ISBN: 978-3-95450-231-8 ISSN: 2673-5490 doi: 10.18429/JACoW-IPAC2023-WEPL183

ADVANCES IN A PERTURBATION THEORY FOR THE MICROBUNCHING INSTABILITY IN FREE-ELECTRON LASER INJECTORS

Philipp Amstutz*, Mathias Vogt
Deutsches Elektronen-Synchrotron DESY, Germany

Abstract

The microbunching instability is one of the most significant effects, which can lead to a severe degradation of the beam quality in the linac section of free-electron lasers. Direct analytical treatment of the microbunching instability is however challenging. In particular when multiple bunch compression stages are considered, an exact closed-form expression for the charge density of the electron bunch typically cannot be derived. As a remedy, perturbative methods might be considered. Here, the instability is investigated by analyzing the propagation of small perturbations to an otherwise stable phase-space density. One such approach is based on the expansion of the collective Perron-Frobenius operator of the collective system into a Fréchet-Taylor series. Applying the expanded Perron-Frobenius operator to a slightly perturbed phase-space density allows to derive closed-form expressions for the propagated perturbation term, potentially to arbitrary order. In this contribution new advances in a perturbation theory based on the Fréchet-Taylor expansion of collective Perron-Frobenius operators are presented.

MICROBUNCHING IN FEL INJECTORS

A major factor limiting the beam quality achievable in the linear accelerators driving free-electron lasers (FELs) is the microbunching instability (MBI). Collective interactions between the electrons, in conjunction with the longitudinal dispersion introduced by the bunch compression chicanes, can lead to an amplification of initial inhomogeneities in the charge density of the bunch. Generally, the MBI leads to an increased transverse emittance and energy spread of the bunch, both of which degrade the performance of the bunch in the subsequent FEL process. The decreased homogeneity of the longitudinal phase-space distribution can be particularly impeding for advanced operation modes, such as external seeding.

Here, we present advances in a perturbation theory for the MBI, based on the Perron-Frobenius (PF) formalism [1, 2], which are an extension of our previous work on the topic [3–5]. This contribution is a brief summary of parts of [6], where the matter is presented in more detail, including proofs which are omitted here for brevity.

Collective effects that influence the dynamics of a bunch in an FEL injector include space-charge forces, interaction via coherent synchrotron radiation (CSR) in the magnetic chicanes, and wake fields, predominantly from the accelerating cavities. Especially the longitudinal space-charge (LSC) forces are a major driver of the MBI. When only the LSC and wake field interactions are considered, the MBI can be treated using a particularly benign model of the longitudinal beam dynamics. Here, the injector beamline is subdivided into bunch compression stages, each comprising a linac section and a subsequent chicane. Generally, the length of the linac section will be much larger than the length of the magnetic chicane. This justifies the model assumption to neglect the influence of the collective effects in the chicane, so that its contribution on the longitudinal dynamics is given to first-order by the drift map $L_\beta : (q,p) \mapsto (q + \beta p, p)$, where the phase-space coordinates $(q,p) \equiv \vec{z}$ describe the deviation of the longitudinal position $q = z - z_0$ and energy $p = E - E_0$ from a reference particle, and the drift parameter related to the well known longitudinal dispersion M_{56} via $\beta = M_{56}/E_0$.

In the ultra-relativistic limit, there is no significant source of longitudinal dispersion in the linac section. Therefore, the longitudinal coordinate of each electron does not change, so that the charge density of the bunch is invariant along the linac section. As a result, also any collective force that depends on the charge density is invariant. A remarkable property of this model is, that this invariance allows to solve the collective equation of motion exactly, as the charge density has to be calculated only once at the start of the linac section. The total force term can then be determined via an effective, integrated Greens function $G : \mathbb{R} \to \mathbb{R}$, which accounts for varying parameters such as the beamsize and energy along the section, so that the collective dynamics are given by the map

$$K_{\mathrm{coll}}[\Psi] : (q,p) \mapsto (q, p + \kappa[\Psi](q)), \tag{1}$$

where $\Psi : \mathbb{R}^2 \to \mathbb{R}$ denotes the phase-space density and the collective kick function is given via an operation we denote with the $\circledast$ symbol, namely a projection along the p coordinate followed by a convolution with Greens function

$$\kappa[\Psi](q) = \iint_{\mathbb{R}^2} \Psi(q',p') G(q - q') \,\mathrm{d}q' \,\mathrm{d}p' \equiv (\Psi \circledast G)\,. \tag{2}$$

