SPATIALLY HYPERBOLIC UNIVERSES WITH FUNDAMENTAL MATTER SOURCES

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Motivated by publications on the signature of negative curvature Universes in the CMB maps, in the present paper, we have analyzed the $k = -1-$FRW Universe, with different types of matter-sources. Thus, the nonrelativistic dust, the false vacuum, the CMB radiation and the stiff matter cases are discussed in order to see the behavior of the Universe characterized by each matter source alone. Finally, we have discussed the Einstein–Gordon Equations for a scalar with spontaneously broken symmetry.

Key words: 4D FRW metric, negative curvature, Einstein-Gordon equations, perfect fluid, scalar field.

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1. INTRODUCTION

In principle, the so-called *Cosmology* deals with some fundamental questions regarding both the origin and the evolution of the Universe, its matter content and the formation of large-scale structures [1]. The main starting point is based on the celebrated *cosmological principle*, which states that there is no special place in the Universe, *i.e.*, more formally, that the Universe is *homogeneous* and *isotropic*. Isotropy means that the Universe looks the same in all directions and it is proved by the universal temperature of the *cosmic microwave background (CMB)*, while homogeneity means that the Universe is the same at every point in space and it can be shown by proving isotropy at every point [2]. As a remark, we should point out that this principle is not exact, but it becomes more and more accurate at larger and larger scales, in which we are particularly interested.

Another crucial observation about the Universe was the fact that everything is moving away and the more distant a galaxy is, the greater is its recession velocity. This effect was determined by using the so called *redshift* z of the galaxies, which means that the frequency of the light spectrum emitted by them is decreasing, *i.e.* the wavelength λ is increasing, given by:

$$1+z = \frac{\lambda_{observed}}{\lambda_{emitted}} \tag{1}$$

This idea was especially explored by the cosmologist E. Hubble, who developed *Hubble law* and lead to the idea of the *expansion of the Universe*. This states that:

$$v = Hr, \tag{2}$$

where v is the recession velocity of the galaxy and r is the distance from the galaxy, introducing also the *Hubble parameter* H. Combining the expansion of the Universe with the cosmological principle, implies that actually every point in the Universe is moving apart from the others with a velocity proportional to the distance between cosmic objects. Hence, one can interpret that further in the past, all the matter was concentrated at a single point, from which everything started to acceleratingly expand. This initial starting point was called *Big Bang singularity* and this model describing the Universe is known as the *Big Bang model* [3], based primarily on the concept of the *accelerated expansion*.

The main goals of the observational cosmology is to actually measure the present value H_0 of the Hubble parameter and also the the density parameter Ω_m of the matter content. Regarding the Hubble parameter there are various methods of experimentally determining its value, described in [4], and was set to $H_0 = 100h$ $\mathrm{km/s/Mpc}$, where $h = 0.71 \pm 0.06$ is a dimensionless constant. For measuring the matter density, one can either analyze the dynamics of the large clusters of galaxies through gravitational lensing [5] or measure the power spectrum of the fluctuations in the CMB radiation [6]. All these experiments show that Ω_m oscillates between 0.1 and 0.4, from which only 0.04 corresponds to *baryonic matter* (protons, neutrons) and the rest is attributed to the so called *cold dark dark matter (CDM)*. "Cold" means that it is nonrelativistic, to allow the formation of large-scale structures and "dark" means that it is weakly interacting with the electromagnetic field, being very hard to detect it.

In order to account for the accelerated expansion of the Universe and to obtain a density parameter very close to unity, characteristic to the observational nearly flat Universe, one has to invoke the presence of another form of matter, called *dark energy*, with $\Omega_\Lambda \approx 0.7$. This can be determined by analyzing the type IA supernovae, which are subject to a very high redshift effect [7]. Another way of compensating the additional anti-gravity force is to postulate the existence of a cosmological constant or false vacuum type of matter, for which some observational evidences can be found in [8].

The radiation that is filling the Universe is a blackbody radiation, with uniform temperature $T = 2.73K$, known as *cosmic microwave background radiation*. The *anisotropies in the CMB*, coming from the different rates at which the photons redshifted, gave rise to small density perturbations, which are the basics for the formation of structures in the Universe, at early times. The peaks in the CMB power spectrum offers us a hint about the value of the total energy density Ω_{total}, between

0.98 and 1.08, according to [9].

Finally, based on the mentioned cosmological principle and the universal expansion, the line element of the Universe is the famous *Robertson–Walker metric*, which can be written in spherical coordinates, into the form

$$ds_4^2 = a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right] - dt^2, \tag{3}$$

where $a(t)$ is the cosmological *scale factor*, describing the expansion of the Universe and k is the *3D-curvature*, describing the shape of the Universe, which can be "1", for spherically closed $t = constant-$hypersurfaces, "-1" for hyperbolically ones and, respectively "0", for the spatially-flat models, as it has been explained, with significant details in [10].

