Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 960 (2020) 115173

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Affine Yangian and 3-Schur functions

Na Wang

School of Mathematics and Statistics, Henan University, Kaifeng, 475001, China

Received 30 June 2020; received in revised form 5 September 2020; accepted 12 September 2020
Available online 16 September 2020
Editor: Hubert Saleur

Abstract

3D (3 dimensional) Young diagram is a generalization of 2D Young diagram. In this paper, from the orthogonality of 3D Young diagrams and the properties in affine Yangian and its MacMahon representation, we obtain the Schur functions corresponding to 3D Young diagrams, which are called 3-Schur functions. 3-Schur functions are a generalization of Schur functions in the sense that when $h_1 = 1, h_2 = -1, h_3 = 0$, the 3-Schur functions of 3D Young diagrams become Schur functions of 2D Young diagrams, which is a special case of $h_1 = h, h_2 = -\frac{1}{h}, h_3 = \frac{1}{h} - h$. When $h_1 = h, h_2 = -\frac{1}{h}, h_3 = \frac{1}{h} - h$, the 3-Schur functions turn into the Jack symmetric polynomials of 2D Young diagrams by multiplying a coefficient. We will see that 3-Schur functions are symmetric about three coordinate axes.

1. Introduction

Let $\mathbf{p} = (p_1, p_2, \cdots)$. The operators $h_n(\mathbf{p})$ are determined by the generating function:

$$H(z) = \sum_{n=0}^{\infty} h_n(\mathbf{p}) z^n = e^{\xi(\mathbf{p},z)}, \quad \xi(\mathbf{p}, z) = \sum_{n=1}^{\infty} \frac{p_n}{n} z^n \tag{1}$$

and set $h_n(\mathbf{x}) = 0$ for $n < 0$. Note that $h_n(\mathbf{p})$ is the complete homogeneous symmetric function by the Miwa transform, i.e., replacing p_i with the power sum $\sum_k x_k^i$. For 2D Young diagram $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$, the Schur function $S_\lambda = S_\lambda(\mathbf{p})$ is a polynomial of variables $\mathbf{p}$ in $\mathbb{C}[\mathbf{p}]$ defined by the Jacobi-Trudi formula [1–3]:

E-mail address: wangna@henu.edu.cn.

https://doi.org/10.1016/j.nuclphysb.2020.115173
0550-3213/

$$S_\lambda(\mathbf{p}) = \det\left(h_{\lambda_i - i + j}(\mathbf{p})\right)_{1\le i,j\le l}. \tag{2}$$

2D Young diagrams and Schur functions are attractive research objects, which were used to determine irreducible characters of highest weight representations of the classical groups [1,2,4]. Recently they appear in mathematical physics, especially in integrable models. In [5], the group in the Kyoto school uses Schur functions in a remarkable way to understand the KP and KdV hierarchies. In [6,7], Tsilevich and Sułkowski, respectively, give the realization of the phase model in the algebra of Schur functions and build the relations between the q-boson model and Hall-Littlewood functions. In [8], we build the relations between the statistical models, such as phase model, and KP hierarchy by using 2D Young diagrams and Schur functions.

3D Young diagram (plane partition) is a generalization of 2D Young diagram, which arose naturally in crystal melting model [9,10]. 3D Young diagrams also have many applications in many fields of mathematics and physics, such as statistical models, number theory, representations of some algebras (Ding-Iohara-Miki algebras, affine Yangian, etc). In this paper, we consider the relations between 3D Young diagrams and affine Yangian. The Yangian of a finite dimensional simple Lie algebra $\mathfrak{g}$ is defined by Drinfeld [11,12] in order to obtain a solution of the Yang-Baxter equation. The Yangian is a quantum group which is the deformation of the current algebra $\mathfrak{g}[z]$. In this paper, we will use the properties of affine Yangian and its MacMahon representation.

We know that the vector space of 2D Young diagrams is isomorphic to that of Schur functions. In many cases, we do not distinguish 2D Young diagram and its corresponding Schur function. For example, let λ, μ be 2D Young diagrams and S_λ, S_μ their corresponding Schur functions, the orthogonality of Schur functions [1,2]

$$\langle S_\lambda | S_\mu\rangle = \delta_{\lambda,\mu} \tag{3}$$

can also be written as the orthogonality of 2D Young diagrams

$$\langle \lambda | \mu\rangle = \delta_{\lambda,\mu}. \tag{4}$$

The 3D Young diagram is a generalization of 2D Young diagram, but 3D Young diagram does not have many structures which are clear on 2D Young diagram. There are some work about these [13,14] recently. In [15], we calculate the orthogonality of 3D Young diagrams according to the properties of affine Yangian and its MacMahon representation. In this paper, we treat the orthogonality of 3D Young diagrams as that of 3-Schur functions corresponding to 3D Young diagrams, then we calculate 3-Schur functions. We will see the 3-Schur functions are symmetric about h_1, h_2, h_3 which are crucial parameters in affine Yangian. From [13], we know that the 3-Schur functions are functions depending on the triangular set of variables P_1, $P_{2,1}$, $P_{2,2}$, $P_{3,1}$, $P_{3,2}$, $P_{3,3}$, $\cdots$, here we use capital P denoting the variables of 3-Schur functions to distinguish that p_n are the variables of Schur functions.

The paper is organized as follows. In section 2, we recall the definition of affine Yangian of $\mathfrak{gl}(1)$ and its MacMahon representation first, then we recall the orthogonality of 3D Young diagrams we calculated. In section 3, we calculate the 3-Schur functions of 3D Young diagrams by the orthogonality of 3D Young diagram. In section 4, we show that the 3-Schur functions obtained in last section are already exist in affine Yangian and its MacMahon representation. In section 5, we show that 3-Schur functions of 3D Young diagrams with more than one layer in the z-axis direction vanish, and 3-Schur functions of 3D Young diagrams with one layer in the z-axis direction become the Schur functions of 2D Young diagrams when $h_1 = 1, h_2 = -1, h_3 = 0$. In section 6, we show that 3-Schur functions of 3D Young diagrams are reduced to the Jack symmetric polynomials of 2D Young diagrams when $h_1 = h, h_2 = -\frac{1}{h}, h_3 = \frac{1}{h} - h$, and the result in last section is the special case $h = 1$ of that in this section.

2. Affine Yangian and the orthogonality of 3D Young diagrams

In this section, we recall the definition of the affine Yangian of $\mathfrak{gl}(1)$ and its MacMahon representation as in papers [16,17] first. Then we recall the orthogonality we given in paper [15]. The affine Yangian $\mathcal{Y}$ of $\mathfrak{gl}(1)$ is an associative algebra with generators e_j, f_j and ψ_j, $j = 0, 1, \ldots$ and the following relations [16,17]

$$\left[\psi_j, \psi_k\right] = 0, \tag{5}$$

$$\left[e_{j+3}, e_k\right] - 3\left[e_{j+2}, e_{k+1}\right] + 3\left[e_{j+1}, e_{k+2}\right] - \left[e_j, e_{k+3}\right] \\ + \sigma_2\left[e_{j+1}, e_k\right] - \sigma_2\left[e_j, e_{k+1}\right] - \sigma_3\left\{e_j, e_k\right\} = 0, \tag{6}$$

$$\left[f_{j+3}, f_k\right] - 3\left[f_{j+2}, f_{k+1}\right] + 3\left[f_{j+1}, f_{k+2}\right] - \left[f_j, f_{k+3}\right] \\ + \sigma_2\left[f_{j+1}, f_k\right] - \sigma_2\left[f_j, f_{k+1}\right] + \sigma_3\left\{f_j, f_k\right\} = 0, \tag{7}$$

$$\left[e_j, f_k\right] = \psi_{j+k}, \tag{8}$$

$$\left[\psi_{j+3}, e_k\right] - 3\left[\psi_{j+2}, e_{k+1}\right] + 3\left[\psi_{j+1}, e_{k+2}\right] - \left[\psi_j, e_{k+3}\right] \\ + \sigma_2\left[\psi_{j+1}, e_k\right] - \sigma_2\left[\psi_j, e_{k+1}\right] - \sigma_3\left\{\psi_j, e_k\right\} = 0, \tag{9}$$

$$\left[\psi_{j+3}, f_k\right] - 3\left[\psi_{j+2}, f_{k+1}\right] + 3\left[\psi_{j+1}, f_{k+2}\right] - \left[\psi_j, f_{k+3}\right] \\ + \sigma_2\left[\psi_{j+1}, f_k\right] - \sigma_2\left[\psi_j, f_{k+1}\right] + \sigma_3\left\{\psi_j, f_k\right\} = 0, \tag{10}$$

together with boundary conditions

$$\left[\psi_0, e_j\right] = 0, \left[\psi_1, e_j\right] = 0, \left[\psi_2, e_j\right] = 2e_j, \tag{11}$$

$$\left[\psi_0, f_j\right] = 0, \left[\psi_1, f_j\right] = 0, \left[\psi_2, f_j\right] = -2f_j, \tag{12}$$

and a generalization of Serre relations

$$\mathrm{Sym}_{(j_1,j_2,j_3)}\left[e_{j_1}, \left[e_{j_2}, e_{j_3+1}\right]\right] = 0, \tag{13}$$

$$\mathrm{Sym}_{(j_1,j_2,j_3)}\left[f_{j_1}, \left[f_{j_2}, f_{j_3+1}\right]\right] = 0, \tag{14}$$

where Sym is the complete symmetrization over all indicated indices which include 6 terms. In this paper, we set $\psi_0 = 1$ with no loss of generality.

The notations σ_2, σ_3 in the definition of affine Yangian are functions of three complex numbers h_1, h_2 and h_3:

$$\sigma_1 = h_1 + h_2 + h_3 = 0,$$

$$\sigma_2 = h_1 h_2 + h_1 h_3 + h_2 h_3,$$

$$\sigma_3 = h_1 h_2 h_3.$$

The affine Yangian $\mathcal{Y}$ has a representation on the plane partitions. A plane partition π is a 2D Young diagram in the first quadrant of plane xOy filled with non-negative integers that form nonincreasing rows and columns [18]. The number in the position (i, j) is denoted by $\pi_{i,j}$

$$\begin{pmatrix} \pi_{1,1} & \pi_{1,2} & \cdots \\ \pi_{2,1} & \pi_{2,2} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}.$$

The integer $\pi_{i,j}$ satisfies

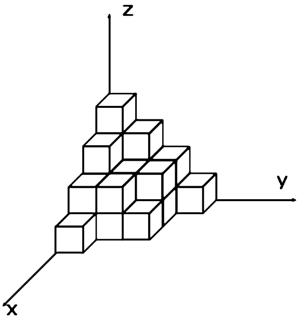

Fig. 1. A 3D Young diagram.

$$\pi_{i,j} \geq \pi_{i+1,j}, \quad \pi_{i,j} \geq \pi_{i,j+1}, \quad \lim_{i\to\infty} \pi_{i,j} = \lim_{j\to\infty} \pi_{i,j} = 0$$

for all integers $i, j \geq 0$.

Piling $\pi_{i,j}$ cubes over position (i, j) gives a three dimensional object in the first octant of space Cartesian coordinate system $O - xyz$. This three dimensional object is called 3D Young diagram. 3D Young diagrams arose naturally in the melting crystal model [9,10]. We always think that 3D Young diagrams and plane partitions are the same thing. For example, the 3D Young diagram in Fig. 1

can also be denoted by plane partition

$$\begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 2 & 2 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The total volume of the 3D Young diagram π is defined by

$$|\pi| = \sum_{i,j=1}^{\infty} \pi_{i,j}. \tag{15}$$

If $|\pi| = n$, we say the 3D Young diagram π is a plane partition of integer n. Let $p(n)$ denote the number of plane partitions π with $|\pi| = n$. The generating function of numbers $p(n)$ was given by MacMahon function:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} \tag{16}$$

For a 3D Young diagram π, the notation $\square \in \pi^+$ means that this box is not in π and can be added to π, here "can be added" means that when this box is added, it is still a 3D Young diagram. The notation $\square \in \pi^-$ means that this box is in π and can be removed from π, here "can be removed" means that when this box is removed, it is still a 3D Young diagram. For a box $\square$, we let

$$h_\square = h_1 y_\square + h_2 x_\square + h_3 z_\square, \tag{17}$$

where $(x_\square, y_\square, z_\square)$ is the coordinate of box $\square$ in coordinate system $O - xyz$. Here we use the order $y_\square, x_\square, z_\square$ to match that in paper [16].

