ARGONNE NATIONAL LABORATORY
Lemont, Illinois

ANLAD-55

HELICAL QUADRUPOLE MAGNETIC FOCUSING SYSTEMS

L. C. Teng
Particle Accelerator Division

February 18, 1959

ABSTRACT

A. V. Crewe pointed out that with air core and high field a helical quadrupole magnetic lens is easier to build than a conventional alternating sectional quadrupole focusing magnet system. The motion of a charged particle in such a helical quadrupole field is studied here under the linear paraxial approximation. The result shows that the focusing action of such a helical quadrupole system is about 10% stronger than that of a corresponding alternating sectional system with the same periodicity and field gradient. With appropriately chosen field gradient, pitch and length, a helical quadrupole magnet can be designed to form point images from point sources located on the axis. Formulas and graphs relating these parameters are derived.

An example of a helical quadrupole lens system is shown in Figure 1. λ is the pitch of the helix. In the analogous case of alternating conventional lenses, within this distance there would normally occur two pairs of focus-defocus elements.

I. Magnetic Field in a General Helical 2n - pole Magnet

The scalar potential for such a field is of the form

$$\phi_n(r, \theta, z) = f_n(r) \sin n\left(\theta - \frac{2\pi z}{\lambda}\right) \tag{1}$$

where r, θ, z are the conventional cylindrical coordinates and λ is the pitch of the helix. Substituting ϕ_n in the Laplace equation we get the radial equation for f_n

$$\frac{d^2 f_n}{dr^2} + \frac{1}{r}\frac{df_n}{dr} - n^2\left(\frac{4\pi^2}{\lambda^2} + \frac{1}{r^2}\right) f_n = 0 \tag{2}$$

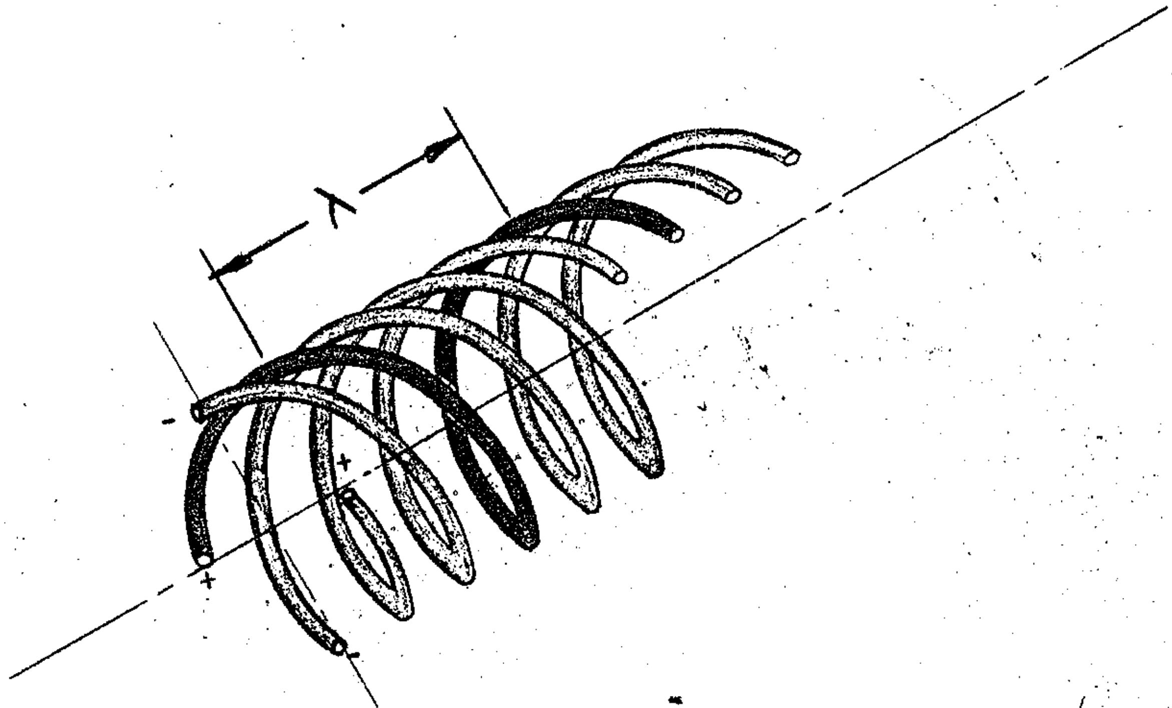


Figure 1
ANLAD-55

Air Core Helical Quadrupole Focusing Magnet

The solution of this equation which is regular at $r = 0$ is

$$f_n(r) \hookrightarrow J_n\left(i \frac{2\pi n}{\lambda} r\right) \tag{3}$$

where J_n is the Bessel function of order n and i is the imaginary unit. Since we will be interested only in the values of $f_n(r)$ for small r we need the expansion formula

$$J_n(Z) = \left(\frac{Z}{2}\right)^n \sum_{\ell=0}^{n} \frac{(-1)^\ell}{\ell!\,(n+\ell)!} \left(\frac{Z}{2}\right)^{2\ell} \qquad n = \text{integer} > 0 \tag{4}$$

Using (4) we get for (3)

$$f_n(r) \hookrightarrow r^n \sum_{\ell=0}^{\infty} \frac{1}{\ell!\,(n+\ell)!} \left(\frac{\pi n r}{\lambda}\right)^{2\ell} \tag{5}$$

and for the scalar potential

$$\phi_n(r,\theta,z) \hookrightarrow r^n \sin n\left(\theta - \frac{2\pi z}{\lambda}\right) \sum_{\ell=0}^{\infty} \frac{1}{\ell!\,(n+\ell)!} \left(\frac{\pi n r}{\lambda}\right)^{2\ell} \tag{6}$$

Whence we see that the lowest term is of degree n in r. For collineation we need a field linear in r or a potential quadratic in r, i.e., a quadrupole field with $n = 2$, and

$$\phi_2(r,\theta,z) \hookrightarrow r^2 \sin 2\left(\theta - \frac{2\pi z}{\lambda}\right) \sum_{\ell=0}^{\infty} \frac{1}{\ell!\,(\ell+2)!} \left(\frac{2\pi r}{\lambda}\right)^{2\ell} \tag{7}$$

We shall be interested only in the lowest degree term and shall write approximately

$$\phi_2 \cong \frac{G}{2} r^2 \sin 2\left(\theta - \frac{2\pi z}{\lambda}\right) \tag{8}$$

where G is the proportionality constant. This approximate potential gives for the magnetic field in the cylindrical coordinates

$$\left\{\begin{aligned} B_r &= \frac{\partial \phi_2}{\partial r} \cong G\, r \sin 2\left(\theta - \frac{2\pi z}{\lambda}\right) \\ B_\theta &= \frac{\partial \phi_2}{r\, \partial \theta} \cong G\, r \cos 2\left(\theta - \frac{2\pi z}{\lambda}\right) \\ B_z &= \frac{\partial \phi_2}{\partial z} \cong -\frac{2\pi G}{\lambda}\, r^2 \cos 2\left(\theta - \frac{2\pi z}{\lambda}\right) \end{aligned}\right. \tag{9}$$

and in the Cartesian coordinates $x = r \cos \theta$, $y = r \sin \theta$, $z = z$,

$$\left\{\begin{aligned} B_x &= \frac{\partial \phi_2}{\partial x} \cong G\left(-x \sin \frac{4\pi z}{\lambda} + y \cos \frac{4\pi z}{\lambda}\right) \\ B_y &= \frac{\partial \phi_2}{\partial y} \cong G\left(y \sin \frac{4\pi z}{\lambda} + x \cos \frac{4\pi z}{\lambda}\right) \\ B_z &= \frac{\partial \phi_2}{\partial z} \cong -\frac{2\pi G}{\lambda}\left[(x^2 - y^2) \cos \frac{4\pi z}{\lambda} + 2xy \sin \frac{4\pi z}{\lambda}\right] \end{aligned}\right. \tag{10}$$