Although the cosmological observations seem to favor the nearly flat Universe, *i.e.* vanishing curvature, there are a number of issues that this model cannot explain. For this reason, it keeps being a good idea to explore the *negative curvature* case, in order to identify and describe the different Universe models and types situated in the verge of the present observational data. For additional characteristics and a discussion regarding the observations in favor of a negatively curved Universe, see paper [11].

Actually, one of the biggest problem with the Big Bang model in Cosmology is the so called *flatness problem*. As discussed above, the observations seem to indicate that the total density parameter Ω_{total} is very close to unity, implying a flat spacetime. But, as we will derive in the next sections, the density parameter Ω is given by the formula:

$$\Omega(t) - 1 = \frac{k}{\dot{a}(t)^2}, \tag{4}$$

such that when $k = 0$, *i.e.* flat Universe, $\Omega = 1$, as previously motivated. Now, from this equation we can deduce that if the density is equal to unity, it should have been like this at any time in the past. On the other hand, if the density would be slightly different from one, then the Universe will become very rapidly more and more curved, since for the known matter the scale factor is an increasing function of time. This situation makes the flat geometry a very unstable state of the Universe and also it gives rise to the peculiar question of why the density would have picked exactly the value of one, out of a very broad range, shortly after the Big Bang.

Regarding the research activity performed within the negative curvature models framework, a detailed analysis was considered in paper [12]. The method used is based on rewriting the *Friedmann equation*, coming from Einstein equations, in a form similar to the classical equation of motion of a particle, *i.e.* conservation of total energy expression. Then, one identifies a generalized potential depending on the scale factor, the curvature, the type of matter and on a cosmological constant term.

By choosing a particular matter-source, a positive, negative or vanishing curvature and cosmological constant, one obtains many different types of Universe models with very distinctive behaviors.

Another attempt to consider FRW Universe models can be found in [13]. Here, one considers a Universe filled with dark energy, corresponding to a negative pressure and a combined type of matter, made of nonrelativistic dust and dark energy. Thus, by applying the same generalized classical potential method, one obtains several cosmological models, characterized by the age of the Universe, the particle horizon and the way the large-scale structures arose. This case includes also different types of dark energy like: the false vacuum, the curvature term and the phantom field, for which also the energy density is negative, being an exotic matter.

Further, one can take a negative curvature FRW metric with an Universe dominated by both dust and relativistic radiation [14]. Hence, by directly trying to solve the Friedmann equation, with positive, negative or null cosmological constant, one can write down the scale factor describing a de Sitter universe or an Einstein-de Sitter universe. Note that all the above discussions were based on using a *perfect fluid* matter-source, described by an equation of state connecting the pressure and the energy density, including all the matter types aforementioned.

Another interesting aspect is to consider the shape of the negative curvature FRW metric in *conformally flat spacetime coordinates*, as in [15]. In this way, one can describe the particle horizon and the Hubble parameter, with respect to the conformal time. Using this method, also the Milne Universe, the combination of dust and radiation and the vacuum energy cases are considered and interpreted. A particular effect, that arises within this framework, is the continual creation of conformal space in the matter-dominated Universe and the continual annihilation of conformal space in the vacuum-dominated Universe.

On the other hand, as in [16], one can use a *massive scalar field* type of matter, *e.g.* a Higgs field, together with the FRW metric with negative curvature. More specifically, a spontaneously Z_2-symmetry breaking scalar field is taken into account, giving rise to an energy-momentum tensor with quartically self-interacting terms. By deriving the Einstein-Gordon equations and solving them for a time independent field, one can obtain either a Milne Universe or an anti-de Sitter Universe.

2. FRW UNIVERSE MODELS WITH NEGATIVE CURVATURE

Consider the 4-dimensional dynamic FRW metric with negative curvature written as

$$ds^2 = a_0^2 e^{2f(t)} \left[d\psi^2 + \sinh^2\psi \, (d\theta^2 + \sin^2\theta \, d\varphi^2) \right] - dt^2, \tag{5}$$

where $a(t) = a_0 e^{f(t)}$ is the scale factor. We use the pseudo-orthonormal frame $e_{a\,(a=\overline{1,4})}$, whose corresponding dual base, $\{\omega^a\}$, is

$$\begin{aligned}
\omega^1 &= a_0 e^{f(t)}\, d\psi,\\
\omega^2 &= a_0 e^{f(t)} \sinh\psi\, d\theta,\\
\omega^3 &= a_0 e^{f(t)} \sinh\psi \sin\theta\, d\varphi,\\
\omega^4 &= dt,
\end{aligned}$$

together with the Cartan equations,

$$\begin{aligned}
d\omega^a &= \Gamma^a_{.[bc]}\, \omega^b \wedge \omega^c\,;\\
\mathcal{R}_{ab} &= d\Gamma_{ab} + \Gamma_{ac} \wedge \Gamma^c_{.b},
\end{aligned}$$

where Γ_{abc} are the connection coefficients, giving the connection one-forms, $\Gamma_{ab} = \Gamma_{abc}\,\omega^c$ and $\mathcal{R}_{ab} = R_{abcd}\,\omega^c \wedge \omega^d$ are the curvature two-forms. From Cartan's first equation, we obtain the one-forms:

$$\begin{aligned}
&\Gamma_{12} = -a_0^{-1} e^{-f} \coth\psi\, \omega^2, \quad \Gamma_{13} = -a_0^{-1} e^{-f} \coth\psi\, \omega^3;\\
&\Gamma_{14} = f_{|4}\, \omega^1, \quad \Gamma_{23} = -a_0^{-1} e^{-f} \sinh^{-1}\psi \cot\theta\, \omega^3;\\
&\Gamma_{24} = f_{|4}\, \omega^2, \quad \Gamma_{34} = f_{|4}\, \omega^3,
\end{aligned} \tag{6}$$

while from Cartan's second equation, we have the two-forms:

$$\begin{aligned}
&\mathcal{R}_{12} = -(a_0^{-2} e^{-2f} - f_{|4}^2)\, \omega^1 \wedge \omega^2, \quad &&\mathcal{R}_{13} = -(a_0^{-2} e^{-2f} - f_{|4}^2)\, \omega^1 \wedge \omega^3;\\
&\mathcal{R}_{14} = -(f_{|4}^2 + f_{|44})\, \omega^1 \wedge \omega^4, \quad &&\mathcal{R}_{23} = -(a_0^{-2} e^{-2f} - f_{|4}^2)\, \omega^2 \wedge \omega^3;\\
&\mathcal{R}_{24} = -(f_{|4}^2 + f_{|44})\, \omega^2 \wedge \omega^4, \quad &&\mathcal{R}_{34} = -(f_{|4}^2 + f_{|44})\, \omega^3 \wedge \omega^4.
\end{aligned} \tag{7}$$

Thus, one is able to compute the Ricci tensor

$$\begin{aligned}
R_{\alpha\alpha} &= -2a_0^{-2} e^{-2f} + 3f_{|4}^2 + f_{|44};\\
R_{44} &= -3(f_{|4}^2 + f_{|44}),
\end{aligned} \tag{8}$$

together with the Ricci scalar

$$R = -6a_0^{-2} e^{-2f} + 12 f_{|4}^2 + 6 f_{|44}, \tag{9}$$

in order to write down the Einstein tensor components, defined by $G_{ab} = R_{ab} - (1/2)R\,g_{ab}$, where $g_{ab} = diag(1,1,1,-1)$, *i.e.*

$$\begin{aligned}
G_{\alpha\alpha} &= -2\ddot{f} - 3\dot{f}^2 + a_0^{-2} e^{-2f};\\
G_{44} &= 3\dot{f}^2 - 3a_0^{-2} e^{-2f},
\end{aligned} \tag{10}$$

where the overdot means the derivative with respect to the cosmic time t.

At this stage, we take as matter-source the perfect fluid of energy-momentum tensor

$$T_{ab} = (\rho + P)\, u_a u_b + P\, g_{ab},$$

with $u_4 = -1$ and $u_\alpha = 0$, such that its components are $T_{\alpha\alpha} = P$, $T_{44} = \rho$.

Putting everything together, from the Einstein's system of equations

$$\begin{aligned} -2\ddot{f} - 3\dot{f}^2 + a_0^{-2} e^{-2f} + \Lambda =& k_0\, P \\ 3\dot{f}^2 - 3a_0^{-2} e^{-2f} - \Lambda =& k_0\, \rho, \end{aligned} \tag{11}$$

where $k_0 = 8\pi G/c^4$ is the Einstein's constant, one gets the corresponding Friedmann equation for $k = -1-$FRW Universe

$$\dot{f}^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} + \frac{e^{-2f}}{a_0^2}, \tag{12}$$

where $\dot{f} = \dot{a}/a \equiv H$ is the Hubble parameter. For a positive Hubble parameter the Universe is expanding, while a negative one implies a contraction. Also, the acceleration parameter is defined as

$$q = \frac{a(t)\,\ddot{a}(t)}{\dot{a}(t)^2} \equiv 1 + \frac{\ddot{f}(t)}{\dot{f}(t)^2}, \tag{13}$$

and is positive for accelerated expansion (contraction) and negative if the expansion (contraction) is decelerated.

Applying the conservation law on the energy-momentum tensor or by combining the two equations in (10) we get the relation

$$\dot{\rho} + 3\dot{f}\,(\rho + P) = 0\,, \tag{14}$$

for a perfect fluid, with the general equation of state $P = w\,\rho$, which leads, by integration, to the energy density

$$\rho = \rho_0\, e^{-3(1+w)f}, \tag{15}$$

where ρ_0 represents the present day value.

In what it follows, we consider each type of matter separately, and apply the derived Einstein equations, in order to identify and describe different Universe models. A similar classification was performed in [12], but using a different method, namely rewriting the Friedmann equation in a form equivalent to the classical equation of motion of a particle, upon introducing a generalized potential.