Following [16,17], we introduce the generating functions as follows:

$$e(u)=\sum_{j=0}^{\infty}\frac{e_j}{u^{j+1}},$$
$$f(u)=\sum_{j=0}^{\infty}\frac{f_j}{u^{j+1}}, \tag{18}$$
$$\psi(u)=1+\sigma_3\sum_{j=0}^{\infty}\frac{\psi_j}{u^{j+1}},$$

where u is a parameter. Introduce

$$\psi_0(u)=\frac{u+\sigma_3}{u} \tag{19}$$

and

$$\varphi(u)=\frac{(u+h_1)(u+h_2)(u+h_3)}{(u-h_1)(u-h_2)(u-h_3)}. \tag{20}$$

For a 3D Young diagram π, define $\psi_\pi(u)$ by

$$\psi_\pi(u)=\psi_0(u)\prod_{\square\in\pi}\varphi(u-h_\square). \tag{21}$$

In the following, we recall the representation of affine Yangian on 3D Young diagrams as in paper [16] by making a slight change. The representation of affine Yangian on 3D Young diagrams is given by

$$\psi(u)|\pi\rangle=\psi_\pi(u)|\pi\rangle, \tag{22}$$
$$e(u)|\pi\rangle=\sum_{\square\in\pi^+}\frac{E(\pi\to\pi+\square)}{u-h_\square}|\pi+\square\rangle, \tag{23}$$
$$f(u)|\pi\rangle=\sum_{\square\in\pi^-}\frac{F(\pi\to\pi-\square)}{u-h_\square}|\pi-\square\rangle \tag{24}$$

where $|\pi\rangle$ means the state characterized by 3D Young diagram π and the coefficients

$$E(\pi\to\pi+\square)=-F(\pi+\square\to\pi)=\sqrt{\frac{1}{\sigma_3}\text{res}_{u\to h_\square}\psi_\pi(u)}. \tag{25}$$

In paper [16], the relations are

$$E(\pi\to\pi+\square)=F(\pi+\square\to\pi)=\sqrt{-\frac{1}{\sigma_3}\text{res}_{u\to h_\square}\psi_\pi(u)},$$

this is the slight change we make. Equations (23) and (24) mean generators $e_j,\ f_j$ acting on 3D Young diagram π by

$$e_j|\pi\rangle=\sum_{\square\in\pi^+}h_\square^j E(\pi\to\pi+\square)|\pi+\square\rangle, \tag{26}$$
$$f_j|\pi\rangle=\sum_{\square\in\pi^-}h_\square^j F(\pi\to\pi-\square)|\pi-\square\rangle. \tag{27}$$

The triangular decomposition of affine Yangian $\mathcal{Y}$ is

$$\mathcal{Y} = \mathcal{Y}^+ \oplus \mathcal{B} \oplus \mathcal{Y}^- \tag{28}$$

where $\mathcal{Y}^+$ is the subalgebra generated by generators e_j with relations (6) and (13), $\mathcal{B}$ is the commutative subalgebra with generators ψ_j, $\mathcal{Y}^-$ is the subalgebra generated by generators f_j with relations (7) and (14).

Define the anti-automorphism $\tilde{a}$ by

$$\tilde{a}(e_j) = -f_j \tag{29}$$

The quadratic form on $\mathcal{Y}^+|0\rangle$ is defined by

$$\tilde{B}(x|0\rangle, y|0\rangle) = \langle 0|\tilde{a}(y)x|0\rangle \tag{30}$$

where $x, y \in \mathcal{Y}^+$. Note that the quadratic form here is different from the Shapovalov form in [16]. For 3D Young diagrams π, π' and let $\pi = x|0\rangle, \pi' = y|0\rangle$ for $x, y \in \mathcal{Y}^+$, define the orthogonality

$$\langle \pi'|\pi\rangle = \langle 0|\tilde{a}(y)x|0\rangle. \tag{31}$$

For a 3D Young diagram π, let n denote its total volume $|\pi|$. We describe the orthogonality in the order of n. When $n = 0$, we set $\langle 0|0\rangle = 1$. When $n = 1$, there is only one 3D Young diagram (1), we have

$$\langle (1)|(1)\rangle = 1.$$

When $n = 2$, from MacMahon function, we know that there are 3 3D Young diagrams, which are $(1, 1)$, (2), $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we have

$$\begin{aligned}
|(1,1)\rangle &= \frac{1}{(h_1-h_2)(h_1-h_3)}\,(e_2 + h_1 e_1 + h_2 h_3 e_0)\, e_0|0\rangle, \\
|\begin{pmatrix} 1 \\ 1 \end{pmatrix}\rangle &= \frac{1}{(h_2-h_1)(h_2-h_3)}\,(e_2 + h_2 e_1 + h_1 h_3 e_0)\, e_0|0\rangle, \\
|(2)\rangle &= \frac{1}{(h_3-h_1)(h_3-h_2)}\,(e_2 + h_3 e_1 + h_1 h_2 e_0)\, e_0|0\rangle.
\end{aligned} \tag{32}$$

From the relations in (8), we have

$$\begin{aligned}
\langle (1,1)|(1,1)\rangle &= \frac{2(1+h_2h_3)}{(h_1-h_2)(h_1-h_3)}, \quad \langle (1,1)|\begin{pmatrix} 1 \\ 1 \end{pmatrix}\rangle = 0, \\
\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}|\begin{pmatrix} 1 \\ 1 \end{pmatrix}\rangle &= \frac{2(1+h_1h_3)}{(h_2-h_1)(h_2-h_3)}, \quad \langle (1,1)|(2)\rangle = 0, \\
\langle (2)|(2)\rangle &= \frac{2(1+h_1h_2)}{(h_3-h_1)(h_3-h_2)}, \quad \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}|(2)\rangle = 0.
\end{aligned} \tag{33}$$

Here, we can see the symmetry of h_1, h_2, h_3 corresponds to rotating 3D Young diagram, for example, changing 3D Young diagram $(1, 1)$ to (2) on the left hand of equations above corresponds to exchanging h_1 and h_3.

When $n = 3$, from MacMahon function, we know that there are 6 3D Young diagrams, which are $(1, 1, 1)$, $(2, 1)$, (3), $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. There are two ways to get 3D Young diagram $(2, 1)$, which are

$$(1) \to (1,1) \to (2,1),$$
$$(1) \to (2) \to (2,1).$$

We denote the state corresponding to $(2,1)$ in the first equation of the two equations above by $|(2,1)\rangle_{h_1,h_3}$, and that in the second equation above by $|(2,1)\rangle_{h_3,h_1}$. We explain the subscripts: let $h_\square = x_\square h_2 + y_\square h_1 + z_\square h_3$, we use $h_\square$-position to represent the position $(x_\square, y_\square, z_\square)$. The notation "$h_1, h_3$" means adding one box to $\square$ in h_1-position first, then adding one box in h_3-position. Even though h_1-position is not unique, for example, h_1-position can be the positions $(1,2,1), (2,3,2), \cdots$ since $h_1+h_2+h_3=0$, but it is unique if we want to get a new 3D Young diagram after adding this box. Therefore, we can read the notation $|\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}\rangle_{h_1,h_2}$, which means the 3D Young diagram $|\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}\rangle$ is obtained from $\square$ by adding one box in h_1-position first, then adding one box in h_2-position. When there is no confusion, we will omit the subscripts.

From the MacMahon representation of affine Yangian, we know that

$$\begin{aligned}
|(1,1,1)\rangle &= \frac{1}{(2h_1-h_2)(2h_1-h_3)}(e_2+h_1e_1+h_2h_3e_0)|(1,1)\rangle,\\
|\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}\rangle_{h_1,h_2} &= \frac{1}{(h_2-2h_1)(h_2-h_3)}(e_2+(h_2-h_1)e_1+2h_1h_3e_0)|(1,1)\rangle,\\
|(2,1)\rangle_{h_1,h_3} &= \frac{1}{(h_3-2h_1)(h_3-h_2)}(e_2+(h_3-h_1)e_1+2h_1h_2e_0)|(1,1)\rangle.
\end{aligned} \tag{34}$$

From the relations in (8), we have

$$\begin{aligned}
\langle(1,1,1)|(1,1,1)\rangle &= \frac{3(2+h_2h_3\psi_0)\psi_0}{(2h_1-h_2)(2h_1-h_3)}\frac{2(1+h_2h_3\psi_0)}{(h_1-h_2)(h_1-h_3)}\psi_0,\\
{}_{h_1,h_2}\langle\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}|\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}\rangle_{h_1,h_2} &= \frac{2(1+h_1h_3\psi_0)(2h_2-h_1)}{(h_2-h_1)(h_2-h_3)(h_2-2h_1)}\frac{2(1+h_2h_3\psi_0)}{(h_1-h_2)(h_1-h_3)}\psi_0,\\
{}_{h_1,h_3}\langle(2,1)|(2,1)\rangle_{h_1,h_3} &= \frac{2(1+h_1h_2\psi_0)(2h_3-h_1)}{(h_3-h_1)(h_3-h_2)(h_3-2h_1)}\frac{2(1+h_2h_3\psi_0)}{(h_1-h_2)(h_1-h_3)}\psi_0,
\end{aligned}$$

and others equal zero. We can see that one changes into another of the last two equations by exchanging h_2 and h_3. By the symmetry of h_1, h_2, h_3 or direct calculations, we can write that

$$\begin{aligned}
\langle\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}|\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\rangle &= \frac{3(2+h_1h_3\psi_0)\psi_0}{(2h_2-h_1)(2h_2-h_3)}\frac{2(1+h_1h_3\psi_0)}{(h_2-h_1)(h_2-h_3)}\psi_0,\\
\langle(3)|(3)\rangle &= \frac{3(2+h_1h_2\psi_0)\psi_0}{(2h_3-h_1)(2h_3-h_2)}\frac{2(1+h_1h_2\psi_0)}{(h_3-h_1)(h_3-h_2)}\psi_0,\\
{}_{h_2,h_1}\langle\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}|\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}\rangle_{h_2,h_1} &= \frac{2(1+h_2h_3\psi_0)(2h_1-h_2)}{(h_1-h_2)(h_1-h_3)(h_1-2h_2)}\frac{2(1+h_1h_3\psi_0)}{(h_2-h_1)(h_2-h_3)}\psi_0,\\
{}_{h_3,h_1}\langle(2,1)|(2,1)\rangle_{h_3,h_1} &= \frac{2(1+h_2h_3\psi_0)(2h_1-h_3)}{(h_1-h_3)(h_1-h_2)(h_1-2h_3)}\frac{2(1+h_1h_2\psi_0)}{(h_3-h_1)(h_3-h_2)}\psi_0,\\
{}_{h_2,h_3}\langle\begin{pmatrix} 2 \\ 1 \end{pmatrix}|\begin{pmatrix} 2 \\ 1 \end{pmatrix}\rangle_{h_2,h_3} &= \frac{2(1+h_1h_2\psi_0)(2h_3-h_2)}{(h_3-h_1)(h_3-h_2)(h_3-2h_2)}\frac{2(1+h_1h_3\psi_0)}{(h_2-h_1)(h_2-h_3)}\psi_0,
\end{aligned}$$

$$ {}_{h_3,h_2}\langle\begin{pmatrix}2\\1\end{pmatrix}|\begin{pmatrix}2\\1\end{pmatrix}\rangle_{h_3,h_2}=\frac{2(1+h_1h_3\psi_0)(2h_2-h_3)}{(h_2-h_3)(h_2-h_1)(h_2-2h_3)}\frac{2(1+h_1h_2\psi_0)}{(h_3-h_1)(h_3-h_2)}\psi_0. $$

We can see that

$$ {}_{h_1,h_2}\langle\begin{pmatrix}1&1\\1&\end{pmatrix}|\begin{pmatrix}1&1\\1&\end{pmatrix}\rangle_{h_1,h_2}\neq {}_{h_2,h_1}\langle\begin{pmatrix}1&1\\1&\end{pmatrix}|\begin{pmatrix}1&1\\1&\end{pmatrix}\rangle_{h_2,h_1} $$

we discuss the relations between them. We know that

$$ |\begin{pmatrix}1&1\\1&\end{pmatrix}\rangle_{h_2,h_1}=\frac{1}{(h_1-2h_2)(h_1-h_3)}(e_2+(h_1-h_2)e_1+2h_2h_3e_0)|\begin{pmatrix}1\\1\end{pmatrix}\rangle. $$

By calculation, we have

$$ {}_{h_1,h_2}\langle\begin{pmatrix}1&1\\1&\end{pmatrix}|\begin{pmatrix}1&1\\1&\end{pmatrix}\rangle_{h_2,h_1}=\frac{2(1+h_1h_3\psi_0)2(1+h_2h_3\psi_0)}{(h_1-h_2)^2(h_2-h_3)(h_1-h_3)}\psi_0, $$

which means that

$$ \begin{aligned} &{}_{h_1,h_2}\langle\begin{pmatrix}1&1\\1&\end{pmatrix}|\begin{pmatrix}1&1\\1&\end{pmatrix}\rangle_{h_2,h_1}\\ &=\sqrt{{}_{h_1,h_2}\langle\begin{pmatrix}1&1\\1&\end{pmatrix}|\begin{pmatrix}1&1\\1&\end{pmatrix}\rangle_{h_1,h_2}\cdot {}_{h_2,h_1}\langle\begin{pmatrix}1&1\\1&\end{pmatrix}|\begin{pmatrix}1&1\\1&\end{pmatrix}\rangle_{h_2,h_1}}. \end{aligned} $$

The result tells us that $|\begin{pmatrix}1&1\\1&\end{pmatrix}\rangle_{h_1,h_2}$ and $|\begin{pmatrix}1&1\\1&\end{pmatrix}\rangle_{h_2,h_1}$ are linearly dependent, actually, they have the following relation. For total volume 3, the relations between the same shaped 3D Young diagrams which are obtained from different steps are

$$ |\begin{pmatrix}1&1\\1&\end{pmatrix}\rangle_{h_1,h_2}=\varphi(h_2-h_1)|\begin{pmatrix}1&1\\1&\end{pmatrix}\rangle_{h_2,h_1}, \tag{35} $$

$$ |\begin{pmatrix}2\\1\end{pmatrix}\rangle_{h_2,h_3}=\varphi(h_3-h_2)|\begin{pmatrix}2\\1\end{pmatrix}\rangle_{h_3,h_2}, \tag{36} $$

$$ |(2,1)\rangle_{h_1,h_3}=\varphi(h_3-h_1)|(2,1)\rangle_{h_3,h_1}, \tag{37} $$

where $\varphi(u)$ is defined in (20).

