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RESEARCH - GENERAL, MATHEMATICAL AND STATISTICAL PHYSICS

Chebyshev polynomials boost the π-series convergence

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Abstract

We present an interesting application of the solution to the simple harmonic oscillator (SHO) that can serve as a computation of π. We begin with a review of a compact teaching strategy for solving its equation of motion through integration in a general physics course, where many students face difficulties with conventional methods for solving differential equations. This integration approach leads to the arcsine function, the inverse of the sine function, ultimately providing the solution to the SHO. We investigate various series for approximating π, focusing on the arcsine series and their difference in convergence speed. We begin with Newton's arcsine series for $\pi = 2\arcsin 1$. We then explore a series based on powers of $\sin\frac{\pi}{2^{k+1}} \ll 1$, where k is a large positive integer and the sine term is computed using nested radicals through half-angle formulas, resembling Viète's formula. The small sine term acts as a power-counting parameter, making the series better convergent to π with reliable error estimation. We extend this approach to a fractional-angle method, generalizing the factor from 1/2 to $1/p'$ for a prime number p', by employing Chebyshev polynomials of the second kind, which commonly arise in physics problems. This leads to a series involving powers of $\sin\frac{\pi}{p}$, where p is an arbitrary integer expressed as a product of prime factors, further enhancing convergence with a smaller power-counting parameter. The power counting allows us to identify significant terms in the Chebyshev polynomials and to truncate numerically insignificant contributions that optimize and simplify the computation of the sine term. Our novel strategies are pedagogical and suitable for advanced physics undergraduates, enabling them to approximate π with high accuracy using techniques covered in physics courses.

Keywords π-series · Chebyshev Polynomials · Convergence of π series

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1 Introduction

The sine function is one of the most important mathematical functions, appearing frequently in physics problems from the elementary level throughout. It describes the solution of the simple harmonic oscillator and it serves as the basis of the Fourier-series expansion of waves. Freshmen taking a general physics course first encounter the sine function when learning about the simple harmonic oscillator. Usually, these students are acquainted with various formulas involving trigonometric functions and take calculus to learn how to differentiate and integrate a sinusoidal function and its inverse function. Unfortunately, in a typical general physics course, the instructor teaches what the solution of the simple harmonic oscillator is but not why. For example, readers are referred to conventional textbooks like Serway [1] or Halliday and Resnick [2]. The primary reason for this omission is that the equation of motion for the simple harmonic oscillator is the second-order linear differential equation and solving the differential equation is out of the scope of the general physics course.

In fact, many authors have developed methods for solving the equation of motion for the harmonic oscillator without relying on the standard approach of solving differential equations. Weinstock [3], Bush [4], Gauthier [5], and Tisdell [6] use the conservation of the total mechanical energy to parameterize the dimensionless velocity and dimensionless displacement at a point on a circle. However, given that the students know the integral representation of the arcsine function that is the inverse of the sine function, it is possible, in principle, to avoid solving the differential equation directly. Since general physics covers the conservation of the total mechanical energy in a simple harmonic oscillator, one can solve for the oscillator's velocity and reorganize the equation so that the variables are separated. Then, finding the velocity reduces to evaluating a definite integral, the arcsine function. Nevertheless, few instructors employ this well-known teaching strategy. For examples of this approach, readers may refer to Eq. (15) of Lenz [7], Eqs. (3.213) and (3.214) on p. 125 of Symon [8] and Eqs. (8.15) and (8.17) on p. 291 of [9].

In this paper, we review a compact teaching strategy for solving the equation of motion for the simple harmonic oscillator by integration. The equation of motion is multiplied by the velocity to construct the total differential form of the Hamiltonian, which represents the total mechanical energy. The quantities with non-vanishing physical dimensions are factored out to find that the product of the characteristic frequency $\omega_0 = \sqrt{k/m}$ multiplied by time t is proportional to a definite integral of a displacement from the equilibrium point, which is an integral representation of the arcsine function. Here, k is the spring constant of the restoring force, and m is the mass of the oscillator.

Since the arcsine function is the inverse of the sine function, it is evident that the definite integral involves π, the one-half period of the sine function. As a result, we obtain $T\sqrt{k/m} = 2\pi$, where T is the period of the oscillation. By choosing all of the physically dimensionless quantities in units, we can identify that the integral can be used to compute $\pi = 2\arcsin 1$ that is indeed what Newton found to compute the fluxion (derivative) of a fluent (variable) as is written on p. 140 of Beckmann [10] in which the history of computing π is reviewed. This is one of the oldest methods of computing π whose convergence is very slow. The Gregory–Leibniz series $\pi = 4\arctan 1 = 4\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots\right] = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ $= 4\sum_{n=0}^{\infty}\frac{2}{4(2n+1)^2-1}$ is an alternating series that converges much faster than $\pi = 2\arcsin 1$. As is stated on p. 141 of Beckmann [10], however, $\pi = 6\arcsin\frac{1}{2}$ converges in comparably more quickly than the Gregory–Leibniz series. This indicates that the arcsine function is still useful in computing π if there is a systematic way to evaluate the value of sine at small angles.

We focus on the fact that freshmen taking a general physics course know how to apply binomial theorem generalized to non-integer powers as developed by Newton around 1665. By employing this method, we can expand the integrand as an infinite power series. Since the definite integral can be evaluated from the initial position to any position, we can vary the upper limit of the integral. By applying the half-angle formulas for the cosine and sine functions that are familiar to freshmen, we compute the numerical value of $\xi = \sin\frac{\pi}{2^{k+1}}$ for a large positive integer k. We demonstrate that the convergence of the power series for $2^{k+1}\arcsin\xi$ is significantly improved compared to that of Newton's formula. The initial term $2^{k+1}\arcsin\xi$ in the series is in agreement with findings by Chang and Chang [11] variated from Vète's formula [12].

As a further improvement in convergence, we employ Chebyshev's polynomials of the second kind $U_{p-1}(\cos\theta) = \sin[p\theta]/\sin\theta$, with a sufficiently large index $p \gg 2$. This generalizes the half-angle method to the fractional-angle method, dividing the angle into p parts. In general, the computation with Chebyshev's polynomials of large indices should be carried out numerically. We provide systematic power-counting rules, a useful tool for systematic expansions in quantum field theory, to significantly reduce computation time.

Perhaps, our strategy would not outperform well-established methods such as Machin-like formulas [13], Chudnovsky's algorithm [14, 15], or Gauss–Legendre methods [16] and not be comparable to state-of-the-art computational algorithms developed by advanced digit hunters. However, our strategies are suitable for advanced freshmen and physics majors, enabling them to achieve an approximate value of π to high accuracies using knowledge obtained in physics courses. The systematic development of computational algorithms is quite pedagogical and can serve as a valuable exercise in computational and mathematical physics courses.