To first order the effect of an rf-cavity is the linear kick map $K_h : (q,p) \mapsto (q, p + h\,q)$, so that the total map of a linac section is $K_{\mathrm{linac}}[\Psi] = K_{\mathrm{coll}}[\Psi] \circ K_h = K_h \circ K_{\mathrm{coll}}[\Psi]$ and that of a single complete bunch compression stage

$$M[\Psi] = L_\beta \circ K_{\mathrm{linac}}[\Psi] = L_\beta \circ K_h \circ K_{\mathrm{coll}}[\Psi]\,. \tag{3}$$

We note that the the non-collective part of the map can also be represented as

$$L_\beta \circ K_h \equiv K_{Ch} \circ S_C \circ L_{C\beta}, \tag{4}$$

with the symplectic scaling map $S_C : (q,p) \mapsto (q/C, pC)$, and the compression factor $C = (1 + h\beta)^{-1}$.

* philipp.amstutz@desy.de

ISBN: 978-3-95450-231-8 ISSN: 2673-5490 doi: 10.18429/JACoW-IPAC2023-WEPL183

THE VLASOV PICTURE

In the Liouville picture, the bunch is described by a phase-space density $\Psi(s;q,p)$, which evolves according to the Liouville equation

$$\frac{\partial \Psi}{\partial s} = \{H, \Psi\} \equiv :H: \Psi \ , \ \Psi(0;\vec{z}) = \Psi_0(\vec{z}), \qquad (5)$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is the Hamiltonian of the system, $\{\cdot,\cdot\}$ is the usual Poisson bracket, and $:H:$ the associated Lie operator. The solution is given by $\Psi(s;\vec{z}) = \exp(s:H:)\Psi_0(\vec{z}) = \Psi_0\left(M^{-1}z\right) = \left[\Psi_0 \circ M^{-1}\right](\vec{z}) \equiv \mathscr{M}\,\Psi_0$, where $M = \exp(-s:H:)$ is the map generated by the Hamiltonian, and $\mathscr{M} \in \mathrm{aut}(\mathscr{W})$ is called the *Perron-Frobenius operator* (PFO) associated with M. If collective interactions can be included by adding a dependence of the Hamiltonian on the phase-space density, this yields the Vlasov equation in time-continuous and equivalently in time-discrete form

$$\begin{cases} \frac{\partial \Psi}{\partial s} - :H[\Psi]: \Psi = 0 \ , \ \Psi(0;\vec{z}) = \Psi_0(\vec{z}) \ , \\ \Psi_{n+1} = \mathscr{M}_{n+1}[\Psi_n]\,\Psi_n \ , \ n \in \mathbb{Z}_+ \ . \end{cases} \qquad (6)$$

Being a non-linear integro- partial differential equation or difference equation in Ψ, respectively, the Vlasov equation generally does not admit a closed-form solution. However, in our ultra-relativistic model motivated above, the collective Hamiltonian of a linac section depends only on the initial value of the phase-space density Ψ_0 at the entrance of the section, which allows us to write the exact solution after the first bunch compression stage as

$$\Psi_1 = \mathscr{M}_1[\Psi_0]\Psi_0 = \mathscr{L}_{\beta_1}\,\mathscr{K}_{h_1}\,\mathscr{K}_{\mathrm{coll},1}[\Psi_0]\Psi_0. \qquad (7)$$