These equations provide a physical meaning for G, namely

$$G = \left(\frac{\partial B_x}{\partial y}\right)_{z=0} = \left(\frac{\partial B_y}{\partial x}\right)_{z=0} = \left(\frac{\partial B_\theta}{\partial r}\right)_{z=\theta=0} \tag{11}$$

II. Orbit Equations

The equations of the orbit of a charged particle moving in a magnetic field have been derived before in many places (see Orbit Theory Notes). With z as the independent variable the orbit equations in the cylindrical coordinates are

$$\begin{cases} \dfrac{d}{dz}\left(\dfrac{r'}{\sqrt{1+r'^2+r^2\theta'^2}}\right) - \dfrac{r\,\theta'^2}{\sqrt{1+r'^2+r^2\,\theta'^2}} = \dfrac{e}{pc}\,(r\,\theta' B_z - B_\theta) \\[2ex] r\,\dfrac{d}{dz}\left(\dfrac{\theta'}{\sqrt{1+r'^2+r^2\,\theta'^2}}\right) + \dfrac{2\,r'\theta'}{\sqrt{1+r'^2+r^2\theta'^2}} = \dfrac{e}{pc}\,(B_r - r'B_z) \end{cases} \tag{12}$$

where prime means $\frac{d}{dz}$, p and e are the momentum and the charge of the particle, and c is the velocity of light. In Cartesian coordinates the corresponding equations are

$$\begin{cases} \dfrac{d}{dz}\left(\dfrac{x'}{\sqrt{1+x'^2+y'^2}}\right) = \dfrac{e}{pc}\,(y' B_z - B_y) \\[2ex] \dfrac{d}{dz}\left(\dfrac{y'}{\sqrt{1+x'^2+y'^2}}\right) = \dfrac{e}{pc}\,(B_x - x' B_z) \end{cases} \tag{13}$$

Since we are interested here only in the paraxial motions, we shall expand (12) and (13) to first degree terms in r, (x,y), and r', (x',y'). The resulting equations after the substitution of (9) and (10) are, for cylindrical coordinates

$$\begin{cases} r'' - r\,\theta'^2 = - \dfrac{eG}{pc}\, r \cos 2\left(\theta - \dfrac{2\pi z}{\lambda}\right) \\[2ex] r\,\theta'' + 2\,r'\theta' = \dfrac{eG}{pc}\, r \sin 2\left(\theta - \dfrac{2\pi z}{\lambda}\right) \end{cases} \tag{14}$$

and for Cartesian coordinates

$$\begin{cases} x'' = -\dfrac{eG}{pc}\left(x \cos \dfrac{4\pi z}{\lambda} + y \sin \dfrac{4\pi z}{\lambda}\right) \\ y'' = -\dfrac{eG}{pc}\left(x \sin \dfrac{4\pi z}{\lambda} - y \cos \dfrac{4\pi z}{\lambda}\right) \end{cases} \tag{15}$$

It should be noted that since the lowest term occurring in B_z is of the second degree in r, B_z is completely absent in these linearized equations. Also, because of the absence of the centripetal and the Coriolis terms in the Cartesian forms (15) these equations are simpler in structure. Henceforth, we shall focus our attention on these equations.

III. Approximate Solutions

Equation (15) can be further simplified by expressing z in units of $\dfrac{\lambda}{4\pi}$ and defining $a \equiv -\dfrac{\lambda^2}{16\pi^2}\dfrac{eG}{pc}$. The resulting equations are

$$\begin{cases} x'' = a(x \cos z + y \sin z) \\ y'' = a(x \sin z - y \cos z) \end{cases} \tag{16}$$

Although this set of coupled Mathieu equations being of a very special type can be solved exactly in terms of closed elementary functions, an approximate iterative solution in power series of "a" serves in the first place, to illustrate a general method applicable to all systems of coupled linear equations and, in the second place, to give a set of approximate formulas which are, in certain cases, more illuminating and easier to handle.

General Method

We shall start with a system of n coupled homogeneous linear equations (since the particular integral for inhomogeneous equations can be obtained by straightforward quadrature from the Green's function formed with the general solutions of the homogeneous part of the equations) which has already been reduced to the first order, namely

$$x_i' = A_{ij}(z)\, x_j \qquad i = 1, 2, 3, \ldots . n \tag{17}$$

These are, in general, n linearly independent sets of solutions which shall be denoted by $x_{i\mu}(z)$ ($\mu = 1, 2, 3, \ldots n$). The coefficients A_{ij} are assumed to be small and shall for the moment be written as ϵA_{ij} where ϵ is the "smallness" parameter. The solutions $x_{i\mu}$ are now written as a power series in ϵ

$$x_{i\mu}(z) = x_{i\mu}^{(0)} + \epsilon x_{i\mu}^{(1)} + \epsilon^2 x_{i\mu}^{(2)} + \ldots . . \tag{18}$$

where we shall put

$$x_{i\mu}^{(0)} = x_{i\mu}(z = 0) = \text{constants.}$$

Substituting (18) in (17) and equating the coefficients of monomials in ϵ, we get

$$\begin{aligned} x_{i\mu}^{(0)\prime} &= 0 \\ x_{i\mu}^{(1)\prime} &= A_{ij}\, x_{j\mu}^{(0)} \\ x_{i\mu}^{(2)\prime} &= A_{ij}\, x_{j\mu}^{(1)} \qquad \text{etc.} \end{aligned} \tag{19}$$

These equations can be integrated to give

$$\begin{aligned} x_{i\mu}^{(0)} &= x_{i\mu}(z = 0) = \text{initial conditions} \\ x_{i\mu}^{(1)} &= \int_o^z dz\; A_{ij}\, x_{j\mu}^{(0)} \\ x_{i\mu}^{(2)} &= \int_0^z dz\, A_{ij}\, x_{j\mu}^{(1)} = \int_0^z dz\, A_{ij} \int_0^z dz\, A_{jk}\, x_{k\mu}^{(0)} \end{aligned} \tag{20}$$

etc.

For a simple complete set of initial conditions we can take

$$\overline{x}_{i\mu}^{(0)} = \delta_{i\mu} = \text{Kronecker } \delta \tag{21}$$

Substituting (21) in (20) we get

$$\overline{x}_{i\mu}^{(1)}(z) = \int_0^z dz\, A_{i\mu}$$

$$\overline{x}_{i\mu}^{(2)}(z) = \int_0^z dz\, A_{ij} \int_0^z dz\, A_{j\mu} \tag{22}$$

etc.

Now if we write the general solution in the matrix form

$$x_i(z) = M_{ij}(z)\ x_j(0) \tag{23}$$

We get immediately

$$M_{ij}(z) = \overline{x}_{ij}(z) = \sum_{k=0}^{\infty} \overline{x}_{ij}^{(k)}(z)$$

$$= \delta_{ij} + \int_0^z dz\, A_{ij} + \int_0^z dz\, A_{ik} \int_0^z dz\, A_{kj}$$

$$+ \int_0^z dz\, A_{ik} \int_0^z dz\, A_{k\ell} \int_0^z dz\, A_{\ell j} + \ldots\ldots \tag{24}$$

Equations (23) and (24) give the general solution of (17) in power series of A_{ij}.