This paper is organized as follows. In Sect. 2, we review a teaching strategy to find the solution of the equation of motion for the simple harmonic oscillator without relying on the conventional approach for solving differential equations. Instead, we find the solution by integration. The integral table illustrated in Sect. 2 is used to compute π in power series expansions in Sect. 3. Arcsine at a tiny argument $\xi = \sin\frac{\pi}{2^{k+1}}$ is determined by making use of the half-angle formulas for the trigonometric functions. In Sect. 4, we generalize this strategy by employing Chebyshev's polynomials of the second kind to construct a series of arcsine that converges much faster than the former series introduced in Sect. 3. A conclusion is given in Sect. 5.

2 Solving simple harmonic motion with arcsine

The arcsine function is the inverse function of the sine function and has the following integral representation:

$$\int_0^{\xi} \frac{du}{\sqrt{1-u^2}} = \arcsin \xi. \tag{1}$$

Typically, students take a calculus-based general physics course alongside calculus often before becoming familiar with solving the second-order linear differential equation. Hence, the integral in Eq. (1) is very convenient in teaching how to solve the equation of motion for the simple harmonic oscillator,

$$m\ddot{x} + kx = 0, \qquad \omega_0 = \sqrt{\frac{k}{m}}, \tag{2}$$

in a general physics class. Here, x is the displacement from the equilibrium position of a particle with mass m attached to a spring with spring constant k and ω_0 is the characteristic angular frequency of the simple harmonic motion.

One can multiply the velocity by Eq. (2) to express it as the total differential of the total mechanical energy (Hamiltonian). The total mechanical energy is given by the sum of the kinetic and potential energies as follows:

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}k\mathscr{A}^2, \tag{3}$$

where the amplitude $\mathscr{A}$ of the oscillation is related to the positions $x = \pm\mathscr{A}$ of the turning points.

Introducing a dimensionless displacement u defined by

$$u = \frac{x}{\mathscr{A}}, \tag{4}$$

we can rewrite Eq. (3) in terms of u. Solving for the velocity $\dot{u} = \dot{x}/\mathscr{A}$, we obtain

$$\dot{u} = \pm\omega_0\sqrt{1-u^2}, \tag{5}$$

where the sign in front of the square root is positive when x varies from $-\mathscr{A}$ to $+\mathscr{A}$. The negative sign corresponds to the motion from $+\mathscr{A}$ to $-\mathscr{A}$.

The energy conservation constraint in Eq. (5) can be used to compute the time interval corresponding to the differential du of the dimensionless displacement as

$$dt = \pm\frac{du}{\omega_0\sqrt{1-u^2}}. \tag{6}$$

Since time flows forward only, $\pm du$ must be positive definite. We can make use of the symmetry of the motion to find that these two time intervals from $\mp\mathscr{A}$ to $\pm\mathscr{A}$ are identical to one-half of the period T:

$$\frac{\omega_0}{2}T = \int_{-1}^{+1}\frac{du}{\sqrt{1-u^2}} = 2\int_0^1\frac{du}{\sqrt{1-u^2}} = 2\arcsin 1 = \pi, \tag{7}$$

where we have made use of the identity $\sin\frac{\pi}{2} = 1 \to \arcsin 1 = \frac{\pi}{2}$. Thus, we have reached the identity

$$T = \frac{2\pi}{\omega_0}. \tag{8}$$

If the initial conditions are given by

$$x_0 = x(0) = \mathscr{A}u_0, \qquad \dot{x} = \dot{x}(0) = \mathscr{A}\dot{u}_0, \tag{9}$$

then Eq. (1) can be modified as

$$\omega_0\int_0^t dt = \int_{u_0}^{u}\frac{du}{\sqrt{1-u^2}} = \arcsin u - \arcsin u_0. \tag{10}$$

The solution is

$$x(t) = \mathscr{A}\sin\left(\omega_0 t + \delta\right), \qquad \delta = \arcsin\frac{x_0}{\mathscr{A}}. \tag{11}$$

We have demonstrated that the arcsine function is highly useful in teaching simple harmonic motion in general physics. The approach described in this section is a well-known strategy. Readers are referred to equation (15) of Lenz [7] for further details. This method is a standard technique to find the solution for the radial equation of the planetary motion that can be found in equations (3.213) and (3.214) on p. 125 of Symon [8] and equations (8.15) and (8.17) on p. 291 of [9], for instance.

3 Computation of π

In this section, we illustrate how to compute π by making use of the integral representation (1) in association with the binomial expansion of the integrand in powers of u. The resulting series for the integral converges slowly to $\arcsin 1 = \frac{\pi}{2}$. By restricting the upper limit of the integral from 1 to $\xi = \sin\frac{\pi}{2^{k+1}}$, where k is a non-negative integer, we improve the convergence of the series by choosing a sufficiently large k. The method is based on the elementary trigonometric identity known as the half-angle formula.

3.1 Binomial expansion

In physics, there are many applications of Newton's generalized binomial theorem. For all $|x| < 1$ and for any constant α, $(1+x)^\alpha$ can be expanded in powers of x as

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots \tag{12}$$

For example, the electrostatic potential or corresponding electric field due to an electric dipole can be approximated using the generalized binomial expansion. In ordinary physics problems, the power α in Eq. (12) is either an integer or a half-integer. If α is a non-negative integer, then the series terminates. If α is a half-integer or a negative integer, then the expansion is an infinite series. When α is a half-integer, as in the integrand in Eq. (1), the general term of the generalized binomial expansion contains double factorials:

$$\begin{aligned}&\frac{\overbrace{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}^{n\text{ factors}}}{n!}\\ &\quad= (-1)^n \overbrace{\frac{1}{2}\cdot\frac{3}{4}\cdot\cdots\frac{2n-1}{2n}}^{n\text{ factors}} = (-1)^n\frac{(2n-1)!!}{(2n)!!}.\end{aligned} \tag{13}$$

The gamma function is closely related to the factorial through the formula $\Gamma[n] = (n-1)!$ providing an extension of the factorial to real and complex numbers [17]. The gamma function satisfies the following recurrence relation (See, for example, chapter 8 of [17].):

$$z\Gamma[z] = \Gamma[z+1], \quad \mathfrak{Re}(z) > 0, \tag{14}$$

This function is convenient for expressing factors that involve factorials or double factorials. Euler's integral definition of the Gamma function is given by

$$\Gamma[z] = \int_0^{\infty} \mathrm{d}u\, u^{z-1} e^{-u}. \tag{15}$$

By multiplying the coefficient in Eq. (13) by $1 = \Gamma[\frac{1}{2}]/\Gamma[\frac{1}{2}]$ and applying the recurrence relation (14), we can rewrite the expression in Eq. (13) as

$$\begin{aligned}&\frac{\overbrace{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}^{n\text{ factors}}}{n!}\times 1\\ &\quad= (-1)^n \frac{\overbrace{\Gamma[\frac{1}{2}]\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots(n-\frac{1}{2})}^{n\text{ factors}}}{\Gamma[\frac{1}{2}]\Gamma[n+1]} = (-1)^n\frac{\Gamma[n+\frac{1}{2}]}{\Gamma[\frac{1}{2}]\Gamma[n+1]}.\end{aligned} \tag{16}$$

This simplification stems from the Legendre duplication formula:

$$\frac{\Gamma[n+\frac{1}{2}]}{\Gamma[\frac{1}{2}]} = \frac{\Gamma[2n+1]}{2^{2n}\Gamma[n+1]} = \frac{(2n-1)!!}{2^n}. \tag{17}$$

The identities given in Eqs. (16) and (17) are quite useful in solving various problems in physics. For example, the power series expansion of a special function using Rodrigues' formula usually contains double factorials that can be simplified using these identities.

