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On some projective unitary qutrit gates

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Abstract

As part of a protocol, we braid in a certain way six anyons of topological charges 222211 in the Kauffman-Jones version of $SU(2)$ Chern-Simons theory at level 4. The gate we obtain is a braid for the usual qutrit 2222 but with respect to a different basis than the usual basis. With respect to that basis, the Freedman group of J. Phys. A: Math. Theor. 47, 285203 (2014) is identical to the D-group $D(18, 1, 1; 2, 1, 1)$ from the Blichfeldt classification. We provide a physical realization of each Blichfeldt generator of the $SU(3)$ finite subgroup $D(18, 1, 1; 2, 1, 1)$. The link of this group to elementary particle physics was already well-known by the physicists. This paper uncovers its realization in quantum physics. Inspired by these new techniques for the qutrit, we are able to make new ancillas, namely $\frac{1}{\sqrt{2}}(|1> +|3>)$ and $\frac{1}{\sqrt{2}}(|1> -|3>)$, for the qubit 1221.

Keywords SU(3) finite subgroups · Topological quantum computation · $SU(2)_4$ anyons

1 Setting

Recently, there has been some interest in finding qutrit gates which are universal for quantum computation. When the group of qutrit gates in the projective unitaries $PU(3)$ acts irreducibly on $\mathbb{C}^3$, a result of [2] provides a sufficient condition for an $SU(3)$-subgroup of single projective unitary qutrit gates to form a dense set of $PU(3)$. This condition finds its origins in a 2002 work [10] by Michael Freedman, Alexei Kitaev and Jacob Lurie. An older result from Jean-Luc Brylinski and Ranee Brylinski [7] implies that such a dense set of 1-qutrit gates, together with a 2-qutrit entangling gate, is universal for quantum computation. Therefore, there have been some attempts and hopes, starting from a finite group of projective unitary qutrit gates obtained by anyonic braiding, to add an extra projective unitary gate which would this time be obtained by braiding and interferometric measurement and would make the group become infinite. Such a group would then satisfy to the condition mentioned above for density. The

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work of the present paper finds its origin in trying to solve the problem of leakage when doing measurements on some specific protocols aimed at finding gates beyond braiding. Since then, the problem of finding an anyonic gate beyond braiding, which would make the finite $SU(3)$ subgroup obtained by braiding only become infinite and dense in $PU(3)$, got solved in [19] for $SU(2)_4$ anyons, where the author realizes a 1-qutrit irrational phase gate by braiding and measuring $SU(2)_4$ anyons. This led to universal quantum computation on the 1-qutrits with $SU(2)_4$ anyons. Additionally, the same author realizes a 2-qutrit entangling gate in [18], thus proving that quantum computation can be made universal with $SU(2)_4$ anyons. Universality for the qutrit was also proven independently by Shawn Cui and Zhenghan Wang in [8], with respect to a different encoding. While motivated by this problem of universality of topological quantum computation, we found out how to realize by means of quantum physics, a finite $SU(3)$ subgroup of interest to the elementary particle physicists, namely the D-group $D(18, 1, 1; 2, 1, 1)$ of order 648. This group plays an important role in neutrino physics [16]. The experimental observation that neutrinos have nonzero mass has opened up a window to physics beyond the Standard Model. Advances in the setup of several neutrino experiments revealed that the mixing in the lepton sector features two large and one small angle. This was explained most plausibly by the existence of an underlying discrete symmetry. Having chosen a preferred non-Abelian finite symmetry group, the three generations of fermions as well as the Higgs sector particles are assigned to irreducible representations of the group. The theory can then be formulated by writing down all the couplings which are allowed by the symmetry group. Assuming the Majorana nature of neutrinos, and after scan of over one million groups, the authors of [14] found out that only three finite discrete groups can yield the neutrino mixing angles allowed by the experimental constraints. Right after the tri-bimaximal mixing got ruled out, $D(18, 1, 1; 2, 1, 1)$ became one of the three groups to predict a different type of leptonic mixing pattern, that is a new starting point about which models could be built. Our work offers a concrete realization of this group; it builds upon [1] and [17].

In [1], we study a finite subgroup of $SU(3)$ arising from anyonic braiding. The anyonic system we consider is the Kauffman-Jones version of $SU(2)$ Chern-Simons theory at level 4. It is also the anyonic system that is considered in the present paper. For the theory of braiding anyons, we refer the reader to [15, 18, 19, 25], as a non-exhaustive list. This group has order 162 and is later enlarged to a group of order 648, namely the Freedman group, by a Freedman Fusion Operation (FFO for future reference) due to Michael Freedman; see [17]. Both groups, the one of order 162 and its extension of order 648, are isomorphic to D-groups in the 1916 classification of finite $SU(3)$-subgroups by Blichfeldt [22], namely to $D(9, 1, 1; 2, 1, 1)$ and to $D(18, 1, 1; 2, 1, 1)$, respectively. Both of these groups serve as models for the mixing patterns of the three generations of quarks and leptons. This connection between qutrit anyonic encoding and elementary particle physics could offer new insights into the structure of matter. For further context, see Tables 3 and 6 of [24]. In [17], it is shown further that the D-group $D(18, 1, 1; 2, 1, 1)$ is the Freedman group, with respect to a different basis, that is both groups are conjugate, instead of simply being isomorphic.

As it is, the Blichfeldt classification, which got later augmented by two new finite $SU(3)$-subgroups, is not a classification up to conjugacy. It is not even a classification up to isomorphism. Some groups of this classification are even identical. Hence, the ongoing attempts to better understand these groups and their respective structures. Historically, the group $D(9, 1, 1; 2, 1, 1)$ was mentioned in [21] as a counter-example that every group of type (D) can be interpreted as an irreducible representation of a group of type $\Delta(6n^2)$. The series $\Delta(6n^2)$ is a subseries of the series (D), and the structure of these Δ-groups got broadly investigated in [9]. Classically and originally, the group $D(18, 1, 1; 2, 1, 1)$ is defined by three matrix generators which first appeared in the 1916 book by Blichfeldt [22] as part of the three generic generators for the groups $D(n, a, b; d, r, s)$ from the series (D). The latter series was more recently extensively studied by Walter Grimus and Patrick Ludl in [13].

In [17], a detailed study of the structures of $D(18, 1, 1; 2, 1, 1)$, as defined by its Blichfeldt generators, and of the Freedman group $Fr(162 \times 4)$ (where "Fr" stands for Freedman), as defined by its generators arising from single braids and FFO, is jointly performed, which includes the study of their respective 3-Sylow subgroups. The latter study allows to exhibit an explicit isomorphism between both groups, instead of simply providing a presentation for each group. It further allows to prove that both groups are conjugate under an orthogonal matrix, which we provide. Since the new basis can be realized physically with the anyons, from the old and standard basis, we thus obtain a physical realization of the D-group. Our approach here is different and is almost reversed. Namely, after noticing the special role played by some of the qutrits when applying to them the above anyonic operations, we decide to work with respect to the basis which they form. We further exploit the nice properties of these qutrits with respect to some anyonic protocol which uses an ancilla pair and braids. This protocol is studied in an attempt to construct a new gate, which would enlarge the Freedman group. The quantum gate arising from this new protocol, together with the generators obtained from the single braids and FFO with respect to the new basis, generates a group, say $\tilde{\mathcal{G}}$, of the same order as $D(18, 1, 1; 2, 1, 1)$ (in fact these two groups are isomorphic and this can be seen by computing their GAP ID [6] from their respective presentations). Moreover, we find out that each Blichfeldt generator can be expressed from the generators of $\tilde{\mathcal{G}}$. Then, the two groups must be equal. We obtain a physical realization for each Blichfeldt generator. Because the first three generators of $\tilde{\mathcal{G}}$ are those from the Freedman group, but with respect to a different basis, the order of the subgroup generated by these generators equals the order of the Freedman group. By [17], the Freedman group has the same order as the D-group. Then, the last generator of $\tilde{\mathcal{G}}$ is redundant. We now deduce that $D(18, 1, 1; 2, 1, 1)$ is the Freedman group with respect to two different bases, namely the one found in [17] using some group theoretic machinery and the one of the present paper. This implies surprisingly that by swapping the first and third rows and the first and third columns of a Freedman matrix, we obtain another Freedman matrix.

