, but we did not say much about the coordinate system itself. In the absence of perturbations there is a preferred coordinate system, which corresponds to comoving frame, in which the observers see the momentum density to be zero at that point and the comoving observers are free falling. The spatial slices (defined as slices of constant time) are orthogonal to the time threading (defined as a constant coordinate $\vec{x}$) and for $K=0$ they are spatially flat. For $K<0$ they correspond to a 3-hyperboloid and for $K>0$ to a 3-sphere. This defines the background space-time.

The choice of coordinates becomes nontrivial once we discuss the perturbations and there is no unique choice. One can, for example, choose the coordinates in which the observers are free-falling (corresponding to the synchronous gauge), but this is no longer the same choice as if the observers are comoving so that the momentum density vanishes (corresponding to the comoving gauge). One can instead choose a spatially flat gauge so that spatial components of the metric vanish, or a gauge with zero metric shear with threading and slicing being orthogonal (corresponding to conformal Newtonian or longitudinal gauge).

To represent the perturbations we must thus make a gauge choice. A gauge transformation is a change from one coordinate choice to another. The most general form is $\tilde{x}_\mu = x_\mu + \delta x_\mu$ or

$$\begin{aligned} \tilde{\tau} &= \tau + T, \\ \tilde{x}_i &= x_i + L_i. \end{aligned} \tag{35}$$

T corresponds to a choice in time slicing and L_i a choice of spatial coordinates. This can be decomposed into Fourier modes

$$\begin{aligned} \tilde{\tau} &= \tau + T^{(0)} Q^{(0)}, \\ \tilde{x}_i &= x_i + \textstyle\sum_{m=-1}^{1} L^{(m)} Q_i^{(m)}. \end{aligned} \tag{36}$$

Since gauge freedom only uses scalar ($m=0$) and vector quantities ($m=\pm 1$), it is clear that tensor modes ($m=\pm 2$) will not change under gauge transformation. Even though the coordinates can change the metric distance ds^2 must remain invariant, $g_{\mu\nu} dx^\mu dx^\nu = \tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu$. Since $g_{\mu\nu}$ is a tensor and transforms in the same way as other tensors we can derive the transformation property of a general tensor

$$\tilde{T}_{\mu\nu}(\tilde{x}_\gamma) = \frac{\partial x^\alpha}{\partial \tilde{x}_\mu} \frac{\partial x^\beta}{\partial \tilde{x}_\nu} T_{\alpha\beta}(\tilde{x}_\gamma - \delta x_\gamma)$$

$$= T_{\mu\nu} \quad - \quad T_{\mu\beta}\partial_\nu \delta x^\beta - T_{\alpha\nu}\partial_\mu \delta x^\alpha - \delta x^\alpha \partial_\alpha T_{\mu\nu}. \tag{37}$$

This transformation law applied to the metric in equation 15 gives [11, 12]

$$\begin{aligned}
\tilde{A}^{(0)} &= A^{(0)} - \dot{T}^{(0)} - \eta T^{(0)}\,, \\
\tilde{B}^{(m)} &= B^{(m)} + \dot{L}^{(m)} + kT^{(m)}\,, \\
\tilde{H}_L^{(0)} &= H_L^{(0)} - \frac{k}{3}L^{(0)} - \eta T^{(0)}\,, \\
\tilde{H}_T^{(m)} &= H_T^{(m)} + kL^{(m)}\,,
\end{aligned} \tag{38}$$

where $m = 0, \pm 1$.

The stress-energy perturbations in different gauges are similarly related by the gauge transformations

$$\begin{aligned}
\tilde{\delta\rho}^{(0)} &= \delta\rho^{(0)} - \dot{\bar{\rho}} T^{(0)}\,, \\
\tilde{\delta p}^{(0)} &= \delta p^{(0)} - \dot{\bar{p}} T^{(0)}\,, \\
\tilde{v}^{(m)} &= v^{(m)} + \dot{L}^{(m)}\,, \\
\tilde{\Pi}^{(m)} &= \Pi^{(m)}\,,
\end{aligned} \tag{39}$$

where $m = 0, \pm 1$ in the velocity equation and $m = 0, \pm 1, \pm 2$ in anisotropic stress equation. The anisotropic stress is gauge-invariant.

A scalar field transforms as

$$\tilde{\delta\phi}^{(0)} = \delta\phi^{(0)} - \dot{\phi} T^{(0)}\,. \tag{40}$$

2.5 Popular gauge choices

The choice of gauge can be governed by the simplicity of equations, numerical stability of solutions, Newtonian intuition or other considerations. As we discussed above there is no gauge ambiguity for tensor modes. For vector modes the choice can either be $B^{(\pm 1)} = 0$ or $H_T^{(\pm 1)} = 0$. The latter specifies the gauge completely, since it fixes $L^{(\pm 1)}$, while the former only fixes $\dot{L}^{(\pm 1)}$ and thus leads to unspecified integration constant in $H_T^{(\pm 1)}$. This however does not lead to any dynamical effects. Since vector modes are unlikely to be generated in the early universe we will not discuss them further. Instead we will look at four popular gauge choices for scalar modes that we will have the chance to use in the rest of the lectures. We will drop the superscripts (0) from now on.

2.5.1 Synchronous gauge

This gauge was very popular in the early development of perturbation theory. Its main advantage from today's perspective is its numerical stability, which is why it is still the gauge of choice in CMBFAST [13] numerical package. It corresponds to setting $A = B = 0$, so that only the spatial metric is perturbed. This implies that slicing is orthogonal to the threading and that a set of freely falling observers remains at a fixed coordinate position. To show this one must show that the spatial part of 4-velocity $u^\mu = dx^\mu/d\lambda$, where λ is affine parameter parametrizing the geodesic, vanishes. Geodesic equation is

$$\frac{Du^\mu}{D\lambda} = u^\mu_{;\nu}\frac{dx^\nu}{d\lambda} = \frac{du^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = 0. \tag{41}$$

Since $A = B^{(m)} = 0$ implies $\Gamma^i_{00} = 0$ it follows that $u^i = 0$ is a geodesic.

The property that the fundamental observers follow geodesics means that the coordinates are Lagrangian and this gauge can only be used while $\delta\rho/\bar{\rho} \ll 1$. In the nonlinear regime where this condition is not satisfied one can have orbit crossings where two observers with different Lagrangian coordinates find themselves at the same real (Eulerian space) position. This can only happen if the metric perturbations diverge and the linear perturbation theory is no longer valid. While this limits the use of this gauge at late times it can still be used successfully in the early universe, as long as the density perturbations are small.

Another shortcoming of this gauge is that the gauge choice does not fully specify it. One can see this by using a gauge transformation from a general gauge. Imposing $\tilde{A} = \tilde{B} = 0$ to the equations 38 we find

$$\begin{aligned} T &= a^{-1}\int aAd\tau + c_1 a^{-1} \\ L &= -\int (B + kT)d\tau + c_2, \end{aligned} \tag{42}$$

where c_1 and c_2 are integration constants. These remain unspecified in this gauge. They lead to unphysical gauge mode solutions for the density perturbations outside horizon. While historically this caused some confusion in their interpretation they are not really a problem since these modes do not show up in any observable quantity and are at any rate decaying faster than the physical modes.

The remaining two scalar variables in this gauge are H_L and H_T. Instead of these one often introduces $\eta \equiv -H_L - H_T/3$ and $h \equiv 6H_L$, but we will not make this replacement here to avoid confusion with comoving Hubble parameter $\eta = \dot{a}/a$. Einstein's equations are

$$\begin{aligned} -(k^2-3K)(H_L+\frac{1}{3}H_T)+3\eta\dot{H}_L &= 4\pi G a^2\delta\rho\,, \\ \dot{H}_L+\frac{1}{3}(1-3K/k^2)\dot{H}_T &= -4\pi G a^2(\bar{\rho}+\bar{p})v/k\,, \\ \ddot{H}_L+\eta\dot{H}_L &= -4\pi G a^2[\frac{1}{3}\delta\rho+\delta p]\,, \\ \ddot{H}_T+\eta\dot{H}_T-k^2(H_L+\frac{1}{3}H_T) &= -8\pi G a^2 p\Pi\,. \end{aligned} \tag{43}$$

Two of these are of course redundant. The conservation equations are

$$\begin{aligned} &\left(\frac{\partial}{\partial\tau}+3\eta\right)\delta\rho+3\eta\delta p=-(\bar{\rho}+\bar{p})(kv+3\dot{H}_L) \\ &\left(\frac{\partial}{\partial\tau}+4\eta\right)([\bar{\rho}+\bar{p})v/k]=\delta p-\tfrac{2}{3}(1-3\tfrac{K}{k^2})p\Pi. \end{aligned} \tag{44}$$

2.5.2 Newtonian gauge

In Newtonian gauge one sets $B = H_T = 0$. The remaining two scalar perturbations are renamed into $A \equiv \Psi$ and $H_L \equiv -\Phi$. A general gauge transformation into Newtonian gauge gives

$$\begin{aligned} \tilde{H}_T &= 0 \to L=-H_T/k \\ \tilde{B} &= 0 \to T=-B/k+\dot{H}_T/k^2. \end{aligned} \tag{45}$$

One can see that there is no remaining gauge freedom, so the gauge is entirely fixed. The main advantage of this gauge is that there is a simple Newtonian correspondence and the equations reduce to Newtonian laws in the limit of small scales. The Einstein's equations are,

$$\begin{aligned} (-k^2+3K)\Phi-3\eta\left(\dot{\Phi}+\eta\Psi\right) &= 4\pi G a^2\delta\rho \\ \dot{\Phi}+\eta\Psi &= 4\pi G a^2\left[(\bar{\rho}+\bar{p})v/k\right] \\ \ddot{\Phi}-K\Phi+\eta(\dot{\Psi}+2\dot{\Phi})+(2\dot{\eta}+\eta^2)\Psi-\tfrac{1}{3}k^2(\Phi-\Psi) &= 4\pi G a^2\delta p\,, \\ k^2(\Phi-\Psi) &= 8\pi G a^2 p\Pi. \end{aligned} \tag{46}$$

On small scales the second term on the left hand side of first equation above becomes negligible compared to the first one. Similarly we can also neglect K relative to k^2, since curvature scale, if present, is of the order of the Hubble length. The result is a Poisson equation in an expanding universe. This means that we can identify metric perturbation Φ with the perturbed Newtonian potential on small scales. From the last of equations above we find $\Phi = \Psi$ in the absence of anisotropic stress, which is a good approximation in the matter era where ideal fluids dominate the energy density. We know that for astrophysical sources gravitational potential is roughly $\Phi \sim v^2$ (in units where $c = 1$), where v is a typical velocity (rotation, dispersion etc.) of the object. This implies $\Phi \sim \Psi \ll 1$ almost everywhere in the universe, except near a black hole. So in this gauge the linear perturbation theory is almost always valid and one can use these equations also to describe the late time nonlinear evolution in the density field, as long as the gravitational potential remains small.

