Exotic Solutions of Yang-Baxter Equations and Yang-Baxterization Approach

Mo-Lin Ge Kang Xue
Theoretical Physics, Nankai Institute of Mathematics, Tianjin, 300071, P.R. China

ABSTRACT

The new solutions of Yang-Baxter equations associated with the fundamental representations of B_n, C_n and D_n are derived through the braid group representations and the trigonometric Yang - Baxterization.

Remarkable progress has been made in the derivation of trigonometric solutions of Yang-Baxter equations (YBE) associated with simple Lie algebras(1,2). The standard approach is to make q-deformation of classical Lie algebras, namely, based on the current formulation of quantum group including its loop extension(3). We call this type of solutions "standard" one. However, on the basis of the same Lie algebraic structure it allows to generate new family of solutions of YBE, which is different from the standard one and is called "exotic" family of solutions of YBE.

Our strategy is stated in the following.

(I) In order to solve YBE

$$\check{R}_{12}(x)\check{R}_{23}(xy)\check{R}_{12}(y) = \check{R}_{23}(y)\check{R}_{12}(xy)\check{R}_{23}(x)$$

where $x=e^{-u}$ is the spectral parameter relating with the rapidity for two-particle collision, the asymptotic behavior $T= R(x)_{u=\infty}$ satisfying

$$T_{12}T_{23}T_{12}=T_{23}T_{12}T_{23}$$

is firstly solved for given Lie algebraic structure. T is reffred to braid group representation (BGR).

(II) By using the trigonometric Yang-Baxterization (4,5) prescription (T-YB) $\check{R}(x)$ can be generated for a given BGR. So far the T-YB has been established for those BGR's which possess distinct eigenvalues being three and four(two, needless to say).

In this talk we only discuss the new solutions of YBE associated with the fundamental representations of Lie algebras B_n, C_n and D_n. As was known that the corresponding BGR's possess three distinct eigenvalues.

First we calculate the BGR's which are given by $(w=q-q^{-1})$

$$T= \sum_{k\neq 0} u_k\, e_{kk} \otimes e_{kk} + w \sum_{\substack{k<m\\ k+m\neq 0}} e_{kk}\otimes e_{mm} + \sum_{\substack{k\neq m\\ k+m\neq 0}} e_{km} \otimes e_{mk} + \sum_{k,m} a_{km}\, e_{k-m}\otimes e_{-km}$$

where $u_k=q$ or $-q^{-1}$ for k=0 and $u_{-k}=u_k$. $k,m \in [(N-1)/2, \ldots, -(N-1)/2]$

where N=2n+1, 2n and 2n for B_n, C_n and

D_n, respectively. The a_{km} are given by

$$
a_{km}=\begin{cases}
1 & (k=m=0)\\
u_k^{-1} & (k=m\neq 0)\\
w\left[1-u_m^{-1}\left(\prod_{j=1}^{m-1} u_j^{-2}\right)\right] & (k=-m<0)\\
(-1)^{k+m+1} w u_{k+m}^{-\frac{1}{2}} \prod_{j=1}^{|k+m|-1} u_j^{-1} & (k=0<m,\ \text{or}\ k<m=0)\\
(-1)^{k+m+1} w u_m^{-\frac{1}{2}} u_k^{-\frac{1}{2}} \left(\prod_{j=|k|+1}^{|m|-1} u_j^{-1}\right) & (0<k<m,\ \text{Or}\ k<m<0)\\
(-1)^{k+m+1} w u_m^{-\frac{1}{2}} u_k^{-\frac{1}{2}} \left(\prod_{j=|k|}^{m-1} u_j^{-1}\right)\cdot\left(\prod_{i=1}^{|k|-1} u_i^{-2}\right) & (k<0,\ m>0,\ k+m\neq 0)
\end{cases}
$$

for $B_n^{(1)}$.

$$
a_{km}=\begin{cases}
u_k^{-1} & (k=m)\\
w\left[1-\varepsilon u_m^{-1}\left(\prod_{j=1}^{m-\frac{1}{2}} u_{j-\frac{1}{2}}^{-2}\right)u_{\frac{1}{2}}\right] & (k=-m\ 0)\\
-w u_m^{-\frac{1}{2}} u_k^{-\frac{1}{2}} \left(\prod_{j=1}^{m-\frac{1}{2}} u_{j-\frac{1}{2}}^{-1}\right) u_k^{-\frac{1}{2}} & (0<k<m,\ \text{or}\ k<m<0)\\
-\varepsilon w u_m^{-\frac{1}{2}} \left(\prod_{j=|k|+\frac{1}{2}}^{m-\frac{1}{2}} u_{j-\frac{1}{2}}^{-1}\right)\cdot\left(\prod_{i=1}^{|k|+\frac{1}{2}} u_{i-\frac{1}{2}}^{-2}\right) u_k^{\frac{1}{2}} u_{\frac{1}{2}}^{\varepsilon} & (k\ 0,\ m\ 0,\ k+m=0)
\end{cases}
$$

with $-\varepsilon=1$ for $C_n^{(1)}$ and $\varepsilon=1$ for $D_n^{(1)}$.