In summary, our paper introduces a new set of four generators for the D-group $D(18, 1, 1; 2, 1, 1)$, but the group is only generated by three of them. These generators all arise from anyonic braiding and FFO. This provides an actual physical realization of this D-group. Additionally, we find a physical realization of the three original Blichfeldt generators. Both the Freedman group of order 648 and our novel physical realization of $D(18, 1, 1; 2, 1, 1)$ contain the center Z_3 of $SU(3)$; hence, we note that the number of projective unitary qutrit gates available remains the same.

In more details now, we consider the qutrit 2222 and a pair of 1's, do some specific braids and fail to obtain a new gate. Of course the number of protocols available is extremely large, so this failure does not imply that by choosing such an ancilla we would not ever be able to obtain an interesting gate by braiding only or by braiding and measurement. Two fundamental facts are enlightened from this protocol. First, when doing a full twist σ_2 on four anyons 2211, it results in swapping the topological charges 0 and 2. Second, when doing a single braid σ_2 on four anyons 2211, we obtain a qubit 2121 with the same proportion of $|1>$ and $|3>$. Since doing σ_1 braids only introduces phases, we can thus make a qubit 1221 with equal norms of $|1>$ and $|3>$. This was unknown fact in [2], where in some protocols using braiding and interferometric measurement on the qubit 1221, we were missing such ancillas which play a critical role for the no-leakage condition. In topological quantum computation, we say that "leakage occurs" when a measurement induces a loss of unitarity. Uncovering how to realize such a balanced ancilla for the qubit 1221 at level 4 constitutes a major progress for quantum computation on the qubit 1221 with $SU(2)_4$ anyons.

The paper is structured as follows. In § 2, we state our main result regarding a physical realization of $D(18, 1, 1; 2, 1, 1)$. Then, in § 3, we explain in detail the protocol used in order to realize a certain quantum gate of interest. The following § 4 is concerned with proving the group theory result announced in § 2. Moreover, we provide a physical realization for the three original Blichfeldt generators of $D(18, 1, 1; 2, 1, 1)$. This is the purpose of forthcoming Theorem 3. Finally, in § 5, we discuss around new ancillas for the qubit 1221.

2 Result

2.1 The anyonic system

In what follows, we will work in the Kauffman-Jones version of $SU(2)$ Chern-Simons theory at level 4. The value of the Kauffman constant is, with the same notation as in [15]:

$$A = i\, e^{-i\,\frac{\pi}{12}}.$$

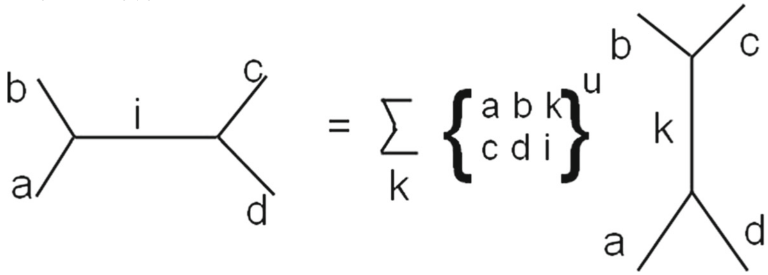
In this theory, there are five quasi-particle types, called anyons, with respective topological charges 0, 1, 2, 3 and 4. The anyons obey fusion rules that are governed by, where a, b and c denote the topological charges of the quasi-particles:

$$\begin{cases} a \bigotimes b = \sum_{c=0}^{4} c, \\ a \leq b + c,\ b \leq c + a,\ c \leq a + b, \\ a + b + c \leq 8, \\ a + b + c \text{ is even.} \end{cases}$$

For basic facts about recoupling theory, we refer the reader to the excellent book of Kauffman and Lins [15]. In order to make the theory unitary, we use unitary theta symbols and unitary 6j-symbols, like defined in Appendix of [17], instead of the original symbols of [15]. The minor modifications resulting from studying $SU(2)_4$ anyons, instead of the Kauffman-Jones version of $SU(2)_4$, get explained in [20].

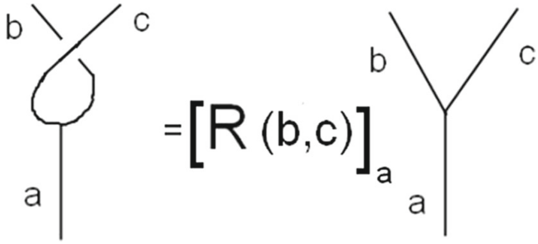
The main four moves which we use throughout the paper are summarized below.

- The "F-move"



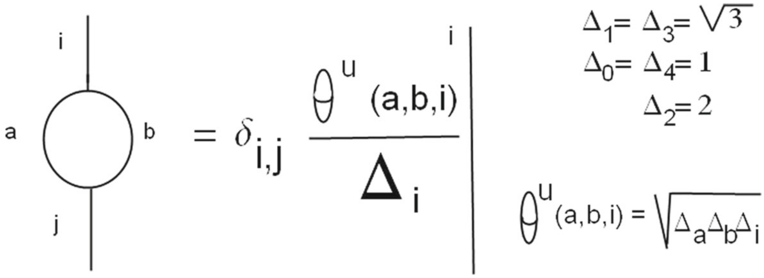
The brackets are called unitary $6j$-symbols.

- The "R-move"



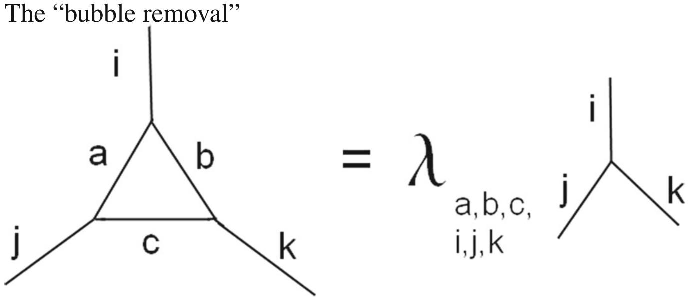
For a definition of these coefficients, see for instance [18].

- The "theta move"



The delta's are the quantum dimensions, and the θ^u are the unitary theta symbols.

- The "bubble removal"



The coefficient is obtained by doing an F-move, followed by a theta move. Note that i, j and k must obey the fusion rules, else the evaluation is zero.