The conservation equations are

$$\begin{aligned} &\left(\frac{\partial}{\partial\tau} + 3\eta\right)\delta\rho + 3\eta\delta p = -(\bar\rho + \bar p)(kv - 3\dot\Phi) \\ &\left(\frac{\partial}{\partial\tau} + 4\eta\right)([\bar\rho + \bar p)v/k] = \delta p - \tfrac{2}{3}(1 - 3\tfrac{K}{k^2})p\Pi + (\bar\rho + \bar p)\Psi. \end{aligned} \tag{47}$$

2.5.3 Comoving gauge

This gauge is convenient because it introduces the curvature perturbation, which is useful when one wants to describe the evolution of perturbations, generated by say inflation, outside the horizon. It turns out that this quantity is conserved for adiabatic perturbations (see more on this below) and so evolution is particularly simple in this case. On the other hand, this gauge is not particularly intuitive inside the horizon, so one is better off to transform into the Newtonian gauge in this limit.

The gauge is defined so that the momentum density T^0_i vanishes. From equation (26) this implies $B = v$. The second constraint can be set to $H_T = 0$. The remaining two scalar perturbations are renamed as $A \equiv \xi$ and $H_L \equiv \zeta$. Gauge transformation from a general gauge into comoving gauge gives

$$\begin{aligned} \tilde v - \tilde B &= 0 \to T = (v - B)/k \\ \tilde H_T &= 0 \to L = -H_T/k. \end{aligned} \tag{48}$$

The gauge is entirely fixed, since these are just algebraic relations between the quantities.

Einstein's equations are

$$\begin{aligned}
(k^2-3K)\,(\zeta+\eta v/k) &= 4\pi G a^2 \delta\rho \\
\eta\xi - \dot\zeta - \tfrac{K}{k}v &= 0 \\
\left(2\tfrac{\ddot a}{a} - 2\eta^2 + \eta\tfrac{\partial}{\partial\tau} - \tfrac{1}{3}k^2\right)\xi - \left(\tfrac{\partial}{\partial\tau}+\eta\right)(\dot\zeta + \tfrac{k}{3}v) &= 4\pi G a^2(\delta p + \tfrac{1}{3}\delta\rho) \\
k^2(\xi+\zeta) + \left([\tfrac{\partial}{\partial\tau} + 2\eta\right) kv &= 8\pi G a^2 p\Pi.
\end{aligned} \tag{49}$$

Corresponding energy-momentum tensor conservation equations are

$$\begin{aligned}
&\left(\tfrac{\partial}{\partial\tau} + 3\eta\right)\delta\rho + 3\eta\delta p = -(\bar\rho+\bar p)(kv+3\dot\zeta) \\
&(\bar\rho+\bar p)\xi = -\delta p + \tfrac{2}{3}(1-3\tfrac{K}{k^2})p\Pi.
\end{aligned} \tag{50}$$

2.5.4 Spatially flat gauge

While comoving gauge is a useful gauge to describe evolution of perturbations after they cross the horizon, it has the shortcoming that the scalar field perturbations vanish in this gauge. It thus cannot be used to calculate scalar fluctuations from inflation, for example. One way to solve this is to calculate them in another gauge and use the gauge transformation to calculate curvature perturbation in comoving gauge. The simplest gauge to choose is the spatially flat gauge, where $H_L = H_T = 0$,

$$\begin{aligned}
&(1-3K/k^2)\eta k B = 4\pi G a^2\left[\delta\rho + 3\eta(\rho+p)(v-B)/k\right] \\
&\eta A - \tfrac{K}{k}B = 4\pi G a^2(\bar\rho+\bar p)(v-B)/k \\
&\left(2\tfrac{\ddot a}{a} - 2\eta^2 + \eta\tfrac{\partial}{\partial\tau} - \tfrac{1}{3}k^2\right)A - \left(\tfrac{\partial}{\partial\tau}+\eta\right)\tfrac{k}{3}B = 4\pi G a^2(\delta p + \tfrac{1}{3}\delta\rho) \\
&k^2 A + \left([\tfrac{\partial}{\partial\tau} + 2\eta\right) kB = 8\pi G a^2 p\Pi.
\end{aligned} \tag{51}$$

Corresponding energy-momentum tensor conservation equations are

$$\begin{aligned}
&\left(\tfrac{\partial}{\partial\tau} + 3\eta\right)\delta\rho + 3\eta\delta p = -(\bar\rho+\bar p)kv \\
&\left(\tfrac{\partial}{\partial\tau} + 4\eta\right)([\bar\rho+\bar p)(v-B)/k] = \delta p - \tfrac{2}{3}(1-3\tfrac{K}{k^2})p\Pi + (\bar\rho+\bar p)A.
\end{aligned} \tag{52}$$

Scalar field equation is

$$\ddot{\delta\phi} + 2\eta\dot{\delta\phi} + (k^2 + a^2V'')\delta\phi = (\dot A - kB)\dot\phi - 2a^2V'A. \tag{53}$$

3 Evolution of perturbations

In previous section we have described several gauge choices encountered in the literature and presented Einstein's equations, which relate metric perturbations to the energy-momentum tensor, which receives contributions from radiation and matter. In this section we derive evolution equations for all the ingredients that contribute to the energy-momentum tensor. Together with the Einstein's equations they form a closed set of equations that can be evolved forward in time from some given initial conditions. We will restrict ourselves to the Newtonian gauge, since the equations are easiest to interpret and the solutions do not suffer from gauge modes or breakdown in the nonlinear regime.

3.1 Fluid equations for scalar perturbations

Matter content of the universe can be divided into two classes. In the first class are matter components, which can be described within the fluid approximation. This class includes cold dark matter, baryons and scalar fields. To specify their evolution one only needs the equations for overdensity $\delta = \delta\rho/\bar{\rho}$ and velocity v, both of which can be obtained from the energy-momentum conservation equations (47). In this section we present the system of evolution equations for the matter. We restrict the application to the scalar fluctuations, given that only these can lead to growth of perturbations by self-gravity and thus to structure formation in the universe.

From equations (47) one obtains the following set of equations,

Cold dark matter —

$$\dot{\delta}_c = -kv_c + 3\dot{\Phi}\,, \quad \dot{v}_c = -\eta\, v_c + k\Psi\,. \tag{54}$$

By assumption cold dark matter (CDM) is cold and its pressure and anisotropic stress are zero.

For baryons one must also include pressure and Thomson scattering between photons and electrons. Thomson scattering has a well specified angular dependence in the rest frame of the electron and rapid scattering leads to isotropic photon distribution in this frame. The change in photon velocity is proportional to the difference between photon and baryon velocities times the scattering probability and so leads to the exchange of momentum between the photons and baryons (see below). Momentum conservation requires an opposite term in the baryon momentum conservation equation, *Baryons—*

$$\begin{aligned}\dot{\delta}_b &= -kv_b + 3\dot{\Phi}\,, \\ \dot{v}_b &= -\eta v_b + c_s^2 k\delta_b + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} a n_e x_e \sigma_T (v_\gamma - v_b) + k\Psi\,,\end{aligned} \tag{55}$$

where n_e is the electron density, x_e the ionization fraction and σ_T Thomson cross section. We also included the pressure term, relating it to density gradients via adiabatic sound speed $c_s^2 = (\partial p/\partial\rho)_S$ and neglecting entropy gradients.

The evolution equation for scalar field perturbations in Newtonian gauge follows from equation (29)

Scalar field—

$$\delta\ddot{\phi} + 2\eta\delta\dot{\phi} + [k^2 + a^2V'']\delta\phi = (\dot{\Psi} + 3\dot{\Phi})\dot{\phi} - 2a^2V'\Psi. \tag{56}$$

We will show below that for $w = -1$ both $\dot{\phi}$ and V' vanish. In this limit we have no gravitational source for $\delta\phi$ and there are no perturbations in the scalar field assuming none existed initially. Cosmological constant and scalar field with $w = -1$ are thus indistinguishable. To solve for perturbations we must specify $\dot{\phi}$, V' and V''. It is often more convenient to express these in terms of mean scalar field density at present Ω_ϕ and equation of state w as a function of time.

3.2 Boltzmann equation

In the second class are components for which the full phase space distribution function $f(\vec{x},\vec{p},\tau)$ is required. This class includes neutrinos (both massless and massive) and photons. The distribution function describes phase space density,

$$dN = f(\vec{x},\vec{p},\tau)d^3\vec{x}d^3\vec{p}, \tag{57}$$

where dN is number of particles inside this volume.

The evolution of distribution function is governed by the Boltzmann equation

$$\frac{df(\vec{x},\vec{p},\tau)}{d\tau} = \frac{\partial f}{\partial\tau} + \frac{\partial f}{\partial x^i}\frac{dx^i}{d\tau} + \frac{\partial f}{\partial p}\frac{dp}{d\tau} + \frac{\partial f}{\partial n^i}\frac{dn^i}{d\tau} = \left(\frac{df(\vec{x},\vec{p},\tau)}{d\tau}\right)_C, \tag{58}$$

where $\vec{p} = p\vec{n}$ is the momentum of the particles and the term on the right hand side is the collision term (absent in the case of collisionless neutrinos).

