# Tutti-Frutti method: Recent developments in the PN/PM/SF treatment of the gravitational two-body problem

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We review some recent results obtained at the sixth Post-Newtonian level of approximation for the Hamiltonian description of a two-body system, by using several methods whose combination has led to the so-called "Tutti-Frutti" approach.

*Keywords*: Two-body system; Post-Newtonian and Post-Minkowskian approximation; Scattering angle

## 1. Introduction

The two-body gravitational interaction is a fundamental physics problem whose descriptions necessitates a fully general relativistic treatment. The latter, in turn, is much difficult and over the years several approximation methods have been implemented for the mathematical treatment of this problem, ranging from weak-field and slow-motion (Post-Newtonian (PN) approximation[1]), weak-field but eventually relativistic motions (Post-Minkowskian (PM) approximation[2,3]), extreme-mass-ratio limit (i.e., the condition in which the mass of one of the two bodies is much larger of the other, discussed in the framework of perturbation theory and gravitational self-force (GSF)[4]), Effective Field Theory (EFT),[5] numerical relativity (NR).[6]

All these analytical (and semi-analytical) treatments have been used to build up a Hamiltonian description of the system. Since as soon as one raises the level of approximation considered the number of terms entering this Hamiltonian raises as well, in 1999 A. Buonanno and T. Damour[7,8] introduced the so-called Effective One-Body (EOB) approach aiming at a partial resummation of the Hamiltonian itself. Indeed, the EOB is especially useful since it condensates in a few gravitational potentials the essential characteristics of the interaction, and can be also continuously (and easily) updated as soon as new results become available in the literature by using whatever approximation scheme.

Let us recall also that 1) the gravitational interaction of two bodies is actually compatible with two basic scenarios: capture (the more massive of the two attracts and then swallows the other; the system in this case spirals undergoing ellipticlike motions) and scattering (the two bodies can be close enough but they have enough energy to resist the attraction; the system in this case undergoes hyperboliclike or paraboliclike motions). 2) Both cases are of particular importance in view of the possibility to detect gravitational wave signals from Earth-based interferometers operating now (e.g., Ligo[9] and Virgo[10]) and also (more likely) by future, forthcoming satellite missions involving space-based interferometers (Lisa[11]). 3) The mathematical description of the dynamics by any available approximation method has strong limitations when the gravitational field becomes too strong, as for example in the case of the capture. In this condition one is only left with NR.

To add fuel to fire one should also take into account that new difficulties arise when taking into account the emission of gravitational radiation (energy, angular momentum and linear momentum) by the system as soon as a purely conservative scenario is no more possible. Indeed, starting from the 2.5PN level of approximation the problem is no more conservative, so that one has to deal with radiation-reaction effects. Furthermore, starting from the 4PN order the Hamiltonian of the system also includes a nonlocal part, which accounts for the past history of the system.

This picture clearly explains the difficulties which one encounters when trying to reach the 5PN (and beyond) level of accuracy in the model. Luckily, the various concomitant effects can be still computed separately so one can decide to limit to the study of the conservative and local part of the Hamiltonian, and later including nonlocal effects and radiation-reaction induced effects. This is the spirit of the recently developed "Tutti-Frutti" (TF) method, which combines several theoretical formalisms: PN, PM, multipolar-post-Minkowskian (MPM), EFT, GSF, EOB, and Delaunay averaging.[12–16]

## 2. EOB Hamiltonian

The EOB approach rewrites the (real, conservative) two-body Hamiltonian $H(r, p_r, p_\phi)$ in terms of an effective Hamiltonian $H_{\rm eff}(r, p_r, p_\phi)$ ($\phi$ is an ignorable coordinate in the conservative case)

$$H = M\sqrt{1 + 2\nu(\hat{H}_{\rm eff} - 1)}\,, \tag{1}$$

with

$$M = m_1 + m_2\,, \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}\,, \qquad \nu = \frac{m_1 m_2}{(m_1 + m_2)^2} \tag{2}$$

the total mass of the system ($M$), the reduced mass ($\mu$) and the symmetric mass ratio ($\nu$). The effective Hamiltonian (per unit of reduced mass, $\mu$) $\hat{H}_{\rm eff}$ written in the standard $p_r$-gauge or DJS gauge[17] involves several potentials: $A$, $\bar{D}$, $\hat{Q}$, etc.,

$$\hat{H}^2_{\rm eff} = A(u,\nu)[1 + p_\phi^2 u^2 + A(u,\nu)\bar{D}(u,\nu)p_r^2 + \hat{Q}(u,\nu;p_r)]\,, \tag{3}$$

where $u = GM/r$ and

$$\hat{Q}(u,\nu;p_r) = p_r^4 q_4(u,\nu) + p_r^6 q_6(u,\nu) + p_r^8 q_8(u,\nu) + \dots\,. \tag{4}$$