The operations we do consist in braiding and measuring the anyons, either by fusion or by interferometric measurements; see [3–5, 11]. An important aspect during a topological quantum computation is to use ancillas. This idea originated in the paper by Carlos Mochon [23]. Ancillas are auxiliary anyons that are used during the computation in order to achieve special quantum gates that would have been impossible to achieve by using the anyons from the input alone. Ancillas thus interact with the input by braiding and or measurement and later vanish at the end of the computation. In this paper, we use an ancilla pair of anyons in an attempt to make a new gate by braiding only. In all generality, ancillas can be qubit ancillas or qutrit ancillas and they can be realized by a series of braids and measurements. Ancillas are much easier to make than quantum gates since in case of a bad measurement outcome, we can simply dispose of the ancilla and try again with fresh anyons, whereas for a quantum gate, having a bad measurement outcome supposes having a recovery procedure. It could for instance be a way of undoing this measurement or a way of getting back to a previous stage in the protocol, among other possibilities. Also, a measurement, by being destructive in nature, does not necessarily preserve unitarity. Making a quantum gate that uses measurements in topological quantum computation is a very difficult, yet not impossible, problem. Thus our attempt using only braids here.

2.2 Main Result

We state below our main result. In the following theorem and throughout the remainder of the paper, the cyclic group of order n will be denoted by C_n.

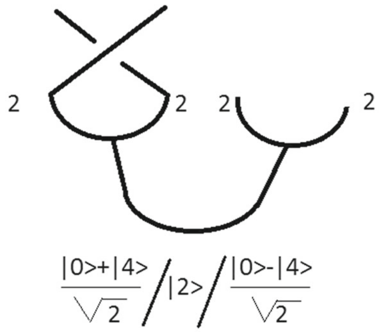
Theorem 1 *The group $\widetilde{\mathcal{G}}$ generated by the four matrices*

$$\widetilde{G_1} = \begin{pmatrix} & & e^{\frac{7i\pi}{9}} \\ & -e^{\frac{4i\pi}{9}} & \\ e^{\frac{7i\pi}{9}} & & \end{pmatrix}, \; \widetilde{G_2} = \begin{pmatrix} & e^{\frac{7i\pi}{9}} & \\ e^{\frac{7i\pi}{9}} & & \\ & & -e^{\frac{4i\pi}{9}} \end{pmatrix},$$

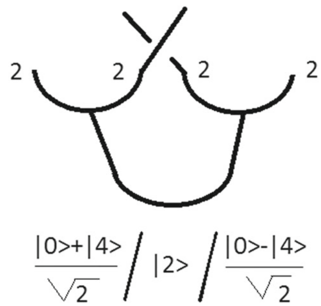
$$\widetilde{FUM} = \begin{pmatrix} -e^{\frac{2i\pi}{3}} & & \\ & -e^{\frac{2i\pi}{3}} & \\ & & e^{\frac{2i\pi}{3}} \end{pmatrix}, \; N = \begin{pmatrix} -e^{-\frac{i\pi}{9}} & & \\ & -e^{-\frac{i\pi}{9}} & \\ & & e^{i\frac{2\pi}{9}} \end{pmatrix}$$

is a finite subgroup of $SU(3)$ of order 648. It is isomorphic to a semi-direct product $(C_6 \times C_{18}) \rtimes S_3$. The generators above are up to phase obtained by the following unitary operations in the Kauffman-Jones version of $SU(2)$ Chern-Simons theory at level 4.

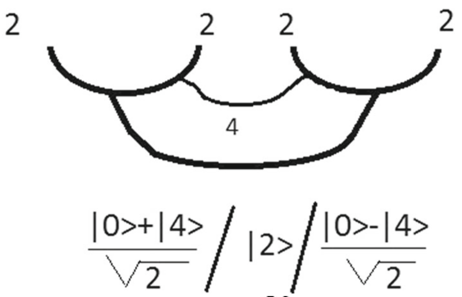
$\widetilde{G1}$



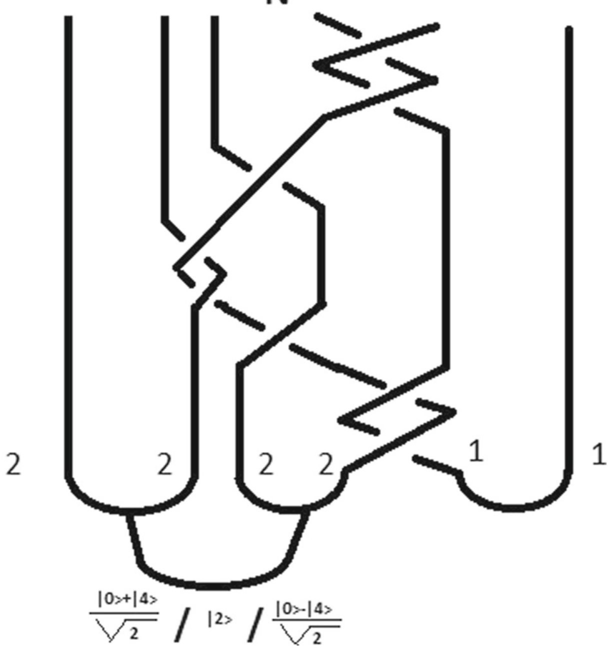
$\widetilde{G2}$



$\widetilde{FUM}$



N



The generator N belongs to the subgroup generated by $\overset{\sim}{G_2}$. Moreover, we have:

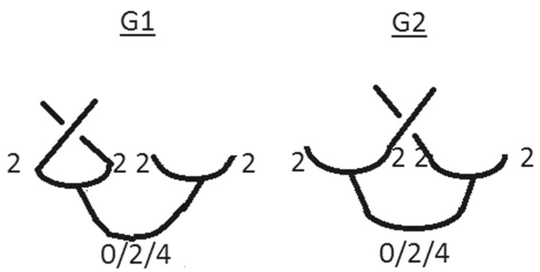
$$\overset{\sim}{\mathcal{G}} =< \overset{\sim}{G_1}, \overset{\sim}{G_2}, F\tilde{U}M >= D(18, 1, 1; 2, 1, 1).$$

In § 3 and § 4, we prove Theorem 1, as well as some additional results of interest.