Application to (16)

Equation (16) can be put in the form (17) by the substitution $x_1 = x$, $x_2 = \frac{x'}{\sqrt{a}}$, $x_3 = y$, $x_4 = \frac{y'}{\sqrt{a}}$.

This gives

$$\begin{cases} x_1' = \sqrt{a}\ x_2 \\ x_2' = \sqrt{a}\,(x_1 \cos z + x_3 \sin z) \\ x_3' = \sqrt{a}\ x_4 \\ x_4' = \sqrt{a}\ (x_1 \sin z - x_3 \cos z) \end{cases} \tag{25}$$

or

$$A_{ij} = \sqrt{a} \begin{pmatrix} 0 & 1 & 0 & 0 \\ \cos z & 0 & \sin z & 0 \\ 0 & 0 & 0 & 1 \\ \sin z & 0 & -\cos z & 0 \end{pmatrix} \tag{26}$$

Straightforward integrations, then, give up to a^4 terms

$$\overline{x}_{ij}^{(1)} = a^{\frac{1}{2}} \begin{pmatrix} 0 & z & 0 & 0 \\ \sin z & 0 & 1 - \cos z & 0 \\ 0 & 0 & 0 & z \\ 1 - \cos z & 0 & -\sin z & 0 \end{pmatrix} \tag{27}$$

$$\bar{x}_{ij}^{(2)} = a \begin{pmatrix} 1-\cos z & 0 & z-\sin z & 0 \\ 0 & -(1-z\sin z-\cos z) & 0 & \sin z - z\cos z \\ z-\sin z & 0 & -(1-\cos z) & 0 \\ 0 & \sin z - z\cos z & 0 & 1-z\sin z-\cos z \end{pmatrix}$$

$$\bar{x}_{ij}^{(3)} = a^{\frac{3}{2}} \begin{pmatrix} 0 & -(z-2\sin z+z\cos z) & 0 & 2-z\sin z-2\cos z \\ -(z-2\sin z+z\cos z) & 0 & -(2-z\sin z-2\cos z) & 0 \\ 0 & 2-z\sin z-2\cos z & 0 & z-2\sin z+z\cos z \\ 2-z\sin z-2\cos z & 0 & -(z-2\sin z+z\cos z) & 0 \end{pmatrix}$$ (2

$$\bar{x}_{ij}^{(4)} = a^2 \begin{pmatrix} 3-\frac{1}{2}z^2-z\sin z-3\cos z & 0 & -(2z-3\sin z+z\cos z) & 0 \\ 0 & 3-\frac{1}{2}z^2-z\sin z-3\cos z & 0 & -(2z-3\sin z+z\cos \\ 2z-3\sin z+z\cos z & 0 & 3-\frac{1}{2}z^2-z\sin z-3\cos z & 0 \\ 0 & 2z-3\sin z+z\cos z & 0 & 3-\frac{1}{2}z^2-z\sin z-3\cos \end{pmatrix}$$

$$
\left\{
\begin{aligned}
&\bar{x}_{11}^{(5)} = \bar{x}_{13}^{(5)} = \bar{x}_{22}^{(5)} = \bar{x}_{24}^{(5)} = \bar{x}_{31}^{(5)} = \bar{x}_{33}^{(5)} = \bar{x}_{42}^{(5)} = \bar{x}_{44}^{(5)} = 0 \\
&\bar{x}_{12}^{(5)} = \bar{x}_{34}^{(5)} = a^{5/2}\,(3z - \frac{1}{6}z^3 - 4\sin z + z\cos z) \\
&\bar{x}_{43}^{(5)} = -\bar{x}_{21}^{(5)} = a^{5/2}\left[3z - (6 - \frac{1}{2}z^2)\sin z + 3z\cos z\right] \\
&\bar{x}_{23}^{(5)} = \bar{x}_{41}^{(5)} = a^{5/2}\left[6 - \frac{1}{2}z^2 - 3z\sin z - (6 - \frac{1}{2}z^2)\cos z\right] \\
&\bar{x}_{14}^{(5)} = -\bar{x}_{32}^{(5)} = a^{5/2}\,(4 - z^2 - z\sin z - 4\cos z)
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
&\bar{x}_{12}^{(6)} = \bar{x}_{14}^{(6)} = \bar{x}_{21}^{(6)} = \bar{x}_{23}^{(6)} = \bar{x}_{32}^{(6)} = \bar{x}_{34}^{(6)} = \bar{x}_{41}^{(6)} = \bar{x}_{43}^{(6)} = 0 \\
&\bar{x}_{11}^{(6)} = -\bar{x}_{33}^{(6)} = a^3\left[10 - \frac{3}{2}z^2 - 4z\sin z - (10 - \frac{1}{2}z^2)\cos z\right] \\
&\bar{x}_{44}^{(6)} = -\bar{x}_{22}^{(6)} = a^3\left[10 - \frac{1}{2}z^2 - (6z - \frac{1}{6}z^3)\sin z - (10 - \frac{3}{2}z^2)\cos z\right] \\
&\bar{x}_{13}^{(6)} = \bar{x}_{31}^{(6)} = a^3\left[6z - \frac{1}{6}z^3 - (10 - \frac{1}{2}z^2)\sin z + 4z\cos z)\right] \\
&-\bar{x}_{24}^{(6)} = -\bar{x}_{42}^{(6)} = a^3\left[4z - (10 - \frac{3}{2}z^2)\sin z + (6z - \frac{1}{6}z^3)\cos z\right]
\end{aligned}
\right. \tag{27}
$$

$$
\left\{
\begin{aligned}
&\bar{x}_{11}^{(7)} = \bar{x}_{13}^{(7)} = \bar{x}_{22}^{(7)} = \bar{x}_{24}^{(7)} = \bar{x}_{31}^{(7)} = \bar{x}_{33}^{(7)} = \bar{x}_{42}^{(7)} = \bar{x}_{44}^{(7)} = 0 \\
&\bar{x}_{14}^{(7)} = \bar{x}_{41}^{(7)} = \bar{x}_{32}^{(7)} = -\bar{x}_{23}^{(7)} \\
&\qquad = a^{7/2} \left[20 - 2z^2 - (10z - \frac{1}{6} z^3) \sin z - (20 - 2z^2) \cos z \right] \\
&\bar{x}_{34}^{(7)} = -\bar{x}_{43}^{(7)} = -\bar{x}_{12}^{(7)} = -\bar{x}_{21}^{(7)} \\
&\qquad = a^{7/2} \left[10z - \frac{1}{6} z^3 - (20 - 2z^2) \sin z + (10z - \frac{1}{6} z^3) \cos z \right]
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
&\bar{x}_{12}^{(8)} = \bar{x}_{14}^{(8)} = \bar{x}_{21}^{(8)} = \bar{x}_{23}^{(8)} = \bar{x}_{32}^{(8)} = \bar{x}_{34}^{(8)} = \bar{x}_{41}^{(8)} = \bar{x}_{43}^{(8)} = 0 \\
&\bar{x}_{11}^{(8)} = \bar{x}_{22}^{(8)} = \bar{x}_{33}^{(8)} = \bar{x}_{44}^{(8)} \\
&\qquad = a^4 \left[35 - 5z^2 + \frac{1}{24} z^4 - (15z - \frac{1}{6} z^3) \sin z - (35 - \frac{5}{2} z^2) \cos z \right] \\
&\bar{x}_{31}^{(8)} = \bar{x}_{42}^{(8)} = -\bar{x}_{13}^{(8)} = -\bar{x}_{24}^{(8)} \\
&\qquad = a^4 \left[20z - \frac{2}{3} z^3 - (35 - \frac{5}{2} z^2) \sin z + (15z - \frac{1}{6} z^3) \cos z \right]
\end{aligned}
\right.
\qquad (27)
$$