CASE 1: Nonrelativistic dust. For $w = 0 \Rightarrow P = 0$, the energy density is

$$\rho = \rho_0\, e^{-3f}. \tag{16}$$

Switching to $a(t) = a_0 e^{f(t)}$, the Einstein equations (11) turn into

$$\begin{aligned} 2a\ddot{a} + \dot{a}^2 - 1 - \Lambda a^2 &= 0 \\ \frac{3}{a^2}\dot{a}^2 - \frac{3}{a^2} - \Lambda &= k_0\,\rho, \end{aligned} \tag{17}$$

with the density given by

$$\rho = \rho_0\,(a/a_0)^{-3} \sim a^{-3}. \tag{18}$$

Now, assuming a zero cosmological constant, the first equation in system (17) leads to the relation

$$\frac{da}{\sqrt{Ca^{-1}+1}} = dt, \tag{19}$$

where C is a positive integration constant. Thus, we get the transcendental equation

$$t - t_* = \sqrt{a(C+a)} - C\,\mathrm{arcsinh}(\sqrt{a/C}), \tag{20}$$

with t_* an integration constant corresponding to a special universal (cosmic) moment.

By comparing the equation (19) with the second equation in (17), we can identify the constant C as being

$$C = \frac{k_0\rho_0 a_0^3}{3}, \tag{21}$$

where we recall that a_0, ρ_0 are the present day values. Furthermore, based on the substitution $\mathrm{arcsinh}\left(\sqrt{a/C}\right) = \chi$, we can derive from eq. (20) the following representation:

$$\begin{aligned} a(\chi) &= C\sinh^2(\chi) \\ t - t_* &= \frac{C}{2}\left[\sinh(2\chi) - 2\chi\right]. \end{aligned} \tag{22}$$

From the system above, we notice that for $\chi_* = 0$ and $t = t_*$ we get $a(\chi_*) = 0$, pointing out the Big Bang singularity at t_*. Also, for small χ, the behavior is $a \sim t^{2/3}$, leading to the conclusion that this model behaves like a dust-dominated Universe with *zero curvature*. This is due to the fact that initially, around the time t_*, the density of matter term dominates the curvature term. On the other hand, for χ large, we get $a \sim t$, recovering the *Milne model* of the Universe, since, as the time evolves, the curvature of space starts to dominate the matter density.

Further, the positive Hubble parameter,

$$H = \frac{1}{C}\frac{\coth\chi}{\sinh^2\chi}, \tag{23}$$

is indicating an expanding Universe from the initial Big Bang, at $t = t_*$, while the negative acceleration parameter (13)

$$q = -\frac{1}{2\cosh^2\chi}, \tag{24}$$

points out that the expansion is decelerated.

CASE 2: False vacuum. For $w = -1 \Rightarrow P = -\rho$, the system (10) becomes

$$\begin{aligned} -2\ddot{f} - 3\dot{f}^2 + a_0^{-2}e^{-2f} + \Lambda &= -\,k_0\,\rho \\ 3\dot{f}^2 - 3a_0^{-2}e^{-2f} - \Lambda &= k_0\,\rho, \end{aligned} \tag{25}$$

with a constant vacuum expectation value of the energy density, $\rho = \rho_0$, in view of the continuity equation (15). Adding the two equations in (25), we obtain

$$\ddot{f}e^{2f} = -a_0^{-2} \Rightarrow f(t) = \ln\left[\frac{b}{a_0}\sin\frac{t}{b}\right], \tag{26}$$

and the periodic scale factor

$$a(t) = b\sin\frac{t}{b}. \tag{27}$$

The expression of the integration constant, b, is obtained by replacing the logarithmic scale function (26) back into the second equation of (25), and therefore

$$\rho = -\frac{1}{k_0}\left(\frac{3}{b^2} + \Lambda\right) = -P \equiv \rho_0, \tag{28}$$

so that

$$b = \sqrt{-\frac{3}{k_0\rho_0 + \Lambda}}, \tag{29}$$

with the reality condition $\kappa_0\rho_0 + \Lambda < 0$.

The negative Λ corresponds to an anti-de Sitter space, which is also confirmed by the negative value of the Ricci scalar curvature (9),

$$R = -\frac{12}{b^2}.$$

The behavior of the scale function (26) around $t_* = 0$, *i.e.*

$$f(0) \to -\infty \Rightarrow a(0) = 0, \tag{30}$$

points out that, at $t_* = 0$, there is a Bang singular event, which is *not* a true *singularity* in spacetime, we mean a *real* Big-Bang, since the curvature is *constant* instead of being infinite. The same behavior of the scale function arises at $t_n = nb\pi$, indicating the presence of a countable periodic set of *Bangs and Crunches*.

As for the Hubble parameter,

$$H \equiv \dot{f} = \frac{1}{b}\cot\frac{t}{b}, \tag{31}$$

this oscillates between unlimited positive and negative values, suggesting the Bang-Crunch cycles and respectively, the expansion and contraction phases. The acceleration parameter

$$q = -\tan^2\left(\frac{t}{b}\right), \tag{32}$$

is always negative, so that, within this model, the Universe has a decelerating expansion followed by a decelerating contraction along the endless cycles of bangs and crunches.