3 Protocol

A starting point is braids on four anyons of topological charge 2 in the Jones-Kauffman version of $SU(2)$ Chern-Simons theory at level 4. We recall below the matrices G_2 for a σ_2-braid and G_1 for a σ_1-braid, also commonly called R-matrix, taken from [1]. All the matrices are defined in $SU(3)$, that is they are defined up to phase.

$$G_1 = \begin{pmatrix} e^{\frac{7i\pi}{9}} & & \\ & -e^{\frac{4i\pi}{9}} & \\ & & -e^{\frac{7i\pi}{9}} \end{pmatrix}, \; G_2 = \begin{pmatrix} -\frac{1}{2}e^{\frac{4i\pi}{9}} & \frac{e^{\frac{7i\pi}{9}}}{\sqrt{2}} & \frac{1}{2}e^{\frac{4i\pi}{9}} \\ \frac{e^{\frac{7i\pi}{9}}}{\sqrt{2}} & 0 & \frac{e^{\frac{7i\pi}{9}}}{\sqrt{2}} \\ \frac{1}{2}e^{\frac{4i\pi}{9}} & \frac{e^{\frac{7i\pi}{9}}}{\sqrt{2}} & -\frac{1}{2}e^{\frac{4i\pi}{9}} \end{pmatrix}$$



On the matrices above, we notice the special roles played by the qutrits $|0>$ and $|4>$ on one hand and $|2>$ on the other hand. Explicitly, braiding anyons 1 and 2 maps the qutrit $|0>$ to itself and the qutrit $|4>$ to the qutrit $-|4>$, up to a common phase. Notice further that

$$G_2(|2>) = e^{\frac{7i\pi}{9}} \; \frac{|0> + |4>}{\sqrt{2}}$$

and

$$G_2\left(\frac{|0> + |4>}{\sqrt{2}}\right) = e^{\frac{7i\pi}{9}} \; |2>$$

and

$$G_2\left(\frac{|0> -|4>}{\sqrt{2}}\right) = -e^{\frac{4i\pi}{9}} \frac{|0> -|4>}{\sqrt{2}}.$$

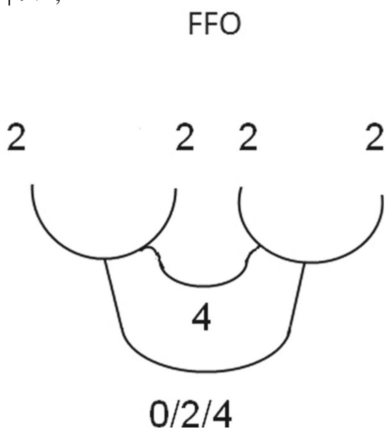
From now on, we will work with respect to a new basis (e_1, e_2, e_3), namely:

$$e_1 = \frac{|0> +|4>}{\sqrt{2}}, \quad e_2 = |2>, \quad e_3 = \frac{|0> -|4>}{\sqrt{2}}.$$

The matrices of the σ_1 and σ_2 braids with respect to this new basis are the following. Again, in all what follows, we write the matrices involved with determinant 1, that is we drop a phase. And so we get:

$$\widetilde{G_1} = \begin{pmatrix} & & e^{\frac{7i\pi}{9}} \\ & -e^{\frac{4i\pi}{9}} & \\ e^{\frac{7i\pi}{9}} & & \end{pmatrix}, \ \widetilde{G_2} = \begin{pmatrix} & e^{\frac{7i\pi}{9}} & \\ e^{\frac{7i\pi}{9}} & & \\ & & -e^{\frac{4i\pi}{9}} \end{pmatrix}.$$

With respect to the new basis, the FFO whose effect is to swap the qutrits $|0>$ and $|4>$,



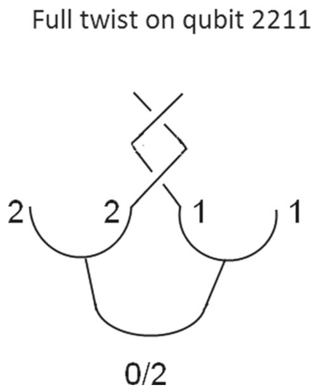
is encoded as follows:

$$F\widetilde{U}M = \begin{pmatrix} -e^{\frac{2i\pi}{3}} & & \\ & -e^{\frac{2i\pi}{3}} & \\ & & e^{\frac{2i\pi}{3}} \end{pmatrix}.$$

Note in $PU(3) = SU(3)/Z_3$, this matrix is simply:

$$\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}.$$

We have an analog for the qubit without fusing any anyons, but simply by using braids. Namely, a full twist like on the figure below has the effect of swapping the qubits $|0>$ and $|2>$. This is a fundamental observation in the protocol which we will soon describe.

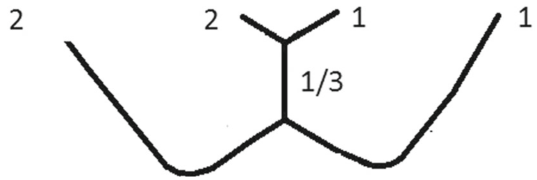


Proof The fact that $|0>$ is mapped to $|2>$ essentially relies on the following two points.

- The quantum dimensions of anyons of topological charge 1 and 3 are the same.
- The two diagonal coefficients of the squared R-matrix $R(2, 1)$ are opposite.

It then follows that $|2>$ is mapped to $|0>$ by unitarity of the matrix.

We will now justify this fact in more detail. Acting on the qubit $|0>$, after doing an F-move with horizontal charge line 0 at the level of the second and the third anyon, followed by two R-moves, we obtain the diagram



We then do an F-move again. When looking at the $|0>$ projection, the two unitary $6j$-symbols involved each contain a "0" which makes them be unitary theta symbols. Using the notations of [25] and [17], the two values $\theta^u(1, 2, 1)$ and $\theta^u(1, 2, 3)$ are identical since the quantum dimensions of anyons of respective topological charge 1 and 3 are the same. □

We will use this swap operation on the qubit 2211 in the protocol below, which physically describes how to generate the quantum gate N by braiding and using an ancilla pair. □

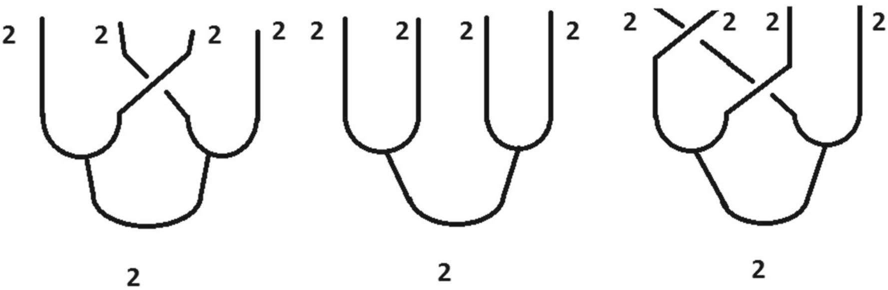
<u>*Protocol*</u>

(i) Take a qutrit 2222 and a pair of 1's out of the vacuum. Number the anyons 1, 2, 3, 4, 5, 6, those from the qutrit being numbered first.
(ii) Prepare the qutrit in one of the states $|2>$ or $\frac{|0>+|4>}{\sqrt{2}}$ or $\frac{|0>-|4>}{\sqrt{2}}$.

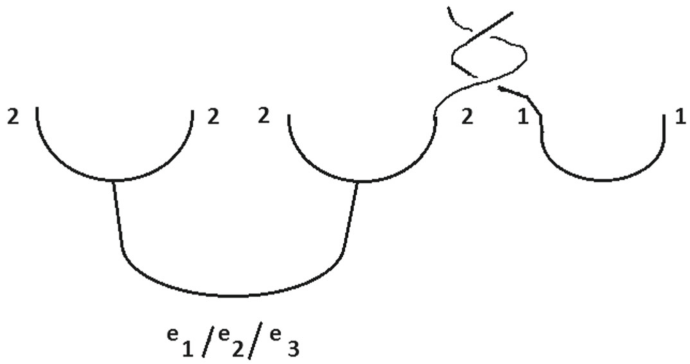
(iii) Do a full twist on anyons 4 and 5 to "create" a 2 charge line in between the qutrit and the pair of 1's.
(iv) Use this "extended" version of the qutrit to make braids in such a way that the outcome is a qutrit 2222 on the first 3 anyons and a qubit 2211 on the last 3.
(v) Go back to the original configuration of qutrit 2222 and pair of 1's by doing a full twist between anyons 4 and 5.

