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Massless particles in five and higher dimensions

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ABSTRACT

We describe a five-dimensional analogue of Wigner's operator equation $\mathbb{W}_a = \lambda P_a$, where $\mathbb{W}_a$ is the Pauli-Lubanski vector, P_a the energy-momentum operator, and λ the helicity of a massless particle. Higher dimensional generalisations are also given.

1. Introduction

The unitary representations of the Poincaré group in four dimensions were classified by Wigner in 1939 [1], see [2] for a recent review. Our modern understanding of elementary particles is based on this classification.

Unitary representations of the Poincaré group $\mathsf{ISO}_0(d-1,1)$ in higher dimensions, $d > 4$, have been studied in the literature, see [3–5] and references therein. However, significant interest in the topic remains due to some unexplored generalisations (including the supersymmetric case). It suffices to mention the recent work by Weinberg [6] in which free massless field equations were derived as an incidental consequence of the condition that the invariant Abelian subgroup of the little group is represented trivially on states of a single massless particle. Unlike the analyses in [3–6], in this paper we do not discuss the massless field equations in higher dimensions. We analyse covariant operator equations that correspond to the irreducible massless representations of $\mathsf{ISO}_0(d-1,1)$ with a finite (discrete) spin.

We recall that the Poincaré algebra $\mathfrak{iso}(d-1,1)$ in d dimensions is characterised by the commutation relations[1]

$$\big[P_a, P_b\big] = 0\,, \tag{1.1a}$$

$$\big[J_{ab}, P_c\big] = \mathrm{i}\eta_{ac} P_b - \mathrm{i}\eta_{bc} P_a\,, \tag{1.1b}$$

$$\big[J_{ab}, J_{cd}\big] = \mathrm{i}\eta_{ac} J_{bd} - \mathrm{i}\eta_{ad} J_{bc} + \mathrm{i}\eta_{bd} J_{ac} - \mathrm{i}\eta_{bc} J_{ad}\,. \tag{1.1c}$$

In any unitary representation of (the universal covering group of) the Poincaré group, the energy-momentum operator P_a and the Lorentz generators J_{ab} are Hermitian. For every dimension d, the operator $P^a P_a$ is a Casimir operator. Other Casimir operators are dimension dependent.

In four dimensions, the second Casimir operator is $\mathbb{W}^a \mathbb{W}_a$, where

$$\mathbb{W}^a = \frac{1}{2}\varepsilon^{abcd} J_{bc} P_d \tag{1.2}$$

is the Pauli-Lubanski vector. Using the commutation relations (1.1), it follows that the Pauli-Lubanski vector is translationally invariant,

$$\big[P_a, \mathbb{W}_b\big] = 0\,, \tag{1.3a}$$

and possesses the following properties:

$$\mathbb{W}^a P_a = 0\,, \tag{1.3b}$$

$$\big[J_{ab}, \mathbb{W}_c\big] = \mathrm{i}\eta_{ac}\mathbb{W}_b - \mathrm{i}\eta_{bc}\mathbb{W}_a\,, \tag{1.3c}$$

$$\big[\mathbb{W}_a, \mathbb{W}_b\big] = \mathrm{i}\varepsilon_{abcd}\mathbb{W}^c P^d\,. \tag{1.3d}$$

The irreducible massive representations are characterised by the conditions

$$P^a P_a = -m^2\,\mathbb{1}\,, \qquad m^2 > 0\,, \qquad \mathrm{sign}\, P^0 > 0\,, \tag{1.4a}$$

$$\mathbb{W}^a \mathbb{W}_a = m^2 s(s+1)\,\mathbb{1}\,, \tag{1.4b}$$

where the quantum number s is called spin. Its possible values in different representations are $s = 0, 1/2, 1, 3/2 \dots$. The massless representations are characterised by the condition $P^a P_a = 0$. For the physically interesting massless representations, it holds that

$$\mathbb{W}_a = \lambda P_a\,, \tag{1.5}$$

where the parameter λ determines the representation and is called the helicity. Its possible values are $0, \pm\frac{1}{2}, \pm 1$, and so on. The parameter $|\lambda|$ is called the spin of a massless particle.

In this paper we present a generalisation of Wigner's equation (1.5) to five and higher dimensions.