For positive values of $\kappa_0\rho_0 + \Lambda$, the relation (29) can be rewritten as

$$b = i\sqrt{\frac{3}{k_0\rho_0 + \Lambda}} \equiv i\,B, \tag{33}$$

where B is real-valued and the cosmological scale factor turns into the better-known expression

$$a(t) = B\sinh\frac{t}{B}, \tag{34}$$

corresponding to the so-called $(k = -1)-$de Sitter Universe, with positive Ricci scalar.

Again, at $t_* = 0$, the scale factor vanishes, indicating the presence of a singular Bang event. Since the Hubble constant and the acceleration parameter are both positive quantities,

$$H = \frac{1}{B}\coth\frac{t}{B}, \tag{35}$$

$$q = \tanh^2\left(\frac{t}{B}\right), \tag{36}$$

it means that, in this model, the Universe is characterized by an asymptotically $q_M = 1$ accelerated expansion, from the initial singular event, towards an excessively (meaning $k = -1$) open Universe. For a classification of the Universe models, corresponding to various types of dark energy, including the cosmologically constant vacuum energy case, see [13].

CASE 3: CMB radiation. Considering $w = 1/3 \Rightarrow P = \rho/3$, the Einstein system of equations (11) with $\Lambda = 0$, becomes

$$\begin{aligned} 2\ddot{f} + 3\dot{f}^2 - \frac{e^{-2f}}{a_0^2} &= -\frac{k_0}{3}\rho \\ \dot{f}^2 - \frac{e^{-2f}}{a_0^2} &= \frac{k_0}{3}\rho\,. \end{aligned} \tag{37}$$

By summing them, it yields the differential equation

$$2\ddot{f} + 4\dot{f}^2 = \frac{2}{a_0^2}e^{-2f}\,,$$

which is nothing else but the very simple one

$$\frac{d^2}{dt^2}a^2(t) = 2\,, \tag{38}$$

since $a(t) = a_0 e^{f(t)}$. Consequently, its solution reads

$$a(t) = \left[t^2 + Kt + a_0^2\right]^{1/2}\,, \tag{39}$$

where $K \in \mathbf{R}_+$ is an integration constant, and we have set the present-day gauge (for the cosmic time t)

$$a(t=0) = a_0 \;\; i.e. \;\; t_0 = 0\,.$$

The CMBR energy density evolution can be derived from the $(k=-1)-$Friedmann equation, *i.e.*

$$\rho(t) = \frac{3}{\kappa_0}\left[\frac{\dot{a}^2}{a^2} - \frac{1}{a^2}\right],$$

which does actually mean

$$\rho(t) = \frac{3}{\kappa_0}\left(\frac{K^2}{4} - a_0^2\right)\left[t^2 + Kt + a_0^2\right]^{-2}. \tag{40}$$

On the other hand, the Hubble function becomes

$$H(t) = \frac{t + K/2}{t^2 + Kt + a_0^2}\,, \tag{41}$$

so that, one can work out the two cosmological parameters, K and a_0, with respect to the "present-day" values (at $t = t_0 \equiv 0$) of ρ and H:

$$\frac{\kappa_0}{3}\rho_0 = \frac{K^2}{4a_0^4} - \frac{1}{a_0^2} \;\; \text{and} \;\; H_0 = \frac{K}{2a_0^2}\,.$$

Thus, inserting $K = 2H_0a_0^2$ (from the second equation) into the first relation (for ρ_0) it yields the cosmological *scale* parameter a_0 and the $K-$integration constant as being

$$\text{(a)}\quad a_0 = \frac{H_0^{-1}}{\sqrt{1-\Omega_R}}\,,$$
$$\text{(b)}\quad K = \frac{2H_0^{-1}}{1-\Omega_R}\,,\ \text{where } \Omega_R = \frac{\kappa_0\rho_0}{3H_0^2}\,. \tag{42}$$

Thence, in full, the cosmological scale function becomes

$$a(t) = \left(t^2 + \frac{2H_0^{-1}}{1-\Omega_R}\,t + \frac{H_0^{-2}}{1-\Omega_R}\right)^{1/2}, \tag{43}$$

leading to the past-directed value, $t = t_* < 0$, of the Big-Bang event, $a(t_*) = 0$, namely

$$t_* = -\frac{H_0^{-1}}{1+\sqrt{\Omega_R}}\,. \tag{44}$$

In modulus, that would be the age of the spatially hyperbolic Universe if it were dominated by the isotropic radiation alone. For a combined matter-source, containing both nonrelativistic dust and radiation, including a cosmological constant, see paper [14]. Note that also the cases of positive and vanishing curvature are considered and described therein.