It is apparent that $\Gamma[\frac{1}{2}]$ reduces to the Gaussian integral by changing the variable $u = x^2$:

$$\Gamma[\tfrac{1}{2}] = \int_0^{\infty} \mathrm{d}u\, u^{-\frac{1}{2}} e^{-u} = 2\int_0^{\infty} e^{-x^2}\mathrm{d}x = \int_{-\infty}^{+\infty} e^{-x^2}\mathrm{d}x. \tag{18}$$

Then, the square of the Gaussian integral can be evaluated in two-dimensional polar coordinates using $x = r\cos\theta$ and $y = r\sin\theta$ as follows:

$$\int_{-\infty}^{+\infty} e^{-x^2}\mathrm{d}x \int_{-\infty}^{+\infty} e^{-y^2}\mathrm{d}y = \int_0^{2\pi}\mathrm{d}\theta \int_0^{\infty} e^{-r^2} r\,\mathrm{d}r = \pi. \tag{19}$$

Since the Gaussian integral is positive definite, we can determine the value of $\Gamma[\frac{1}{2}]$ as

$$\Gamma[\tfrac{1}{2}] = \sqrt{\pi}. \tag{20}$$

Hence, a direct numerical evaluation of the definite integral (18) can determine the numerical value of $\sqrt{\pi}$.

By employing the identity in Eq. (12), we can expand the integrand in Eq. (1) in powers of t and evaluate the integral over t as

$$\int_0^{\xi} \mathrm{d}t\, t^{2n} = \frac{\xi^{2n+1}}{2n+1}. \tag{21}$$

The resultant expression for $\arcsin\xi$ of the power series in ξ is obtained as

$$\begin{aligned}\arcsin\xi &= \int_0^{\xi}\frac{\mathrm{d}t}{\sqrt{1-t^2}}\\ &= \int_0^{\xi}\mathrm{d}t\sum_{n=0}^{\infty} t^{2n}\frac{\Gamma[n+\frac{1}{2}]}{\Gamma[\frac{1}{2}]\Gamma[n+1]}\\ &= \sum_{n=0}^{\infty}\frac{\Gamma[2n+1]}{2^{2n}\Gamma^2[n+1]}\times\int_0^{\xi}\mathrm{d}t\, t^{2n}\\ &= \sum_{n=0}^{\infty}\xi^{2n+1}\frac{\Gamma[2n+1]}{2^{2n}(2n+1)\Gamma^2[n+1]}.\end{aligned} \tag{22}$$

When transitioning from the second to the third line, the summation symbol and the Gamma function terms are moved out of the integration because they are independent of t. Equations (17) and (20) can be employed to rewrite Eq. (22) as

$$\sqrt{\pi}\arcsin\xi = \sum_{n=0}^{\infty}\xi^{2n+1}\frac{\Gamma[n+\frac{1}{2}]}{(2n+1)\Gamma[n+1]}. \tag{23}$$

However, Eq. (23) is not useful to determine π because $\Gamma[n+\frac{1}{2}]$ in the numerator contains a factor of $\Gamma[\frac{1}{2}] = \sqrt{\pi}$.

As a byproduct, we can use Eq. (22) to compute π. While various efficient computation strategies exist, our primary focus in the remainder of this paper will be on the arcsine function. For example, the Gregory–Leibniz series for the arctangent function $\frac{\pi}{4} = \arctan 1$ converges faster than the expression in Eq. (22). Readers are referred to Frame [18], Kobayashi [19], and Alzer [20] for more details about the Leibniz series. Machin-like formulas, based on the inverse tangent function, offer rapid convergence [13]:

$$\frac{\pi}{4} = \sum_k c_k \arctan\frac{1}{m_k},$$

where c_k are small integers (often positive or negative) and m_k are integers chosen such that their reciprocals provide rapidly converging series for π. Chudnovsky's algorithm, rooted in elliptic curve theory and based on Ramanujan-like series, provides extremely fast convergence and is used in modern record computations [14, 15]:

$$\frac{1}{\pi} = 12\sum_{k=0}^{\infty}\frac{(-1)^k(6k)!(545140134k+13591409)}{(3k)!(k!)^3(640320)^{3k+\frac{3}{2}}}.$$

Additionally, Gauss–Legendre methods, which are iterative algorithms, exhibit quadratic convergence [16]. These methods significantly outperform the series in Eq. (23) in terms of computational efficiency. We refer readers to Castellano's reviews [21, 22] and to Beckmann's book [10] regarding various methods for computing π.

While such state-of-the-art techniques listed above are based on high-performance numerical algorithms, they are not suitable for teaching physics-major undergraduate students. Instead, we introduce more heuristically meaningful approaches that are based on the building blocks that are already familiar to physics-major students.

3.2 Approach using arcsine 1

Equation (22) serves as our master formula from which we compute approximate values of π with quantitatively controlled errors. Since the maximum value of the arcsine function is $\arcsin\xi = \frac{\pi}{2}$ at $\xi = 1$, we multiply both sides of Eq. (22) by 2 and substitute $\xi = 1$ to compute π as follows:

$$\pi = 2\arcsin 1 = \lim_{N\to\infty} S_N, \tag{24a}$$

$$S_N = 2\sum_{n=0}^{N}\frac{\Gamma[2n+1]}{2^{2n}(2n+1)\Gamma^2[n+1]}. \tag{24b}$$

The convergence of the series in Eq. (24) is very poor. For example, the partial sum S_N converges to π very slowly:

$$S_0 = 2, \tag{25a}$$

$$S_1 = 2.3333\cdots, \tag{25b}$$

$$S_{10} = 2.8002\cdots, \tag{25c}$$

$$S_{10^2} = 3.0293\cdots, \tag{25d}$$

$$S_{10^3} = 3.1059\cdots, \tag{25e}$$

$$S_{10^4} = 3.1303\cdots, \tag{25f}$$

$$S_{10^5} = 3.1380\cdots. \tag{25g}$$

We observe that the correct values for the first, second, and third digits are achieved by adding terms of order 10^2, 10^3, and 10^5, respectively.[1]

3.3 Approach using half-angle formula

Another way to determine π using the arcsine function is to apply L'Hôpital's theorem to the sine function:

$$\pi = \lim_{x\to 0}\frac{\sin\pi x}{x}. \tag{26}$$

At first glance, equation (26) appears to suggest that we need to know the value of π to compute π. However, if we choose a tiny number πx whose sine is known from the half-angle formula, then we do not need the explicit value of π to evaluate the right-hand side of Eq. (26).