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[1] We make use of the mostly plus Minkowski metric η_{ab} and normalise the Levi-Civita tensor $\varepsilon_{a_1 \dots a_d}$ by $\varepsilon_{01\dots d-1} = 1$.

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2. Unitary representations of $\mathsf{ISO}_0(4,1)$

The five-dimensional analogue of (1.2) is the Pauli-Lubanski tensor

$$\mathbb{W}^{ab} = \frac{1}{2}\varepsilon^{abcde} J_{cd} P_e \, . \tag{2.1}$$

It is translationally invariant,

$$\big[\mathbb{W}_{ab}, P_c\big] = 0 \, , \tag{2.2}$$

and possesses the following properties:

$$\mathbb{W}_{ab} P^b = 0 \, , \tag{2.3a}$$

$$\big[\mathbb{W}_{ab}, J_{cd}\big] = \mathrm{i}\eta_{ac}\mathbb{W}_{bd} - \mathrm{i}\eta_{ad}\mathbb{W}_{bc} - \mathrm{i}\eta_{bc}\mathbb{W}_{ad} + \mathrm{i}\eta_{bd}\mathbb{W}_{ac} \, , \tag{2.3b}$$

$$\big[\mathbb{W}_{ab}, \mathbb{W}_{cd}\big] = \mathrm{i}\varepsilon_{acdfg}\mathbb{W}_b{}^f P^g - \mathrm{i}\varepsilon_{bcdfg}\mathbb{W}_a{}^f P^g \, . \tag{2.3c}$$

Making use of $\mathbb{W}_{ab}$ allows one to construct two Casimir operators, which are

$$\mathbb{W}_{ab}\mathbb{W}^{ab} \, , \qquad \mathbb{H} := \mathbb{W}^{ab} J_{ab} \, . \tag{2.4}$$

2.1. Irreducible massive representations

The irreducible massive representations of the Poincaré group $\mathsf{ISO}_0(4,1)$ are characterised by two conditions

$$\frac{1}{8}\Big(\mathbb{W}^{ab}\mathbb{W}_{ab} + m\mathbb{H}\Big) = m^2 s_1(s_1+1)\mathbb{1} \, , \tag{2.5a}$$

$$\frac{1}{8}\Big(\mathbb{W}^{ab}\mathbb{W}_{ab} - m\mathbb{H}\Big) = m^2 s_2(s_2+1)\mathbb{1} \, , \tag{2.5b}$$

in addition to (1.4a). Here s_1 and s_2 are two spin values corresponding to the two $\mathsf{SU}(2)$ subgroups of the universal covering group $\mathsf{Spin}(4) \cong \mathsf{SU}(2) \times \mathsf{SU}(2)$ of the little group.[2]

2.2. Irreducible massless representations

It turns out that all irreducible massless representations of $\mathsf{ISO}_0(4,1)$ with a finite spin are characterised by the condition

$$\varepsilon_{abcde} P^c \mathbb{W}^{de} = 0 \quad \Longleftrightarrow \quad P^{[a}\mathbb{W}^{bc]} = 0 \, . \tag{2.6}$$

Both Casimir operators (2.4) are equal to zero in these representations, $\mathbb{W}_{ab}\mathbb{W}^{ab} = 0$ and $\mathbb{W}^{ab}J_{ab} = 0$.

Let $|p,\sigma\rangle$ be an orthonormal basis in the Hilbert space of one-particle states, where p^a denotes the momentum of a particle, $P^a|p,\sigma\rangle = p^a|p,\sigma\rangle$, and σ stands for the spin degrees of freedom. For a massless particle, we choose as our standard 5-momentum $k^a = (E,0,0,0,E)$. On this eigenstate:

$$\begin{aligned} \mathbb{W}^{ab}|k,\sigma\rangle &= \frac{1}{2}\varepsilon^{abcde} J_{cd} P_e |k,\sigma\rangle \\ &= \frac{E}{2}\Big(\varepsilon^{abcd4} J_{cd} - \varepsilon^{abcd0} J_{cd}\Big)|k,\sigma\rangle \, . \end{aligned} \tag{2.7}$$