Step (ii) is summarized in the following figure, and Step (iii) is represented in the figure below it.

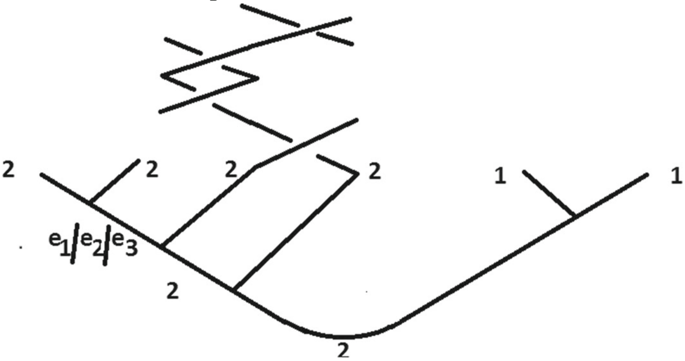
Preparation of the qutrit



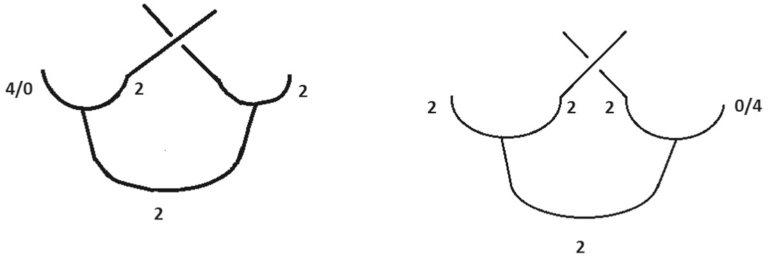
Step (iii) of protocol



The braids of step (iv) are now described below.



It is a consequence of the fusion rules that a "middle braid" on particles 4222 or 0222 or their respective vertical mirror images will map $\mathbb{C}|2>$ into $\mathbb{C}|2>$. Moreover, the braiding simply introduces the same phase $e^{\frac{4i\pi}{3}}$, whether dealing with 4222 or with 0222.



In light of this, it makes sense to do a full twist between anyons 2 and 3 on the figure above. It namely allows the charge line adjacent to the input to carry the charge 2 at the end of the braiding process in order for step (v) to be successful independently from the input. After completing the whole protocol, we obtain a new matrix in $SU(3)$, namely:

$$N = \begin{pmatrix} e^{\frac{8i\pi}{9}} & & \\ & e^{\frac{8i\pi}{9}} & \\ & & e^{\frac{2i\pi}{9}} \end{pmatrix}.$$

The next part addresses the group structure of $< \widetilde{G}_1, \widetilde{G}_2, F\widetilde{U}M, N >$.

4 Group structure

We will show the following result.

Theorem 2 *The group* $\widetilde{\mathcal{G}}$ *generated by the matrices* $\widetilde{G}_1$, $\widetilde{G}_2$, $F\widetilde{U}M$ *and* N *has order* 648 *and is isomorphic to a semi-direct product* $(C_6 \times C_{18}) \rtimes S_3$ *with respect to conjugation, for the action provided in Lemma* 1 *below. Moreover, it is the group* $D(18, 1, 1; 2, 1, 1)$.

Proof There is in $\widetilde{\mathcal{G}}$ a normal subgroup, say Δ, generated by all the diagonal matrices. Moreover, there is a Klein group inside Δ generated by the two matrices $(F\widetilde{U}M)^3$ and its $\widetilde{G}_1$-conjugate $\widetilde{G}_1(F\widetilde{U}M)^3\widetilde{G}_1^{-1}$. Indeed, we have:

$$(F\widetilde{U}M)^3 = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix},\ \widetilde{G}_1\,(F\widetilde{U}M)^3\,\widetilde{G}_1^{-1} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}.$$

□

Lemma 1 *Our group* $\widetilde{\mathcal{G}} = \left\langle \widetilde{G_1},\ \widetilde{G_2},\ F\widetilde{U}M,\ N \right\rangle$ *is isomorphic to*

$$\left(< N^2\widetilde{G_1}^2 > \times\ < N > \times\ < (F\widetilde{U}M)^3, \widetilde{G_1}(F\widetilde{U}M)^3\widetilde{G_1}^{-1} > \right) \rtimes S_3,$$

with

$$S_3 = \left\{ \widetilde{G_1}^9,\ \widetilde{G_2}^9,\ \widetilde{G_1}^9\widetilde{G_2}^9,\ \widetilde{G_2}^9\widetilde{G_1}^9,\ \widetilde{G_2}\widetilde{G_1}\widetilde{G_2}, I_3 \right\}.$$

Denoting the latter set by $\{t_3,\ t_1,\ c_1,\ c_2,\ t_2, id\}$ *and the generators from the direct product of two cyclic groups* $C_6 \times C_{18}$ *by*

$$\left| \begin{array}{l} x_6 = \ N^2\widetilde{G_1}^2(F\widetilde{U}M)^3 \\ x_{18} = N\,\widetilde{G_1}(F\widetilde{U}M)^3\widetilde{G_1}^{-1} \end{array} \right. ,$$

a presentation for this group is given by:

$$\left\langle x_6, x_{18}, t_1, t_2 \middle| \begin{array}{c} t_1^2 = t_2^2 = 1 = x_6^6 = x_{18}^{18} = [x_6, x_{18}] = (t_1t_2)^3, \\ t_1\,x_6\,t_1 \ = \ x_6^{-1}, \quad t_2\,x_6\,t_2 \ = x_6^5\,x_{18}^3 \\ t_1\,x_{18}\,t_1 = x_6^3\,x_{18}\ ,\ \ t_2\,x_{18}\,t_2 = x_6^4\,x_{18}^{13} \end{array} \right\rangle.$$

In the semi-direct product above, $N^2\widetilde{G_1}^2$ *is the matrix* B^2 *of [1] with respect to the basis* $\left(e_1 = \frac{|0>+|4>}{\sqrt{2}},\ e_2 = |2>,\ e_3 = \frac{|0>-|4>}{\sqrt{2}} \right)$. *The GAP ID following [6] for the presentation given above is:*

$$[648, 259].$$

That is our group is the 259*-th group of order* 648 *in the SmallGroups library by H. Besche, B. Eick and E. O'Brien dating from the beginning of the* 2000 *millennium. This is the same GAP ID as the one of* $D(18, 1, 1; 2, 1, 1)$. *Further, the group* $\widetilde{\mathcal{G}}$ *is precisely the group*

$$D(18, 1, 1; 2, 1, 1),$$

defined by matrix generators by Blichfeldt in 1916.