We shall find that the sine function of $\frac{\pi}{2^{k+1}}$ for any integer $k \geq 0$ can be computed and the corresponding sequence π_k approaches π as $k \to \infty$:

$$\pi_k \equiv \frac{\sin\frac{\pi}{2^{k+1}}}{\frac{1}{2^{k+1}}} = 2^{k+1}\sin\frac{\pi}{2^{k+1}}, \qquad k = 0,\ 1,\ 2,\ \cdots \tag{27}$$

Leading entries of the sequence π_k are

$$\pi_0 = 2, \quad \pi_1 \approx 2.83, \quad \pi_2 \approx 3.06, \quad \pi_3 \approx 3.12, \quad \pi_4 \approx 3.14. \tag{28}$$

[1] Rounded to the second decimal place.

The value of $\sin\frac{\pi}{2^{k+1}}$ can be computed by applying the half-angle formula for the sine function recursively. For any $\theta \in [0, \frac{\pi}{2}]$, the half-angle formulas are given by

$$\left[\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right] = \left[\sqrt{\frac{1}{2}(1+\cos\theta)}, \sqrt{\frac{1}{2}(1-\cos\theta)}\right]. \tag{29}$$

It is straightforward to construct the following recurrence relations:

$$\left[\cos\frac{\theta}{2^k}, \sin\frac{\theta}{2^k}\right] = \left[\sqrt{\frac{1}{2}\left(1+\cos\frac{\theta}{2^{k-1}}\right)}, \sqrt{\frac{1}{2}\left(1-\cos\frac{\theta}{2^{k-1}}\right)}\right]. \tag{30}$$

Substituting $\frac{\pi}{2}$ for θ in (30) and varying k recursively, we find that

$$\left(\cos\frac{\pi}{2}, \sin\frac{\pi}{2}\right) = (0, 1), \tag{31a}$$

$$\left(\cos\frac{\pi}{2^2}, \sin\frac{\pi}{2^2}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \tag{31b}$$

$$\left(\cos\frac{\pi}{2^3}, \sin\frac{\pi}{2^3}\right) = \left(\frac{\sqrt{2+\sqrt{2}}}{2}, \frac{\sqrt{2-\sqrt{2}}}{2}\right), \tag{31c}$$

$$\left(\cos\frac{\pi}{2^4}, \sin\frac{\pi}{2^4}\right) = \left(\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}, \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}\right), \tag{31d}$$

$$\vdots$$

$$\left(\cos\frac{\pi}{2^{k+1}}, \sin\frac{\pi}{2^{k+1}}\right) = \left(\frac{\underbrace{\sqrt{2+\sqrt{2+\cdots\sqrt{2+\sqrt{2}}}}}_{k\text{ roots}}}{2}, \frac{\underbrace{\sqrt{2-\sqrt{2+\cdots\sqrt{2+\sqrt{2}}}}}_{k\text{ roots}}}{2}\right). \tag{31e}$$

The expression for the sine in Eq. (31e) agrees with the theorem given by Chang and Chang [11], which is a variation of Viète's original formula [12].

The numerical value for $\pi_k = 2^{k+1}\sin\frac{\pi}{2^{k+1}}$ in Eq. (27) can be computed by making use of the formula given in the last line of Eq. (31). At $k = 80$, π_k reproduces π up to 50 digits:

$$\pi = 3.14159265358979323846264338327950288419716939937 4222\cdots,$$

The error is $\pi - \pi_k \approx \frac{\pi^3}{3!\,4^{k+1}} \approx 8.8\times10^{-49}$, where we Taylor expanded π_k to identify the dominant contribution to $\pi - \pi_k$. Indeed, it is the single-term contribution $S_0^{(k)}$ of the series:

$$\pi = \lim_{N\to\infty} S_N^{(k)}, \tag{32a}$$

$$S_N^{(k)} \equiv 2^{k+1}\sum_{n=0}^{N}\left[\sin\frac{\pi}{2^{k+1}}\right]^{2n+1}\frac{\Gamma[2n+1]}{2^{2n}(2n+1)\Gamma^2[n+1]}, \tag{32b}$$

where $\sin\frac{\pi}{2^{k+1}}$ is given in Eq. (31). Equation (32) derives from Eq. (22) by substituting $\sin\frac{\pi}{2^{k+1}}$ for ξ and multiplying by 2^{k+1}. A similar series based on the half-angle formula is presented in [23], but its alternating nature, arising from the expansion of an arctangent term, distinguishes it from our series. The numerical error of $S_N^{(k)}$ decreases exponentially as we increase the number of terms N. The ratio of the $(N+1)$th entry of the sequence to the Nth entry scales like

$$\sin^2\frac{\pi}{2^{k+1}} \sim \frac{1}{4^k}, \qquad \text{for} \qquad k \gg 1, \tag{33}$$

which is a power-counting parameter that indicates the size of the relative correction of the series sum as N increases by 1, thus representing the relative error of $S_N^{(k)}$. Consequently, the error of $S_N^{(k)}$ scales like

$$\Delta S_N^{(k)} = \pi - S_N^{(k)} \approx S_{N+1}^{(k)} - S_N^{(k)} \sim \frac{1}{4^{k(N+1)}}. \tag{34}$$