The various potentials 1) have a polynomial structure in terms of the symmetric mass-ratio $\nu$, e.g.,

$$\begin{aligned} A(u,\nu) &= 1 - 2u + \nu a^{\nu^1}(u) + \nu^2 a^{\nu^2}(u) + \nu^3 a^{\nu^3}(u) + \dots\,, \\ \bar{D}(u,\nu) &= 1 + \nu \bar{d}^{\nu^1}(u) + \nu^2 \bar{d}^{\nu^2}(u) + \nu^3 \bar{d}^{\nu^3}(u) + \dots\,, \\ q_4(u,\nu) &= 1 + \nu q_4^{\nu^1}(u) + \nu^2 q_4^{\nu^2}(u) + \nu^3 q_4^{\nu^3}(u) + \dots\,, \end{aligned} \tag{5}$$

etc., with the degree of the polynom increasing with PN approximation order; 2) have both a local part and a nonlocal part

$$\begin{aligned} A(u,\nu) &= A_{\rm loc}(u,\nu) + A_{\rm nl}(u,\nu)\,, \\ \bar{D}(u,\nu) &= \bar{D}_{\rm loc}(u,\nu) + \bar{D}_{\rm nl}(u,\nu)\,, \\ \hat{Q}(u,\nu;p_r) &= \hat{Q}_{\rm loc}(u,\nu;p_r) + \hat{Q}_{\rm nl}(u,\nu;p_r)\,, \end{aligned} \tag{6}$$

the nonlocal part starting at 4PN.

## 3. The Tutti-Frutti approach and the determination of the conservative dynamics at 6PN

The starting point of the TF approach is to consider the total two-body conservative action, which at the 6PN accuracy has both a local-in-time part and a nonlocal-in-time part (starting at 4PN)

$$S_{\rm tot}^{\leq 6{\rm PN}} = S_{\rm loc}^{\leq 6{\rm PN}} + S_{\rm nonloc}^{\leq 6{\rm PN}}\,. \tag{7}$$

The main steps are summarized below.

(1) Fix completely the nonlocal part of the Hamiltonian by using Delaunay averaging along ellipticlike orbits.
The nonlocal action is given by

$$S_{\rm nonloc}^{4+5+6{\rm PN}} = -\int dt\, H_{\rm nonloc}^{4+5+6{\rm PN}}(t)\,, \tag{8}$$

with

$$H_{\rm nonloc}^{4+5+6{\rm PN}}(t) = \frac{G\mathcal{M}}{c^3}{\rm Pf}_{2r_{12}/c}\int \frac{dt'}{|t-t'|}\mathcal{F}_{\rm 2PN}^{\rm split}(t,t')\,. \tag{9}$$

Here, $\mathcal{M}$ denotes the total ADM conserved mass-energy of the binary system, $r_{12}$ entering the timescale of the the *partie finie* (Pf) operation is the harmonic-coordinate radial distance, and $\mathcal{F}_{\rm 2PN}^{\rm split}(t,t')$ is the time-split version of the fractionally 2PN-accurate gravitational-wave energy flux, i.e.,

$$\mathcal{F}_{\rm 2PN}^{\rm GW}(t) \propto I_{ab}^{(3)}(t) I_{ab}^{(3)}(t) + O(\eta^2) \quad \to \quad \mathcal{F}_{\rm 2PN}^{\rm split}(t,t') \propto I_{ab}^{(3)}(t) I_{ab}^{(3)}(t') + O(\eta^2)\,, \tag{10}$$

with $\eta \equiv \frac{1}{c}$, the superscript in parenthesis denoting repeated time-derivatives of the quadrupole moment $I_{ab}$. Taking then the (Delaunay) time average of the harmonic-coordinates Hamiltonian (9) gives a gauge-invariant function of two orbital parameters

$$\langle \delta H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,h}} \rangle = \frac{1}{\oint dt_h} \oint \delta H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,h}}(t_h) dt_h \,. \tag{11}$$

Next, do the same computation in EOB coordinates. Comparison between the two results allows fixing the nonlocal parts of the EOB potentials.