3. 3-Schur functions

In this section, we calculate the 3-Schur functions of 3D Young diagrams by the orthogonality of 3D Young diagram. For a 3D Young diagram π, we denote the 3-Schur function corresponding to π by S_π, which is a function of variables $P_1, P_{2,1}, P_{2,2}, P_{3,1}, P_{3,2}, P_{3,3}, \cdots$. For 3D Young diagrams π, π', define

$$ \langle S_\pi|S_{\pi'}\rangle=\langle\pi|\pi'\rangle. \tag{38} $$

From the properties of $\langle\pi|\pi'\rangle$, we have that $\langle S_\pi|S_{\pi'}\rangle=\langle S_{\pi'}|S_\pi\rangle$, and if π and π' do not have the same shape, then $\langle S_\pi|S'_\pi\rangle=0$.

We set $S_0=1$, which is the 3-Schur function corresponding to empty, and $S_\square=S_{(1)}=P_1$, which is the 3-Schur function corresponding to 3D Young diagram (1). We have $\langle S_\square|S_\square\rangle=$

$\langle P_1|P_1\rangle = 1$, which is the same as in Schur functions and 2D Young diagrams. Actually, we want to keep $\langle\cdot|\cdot\rangle$ of p_n, where p_n are variables of Schur functions corresponding to 2D Young diagrams.

Let

$$\begin{aligned}&\langle P_1^2|P_1^2\rangle = 2,\ \ \langle P_1^2|P_{2,1}\rangle = 0,\ \ \langle P_1^2|P_{2,2}\rangle = 0,\\&\langle P_{2,1}|P_{2,1}\rangle = 2,\ \ \langle P_{2,1}|P_{2,2}\rangle = 0,\ \ \langle P_{2,2}|P_{2,2}\rangle = -2.\end{aligned} \tag{39}$$

We give a remark here to explain the values of these scalar products. I give them for the following reasons: firstly, I want to keep the values and properties of the scalar products corresponding to Schur functions of 2D Young diagram unchanged, for example, $\langle P_1^2|P_1^2\rangle = 2$ is the same as that in the case of Schur functions of 2D Young diagrams; Secondly, I want the 3-Schur functions have good properties, for example, exchanging h_1, h_2, h_3 corresponds to rotating 3D Young diagram; Thirdly, I want the values of these scalar products match the values of the scalar products of $e_ie_0|0\rangle, e_ie_je_0|0\rangle$ respectively which we will discuss in the following section. We will give the values of other scalar products in the same way.

We know that 3-Schur functions $S_{(1,1)},\ S_{\begin{pmatrix}1\\1\end{pmatrix}},\ S_{(2)}$ are functions of $P_1^2,\ P_{2,1},\ P_{2,2}$. Let

$$\begin{aligned}S_{(1,1)} &= a_1P_1^2 + a_2P_{2,1} + a_3P_{2,2},\\S_{\begin{pmatrix}1\\1\end{pmatrix}} &= b_1P_1^2 + b_2P_{2,1} + b_3P_{2,2},\\S_{(1,1)} &= a_1P_1^2 + a_2P_{2,1} + a_3P_{2,2},\end{aligned}$$

they satisfy the relations in (33) and

$$P_1^2 = S_{(1,1)} + S_{\begin{pmatrix}1\\1\end{pmatrix}} + S_{(2)},$$

then we have

$$\begin{aligned}a_1^2 + a_2^2 - a_3^2 &= \frac{(1+h_2h_3)}{(h_1-h_2)(h_1-h_3)},\\b_1^2 + b_2^2 - b_3^2 &= \frac{(1+h_1h_3)}{(h_2-h_1)(h_2-h_3)},\\c_1^2 + c_2^2 - c_3^2 &= \frac{(1+h_1h_2)}{(h_3-h_1)(h_3-h_2)},\\a_1b_1 + a_2b_2 + a_3b_3 &= 0,\\a_1c_1 + a_2c_2 + a_3c_3 &= 0,\\b_1c_1 + b_2c_2 + b_3c_3 &= 0,\\a_1 + b_1 + c_1 &= 1,\\a_2 + b_2 + c_2 &= 0,\\a_3 + b_3 + c_3 &= 0.\end{aligned}$$

Solve these equations, we get

$$a_1 = \frac{(1+h_2h_3)}{(h_1-h_2)(h_1-h_3)}, b_1 = \frac{(1+h_1h_3)}{(h_2-h_1)(h_2-h_3)},\ c_1 = \frac{(1+h_1h_2)}{(h_3-h_1)(h_3-h_2)},$$
$$a_2 = \frac{(1+h_2h_3)}{(h_1-h_2)(h_1-h_3)}h_1, b_1 = \frac{(1+h_1h_3)}{(h_2-h_1)(h_2-h_3)}h_2,\ c_1 = \frac{(1+h_1h_2)}{(h_3-h_1)(h_3-h_2)}h_3,$$
$$a_3 = \frac{\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}}{(h_1-h_2)(h_1-h_3)}, b_3 = \frac{\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}}{(h_2-h_1)(h_2-h_3)},$$
$$c_3 = \frac{\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}}{(h_3-h_1)(h_3-h_2)},$$

that is

$$S_{(1,1)} = \frac{(1+h_2h_3)}{(h_1-h_2)(h_1-h_3)}P_1^2 + \frac{(1+h_2h_3)}{(h_1-h_2)(h_1-h_3)}h_1P_{2,1} + \frac{\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}}{(h_1-h_2)(h_1-h_3)}P_{2,2}, \tag{40}$$

$$S_{\begin{pmatrix}1\\1\end{pmatrix}} = \frac{(1+h_1h_3)}{(h_2-h_1)(h_2-h_3)}P_1^2 + \frac{(1+h_1h_3)}{(h_2-h_1)(h_2-h_3)}h_2P_{2,1} + \frac{\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}}{(h_2-h_1)(h_2-h_3)}P_{2,2}, \tag{41}$$

$$S_{(2)} = \frac{(1+h_1h_2)}{(h_3-h_1)(h_3-h_2)}P_1^2 + \frac{(1+h_1h_2)}{(h_3-h_1)(h_3-h_2)}h_3P_{2,1} + \frac{\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}}{(h_3-h_1)(h_3-h_2)}P_{2,2}. \tag{42}$$

From the expression above, we can see these 3-Schur functions are symmetric about three coordinate axes, i.e., changing 3D Young diagram $(1,1)$ to $\begin{pmatrix}1\\1\end{pmatrix}$ corresponds to exchanging $h_1 \leftrightarrow h_2$ in 3-Schur functions, changing 3D Young diagram $(1,1)$ to (2) corresponds to exchanging $h_1 \leftrightarrow h_3$ in 3-Schur functions, changing 3D Young diagram $\begin{pmatrix}1\\1\end{pmatrix}$ to (2) corresponds to exchanging $h_2 \leftrightarrow h_3$ in 3-Schur functions.

Let $P_1^3, P_1P_{2,1}, P_1P_{2,2}, P_{3,1}, P_{3,2}, P_{3,3}$ satisfy

$$\begin{aligned}
&\langle P_1^3|P_1^3\rangle = 6,\ \langle P_1^3|P_1P_{2,1}\rangle = 0,\ \langle P_1^3|P_1P_{2,2}\rangle = 0,\ \langle P_1^3|P_{3,1}\rangle = 0,\ \langle P_1^3|P_{3,2}\rangle = 0,\\
&\langle P_1^3|P_{3,3}\rangle = 0,\ \langle P_1P_{2,1}|P_1P_{2,1}\rangle = 2,\ \ \langle P_1P_{2,1}|P_1P_{2,2}\rangle = 0,\ \langle P_1P_{2,1}|P_{3,1}\rangle = 0,\\
&\langle P_1P_{2,1}|P_{3,2}\rangle = 0, \langle P_1P_{2,1}|P_{3,3}\rangle = 0, \langle P_1P_{2,2}|P_1P_{2,2}\rangle = -2,\ \langle P_1P_{2,1}|P_{3,1}\rangle = 0,\\
&\langle P_1P_{2,2}|P_{3,2}\rangle = 0, \langle P_1P_{2,2}|P_{3,3}\rangle = 0, \langle P_{3,1}|P_{3,1}\rangle = 3, \langle P_{3,1}|P_{3,2}\rangle = 0,\\
&\langle P_{3,1}|P_{3,3}\rangle = 0, \langle P_{3,2}|P_{3,2}\rangle = -2, \langle P_{3,2}|P_{3,3}\rangle = 0, \langle P_{3,3}|P_{3,3}\rangle = 6.
\end{aligned} \tag{43}$$

We know that 3-Schur functions $S_{(1,1,1)}, S_{(2,1)}, S_{(3)}, S_{\begin{pmatrix}2\\1\end{pmatrix}}, S_{\begin{pmatrix}1&1\\1&\end{pmatrix}}$ and $S_{\begin{pmatrix}1\\1\\1\end{pmatrix}}$ are functions of $P_1^3, P_1P_{2,1}, P_1P_{2,2}, P_{3,1}, P_{3,2}, P_{3,3}$. Let

$$
\begin{aligned}
S_{(1,1,1)} &= a_1 P_1^3 + a_2 P_1 P_{2,1} + a_3 P_1 P_{2,2} + a_4 P_{3,1} + a_5 P_{3,2} + a_6 P_{3,3},\\
S_{\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}_{h_1,h_2}} &= b_1 P_1^3 + b_2 P_1 P_{2,1} + b_3 P_1 P_{2,2} + b_4 P_{3,1} + b_5 P_{3,2} + b_6 P_{3,3},\\
S_{(2,1)_{h_1,h_3}} &= c_1 P_1^3 + c_2 P_1 P_{2,1} + c_3 P_1 P_{2,2} + c_4 P_{3,1} + c_5 P_{3,2} + c_6 P_{3,3},\\
S_{(3)} &= d_1 P_1^3 + d_2 P_1 P_{2,1} + d_3 P_1 P_{2,2} + d_4 P_{3,1} + d_5 P_{3,2} + d_6 P_{3,3},\\
S_{\begin{pmatrix} 2 \\ 1 \end{pmatrix}_{h_2,h_3}} &= g_1 P_1^3 + g_2 P_1 P_{2,1} + g_3 P_1 P_{2,2} + g_4 P_{3,1} + g_5 P_{3,2} + g_6 P_{3,3},\\
S_{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} &= h_1 P_1^3 + h_2 P_1 P_{2,1} + h_3 P_1 P_{2,2} + h_4 P_{3,1} + h_5 P_{3,2} + h_6 P_{3,3}
\end{aligned}
$$

and

$$
S_{\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}_{h_1,h_2}} = \varphi(h_2 - h_1) S_{\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}_{h_2,h_1}}, \tag{44}
$$

$$
S_{(2,1)_{h_1,h_3}} = \varphi(h_3 - h_1) S_{(2,1)_{h_3,h_1}}, \tag{45}
$$

$$
S_{\begin{pmatrix} 2 \\ 1 \end{pmatrix}_{h_2,h_3}} = \varphi(h_3 - h_2) S_{\begin{pmatrix} 2 \\ 1 \end{pmatrix}_{h_3,h_2}}, \tag{46}
$$