The distribution function f can be expanded to the first order,

$$f(\vec{x},\vec{p},\tau) = f_0(p) + \delta f(\vec{x},\vec{p},\tau). \tag{59}$$

The zero-order phase space distribution is the solution to the Boltzmann equation in the absence of perturbations. There is no spatial or momentum direction dependence in this case and it is given by Planck distribution for photons (bosons) or Fermi-Dirac distribution for neutrinos (fermions)

$$f_0(q) \propto [4\pi^3(e^{\hbar q/kT_0} \pm 1)]^{-1}, \tag{60}$$

where + is for fermions and - for bosons and T_0 is the temperature today. We have introduced comoving momentum $q = ap$, which accounts for the redshifting of temperature. Similarly we can also introduce comoving energy $\epsilon = aE = (q^2 + m^2a^2)^{1/2}$. In principle chemical potential term μ should be subtracted from $\hbar q$, but in standard cosmological model it is not generated and can be set to zero.

Let us now look at the perturbed Boltzmann equation, keeping only the first order terms. Since zero order solution does not depend on momentum direction the term $\frac{\partial f}{\partial n^i}$ in equation (58) is of first order. Similarly, $\frac{dn^i}{d\tau}$ is also of first order, since it involves a change in direction. So the product of the two is second order and one may drop this term from perturbed Boltzmann equation. The remaining terms are

$$\frac{\partial \delta f}{\partial \tau} + \frac{q}{\epsilon} n^i \frac{\partial \delta f}{\partial x^i} + \frac{df_0}{dq}\frac{dq}{d\tau} = \left(\frac{df}{d\tau}\right)_C, \tag{61}$$

where we used

$$\frac{dx^i}{d\tau} = \frac{dx^i}{d\lambda}\frac{d\lambda}{d\tau} = \frac{p^i}{p^0} = \frac{q}{\epsilon}n^i \tag{62}$$

in the lowest order and we used the fact that 4-momentum p^i is proportional to 4-velocity u^i (with mass being the proportionality constant in the case of massive particles).

We are left to evaluate $dq/d\tau$. For this we will use geodesic equation already introduced in equation 41. It is a straightforward algebraic exercise to show that applied to the perturbed metric in Newtonian gauge the zero component of the geodesic gives

$$\frac{dq}{d\tau} = q\dot{\Phi} - \epsilon n^i \partial_i \Psi, \tag{63}$$

where only scalar perturbations have been included. The resulting Boltzmann equation in Fourier space for $K = 0$ case is

$$\frac{\partial \delta f}{\partial \tau} + i\frac{qk}{\epsilon}\mu\delta f + \frac{df_0}{d\ln q}\left(\dot{\Phi} - i\frac{\epsilon k}{q}\mu\Psi\right) = \left(\frac{df}{d\tau}\right)_C, \tag{64}$$

where $\mu = \hat{k}\cdot\hat{n}$ is cosine of the angle between the wavevector and the particle direction.

To compute components of energy-momentum tensor we must integrate the distribution function over the momentum $d^3\vec{p}$ times the components of appropriate 4-momenta. This is however not Lorentz invariant or GR covariant and the appropriate generalization is

$$T_{\mu\nu} = \int (-g)^{1/2} f p_\mu p_\nu \frac{d^3\vec{p}}{p^0}, \tag{65}$$

where g is the determinant of the metric and we inserted this term to make it valid in general relativity, although in the lowest order that we work in it does not affect the results. For individual components of energy-momentum tensor this gives

$$\begin{aligned} \delta T^0{}_0 &= -a^{-4}\int q^2 dq\, d\Omega\, \epsilon\delta f\,, \\ \delta T^0{}_i &= a^{-4}\int q^2 dq\, d\Omega\, q\, n_i\, \delta f(q)\,, \\ \delta T^i{}_j &= a^{-4}\int q^2 dq d\Omega\, \frac{q^2 n_i n_j}{\epsilon}\,\delta f(q). \end{aligned} \tag{66}$$

3.2.1 Massive neutrinos

For massive neutrinos the collision term on the right hand side of equation (58) is zero. It is useful to expand the perturbed distribution function in a generalized Legendre series,

$$\delta f(\vec{n},\tau) = \sum_{l=0}^{\infty}(2l+1)\delta f_l(-ik)^{-l}Q_{;i\ldots j}P_l^{i\ldots j}, \tag{67}$$

where $\vec{n}$ is the direction of the photon and $P_l^{i\ldots j}$ is defined recursively as

$$P_0 = 1,\, P_1^i = n^i,\, P_{l+1}^{ijk\ldots m} = \frac{2l+1}{l+1}n^{(i}P_l^{jk\ldots m)} - \frac{l}{l+1}\gamma^{(ij}P_{l-1}^{k\ldots m)}. \tag{68}$$

The brackets denote symmetrization with respect to all the indices. In the flat space this reduces to the usual Legendre expansion $\delta f = \sum_l (2l+1)(-i)^l \delta f_l P_l(\mu)$, where P_l are the Legendre polynomials of order l.

After integration over μ one obtains the following hierarchy of coupled evolution equations,
Massive neutrinos—

$$\begin{aligned} \delta\dot{f}_0 &= -\frac{qk}{\epsilon}\delta f_1 + \dot{\Phi}\frac{df_0}{d\ln q}, \\ \delta\dot{f}_1 &= \frac{qk}{3\epsilon}(\delta f_0 - 2\delta f_2) - \frac{\epsilon\, k}{3q}\Psi\frac{df_0}{d\ln q}, \\ \delta\dot{f}_l &= \frac{qk}{(2l+1)\epsilon}\left[l\delta f_{l-1} - (l+1)\delta f_{l+1}\right], \quad l \geq 2. \end{aligned} \tag{69}$$

Because of q dependence one must discretize these equations and solve the hierarchy for each value of q. In the end one must integrate over all values of q to obtain the perturbed energy-momentum tensor,

$$\begin{aligned} \delta\rho &= 4\pi a^{-4}\int q^2 dq\, \epsilon f_0(q)\delta f_0, \\ \delta p &= \frac{4\pi}{3}a^{-4}\int q^2 dq\, \frac{q^2}{\epsilon}\delta f_0, \\ (\bar{\rho}+\bar{p})v &= 4\pi a^{-4}\int q^2 dq\, q f_0(q)\delta f_1, \\ \Pi &= a^{-4}\int q^2 dq\, \frac{q^2}{\epsilon}\delta f_2. \end{aligned} \tag{70}$$

3.2.2 Massless neutrinos

When the particles are massless it is useful to introduce temperature perturbation Δ, such that $T = T_0(1+\Delta)$. One can easily show that

$$\delta f = -q\frac{df_0}{dq}\Delta, \tag{71}$$

which together with $\epsilon = q$ allows one to eliminate q dependence. We can again expand Δ in a generalized Legendre series,

$$\Delta(\vec{n},\tau) = \sum_{l=0}^{\infty}(2l+1)\Delta_l(-ik)^{-l}Q_{;i\dots j}P_l^{i\dots j}. \tag{72}$$

As before in the flat space this reduces to the usual Legendre expansion $\Delta = \sum_l (2l+1)(-i)^l \Delta_l P_l(\mu)$.

After angular integration the Boltzmann equation for massless neutrinos becomes,

$$\dot{\Delta}_\nu + k\mu\Delta_\nu = \dot{\Phi} - ik\mu\Psi. \tag{73}$$

From this one obtains the following hierarchy of coupled evolution equations,
Massless neutrinos—

$$\begin{aligned}
\dot{\delta}_\nu \equiv 4\Delta_{\nu,0} &= -\frac{4}{3}kv_\nu + 4\dot{\Phi}\,, \\
\dot{v}_\nu \equiv 3\Delta_{\nu,1} &= k\left[\Delta_{\nu,0} - 2\left(1 - \frac{3K}{k^2}\right)\Delta_{\nu,2} + \Psi\right] \\
\dot{\Delta}_{\nu,2} &= \frac{k}{5}\left[\frac{2}{3}v_\nu - \frac{3}{5}\left(1 - \frac{8K}{k^2}\right)\Delta_{\nu,3}\right]\,, \\
\dot{\Delta}_{\nu,l} &= \frac{k}{2l+1}\left[l\Delta_{\nu,l-1} - (l+1)\left(1 - \frac{l(l+1)K}{k^2}\right)\Delta_{\nu,l+1}\right]\,.
\end{aligned} \tag{74}$$

3.2.3 Photons

Photons interact with electrons and one needs to add the collision term in equation (58). The collision term depends on the angular distribution of incoming photons and on their polarization. Since photons are not the main topic of these lectures we will not present the details of Thomson scattering calculation. Instead we refer the reader to [14] for more details.

The hierarchy is
Photons—

$$\begin{aligned}
\dot{\delta}_\gamma \equiv 4\Delta_{\gamma,0} &= -\frac{4}{3}kv_\gamma + 4\dot{\Phi}\,, \\
\dot{v}_\gamma \equiv 3\Delta_{\gamma,1} &= k\left[\Delta_{\gamma,0} - 2\Delta_{\gamma,2}\left(1 - \frac{3K}{k^2}\right) + \Psi\right] + an_e\sigma_T(v_b - v_\gamma)\,, \\
\dot{\Delta}_{\gamma 2} &= \frac{k}{5}\left[\frac{2}{3}v_\gamma - 3\left(1 - \frac{8K}{k^2}\right)\Delta_{\gamma,3}\right] - an_e\sigma_T[\Delta_{\gamma,2} - \Pi/10]\,, \\
\dot{\Delta}_{\gamma,l} &= \frac{k}{2l+1}\left[l\Delta_{\gamma,l-1} - (l+1)\left(1 - \frac{l(l+1)K}{k^2}\right)\Delta_{\gamma,l+1}\right] - an_e\sigma_T\Delta_{\gamma,l}.
\end{aligned} \tag{75}$$

Here $\Pi = \Delta_{\gamma,2} - 12E_2$, where E_2 is the $l = 2$ multipole moment of polarization expansion [15]. It can be neglected if a few percent accuracy suffices.

4 Initial conditions: inflation

Inflation is our primary mechanism to produce fluctuations. Historically, inflation was proposed to explain flatness of the universe and horizon problem.

Flatness problem can be explained by revisiting Friedmann's equation (8), which can be rewritten as $\Omega - 1 = K/\eta^2$, where Ω is now time dependent. We know it is not far from unity today. However, since $\eta \propto \tau^{-1}$ in both matter and radiation epochs we see that Ω was extremely close to unity in the past. The flatness problem states that such initial conditions are unlikely, unless there is a mechanism that drives the universe to $\Omega = 1$, in which case it stays there.