CASE 4: Stiff matter. For $w = 1 \Rightarrow P = \rho$, the energy density is

$$\rho = \rho_0\,(a/a_0)^{-6} \tag{45}$$

and we get, from the corresponding Einstein equations, the relation

$$a\ddot{a} + 2\dot{a}^2 - 2 = 0, \tag{46}$$

so that

$$\frac{da}{\sqrt{Ca^{-4}+1}} = dt\,, \tag{47}$$

leading to an elliptic integral of the second kind. However, in the limit $a \to 0$, we obtain the following dependence of the scale function on time,

$$a(t) = (3\sqrt{C})^{1/3} t^{1/3}\,, \tag{48}$$

where the explicit form of the integration constant is

$$C = \frac{k_0\rho_0 a_0^6}{3}, \tag{49}$$

pointing out the Big Bang singularity, at $t_* = 0$, from which the Universe evolves with a decelerated expansion, due to the negative acceleration parameter.

Once we turn to the *conformal time* τ [15, 17], by

$$d\tau = dt/a(t), \tag{50}$$

the Einstein equations have the very simple form

$$\begin{aligned} &h' + 2h^2 - 2 = 0 \\ &3(h^2 - 1) = k_0 \rho a^2, \end{aligned} \tag{51}$$

where $h(\tau) \equiv a'(\tau)/a(\tau)$, with differentiation with respect to τ. By inspecting the solutions

$$\begin{aligned} &h(\tau) = \tanh(2\tau)\ ,\ a(\tau) = \sqrt{\cosh(2\tau)}\ ,\ \text{for } h^2 < 1\,; \\ &h(\tau) = \coth(2\tau)\ ,\ a(\tau) = \sqrt{\sinh(2\tau)}\ ,\ \text{for } h^2 > 1, \end{aligned} \tag{52}$$

both corresponding to a de Sitter ($k = -1$) Universe, one may notice the Big Bang singularity only for $h^2 > 1$ $a(\tau = 0) = 0$ and no singularity for $h^2 < 1$ ($a(\tau = 0) = 1$).

In what it concerns the Hubble constant and the acceleration parameter, these are given by

$$\begin{aligned} &H = \frac{\sinh(2\tau)}{\cosh^{3/2}(2\tau)}\ ,\ q = \frac{2}{\sinh^2(2\tau)}\ ,\ \text{for } h^2 < 1\ ; \\ &H = \frac{\cosh(2\tau)}{\sinh^{3/2}(2\tau)}\ ,\ q = -\frac{2}{\cosh^2(2\tau)}\ ,\ \text{for } h^2 > 1 \end{aligned} \tag{53}$$

signaling the decelerated expansion of the Universe from the Big Bang, for $h^2 > 1$, and the acceleratingly expanding never ending Universe, for $h^2 < 1$.

3. EINSTEIN-GORDON EQUATIONS IN $(k = -1)-$FRW COSMOLOGIES

In this section, we deal with the case where the $(k = -1)-$FRW geometry is driven by the spontaneously Z_2-symmetry breaking massive scalar field alone, ϕ. A connection between the structure of spacetime in Cosmology and the spontaneous symmetry breaking of vacua is given in [18].

The inner parity invariant Lagrangian density, describing the real scalar field, is:

$$\mathcal{L}[\Phi] = -\frac{1}{2}\eta^{ab}\,\Phi_{|a}\Phi_{|b} + \frac{1}{2}\mu^2\,\Phi^2 - \frac{\lambda}{24}\,\Phi^4\,, \tag{54}$$

where μ^2 and λ are the two positive parameters that accommodate the spontaneously symmetry breaking mechanism. The functional expression

$$\boldsymbol{T} = -\frac{2}{\sqrt{-g}}\,\frac{\delta}{\delta \boldsymbol{g}}\left[\sqrt{-g}\,\mathcal{L}\right] \tag{55}$$

leads to the covariant components of the conservative energy-momentum tensor

$$T_{ab} = \Phi_{|a}\,\Phi_{|b} - \frac{1}{2}\,\eta_{ab}\left[\Phi^{|c}\,\Phi_{|c} - \mu^2\,\Phi^2 + \frac{\lambda}{12}\,\Phi^4\right] \tag{56}$$

so that the Hamiltonian density

$$\mathcal{H} = T_{44} = \frac{1}{2}\,\delta^{ab}\,\Phi_{|a}\,\Phi_{|b} - \frac{\mu^2}{2}\,\Phi^2 + \frac{\lambda}{24}\,\Phi^4 \tag{57}$$

has the extrema given by the equation

$$\frac{\partial\mathcal{H}}{\partial\Phi}\,(\,\Phi_0\,) = \Phi_0\left(\frac{\lambda}{6}\,\Phi_0^2 - \mu^2\right) = 0 \tag{58}$$

Inspecting the sign of the second derivative for each of the three roots, it results that $\Phi_0^0 = 0$ is an *unstable* fixed point, while $\Phi_0^{\pm} = \pm\mu\sqrt{6/\lambda}$ are the "real" *minima* which correspond to the two possible ground states of the (initially fictitious) scalar field Φ. Choosing $v = \Phi_0^+ = \mu\sqrt{\frac{6}{\lambda}}$ as the vacuum expectation value of Φ and accordingly shifting the field

$$\Phi = v + \phi\,, \text{ where } \phi : M_4 \to \mathbf{R}\,, \tag{59}$$

we get the spontaneously $\boldsymbol{Z}_2-$broken Lagrangian density

$$\mathcal{L}[\phi] = -\frac{1}{2}\,\phi^{|c}\,\phi_{|c} - \frac{1}{2}(2\mu^2)\phi^2 - \mu\sqrt{\frac{\lambda}{6}}\,\phi^3 - \frac{\lambda}{24}\,\phi^4 + \frac{3\,\mu^4}{2\,\lambda} \tag{60}$$