For a collective PFO $\mathscr{M}[\cdot]$ its Fréchet derivative at a given Ψ is defined as the linear operator $\mathrm{D}\mathscr{M}[\Psi]\cdot$ which fulfills the condition

$$\lim_{\|\phi\|_{\mathscr{W}}\to 0} \frac{\|\mathscr{M}[\Psi+\phi] - \mathscr{M}[\Psi] - \mathrm{D}\mathscr{M}[\Psi]\phi\|_{\mathrm{op}}}{\|\phi\|_{\mathscr{W}}} = 0, \qquad (8)$$

where $\|\cdot\|_{\mathrm{op}}$ is an operator norm, compatible with $\|\cdot\|_{\mathscr{W}}$. Higher-order derivatives are given by $\mathrm{D}^n\mathscr{M}[\Psi] \equiv \mathrm{D}^{n-1}(\mathrm{D}\mathscr{M}[\Psi]\cdot)$, which define a multilinear map $\mathrm{D}^n\mathscr{M}[\Psi] : \mathscr{W}^n \to \mathrm{aut}(\mathscr{W})$. For convenience we write $\mathrm{D}^n\mathscr{M}[\Psi]\phi^n \equiv \mathrm{D}^n\mathscr{M}[\Psi]\phi^{\times n}$ which allows to expand the PFO into a Fréchet-Taylor series

$$\mathscr{M}[\Psi+\phi] = \sum_{n=0}^{N} \frac{1}{n!}\mathrm{D}^n\mathscr{M}[\Psi]\phi^n + o\left(\|\phi\|^N\right). \qquad (9)$$

With this, the evolution of an initial density Ψ_0 plus a perturbation density $\epsilon\phi_0$ can be written in form of the perturbative series

$$\mathscr{M}[\Psi+\epsilon\phi](\Psi+\epsilon\phi) = \sum_{n=0}^{N} \frac{\epsilon^n}{n!}\phi_{1,n} + o\left(\epsilon^N\right), \qquad (10)$$

where the nth order perturbation density is given by

$$\phi_{1,n} = \mathrm{D}^n\mathscr{M}[\Psi_0]\phi_0^n\Psi_0 + n\,\mathrm{D}^{n-1}\mathscr{M}[\Psi_0]\phi_0^n. \qquad (11)$$

Given a non-collective PFO $\mathscr{N}$ and a collective PFO $M[\Psi]$, it can be shown that $\mathrm{D}\left(\mathscr{N}\,\mathscr{M}[\Psi]\right) = \mathscr{N}\,\mathrm{D}\mathscr{M}[\Psi]$.

Homomorphic Collective PF Operators

If a collective PF operator fulfills the condition $\mathscr{M}[\Psi+\phi] = \mathscr{M}[\Psi]\mathscr{M}[\phi]$, we call the operator *homomorphic*. In particular, it can be shown that PF operators that are generated by collective Hamiltonians $\mathscr{M}[\Psi] = \exp(:H[\Psi]:)$ are homomorphic if the Poisson bracket of the Hamiltonian vanishes for all phase-space densities $\{H[\Psi], H[\phi]\} = 0 \quad \forall\,\Psi,\phi \in \mathscr{W}$. For this type of PF operators, it can be shown that their Fréchet derivatives are given by

$$\mathrm{D}^n\mathscr{M}[\Psi]\phi^n = \mathscr{M}[\Psi]\,:H[\phi]:^n \ . \qquad (12)$$

In this case, the expression in Eq. (11) for the nth order perturbation density simplifies to

$$\phi_{1,n} = \mathscr{M}[\Psi_0]\,:H[\phi_0]:^{n-1}\,(:H[\phi_0]:\Psi_0 + n\,\phi_0)\,. \qquad (13)$$

It can be seen that the collective kick operator $\mathscr{K}_{\mathrm{coll},1}$ in Eq. (7), which is associated to a kick map of the form shown in Eq. (1) is generated by the Hamiltonian

$$H_{\mathrm{coll},1}[\Psi](q,p) = -\left[\Gamma_1(q-q') \circledast \Psi(q,p)\right](q), \qquad (14)$$

where $\frac{\partial}{\partial q}\Gamma_1(q) = G_1(q)$ and G_1 is the Greens function of the collective kick in the first stage. Collective Hamiltonians of this type are indeed homomorphic in the afforementioned sense.