Horizon problem involves CMB in directions in the sky separated by more than a few degrees. These should not be in causal contact in the past if one uses standard cosmological expansion, either matter or radiation domination. However, the uniformity of CMB suggests either something drove the CMB to the same temperature or it was part of initial conditions. In the former case these regions must have been in causal contact in the past.

Inflation suggests a solution by postulating a period of rapid expansion during which the universe is accelerating. The goal is to make comoving Hubble length η^{-1} to decrease in time, $d\eta^{-1}/dt < 0$. In this case the size of observable universe decreases with time and after the end of this period the observable size is in fact much smaller than the size of the causally connected region. Since

$$\frac{d\eta^{-1}}{dt} = -\frac{\ddot{a}}{\eta^2} < 0 \tag{76}$$

this indeed implies accelerating universe with $\ddot{a} > 0$. From Friedmann's equations (8) we find that this condition gives $\bar{\rho} + 3\bar{p} < 0$ or $\bar{p} < -\bar{\rho}/3$. So only exotic matter with negative large pressure can lead to inflation. Fortunately there is no shortage of such matter in the early universe. The simplest example is a scalar field, which was introduced in §2. One can see from equations (28) that both energy and pressure have contributions from kinetic and potential energy, with opposite sign for the latter. So to achieve $\bar{p} < -\bar{\rho}/3$ one must have a scalar field with a small kinetic term compared to the potential term. In other words, the field must be slowly moving in an external potential, which cannot be close to its zero value.

4.1 Slow-roll parameters

To further describe inflation it is useful to introduce two slow-roll parameters, ϵ and δ (note that here ϵ no longer describes comoving energy as when we discussed Boltzmann equation). First parameter simply describes deviations from the $w = -1$ case,

$$\epsilon \equiv \frac{3}{2}(1+w), \tag{77}$$

where $w = \bar{p}/\bar{\rho}$. Second parameter quantifies the requirement that the field is slowly rolling, $d^2\phi/dt^2 \sim 0$. Note that this requirement is in terms of proper time. When expressed in terms of conformal time it becomes $\ddot{\phi} \sim \eta\dot{\phi}$ and second slow-roll parameter is defined as

$$\delta \equiv \frac{\ddot{\phi}}{\eta\dot{\phi}} - 1. \tag{78}$$

The condition for successful inflation is that both parameters are small compared to unity. When this condition breaks down inflation stops.

We may relate both parameters to the local shape of potential. Small ϵ means we can Taylor expand $w = \bar{p}/\bar{\rho}$ to obtain $w = -1 + \dot{\phi}^2/(a^2V)$ this leads to

$$\epsilon = \frac{3\dot{\phi}^2}{2a^2V} = \frac{3a^4V'^2}{18\eta^2a^2V}, \tag{79}$$

where we used

$$3\eta\dot{\phi} = -a^2V' \tag{80}$$

obtained in the lowest order combining second slow-roll condition (equation 78) and evolution equation 13. In addition, since the energy is dominated by potential term we can write Friedmann's equation in the lowest order as

$$\eta^2 = 8\pi Ga^2V/3 \tag{81}$$

(we are ignoring curvature here since inflation will make it negligible). This gives

$$\epsilon = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2. \tag{82}$$

To relate δ to the potential we take time derivative of equation (80) to obtain

$$\begin{aligned}\ddot{\phi} &= \frac{1}{3}[-2a^2V' + a^2V'(1-\epsilon) - a^2\eta^{-1}V''\dot{\phi}] \\ &= -\frac{a^2V'(1+\epsilon)}{3} + \frac{a^4V'V''}{9\eta^2},\end{aligned} \tag{83}$$

where in the first line we used $d(\eta^{-1})/d\tau = \eta^2(1-\epsilon)$ following from second Friedmann's equation (8) and $w = 2\epsilon/3 - 1$. In second line we again used equation (80). From the definition of δ follows

$$\delta = \epsilon - \frac{V''}{8\pi G V}. \tag{84}$$

The two parameters are thus related to the first and second derivative of V. Since the definition of Planck mass is $M_{Pl}^{-2} \equiv 8\pi G$ one can see that the scale in ϕ over which potential changes must be large in Planck mass units for inflation to work. Note that this says nothing about the overall energy scale of inflation potential, which can be well below the Planck scale (in fact observationally it is required to be small in the simplest inflationary models).

4.2 Perturbations

Perhaps even more important than solution to the flatness and horizon problem is the ability of inflation to generate perturbations, which could act as seeds for the structure formation in our universe. The perturbations have a quantum mechanical origin, but because of the rapid expansion they become classical as they cross the horizon. Once they are outside horizon their amplitude does not change if appropriate gauge is used.

To demonstrate the statements above one must proceed in several steps. First one must demonstrate that the curvature perturbation in comoving gauge indeed remains unchanged outside the horizon regardless of the underlying equation of state. Second, one must compute perturbations generated by quantum fluctuations and evaluate their time evolution. The approach here is to write the perturbation equation in a harmonic oscillator form, which we know how to quantize, and then use the solution to the classical evolution equation to evolve the quantum fluctuations forward in time. These perturbations actually vanish in comoving gauge, making it inappropriate for this purpose, so instead we will calculate them in a spatially flat gauge and use gauge transformation to obtain curvature perturbation in comoving gauge. Finally, long after inflation we can make another gauge transformation and evaluate potential fluctuations in either synchronous or Newtonian gauge for computational purposes.

4.3 Curvature perturbation

Curvature perturbation ζ defined in comoving gauge (section 2.5.3) is a useful quantity because, as shown below, it is constant outside the horizon. It also relates simply to the Newtonian perturbation Φ, which allows one to use ζ to evaluate Φ outside the horizon even if the underlying equation of state changes. To relate the two we first use the gauge transformation between the Newtonian and comoving density,

$$\delta\rho_{\rm com} = \delta\rho_N - \dot{\bar{\rho}} v_N/k = \delta\rho_N + 3\eta(\bar{\rho}+\bar{p})v_N/k, \tag{85}$$

where the quantities on the left are in comoving gauge and those on the right in Newtonian gauge. Poisson's equation in Newtonian gauge gives

$$(k^2 - 3K)\Phi = 4\pi G a^2 \delta\rho_{\rm com}. \tag{86}$$

Inserting this into Poisson's equation in comoving gauge leads to

$$\zeta + \eta v_{\rm com}/k = \Phi. \tag{87}$$

But since $H_T = 0$ in Newtonian gauge $v_{\rm com} = v_N$, which follows from the gauge transformation in equations (39) and (48). The curvature perturbation is thus related to Φ as

$$\zeta + \eta v_N/k = \frac{2}{3}\frac{\eta^{-1}a\dot{\Phi} + \Psi}{1+w} + \Phi, \tag{88}$$

where the latter relation follows from velocity equation (46). As we will show later outside the horizon Φ does not change in adiabatic case in matter and radiation epochs. In the absence of anisotropic stress we have $\Phi = \Psi$. Then

$$\Phi = \frac{3+3w}{5+3w}\zeta. \tag{89}$$

This shows that assuming ζ is constant we can evaluate Φ for any w. For example, changing from $w = 1/3$ to $w = 0$ from radiation to matter domination produces a 10% decrease in Φ as can easily be verified using the above equation.

Curvature perturbation ζ obeys the evolution equation (49), which in the absence of curvature and anisotropic stress becomes

$$\dot{\zeta} = \eta\xi = -\frac{\eta\delta p}{(\bar{\rho}+\bar{p})}, \tag{90}$$

where the latter relation follows from Euler's equation (50) in comoving gauge. To show that this quantity is constant outside horizon one must show $\dot{\zeta} \ll \eta\zeta$. In the absence of entropy perturbations we can relate pressure perturbations to density perturbations through the speed of sound, which is less than unity. Then

$$\begin{aligned}\dot{\zeta} &= -\frac{\eta\delta p_{\rm com}}{(\bar{\rho}+\bar{p})} = -\frac{c_s^2\eta\delta\rho_{\rm com}}{(\bar{\rho}+\bar{p})} \\ &= \frac{c_s^2k^2\eta\Phi}{4\pi Ga^2(\bar{\rho}+\bar{p})} = \frac{2}{3}\left(\frac{c_sk}{\eta}\right)^2\frac{\eta\Phi}{1+w} = \left(\frac{c_sk}{\eta}\right)^2\frac{2\eta\zeta}{5+3w},\end{aligned} \tag{91}$$

which is indeed much smaller than $\eta\zeta$ outside the horizon, i.e. in the limit $k/\eta \to 0$.

As noted above and evident from equation 28 in comoving gauge $v-B=0$ and so $\delta\phi=0$. We must thus use a different gauge to evaluate $\delta\phi$ and then use a gauge transform into the comoving gauge to evaluate ζ. Gauge transformation of scalar field in equation (40) gives $T=\delta\phi/\dot{\phi}$, where $\delta\phi$ is evaluated in a general (different from comoving) gauge. Gauge transformation of metric component H_L (equation 38) gives

$$\zeta = H_L + \frac{H_T}{3} - \eta\frac{\delta\phi}{\dot{\phi}}. \tag{92}$$

One can see that the simplest gauge choice to evaluate $\delta\phi$ is spatially flat gauge with $H_L=H_T=0$, in which case

$$\zeta = -\eta\frac{\delta\phi}{\dot{\phi}}, \tag{93}$$

where $\delta\phi$ is evaluated in spatially flat gauge.