they satisfy

$$
\begin{aligned}
\langle S_{(1,1,1)} | S_{(1,1,1)} \rangle &= \frac{3(2 + h_2 h_3 \psi_0)\psi_0}{(2h_1 - h_2)(2h_1 - h_3)} \frac{2(1 + h_2 h_3 \psi_0)}{(h_1 - h_2)(h_1 - h_3)} \psi_0,\\
{}_{h_1,h_2}\langle S_{\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}} | S_{\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}} \rangle_{h_1,h_2} &= \frac{2(1 + h_1 h_3 \psi_0)(2h_2 - h_1)}{(h_2 - h_1)(h_2 - h_3)(h_2 - 2h_1)} \frac{2(1 + h_2 h_3 \psi_0)}{(h_1 - h_2)(h_1 - h_3)} \psi_0,\\
{}_{h_1,h_3}\langle S_{(2,1)} | S_{(2,1)} \rangle_{h_1,h_3} &= \frac{2(1 + h_1 h_2 \psi_0)(2h_3 - h_1)}{(h_3 - h_1)(h_3 - h_2)(h_3 - 2h_1)} \frac{2(1 + h_2 h_3 \psi_0)}{(h_1 - h_2)(h_1 - h_3)} \psi_0,\\
\langle S_{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} | S_{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \rangle &= \frac{3(2 + h_1 h_3 \psi_0)\psi_0}{(2h_2 - h_1)(2h_2 - h_3)} \frac{2(1 + h_1 h_3 \psi_0)}{(h_2 - h_1)(h_2 - h_3)} \psi_0,\\
\langle S_{(3)} | S_{(3)} \rangle &= \frac{3(2 + h_1 h_2 \psi_0)\psi_0}{(2h_3 - h_1)(2h_3 - h_2)} \frac{2(1 + h_1 h_2 \psi_0)}{(h_3 - h_1)(h_3 - h_2)} \psi_0,\\
{}_{h_2,h_3}\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix} | \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rangle_{h_2,h_3} &= \frac{2(1 + h_1 h_2 \psi_0)(2h_3 - h_2)}{(h_3 - h_1)(h_3 - h_2)(h_3 - 2h_2)} \frac{2(1 + h_1 h_3 \psi_0)}{(h_2 - h_1)(h_2 - h_3)} \psi_0,\\
\langle S_{(1,1,1)} | S_{(2,1)} \rangle = 0, \quad & \langle S_{(1,1,1)} | S_{\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}} \rangle = 0, \langle S_{(1,1,1)} | S_{\begin{pmatrix} 2 \\ 1 \end{pmatrix}} \rangle = 0,\\
\langle S_{(1,1,1)} | S_{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \rangle = 0, & \langle S_{(1,1,1)} | S_{(3)} \rangle = 0, \langle S_{(2,1)} | S_{(3)} \rangle = 0,
\end{aligned}
$$

$$\langle S_{(2,1)}|S_{\begin{pmatrix}1&1\\1&\end{pmatrix}}\rangle=0,\ \langle S_{(2,1)}|S_{\begin{pmatrix}2\\1\end{pmatrix}}\rangle=0,\ \langle S_{(2,1)}|S_{\begin{pmatrix}1\\1\\1\end{pmatrix}}\rangle=0,$$

$$\langle S_{\begin{pmatrix}2\\1\end{pmatrix}}|S_{\begin{pmatrix}1&1\\1&\end{pmatrix}}\rangle=0,\ \langle S_{\begin{pmatrix}2\\1\end{pmatrix}}|S_{(3)}\rangle=0,\ \langle S_{\begin{pmatrix}2\\1\end{pmatrix}}|S_{\begin{pmatrix}1\\1\\1\end{pmatrix}}\rangle=0,$$

$$\langle S_{\begin{pmatrix}1&1\\1&\end{pmatrix}}|S_{(3)}\rangle=0,\ \langle S_{\begin{pmatrix}1&1\\1&\end{pmatrix}}|S_{\begin{pmatrix}1\\1\\1\end{pmatrix}}\rangle=0,\ \langle S_{(3)}|S_{\begin{pmatrix}1\\1\\1\end{pmatrix}}\rangle=0,$$

and

$$\begin{aligned}
P_1S_{(1,1)}&=S_{(1,1,1)}+S_{\begin{pmatrix}1&1\\1&\end{pmatrix}_{h_1,h_2}}+S_{(2,1)_{h_1,h_3}},\\
P_1S_{\begin{pmatrix}1\\1\end{pmatrix}}&=S_{\begin{pmatrix}1\\1\\1\end{pmatrix}}+S_{\begin{pmatrix}1&1\\1&\end{pmatrix}_{h_2,h_1}}+S_{\begin{pmatrix}2\\1\end{pmatrix}_{h_2,h_3}},\\
P_1S_{(2)}&=S_{(2,1)_{h_3,h_1}}+S_{\begin{pmatrix}2\\1\end{pmatrix}_{h_3,h_2}}+S_{(3)},
\end{aligned}$$

then we have

$$\begin{aligned}
&6a_1^2+2a_2^2-2a_3^2+3a_4^2-2a_5^2+6a_6^2=\frac{3(2+h_2h_3\psi_0)\psi_0}{(2h_1-h_2)(2h_1-h_3)}\frac{2(1+h_2h_3\psi_0)}{(h_1-h_2)(h_1-h_3)}\psi_0,\\
&6b_1^2+2b_2^2-2b_3^2+3b_4^2-2b_5^2+6b_6^2\\
&\quad=\frac{2(1+h_1h_3\psi_0)(2h_2-h_1)}{(h_2-h_1)(h_2-h_3)(h_2-2h_1)}\frac{2(1+h_2h_3\psi_0)}{(h_1-h_2)(h_1-h_3)}\psi_0,\\
&6c_1^2+2c_2^2-2c_3^2+3c_4^2-2c_5^2+6c_6^2\\
&\quad=\frac{2(1+h_1h_2\psi_0)(2h_3-h_1)}{(h_3-h_1)(h_3-h_2)(h_3-2h_1)}\frac{2(1+h_2h_3\psi_0)}{(h_1-h_2)(h_1-h_3)}\psi_0,\\
&6d_1^2+2d_2^2-2d_3^2+3d_4^2-2d_5^2+6d_6^2=\frac{3(2+h_1h_2\psi_0)\psi_0}{(2h_3-h_1)(2h_3-h_2)}\frac{2(1+h_1h_2\psi_0)}{(h_3-h_1)(h_3-h_2)}\psi_0,\\
&6g_1^2+2g_2^2-2g_3^2+3g_4^2-2g_5^2+6g_6^2\\
&\quad=\frac{2(1+h_1h_2\psi_0)(2h_3-h_2)}{(h_3-h_1)(h_3-h_2)(h_3-2h_2)}\frac{2(1+h_1h_3\psi_0)}{(h_2-h_1)(h_2-h_3)}\psi_0,\\
&6h_1^2+2h_2^2-2h_3^2+3h_4^2-2h_5^2+6h_6^2=\frac{3(2+h_1h_3\psi_0)\psi_0}{(2h_2-h_1)(2h_2-h_3)}\frac{2(1+h_1h_3\psi_0)}{(h_2-h_1)(h_2-h_3)}\psi_0,\\
&6a_1b_1+2a_2b_2-2a_3b_3+3a_4b_4-2a_5b_5+6a_6b_6=0,\\
&6a_1c_1+2a_2c_2-2a_3c_3+3a_4c_4-2a_5c_5+6a_6c_6=0,\\
&6a_1d_1+2a_2d_2-2a_3d_3+3a_4d_4-2a_5d_5+6a_6d_6=0,\\
&6a_1g_1+2a_2g_2-2a_3g_3+3a_4g_4-2a_5g_5+6a_6g_6=0,
\end{aligned}$$

$$
\begin{aligned}
&6a_1h_1 + 2a_2h_2 - 2a_3h_3 + 3a_4h_4 - 2a_5h_5 + 6a_6h_6 = 0,\\
&6b_1c_1 + 2b_2c_2 - 2b_3c_3 + 3b_4c_4 - 2b_5c_5 + 6b_6c_6 = 0,\\
&6b_1d_1 + 2b_2d_2 - 2b_3d_3 + 3b_4d_4 - 2b_5d_5 + 6b_6d_6 = 0,\\
&6b_1g_1 + 2b_2g_2 - 2b_3g_3 + 3b_4g_4 - 2b_5g_5 + 6b_6g_6 = 0,\\
&6b_1h_1 + 2b_2h_2 - 2b_3h_3 + 3b_4h_4 - 2b_5h_5 + 6b_6h_6 = 0,\\
&6c_1d_1 + 2c_2d_2 - 2c_3d_3 + 3c_4d_4 - 2c_5d_5 + 6c_6d_6 = 0,\\
&6c_1g_1 + 2c_2g_2 - 2c_3g_3 + 3c_4g_4 - 2c_5g_5 + 6c_6g_6 = 0,\\
&6c_1h_1 + 2c_2h_2 - 2c_3h_3 + 3c_4h_4 - 2c_5h_5 + 6c_6h_6 = 0,\\
&6d_1g_1 + 2d_2g_2 - 2d_3g_3 + 3d_4g_4 - 2d_5g_5 + 6d_6g_6 = 0,\\
&6d_1h_1 + 2d_2h_2 - 2d_3h_3 + 3d_4h_4 - 2d_5h_5 + 6d_6h_6 = 0,\\
&6g_1h_1 + 2g_2h_2 - 2g_3h_3 + 3g_4h_4 - 2g_5h_5 + 6g_6h_6 = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&a_1 + b_1 + c_1 = \frac{(1+h_2h_3)}{(h_1-h_2)(h_1-h_3)},\\
&a_2 + b_2 + c_2 = \frac{(1+h_2h_3)h_1}{(h_1-h_2)(h_1-h_3)},\\
&a_3 + b_3 + c_3 = \frac{\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}}{(h_1-h_2)(h_1-h_3)},\\
&a_4 + b_4 + c_4 = 0, a_5 + b_5 + c_5 = 0, a_6 + b_6 + c_6 = 0,\\
&\varphi(h_1-h_2)b_1 + g_1 + h_1 = \frac{(1+h_1h_3)}{(h_2-h_1)(h_2-h_3)},\\
&\varphi(h_1-h_2)b_2 + g_2 + h_2 = \frac{(1+h_1h_3)h_2}{(h_2-h_1)(h_2-h_3)},\\
&\varphi(h_1-h_2)b_3 + g_3 + h_3 = \frac{\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}}{(h_2-h_1)(h_2-h_3)},\\
&\varphi(h_1-h_2)b_4 + g_4 + h_4 = 0, \varphi(h_1-h_2)b_5 + g_5 + h_5 = 0, \varphi(h_1-h_2)b_6 + g_6 + h_6 = 0,\\
&\varphi(h_1-h_3)c_1 + \varphi(h_2-h_3)g_1 + d_1 = \frac{(1+h_1h_2)}{(h_3-h_1)(h_3-h_2)},\\
&\varphi(h_1-h_3)c_2 + \varphi(h_2-h_3)g_2 + d_2 = \frac{(1+h_1h_2)h_3}{(h_3-h_1)(h_3-h_2)},\\
&\varphi(h_1-h_3)c_3 + \varphi(h_2-h_3)g_3 + d_3 = \frac{\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}}{(h_3-h_1)(h_3-h_2)},\\
&\varphi(h_1-h_3)c_4 + \varphi(h_2-h_3)g_4 + d_4 = 0, \varphi(h_1-h_3)c_5 + \varphi(h_2-h_3)g_5 + d_5 = 0,\\
&\varphi(h_1-h_3)c_6 + \varphi(h_2-h_3)g_6 + d_6 = 0.
\end{aligned}
$$