Knowing the definite scaling of the error is useful since it immediately tells us how many sequences are needed to achieve the desired accuracy. Numerically, it scales like $10^{-6(N+1)}$, $10^{-30(N+1)}$, and $10^{-60(N+1)}$ for $k = 10, 50$, and 100, respectively. We verify the scaling behavior from an explicit example below. It is convenient to define the error vector $\Delta\boldsymbol{S}^{(k)}$ as

$$\Delta\boldsymbol{S}^{(k)} = \left[\pi - S_0^{(k)}, \pi - S_1^{(k)}, \pi - S_2^{(k)}, \pi - S_3^{(k)}, \pi - S_4^{(k)}, \cdots\right], \tag{35}$$

where the vector remains positive because each term in the series (32) is positive definite. An explicit computation of the partial sum (32) for various values of k reveals that the error decreases dramatically as both k and N increase:

$$\Delta\boldsymbol{S}^{(0)} = [1.1, 0.81, 0.66, 0.57, 0.51, \cdots], \tag{36a}$$

$$\Delta S^{(10)} = [1.2\times10^{-6}, 1.3\times10^{-12}, 1.8\times10^{-18}, 2.9\times10^{-24}, 5.1\times10^{-30}, \cdots], \tag{36b}$$

$$\Delta S^{(20)} = [1.2\times10^{-12}, 1.2\times10^{-24}, 1.6\times10^{-36}, 2.4\times 10^{-48}, 4.0\times10^{-60}, \cdots], \tag{36c}$$

$$\Delta S^{(30)} = [1.1\times10^{-18}, 1.1\times10^{-36}, 1.4\times10^{-54}, 2.0\times10^{-72}, 3.2\times10^{-90}, \cdots], \tag{36d}$$

$$\Delta S^{(40)} = [1.1\times10^{-24}, 9.8\times10^{-49}, 1.2\times10^{-72}, 1.7\times10^{-96}, 2.5\times10^{-120}, \cdots], \tag{36e}$$

$$\Delta S^{(50)} = [1.0\times10^{-30}, 8.9\times10^{-61}, 1.0\times10^{-90}, 1.4\times10^{-120}, 2.0\times10^{-150}, \cdots], \tag{36f}$$

$$\Delta S^{(60)} = [9.7\times10^{-37}, 8.1\times10^{-73}, 9.0\times10^{-109}, 1.1\times10^{-144}, 1.5\times10^{-180}, \cdots], \tag{36g}$$

$$\Delta S^{(70)} = [9.3\times10^{-43}, 7.4\times10^{-85}, 7.8\times10^{-127}, 9.4\times10^{-169}, 1.2\times10^{-210}, \cdots], \tag{36h}$$

$$\Delta S^{(80)} = [8.8\times10^{-49}, 6.7\times10^{-97}, 6.7\times10^{-145}, 7.8\times10^{-193}, 9.6\times10^{-241}, \cdots], \tag{36i}$$

$$\Delta S^{(90)} = [8.4\times10^{-55}, 6.1\times10^{-109}, 5.9\times10^{-163}, 6.4\times10^{-217}, 7.6\times10^{-271}, \cdots], \tag{36j}$$

$$\Delta S^{(100)} = [8.0\times10^{-61}, 5.6\times10^{-121}, 5.1\times10^{-181}, 5.3\times10^{-241}, 6.0\times10^{-301}, \cdots]. \tag{36k}$$

The size of the errors agrees with the predictions for $k = 10,\ 50,$ and 100 from the scaling behavior in Eq. (34). According to the errors in Eq. (36) for the series in Eq. (35), we conclude that the evaluation at $\xi = \sin\frac{\pi}{2^{k+1}}$ is far more efficient than the original series expansion in Eq. (24), whose errors can be determined from Eq. (25). There is a trade-off in achieving efficiency. While each sequence in Eq. (24) consists of rational numbers with infinite precision, the factor $\sin\frac{\pi}{2^{k+1}}$ in Eq. (35) is irrational and must be computed numerically using Eq. (31e) to the desired accuracy, which can be costly.

While the expression for the sine in Eq. (31e) corresponds to the theorem established by Chang and Chang [11], which pertains only to the first term, to the best of our knowledge, the complete series presented in Eq. (32) and the scaling of the error in Eq. (34) is novel.

4 Fractional-angle method

One might wonder whether the convergence of the series in Eq. (32) could be further improved by generalizing the factor 2^{k+1} to an arbitrarily large integer p, expressed as a product of prime factors:

$$p = \prod_i p_i'^{k_i}, \tag{37}$$

where p_i' are prime numbers and k_i are non-negative integer powers, respectively. Then, we can replace 2^{k+1}'s in Eq. (32b) with p's to find that

$$S_N^{\{k_i\}} \equiv p \sum_{n=0}^{N} \left[\sin\frac{\pi}{p}\right]^{2n+1} \frac{\Gamma[2n+1]}{2^{2n}(2n+1)\Gamma^2[n+1]}. \tag{38}$$

Now, the power-counting parameter 2^{k+1} in Eq. (33) is replaced by the factor p:

$$\sin^2\frac{\pi}{p} \sim \frac{1}{p^2} = \frac{1}{(p_1'^{2k_1})(p_2'^{2k_2})(p_2'^{2k_3})\cdots}. \tag{39}$$

The corresponding error scales like

$$\begin{aligned}\Delta S_N^{\{k_i\}} &= \pi - S_N^{\{k_i\}} \sim \frac{1}{p^{2(N+1)}} \\ &= \frac{1}{(p_1'^{2k_1(N+1)})(p_2'^{2k_2(N+1)})(p_2'^{2k_3(N+1)})\cdots}.\end{aligned} \tag{40}$$

The expression in Eq. (38) is fully determined once an expression for the sine of the fractional angle, $\sin\frac{\pi}{p}$, is found. For instance, when $p = 2^k$, the formulation naturally aligns with a Viète-like formula due to the recursive application of the half-angle formula, resulting in a nested square root structure with a convergence rate of $(1/2)^{2k(N+1)}$. For $p = 3^k$, the formulation extends to triple-angle formulas, introducing cubic polynomial dependencies instead of simple nested radicals, leading to an alternative rapidly converging series with a rate of $(1/3)^{2k(N+1)}$. For higher prime numbers, similar formulations and convergence behaviors can be inferred.

4.1 Chebyshev polynomials of the second kind

We demonstrate an angle-chopping strategy for an arbitrary integer p by making use of Chebyshev polynomials of the second kind, $U_n(x)$. In physics, Chebyshev polynomials are orthogonal polynomials that serve as basis functions for the Fourier expansion of waves generated by a vibrating string (see, for example, Jung et al. [24]) and appear in various other physics applications, including charge distribution on a conducting disk or thin wire in electrostatics, approximations of wavefunctions and eigenvalues in numerical quantum mechanics, and rapid numerical approximations for Lorentz factor-dependent functions in high-energy physics. They have the geometric implication that

$$U_n(\cos\theta) = \frac{\sin[(n+1)\theta]}{\sin\theta}. \tag{41}$$

An elementary way to verify the factorization formula in Eq. (41) is to apply the addition formula for the sinusoidal functions recursively:

$$\cos[(n+1)\theta] = \cos n\theta \cos\theta - \sin n\theta \sin\theta, \tag{42a}$$

$$\sin[(n+1)\theta] = \sin n\theta \cos\theta + \cos n\theta \sin\theta. \tag{42b}$$