We shall reserve the discussions of this approximate solution until after we obtain the exact solution. For later reference we shall rewrite the displacement-slope vector as $\begin{pmatrix} x \\ \frac{x'}{2a} \\ y \\ \frac{y'}{2a} \end{pmatrix}$. The corresponding transfer matrix $M_{ij}(z)$ is, then,

$$
M = \begin{pmatrix}
1 + aA + a^2 B + a^3 C + a^4 D & 2(az + a^2 J + a^3 P + a^4 L) & aE - a^2 F + a^3 G - a^4 H & 2(a^2 M - a^3 Q + a^4 O) \\
\frac{1}{2}(I + aJ + a^2 K + a^3 L) & 1 + aR + a^2 B + a^3 S + a^4 D & \frac{1}{2}(A - aM + a^2 N - a^3 O) & aT - a^2 F + a^3 U - a^4 H \\
aE + a^2 F + a^3 G + a^4 H & 2(a^2 M + a^3 Q + a^4 O) & 1 - aA + a^2 B - a^3 C + a^4 D & 2(az - a^2 J + a^3 P - a^4 L) \\
\frac{1}{2}(A + aM + a^2 N + a^3 O) & aT + a^2 F + a^3 U + a^4 H & -\frac{1}{2}(I - aJ + a^2 K - a^3 L) & 1 - ak + a^2 B - a^3 S + a^4 D
\end{pmatrix} \tag{28}
$$

where

$A = 1 - \cos z$

$B = (3 - \frac{1}{2} z^2) - z \sin z - z \cos z$

$C = (10 - \frac{3}{2} z^2) - 4 z \sin z + (10 + \frac{1}{2} z^2) \cos z$

$D = (35 - 5z^2 + \frac{1}{24} z^4) + (-15 z + \frac{1}{6} z^3) \sin z + (-35 + \frac{5}{2} z^2) \cos z$

$E = z - \sin z$

$F = 2z - 3 \sin z + z \cos z$

$G = (6 z - \frac{1}{6} z^3) + (-10 + \frac{1}{2} z^2) \sin z + 4 z \cos z$

$H = (20 z - \frac{2}{3} z^3) + (-35 + \frac{5}{2} z^2) \sin z + (15 z - \frac{1}{6} z^3) \cos z$

$I = \sin z$

$J = -z + 2 \sin z - z \cos z$

$K = -3 z + (6 - \frac{1}{2} z^2) \sin z - 3 z \cos z$

$L = (-10 z + \frac{1}{6} z^3) + (20 - 2z^2) \sin z + (-10 z + \frac{1}{6} z^3) \cos z$

$T = \sin z - z \cos z$

$M = 2 - z \sin z - 2 \cos z$

$N = (6 - \frac{1}{2} z^2) - 3 z \sin z + (-6 + \frac{1}{2} z^2) \cos z$

$O = (20 - 2 z^2) + (-10 z + \frac{1}{6} z^3) \sin z + (-20 + 2 z^2) \cos z$

$R = -1 + z \sin z + \cos z$

$Q = (-4 + z^2) + z \sin z + 4 \cos z$

$P = (3 z - \frac{1}{6} z^3) - 4 \sin z + z \cos z$

$S = (-10 + \frac{1}{2} z^2) + (6 z - \frac{1}{6} z^3) \sin z + (10 - \frac{3}{2} z^2) \cos z$

$U = -4z + (10 - \frac{3}{2} z^2) \sin z + (-6z + \frac{1}{6} z^3) \cos z$

(NOTE: The capital letters A to U used here should not be confused with similar ones used elsewhere in this paper.)

IV. Exact Solutions

Multiplying the second of (16) by i and adding to the first we obtain

$$Z'' = a\, e^{iz}\, Z^* \tag{29}$$

where $Z = x + i\,y$. Similarly we can obtain the complex conjugate equation

$$Z^{*''} = a\, e^{-iz}\, Z \tag{30}$$

Solving (29) for Z^* and differentiating with respect to z twice we get

$$Z^{*''} = \frac{1}{a}\,(Z'''' - 2i\,z''' - Z'')\,e^{-iz} \tag{31}$$

Combining (30) and (31) we get

$$Z'''' - 2i\,Z''' - Z'' - a^2 Z = 0 \quad , \tag{32}$$

a linear equation with constant coefficient. The index equation

$$\alpha^4 - 2i\,\alpha^3 - \alpha^2 - a^2 = 0$$

or

$$\alpha^2\,(\alpha - i)^2 = a^2$$

gives the four roots

$$\alpha = \frac{i}{2}\,(\,1 \pm \sqrt{1 \pm 4\,a}\,) \tag{33}$$

We shall be interested only in the case when all four roots are purely imaginary so that the motion is purely oscillatory. This means that we shall limit a so that

$$-\,\frac{1}{4} < a < \frac{1}{4} \tag{34}$$

The solution can now be written as

$$Z = Z_1\, e^{\frac{i}{2}(1+\sqrt{1+4a})z} + Z_2\, e^{\frac{i}{2}(1-\sqrt{1+4a})z} + Z_3\, e^{\frac{i}{2}(1+\sqrt{1-4a})z} + Z_4\, e^{\frac{i}{2}(1-\sqrt{1-4a})z} \tag{35}$$

where the 4 complex constants Z_1 Z_2 Z_3 and Z_4 are, however, not all independently arbitrary. Substituting (35) in the second order equation (29) or (30) to get the 2 complex relations between Z_1 Z_2 Z_3 and Z_4, expressing these complex constants in component forms (with 4 real arbitrary constants) such that these relations are identically satisfied, then taking the real and the imaginary parts of Z we obtain

$$
\left\{
\begin{aligned}
x = {} & A\left[\frac{\cos\frac{z}{2}(1+\sqrt{1+4a})}{1+\sqrt{1+4a}} + \frac{\cos\frac{z}{2}(1-\sqrt{1+4a})}{1-\sqrt{1+4a}}\right] \\
& -B\left[\frac{\sin\frac{z}{2}(1+\sqrt{1+4a})}{1+\sqrt{1+4a}} - \frac{\sin\frac{z}{2}(1-\sqrt{1+4a})}{1-\sqrt{1+4a}}\right] \\
& +C\left[\frac{\cos\frac{z}{2}(1+\sqrt{1-4a})}{1+\sqrt{1-4a}} - \frac{\cos\frac{z}{2}(1-\sqrt{1-4a})}{1-\sqrt{1-4a}}\right] \\
& -D\left[\frac{\sin\frac{z}{2}(1+\sqrt{1-4a})}{1+\sqrt{1-4a}} + \frac{\sin\frac{z}{2}(1-\sqrt{1-4a})}{1-\sqrt{1-4a}}\right] \\
y = {} & A\left[\frac{\sin\frac{z}{2}(1+\sqrt{1+4a})}{1+\sqrt{1+4a}} + \frac{\sin\frac{z}{2}(1-\sqrt{1+4a})}{1-\sqrt{1+4a}}\right] \\
& +B\left[\frac{\cos\frac{z}{2}(1+\sqrt{1+4a})}{1+\sqrt{1+4a}} - \frac{\cos\frac{z}{2}(1-\sqrt{1+4a})}{1-\sqrt{1+4a}}\right] \\
& +C\left[\frac{\sin\frac{z}{2}(1+\sqrt{1-4a})}{1+\sqrt{1-4a}} - \frac{\sin\frac{z}{2}(1-\sqrt{1-4a})}{1-\sqrt{1-4a}}\right] \\
& +D\left[\frac{\cos\frac{z}{2}(1+\sqrt{1-4a})}{1+\sqrt{1-4a}} + \frac{\cos\frac{z}{2}(1-\sqrt{1-4a})}{1-\sqrt{1-4a}}\right]
\end{aligned}
\right. \tag{36}
$$