Proof of Lemma. We look for more cyclic groups generated by diagonal matrices, and whose mutual intersections and intersection with the Klein group are trivial. Begin obviously with the subgroup of $\widetilde{\mathcal{G}}$ generated by the matrix N. Notice also $\widetilde{G_1}$ squared is a diagonal matrix. We have:

$$\tilde{G_1}^2 = \begin{pmatrix} e^{-\frac{4i\pi}{9}} & & \\ & e^{\frac{8i\pi}{9}} & \\ & & e^{-\frac{4i\pi}{9}}, \end{pmatrix}, \; N = \begin{pmatrix} e^{\frac{8i\pi}{9}} & e^{\frac{8i\pi}{9}} & e^{\frac{2i\pi}{9}} \end{pmatrix}.$$

Now stare at these matrices. Both matrices have order 9. Because their diagonal phases in position (2, 2) are identical, we see that the two subgroups $< \tilde{G_1}^2 >$ and $< N >$ intersect non-trivially only for the k-th powers of the generators with k satisfying to $1 \leq k \leq 8$ and

$$2k \equiv -4k \equiv 8k \;\; (mod\, 18).$$

This implies that 3 must divide k. Then $k = 3$ or $k = 6$. In order to solve this unpleasant issue, we must "mix" the generators instead. We have

$$N^2\tilde{G_1}^2 = \begin{pmatrix} e^{\frac{4i\pi}{3}} & e^{\frac{2i\pi}{3}} & 1 \end{pmatrix}$$

and

$$\left(N^2\tilde{G_1}^2\right)^2 = \begin{pmatrix} e^{\frac{2i\pi}{3}} & & \\ & e^{\frac{4i\pi}{3}} & \\ & & 1 \end{pmatrix}.$$

And so, we have:

$$< N^2\tilde{G_1}^2 > \; \cap \; < N >= \{I_3\}.$$

In the Klein group, all the elements have order 2 and in a cyclic group of odd order, all the elements have an odd order. Hence $< N^2\tilde{G_1}^2 >$ and $< N >$ do not intersect with the Klein group.

We now exhibit a symmetric group S_3 inside $\tilde{\mathcal{G}}$. It suffices to notice that $\tilde{G_1}^9 \tilde{G_2}^9$ and $\tilde{G_2}^9 \tilde{G_1}^9$ are the two usual permutation matrices associated with the respective two cycles of $Sym(3)$. Further, we have

$$\tilde{G_1}^9 = \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}, \; \tilde{G_2}^9 = \begin{pmatrix} & -1 & \\ -1 & & \\ & & -1 \end{pmatrix},$$

$$\tilde{G_2}\tilde{G_1}\tilde{G_2} = \begin{pmatrix} -1 & & \\ & & -1 \\ & -1 & \end{pmatrix}.$$

These matrices provide the additional matrices, respectively, associated with the three transpositions (13), (12) and (23) of $Sym(3)$.

It remains to show that each of the $\widetilde{\mathcal{G}}$-generators N, $F\widetilde{U}M$, $\widetilde{G_1}$ and $\widetilde{G_2}$ can be written as a product of an element of the direct product and a group element of S_3. The result from the lemma will then classically follow.

First and foremost, we are able to write, using the fact that $\widetilde{G_1}$ has order 18,

$$\widetilde{G_1} = N^{-10}\left(N^2\widetilde{G_1}^2\right)^5\widetilde{G_1}^9.$$

Next, it suffices to notice that

$$(F\widetilde{U}M)^4 = N^3$$

and so,

$$F\widetilde{U}M = N^3(F\widetilde{U}M)^{-3}.$$

In particular, we see that the matrix corresponding to the FFO is in the direct product. This was expected since it is a diagonal matrix. Further, we have:

$$N^2\widetilde{G_2}^2 = (F\widetilde{U}M)^2.$$

We derive,

$$\widetilde{G_2}^2 = N^{-2}(F\widetilde{U}M)^2.$$

Now write

$$\widetilde{G_2} = \left(\widetilde{G_2}^2\right)^5\widetilde{G_2}^9$$

in order to conclude.

Finally, it is straightforward to see that $\widetilde{\mathcal{G}} = D(18, 1, 1; 2, 1, 1)$. Namely, recall below the Blichfeldt generators of $D(18, 1, 1; 2, 1, 1)$:

$$F(18, 1, 1) = \begin{pmatrix} e^{\frac{i\pi}{9}} & & \\ & e^{\frac{i\pi}{9}} & \\ & & e^{\frac{-2i\pi}{9}} \end{pmatrix},$$

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \widetilde{B} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

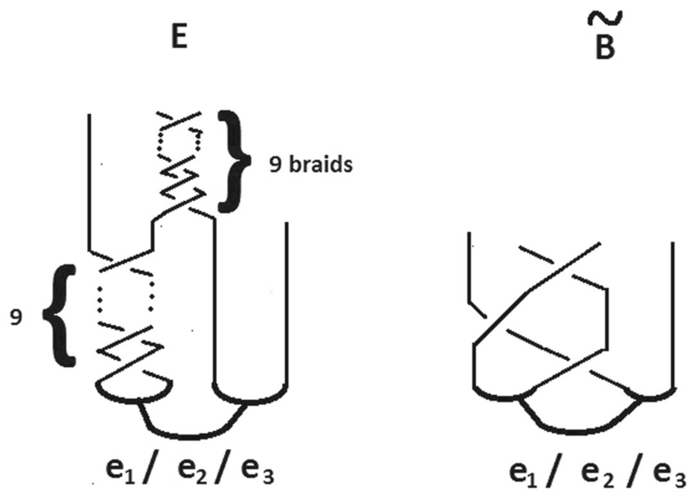
We see that $E = \widetilde{G}_2^{\,9}\,\widetilde{G}_1^{\,9}$ and $\widetilde{B} = \widetilde{G}_2\,\widetilde{G}_1\,\widetilde{G}_2$; hence, E and $\widetilde{B}$ both belong to $\widetilde{\mathcal{G}}$. Further, we have:

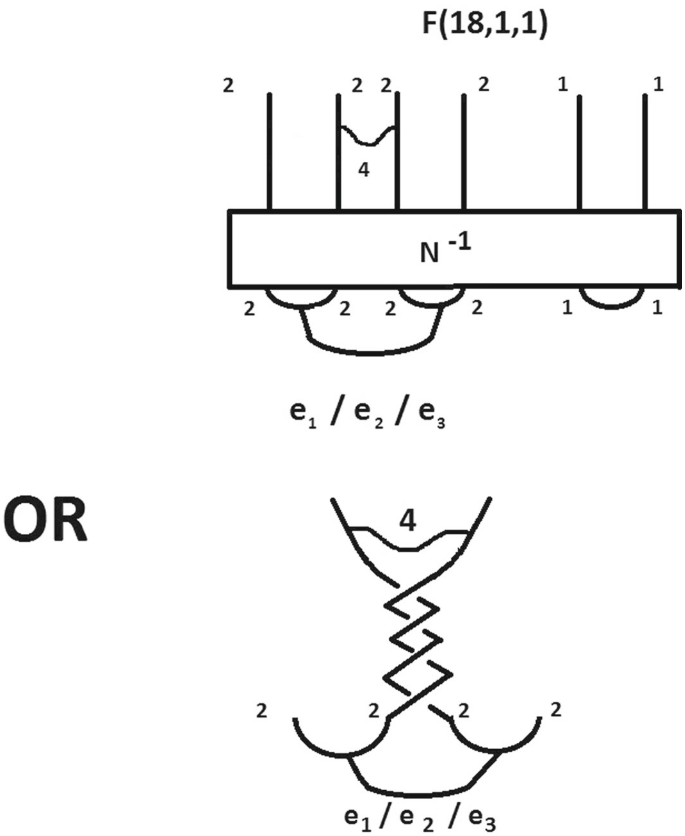
$$F(18, 1, 1) = (F\widetilde{U}M)^3\, N^{-1}. \tag{1}$$

Thus, we see that $F(18, 1, 1)$ also belongs to $\widetilde{\mathcal{G}}$ and the Blichfeldt generator $F(18, 1, 1)$ can be expressed in terms of the FFO matrix and the N gate. We conclude that the groups $\widetilde{\mathcal{G}}$ and $D(18, 1, 1; 2, 1, 1,)$ are identical since by [17] and the current work, they have the same order. This ends the proofs of Lemma 1 and Theorem 2. □

We deduce below a theorem about a physical realization of the original Blichfeldt generators of $D(18, 1, 1; 2, 1, 1)$.