One can also employ de Moivre's theorem to confirm that $\sin[(n+1)\theta]$ can always be factorized into the product of $\sin\theta$ and a polynomial in $\cos\theta$:

$$\begin{aligned}\sin[(n+1)\theta] &= \mathfrak{Im}\left[(\cos\theta + i\sin\theta)^{n+1}\right]\\ &= \sum_{k=0}^{\lfloor\frac{1}{2}(n+1)\rfloor} (-1)^k \frac{(n+1)!}{(2k+1)!(n-2k)!} \cos^{n-2k}\theta \sin^{2k+1}\theta\\ &= \sin\theta \times \sum_{k=0}^{\lfloor\frac{1}{2}(n+1)\rfloor} (-1)^k \frac{(n+1)!}{(2k+1)!(n-2k)!}\\ &\quad\times \cos^{n-2k}\theta(1-\cos^2\theta)^k,\end{aligned} \tag{43}$$

where $\lfloor x \rfloor$ is the floor function, which gives the greatest integer less than or equal to x:

$$\lfloor x \rfloor = m, \quad \text{if and only if} \quad m \le x < m+1, \quad m \in \mathbb{Z}. \tag{44}$$

According to Eqs. (41) and (43), the Chebyshev polynomials of the second kind $U_n(\cos\theta)$ is determined as

$$\begin{aligned}U_n(\cos\theta) = &\sum_{k=0}^{\lfloor\frac{1}{2}(n+1)\rfloor} (-1)^k \frac{(n+1)!}{(2k+1)!(n-2k)!}\\ &\times \cos^{n-2k}\theta(1-\cos^2\theta)^k.\end{aligned} \tag{45}$$

One could also extract the polynomial by expanding the corresponding generating function:

$$\frac{1}{1-2xt+t^2} = \sum_{t=0}^{\infty} U_n(x)t^n. \tag{46}$$

The first ten entries of the polynomials are given by

$$U_0(x) = 1, \tag{47a}$$

$$U_1(x) = 2x, \tag{47b}$$

$$U_2(x) = 4x^2 - 1, \tag{47c}$$

$$U_3(x) = 8x^3 - 4x, \tag{47d}$$

$$U_4(x) = 16x^4 - 12x^2 + 1, \tag{47e}$$

$$U_5(x) = 32x^5 - 32x^3 + 6x, \tag{47f}$$

$$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1, \tag{47g}$$

$$U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x, \tag{47h}$$

$$U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1, \tag{47i}$$

$$U_9(x) = 512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x. \tag{47j}$$

4.2 Special values of sine

The identity in Eq. (41) is particularly useful for expressing the sine of an angle θ in terms of the sine and cosine of the fractional angle $\theta/(n+1)$:

$$U_n(\cos\theta)\sin\theta = U_n\left[\sqrt{1-\sin^2\theta}\right]\sin\theta = \sin[(n+1)\theta]. \tag{48}$$

Since $(n+1)\theta \in [0, \frac{\pi}{2}]$, Eq. (48) has only a single solution for θ.

For an integer $p = 2^k p'$, where p' is a prime number greater than or equal to 2, we can find $\sin\frac{\pi}{2^k p'}$ by solving the following equation:

$$\begin{aligned}\sin\frac{\pi}{2^k} &= \frac{1}{2}\underbrace{\sqrt{2-\sqrt{2+\cdots\sqrt{2+\sqrt{2}}}}}_{k-1 \text{ roots}}\\ &= U_{p'-1}\left[\sqrt{1-\sin^2\frac{\pi}{2^k p'}}\right]\sin\frac{\pi}{2^k p'}, \qquad k \ge 0.\end{aligned} \tag{49}$$

For $p' = 2$, the equation simplifies to one whose analytic solution is given by the half-angle formula in Eq. (31e). Thus, the half-angle formula represents a special case of the fractional-angle method in Eq. (49). For higher prime numbers, $U_{p'-1}$ in Eq. (49) becomes a cubic or higher order polynomial, and analytic solutions exist only for a few specific cases. In general, a numerical approach is required, which is discussed in the next subsection.

To obtain $\sin\frac{\pi}{2^k p'^{\ell+1}}$, one can repeatedly solve the following equation:

$$\sin\frac{\pi}{2^k p'^{\ell}} = U_{p'-1}\left[\sqrt{1-\sin^2\frac{\pi}{2^k p'^{\ell+1}}}\right]\sin\frac{\pi}{2^k p'^{\ell+1}}, \qquad k \ge 0, \tag{50}$$

where ℓ is an integer starting from 0. In the same way, we can find $\sin\frac{\pi}{p}$ for an arbitrary integer p in Eq. (37).

Although Eqs. (49) and (50) do not have analytic solutions except for $p = 2^{k+1}$, in many cases, an integer of form $p = 2p'$ has analytic result for $\sin\frac{\pi}{2p'}$. The leading entries of Eq. (49) for $k = 0$ are

$$x = \sin\frac{\pi}{4}: \quad 1 = 2x\sqrt{1-x^2}, \tag{51a}$$

$$x = \sin\frac{\pi}{6}: \quad 1 = 3x - 4x^3, \tag{51b}$$

$$x = \sin\frac{\pi}{8}: \quad 1 = 4x(1-2x^2)\sqrt{1-x^2}, \tag{51c}$$

$$x = \sin\frac{\pi}{10}: \quad 1 = x(5 - 20x^2 + 16x^4), \tag{51d}$$

$$x = \sin\frac{\pi}{12}: \quad 1 = 2x(3 - 16x^2 + 16x^4)\sqrt{1-x^2}, \tag{51e}$$

$$x = \sin\frac{\pi}{14}: \quad 1 = 7x - 56x^3 + 112x^5 - 64x^7, \tag{51f}$$

$$x = \sin\frac{\pi}{16}: \quad 1 = 8x(1 - 10x^2 + 24x^4 - 16x^6)\sqrt{1-x^2}, \tag{51g}$$

$$x = \sin\frac{\pi}{18}: \quad 1 = x(9 - 120x^2 + 432x^4 - 576x^6 + 256x^8), \tag{51h}$$

$$x = \sin\frac{\pi}{20}: \quad 1 = 2x(5 - 80x^2 + 336x^4 - 512x^6 + 256x^8)\sqrt{1-x^2}, \tag{51i}$$

$$x = \sin\frac{\pi}{22}: \quad 1 = 11x - 220x^3 + 1232x^5 - 2816x^7 + 2816x^9 - 1024x^{11}, \tag{51j}$$

where, for completeness, non-prime numbers $p' \in [2, 11]$ are included. Except for $\sin\frac{\pi}{14}$, $\sin\frac{\pi}{18}$, and $\sin\frac{\pi}{22}$, the remaining equations in Eq. (51) are exactly solvable:

$$\sin\frac{\pi}{2} = 1, \tag{52a}$$

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = 0.70710678118654752440\cdots, \tag{52b}$$

$$\sin\frac{\pi}{6} = \frac{1}{2} = 0.5, \tag{52c}$$

$$\sin\frac{\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}} = 0.38268343236508977173\cdots, \tag{52d}$$

$$\sin\frac{\pi}{10} = \frac{1}{4}\left(\sqrt{5}-1\right) = 0.30901699437494742410, \tag{52e}$$

$$\sin\frac{\pi}{12} = \frac{1}{2}\sqrt{2-\sqrt{3}} = 0.25881904510252076235\cdots, \tag{52f}$$

$$\sin\frac{\pi}{16} = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}} = 0.19509032201612826785\cdots, \tag{52g}$$

$$\sin\frac{\pi}{20} = \frac{1}{2}\sqrt{\frac{1}{2}\left[4-\sqrt{2\left(5+\sqrt{5}\right)}\right]} = 0.15643446504023086901\cdots. \tag{52h}$$