Solve these equations, we get

$$
a_1 = \frac{(1+h_2h_3)(2+h_2h_3)}{(h_1-h_2)(h_1-h_3)(2h_1-h_2)(2h_1-h_3)},
$$

$$a_2 = \frac{3h_1(1+h_2h_3)(2+h_2h_3)}{(h_1-h_2)(h_1-h_3)(2h_1-h_2)(2h_1-h_3)},$$
$$a_3 = \frac{3(2+h_2h_3)\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}}{(h_1-h_2)(h_1-h_3)(2h_1-h_2)(2h_1-h_3)},$$
$$a_4 = \frac{2h_1^2(1+h_2h_3)(2+h_2h_3)}{(h_1-h_2)(h_1-h_3)(2h_1-h_2)(2h_1-h_3)},$$
$$a_5 = \frac{3h_1(2+h_2h_3)\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}}{(h_1-h_2)(h_1-h_3)(2h_1-h_2)(2h_1-h_3)},$$
$$a_6 = \frac{\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)(2+h_2h_3)(2+h_1h_3)(2+h_1h_2)}}{(h_1-h_2)(h_1-h_3)(2h_1-h_2)(2h_1-h_3)},$$

that is,

$$\begin{aligned} S_{(1,1,1)} = \frac{1}{(h_1-h_2)(h_1-h_3)(2h_1-h_2)(2h_1-h_3)} \Big(& (1+h_2h_3)(2+h_2h_3)P_1^3 \\ & + 3h_1(1+h_2h_3)(2+h_2h_3)P_1P_{2,1} \\ & + 3(2+h_2h_3)\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}P_1P_{2,2} \\ & + 2h_1^2(1+h_2h_3)(2+h_2h_3)P_{3,1} \\ & + 3h_1(2+h_2h_3)\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}P_{3,2} \\ & + \sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)(2+h_2h_3)(2+h_1h_3)(2+h_1h_2)}P_{3,3}\Big). \end{aligned} \tag{47}$$

We also get b_i, c_i, d_i, g_i, h_i by solving the equations above which we do not list, instead we list the following result:

$$\begin{aligned} S_{\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}_{h_1,h_2}} = \frac{1}{(h_2-2h_1)(h_2-h_3)(h_1-h_2)(h_1-h_3)} \Big(& 2(1+h_1h_3)(1+h_2h_3)P_1^3 \\ & + (-2h_3)(1+h_2h_3)(1+h_1h_3)P_1P_{2,1} \\ & + (6-2h_3^2)\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}P_1P_{2,2} \\ & + 2h_1h_2(1+h_2h_3)(1+h_1h_3)P_{3,1} \\ & + (-2h_3+3\sigma_3)\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}P_{3,2} \\ & + \sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)(2+h_2h_3)(2+h_1h_3)(2+h_1h_2)}P_{3,3}\Big). \end{aligned} \tag{48}$$

$$\begin{aligned} S_{(2,1)_{h_1,h_3}} = \frac{1}{(h_3-2h_1)(h_3-h_2)(h_1-h_2)(h_1-h_3)} \Big(& 2(1+h_1h_2)(1+h_2h_3)P_1^3 \\ & + (-2h_2)(1+h_2h_3)(1+h_1h_2)P_1P_{2,1} \\ & + (6-2h_2^2)\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}P_1P_{2,2} \\ & + 2h_1h_3(1+h_2h_3)(1+h_1h_2)P_{3,1} \\ & + (-2h_2+3\sigma_3)\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}P_{3,2} \\ & + \sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)(2+h_2h_3)(2+h_1h_3)(2+h_1h_2)}P_{3,3}\Big). \end{aligned} \tag{49}$$

$$\begin{aligned} S_{\begin{pmatrix} 2 \\ 1 \end{pmatrix}_{h_2,h_3}} = \frac{1}{(h_3-2h_2)(h_3-h_1)(h_2-h_1)(h_2-h_3)} \Big(& 2(1+h_1h_2)(1+h_1h_3)P_1^3 \\ & + (-2h_1)(1+h_1h_3)(1+h_1h_2)P_1P_{2,1} \end{aligned}$$

$$+ (6-2h_1^2)\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}P_1P_{2,2}$$
$$+ 2h_2h_3(1+h_1h_3)(1+h_1h_2)P_{3,1} \tag{50}$$
$$+ (-2h_1+3\sigma_3)\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}P_{3,2}$$
$$+\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)(2+h_2h_3)(2+h_1h_3)(2+h_1h_2)}P_{3,3}\Big).$$

$$S_{(3)} = \frac{1}{(h_3-h_2)(h_3-h_1)(2h_3-h_2)(2h_3-h_1)}\Big((1+h_1h_2)(2+h_1h_2)P_1^3$$
$$+ 3h_3(1+h_1h_2)(2+h_1h_2)P_1P_{2,1}$$
$$+ 3(2+h_1h_2)\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}P_1P_{2,2}$$
$$+ 2h_3^2(1+h_1h_2)(2+h_1h_2)P_{3,1} \tag{51}$$
$$+ 3h_3(2+h_1h_2)\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}P_{3,2}$$
$$+\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)(2+h_2h_3)(2+h_1h_3)(2+h_1h_2)}P_{3,3}\Big),$$

and

$$S_{\begin{pmatrix}1\\1\\1\end{pmatrix}} = \frac{1}{(h_2-h_1)(h_2-h_3)(2h_2-h_1)(2h_2-h_3)}\Big((1+h_1h_3)(2+h_1h_3)P_1^3$$
$$+ 3h_2(1+h_1h_3)(2+h_1h_3)P_1P_{2,1}$$
$$+ 3(2+h_1h_3)\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}P_1P_{2,2}$$
$$+ 2h_2^2(1+h_1h_3)(2+h_1h_3)P_{3,1} \tag{52}$$
$$+ 3h_2(2+h_1h_3)\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)}P_{3,2}$$
$$+\sqrt{(1+h_2h_3)(1+h_1h_3)(1+h_1h_2)(2+h_2h_3)(2+h_1h_3)(2+h_1h_2)}P_{3,3}\Big).$$

From the expressions above, we also can see these 3-Schur functions are also symmetric about three coordinate axes, for example, changing 3D Young diagram $(1,1,1)$ to $\begin{pmatrix}1\\1\\1\end{pmatrix}$ corresponds to exchanging $h_1 \leftrightarrow h_2$ in 3-Schur functions, changing 3D Young diagram $(2,1)_{h_1,h_3}$ to $\begin{pmatrix}1 & 1\\1 & \end{pmatrix}_{h_1,h_2}$ corresponds to exchanging $h_2 \leftrightarrow h_3$ in 3-Schur functions, and so on.

We can calculate 3-Schur functions corresponding to other 3D Young diagrams in the same way.

4. 3-Schur functions and MacMahon representation

In this section, we show that the 3-Schur functions obtained in last section are already exist in affine Yangian and its MacMahon representation.

From

$$\langle 0|\tilde{a}(e_0)e_0|0\rangle = \langle 0|(-f_0)e_0|0\rangle = 1$$

we set $S_{(1)} = P_1 = e_0|0\rangle$.

From the relation (8) in affine Yangian, we have

$$\langle f_0 f_0 e_0 e_0|0\rangle = 2, \quad \langle f_0 f_0 e_1 e_0|0\rangle = 0, \quad \langle f_0 f_1 e_1 e_0|0\rangle = 2,$$

then we set $P_1^2 = e_0 e_0|0\rangle$, $P_{2,1} = e_1 e_0|0\rangle$. By calculation, we have

$$\langle f_0 f_0 e_2 e_0|0\rangle = 2, \quad \langle f_0 f_1 e_2 e_0|0\rangle = 2\sigma_3,$$

then

$$\langle f_0 f_0 (e_2 - \sigma_3 e_1 - e_0) e_0|0\rangle = 0, \quad \langle f_0 f_1 (e_2 - \sigma_3 e_1 - e_0) e_0|0\rangle = 0,$$

and from

$$\langle f_0 (f_2 - \sigma_3 f_1 - f_0)(e_2 - \sigma_3 e_1 - e_0) e_0|0\rangle = -2(1 + h_1 h_2)(1 + h_1 h_3)(1 + h_2 h_3),$$

we set

$$\sqrt{(1 + h_1 h_2)(1 + h_1 h_3)(1 + h_2 h_3)}\, P_{2,2} = (e_2 - \sigma_3 e_1 - e_0) e_0|0\rangle. \tag{53}$$

We can see that equations ((40)-(42)) become

$$S_{(1,1)} = \frac{1}{(h_1 - h_2)(h_1 - h_3)} \left((1 + h_2 h_3) e_0 e_0|0\rangle + (1 + h_2 h_3) h_1 e_1 e_0|0\rangle \right. \\ \left. + (e_2 - \sigma_3 e_1 - e_0) e_0|0\rangle\right), \tag{54}$$

$$S_{\begin{pmatrix}1\\1\end{pmatrix}} = \frac{1}{(h_2 - h_1)(h_2 - h_3)} \left((1 + h_1 h_3) e_0 e_0|0\rangle + (1 + h_1 h_3) h_2 e_1 e_0|0\rangle \right. \\ \left. + (e_2 - \sigma_3 e_1 - e_0) e_0|0\rangle\right), \tag{55}$$

$$S_{(2)} = \frac{1}{(h_3 - h_1)(h_3 - h_2)} \left((1 + h_1 h_2) e_0 e_0|0\rangle + (1 + h_1 h_2) h_3 e_1 e_0|0\rangle \right. \\ \left. + (e_2 - \sigma_3 e_1 - e_0) e_0|0\rangle\right), \tag{56}$$

which match the results in (32) by direct calculation.

From the relation (8) in affine Yangian, we calculate the following results

$$\langle f_0 f_0 f_0 e_0 e_0 e_0|0\rangle = -6, \quad \langle f_0 f_0 f_0 e_0 e_1 e_0|0\rangle = 0, \quad \langle f_0 f_0 f_0 e_0 e_2 e_0|0\rangle = -6,$$
$$\langle f_0 f_0 f_0 e_1 e_0 e_0|0\rangle = 0, \quad \langle f_0 f_0 f_0 e_2 e_0 e_0|0\rangle = -12, \quad \langle f_0 f_0 f_0 e_1 e_1 e_0|0\rangle = 0,$$
$$\langle f_0 f_0 f_0 e_2 e_1 e_0|0\rangle = 0, \quad \langle f_0 f_0 f_0 e_1 e_2 e_0|0\rangle = 0, \quad \langle f_0 f_0 f_0 e_2 e_2 e_0|0\rangle = -12,$$

and

$$\langle f_0 f_1 f_0 e_0 e_0 e_0|0\rangle = 0, \quad \langle f_0 f_1 f_0 e_0 e_1 e_0|0\rangle = -2, \quad \langle f_0 f_1 f_0 e_0 e_2 e_0|0\rangle = -2\sigma_3,$$
$$\langle f_0 f_1 f_0 e_1 e_0 e_0|0\rangle = -4, \quad \langle f_0 f_1 f_0 e_2 e_0 e_0|0\rangle = -4\sigma_3, \quad \langle f_0 f_1 f_0 e_1 e_1 e_0|0\rangle = 0,$$
$$\langle f_0 f_1 f_0 e_2 e_1 e_0|0\rangle = -8, \quad \langle f_0 f_1 f_0 e_1 e_2 e_0|0\rangle = -4, \quad \langle f_0 f_1 f_0 e_2 e_2 e_0|0\rangle = -12\sigma_3,$$

and

$$\langle f_0(f_2-\sigma_3 f_1-f_0)f_0e_0e_0e_0|0\rangle=0,\ \ \langle f_0(f_2-\sigma_3 f_1-f_0)f_0e_0e_1e_0|0\rangle=0,$$
$$\langle f_0(f_2-\sigma_3 f_1-f_0)f_0e_0e_2e_0|0\rangle=2(1+h_1h_2)(1+h_1h_3)(1+h_2h_3),$$
$$\langle f_0(f_2-\sigma_3 f_1-f_0)f_0e_1e_0e_0|0\rangle=0,\ \ \langle f_0(f_2-\sigma_3 f_1-f_0)f_0e_1e_1e_0|0\rangle=0,$$
$$\langle f_0(f_2-\sigma_3 f_1-f_0)f_0e_2e_0e_0|0\rangle=4(1+h_1h_2)(1+h_1h_3)(1+h_2h_3),$$
$$\langle f_0(f_2-\sigma_3 f_1-f_0)f_0e_2e_1e_0|0\rangle=0,\ \ \langle f_0(f_2-\sigma_3 f_1-f_0)f_0e_1e_2e_0|0\rangle=0,$$
$$\langle f_0(f_2-\sigma_3 f_1-f_0)f_0e_2e_2e_0|0\rangle=12(1+h_1h_2)(1+h_1h_3)(1+h_2h_3).$$

From the equations above, we see that $e_0e_0e_0|0\rangle, e_0e_0e_0|0\rangle, e_0(e_2-\sigma_3e_1-e_0)e_0|0\rangle, e_1e_1e_0|0\rangle$ are orthogonal to each other under the quadratic form we introduced in equations (30) and (31), in the following, we want to find other vectors who are orthogonal to every one in the set $\{e_0e_0e_0|0\rangle, e_0e_0e_0|0\rangle, e_0(e_2-\sigma_3e_1-e_0)e_0|0\rangle, e_1e_1e_0|0\rangle\}$. We calculate

$$\langle f_0f_1f_1e_0e_0e_0|0\rangle=0,\ \ \langle f_0f_1f_1e_0e_1e_0|0\rangle=0,\ \ \langle f_0f_1f_1e_0e_2e_0|0\rangle=0,$$
$$\langle f_0f_1f_1e_1e_0e_0|0\rangle=0,\ \ \langle f_0f_1f_1e_2e_0e_0|0\rangle=-12,\ \ \langle f_0f_1f_1e_1e_1e_0|0\rangle=-12,$$
$$\langle f_0f_1f_1e_2e_1e_0|0\rangle=-24\sigma_3,\ \ \langle f_0f_1f_1e_1e_2e_0|0\rangle=-12\sigma_3,$$
$$\langle f_0f_1f_1e_2e_2e_0|0\rangle=12\sigma_2-12\sigma_3^2.$$