These are the exact solutions of (16). The most convenient manner to present the solution is, again, to write it in the matrix form (23). To do this, we first express the constants A, B, C, D in terms of $x(z=0)$, $x'(z=0)$, $y(z=0)$, $y'(z=0)$. The coefficients of these 4 quantities in the expressions for x, x', y, and y' are then the matrix elements of M_{ij}. If we write the vector x_i as

$$x_i = \begin{pmatrix} x \\ \frac{x'}{2a} \\ y \\ \frac{y'}{2a} \end{pmatrix} \tag{37}$$

the matrix M_{ij} is

$$
\begin{pmatrix}
\frac{\sin\psi}{\sqrt{1-4a}}\sin\frac{z}{2} + \cos\psi\cos\frac{z}{2} & -(\cos\phi-\cos\psi)\sin\frac{z}{2} + \left[\frac{\sin\phi}{\sqrt{1+4a}} - (1-4a)\frac{\sin\psi}{\sqrt{1+4a}}\right]\cos\frac{z}{2} & -\cos\phi\sin\frac{z}{2} + \frac{\sin\phi}{\sqrt{1+4a}}\cos\frac{z}{2} & \left[\frac{\sin\psi}{\sqrt{1-4a}} - (1+4a)\frac{\sin\phi}{\sqrt{1+4a}}\right]\sin\frac{z}{2} + (\cos\psi-\cos\phi)\cos\frac{z}{2} \\
\frac{\sin\psi}{\sqrt{1-4a}}\cos\frac{z}{2} & \frac{\sin\phi}{\sqrt{1+4a}}\sin\frac{z}{2} + \cos\psi\cos\frac{z}{2} & \frac{\sin\phi}{\sqrt{1+4a}}\sin\frac{z}{2} & -\cos\phi\sin\frac{z}{2} + \frac{\sin\psi}{\sqrt{1-4a}}\cos\frac{z}{2} \\
\cos\psi\sin\frac{z}{2} - \frac{\sin\psi}{\sqrt{1-4a}}\cos\frac{z}{2} & \left[\frac{\sin\phi}{\sqrt{1+4a}} - (1-4a)\frac{\sin\psi}{\sqrt{1-4a}}\right]\sin\frac{z}{2} + (\cos\phi-\cos\psi)\cos\frac{z}{2} & \frac{\sin\phi}{\sqrt{1+4a}}\sin\frac{z}{2} + \cos\phi\cos\frac{z}{2} & (\cos\psi-\cos\phi)\sin\frac{z}{2} - \left[\frac{\sin\psi}{\sqrt{1-4a}} - (1+4a)\frac{\sin\phi}{\sqrt{1+4a}}\right]\cos\frac{z}{2} \\
\frac{\sin\psi}{\sqrt{1-4a}}\sin\frac{z}{2} & \cos\psi\sin\frac{z}{2} - \frac{\sin\phi}{\sqrt{1+4a}}\cos\frac{z}{2} & -\frac{\sin\phi}{\sqrt{1+4a}}\cos\frac{z}{2} & \cos\phi\cos\frac{z}{2} + \frac{\sin\psi}{\sqrt{1-4a}}\sin\frac{z}{2}
\end{pmatrix}
\tag{38}
$$

where $\phi \equiv \frac{z}{2}\sqrt{1+4a}$, and $\psi \equiv \frac{z}{2}\sqrt{1-4a}$. It can now be checked that the matrix elements given in (38) when expanded in power series of "a" agree with those given in (28).

A further simplification in the representation of the solution in matrix form is accomplished by choosing at each location z a set of quasistationary co-ordinate axes X and Y rotated about the z-axis from the x and y axes so as to be aligned to the orientation of the helix at that z location. These X and Y axes are to be quasistationary in the sense that once they are aligned at a given location z they are, then, assumed to be stationary as z takes on increments instead of to be following the rotation of the helix. This means that the relations between X, X', Y, Y', and x, x', y, y' are given by

$$\begin{cases} X = x\cos\frac{z}{2} + y\sin\frac{z}{2} \\ X' = x'\cos\frac{z}{2} + y'\sin\frac{z}{2} \\ Y = -x\sin\frac{z}{2} + y\cos\frac{z}{2} \\ Y' = -x'\sin\frac{z}{2} + y'\cos\frac{z}{2} \end{cases} \qquad \left(z \text{ in units of } \frac{\lambda}{4\pi}\right) \qquad (39)$$

or

$$\begin{pmatrix} X \\ \frac{X'}{2a} \\ Y \\ \frac{Y'}{2a} \end{pmatrix}_z = \begin{pmatrix} \cos\frac{z}{2} & 0 & \sin\frac{z}{2} & 0 \\ 0 & \cos\frac{z}{2} & 0 & \sin\frac{z}{2} \\ -\sin\frac{z}{2} & 0 & \cos\frac{z}{2} & 0 \\ 0 & -\sin\frac{z}{2} & 0 & \cos\frac{z}{2} \end{pmatrix} \begin{pmatrix} x \\ \frac{x'}{2a} \\ y \\ \frac{y'}{2a} \end{pmatrix}_z$$

$$= \begin{pmatrix} \cos\frac{z}{2} & 0 & \sin\frac{z}{2} & 0 \\ 0 & \cos\frac{z}{2} & 0 & \sin\frac{z}{2} \\ -\sin\frac{z}{2} & 0 & \cos\frac{z}{2} & 0 \\ 0 & -\sin\frac{z}{2} & 0 & \cos\frac{z}{2} \end{pmatrix} (M_{ij}) \begin{pmatrix} x \\ \frac{x'}{2a} \\ y \\ \frac{y'}{2a} \end{pmatrix}_o$$

$$\equiv (N_{ij}) \begin{pmatrix} x \\ \frac{x'}{2a} \\ y \\ \frac{y'}{2a} \end{pmatrix}_o \tag{40}$$

The elements of N_{ij} are given by

$$\begin{pmatrix} \cos\psi & \frac{\sin\phi}{\sqrt{1+4a}} - (1-4a)\frac{\sin\psi}{\sqrt{1-4a}} & \frac{\sin\phi}{\sqrt{1+4a}} & \cos\psi - \cos\phi \\ \frac{\sin\psi}{\sqrt{1-4a}} & \cos\psi & 0 & \frac{\sin\psi}{\sqrt{1-4a}} \\ -\frac{\sin\psi}{\sqrt{1-4a}} & \cos\phi - \cos\psi & \cos\phi & -\frac{\sin\psi}{\sqrt{1-4a}} + (1+4a)\frac{\sin\phi}{\sqrt{1+4a}} \\ 0 & -\frac{\sin\phi}{\sqrt{1+4a}} & -\frac{\sin\phi}{\sqrt{1+4a}} & \cos\phi \end{pmatrix} \tag{41}$$

V. Discussions

A. Radial Oscillations

Let σ_1 and σ_2 be the phase shifts per periodic length $\frac{\lambda}{2}$, of the two normal modes of radial oscillation. Then we have, in general,

$$\cos\sigma_1 + \cos\sigma_2 = \frac{1}{2}\,\mathrm{Trace}\,\left[M(z = 2\pi)\right] = -\frac{1}{2}\,\mathrm{Trace}\,\left[N(z = 2\pi)\right] \tag{42}$$

where the negative sign infront of Trace (N) corresponds to the fact that at $z = 2\pi$ the local quasistationary coordinates X and Y are just the inverse of the initial fixed coordinates x and y.