(2) Use SF information about small-eccentricity ellipticlike motion to determine (part of) the local EOB Hamiltonian.
Compute the averaged redshift factor, $z_1$, i.e., a (first) gauge-invariant variable associated with the conservative dynamics of the two-body system, along ellipticlike orbits by using first-order SF techniques in a small-eccentricity expansion limit, including but high powers of the eccentricity (up to the eighth order). $z_1 \sim \langle \partial_{m_1} H \rangle$ incorporates both local and nonlocal contributions in the Hamiltonian, but it is limited from the fact that analytic computations are possible only at first-order in the symmetric mass-ratio $\nu$. Combining this result with the information about the nonlocal part of the EOB Hamiltonian specified at the point 1 will allow one to determine all the linear-in-$\nu$ local EOB potentials. The remaining (i.e., higher order in $\nu$) coefficients are still undetermined.

(3) Compute another gauge-invariant quantity, the scattering angle, $\chi$, along hyperboliclike orbits to determine most of the remaining coefficients.
The total scattering angle $\chi^{\mathrm{tot}} = \chi^{\mathrm{loc}} + \chi^{\mathrm{nonloc}}$ can be expressed as a large-angular momentum (or equivalently large-eccentricity) expansion, with coefficients having a precise mass-structure (or $\nu$-structure) as recently shown by Damour.[18] Therefore, one needs to separately compute both the local and the nonlocal contributions to the scattering angle. The latter is defined by

$$\chi^{\mathrm{nonloc}}(E, J, \nu) = \frac{\partial W^{\mathrm{nonloc}}(E, J, \nu)}{\partial J} \,, \tag{12}$$

where $E$ and $J$ are the total energy and angular momentum in the c.m. frame, respectively, and

$$W^{\mathrm{nonloc}}(E, J; \nu) \equiv \int_{-\infty}^{+\infty} dt\, H^{\mathrm{nonloc}}(t) \,, \tag{13}$$

is the integrated nonlocal action. In order to compute the local contribution $\chi^{\mathrm{loc}}$, instead, it is convenient to convert the local EOB Hamiltonian into the so-called energy-gauge.[19] Imposing then that $\chi^{\mathrm{tot}}$ satisfies the prescribed $\nu$-structure will fix most of the parametrizing coefficients of the EOB potentials.

All the above steps used jointly have lead to the determination of most of the coefficients entering the conservative Hamiltonian of the two-body system. More precisely at 5PN there remain only 2 quantities to be determined ($\bar{d}_5^{\nu^2}$ and $a_6^{\nu^2}$) and at 6PN there remain only 4 more quantities to be determined ($q_{45}^{\nu^2}$, $\bar{d}_6^{\nu^2}$, $a_7^{\nu^2}$, and $a_7^{\nu^3}$).

Within the EFT approach, the conservative scattering angle can be decomposed into a potential-graviton contribution, and a radiation-graviton one.[5] The $O(G^4)$ potential-graviton contribution to the radial action has been recently derived in Ref.,[20] whereas Ref.[21] has computed the potential-graviton contribution to the 5PN two-body Hamiltonian. Although the TF decomposition of the two-body dynamics into local-in-time and non-local-in-time parts is closely linked to the EFT decomposition, one cannot simply identify the TF time-symmetric local-in-time dynamics to the EFT time-symmetric potential-graviton dynamics. However, one can compare the total conservative scattering angle derived within the two approaches. This allows one to complete the result of Ref.[20] by providing the explicit expression (at the 6PN accuracy) of the complementary radiation-graviton contributions to the scattering angle, or equivalently, to the radial action. Furthermore, adding the radiation-graviton contributions to the 5PN Hamiltonian obtained in Ref.[22] yields explicit expressions for the two 5PN undetermined $O(G^5)$ and $O(G^6)$ TF parameters $\bar{d}_5^{\nu^2}$ and $a_6^{\nu^2}$, involving either $\pi^2$ terms and rational coefficients entering various (local-in-time) radiation-graviton contributions to the conservative effective 5PN action.[22] Our results are in agreement (for the $\pi^2$ contributions) with those of Ref.[21] A comparison with the results of Ref.[22] is currently under consideration.