Running through the elements of $\mathbb{W}^{ab}$, one finds:

$$\begin{aligned} &\mathbb{W}^{01} = \mathbb{W}^{41} = -E J_{23} \, , \qquad \mathbb{W}^{12} = E(J_{30} + J_{34}) \, , \\ &\mathbb{W}^{02} = \mathbb{W}^{42} = -E J_{31} \, , \qquad \mathbb{W}^{23} = E(J_{10} + J_{14}) \, , \\ &\mathbb{W}^{03} = \mathbb{W}^{43} = -E J_{12} \, , \qquad \mathbb{W}^{31} = E(J_{20} + J_{24}) \, , \\ &\qquad\qquad \mathbb{W}^{04} = 0 \, . \end{aligned} \tag{2.8}$$

If we rescale these generators and define:

$$\begin{gathered} \mathbb{R}_1 \equiv \frac{1}{E}\mathbb{W}^{23} \, , \qquad \mathbb{R}_2 \equiv \frac{1}{E}\mathbb{W}^{31} \, , \qquad \mathbb{R}_3 \equiv \frac{1}{E}\mathbb{W}^{12} \, , \\ \mathcal{J}_i \equiv -\frac{1}{E}\mathbb{W}^{0i} \, , \end{gathered} \tag{2.9}$$

then these new operators satisfy:

$$\big[\mathcal{J}_i, \mathcal{J}_j\big] = \mathrm{i}\varepsilon_{ijk}\mathcal{J}_k \, , \qquad \big[\mathcal{J}_i, \mathbb{R}_j\big] = \mathrm{i}\varepsilon_{ijk}\mathbb{R}_k \, , \qquad \big[\mathbb{R}_i, \mathbb{R}_j\big] = 0 \, . \tag{2.10}$$

These are the commutation relations for the three-dimensional Euclidean algebra, $\mathfrak{iso}(3)$. The irreducible unitary representations of $\mathfrak{iso}(3)$ are labelled by a continuous parameter μ^2, corresponding to the value the Casimir operator $\mathbb{R}^i\mathbb{R}_i$ takes. Since $\mathbb{R}_i$ commute among themselves the operators can be simultaneously diagonalised, and the eigenvectors $|r_i\rangle$ taken as a basis. However the only restriction on these is that $r_i r^i = \mu^2$, which for non-zero μ^2 permits a continuous basis and is thus an infinite dimensional representation. Because we want only finite-dimensional representations, we must take:

$$\mu^2 = 0 \quad \Longrightarrow \quad \mathbb{R}_i = 0 \quad \Longleftrightarrow \quad J_{0i} = -J_{4i} \, . \tag{2.11}$$

We are therefore restricted to those representations in which the translation component is trivial, and so only the generators $\mathcal{J}_i$ remain, which generate the algebra $\mathfrak{so}(3)$. The algebra of the little group on massless representations is thus $\mathfrak{so}(3)$ which is isomorphic to $\mathfrak{su}(2)$. As stated previously, the irreducible representations of $\mathfrak{su}(2)$ are labelled by a non-negative (half) integer s and have a single Casimir operator $\mathcal{J}^i\mathcal{J}_i$ which takes the value $s(s+1)\mathbb{1}$. This analysis leads to (2.6).

The spin value of a massless representation can still be found using a ‘spin’ operator. The following relation holds on massless representations:

$$\mathbb{S}_a := -\frac{1}{4}\varepsilon_{abcde} J^{bc}\mathbb{W}^{de} = \mathcal{J}^2 P_a = s(s+1)P_a \, , \tag{2.12}$$

where $\mathcal{J}^2 = \mathcal{J}^i\mathcal{J}_i$ is the Casimir operator for the $\mathfrak{so}(3)$ generators in (2.9). The parameter s is the spin of a massless particle. Its possible values in different representations are $s = 0, 1/2, 1$, and so on. Equation (2.12) naturally holds for massless spinor and vector fields [7].