In our present case it is obvious that $\sigma_1 = \sigma_2 \equiv \sigma$. Therefore, we have

$$\cos\sigma = \frac{1}{4}\,\text{Trace}\,\left[M(z = 2\pi)\right] = -\frac{1}{4}\,\text{Trace}\,\left[N(z=2\pi)\right]$$

$$= -\frac{1}{2}\left(\cos\pi\sqrt{1+4a} + \cos\pi\sqrt{1-4a}\right) \qquad (43)$$

$$= 1 - 2\pi^2 a^2 - \left(10 - \frac{2}{3}\pi^2\right)\pi^2 a^4 + O\left(a^6\right)$$

where the next to the last expression is the exact result obtained from (41), and the last expression is the approximate result correct to a^5 terms obtained from (28) and can be further simplified to give

$$\sigma = 2\pi a + 5\pi a^3 + O\left(a^5\right) \qquad (44)$$

It is interesting to compare these results with those for a conventional linear-alternating-sectioned quadrupole magnet system with the same field gradient and the same repetition length $\frac{\lambda}{2}$ (length of each magnet = $\frac{\lambda}{4}$). They are

Exact

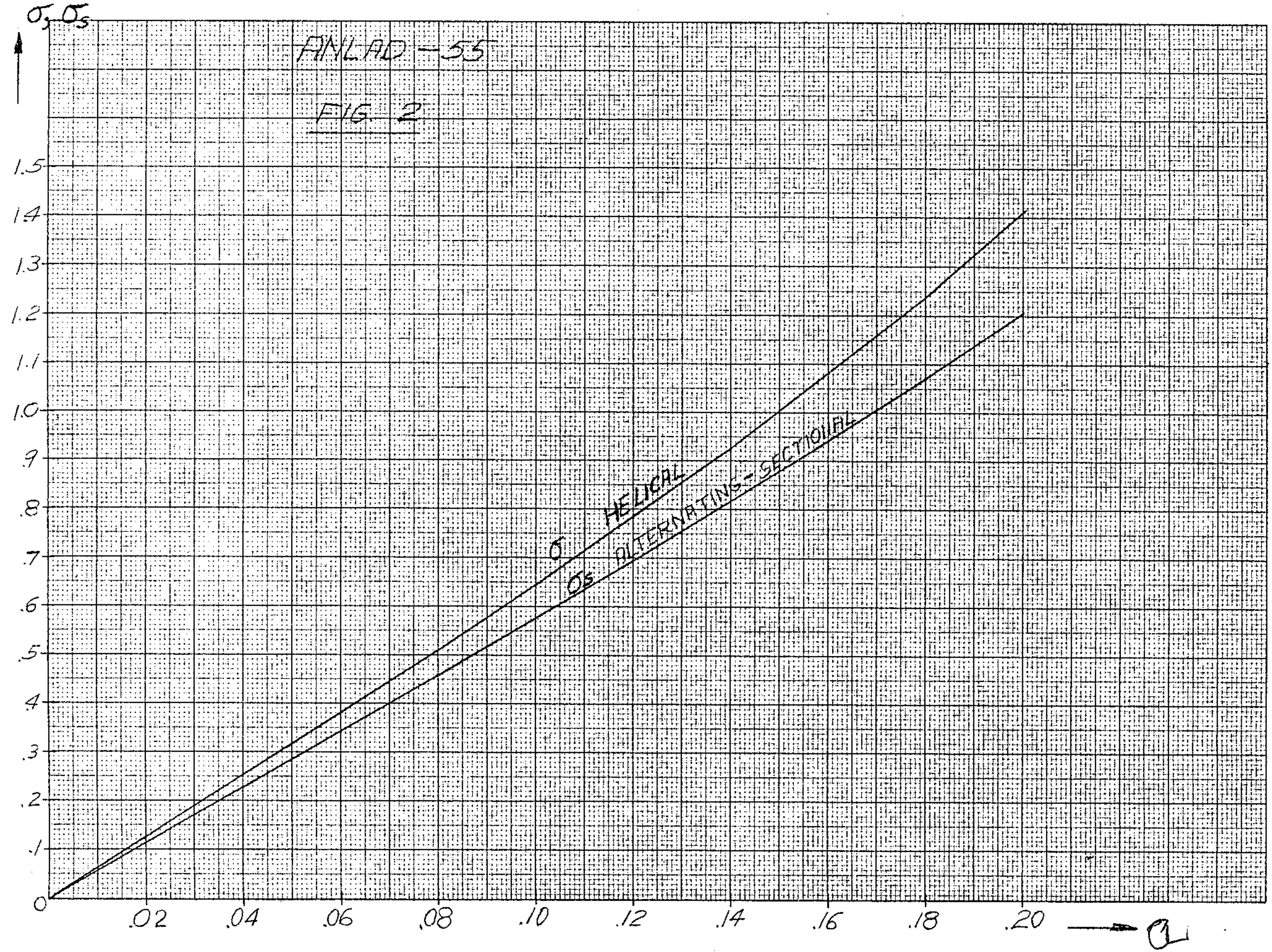
$$\cos\sigma_s = \cosh\pi\sqrt{a}\;\cos\pi\sqrt{a} \qquad (45)$$

Approximate

$$\sigma_s = \frac{\pi^2}{\sqrt{3}}\,a + \frac{4\,\pi^6}{315\sqrt{3}}\,a^3 + O\left(a^5\right)$$

$$= \frac{\pi}{2\sqrt{3}}\,(2\pi a) + \frac{4\,\pi^5}{1575\sqrt{3}}\,(5\pi a^3) + O\left(a^5\right) \qquad (46)$$

$$= 0.9069\,(2\pi a) + 0.4487\,(5\pi a^3) + O\left(a^5\right)$$


ANLAD-55
FIG. 2
σ, σs
1.5
1.4
1.3
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
.02
.04
.06
.08
.10
.12
.14
.16
.18
.20
a
σ HELICAL
σs ALTERNATING-SECTIONAL

The comparison shows that for a $\ll 1$ (in practice, for protons with energy above 1 Bev it is very difficult to get values of a > 0.01) the strength of focusing (measured by σ) is linearly proportional to "a" in both cases, being about 10% larger for the helical quadrupole system. The exact σ and σ_s for large values of "a" are plotted in Fig. 2, where it can be seen that the advantage of stronger focusing of the helical quadrupole system becomes more prominent for larger values of a.