4.4 Quantum fluctuations of scalar field

Equation (53) describes evolution of $\delta\phi$ in a spatially flat gauge. This equation can be further simplified in the slow-roll approximation by noting that all the terms on the right hand side as well as the last term on the left hand side are of order $\epsilon\eta^2\delta\phi$ or $\delta\eta^2\delta\phi$ and so small compared to the other terms on the left hand side. This is because from Einstein's equations and the energy-momentum tensor for a scalar field metric perturbations scale as $\dot{\phi}\delta\phi$, which combined with another $\dot{\phi}$ or V' gives rise to $\epsilon\eta^2\delta\phi$. Similarly a^2V'' term gives rise to $\epsilon\eta^2$ and $\delta\eta^2$ terms, which again justify neglecting it in comparison to the other terms on the left hand side. The resulting equation is

$$\delta\ddot{\phi} + 2\eta\delta\dot{\phi} + k^2\delta\phi = 0. \tag{94}$$

This equation can be further simplified by introducing a new variable $u = a\delta\phi$, so that

$$\ddot{u} + (k^2 - 2\eta^2)u = 0. \tag{95}$$

It is convenient at this point to introduce conformal time relative to the end of inflation,

$$\hat{\tau} = \int_{\tau_e}^{\tau} d\tau = \int_{a_e}^{a} \frac{da}{\eta a}. \tag{96}$$

Since in the lowest order of slow-roll approximation $\eta^2 = 8\pi G a^2 V/3$ with V being approximately constant it follows that η/a is approximately constant and can be taken out of the integral above, so that

$$\hat{\tau} \approx -\eta^{-1}, \tag{97}$$

where we ignored the comoving Hubble length at the end of inflation assuming it is small. This leads to

$$\ddot{u} + \left(k^2 - \frac{2}{\hat{\tau}^2}\right) u = 0, \tag{98}$$

where the derivative can now be taken with respect to $\hat{\tau}$, since it differs from τ only by a constant.

On small scales ($k\hat{\tau} \gg 1$) we can drop the last term in equation (98) giving an equation for a simple harmonic oscillator. We know how to solve this equation: we first solve the equation classically,

$$u = w_k(\hat{\tau})a_k + w_k^*(\hat{\tau})a_k^*, \tag{99}$$

where on small scales $w_k = (2k)^{-1/2} exp(ik\hat{\tau})$. The full solution is

$$w_k = \left(1 - \frac{i}{k\hat{\tau}}\right) \frac{1}{\sqrt{2k}} e^{-ik\hat{\tau}}. \tag{100}$$

Quantization changes a_k^* and a_k into $\hat{a}_k^\dagger$ and $\hat{a}_k$, creation and annihilation operators, which obey the following commutation relations

$$\left[\hat{a}_k, \hat{a}_{k'}^\dagger\right] = \delta_{kk'}, \ [\hat{a}_k, \hat{a}_{k'}] = \left[\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger\right] = 0. \tag{101}$$

Creation and annihilation operators create or destroy a particle in a given state. The state can be described with the number of particles for each wavevector k. Annihilation operator acting on the ground state is zero,

$$\hat{a}_k|0> = <0|\hat{a}_k^\dagger = 0. \tag{102}$$

As is well known the ground state of the harmonic oscillator system also has fluctuations. The expectation value for these is

$$\begin{aligned} < 0|\hat{u}_k \hat{u}_k^\dagger|0> &= < 0||w_k(\hat{\tau})\hat{a}_k + w_k^*(\hat{\tau})\hat{a}_k^\dagger|^2|0> \\ &= |w_k|^2 < 0|\hat{a}_k\hat{a}_k^\dagger|0>= |w_k|^2 < 0|[\hat{a}_k,\hat{a}_k^\dagger] + \hat{a}_k^\dagger\hat{a}_k|0>= |w_k|^2, \end{aligned} \tag{103}$$

while if $k \neq k'$ the expectation value vanishes,

$$< 0|\hat{u}_k \hat{u}_{k'}^\dagger|0>= 0. \tag{104}$$

In the limit $-k\hat{\tau} \sim k/\eta \to 0$ the solution is

$$w_k = -\frac{i}{(2k^3)^{1/2}\hat{\tau}} = \frac{i\eta}{(2k^3)^{1/2}} \tag{105}$$

and so from equation (93)

$$\zeta = \frac{-i\eta^2}{(2k^3)^{1/2}a\dot{\phi}}. \tag{106}$$

Despite the apparent time dependence we have shown in equation (91) that ζ does not change in time for $k/\eta \to 0$. The spectrum of fluctuations is

$$\begin{aligned} k^3|\zeta|^2 &= k^3\frac{\eta^2 < 0|\hat{u}_k\hat{u}_k^\dagger|0>}{a^2\dot{\phi}^2} \\ &= \frac{\eta^4}{2a^2\dot{\phi}^2} = \frac{2\pi G\eta^2}{\epsilon a^2}, \end{aligned} \tag{107}$$

where we used $4\pi G\dot{\phi}^2 = \epsilon\eta^2$.

Variance in curvature is defined as $\Delta_\zeta^2 = k^3|\zeta|^2/2\pi^2$. Since $H = \eta/a$ is approximately constant during inflation the variance of curvature perturbations is nearly scale invariant. To quantify the departures from the scale invariance we need to evaluate logarithmic derivative of the amplitude of perturbations outside the horizon. To do this we use the fact that curvature perturbation is frozen to a constant value outside horizon. We may thus evaluate all the quantities at the time of horizon crossing $k/\eta = -k\hat{\tau} = 1$. Logarithmic derivative with respect to wavevector is

$$\frac{d}{d\ln k} = k\left(\frac{d\hat{\tau}}{dk}\right)_{k\hat{\tau}=-1}\frac{d}{d\hat{\tau}} = \frac{1}{k}\frac{d}{d\hat{\tau}} = \frac{\dot{\phi}}{k}\frac{d}{\phi}. \tag{108}$$

Using relations (80) and (81) this leads to

$$\frac{d}{d\ln k} = -\frac{V'}{8\pi GV}\frac{d}{d\phi}. \tag{109}$$

Applying equation (109) to equation (107) and using

$$\Delta_\zeta^2 \propto \frac{\eta^2}{a^2\epsilon} \propto \frac{V^3}{V'^2} \tag{110}$$

gives

$$\frac{d\ln\Delta_\zeta^2}{d\ln k} \equiv n_s - 1 = -\frac{V'}{8\pi GV}\left(\frac{3V'}{V} - \frac{2V''}{V'}\right) = -4\epsilon - 2\delta. \tag{111}$$

So the deviations from the scale invariance can be expressed in the lowest order of slow-roll parameters as $n_s = 1 - 4\epsilon - 2\delta$.

4.5 Gravity wave fluctuations

The calculation for gravity waves (tensor modes) follows the same procedure as for the scalar modes. Since tensor modes are gauge invariant there are no gauge ambiguities and one must simply quantize equation (34) in the absence of a source and curvature. This equation is in fact the same as equation (94), for which we already know the solution given in equation (100). The resulting spectrum of fluctuations is

$$\Delta_h^2 = \frac{4G\eta^2}{\pi a^2}, \tag{112}$$

where we inserted normalization factor $(16\pi G)^{-1/2}$ in the amplitude of tensor modes to make it consistent with the Einstein-Hilbert action formulation.

Since $\Delta_h^2 \propto V$ the slope of tensor fluctuation spectrum is

$$\frac{d\ln\Delta_h^2}{d\ln k} \equiv n_T = -\frac{V'}{8\pi GV}\frac{V'}{V} = -2\epsilon. \tag{113}$$

Here again the spectrum is nearly scale invariant if slow-roll conditions are satisfied.

Finally, the ratio of tensor to scalar amplitude is

$$\frac{\Delta_h^2}{\Delta_\zeta^2} = 4\epsilon = -2n_T. \tag{114}$$

This is called the consistency relation of inflation and is in principle testable with CMB if tensor modes are of comparable amplitude to the scalar modes.

We may pursue departures from scale invariance one step further and investigate deviations from a power law. This will be even smaller, being of second order in the slow-roll approximation. We need to introduce another slow-roll parameter ξ

$$\xi^2 = \frac{V'V'''}{(8\pi GV)^2}, \tag{115}$$

which is of the same order as the other two parameters. In terms of these we find

$$\begin{aligned} \frac{dn_s}{d\ln k} &= -8\epsilon^2 - 16\epsilon\delta - 2\xi^2 \\ \frac{dn_T}{d\ln k} &= -4\epsilon^2 - 4\epsilon\delta, \end{aligned} \tag{116}$$

where we used $\epsilon \propto (V'/V)^2$ and $\epsilon - \delta \propto V''/V$. While the first relation can at least in principle be tested the second relation is unlikely to be accessible to observational test given the existing constraints on the amplitude of tensor modes.

The resulting predictions from the simplest models of inflation are very simple: inflation predicts nearly scale-free spectrum of initial fluctuations for both scalar and tensor modes. Because the Fourier modes are independent the distribution is gaussian by central limit theorem regardless of the distribution for each Fourier mode (which is in fact also gaussian). Finally, since only inflation carries the energy density only curvature perturbations are generated (there will be no entropy perturbations). After the end of inflation other particles will be generated through the process of reheating and their perturbations will remain adiabatic. This provides a very specific set of initial conditions, such that all the modes of a given wavelength amplitude start with the same value. This is a remarkable prediction and leads to the coherent evolution of structure: all modes with the same amplitude of the wavevector evolve in the same way while in the linear regime.

5 Solutions for density perturbations

In previous section we derived initial conditions for matter perturbations, which determine the initial power spectrum. This is however not what is observed, because subsequent evolution modifies the mode amplitude as a function of scale.

To obtain solutions for dark matter evolution we can combine the continuity and Euler's equations (47) to obtain

$$\ddot{\delta}_c + \eta\dot{\delta}_c = k^2\Psi - 3\eta\dot{\Phi} - 3\ddot{\Phi}. \tag{117}$$

If we further assume anisotropic stress is negligible, which is almost always a valid approximation except on very large scales where neutrinos make a non-negligible contribution, we have $\Psi = \Phi$, which takes the role of Newtonian potential. This equation can be solved analytically for two cases of interest, matter domination and radiation domination. In the first case the source of potential are CDM fluctuations themselves, while in the second case the source of potential is coming from photons and can be treated as external if matter evolution is studied. We discuss both cases next.

5.1 Solutions inside horizon: matter domination

On scales smaller than the Hubble length we can ignore time derivatives of potential relative to the spatial derivatives. This only works if the solution is not oscillating and must be justified aposteriori from the obtained solution. Under this assumption we drop $\dot{\Phi}$ and $\ddot{\Phi}$ terms in equation (117). To relate potential to density perturbation we use Poisson's equation (46), ignoring $\eta\Psi$ and K terms in addition to $\dot{\Phi}$. We thus obtain a second order differential equation,

$$\ddot{\delta}_m + \eta\dot{\delta}_m = 4\pi G\rho_m a^2\delta_m. \tag{118}$$

While equation (118) was derived for CDM it can also be used to describe baryons on large scales and after the recombination, where the pressure term from baryons and the coupling between baryons and photons can be neglected. We will denote with δ_m matter perturbation when it applies both to CDM and baryons.