The *real* massive scalar field ϕ obeys the generalized Gordon equation

$$\Box\phi - \left(2\,\mu^2\right)\,\phi = 3\,\mu\sqrt{\frac{\lambda}{6}}\,\phi^2 + \frac{\lambda}{6}\,\phi^3\,, \tag{61}$$

where

$$\Box\phi = \frac{1}{\sqrt{-g}}\,\frac{\partial}{\partial x^i}\left[\sqrt{-g}\,g^{ik}\,\frac{\partial\phi}{\partial x^k}\right] \tag{62}$$

is the d'Alembert operator on M_4, in terms of some local coordinates $\{x^i\}_{i=\overline{1,4}}$.

From (56) with the shift (59), it yields the components of the energy-momentum tensor $\mathbf{T}[\phi]$

$$T_{ab} = \phi_{|a}\,\phi_{|b} - \frac{1}{2}\,\eta_{ab}\left[\phi^{|c}\,\phi_{|c} + 2\,V(\phi) - \frac{3\,\mu^4}{\lambda}\right]\,, \tag{63}$$

or, in the case of a coherent (spatially homogeneous) field $\phi(t)$,

$$\begin{aligned} &(a)\quad T_{\alpha\beta} = -\frac{1}{2}\left[-\left(\phi_{|4}\right)^2 + 2\,V(\phi) - \frac{3\mu^4}{\lambda}\right]\,\delta_{\alpha\beta} \\ &(b)\quad T_{44} = \frac{1}{2}\left[\left(\phi_{|4}\right)^2 + 2\,V(\phi) - \frac{3\mu^4}{\lambda}\right], \end{aligned} \tag{64}$$

where the total, semi-classical, potential

$$V(\phi) = \mu^2 \phi^2 + \mu\sqrt{\frac{\lambda}{6}}\phi^3 + \frac{\lambda}{24}\phi^4 \tag{65}$$

is clearly no longer invariant under the discrete transformation $\phi \to -\phi$.

The whole set of Einstein-Gordon equations

$$G_{ab}[f] = \kappa_0 T_{ab}[\phi]$$

leads to the following functionally 2-dimensional nonlinear differential system

$$\begin{aligned} &(a) \quad 2 f_{,44} + 3\,(f_{,4})^2 - \frac{e^{-2f}}{a_0^2} = \frac{\kappa_0}{2}\left[-\,(\phi_{,4})^2 + 2\,V(\phi) - \frac{3\mu^4}{\lambda}\right]\,, \\ &(b) \quad 3\left[(f_{,4})^2 - \frac{e^{-2f}}{a_0^2}\right] = \frac{\kappa_0}{2}\left[(\phi_{,4})^2 + 2\,V(\phi) - \frac{3\mu^4}{\lambda}\right]\,, \end{aligned} \tag{66}$$

where the semi-classical potential V is given by (65).

The particular case $\phi_{|4} = 0$ is of certain interest for it reveals the simplest $(k = -1)-$FRW spacetime dynamics in the fixed points of the nonlinear Gordon equation for the physical field ϕ left-over by the spontaneous breakdown of the Z_2-invariance. Once we observe that, for $\phi_{,4} = 0$, the right-hand-side of the two Einstein equations (66) has the same form, we get, by subtracting them, the following essential equation

$$f_{,44} + \frac{e^{-2f}}{a_0^2} = 0\,. \tag{67}$$

Inserting it back into equation (a) in (66), it gives the (same) equation (b) in (66). Hence, among the three Einstein-Gordon equations (61), (66), we just have to solve the very simple system

$$\begin{aligned} &(a)\ (f_{,4})^2 - \frac{e^{-2f}}{a_0^2} = \frac{\kappa_0}{3}V(\phi) - \frac{\kappa_0\mu^4}{2\lambda} \\ &(b)\ \phi\left[2\mu^2 + 3\mu\sqrt{\frac{\lambda}{6}}\phi + \frac{\lambda}{6}\phi^2\right] = 0\,, \end{aligned} \tag{68}$$

where $V(\phi)$ has concretely taken the form

$$V(\phi) = \frac{\lambda}{24}\phi^2\left[\phi^2 + 4\mu\sqrt{\frac{6}{\lambda}}\phi + \frac{24\mu^2}{\lambda}\right] \tag{69}$$