B. Stigmatic Imaging

For a point source located on the axis at a distance p (in units of $\frac{\lambda}{4\pi}$) from the entrance (z=0) to the helical quadrupole magnet,

$$\begin{cases} x_o = p\, x'_o = 2\, a\, p\, \dfrac{x'_o}{2a} \equiv P\, \dfrac{x'_o}{2a} \\ y_o = p\, y'_o = 2\, a\, p\, \dfrac{y'_o}{2a} \equiv P\, \dfrac{y'_o}{2a} \end{cases} \tag{47}$$

At the exit of the magnet (z = z) we then have

$$\begin{cases} x = (M_{11} P + M_{12}) \dfrac{x'_o}{2a} + (M_{13} P + M_{14}) \dfrac{y'_o}{2a} \\ \dfrac{x'}{2a} = (M_{21} P + M_{22}) \dfrac{x'_o}{2a} + (M_{23} P + M_{24}) \dfrac{y'_o}{2a} \\ y = (M_{31} P + M_{32}) \dfrac{x'_o}{2a} + (M_{33} P + M_{34}) \dfrac{y'_o}{2a} \\ \dfrac{y'}{2a} = (M_{41} P + M_{42}) \dfrac{x'_o}{2a} + (M_{43} P + M_{44}) \dfrac{y'_o}{2a} \end{cases} \tag{48}$$

If a point image is to be formed at a distance (negative) q (in units of $\frac{\lambda}{4\pi}$) from the exit we must have both

$$Q \equiv 2aq = \frac{x}{\frac{x'}{2a}} = \frac{(M_{11}P + M_{12}) + (M_{13}P + M_{14})\frac{y'_o}{x'_o}}{(M_{21}P + M_{22}) + (M_{23}P + M_{24})\frac{y'_o}{x'_o}} \tag{49}$$

and

$$Q \equiv 2aq = \frac{y}{\frac{y'}{2a}} = \frac{(M_{31}P + M_{32}) + (M_{33}P + M_{34})\frac{y'_o}{x'_o}}{(M_{41}P + M_{42}) + (M_{43}P + M_{44})\frac{y'_o}{x'_o}} \tag{50}$$

be independent of $\frac{y'_o}{x'_o}$ and equal to each other. These conditions when written out are

$$\begin{vmatrix} M_{11}P + M_{12} & M_{13}P + M_{14} \\ M_{21}P + M_{22} & M_{23}P + M_{24} \end{vmatrix} = 0 \tag{51}$$

$$\begin{vmatrix} M_{31}P + M_{32} & M_{33}P + M_{34} \\ M_{41}P + M_{42} & M_{43}P + M_{44} \end{vmatrix} = 0 \tag{52}$$

and

$$\begin{vmatrix} M_{11}P + M_{12} & M_{31}P + M_{32} \\ M_{21}P + M_{22} & M_{41}P + M_{42} \end{vmatrix} = 0 \tag{53}$$

It is clear that we could just as well have based the same argument and reasoning on the X Y coordinates at the exit, instead of the original x y coordinates, and would have arrived at a set of conditions identical to (51), (52) and (53) except that the matrix elements M_{ij} are replaced by the corresponding simpler elements N_{ij}. Furthermore, these conditions are not all mutually independent. As a matter of fact, direct substitutions of M_{ij} or N_{ij} into (51) and (52) or their equivalents show that they are identical conditions. Thus we are left with only two conditions, say, (51) and (53) or their equivalents in the X Y coordinates.

$$\begin{vmatrix} N_{11} P + N_{12} & N_{13} P + N_{14} \\ N_{21} P + N_{22} & N_{23} P + N_{24} \end{vmatrix}$$

$$= \begin{vmatrix} N_{11} & N_{13} \\ N_{21} & N_{23} \end{vmatrix} P^2 + \left(\begin{vmatrix} N_{11} & N_{14} \\ N_{21} & N_{24} \end{vmatrix} + \begin{vmatrix} N_{12} & N_{13} \\ N_{22} & N_{23} \end{vmatrix} \right) P + \begin{vmatrix} N_{12} & N_{14} \\ N_{22} & N_{24} \end{vmatrix} \tag{54}$$

$$\equiv A P^2 + B P + C = 0$$

$$\begin{vmatrix} N_{11} P + N_{12} & N_{31} P + N_{32} \\ N_{21} P + N_{22} & N_{41} P + N_{42} \end{vmatrix}$$

$$= \begin{vmatrix} N_{11} & N_{31} \\ N_{21} & N_{41} \end{vmatrix} P^2 + \left(\begin{vmatrix} N_{11} & N_{32} \\ N_{21} & N_{42} \end{vmatrix} + \begin{vmatrix} N_{12} & N_{31} \\ N_{22} & N_{41} \end{vmatrix} \right) P + \begin{vmatrix} N_{12} & N_{32} \\ N_{22} & N_{42} \end{vmatrix}$$

$$\equiv \alpha P^2 + \beta P + \gamma = 0 \tag{55}$$

Straightforward computation gives

$$
\left\{
\begin{aligned}
A &= \frac{\sin\phi}{\sqrt{1+4a}}\,\frac{\sin\psi}{\sqrt{1-4a}} + \cos\phi\cos\psi - 1 \\
B &= \frac{\sin\psi}{\sqrt{1-4a}}\cos\phi - \frac{\sin\phi}{\sqrt{1+4a}}\cos\psi \\
C &= -\frac{\sin\phi}{\sqrt{1+4a}}\,\frac{\sin\psi}{\sqrt{1-4a}}
\end{aligned}
\right. \tag{56}
$$

$$
\left\{
\begin{aligned}
\alpha &= (1-4a)\frac{\sin\phi}{\sqrt{1+4a}}\,\frac{\sin\psi}{\sqrt{1-4a}} - \left(\frac{\sin\phi}{\sqrt{1+4a}}\right)^2 - \cos\phi\cos\psi + \cos^2\psi \\
\beta &= 2\frac{\sin\psi}{\sqrt{1-4a}}\cos\psi - \left(\frac{\sin\phi}{\sqrt{1+4a}}\cos\psi + \frac{\sin\psi}{\sqrt{1-4a}}\cos\phi\right) \\
\gamma &= \left(\frac{\sin\psi}{\sqrt{1-4a}}\right)^2
\end{aligned}
\right. \tag{57}
$$

Eliminating P from (54) and (55) we get

$$
(A\gamma - C\alpha)^2 - (B\gamma - C\beta)(A\beta - B\alpha) = 0 \tag{58}
$$

The magnet parameters a and z should be adjusted to satisfy (58). The corresponding values of P and Q are then given by

$$
P = -\frac{A\gamma - C\alpha}{A\beta - B\alpha} \tag{59}
$$

and

TABLE I (Continued)

a = .0075		a = .0050		a = .0025		a = .0010	
z	P = -Q	z	P = -Q	z	P = -Q	z	P = -Q
6.285307	42.4332	6.284128	63.6616	6.283421	127.32	6.283223	318.310
12.570611	21.2400	12.568255	31.8440	12.566842	63.668	12.566446	159.158
18.855908	14.1857	18.852381	21.2466	18.850263	42.454	18.849669	106.109
25.141197	10.6659	25.136506	15.9530	25.133684	31.850	25.132892	79.5854
31.426474	8.5594	31.420628	12.7808	31.417104	25.4893	31.416115	63.6720
37.711738	7.15933	37.704747	10.6691	37.700525	21.2506	37.699338	53.0639
43.996986	6.16254	43.988863	9.16344	43.983945	18.2244	43.982561	45.4873
50.282218	5.41748	50.272974	8.03633	50.267365	15.9560	50.265784	39.8054
56.567432	4.83993	56.557082	7.16151	56.550784	14.1927	56.549007	35.3864
62.852629	4.37935	62.841185	6.46317	62.834204	12.7832	62.832231	31.8518
69.137808	4.00356	69.125283	5.89310	69.117623	11.6308	69.115454	28.9605
75.422970	3.69113	75.409377	5.41911	75.401041	10.6712	75.398677	26.5510
81.708114	3.42720	81.693464	5.01893	81.684459	9.85987	81.681899	24.5124
87.993243	3.20119	87.977548	4.67666	87.967877	9.16517	87.965122	22.7656
94.278358	3.00532	94.261625	4.38064	94.251294	8.56363	94.248345	21.2520
100.563458	2.83377	100.545697	4.12210	100.534711	8.03783	100.531568	19.9276
106.848549	2.68209	106.829764	3.89435	106.818127	7.57435	106.814791	18.7594
113.133630	2.54685	113.113827	3.69220	113.101542	7.16283	113.098014	17.7212
119.418704	2.42531	119.397884	3.51153	119.384957	6.79502	119.381237	16.7926
125.703772	2.31530	125.681936	3.34905	125.668372	6.46436	125.664460	15.9570