Monochromatic Perturbation

As a first application of the perturbation theory, one might consider the case of an arbitrarily long bunch, with the initial phase-space density $\Psi_0(q,p) = \psi(p)$, with $\int_{\mathbb{R}}\psi(p)\mathrm{d}p = \rho_0$, in conjunction with a monochromatic perturbation $\phi_0(q,p) = [a_k(q) + \bar{a}_k(q)]\psi(p)$ with $a_k(q) \equiv \frac{1}{2}\exp(ikq)\psi(p)$. It can be seen that the Lie operator of the collective Hamiltonian of the perturbation is $:H_{\mathrm{coll},1}[a_k\psi]: = W_1(k)\,a_k(q)\,\frac{\partial}{\partial p}$, with the impedance-like function $W_1(k) = ik\rho_0\tilde{\Gamma}_1(k)$, where $\tilde{\Gamma}_1$ being the Fourier transform of Γ_1.

First Order With the above, it can be seen that the first term in the perturbation series (11) of the system (7) is given by

$$\phi_{1,1} = \mathscr{M}_{\beta_1,h_1}\left(a_k(q)\left[1 + W_1(k)\frac{\partial}{\partial p}\right]\psi(p)\right) + \mathrm{c.c.}, \qquad (15)$$

where we introduce a shorthand for the non-collective part $\mathscr{M}_{\beta_1,h_1} \equiv \mathscr{L}_{\beta_1}\,\mathscr{K}_{h_1}$. A remarkable property of this expression is, that the part on which $\mathscr{M}_{\beta_1,h_1}$ acts, is a factorizable function with respect to the variables q and p. Using Eq. (4) for the non-collective map, it can be seen that for a factorizable function of the form $a_k(q)f(p)$ its charge density after applying $\mathscr{M}_{\beta_1,h_1}$ is given explicitly by

$$\int_{\mathbb{R}} \mathscr{K}_{Ch}\mathscr{S}_C\mathscr{L}_{C\beta}\left(a_k(q)f(p)\right)\mathrm{d}p = C\,a_k(qC)\tilde{f}(kC\beta). \qquad (16)$$

This yields for the first-order charge density

$$\rho_{1,1} = C_1\,a_{Ck}\left[1 + ikC_1\beta_1 W_1(k)\right]\tilde{\psi}(kC_1\beta_1) + \mathrm{c.c.}. \qquad (17)$$

ISBN: 978-3-95450-231-8 ISSN: 2673-5490 doi: 10.18429/JACoW-IPAC2023-WEPL183

Second Order The second-order perturbation density is

$$\phi_{1,2} = \mathscr{M}_{\beta_1,h_1}\Big(a_{2k}(q)\left[\frac{W_1(k)^2}{2}\frac{\partial^2}{\partial p^2} + W_1(k)\frac{\partial}{\partial p}\right] + \left[\frac{|W_1(k)|^2}{4}\frac{\partial^2}{\partial p^2} + \frac{W_1(k)}{2}\frac{\partial}{\partial p}\right]\Big)\psi + \text{c.c.}, \quad (18)$$

which, again, is a factorizable function before application of the non-collective PF operator. We see that this density consists of a q-independent term, plus a term with double the periodicity of the initial perturbation. Using again Eq. (16), we obtain for the second-order charge density

$$\rho_{1,2} = C\, a_{2Ck}(q)\ \tilde{\psi}(2kC\beta)\ \Big[ikC_1\beta_1 W_1(k) - \frac{1}{2}\left\{kC_1\beta_1 W_1(k)\right\}^2\Big] + \text{c.c.}. \quad (19)$$

Higher Orders It becomes apparent that in the same manner perturbation densities of arbitrary order can be derived. At all orders, the perturbation densities are factorizable in the sense introduced above, so that their associated charge densities can be determined explicitly. A general formula for the higher-order densities is given in Ref. [6].