Theorem 3 *The Blichfeldt generators from* $D(18, 1, 1; 2, 1, 1)$ *can be physically realized as follows.*





Proof We have seen at the end of the proof of Theorem 2 that each Blichfeldt generator can be expressed algebraically as a product involving the generators of the group $\tilde{\mathcal{G}}$. Since by § 2, each generator of $\tilde{\mathcal{G}}$ has a physical realization, the physical realization of a Blichfeldt generator is obtained by stacking from bottom to top each corresponding action diagram in the reverse order of the matricial products. For the third diagram, it will be useful to recall that the Freedman fusion operation results in swapping $|0>$ and $|4>$. It follows immediately that $F\tilde{U}M^3$ acts exactly like $F\tilde{U}M$ on the basis (e_1, e_2, e_3). Finally, the fourth and last diagram arises simply from expressing N as a braid matrix; see the result of forthcoming Theorem 4. □

The next theorem states that three generators suffice to generate $\tilde{\mathcal{G}}$. Our results are summarized below. □

Theorem 4 *(i)*

$$\widetilde{\mathcal{G}} = < \widetilde{G}_1, \widetilde{G}_2, F\widetilde{U}M > .$$

(ii) *The matrix* N *is obtained by braiding in an adequate way* 4 *anyons* 2222 *with respect to the basis* (e_1, e_2, e_3). *Explicitly, we have:*

$$N = \overset{\sim}{G_2}^{-4}.$$

Proof For (i), notice that the subgroup $< \overset{\sim}{G_1}, \overset{\sim}{G_2}, F\tilde{U}M >$ has order 648 since it is conjugate to the Freedman group $< G_1, G_2, FUM >$, which has order 648 by [17]. Hence it is actually the whole group $\overset{\sim}{\mathcal{G}}$ since, as part of our work, we showed that $\overset{\sim}{\mathcal{G}}$ has order 648. Then the matrix N must be obtained by braiding and FFO. In fact, it is simply obtained by braiding, like claimed in point (ii). Indeed, simply notice that

$$N = F(9, 1, 1)^4$$

and

$$F(9, 1, 1)^{-1} = \overset{\sim}{G_2}\, t_1.$$

Recall:

$$t_1 = \overset{\sim}{G_2}^{9}.$$

Hence,

$$F(9, 1, 1) = \overset{\sim}{G_2}^{8}.$$

□

Last, we comment on the two groups $Fr(162 \times 4)$ and $\overset{\sim}{\mathcal{G}} = D(18, 1, 1; 2, 1, 1)$. By [17],

$$O^T\, Fr(162 \times 4)\, O = \overset{\sim}{\mathcal{G}}$$

with

$$O = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix},$$

where $Fr(162 \times 4)$ denotes the Freedman group. We read that O is the transition matrix from

$$(|0 >, |2 >, |4 >)$$

to

$$\left(\frac{|4>+|0>}{\sqrt{2}}, |2>, \frac{|4>-|0>}{\sqrt{2}}\right).$$

Thus, we see that $\widetilde{\mathcal{G}} = D(18, 1, 1; 2, 1, 1)$ encodes the σ_1 and σ_2- braids and FFO on 4 anyons of topological charge 2, with respect to either basis:

$$\left(\frac{|4>+|0>}{\sqrt{2}}, |2>, \frac{|4>-|0>}{\sqrt{2}}\right),$$

$$\left(\frac{|0>+|4>}{\sqrt{2}}, |2>, \frac{|0>-|4>}{\sqrt{2}}\right).$$

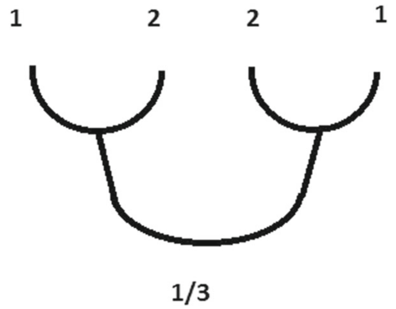
In other words, we have:

$$Fr(162\times 4) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} Fr(162\times 4) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

That is, if we swap the first row and third row and the first column and the third column of a Freedman matrix, we again obtain a Freedman matrix.

5 New ancillas for the qubit 1221

In [2], we seek ancillas of the form $x\,|1> + y\,|3>$ with $|x| = |y|$ for the qubit 1221.



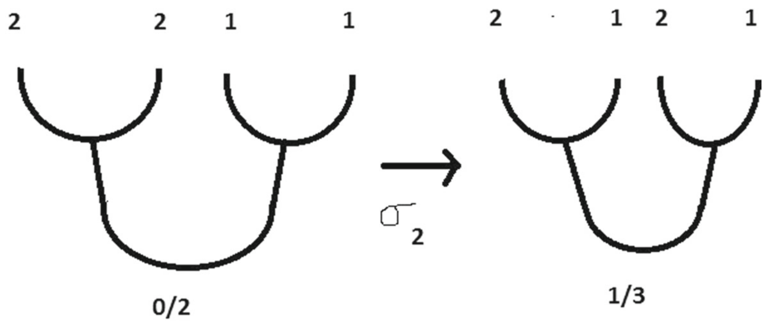
The fact that the norms in $|1>$ and $|3>$ are equal is a necessary condition for no-leakage on some protocols we test which use a combination of braiding and interferometric measurements. Such an ancilla cannot be realized by a combination of σ_1- and σ_2-braids on the qubit 1221. Indeed, the matrix for a σ_2-braid is the following.

$$\begin{pmatrix} -\frac{1}{2} & \frac{i\sqrt{3}}{2} \\ \frac{i\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

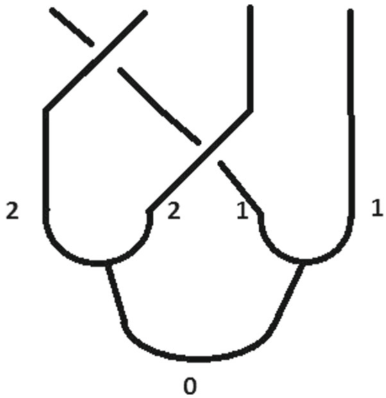
And the matrix for a σ_1-braid is simply a diagonal matrix with phases on the diagonal. Thus, an idea to create such ancillas is to start with the qubit 2211 instead. We have seen when working on the qutrit that a full twist in the center has the effect of swapping $|0>$ and $|2>$. If instead we do a single braid in the center, we obtain the following matrix

$$\begin{array}{cc} & \begin{array}{cc} |0> & |2> \end{array} \\ \begin{array}{c} |1> \\ |3> \end{array} & \begin{pmatrix} \frac{1}{\sqrt{2}}\, e^{i\frac{2\pi}{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\, e^{-i\frac{5\pi}{6}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \end{array}$$