The general solution to equation (117) consists of a growing and decaying solutions. It is instructive to look at the solutions in some limits. In matter domination for $\Omega_m = 1$ we have $a \propto \tau^2$ from equations (8) and so $4\pi G\rho_m a^2 = 6/\tau^2$. Setting the solution as a power law $\delta = \tau^\alpha$ one finds $\alpha = 2, -3$. This means that the growing mode solution grows as a scale factor,

$$\delta_m \propto \tau^2 \propto a \quad \Omega_m = 1. \tag{119}$$

From Poisson's equation (46) one finds that gravitational potential remains constant on small scales, $\Phi = \text{const}$.

In a universe filled with a cosmological constant, curvature or dark energy the growth will be slowed down relative to $\Omega_m = 1$. The effect is only important at late times when the additional component can be dynamically important. No analytic solutions exist in this case, but a very good approximation to the growth suppression relative to $\Omega_m = 1$ case is [16]

$$\frac{\delta(\Omega)}{\delta(\Omega=1)} \approx \frac{5\Omega_m}{2\left[\Omega_m^{4/7} - \Omega_\lambda + 1 + \frac{1}{2}\Omega_m\right]}. \tag{120}$$

This and other solutions such as the one for dark energy with varying equation of state can be obtained numerically from CMBFAST code.

5.2 Solutions inside horizon: radiation domination

In radiation era the photons (and neutrinos, which we ignore here) are the dominant component to the energy density. We show first that because of pressure effects photon perturbations do not grow. To show this one takes pressure equation of Einstein's equations (46) and subtracts it from one third of the density equation. Since $\delta p = \delta\rho\dot{p}/\dot{\rho} = \delta\rho/3$ this gives the following equation,

$$\ddot{\Phi} + 4\eta\dot{\Phi} + \left[2\dot{\eta} + 2\eta^2\right]\Phi = -\frac{k^2}{3}\Phi. \tag{121}$$

We have again ignored the curvature terms. In radiation epoch one has $a \propto \tau$, from which follows $\eta = 1/\tau$ and so the last term on the left hand side vanishes. Then the equation is

$$\ddot{\Phi} + \frac{4}{\tau}\dot{\Phi} = -\frac{k^2}{3}\Phi \tag{122}$$

and has the growing solution

$$\Phi = 3\Phi_i\left(\frac{\sin z}{z^3} - \frac{\cos z}{z^2}\right), \tag{123}$$

where $z = k\tau/\sqrt{3}$ and the solution was normalized to the initial potential value Φ_i. As expected the photon pressure causes the potential to oscillate and decay away as $(kc_s\tau)^{-2}$ inside the horizon, with $c_s = 3^{-1/2}$. While the inclusion of baryons prior to recombination complicates the equations and changes the sound speed somewhat this conclusion does not change significantly.

To solve for CDM we take equation (117) and evaluate it in radiation epoch,

$$\ddot{\delta}_c + \frac{1}{\tau}\dot{\delta}_c = k^2\Psi - 3\eta\dot{\Phi} - 3\ddot{\Phi}. \tag{124}$$

The full solution consists of a homogeneous solution (i.e. solution to the above equation without the source) and a particular solution, which can be written as an integral of Green's function over the source term. The latter decays away on small scales and so we can ignore the particular solution. One is left with the homogeneous solution, which has the growing mode

$$\delta_c = C + \ln(\tau), \tag{125}$$

where C is a constant. So the CDM density grows only logarithmically inside the horizon in radiation era. This has a simple physical interpretation. Prior to horizon crossing both radiation and CDM evolve similarly and acquire a velocity term as they cross the horizon. After that CDM decouples from radiation and its velocity decays as τ^{-1} due to the Hubble drag (damping term). This gives rise to the logarithmic growth of density perturbation.

5.3 Solutions outside horizon

Prior to horizon crossing the solutions depend on the adopted gauge. As above we will use equations in Newtonian gauge, but we emphasize that the solutions can differ drastically if a different gauge is adopted. For example, while in Newtonian gauge adopted here density perturbations do not grow outside horizon, they in fact do grow in synchronous and conformal gauge. If we ignore anisotropic stress (making $\Phi = \Psi$), spatial derivatives and curvature K then we again combine first and third of Einstein's equations (46) to obtain

$$\ddot{\Phi} + \dot{\Phi}\left[3\eta(1 + c_{s,mr}^2)\right] + \Phi\left[2\dot{\eta} + \eta^2(1 + 3c_{s,mr}^2)\right] = \frac{3}{2}\eta^2\left[\delta p - c_{s,mr}^2\delta\rho\right], \tag{126}$$

where effective matter-radiation sound speed is

$$c_{s,mr}^2 = \frac{1}{3\left[1 + 3y/4\right]}. \tag{127}$$

Here

$$y = \frac{\bar{\rho}_m}{\bar{\rho}_r} \equiv \frac{a}{a_{eq}} = x^2 + 2x, \qquad x = \left(\frac{\Omega_m}{4a_{eq}}\right)^{1/2} H_0\tau, \tag{128}$$

and a_{eq} is expansion factor when matter and radiation densities are equal.

The right-hand side of equation (126) is proportional to specific entropy fluctuation $\sigma = \frac{3}{4}\delta_r - \delta_m$,

$$\delta p - c_{s,mr}^2 \delta\rho = \bar{\rho} c_{s,mr}^2 \sigma. \tag{129}$$

If we restrict ourselves to the adiabatic initial conditions where entropy fluctuations vanish initially they cannot be generated on scales outside the horizon. In that case we can solve equation (126) analytically obtaining the growing and decaying solution [12]

$$\Phi_+ = 1 + \frac{2}{9y} - \frac{8}{9y^2} + \frac{16x}{9y^3}, \qquad \Phi_- = \frac{1+x}{y^3}. \tag{130}$$

In the radiation era ($y \ll 1$) the solution is $\Phi_+ = 10/9(1 - y/16) = \Phi_i$, where Φ_i is the initial perturbation, while in the matter era $\Phi_+ = 1$. In general thus

$$\Phi = \frac{9}{10}\Phi_+ \Phi_i. \tag{131}$$

One can see that on scales outside horizon the gravitational potential Φ does not change both in radiation and in matter epochs, but between the two it changes the amplitude by 10% (this gives rise to so called early integrated Sachs-Wolfe effect in CMB). This is the same result we obtained from the constancy of curvature ζ, but here we also solved for how Φ is changing in time from radiation to matter domination.

Density perturbations are related to potential Φ through equation Poisson's equation (46). On large scales one can ignore spatial derivatives and curvature terms to obtain

$$\delta = -2(\Phi + \dot{\Phi}/\eta). \tag{132}$$

In the matter and radiation domination limits this gives $\delta = -2\Phi$. Since

$$\delta = \frac{\delta_\gamma + y\delta_c}{1+y} \tag{133}$$

one finds $\delta_c = -3\Phi/2$ in radiation era and $\delta_c = -2\Phi$ in the matter era. Density perturbations are therefore also constant on large scales. We stress again that this is a gauge dependent statement. For example, in synchronous and conformal gauge density perturbations grow even outside horizon. Of course, for any quantity that is directly observable such as CMB anisotropies the predictions are independent of the gauge choice.

5.4 Transfer functions

Any initial power spectrum of density perturbations gets modified because of the different growth of perturbations in different regimes. This can be conveniently expressed in terms of the transfer function, which is defined as how much δ of a given mode grows (or decays) relative to initial value. For convenience this is divided by k^2 and normalized relative to $k = k_i$ mode, where $k_i\tau_0 \ll 1$,

$$T(k) = \frac{k_i^2 \delta(k,\tau_0)\delta(k=k_i,\tau_i)}{k^2\delta(k,\tau_i)\delta(k=k_i,\tau_0)}, \tag{134}$$

so that the transfer function on large scales is unity, $T(k=k_i) = 1$. Today δ is dominated by CDM, $\delta \sim \delta_c$, but it also has contributions from baryons and massive neutrinos, so one should view is as an average over all the species weighted by their mean density as in equation 133. To obtain the processed power spectrum one multiplies the square of the transfer function with the primordial power spectrum $P(k) = P_i(k)T^2(k)$.

Outside the horizon the modes do not grow in both matter and radiation era and δ_c is constant. When the modes enter horizon in radiation era the density only grows logarithmically until the matter era as discussed above. If they enter in the matter era they begin to grow immediately as τ^2. By definition the modes enter the horizon when $k \sim \tau^{-1}$, so the growth is just proportional to k^2 as long as the mode entered in the matter era. Since in our definition of transfer function we divide by k^2 the transfer function remains unity for all the modes entering horizon after the matter-radiation equality. The transfer function for modes entering prior to that will suffer suppression in the transfer function which scales as $(k\tau_{eq})^{-2}[\ln(k\tau_{eq})+1]$ on small scales, where τ_{eq} is the conformal time at matter-radiation equality.

The most important parameter that determines the transfer function shape is τ_{eq}, which depends on $\Omega_m h$ when the power spectrum is expressed against k/h. For CDM models the asymptotic slope of the transfer function is the same regardless of the value of cosmological parameters. Below we discuss some of the other parameters which can affect the transfer function.