We want to find the basis of the vector space spanned by $e_je_ke_0|0\rangle$ with the relations (6) and (13). In paper [15], we have proved that the following relations hold by Serre relations (13): let l be an integer and $l\geq 0$,

$$e_{l+1}e_le_0|0\rangle=2e_le_{l+1}e_0|0\rangle, \tag{57}$$
$$\begin{aligned}e_{l+m+1}e_{l-m}e_0|0\rangle=&2e_{l-m}e_{l+m+1}e_0|0\rangle+3e_{l-m+1}e_{l+m}e_0|0\rangle+2\times 3e_{l-m+2}e_{l+m-1}e_0|0\rangle\\&+\cdots+2^{m-1}\times 3e_le_{l+1}e_0|0\rangle,\ \ 0<m\leq l,\end{aligned} \tag{58}$$

and

$$e_{l+2}e_le_0|0\rangle=2e_le_{l+2}e_0|0\rangle+e_{l+1}e_{l+1}e_0|0\rangle, \tag{59}$$
$$\begin{aligned}e_{l+m+2}e_{l-m}e_0|0\rangle=&2e_{l-m}e_{l+m+2}e_0|0\rangle+3e_{l-m+1}e_{l+m+1}e_0|0\rangle+2\times 3e_{l-m+2}e_{l+m}e_0|0\rangle\\&+\cdots+2^{m-1}\times 3e_le_{l+2}e_0|0\rangle+2^me_{l+1}e_{l+1}e_0|0\rangle,\ \ 0<m\leq l.\end{aligned} \tag{60}$$

From (6), if $k=3$, we have

$$[e_3,e_0]-3[e_2,e_1]+\sigma_2[e_1,e_0]-\sigma_3e_0e_0=0,$$

if $k>3$, we have

$$\begin{aligned}&[e_k,e_0]-3[e_{k-1},e_1]+3[e_{k-2},e_2]-[e_{k-3},e_3]\\&+\sigma_2[e_{k-2},e_0]-\sigma_2[e_{k-3},e_1]-\sigma_3\{e_{k-3},e_0\}=0,\end{aligned}$$

acting on $|0\rangle$, we obtain

$$e_ke_0|0\rangle=-\sigma_2e_{k-2}e_0|0\rangle+\sigma_3e_{k-3}e_0|0\rangle\ \text{ for }\ k\geq 3. \tag{61}$$

From the results in ((57)-(60)) and (61), we get that the vector space spanned by $e_je_ke_0|0\rangle$ with the relations (6) and (13) has a set of basis

$$\{e_2e_2e_0|0\rangle,\ e_2e_1e_0|0\rangle,\ e_2e_0e_0|0\rangle, e_1e_1e_0|0\rangle,\ e_1e_0e_0|0\rangle,\ e_0e_0e_0|0\rangle\}$$

Suppose

$$a_{22}e_2e_2 + a_{21}e_2e_1 + a_{20}e_2e_0 + a_{11}e_1e_1 + a_{10}e_1e_0 + a_{00}e_0e_0)e_0|0\rangle \tag{62}$$

is orthogonal to every vector in the set $\{e_0e_0e_0|0\rangle, e_0e_0e_0|0\rangle, e_0(e_2-\sigma_3e_1-e_0)e_0|0\rangle, e_1e_1e_0|0\rangle\}$. If $a_{22}=0$, by orthogonality, we get

$$\begin{aligned} 2a_{20} + a_{00} &= 0, \\ 2a_{21} + \sigma_3 a_{20} + a_{10} &= 0, \\ a_{20} &= 0, \\ 2a_{21}\sigma_3 + a_{20} + a_{11} &= 0, \end{aligned}$$

that is, $a_{00}=0,\ a_{10}=-2a_{21},\ a_{20}=0,\ a_{11}=-2a_{21}\sigma_3$, then we get the vector

$$(e_2 - 2\sigma_3e_1 - 4e_0)e_1e_0|0\rangle$$

and

$$\langle 0|f_0f_1(f_2-2\sigma_3f_1-4f_0)(e_2-2\sigma_3e_1-4e_0)e_1e_0|0\rangle = 32(1+h_1h_2)(1+h_1h_3)(1+h_2h_3).$$

If a_{22} in (62) is not equal to zero, let $a_{22}=1$ without loss of generality. That the vector in (62) is orthogonal to every one in the set $\{e_0e_0e_0|0\rangle, e_0e_0e_0|0\rangle, e_0(e_2-\sigma_3e_1-e_0)e_0|0\rangle, e_1e_1e_0|0\rangle, (e_2-2\sigma_3e_1-4e_0)e_1e_0|0\rangle\}$ tells us

$$a_{00}=4, a_{10}=\frac{3}{2}\sigma_3, a_{20}=-3, a_{11}=\sigma_2-\sigma_3^2+\frac{3}{2}\sigma_3+3, a_{21}=-\frac{3}{4}\sigma_3,$$

then we get the vector

$$e_2e_2 - \frac{3}{4}\sigma_3e_2e_1 - 3e_2e_0 + (\sigma_2+\frac{1}{2}\sigma_3^2+3)e_1e_1 + \frac{3}{2}\sigma_3e_1e_0 + 4e_0e_0)e_0|0\rangle.$$

By calculation, we have

$$\begin{aligned} &\langle 0|f_0(f_2f_2 - \frac{3}{4}\sigma_3f_2f_1 - 3f_2f_0 + (\sigma_2+\frac{1}{2}\sigma_3^2+3)f_1f_1 + \frac{3}{2}\sigma_3f_1f_0 + 4f_0f_0) \\ &\quad (e_2e_2 - \frac{3}{4}\sigma_3e_2e_1 - 3e_2e_0 + (\sigma_2+\frac{1}{2}\sigma_3^2+3)e_1e_1 + \frac{3}{2}\sigma_3e_1e_0 + 4e_0e_0)e_0|0\rangle \\ &= 6(1+h_1h_2)(1+h_1h_3)(1+h_2h_3)(2+h_1h_2)(2+h_1h_3)(2+h_2h_3). \end{aligned}$$

Therefore, the vector space spanned by $e_je_ke_0|0\rangle$ with the relations (6) and (13) has a set of orthogonal basis

$$\begin{aligned} &e_0e_0e_0|0\rangle, e_0e_1e_0|0\rangle, e_0(e_2-\sigma_3e_1-e_0)e_0|0\rangle, e_1e_1e_0|0\rangle, (e_2-2\sigma_3e_1-4e_0)e_1e_0|0\rangle, \\ &(e_2e_2 - \frac{3}{4}\sigma_3e_2e_1 - 3e_2e_0 + (\sigma_2+\frac{1}{2}\sigma_3^2+3)e_1e_1 + \frac{3}{2}\sigma_3e_1e_0 + 4e_0e_0)e_0|0\rangle. \end{aligned}$$

The results in (34) can be rewritten by the set of orthogonal basis above:

$$\begin{aligned} |(1,1,1)\rangle &= \frac{1}{(h_1-h_2)(h_1-h_3)(2h_1-h_2)(2h_1-h_3)}(e_2+h_1e_1+h_2h_3e_0) \\ &\quad (e_2+h_1e_1+h_2h_3e_0)e_0|0\rangle \\ &= \frac{1}{(h_1-h_2)(h_1-h_3)(2h_1-h_2)(2h_1-h_3)}\Big((1+h_2h_3)(2+h_2h_3)e_0e_0e_0|0\rangle \end{aligned}$$

$$
\begin{aligned}
&+3h_1(1+h_2h_3)(2+h_2h_3)e_0e_1e_0|0\rangle+3(2+h_2h_3)e_0(e_2-\sigma_3e_1-e_0)e_0|0\rangle\\
&+h_1^2(1+h_2h_3)(2+h_2h_3)e_1e_1e_0|0\rangle \\
&+\frac{3}{4}h_1(2+h_2h_3)(e_2-2\sigma_3e_1-4e_0)e_1e_0|0\rangle\\
&+(e_2e_2-\frac{3}{4}\sigma_3e_2e_1-3e_2e_0+(\sigma_2+\frac{1}{2}\sigma_3^2+3)e_1e_1\\
&+\frac{3}{2}\sigma_3e_1e_0+4e_0e_0)e_0|0\rangle\Big),
\end{aligned} \tag{63}
$$

$$
\begin{aligned}
|\begin{pmatrix}1&1\\1&\end{pmatrix}_{h_1,h_2}\rangle&=\frac{1}{(h_2-2h_1)(h_2-h_3)(h_1-h_2)(h_1-h_3)}(e_2+(h_2-h_1)e_1+2h_1h_3e_0)\\
&\quad(e_2+h_1e_1+h_2h_3e_0)e_0|0\rangle\\
&=\frac{1}{(h_2-2h_1)(h_2-h_3)(h_1-h_2)(h_1-h_3)}\\
&\quad\Big(2(1+h_1h_3)(1+h_2h_3)e_0e_0e_0|0\rangle\\
&\quad-2h_3(1+h_2h_3)(1+h_1h_3)e_0e_1e_0|0\rangle\\
&\quad+(6-2h_3^2)e_0(e_2-\sigma_3e_1-e_0)e_0|0\rangle\\
&\quad+h_1h_2(1+h_2h_3)(1+h_1h_3)e_1e_1e_0|0\rangle\\
&\quad+(\frac{-h_3}{2}+\frac{3}{4}\sigma_3)(e_2-2\sigma_3e_1-4e_0)e_1e_0|0\rangle\\
&\quad+(e_2e_2-\frac{3}{4}\sigma_3e_2e_1-3e_2e_0+(\sigma_2+\frac{1}{2}\sigma_3^2+3)e_1e_1\\
&\quad+\frac{3}{2}\sigma_3e_1e_0+4e_0e_0)e_0|0\rangle\Big),
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
|(2,1)_{h_1,h_3}\rangle&=\frac{1}{(h_3-2h_1)(h_3-h_2)(h_1-h_2)(h_1-h_3)}(e_2+(h_3-h_1)e_1+2h_1h_2e_0)\\
&\quad(e_2+h_1e_1+h_2h_3e_0)e_0|0\rangle\\
&=\frac{1}{(h_3-2h_1)(h_3-h_2)(h_1-h_2)(h_1-h_3)}\Big(2(1+h_1h_2)(1+h_2h_3)e_0e_0e_0|0\rangle\\
&\quad-2h_2(1+h_2h_3)(1+h_1h_2)e_0e_1e_0|0\rangle+(6-2h_2^2)e_0(e_2-\sigma_3e_1-e_0)e_0|0\rangle\\
&\quad+h_1h_3(1+h_2h_3)(1+h_1h_2)e_1e_1e_0|0\rangle\\
&\quad+(\frac{-h_2}{2}+\frac{3}{4}\sigma_3)(e_2-2\sigma_3e_1-4e_0)e_1e_0|0\rangle\\
&\quad+(e_2e_2-\frac{3}{4}\sigma_3e_2e_1-3e_2e_0+(\sigma_2+\frac{1}{2}\sigma_3^2+3)e_1e_1\\
&\quad+\frac{3}{2}\sigma_3e_1e_0+4e_0e_0)e_0|0\rangle\Big),
\end{aligned} \tag{65}
$$

$$
\begin{aligned}
|\begin{pmatrix}1&1\\1&\end{pmatrix}_{h_2,h_1}\rangle&=\frac{1}{(h_1-2h_2)(h_1-h_3)(h_2-h_1)(h_2-h_3)}(e_2+(h_1-h_2)e_1+2h_2h_3e_0)\\
&\quad(e_2+h_2e_1+h_1h_3e_0)e_0|0\rangle\\
&=\frac{1}{(h_1-2h_2)(h_1-h_3)(h_2-h_1)(h_2-h_3)}
\end{aligned}
$$