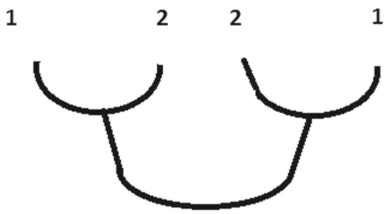
for the action



Thus, by doing



we obtain



$$\frac{1}{\sqrt{2}}\left(|1> + |3> \right)$$

Note that if you braid a $|2>$ instead, you can make the ancilla:

$$\frac{1}{\sqrt{2}}(|1> - |3>).$$

6 Discussion

In the present paper, we provide a physical realization of the actual original $D(18, 1, 1; 2, 1, 1)$ as defined by generators in [22], while in [17] we only provide a physical interpretation of $D(18, 1, 1; 2, 1, 1)$ since we only provide a physical realization of an isomorphic copy of that group. While finite subgroups of SU(3) are frequently used in particle physics, our current work continues shedding light on the fact that these groups also arise in quantum physics. Little remains known on how they arise in quantum physics, and there is hope that more groups will be uncovered in the years to come, which are either already in the incomplete Blichfeldt classification, or possibly not.

From a quantum computing perspective, it is disappointing, but not surprising, that we did not succeed to increase the number of braiding qutrit gates by doing our protocol which uses an ancilla pair. Enlarging such a number is not an easy problem. In fact, even using protocols with both braiding and interferometric measurement does not easily lead to finding additional gates which are not issued from braids we already have (cf. Bela Bauer's programming in [2] to test a few such protocols by brute computer force).

Our configuration of qutrit input and pair of ancilla can be viewed as a special case of a fusion tree. By a result of Vaughan Jones, at level 4, the image of the braid group representation arising from any such fusion tree must be finite. However, we ask the relevant question whether or not it would be possible to enlarge the Freedman group using braids only, by a similar protocol as the one presented here. This would give yet more insights into which $SU(3)$ finite subgroups can be realized from anyonic systems. In [19], we succeeded to make an irrational qutrit phase gate by using ancillas, braiding and measurement. This gate added to the Freedman group makes it become infinite, hence dense. Aiming at better understanding the finite $SU(3)$ subgroups, instead of aiming at finding a universal gate set, we could raise the following question: using ancillas, braiding and measurement, could we make a gate which does not already belong to the Freedman group and which added to it would yield a larger group, still of finite order ? The protocol of this paper could be a starting point in this direction. While keeping the same new basis, we could try more sophisticated protocols, which also involve measurements, with the hope of uncovering a finite $SU(3)$ subgroup of larger order. In case of success, three possibilities could arise. Either the group is identical to one already defined in the extended Blichfeldt classification, or it is an isomorphic copy of such group, or it is a new finite $SU(3)$ subgroup. We could conjecture that there exist such $SU(3)$ finite subgroups that are not in Blichfeldt's original classification nor in its extended version. It is interesting to note that within the existing classification for finite $SU(3)$ subgroups, there exist some non-isomorphic extensions of $D(9, 1, 1; 2, 1, 1)$. This is namely the case of $D(18, 1, 1; 2, 1, 1)$ and of the exceptional group $\Sigma(216 \times 3)$ of same order 648. While studying the principal series of such exceptional groups, Walter Grimus and Patrick Ludl showed in [12] that the latter group is isomorphic to

$$\Delta(54) \rtimes A_4 = (C_3 \times C_3 \rtimes S_3) \rtimes (V_4 \rtimes C_3).$$

Inspired by both principal series, the one provided by Grimus and Ludl for the exceptional group and the one provided in [17] for the Freedman group, and using Grimus and Ludl's fact that

$$\Sigma(72 \times 3)/\Delta(27) \simeq Q_8,$$

it is shown in [17] that if the Freedman group $Fr(162 \times 4)$ contains a proper subgroup H such that H has a quotient H/K with $H/K \simeq Q_8$, then all the 2-Sylow subgroups of $Fr(162 \times 4)$ are isomorphic to Q_8. This is Lemma 9 of [17]. However, there does exist a 2-Sylow subgroup of $Fr(162 \times 4)$ which is not isomorphic to the quaternion group Q_8, as it contains a subgroup which is not normal; see Lemma 10 of [17]. Therefore, the Freedman group and the exceptional group of the same order are not isomorphic, though there are both extensions of $D(9, 1, 1; 2, 1, 1)$. Thus, it is not unreasonable to consider that there could exist an anyonic extension of $D(18, 1, 1; 2, 1, 1)$ which is neither of D-type, nor lies in the updated classification.

As part of his many contributions, particle physicist Patrick Ludl showed in [21] that any SU(3) finite subgroup belonging to the D series has a structure of the form $(C_n \times C_m) \rtimes S_3$. Moreover, in the latter semi-direct product, the Abelian group is the normal subgroup of all the diagonal matrices of the D-group. However, among these semi-direct products, we do not know which ones actually occur as D-groups. More generally, we do not know for which values of n and m, we can have an $SU(3)$ subgroup. Among those semi-direct products that are isomorphic to an $SU(3)$ subgroup, we also do not know which ones may be realized using an anyonic system. And even in that case, we further do not know whether the other non-isomorphic semi-direct products could also occur as $SU(3)$ subgroups and if so, whether they could arise in anyonic physics. This is for instance the case of $(C_6 \times C_{18}) \rtimes S_3$. From [6], we know that there are 757 non-isomorphic groups of order 648, which is a relatively high number when compared to the nearby orders: 21541 of order 640, but only 1 of respective orders 641, 643, 647, 649, only 4 of respective orders 642 and 646 and only 9 of order 644. We wrote a program in GAP which lists all the semi-direct products $(C_6 \times C_{18}) \rtimes S_3$ from the list of all the homomorphisms from S_3 to $Aut(C_6 \times C_{18})$. The program returns the list of their GAP ID. It turns out that out of the 757 groups of order 648, there are 36 non-isomorphic semi-direct products $(C_6 \times C_{18}) \rtimes S_3$, one of which is the Freedman group, and it has 108 distinct semi-direct products isomorphic to it (one of which is $D(18, 1, 1; 2, 1, 1)$ of same GAP ID [648,259]). It is natural to ask: which of these 35 other semi-direct products do occur as $SU(3)$ subgroups, if any ? Could they possibly occur with anyonic braiding and measurements and how ? It seems that we are only at the beginning of such thrilling exploration.

Last, we ask: could the associativity of the semi-direct product, like stated in Remark 1 of [19], and produced by our own study of the structures of the different groups in [1] and [17] confronted with the results obtained from the GAP software, play a role in the fact that both groups $D(9, 1, 1; 2, 1, 1)$ and $D(18, 1, 1; 2, 1, 1)$ can be realized with anyonic systems ?

There is a lot more to understand in this exploration of $SU(3)$ finite subgroups, from the perspective of finite group theory, particle physics, quantum physics and their respective intrications.

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