The results in Eq. (52) illustrate that the special values of the sine function derive from $\sin\frac{\pi}{2} = 1$. This provides a unique elementary demonstration of the usefulness of Chebyshev polynomials of the second kind in verifying trigonometric identities.

4.3 Power counting and convergence

Unlike the case of $\sin\frac{\pi}{2^{k+1}}$, which can be expressed analytically in Eq. (31e) $\sin\frac{\pi}{2^k p'}$ with $p' > 2$ in Eq. (49) should be solved numerically to the desired accuracy. Fortunately, the initial positioning of an approximate trial solution for $\sin\frac{\pi}{2^k p'}$ is relatively straightforward, making it easier to meet the accuracy requirements. For better convergence, the prime number p' can be chosen to be arbitrarily large, provided that the internal precision of a numerical package allows it.

However, this leads to a substantial increase in the number of terms in the Chebyshev polynomials, which grows rapidly as p' increases. By applying the power-counting method described in Eq. (39), we can safely neglect insignificant contributions in the polynomials, such as higher order terms in power-counting parameter that are negligible to the desired accuracy. This will be discussed in greater detail shortly. One can recursively apply the fractional-angle method and the power-counting method by solving Eq. (50) multiple times to obtain $\sin\frac{\pi}{2^k p'^{\ell+1}}$. In the same way, the angle can also be arbitrarily chopped, yielding $\sin\frac{\pi}{p}$. Among various ways to divide the angle, we focus on the following case to simplify our discussion:

$$p = 2^{k+1}p'^{\ell+1}. \tag{53}$$

In order to discuss the truncation of the Chebyshev polynomials using the power-counting method, let us expand the right side of Eq. (49) and truncate the terms beyond the nth order

$$\epsilon = c_0 x + c_1 x^3 + c_2 x^5 + \cdots + c_n x^{2n+1}, \tag{54}$$

where we introduce the simplified notation $\epsilon = \sin \frac{\pi}{2^{k+1}}$ and $x = \sin \frac{\pi}{2^{k+1} p'}$, which is the power-counting parameter in this equation and scales like ϵ/p'. The coefficient c_n is obtained by expanding Chebyshev polynomials and for $n = 0$, it is $c_0 = p'$. Let us denote x_n as the solution to the truncated equation (54). Then, x_{n+1} is the solution to the equation of one-order higher and can be expressed as $x_{n+1} = x_n + \delta_n$, where δ_n is a truncation error scaling like $(\epsilon/p')^{2n+3}$. Inserting the relation into the equation for x_{n+1} and keeping dominant contributions of order $(\epsilon/p')^{2n+3}$, one finds a linear equation $c_0 \delta_n + c_{n+1} x_{n+1}^{2n+3} = 0$, which gives $\delta_n = -\frac{c_{n+1}}{p'^{2n+4}} \epsilon^{2n+3}$. By replacing ϵ with $\epsilon_\ell = \sin \frac{\pi}{2^{k+1} p'^{\ell}}$ and by solving the equation $\ell + 1$ times from $\ell = 0$, we obtain the solution for $\sin \frac{\pi}{2^{k+1} p'^{\ell+1}}$. According to the expression for δ_n, the error is dominated by the contribution at $\ell = 0$: $\delta_n \approx -\frac{c_{n+1}}{p'^{2n+4}} \epsilon_0^{2n+3}$. Thus, the truncation error in $S_N^{(k,\ell)}$ scales as follows:

$$\Delta^{\text{trunc}} S^{(k,\ell)} \approx 2^{k+1} p'^{\ell+1} |\delta_n| \sim |c_{n+1}| \frac{p'^{\ell-2n-3}}{4^{k(n+1)}}. \tag{55}$$

An optimal truncation, which minimizes the number of terms, requires the truncation error to be comparable to the summation errors that follow the scaling in Eq. (40). By equating (40) and (55) and solving for n, we obtain

$$n \approx N + \frac{(N+1)\ell \,\ln p' + \ln \dfrac{|c_{n+1}|}{p'}}{\ln(2^k p')}. \tag{56}$$

Note that the relation (56) depends on the coefficient c_{n+1}, as well as the parameters p', k, ℓ, and N. If $\ell = 0$, the relation simplifies and n increases at the same rate as N. However, for a sufficiently large ℓ, the rate of increase in n grows larger than that of N. For a sufficiently large k, n approaches N because the second term on the right-hand side of Eq. (56) is proportional to $1/k$. In contrast, if k is small, the second term becomes significant, making n sensitive to the values of other parameters such as ℓ, p', and c_{n+1} as well as N. The calculation of n is illustrated in the following example.

To illustrate the efficiency of numerical computation using power counting, we will first retain the higher-order terms by choosing a moderately large p'. For this demonstration, we choose $p' = 101$ in Eq. (53). Then, Eq. (38) becomes

$$S_N^{(k,\ell)} \equiv 2^{k+1} \cdot 101^{\ell+1} \sum_{n=0}^{N} \times \left[\sin \frac{\pi}{2^{k+1} \cdot 101^{\ell+1}}\right]^{2n+1} \frac{\Gamma[2n+1]}{2^{2n}(2n+1)\Gamma^2[n+1]}. \tag{57}$$

The value of $\sin \frac{\pi}{2^{k+1} \cdot 101^{\ell+1}}$ can be computed numerically by making use of the following equation:

$$\sin \frac{\pi}{2^{k+1} \cdot 101^{\ell}} = U_{100}\left[\sqrt{1 - \sin^2 \frac{\pi}{2^{k+1} \cdot 101^{\ell+1}}}\right] \times \sin \frac{\pi}{2^{k+1} \cdot 101^{\ell+1}}. \tag{58}$$

By solving Eq. (58) $\ell + 1$ times starting from $\ell = 0$ and increasing it by 1, we can determine the value of $\sin \frac{\pi}{2^{k+1} \cdot 101^{\ell+1}}$.