$$\begin{aligned}
&\Big(2(1+h_1h_3)(1+h_2h_3)e_0e_0e_0|0\rangle \\
&-2h_3(1+h_2h_3)(1+h_1h_3)e_0e_1e_0|0\rangle \\
&+(6-2h_3^2)e_0(e_2-\sigma_3e_1-e_0)e_0|0\rangle \\
&+h_1h_2(1+h_2h_3)(1+h_1h_3)e_1e_1e_0|0\rangle \\
&+(\frac{-h_3}{2}+\frac{3}{4}\sigma_3)(e_2-2\sigma_3e_1-4e_0)e_1e_0|0\rangle \\
&+(e_2e_2-\frac{3}{4}\sigma_3e_2e_1-3e_2e_0+(\sigma_2+\frac{1}{2}\sigma_3^2+3)e_1e_1 \\
&+\frac{3}{2}\sigma_3e_1e_0+4e_0e_0)e_0|0\rangle\Big),
\end{aligned} \tag{66}$$

$$\begin{aligned}
|\begin{pmatrix}2\\1\end{pmatrix}_{h_2,h_3}\rangle &= \frac{1}{(h_3-2h_2)(h_3-h_1)(h_2-h_1)(h_2-h_3)}(e_2+(h_3-h_2)e_1+2h_1h_2e_0) \\
&\quad (e_2+h_2e_1+h_1h_3e_0)e_0|0\rangle \\
&= \frac{1}{(h_3-2h_2)(h_3-h_1)(h_2-h_1)(h_2-h_3)}\Big(2(1+h_1h_3)(1+h_1h_2)e_0e_0e_0|0\rangle \\
&\quad -2h_1(1+h_1h_2)(1+h_1h_3)e_0e_1e_0|0\rangle+(6-2h_1^2)e_0(e_2-\sigma_3e_1-e_0)e_0|0\rangle \\
&\quad +h_2h_3(1+h_1h_2)(1+h_1h_3)e_1e_1e_0|0\rangle \\
&\quad +(\frac{-h_1}{2}+\frac{3}{4}\sigma_3)(e_2-2\sigma_3e_1-4e_0)e_1e_0|0\rangle \\
&\quad +(e_2e_2-\frac{3}{4}\sigma_3e_2e_1-3e_2e_0+(\sigma_2+\frac{1}{2}\sigma_3^2+3)e_1e_1 \\
&\quad +\frac{3}{2}\sigma_3e_1e_0+4e_0e_0)e_0|0\rangle\Big),
\end{aligned} \tag{67}$$

$$\begin{aligned}
|(3)\rangle &= \frac{1}{(h_3-h_2)(h_3-h_1)(2h_3-h_2)(2h_3-h_1)}(e_2+h_3e_1+h_1h_2e_0) \\
&\quad (e_2+h_3e_1+h_1h_2e_0)e_0|0\rangle \\
&= \frac{1}{(h_3-h_2)(h_3-h_1)(2h_3-h_2)(2h_3-h_1)}\Big((1+h_1h_2)(2+h_1h_2)e_0e_0e_0|0\rangle \\
&\quad +3h_3(1+h_1h_2)(2+h_1h_2)e_0e_1e_0|0\rangle+3(2+h_1h_2)e_0(e_2-\sigma_3e_1-e_0)e_0|0\rangle \\
&\quad +h_3^2(1+h_1h_2)(2+h_1h_2)e_1e_1e_0|0\rangle \\
&\quad +(\frac{3h_3}{2}+\frac{3}{4}\sigma_3)(e_2-2\sigma_3e_1-4e_0)e_1e_0|0\rangle \\
&\quad +(e_2e_2-\frac{3}{4}\sigma_3e_2e_1-3e_2e_0+(\sigma_2+\frac{1}{2}\sigma_3^2+3)e_1e_1+\frac{3}{2}\sigma_3e_1e_0+4e_0e_0)e_0|0\rangle\Big),
\end{aligned} \tag{68}$$

$$\begin{aligned}
|\begin{pmatrix}1\\1\\1\end{pmatrix}\rangle &= \frac{1}{(h_2-h_1)(h_2-h_3)(2h_2-h_1)(2h_2-h_3)}(e_2+h_2e_1+h_1h_3e_0) \\
&\quad (e_2+h_2e_1+h_1h_3e_0)e_0|0\rangle \\
&= \frac{1}{(h_2-h_1)(h_2-h_3)(2h_2-h_1)(2h_2-h_3)}\Big((1+h_1h_3)(2+h_1h_3)e_0e_0e_0|0\rangle
\end{aligned}$$

$$
\begin{aligned}
&+3h_2(1+h_1h_3)(2+h_1h_3)e_0e_1e_0|0\rangle+3(2+h_1h_3)e_0(e_2-\sigma_3e_1-e_0)e_0|0\rangle\\
&+h_2^2(1+h_1h_3)(2+h_1h_3)e_1e_1e_0|0\rangle \qquad (69)\\
&+(\frac{3h_2}{2}+\frac{3}{4}\sigma_3)(e_2-2\sigma_3e_1-4e_0)e_1e_0|0\rangle\\
&+(e_2e_2-\frac{3}{4}\sigma_3e_2e_1-3e_2e_0+(\sigma_2+\frac{1}{2}\sigma_3^2+3)e_1e_1\\
&+\frac{3}{2}\sigma_3e_1e_0+4e_0e_0)e_0|0\rangle\Big),
\end{aligned}
$$

$$
\begin{aligned}
|(2,1)_{h_3,h_1}\rangle &= \frac{1}{(h_1-2h_3)(h_1-h_2)(h_3-h_2)(h_3-h_1)}(e_2+(h_1-h_3)e_1+2h_2h_3e_0)\\
&\quad(e_2+h_3e_1+h_1h_2e_0)e_0|0\rangle\\
&=\frac{1}{(h_1-2h_3)(h_1-h_2)(h_3-h_2)(h_3-h_1)}\Big(2(1+h_1h_2)(1+h_2h_3)e_0e_0e_0|0\rangle\\
&\quad-2h_2(1+h_2h_3)(1+h_1h_2)e_0e_1e_0|0\rangle+(6-2h_2^2)e_0(e_2-\sigma_3e_1-e_0)e_0|0\rangle\\
&\quad+h_1h_3(1+h_2h_3)(1+h_1h_2)e_1e_1e_0|0\rangle \qquad (70)\\
&\quad+(\frac{-h_2}{2}+\frac{3}{4}\sigma_3)(e_2-2\sigma_3e_1-4e_0)e_1e_0|0\rangle\\
&\quad+(e_2e_2-\frac{3}{4}\sigma_3e_2e_1-3e_2e_0+(\sigma_2+\frac{1}{2}\sigma_3^2+3)e_1e_1\\
&\quad+\frac{3}{2}\sigma_3e_1e_0+4e_0e_0)e_0|0\rangle\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
|\begin{pmatrix}2\\1\end{pmatrix}_{h_3,h_2}\rangle &= \frac{1}{(h_2-2h_3)(h_2-h_1)(h_3-h_1)(h_3-h_2)}(e_2+(h_2-h_3)e_1+2h_1h_3e_0)\\
&\quad(e_2+h_3e_1+h_1h_2e_0)e_0|0\rangle\\
&=\frac{1}{(h_2-2h_3)(h_2-h_1)(h_3-h_1)(h_3-h_2)}\Big(2(1+h_1h_3)(1+h_1h_2)e_0e_0e_0|0\rangle\\
&\quad-2h_1(1+h_1h_2)(1+h_1h_3)e_0e_1e_0|0\rangle+(6-2h_1^2)e_0(e_2-\sigma_3e_1-e_0)e_0|0\rangle\\
&\quad+h_2h_3(1+h_1h_2)(1+h_1h_3)e_1e_1e_0|0\rangle \qquad (71)\\
&\quad+(\frac{-h_1}{2}+\frac{3}{4}\sigma_3)(e_2-2\sigma_3e_1-4e_0)e_1e_0|0\rangle\\
&\quad+(e_2e_2-\frac{3}{4}\sigma_3e_2e_1-3e_2e_0+(\sigma_2+\frac{1}{2}\sigma_3^2+3)e_1e_1\\
&\quad+\frac{3}{2}\sigma_3e_1e_0+4e_0e_0)e_0|0\rangle\Big).
\end{aligned}
$$

From the expressions above, we clearly get the relations in equations ((35)-(37)) which match the relations in equation ((44)-(46)) of 3-Schur functions.

We set

$$
\begin{aligned}
&P_1^3=e_0e_0e_0|0\rangle,\ \ P_1P_{2,1}=e_0e_1e_0|0\rangle,\ \ 2P_{3,1}=e_1e_1e_0|0\rangle,\\
&\sqrt{(1+h_1h_2)(1+h_1h_3)(1+h_2h_3)}P_1P_{2,2}=e_0(e_2-\sigma_3e_1-e_0)e_0|0\rangle,
\end{aligned}
$$

$$4\sqrt{(1+h_1h_2)(1+h_1h_3)(1+h_2h_3)}P_{3,2} = (e_2 - 2\sigma_3 e_1 - 4e_0)e_1e_0|0\rangle,$$
$$\sqrt{(1+h_1h_2)(1+h_1h_3)(1+h_2h_3)(2+h_1h_2)(2+h_1h_3)(2+h_2h_3)}P_{3,3} =$$
$$(e_2e_2 - \frac{3}{4}\sigma_3e_2e_1 - 3e_2e_0 + (\sigma_2 + \frac{1}{2}\sigma_3^2 + 3)e_1e_1 + \frac{3}{2}\sigma_3e_1e_0 + 4e_0e_0)e_0|0\rangle,$$

clearly, we get that the orthogonality (43) match the quadratic form (30) and (31), and the expressions in equations ((63)-(71)) match the expressions in equations ((47)-(52)).

5. Back to the Schur functions of 2D Young diagrams

Here, we consider 2D Young diagrams as a special case of 3D Young diagrams which has only one layer in the z-axis direction. We see that $S_{\square} = P_1$. From ((40)-(42)), when $(h_1, h_2, h_3) = (1, -1, 0)$,

$$S_{(1,1)} = \frac{1}{2}P_1^2 + \frac{1}{2}P_{2,1},\ S_{\begin{pmatrix}1\\1\end{pmatrix}} = \frac{1}{2}P_1^2 - \frac{1}{2}P_{2,1},\ S_{(2)} = 0.$$

From the expressions in equations ((47)-(52)), when $(h_1, h_2, h_3) = (1, -1, 0)$, we have

$$S_{(1,1,1)} = \frac{1}{12}\left(2P_1^3 + 6P_1P_{2,1} + 4P_{3,1}\right),\ S_{\begin{pmatrix}1 & 1\\1 & \end{pmatrix}_{h_1,h_2}} = \frac{1}{6}\left(2P_1^3 - 2P_{3,1}\right),$$
$$S_{(2,1)_{h_1,h_3}} = 0,\ S_{\begin{pmatrix}2\\1\end{pmatrix}_{h_2,h_3}} = 0,\ S_{(3)} = 0,$$
$$S_{\begin{pmatrix}1\\1\\1\end{pmatrix}} = \frac{1}{12}\left(2P_1^3 - 6P_1P_{2,1} + 4P_{3,1}\right),$$

and

$$S_{\begin{pmatrix}1 & 1\\1 & \end{pmatrix}_{h_1,h_2}} = S_{\begin{pmatrix}1 & 1\\1 & \end{pmatrix}_{h_2,h_1}},\ S_{(2,1)_{h_1,h_3}} = S_{(2,1)_{h_3,h_1}},\ S_{\begin{pmatrix}2\\1\end{pmatrix}_{h_2,h_3}} = S_{\begin{pmatrix}2\\1\end{pmatrix}_{h_3,h_2}},$$

We see that when $(h_1, h_2, h_3) = (1, -1, 0)$, 3-Schur functions of 3D Young diagrams with more than one layer in the z-axis direction vanish, and 3-Schur functions of 3D Young diagrams with one layer in the z-axis direction become the Schur functions of 2D Young diagrams if we let $P_{n,1} = p_n$, where p_n are the variables of Schur functions in equations (1) and (2).