5.5 Massive neutrinos: mixed dark matter

So far we have ignored neutrinos in our discussion. The corresponding equations for massless neutrinos have been presented above. They contribute to the relativistic energy density and create anisotropic stress on large scales, but do not qualitatively change the results. Massive neutrinos however can have a more important effect on the transfer function. Their mass is related to the density parameter via

$$\Omega_\nu h^2 = \frac{m_\nu}{94eV}\frac{10.75}{g_*}\frac{g_X}{1.5}, \quad (135)$$

where g_* is effective number of relativistic species at decoupling and g_X the number of spin degrees of freedom, which is 1.5 for 2-component fermions. At the time of decoupling such neutrinos are still relativistic and become nonrelativistic around the epoch of matter-radiation equality for $\Omega_\nu \sim 1$, assuming standard value of g_*. Free streaming neutrinos can erase perturbations on scales below the free-streaming distance, defined as the distance at which a neutrino of a given rms velocity $v_{\rm th}$ can still escape against gravity. The velocity is c when neutrinos are relativistic and drops as $1/a$ afterwards because of momentum conservation,

$$v_{\rm th} \sim \frac{k_BT}{m_\nu} = 50(1+z)(m_\nu/{\rm eV})^{-1}{\rm km/s}. \quad (136)$$

Since the Hubble time is proportional to $t_H \sim [(1+z)\Omega_m]^{-1/2}H_0^{-1}$ the product of the two gives an estimate of the free-streaming length. The resulting comoving free-streaming wavevector is

$$k_{fs} = 0.4(\Omega_m h^2)^{1/2}(1+z)^{-1/2}\frac{m_\nu}{1{\rm eV}}{\rm Mpc}^{-1}. \quad (137)$$

For a given k neutrino perturbations δ_ν are suppressed while $k > k_{fs}(a)$. After that they can grow again and catch up with cold dark matter perturbations δ_c. This happens on large scales and as a consequence the transfer function is the same as in CDM models. When neutrinos are dynamically important the damping also affects δ_c, decreasing its amplitude on small scales compared to pure CDM models.

Recent Super-Kamionkande results find $\mu - \tau$ neutrino mass squared difference of $3\times10^{-3}eV^2$ [10]. In the most conservative scenario this gives one neutrino family with $m_\nu \sim 0.06eV$ or $\Omega_\nu \sim 0.15\%$. Since only the mass difference is measured it is in principle possible that the total density of neutrinos is larger than that.

While small, this neutrino mass nevertheless leaves potentially observable effects on the dark matter power spectrum. It changes the transfer function by suppressing the power below the free-streaming length. Because of small masses this suppression is on relatively large scales, where complicating issues of bias are potentially less problematic. There is hope one can measure this directly from the galaxy clustering data in surveys such as SDSS with an accuracy of 0.1eV [9]. The signature imprinted by neutrinos is quite unique and it is unlikely to be mimicked by other cosmological parameters.

Second possibility to detect massive neutrinos is through their suppression of the growth rate below the free-streaming length. If one normalizes the fluctuations from CMB (at $z \sim 1000$) one finds that the small scale normalization of the dark matter power spectrum is affected by neutrinos. This is often parametrized with σ_8, denoting rms mass fluctuations in spheres of $8h^{-1}$Mpc radius. In principle σ_8 can be a well measured number: the abundance of clusters is very sensitive to this number and can change exponentially with it. Current estimates give $\sigma_8 \Omega^{-0.6} \sim 0.5 \pm 0.1$, where the errors are dominated by systematics, but a significant improvement could be possible in the future. Second possibility to measure σ_8 is by measuring mass fluctuations directly using weak gravitational lensing and there are significant observational efforts underway to measure this. The effect on σ_8 is rather small and addition of 0.06eV neutrino changes its value by 1.5% if normalized to COBE. Moreover, this effect is degenerate with many other effects that also change the growth rate, such as a change in equation of state. They can only be distinguished if additional information is used, such as one from supernovae or CMB. It remains to be seen how accurately can one ultimately extract neutrino mass from such observations, but it is clear that dark matter clustering provides one of the most stringent observational tests of neutrino mass.

5.6 Warm dark matter

The scenario above applies to the neutrinos with standard decoupling temperature $T \sim$ MeV, where $g_* = 10.75$. Recently there has been a lot of attention devoted to warm dark matter, in which the decoupling occurs at a higher energy when $g_* > 100$. It is in fact not clear how many degrees of freedom exist at higher temperatures. Standard model predicts $g_* \sim 100$, while the supersymmetric extension doubles that to $g_* \sim 200$. This is however still not sufficient for currently popular models, which require mass of WDM candidate around 1keV, implying $g_* \sim 800$ for $\Omega_\nu h^2 \sim 0.3$. Additional degrees of freedom must therefore be postulated. Alternative possibility is that entropy is released after the decoupling of warm dark matter and before the nucleosynthesis epoch. Another interesting option is that ordinary neutrinos could have a lower density because of low reheating temperature after inflation. Whatever the mechanism, one would like to have a suppression of power on scales below 1Mpc. If one assumes the particle mass of order 1keV one has the free-streaming wavevector of $k_{fs}20 \sim h\text{Mpc}^{-1}$. On large scales the transfer functions are the same as CDM, while on small scales they lead to complete suppression of power below the free-streaming length, just like in the case of ordinary massive neutrinos, except that in this case the free-streaming length is significantly shorter.

5.7 Baryons

Prior to decoupling baryons are tightly coupled with photons and oscillate for modes inside the horizon. After decoupling their sound speed drops significantly and the Thomson scattering term in equation 55 can be dropped. If one combines continuity and Euler's equations into a single second order equation and subtracts it from the same equation for CDM one finds,

$$\ddot{\Delta}_{bc} + 2\eta\dot{\Delta}_{bc} = 0, \tag{138}$$

where $\Delta_{bc} = \delta_b - \delta_c$. This equation is valid on large scales, where baryons pressure can be neglected. The growing mode solution for this equation is a constant, so the difference between baryon and CDM density perturbation does not change in time. However, the CDM density perturbation grows as $\delta_c \propto \tau^2$ in the matter domination, so the relative difference between baryon and CDM density contrast decreases as τ^{-2}. So δ_b catches up with δ_c after decoupling and then continues to grow at an equal rate.

We have assumed that CDM grows following the solution to equation 118. This is valid if CDM is the dominant component in the matter epoch. If baryon density is not negligible compared to CDM density then baryons will also contribute to the gravitational potential from Poisson's equation (46). But baryons after decoupling still reflects acoustic oscillations from the epoch before decoupling, when they follow photon oscillations (equation 123). As a result the total potential will reflect these oscillations and CDM evolution will be modified because of this. An increase in baryon density leads to acoustic oscillations in the transfer function. Second effect of baryons is that they suppress the transfer function on small scales. This is again expected, since baryons are damped prior to decoupling on small scales and if they are dynamically important they lead to a suppression of the gravitational potential. The latter is the source for CDM density fluctuations, which are thus suppressed as well.

6 Nonlinear evolution and bias

Determining the linear power spectrum of dark matter is one of the main goals of modern cosmology. There are several complications that may prevent us at present from reaching this goal. First, on small scales the linear power spectrum is modified by nonlinear evolution which enhances its amplitude over the linear spectrum. It is important to understand this process, so that one can predict the relation between the two. This is necessary both to reconstruct the linear spectrum from a measured nonlinear one and to verify whether there are other mechanisms besides gravity that modify the clustering of dark matter on small scales. Second, it is difficult to observe correlations in dark matter directly. Direct tracers such as peculiar velocity flows or weak lensing still suffer from low statistics and poorly understood systematics. Instead it is much easier to observe correlations between galaxies or correlations between galaxies and dark matter. While these are related to the dark matter correlations, the relation may not be simple.

A simple picture which incorporates both of these effects has been developed recently [17]. It sheds some light on what is or is not possible to extract from galaxy and dark matter clustering in the nonlinear regime. The picture is based on the Press & Schechter model [18], which assumes that at any given time all the matter in the universe is divided into virialized halos. These halos are correlated and have some internal density profile, which can be a function of halo mass. By specifying the halo mass function, their clustering strength and the halo profile one can determine the dark matter correlation function. By additionally modelling the number of galaxies inside the dark matter halo as a function of halo mass one can extend this picture to the galaxy clustering. It provides a natural explanation for why is galaxy correlation function a power law over a wide range of scales and how this ties with the other observations of large scale structure.

The correlation function consists of two terms. On large scales the halos are correlated with each other. One can assume the halo-halo correlation function follows the linear correlation function. Its amplitude depends on the bias for each halo. Halos more massive than the nonlinear mass scale M_* are more strongly clustered than the matter, while those with masses below M_* are less strongly clustered than the matter. Since halos are not pointlike one needs to convolve the halo-halo correlation function with the halo profiles of both halos to obtain the correlation function.

In addition to the halo-halo correlation term there are also correlations between dark matter particles within the same halo. These are also determined by mass function and dark matter profile and are expected to dominate on small scales. The main difference between these two terms is that there is an additional mass weighting for the latter. This makes the dominant contribution to this term to come from the higher mass halos relative to the halo-halo term. The total power spectrum is the sum of the two contributions. Figure 1 shows the individual contributions and the sum in comparison to the linear power spectrum and the nonlinear predictions, which are in agreement with N-body simulations. The model is in excellent agreement with simulation results, indicating that despite its simplicity it includes most of the physics relevant for nonlinear clustering of dark matter.

One can extend this model to include galaxy clustering. Here again one assumes all the galaxies form in halos, which is a reasonable assumption given that only very dense environments, which have undergone nonlinear collapse, allow the gas to cool and to form stars. The key new parameters that need to be introduced are the mean number of galaxies per halo as a function of halo mass and the mean pair weighted number of galaxies per halo. One must further assume that each halo has one galaxy at its center, which was a result of the gas cooling in this halo, while the rest of the galaxies in the halos are distributed in the same way as the dark matter. This is only the simplest assumption and one can easily generalize it to profiles that differ from the dark matter. If only galaxies brighter than a certain luminosity are counted then galaxies in small halos will not be included in the sample. This is in fact one of the reasons why galaxy clustering differs from that of the dark matter on small scales. Second comes from the fact that there is one galaxy which is at the center of the halo and the pair-weighting is enhanced because of that, leading to stronger correlations. Third reason is that number of galaxies inside the halo does not scale linearly with halo mass. Larger halos have higher temperatures and gas takes longer to cool. Thus on average there will be fewer galaxies of a given luminosity formed in such a halo. This also enhances the correlations on small scales, since there will be more galaxies in small halos relative to dark matter. The resulting predictions are shown in figure 2, showing that the predicted model is in good agreement with the data and shows significant differences from the dark matter power spectrum on small scales.