In general, the operator $\mathbb{S}_a$ is not translationally invariant,

$$\big[\mathbb{S}_b, P_a\big] = \frac{\mathrm{i}}{2}\varepsilon_{abcde} P^c \mathbb{W}^{de} \, . \tag{2.13}$$

It is only for the massless representations with finite spin that the quantity on the right vanishes so that the spin operator commutes with the momentum operators. Equation (2.12) is the five-dimensional analogue of the operator equation (1.5). Its consistency condition is (2.6).[3]

3. Generalisations

The results of section 2.2 can be generalised to $d > 5$ dimensions. The Pauli-Lubanski tensor (2.1) turns into

$$\mathbb{W}^{a_1 \dots a_{d-3}} = \frac{1}{2}\varepsilon^{a_1 \dots a_{d-3} bce} J_{bc} P_e \, . \tag{3.1}$$

The condition (2.6) is replaced with

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[2] The equations (2.5) were independently derived during the academic year 1992-93 by Arkady Segal and David Zinger, who were undergraduates at Tomsk State University at the time.

[3] The consistency condition for (1.5) is $P^{[a}\mathbb{W}^{b]} = 0$, which is the four-dimensional counterpart of (2.6).

$$P^{[a}\mathbb{W}^{b_1\dots b_{d-3}]} = 0\,. \tag{3.2}$$

This equation is very similar to another that has appeared in the literature using the considerations of conformal invariance [8–11]. One readily checks that (3.2) is equivalent to

$$J_{ab}P^2 + 2J_{c[a}P_{b]}P^c = 0 \quad\Longrightarrow\quad J_{c[a}P_{b]}P^c = 0\,. \tag{3.3}$$

The latter is solved on the momentum eigenstates by $J_{ab}p^b \propto p_a$, which is of the form considered in [8–11].[4]

Equation (3.2) characterises all irreducible massless representations of $\mathsf{ISO}_0(d-1,1)$ with a finite (discrete) spin. Finally, the spin equation (2.12) turns into

$$\mathbb{S}_a := \frac{(-1)^d}{2(d-3)!}\varepsilon_{abce_1\dots e_{d-3}}J^{bc}\mathbb{W}^{e_1\dots e_{d-3}} = \mathcal{J}^2 P_a\,, \tag{3.4}$$

where $\mathcal{J}^2 = \frac{1}{2}\mathcal{J}^{ij}\mathcal{J}_{ij}$ is the quadratic Casimir operator of the algebra $\mathfrak{so}(d-2)$, with $i, j = 1, \dots, d-2$. For every irreducible massless representation of $\mathsf{ISO}_0(d-1,1)$ with a finite spin, it holds that $\mathcal{J}^2 \propto \mathbb{1}$.

We can extend this further to higher-order Casimir operators of $\mathfrak{so}(d-2)$. As a generalisation of (3.1), we introduce the nth Pauli-Lubanski tensor

$$\mathbb{W}^{(n)}{}_{a_1\dots a_{d-2n-1}} = \frac{1}{2^n}\varepsilon_{a_1\dots a_d}J^{a_{d-2n}a_{d-2n+1}}\dots J^{a_{d-2}a_{d-1}}P^{a_d}\,,\qquad 1 \le n \le \lfloor\frac{d-2}{2}\rfloor \tag{3.5}$$

which is order n in the Lorentz generators (the operator (3.1) coincides with $\mathbb{W}^{(1)}$).[5] Then higher-order spin operators can be defined as

$$\mathbb{S}^{(n)}{}_{a_1} = \frac{(-1)^d}{2(d-2n-1)!}\varepsilon_{a_1\dots a_d}J^{a_2a_3}\dots J^{a_{2n}a_{2n+1}}\mathbb{W}^{(n)a_{2n+2}\dots a_d}\,, \tag{3.6}$$

which are order $2n$ in the Lorentz generators. Using the fact that $J^{a0} = J^{ad-1}$ in the frame with a standard d-momentum $k^a = (E, 0, \dots, 0, E)$, one can show that

$$\mathbb{S}^{(n)}{}_a = \mathcal{C}^{(n)}P_a \tag{3.7}$$

where $\mathcal{C}^{(n)}$ is an order $2n$ Casimir operator for $\mathfrak{so}(d-2)$ defined by

$$\begin{aligned}\mathcal{C}^{(n)} &= \frac{-1}{2^{n+1}(d-2n-2)!}\varepsilon_{0\,i_1\dots i_{d-2}\,d-1}\varepsilon^{0\,j_1\dots j_{2n}i_{2n+1}\dots i_{d-2}\,d-1}\times\\ &\qquad\qquad J^{i_1i_2}\dots J^{i_{2n-1}i_{2n}}J_{j_1j_2}\dots J_{j_{2n-1}j_{2n}}\\ &= \frac{(2n)!}{2^{n+1}}J^{i_1i_2}\dots J^{i_{2n-1}i_{2n}}J_{[i_1i_2}\dots J_{i_{2n-1}i_{2n}]}\,.\end{aligned} \tag{3.8}$$