Note that the 3-Schur functions of 3D Young diagrams will become Schur functions of 2D Young diagrams which are in different plane if we take different value of h_1, h_2, h_3. From above, we know that if $(h_1, h_2, h_3) = (1, -1, 0)$, 3-Schur functions of 3D Young diagrams become the Schur functions of 2D Young diagrams which are in plane xOy. We can see that if $(h_1, h_2, h_3) = (1, 0, -1)$, 3-Schur functions of 3D Young diagrams become the Schur functions of 2D Young diagrams which are in plane yOz, and if $(h_1, h_2, h_3) = (0, 1, -1)$, 3-Schur functions of 3D Young diagrams become the Schur functions of 2D Young diagrams which are in plane xOy. The exchange of 1 and -1 corresponds to the transpose of 2D Young diagram. For example, if $(h_1, h_2, h_3) = (1, -1, 0)$,

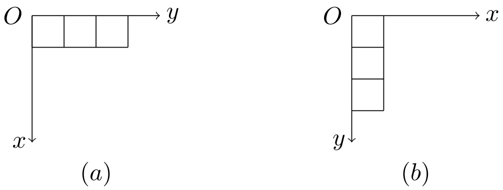

Fig. 2. Young diagram in plane xOy.

$$S_{(1,1,1)} = \frac{1}{12}\left(2P_1^3 + 6P_1P_{2,1} + 4P_{3,1}\right),$$
$$S_{\begin{pmatrix}1\\1\\1\end{pmatrix}} = \frac{1}{12}\left(2P_1^3 - 6P_1P_{2,1} + 4P_{3,1}\right),$$

and if $(h_1, h_2, h_3) = (-1, 1, 0)$,

$$S_{(1,1,1)} = \frac{1}{12}\left(2P_1^3 - 6P_1P_{2,1} + 4P_{3,1}\right),$$
$$S_{\begin{pmatrix}1\\1\\1\end{pmatrix}} = \frac{1}{12}\left(2P_1^3 + 6P_1P_{2,1} + 4P_{3,1}\right),$$

that is, if $(h_1, h_2, h_3) = (1, -1, 0)$, we look at the 2D Young diagrams in plane xOy from top downwards, as in (a) of Fig. 2, and if $(h_1, h_2, h_3) = (1, -1, 0)$, we look at the 2D Young diagrams in plane xOy from bottom up, as in (b) of Fig. 2. We can see that this is the transpose of 2D Young diagrams.

6. The relations with the Jack symmetric polynomials

In this section, we give the relations between the 3-Schur functions we defined and the Jack symmetric polynomials. This section is given under the reviewer's advice. We recall the Jack symmetric polynomial first [2]. Let λ and μ be 2D Young diagrams, for Jack symmetric functions,

$$\langle p_\lambda, p_\mu\rangle_\alpha = \delta_{\lambda\mu}\alpha^{l(\lambda)}z_\lambda, \text{ where } z_\lambda = \prod_i i^{m_i} m_i!,$$

some examples are $\langle p_1^n, p_1^n\rangle = \alpha^n n!$, $\langle p_n, p_n\rangle = \alpha n$. Let

$$b_\lambda^{(\alpha)} = \prod_{s\in\lambda} \frac{\alpha a(s) + l(s) + 1}{\alpha a(s) + l(s) + \alpha},$$

where $a(s)$ and $l(s)$ are arm-length and leg-length of s respectively. The Jack symmetric polynomials are denoted by $P_\lambda^{(\alpha)}$, $Q_\lambda^{(\alpha)}$, the relation between them is

$$Q_\lambda^{(\alpha)} = b_\lambda^{(\alpha)} P_\lambda^{(\alpha)}.$$

The following are some examples.

$$P^{(\alpha)}_{(1)}=p_1,\quad Q^{(\alpha)}_{(1)}=\frac{1}{\alpha}p_1,$$
$$P^{(\alpha)}_{(2)}=\frac{1}{\alpha+1}p_1^2+\frac{\alpha}{\alpha+1}p_2,\quad Q^{(\alpha)}_{(2)}=\frac{1}{2\alpha^2}p_1^2+\frac{1}{2\alpha}p_2,$$
$$P^{(\alpha)}_{(1,1)}=\frac{1}{2}p_1^2-\frac{1}{2}p_2,\quad Q^{(\alpha)}_{(1,1)}=\frac{1}{\alpha(\alpha+1)}p_1^2-\frac{1}{\alpha(\alpha+1)}p_2,$$
$$P^{(\alpha)}_{(3)}=\frac{2\alpha^2}{(2\alpha+1)(\alpha+1)}p_3+\frac{3\alpha}{(2\alpha+1)(\alpha+1)}p_1p_2+\frac{1}{(2\alpha+1)(\alpha+1)}p_1^3,$$
$$Q^{(\alpha)}_{(3)}=\frac{1}{3\alpha}p_3+\frac{1}{2\alpha^2}p_1p_2+\frac{1}{6\alpha^3}p_1^3,$$
$$P^{(\alpha)}_{(2,1)}=\frac{1}{\alpha+2}p_1^3+\frac{\alpha-1}{\alpha+2}p_1p_2-\frac{\alpha}{\alpha+2}p_3,$$
$$Q^{(\alpha)}_{(2,1)}=\frac{1}{\alpha^2(2\alpha+1)}p_1^3+\frac{\alpha-1}{\alpha^2(2\alpha+1)}p_1p_2-\frac{1}{\alpha(2\alpha+1)}p_1^3,$$
$$P^{(\alpha)}_{(1,1,1)}=\frac{1}{6}p_1^3-\frac{1}{2}p_1p_2+\frac{1}{3}p_3,$$
$$Q^{(\alpha)}_{(1,1,1)}=\frac{1}{\alpha(\alpha+1)(\alpha+2)}p_1^3-\frac{3}{\alpha(\alpha+1)(\alpha+2)}p_1p_2+\frac{2}{\alpha(\alpha+1)(\alpha+2)}p_1^3.$$

In the following, we will show what the 3-Schur functions become when $(h_1,h_2,h_3)=(h,-\frac{1}{h},\frac{1}{h}-h)$. When $(h_1,h_2,h_3)=(h,-\frac{1}{h},\frac{1}{h}-h)$, we get $1+h_1h_2=0$, then the 3-Schur functions corresponding to 3D Young diagrams which are more than one layer in z-axis direction vanish, and the 3-Schur functions corresponding to 3D Young diagrams which are one layer in z-axis direction become:

$$S_{(1)}=p_1,\ S_{(1,1)}=\frac{1}{1+h^2}P_1^2+\frac{h}{1+h^2}P_{2,1},\ S_{\begin{pmatrix}1\\1\end{pmatrix}}=\frac{h^2}{1+h^2}P_1^2-\frac{h}{1+h^2}P_{2,1},$$

$$S_{(1,1,1)}=\frac{1}{(1+h^2)(1+2h^2)}(P_1^3+3hP_1P_{2,1}+2h^2P_{3,1}),$$
$$S_{\begin{pmatrix}1&1\\1&\end{pmatrix}_{h_1,h_2}}=\frac{1}{(1+h^2)(1+2h^2)}(2h^2P_1^3+2h(h^2-1)P_1P_{2,1}-2h^2P_{3,1}),$$
$$S_{\begin{pmatrix}1\\1\\1\end{pmatrix}}=\frac{1}{(1+h^2)(2+h^2)}(h^4P_1^3-3h^3P_1P_{2,1}+2h^2P_{3,1}).$$

We change p_n to be $\sqrt{\alpha}p_n$ in Jack symmetric polynomials (this is because $\langle p_n,p_n\rangle_\alpha=n\alpha$ in Jack symmetric polynomials and $\langle p_n,p_n\rangle=n$ in Schur functions), let $\alpha=h^2$, we get

$$P^{(\alpha)}_{(1)}=\sqrt{\alpha}S_{(1)}|_{(h,-\frac{1}{h},\frac{1}{h}-h)},\tag{72}$$
$$P^{(\alpha)}_{(2)}=(\sqrt{\alpha})^2S_{(1,1)}|_{(h,-\frac{1}{h},\frac{1}{h}-h)},\tag{73}$$
$$Q^{(\alpha)}_{(1,1)}=\frac{1}{(\sqrt{\alpha})^2}S_{\begin{pmatrix}1\\1\end{pmatrix}}|_{(h,-\frac{1}{h},\frac{1}{h}-h)},\tag{74}$$

$$P^{(\alpha)}_{(3)} = (\sqrt{\alpha})^3 S_{(1,1,1)}|_{(h,-\frac{1}{h},\frac{1}{h}-h)}, \tag{75}$$

$$Q^{(\alpha)}_{(2,1)} = \frac{1+\alpha}{2(\sqrt{\alpha})^3} S_{\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}_{h_1,h_2}} |_{(h,-\frac{1}{h},\frac{1}{h}-h)}, \tag{76}$$

$$Q^{(\alpha)}_{(1,1,1)} = \frac{1}{(\sqrt{\alpha})^3} S_{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} |_{(h,-\frac{1}{h},\frac{1}{h}-h)}, \tag{77}$$

we can see some rule in them. Therefore, we get that the 3-Schur functions corresponding to 3D Young diagrams which are one layer in z-axis direction become Jack symmetric polynomials by multiplying a coefficient when $(h_1, h_2, h_3) = (h, -\frac{1}{h}, \frac{1}{h} - h)$. The results in last section is the special case $h = 1$ of that in this section, at this point, we should call 3-Schur function S_π 3-Jack polynomial under the reviewer's advice, where π is a 3D Young diagram. One can similarly discuss other results as in last section.

7. Concluding remarks

In this paper, we give the 3-Schur functions of 3D Young diagrams by the orthogonality we have calculated. 3-Schur functions are functions of variables P_1, $P_{2,1}$, $P_{2,2}$, $P_{3,1}$, $P_{3,2}$, $P_{3,3}, \cdots$, whose coefficients are functions of h_1, h_2, h_3 with relation $h_1 + h_2 + h_3 = 0$. The expressions of 3-Schur functions are symmetric about three coordinate axes. We also show that 3-Schur functions we given match that in affine Yangian and its MacMahon representation. When $(h_1, h_2, h_3) = (h, -\frac{1}{h}, \frac{1}{h} - h)$ 3-Schur functions of 3D Young diagrams become the Jack symmetric polynomials of 2D Young diagrams, specially, when $h = 1$, that is, $(h_1, h_2, h_3) = (1, -1, 0)$, 3-Schur functions of 3D Young diagrams become the Schur functions of 2D Young diagrams.

Actually, we give a method to calculate the 3-Schur functions in this paper, next we want to give an expression of 3-Schur functions for all 3D Young diagrams, like the expression in (2) of Schur functions for all 2D Young diagrams.

CRediT authorship contribution statement

Wang Na (the only author): Methodology, Calculation, Writing. Section 6 is given under the reviewer's advice.

Declaration of competing interest

The author (Wang Na) declares that there is no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The author gratefully acknowledges the support of Professors Ke Wu, Shuxia Feng. The author is particularly indebted to the anonymous reviewer of this paper for various kind suggestions and comments.

References

[1] W. Fulton, J. Harris, Representation Theory, a First Course, Springer-Verlag, New York, 1991.
[2] I.G. Macdonald, Symmetric Functions and Hall Polynomials, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1979.
[3] K. Koike, On the decomposition of tensor products of the representations of the classical groups: by means of the universal characters, Adv. Math. 74 (1989) 57–86.
[4] H. Weyl, The Classical Groups; Their Invariants and Representations, Princeton Univ. Press, Princeton, 1946.
[5] T. Miwa, M. Jimbo, E. Date, Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras, Cambridge University Press, Cambridge, 2000.
[6] N. Tsilevich, Quantum inverse scattering method for the q-boson model and symmetric functions, Funct. Anal. Appl. 40 (3) (2006) 207–217, arXiv:math-ph/0510073.
[7] P. Sułkowski, Deformed boson-fermion correspondence, Q-bosons, and topological strings on the conifold, J. High Energy Phys. 10 (2008) 1127, arXiv:0808.2327 [hep-th].
[8] N. Wang, Young diagrams in an $N \times M$ box and the KP hierarchy, Nucl. Phys. B 937 (2018) 478–501.
[9] A. Okounkov, N. Reshetikhin, C. Vafa, Quantum Calabi-Yau and classical crystals, arXiv:hep-th/0309208.
[10] T. Nakatsu, K. Takasaki, Integrable structure of melting crystal model with external potentials, Adv. Stud. Pure Math. 59 (2010) 201–223.
[11] V.G. Drinfeld, Hopf algebras and the quantum Yang-Baxter equation, Dokl. Akad. Nauk SSSR 283 (1985) 1060–1064.
[12] V.G. Drinfeld, A new realization of Yangians and of quantum affine algebras, Dokl. Akad. Nauk SSSR 296 (1987) 13–17.
[13] Y. Zenkevich, $3d$ field theory, plane partitions and triple Macdonald polynomials, arXiv:1712.10300.
[14] A. Morozov, An analogue of Schur functions for the plane partitions, arXiv:1808.01059.
[15] N. Wang, Affine Yangian and the orthogonality of 3D Young diagrams, submitted for publication.
[16] T. Procházka, $\mathcal{W}$-symmetry, topological vertex and affine Yangian, J. High Energy Phys. 10 (2016) 077.
[17] A. Tsymbaliuk, The affine Yangian of gl_1 revisited, arXiv:1404.5240.
[18] O. Foda, M. Wheeler, Hall-Littlewood plane partitions and KP, Int. Math. Res. Not. (2009) 2597–2619, arXiv:0809.2138 [math-ph].