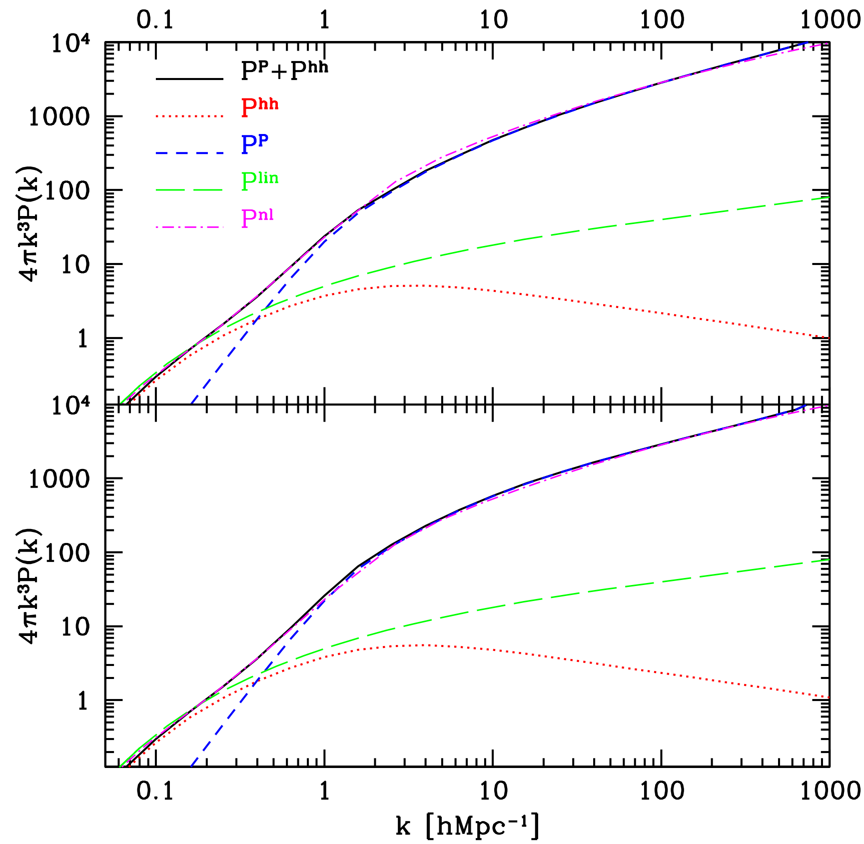

Figure 1: Comparison between the power spectrum predicted from the halo model (P^P+P^{hh}, solid) and the nonlinear power spectrum for ΛCDM model (P^{nl}, dash-dotted). Also shown are the linear power spectrum (P^{lin}, thin solid) and the two individual contributions, single halo P^P and halo-halo P^{hh}. Top plot is for dark matter profile with inner slope $\alpha = -1$, while bottom is for $\alpha = -1.5$. Both profiles give an excellent fit to the nonlinear power spectrum.

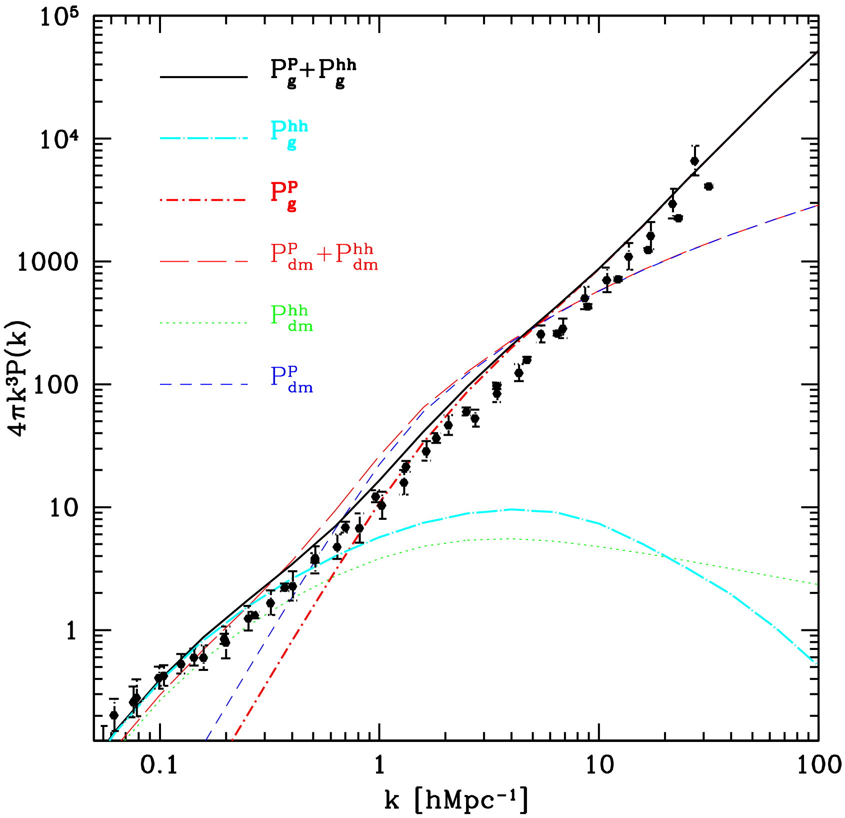

Figure 2: Comparison between galaxy and dark matter power spectrum predictions for galaxies selected by absolute magnitude $M_B < -19.5$. Also shown is the measured power spectrum of galaxies. On large scales dark matter and galaxy spectra agree, while on small scales galaxy power spectrum exceeds that of dark matter, in agreement with the observations.

7 Conclusions

In these lectures we have reviewed the physics of dark matter, with the emphasis on their clustering properties. We have limited the discussion to the simplest class of models produced by inflation, for which the initial fluctuations are gaussian. In this case the linear power spectrum contains all the information on the underlying cosmology and its determination becomes the main goal of observational studies.

In general there are two ways to extract the cosmological information from the dark matter clustering properties. One is to obtain the time dependence of the amplitude of the power spectrum. This can be related to the growth factor, which as discussed above is a function of Ω_m, Ω_λ, curvature, dark energy, massive neutrinos etc. There are several probes of the dark matter clustering at high redshift, which combined with the local clustering amplitude provide constraints on the clustering evolution. Among these are CMB, providing constraints at $z \sim 1100$, Ly-α forest providing constraints at $z \sim 3$, weak lensing, which probes structure around $z \sim 0.5 - 1.5$, galaxy clustering, which can be observed for $z < 4$ and quasar clustering spanning a similar range in z. Other probes such as X-ray, Sunyaev-Zeldovich or far infra-red background are also sensitive to the high redshift clustering.

Second method is through the shape of the power spectrum, which is also affected by a number of cosmological parameters, such as matter-radiation equality, massive neutrinos, baryon to CDM ratio, primordial spectrum of fluctuations etc. The shape can be obtained from the same probes as mentioned above. Even more important in this case are galaxy surveys, both existing such as PSCz and upcoming such as 2dF and SDSS. They probe mostly local universe and so are not very sensitive to the evolution of the growth factor. On the other hand, because of the large number of galaxies measured, they achieve a much more accurate determination of the galaxy power spectrum than probes at a higher redshift. An important issue we briefly discussed in §6 is bias, which relates galaxy clustering to that of dark matter. Although biasing can be complicated on small scales it is likely to be constant on large scales, where many of the most important cosmological parameter constraints are coming from. Alternative approach, which does not require knowledge of bias, is weak lensing. While this approach will not achieve sensitivities of galaxy surveys it can provide an independent measurement of the dark matter power spectrum and should serve as an important check of the different systematics for the two approaches.

Although we only discussed dark matter clustering here the constraints that can be obtained from it can be significantly enhanced if additional observations are included. Most important among these is CMB, which probes much of the same physics at discussed here, but measures it at $z \sim 1100$. Other important constraints will come from supernovae, measuring redshift-luminosity distance relation and from direct Hubble constant determination using a variety of techniques. This should enable one to extract neutrino mass with an accuracy of 0.1eV, equation of state with an accuracy of a few percent, determine primordial slope and amplitude of the spectrum to a few percent, test deviations from the primordial power law, place very accurate limits on dark matter density, test nucleosynthesis predictions for baryon density etc. Dark matter clustering measurements are entering high precision era and theoretical understanding of the physics behind it is needed if we are to extract all the information from it. These lectures should hopefully provide, with sufficient detail, the current theoretical understanding of the physics behind the dark matter clustering, which should help one achieve this goal.

Acknowledgments

I would like to thank the organizers of the ICTP cosmology summer school for inviting me to deliver these lectures and for all the hospitality. Support by Sloan and Packard foundations is greatly appreciated.

References

[1] A. Miller, *et al.*, Ap.J. 524, L1, 1999.

[2] de Bernardis, P. *et al.*, Nature, 404:955, 2000.

[3] S. Hanany *et al.*, astro-ph/0005123, 2000.

[4] N.A. Bahcall, J.P. Ostriker, S. Perlmutter, & P.J. Steinhardt, Science, 284, 1481, 1999.

[5] S. Perlmutter, *et al.*, ApJ, 517, 565, 1999.

[6] B. Schmidt, *et al.*, ApJ, 507, 46, 1998.

[7] R.R. Caldwell, R. Dave & P.J. Steinhardt, ApS&S, 261, 303, 1998.

[8] National Academy of Sciences Report of the Committee on Physics of the Universe, (http://www.nationalacademies.org/bpa/reports/cpu/index.html)

[9] Hu, W., Eisenstein, D. J., Tegmark, M. PRL, 80 5255, 1998.

[10] K. Scholberg, *et al.*, hep-ph/9905016 (Super-Kamiokande collaboration).

[11] Kodama, H., & Sasaki, M., Prog. Theor. Phys. Suppl. 78, 1, 1984.

[12] Kodama, H., & Sasaki, M., Int. J. Mod. Phys. A1, 265, 1986.

[13] U. Seljak and M. Zaldarriaga, ApJ 469, 437, 1996.

[14] M. Zaldarriaga, U. Seljak and E. Bertschinger, ApJ, 494, 491, 1998.

[15] W. Hu, U. Seljak, M. White & M. Zaldarriaga, Phys. Rev. D 57, 3290, 1998.

[16] S. M. Carroll, W. H. Press, & E. L. Turner, ARAA 30, 499, 1992.

[17] Scoccimarro, R., Sheth, R. K., Hui, L., Jain, B. 2000, preprint [astro-ph/0006319]; U. Seljak, 2000, preprint [astro-ph/0001493]; J. A. Peacock, R. E. Smith, 2000, preprint [astro-ph/0005010]; C.-P. Ma, J. N. Fry, 2000, preprint [astro-ph/0001347].

[18] W. H. Press and P. Schechter, Ap. J. 187, 425, 1974.