Second-Order Two-Color Perturbations

Using the perturbation theory it is also possible to treat perturbations on more than one wavelength with relative ease. Consider the two-color perturbation $\phi_0(q,p) = \frac{1}{2}\left[c_1 a_{k_1}(q) + c_2 a_{k_2}(q)\right]\psi(p) + \text{c.c.}$ with two wavenumbers k_1, k_2, phase factors c_1, c_2, and Ψ_0 as before. Plugging this into Eq. (11), it becomes apparent that the second-order perturbation density contains terms corresponding to the individual evolution of the two perturbation colors. This term is equal to a quarter of the sum of Eq. (18) for k_1 and k_2 and is omitted here for the sake of brevity. The remaining term, which encapsulates the interaction of the two perturbations reads

$$\phi^{II}_{1,2} = \text{c.c.} + \mathscr{M}_{\beta_1,h_1}\Big(c_1c_2\Big[W(k_1)W(k_2)\frac{\partial^2\psi}{\partial p^2} + \{W(k_1)+W(k_2)\}\frac{\partial\psi}{\partial p}\Big]a_{k_1+k_2} + c_1\bar{c}_2\Big[W(k_1)\overline{W(k_2)}\frac{\partial^2\psi}{\partial p^2} + \{W(k_1)+\overline{W(k_2)}\}\frac{\partial\psi}{\partial p}\Big]a_{k_1-k_2}\Big). \quad (20)$$

It can be seen that due to the interaction two additional terms are created, one with a frequency equal to the sum of the two perturbation frequencies and one with the difference-frequency.

Two Microbunching Stages

In many experimental setups, the target compression is achieved by two bunch compression stages. After the second stage, the phase-space density is given via discrete Vlasov equation Eq. (5), which reads $\Psi_2 = \mathscr{M}_2[\mathscr{M}_1[\Psi_0]\Psi_0]\,\mathscr{M}_1[\Psi_0]\,\Psi_0$. A perturbation series for this expression can be obtained by expanding all collective

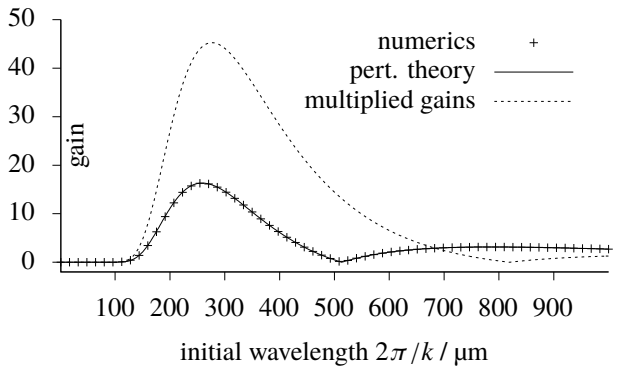

Figure 1: Two-stage gain function evaluated for a parameter set similar to a projected workingpoint of the FLASH after the FLASH2020+ upgrade [7, 8], with $I_0 = 25\,\text{A}$, $C_1 = C_2/2 = 4$, assuming an initially Gaussian energy distribution with $\sigma_p = 5\,\text{keV}$ and using a one-dimensional model for the LSC impedance [9, 10]. CSR effects are excluded.

PF operators as before. Assuming again a monochromatic perturbation, this yields for the first-order perturbation density after the second stage

$$\phi_{2,1} = \text{c.c.} + \mathscr{M}_{\beta_2,h_2}\mathscr{M}_{\beta_1,h_1}\Big(a_k\left[1 + \left\{W_1(k) + g_1\, W_2(C_1\,k)\ \exp(ik\,C_1\beta_1 p)\right\}\frac{\partial}{\partial p}\right]\psi\Big), \quad (21)$$

with $g_1 \equiv \frac{1}{\rho_0}\left[1 + i\,kW_1(k)C_1\beta_1\right]\tilde{\psi}(kC_1\beta_1)$ and $\mathscr{M}_{\beta_2,h_2}$ being the non-collective part of the second stage PF operator. It can be shown, that the combination of the two non-collective PF operators can be written as $\mathscr{M}_{\beta_2,h_2}\,\mathscr{M}_{\beta_1,h_1} = \mathscr{H}_{C_2^*h_2^*}\,\mathscr{S}_{C_2^\dagger}\,\mathscr{D}_{\beta_2^\dagger}$, with $h_2^* = h_2 + C_1h_1$, $C_2^* = (1+h_2^*\beta_2)^{-1}$, $\beta_2^* = C_2^*\beta_2$, $C_2^\dagger = C_2^*C_1$, and $\beta_2^\dagger = \beta_1 + C_1^2\beta_2^*$. Using this identity, the charge density of $\phi_{2,1}$ can be obtained via Eq. (16), which yields $\rho_{2,1} = C_2^\dagger\, a_{C_2^\dagger k}(q)\ g_2^\dagger + \text{c.c.}$. with