## 4. Radiation-reaction effects

The presence of a radiation-reaction force implies a modification of Hamilton equations as

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}\,, \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} + \boldsymbol{\mathcal{F}}_{\rm rr}\,. \tag{14}$$

The work done by this force on the system implies energy, angular momentum and linear momentum losses by the system itself, as well as to the fact that the four-velocity of the center-of-mass is no more conserved, leading to recoil effects. For instance, the scattering angle will be modified as

$$\chi = \chi^{\rm cons} + \delta^{\rm rr}\chi\,, \tag{15}$$

where the radiation-reaction corrections to the conservative value can be written as

$$\delta^{\rm rr}\chi = -\frac{1}{2}\left[\frac{\partial\chi^{\rm cons}}{\partial E}E^{\rm rad} + \frac{\partial\chi^{\rm cons}}{\partial J}J^{\rm rad}\right]\,, \tag{16}$$

in terms of the energy and angular momentum losses $E^{\rm rad}$ and $J^{\rm rad}$. The latter include instantaneous and tail (i.e. hereditary) contributions, and admit a double PM and PN expansion. At the fractional 2PN order they read

$$\begin{aligned} E^{\rm rad}_{\rm 2PN} &= \nu^2\left(E_{\rm N} + \eta^2 E_{\rm 1PN} + \eta^3 E^{\rm tail}_{\rm 1.5PN} + \eta^4 E_{\rm 2PN}\right)\,,\\ J^{\rm rad}_{\rm 2PN} &= \nu^2\left(J_{\rm N} + \eta^2 J_{\rm 1PN} + \eta^3 J^{\rm tail}_{\rm 1.5PN} + \eta^4 J_{\rm 2PN}\right)\,, \end{aligned} \tag{17}$$

where tails start at the 1.5PN order. For example, the Newtonian values are given by

$$E_{\rm N} = \frac{37}{15}\pi\frac{p_\infty^4}{j^3} + \frac{1568}{45}\frac{p_\infty^3}{j^4} + \frac{122}{5}\pi\frac{p_\infty^2}{j^5} + O\left(\frac{1}{j^6}\right) ,$$
$$J_{\rm N} = \frac{16}{5}\frac{p_\infty^3}{j} + \frac{28}{5}\pi\frac{p_\infty^2}{j^2} + \frac{176}{5}\frac{p_\infty}{j^3} + O\left(\frac{1}{j^4}\right) , \qquad (18)$$

where $j \equiv cJ/(GM\mu)$ is a dimensionless angular momentum parameter, and the linear momentum at infinity, $p_\infty$, is related to the binding energy by $\bar{E} \equiv (E - Mc^2)/(\mu c^2) = \frac{1}{2}p_\infty^2$, at the Newtonian level. At higher PN orders the coefficients of the above expansion become functions of $\nu$: linear functions of $\nu$ at 1PN, quadratic at 2PN, etc., but the structure of the PM expansion is exactly the same.

We have computed in Ref.[23] the changes of 4-momentum during the scattering process (between the two asymptotic states of the two bodies labelled by $a = 1, 2$) which are linear order in radiation-reaction, i.e.,

$$\Delta p_{a\mu} \equiv p^+_{a\mu} - p^-_{a\mu} = \Delta p^{\rm cons}_{a\mu} + \Delta p^{\rm rr}_{a\mu} , \qquad (19)$$

where the radiation-reacted contribution $\Delta p^{\rm rr}_{a\mu}$ is the sum of a relative-motion term and a recoil one, which are linear in the radiative losses of energy, linear-momentum, and angular momentum. We have also shown how the polynomial dependence of $\Delta p^{\rm rr}_{a\mu}$ can be exploited to yield some identity relating the various radiative losses. Adding radiation-reaction effects at higher PN orders is an open issue for challenging future works.

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