TABLE I

a = 0.100		a = 0.075		a = 0.050		a = 0.025		a = .0100	
z	P = -Q	z	P = -Q	z	P = -Q	z	P = -Q	z	P = -Q
6.694600	3.03366	6.505087	4.14561	6.379246	6.30680	6.306856	12.7046	6.286958	31.8201
13.363289	1.60359	12.993933	2.19787	12.753889	3.27237	12.613353	6.42544	12.573906	15.9411
16.077000	27.3319	19.484770	1.49233	19.124186	2.26429	18.919257	4.35577	18.860835	10.6613
20.150791	0.959306	22.286382	16.3760	25.494156	1.73442	25.224506	3.33055	25.147738	8.03042
21.241043	1.13447	26.027331	1.03764	31.870373	1.37670	31.529191	2.71664	31.434606	6.45846
26.551304	0.354542	27.553590	1.46147	34.769627	11.1726	37.833513	2.30350	37.721437	5.41520
32.153693	13.7312	32.830458	0.609239	38.264421	1.08816	44.137742	2.00136	44.008225	4.67333
38.839611	2.55487	33.064309	0.613502	40.150161	1.86622	50.442208	1.76567	50.294972	4.11919
45.514306	1.44739	38.575027	0.182324	44.703919	0.819774	56.747313	1.57181	56.581677	3.68963
48.232277	9.02031	44.571713	8.29756	45.885898	0.989103	63.053571	1.40495	62.868343	3.34674
52.383060	0.836662	51.068602	2.87370	51.621636	0.545152	66.064613	24.0460	69.154972	3.06642
53.102608	0.897009	57.554528	1.79777	57.357373	0.237102	69.361681	1.25551	75.441569	2.83264
58.412869	0.253284	64.061050	1.25777	63.158220	61.4439	71.889385	4.07568	81.728140	2.63429
64.306236	6.93407	66.128616	2.65409	69.537146	5.74613	75.672657	1.11683	88.014689	2.46347
70.983704	2.21306	70.675713	0.835261	75.911255	3.12774	77.880168	2.38452	94.301226	2.31440
77.671440	1.30763	71.639335	0.947180	82.281294	2.19730	81.988092	0.983919	100.587754	2.18274
79.653912	2.66313	77.150053	0.382827	88.651547	1.69294	83.870950	1.62432	106.874284	2.06520
84.680014	0.707422	78.326204	0.330027	95.028907	1.34546	88.310676	0.852832	113.160824	1.95923
84.964174	0.714809	82.660771	0.0091879	97.928097	9.40616	89.861732	1.17660	119.447381	1.86282
90.274435	0.159285	89.139964	4.25078	101.425828	1.06074	94.645392	0.720221	125.733967	1.77434
96.456825	4.69152	95.629128	2.22435	103.243272	1.73474	95.852515	0.870192		
103.127820	1.95281	102.119465	1.50617	107.873079	0.792440	101.002663	0.583188		
109.837907	1.17844	104.920852	18.3706	108.979009	0.932999	101.843297	0.638380		
111.515477	1.85132	108.659457	1.04857	114.714746	0.510045	107.408714	0.439157		
116.825739	0.567215	110.214361	1.49942	120.450483	0.209432	107.834080	0.449355		
117.130509	0.560886	115.446924	0.621560	126.316433	30.7380	113.824862	0.285579		
122.136001	0.0695822	115.725080	0.627745			113.956865	0.284962		
128.605165	3.58114	121.235799	0.191971			119.815643	0.136053		
		127.206337	8.76226			125.822272	252.444		

$$Q = \frac{M_{11} P + M_{12}}{M_{21} P + M_{22}} \tag{60}$$

However, (60) may not be necessary at all. From symmetry arguments one may immediately come to the conclusion that $Q = -P$. Equation (58) is solved numerically on the IBM-704 (Fortran program written by E. A. Crosbie). The results are given in Table 1. For a given value of "a" there exist infinitely many values for the length z of the helical quadrupole magnet which will satisfy condition (58). Those smaller than $z \cong 40\pi$ are given in the table together with the corresponding object and image distances P and Q. One observes from this table that for small values of a (< 0.05) the roots of (58) are approximately given by $2n\pi$ ($n = 1, 2, 3, \ldots$). As a matter of fact, when the expanded forms (28) for the elements of M_{ij} are substituted in (58) (with M_{ij} replacing N_{ij}) we see that (58) is satisfied by $z = 2n\pi$ to terms of order a^8 while the lowest non-identically vanishing terms on the left hand side are of order a^6. The lowest order non-zero terms in A, B, and C for $z = 2n\pi$ are

$$\begin{cases} A = 4n^2\pi^2a^2 \\ B = -4n\pi a \\ C = -8n^2\pi^2a^2 \end{cases} \tag{61}$$

which when substituted in (54) and (60) give for P and Q

$$P = -Q \cong \frac{1}{n\pi a} \quad \text{for } z = 2n\pi \quad (2n\pi a \ll 1) \tag{62}$$

However, this approximation is valid only when $2n\pi a \ll 1$. This checks also with the values given in Table 1. For numerical computation "a" is related to the magnet particle parameters by

$$\frac{\lambda^2 G}{\eta} = 4,942 \times 10^5 \text{ a (kilogauss cm)} \tag{63}$$

$\eta = \beta\gamma$

The general relations between the object and image distances in real units p and q, and the parameters P and Q are

$$\frac{p}{\lambda} = \frac{P}{8\pi a}, \quad \frac{q}{\lambda} = \frac{Q}{8\pi a} \tag{64}$$

The approximate formula (62) gives, in real units

$$\text{and} \quad \begin{cases} \dfrac{z}{\lambda} = \dfrac{n}{2} & (a \ll 1) \\ \\ \dfrac{p}{\lambda} = -\dfrac{q}{\lambda} = \dfrac{1}{8\, n\, \pi^2 a^2} & (2\, n\pi a \ll 1) \end{cases} \tag{65}$$

As an example, for 12.5 Bev protons ($\eta = 14.288$) and $\lambda = 30$ cm, a high but still reasonable value of G = 39.23 kilogauss/cm will give a = 0.005. With these parameters a helical magnet of length $\frac{z}{\lambda} = 10.0015$ (Table 1) or $z = 10.0015\,\lambda \cong 3.0$ m will give $\frac{p}{\lambda} = -\frac{q}{\lambda} = \frac{3.3490}{8\,\pi \times 0.005} = 26.65$ (Table 1) or $p = -q = 26.65\,\lambda \cong 8.0$ m. The approximate formulas (65) give $\frac{z}{\lambda} = 10.0$ (n = 20) (a good approximation because the condition $a = 0.005 \ll 1$ is satisfied) and

$$\frac{p}{\lambda} = -\frac{q}{\lambda} = \frac{1}{160\,\pi^2 (0.005)^2} = 25.33 \text{ or } p = -q = 25.33\,\lambda = 7.6 \text{ m. (not a very}$$

good approximation because the condition $2n\pi a = 0.6283 \ll 1$ is not satisfied). This example shows that even with quite high magnetic field gradient available large spaces are required to image a beam of 12.5 Bev protons.

XCT