$$g_2^\dagger \equiv \left[1 + ik\,\beta_2^\dagger W_1(k)\right]\tilde{\psi}(k\,\beta_2^\dagger) + i\,g_1\,C_2^\dagger C_1\beta_2 k\ W_2(C_1\,k)\ \tilde{\psi}(C_2^\dagger C_1\beta_2 k). \quad (22)$$

Figure 1 shows a comparison of the two-stage gain function $|g_2^\dagger|/\rho_0$, numerical results from a semi-Lagrangian Vlasov code [11], and a commonly encountered model, in which the two-stage gain function is estimated by multiplying the single-stage gain functions. It is apparent that while the theoretical gain function agrees excellently with the numerics, the multiplied-gains estimation yields deviating results.

ACKNOWLEDGEMENTS

We dedicate this article to the memory of Robert L. Warnock (*1930 – †2022), who greatly inspired our work on Vlasov systems.

REFERENCES

[1] R. L. Warnock and J. A. Ellison. "A General method for propagation of the phase space distribution, with application

ISBN: 978-3-95450-231-8 ISSN: 2673-5490 doi: 10.18429/JACoW-IPAC2023-WEPL183

to the sawtooth instability", in *The physics of high brightness beams*, World Scientific, 1999, pp. 322–348. doi:10.1142/9789812792181_0020

[2] M. Vogt, J. A. Ellison, T. Sen, and R. L. Warnock. "Two methods for simulating the strong-strong beam-beam interaction in hadron colliders", SLAC, CA, USA, SLAC-PUB-9576, 2002. doi:10.2172/808676

[3] M. Vogt and Ph. Amstutz, "Arbitrary Order Perturbation Theory for Time-Discrete Vlasov Systems with Drift Maps and Poisson Type Collective Kick Maps", in *Nonlinear Dynamics and Collective Effects in Particle Beam Physics*, World Scientific, 2019, pp. 169–181. doi:10.1142/9789813279612_0015

[4] Ph. Amstutz and M. Vogt, "Arbitrary Order Perturbation Theory for a Time-Discrete Model of Micro-Bunching Driven by Longitudinal Space Charge", in *Proc. FEL'19*, Hamburg, Germany, Aug. 2019, pp. 596–598. doi:10.18429/JACoW-FEL2019-THP003

[5] Ph. Amstutz and M. Vogt, "Investigation of Vlasov Systems with a Certain Class of Linearly-Collective Hamiltonians", in *Proc. IPAC'21*, Campinas, Brazil, May 2021, pp. 3157–3160. doi:10.18429/JACoW-IPAC2021-WEPAB226

[6] Ph. Amstutz, PhD thesis, 2023, in preparation.

[7] M. Beye *et al.*, "FLASH2020+, Conceptual Design Report", DESY, Hamburg, 2020. doi:10.3204/PUBDB-2020-00465

[8] Ph. Amstutz and M. Vogt, "Microbunching Studies for the FLASH2020+ Upgrade Using a Semi-Lagrangian Vlasov Solver", in *Proc. IPAC'22*, Bangkok, Thailand, Jun. 2022, pp. 2334–2337. doi:10.18429/JACoW-IPAC2022-WEPOMS037

[9] M. Venturini, "Models of longitudinal space-charge impedance for microbunching instability", *Phys. Rev. Spec. Top. Accel. Beams*, vol. 11, p. 034401, 2008. doi:10.1103/PhysRevSTAB.11.034401

[10] Ph. Amstutz, "Vlasov Simulation of Exotic Phase-Space Densities via Tree-Based Domain-Decomposition with an Application to Microbunchingin Free-Electron Laser Injectors", M.Sc. thesis, Universitat Hamburg, Germany, 2018.

[11] SelaV_{1D}, a semi-Lagrangian Vlasov simulation code in one degree of freedom, https://selav.desy.de