The expression in Eq. (57) approaches π arbitrarily closely as N increases. The rate of the convergence is enhanced as k or ℓ or both increase. We define the error vector $\Delta \boldsymbol{S}^{(k,\ell)}$ as

$$\Delta \boldsymbol{S}^{(k,\ell)} = [\pi - S_0^{(k,\ell)}, \pi - S_1^{(k,\ell)}, \pi - S_2^{(k,\ell)}, \pi - S_3^{(k,\ell)}, \pi - S_4^{(k,\ell)}, \cdots], \tag{59}$$

where $S_N^{(k,\ell)}$ is defined in Eq. (57). For $k = 100$, we find that

$$\Delta \boldsymbol{S}^{(100,0)} = [7.9 \times 10^{-65}, 5.3 \times 10^{-129}, 4.8 \times 10^{-193}, 4.9 \times 10^{-257}, 5.4 \times 10^{-321}, \cdots], \tag{60a}$$

$$\Delta \boldsymbol{S}^{(100,1)} = [7.7 \times 10^{-69}, 5.1 \times 10^{-137}, 4.5 \times 10^{-205}, 4.5 \times 10^{-273}, 4.9 \times 10^{-341}, \cdots], \tag{60b}$$

$$\Delta \boldsymbol{S}^{(100,2)} = [7.6 \times 10^{-73}, 5.1 \times 10^{-145}, 4.2 \times 10^{-217}, 4.2 \times 10^{-289}, 4.5 \times 10^{-361}, \cdots], \tag{60c}$$

$$\Delta \boldsymbol{S}^{(100,3)} = [7.4 \times 10^{-77}, 4.7 \times 10^{-153}, 4.0 \times 10^{-229}, 3.9 \times 10^{-305}, 4.0 \times 10^{-381}, \cdots], \tag{60d}$$

$$\Delta \boldsymbol{S}^{(100,4)} = [7.3 \times 10^{-81}, 4.6 \times 10^{-161}, 3.8 \times 10^{-241}, 3.6 \times 10^{-321}, 3.6 \times 10^{-401}, \cdots], \tag{60e}$$

$$\Delta \boldsymbol{S}^{(100,5)} = [7.1 \times 10^{-85}, 4.4 \times 10^{-169}, 3.5 \times 10^{-253}, 3.3 \times 10^{-337}, 3.3 \times 10^{-421}, \cdots], \tag{60f}$$

$$\Delta \boldsymbol{S}^{(100,6)} = [7.0 \times 10^{-89}, 4.2 \times 10^{-177}, 3.3 \times 10^{-265}, 3.0 \times 10^{-353}, 3.0 \times 10^{-441}, \cdots], \tag{60g}$$

$$\Delta S^{(100,7)} = [6.9\times10^{-93}, 4.0\times10^{-185}, 3.1\times10^{-277}, 2.8\times10^{-369}, 2.7\times10^{-461}, \cdots], \tag{60h}$$

$$\Delta S^{(100,8)} = [6.7\times10^{-97}, 3.9\times10^{-193}, 3.0\times10^{-289}, 2.6\times10^{-385}, 2.4\times10^{-481}, \cdots], \tag{60i}$$

$$\Delta S^{(100,9)} = [6.6\times10^{-101}, 3.7\times10^{-201}, 2.8\times10^{-301}, 2.4\times10^{-401}, 2.2\times10^{-501}, \cdots], \tag{60j}$$

$$\Delta S^{(100,10)} = [6.5\times10^{-105}, 3.6\times10^{-209}, 2.6\times10^{-313}, 2.2\times10^{-417}, 2.0\times10^{-521}, \cdots]. \tag{60k}$$

The computed errors are consistent with the scaling prediction in Eq. (40). Note that every component is positive definite because the series in Eq. (57) is monotonically increasing. The result in Eq. (60) demonstrates that the accuracy increases systematically as the parameters ℓ and N increase.

According to Eq. (56), we can find the minimal number of terms n needed in the Chebyshev polynomials for $k = 100$ and $p' = 101$. For the coefficient c_n, we take an empirical scaling obtained from a fit: $c_n \sim 10^{2n}$. Then, Eq. (56) simplifies as $n \approx [N + (N+1)\ell\alpha]/(1-\alpha)$, where $\alpha = \ln p' / \ln(2^k p') \approx 0.06$. For the case $\ell = 0$, this yields $n \approx 1.07N$, and we can choose n to be the integer greater than this value, $n = N + 1$. For $\ell = 10$, $n \approx 1.7N + 0.7$, resulting in corresponding integers n is $n = \{1, 3, 5, 6, 8\}$ for $N = \{0, 1, 2, 3, 4\}$, respectively. Note that these values of n are a small fraction of 50, the total number of polynomials in $U_{100}(x)$. We confirmed that the errors computed using the equations with n terms agree with those errors in Eqs. (60a) and (60k).

To our best knowledge, the expressions in Eqs. (32) and (57) are new. Note that $p' = 101$ in Eq. (57) is a simple choice for demonstration purposes; however, the large integer p' can be any sufficiently large value.

5 Conclusion

Starting from the teaching strategy to solve the equation of motion for the simple harmonic oscillator through the process of integration, we have investigated the mathematical properties of the integral representation for the arcsine function, which Newton used for computing π. The power-series expansion of the integral was carried out by making use of the binomial theorem generalized for non-integer powers. Although the original series for 2 arcsin 1 in Eq. (24) converges very slowly to π, the novel power series of $\sin \frac{\pi}{2^{k+1}}$ given in Eq. (32), expressed as nested radicals in Eq. (31e) by applying the half-angle formulas, shows boosting the convergence as k increases, as demonstrated in Eq. (36). However, there is a trade-off in boosting convergence. While the original series consists of sums of rational numbers with infinite precision, the term $\sin \frac{\pi}{2^{k+1}}$ turns the expression into a sum of irrational numbers that must be numerically computed to the desired accuracy and costly. Furthermore, our fractional-angle method generalizes the half-angle method from a factor of 1/2 to $1/p'$ with a prime number p', thereby enhancing convergence significantly for a large value of p'. This is achieved through the elementary property of Chebyshev polynomials of the second kind, as given in Eq. (41). The resulting series in Eq. (38) is expressed in powers of $\sin \frac{\pi}{p}$, where p is an arbitrary integer written as a product of prime factors, with its value computed numerically using Eq. (50). Systematic power-counting rules, as provided in Eqs. (33) and (39), allow for the estimation of errors in the series. This enables the safe omission of numerically suppressed contributions, improving computational efficiency without affecting the desired level of accuracy. Nevertheless, our strategy is unlikely to outperform well-established methods and may not be directly comparable to the state-of-the-art algorithms used by advanced π hunters. However, we think that its true strength lies in its pedagogical value, as it provides advanced freshmen and physics majors with a concrete way to engage with and reinforce key concepts and techniques learned in physics courses through π computation.

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Declarations

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