If d is odd, the operators (3.8) can be constructed up to $n = \frac{d-3}{2}$ (the order $n = \frac{d-1}{2}$ Pauli-Lubanski tensor is a scalar). If d is even, it suffices to restrict n to run from 1 to $n = \frac{d-4}{2}$, since the Pauli-Lubanski tensor of order $\frac{d-2}{2}$,

$$\mathbb{W}^{(\frac{d}{2}-1)}{}_{a_1} = \frac{1}{2^{\frac{1}{2}d-1}}\varepsilon_{a_1\dots a_d}J^{a_2a_3}\dots J^{a_{d-2}a_{d-1}}P^{a_d}\,, \tag{3.9}$$

is itself a 'spin operator' with the property

$$\mathbb{W}^{(\frac{d}{2}-1)}{}_a = \Lambda^{(\frac{d}{2}-1)}P_a\,, \tag{3.10}$$

where

$$\Lambda^{(\frac{d}{2}-1)} = -\frac{1}{2^{\frac{1}{2}d-1}}\varepsilon_{0i_1\dots i_{d-2}d-1}J^{i_1i_2}\dots J^{i_{d-3}i_{d-2}} \tag{3.11}$$

Note that the $d = 4$ case corresponds to (1.5). In the $d = 6$ case, the equation (3.10) was pointed out in [12].

For every irreducible massless representation of $\mathsf{ISO}_0(d-1,1)$ with a finite spin, the operator $\mathcal{C}^{(n)}$ in (3.7) is a multiple of the identity operator, $\mathcal{C}^{(n)} \propto \mathbb{1}$. Then the translational invariance of the equations (3.7) implies (3.2) and the relation

$$\mathbb{W}^{(n-1)}{}_{a_1a_2b_1\dots b_{d-2n-1}}\mathbb{W}^{(n)b_1\dots b_{d-2n-1}} = 0\,. \tag{3.12}$$

It is possible to derive a five-dimensional analogue of the operator equation defining the $\mathcal{N} = 1$ superhelicity κ in four dimensions [13]. The latter has the form[6]

$$\mathbb{L}_a = \Big(\kappa + \frac{1}{4}\Big)P_a\,, \tag{3.13}$$

where the operator $\mathbb{L}_a$ is defined by

$$\mathbb{L}_a = \mathbb{W}_a - \frac{1}{16}(\tilde{\sigma}_a)^{\dot{\alpha}\alpha}\big[Q_\alpha, \bar{Q}_{\dot{\alpha}}\big]\,. \tag{3.14}$$

The fundamental properties of the operator $\mathbb{L}_a$ (the latter differs from the supersymmetric Pauli-Lubanski vector [14]) are that it is translationally invariant and commutes with the supercharges Q_α and $\bar{Q}_{\dot{\alpha}}$ in the massless representations of the $\mathcal{N} = 1$ super-Poincaré group.[7] The superhelicity operator (3.14) was generalised to higher dimensions in [15,16]. Generalisations of (3.13) to five and higher dimensions will be discussed elsewhere.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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[4] We are grateful to Warren Siegel for useful comments.

[5] In the massless case, all Casimir operators of the Poincaré group $(\mathbb{W}^{(n)}{}_{a_1\dots a_{d-2n-1}})^2$ vanish, and so does the scalar operator $\mathbb{W}^{(\frac{d-1}{2})}$, which is defined when d is odd.

[6] In the supersymmetric case, the conventions of [13] are used, in particular the Levi-Civita tensor ε_{abcd} is normalised by $\varepsilon_{0123} = -1$.

[7] The irreducible massless representation of superhelicity κ is the direct sum of two irreducible massless Poincaré representations corresponding to the helicity values κ and $\kappa + \frac{1}{2}$.

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