Ordered by their magnitudes, the roots $-$ meaning the matter-field fixed-point values $-$ read

$$\phi_L = -2\mu\sqrt{\frac{6}{\lambda}}\,,\ \phi_M = -\mu\sqrt{\frac{6}{\lambda}}\,,\ \phi_R = 0\,, \tag{70}$$

where the indices L, M, R come from *left*, *Milne*, *right*, respectively. By their insertion in (69), one gets the corresponding values of the semi-classical potential

$$V_L = 0\,,\; V_M = \frac{3\mu^4}{2\lambda}\,,\; V_R = 0\,, \tag{71}$$

so that, the nonlinear first-order differential equation (a) in (68) does only take the following two particular forms

$$\begin{aligned} &(a)\; (f_{,4})^2 - \frac{e^{-2f}}{a_0^2} = -\frac{\kappa_0\mu^4}{2\lambda}\,,\ \text{for}\ V_L = V_R = 0\,,\\ &(b)\; (f_{,4})^2 - \frac{e^{-2f}}{a_0^2} = 0\,,\ \text{for}\ V_M = \frac{3\mu^4}{2\lambda} \end{aligned} \tag{72}$$

In the simpler case (b) in (72), it also results that

$$f_{,44} + (f_{,4})^2 = 0 \tag{73}$$

and thus, having a look at the components (64), we realize that all of them vanish. Hence, the spacetime corresponding to (72) (b), supported by the static physical field $\phi_M = -\mu\sqrt{6/\lambda}$, is *flat*, being basically (geometrically) the Minkowski spacetime. However, especially from a cosmological perspective, the difference is that this spacetime is patched in a different coordinate-system, namely the Milne's one, which sharply presents the evolution of the H^3-spacelike-foliation, instead of the static picture of the Cartesian $\mathbf{R}^3$-foliations (of constant Minkowskian time, $x^4 = t$).

The other $(k=-1)-$FRW model, (a) in (72), together with the result

$$f_{,44} + (f_{,4})^2 = -\frac{\kappa_0\mu^4}{2\lambda}\,, \tag{74}$$

lead to the expressions

$$\begin{aligned} &(a)\; R_{\alpha\beta\alpha\beta} = -R_{\alpha4\alpha4} = -\frac{\kappa_0\mu^4}{2\lambda}\,,\;\; \alpha,\beta = \overline{1,3}\,,\\ &(b)\; R = -12\left(\frac{\kappa_0\mu^4}{2\lambda}\right), \end{aligned} \tag{75}$$

which point out that the solution to the "basic" equation (72) (a) does clearly describe a $(k=-1)-$FRW Universe of constant negative (4D-)curvature and that can only be the anti-de Sitter spacetime.

An investigation on its possibly observable cosmological consequences, related to the spontaneous breaking of the field-reflection symmetry, has been performed in paper [16].

4. CONCLUSIONS

The present paper deals with FRW Universe models with negative curvature.

For an Universe filled with nonrelativistic dust, the transcendental equation (20) is suggesting that the age of the Universe cannot be exactly, analytically computed. However, using a suitable parametrization, we have come to a scale factor describing an Universe with a decelerated expansion from a Big Bang point, t_*, tending towards a Milne Universe, with t_* being the actual *age of the Universe*, given by the present value H_0 in eq. (23).

The false vacuum case ($P = -\rho$) is the most intriguing since, depending on the sign of the cosmological constant, one has an Anti-de Sitter space ($\Lambda < 0$) with decelerated expansions/contractions between endless cycles of Bangs and Crunches or a de Sitter space ($\Lambda > 0$) with accelerated expansion.

In the case of a CMB radiation, the cosmological scale function (43) is leading to the past directed value $t_* < 0$ of the Big Bang event and to the age of the spatially hyperbolic Universe dominated by radiation alone.

Since for $P = \rho$ (stiff matter), the relation (46) coming from the Einstein system of equations is leading to a scale factor expressed as an elliptic integral of the second kind, we have switched to the conformal time. As a remark, as $a \to 0$, corresponding to $h^2 > 1$, recall that $a(t) \sim t^{1/3}$, implying that $\tau = (3/2)t^{2/3}$, both the proper time and the conformal time go simultaneously to zero and the existence of the Big Bang point is confirmed.

In the final section, we have started by by noticing that, unlike for $k = \{0,1\}$, where the G_{44}-component does always stand non-negative, in the case $k = -1$, the sign of G_{44} (given in (10)) and the one of the energy density are undefined, unless $f_{|4} = 0$, when the Einstein equations demand a deeply exotic kind of matter. Thus, as in [16], we have considered the quartically self-interacting massive scalar field, arisen from the inner parity spontaneous breaking, as an exact solution to the Einstein-Gordon equations. In the system fixed points, *i.e.* at the three Z_2-symmetric extrema of the Hamiltonian density, it turns out that the absolute minima representing the matter-source degenerate vacua, is (geometro-dynamically) consistent *only* with the $(k = -1)-$*family* of FRW manifolds, which actually are either Milne or anti-de Sitter Universes. The possibility of a (quasi)Milne stage has actually not been completely ruled out and some experimental evidences sustaining the existence of a ($k = -1$) FRW Universe with linearly expanding scale factor are presented in [19].

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