The distinct eigenvalues are given by

$$(T-\lambda_1)(T-\lambda_2)(T-\lambda_3)=0$$

where

	λ_1	λ_2	λ_3
B_n	q	$-q^{-1}$	$\left(\prod_{j=1}^{n} u_j^{-2}\right)$
C_n	$-q^{-1}$	q	$-\left(\prod_{j=1}^{n} u_{j-\frac{1}{2}}^{-2}\right)u_{\frac{1}{2}}^{-1}$
D_n	q	$-q^{-1}$	$\left(\prod_{j=1}^{n} u_{j-\frac{1}{2}}^{-2}\right)u_{\frac{1}{2}}$.

Next we Yang-Baxterize the solutions derived above to give the corresponding $\check{R}(x)$s. It has been proved that if BGR T satisfies the relation (5)

$$
\lambda_3^{-1}(T_{12}T_{23}^{-1}T_{12}-T_{23}T_{12}^{-1}T_{23})-\lambda_1(T_{12}^{-1}T_{23}T_{12}^{-1}-T_{23}^{-1}T_{12}T_{23}^{-1})-\left(1+\frac{\lambda_1}{\lambda_2}+\frac{\lambda_1}{\lambda_3}+\frac{\lambda_2}{\lambda_3}\right)
$$

$$
-\lambda_2^{-1}(T_{12}-T_{23})+\lambda_2(T_{12}^{-1}-T_{23}^{-1})=0,
$$

then it can be T-YB to

$$
R(x)=\lambda_1 x(x-1)T^{-1}+\left(1+\frac{\lambda_1}{\lambda_2}+\frac{\lambda_1}{\lambda_3}+\frac{\lambda_2}{\lambda_3}\right)\cdot xI-\lambda_3^{-1}(x-1)T.
$$

Now the calculations convince the validity (for the derived T) of the statement. Hence for the considered cases we obtain the general solutions of YBE.

The result is shown in the following form.

$$R(x)=\sum_{k\neq 0} u_k\, e_{kk}\otimes e_{kk}-(q^2-1)(x-\xi)$$

$$\cdot\Big(\sum_{\substack{k<m\\ k+m\neq 0}} +x \sum_{\substack{k>m\\ k+m\neq 0}}\Big) e_{kk}\otimes e_{mm}$$

$$+q(x-1)(x-\xi)\sum_{\substack{k\neq m\\ k+m\neq 0}} e_{km}\otimes e_{mk}$$

$$+\sum_{k,m} a_{km}(x)\; e_{k-m}\otimes e_{-km}$$

where

$$u_k(x)=\begin{cases}(x-q^2)(x-\xi) & \text{when } u_k=q\\ -q^2(x-q^{-2})(x-\xi) & \text{when } u_k=-q^{-1}\end{cases}$$

and

$$a_{km}=q(x-1)(x\tilde{a}_{km}-\xi a_{km})+(\xi-1)(q^2-1)x\,\delta_{k-m}$$

$$\xi=\begin{cases}q^{-1}\lambda_3^{-1} & \text{for } B_n^{(1)} \text{ and } D_n^{(1)}\\ -q\quad\lambda_3^{-1} & \text{for } C_n^{(1)},\end{cases}$$

$$\tilde{a}_{km}(u_k)=a_{mk}(u_k^{-1}).$$

The other permitted solution is obtained by the interchange $q \leftrightarrow -q^{-1}$ and keeping λ_3 unchanged. Such a solution corresponds to the "twisted" one. For instance, it gives rise to $A_{2n}^{(2)}$ or $A_{2n-1}^{(2)}$ corresponding to $B_n^{(1)}$ or $D_n^{(1)}$, respectively. As for the correspondence of $C_n^{(1)}$ it deserves to be understood.

We would like to make comments to the above discussion.

(a) Taking $u_k=q$ for all k our solutions go back to those derived by Jimbo (6). The other choice leads to new solutions which are called exotic ones. Nothing is surprise to appear such a new type of solutions of YBE because the usual classical limit is not required here. Actually the new solution for A_n had been found by Gervais et al(7) in terms of different approach.

(b) In contrast with the standard solutions the exotic ones possess very different properties such as without the usual classical limits, different Hopf algebraic structure due to Faddeev-Reshetikhin-Takhtajan approach (8), some of them even cannot be diagonalized and so on.

(c) We can prove that the exotic solutions still satisfy Birman-Wenzl algebra (9). This fact is determined by the fundamental representations of B_n, C_n and D_n. In a sense our discussion provides another explicit example for Jones' theory(4).

M.L.Ge wishes to thank Professor C.N.Yang for many enlightening discussions and encouragements.

References

(1) M.Jimbo(ed), Yang-Baxter Equation in Integrable Systems. Word Scientific
(2) C.N.Yang and M.L.Ge(eds), Braid Group, Knot Theory and Statistical Mechanics, World Scientific.
(3) M.Jimbo, pp111 in ref.(2).
(4) V.Jones, Commn Math Phys125(89)459
(5) M.L.Ge, Y.S.Wu, K.Xue, ITP-SB-90-02
(6) M.Jimbo, Commn Math Phys102(86)537
(7) E.Cremmer, Gervais, LPTEN 89/19
(8) L.D.Faddeev et al, Alg.Anal, 1(89)
(9) Y.Cheng, M.L.Ge and K.Xue, ITP-SB, 90-24.
(10) About the calculation method for deriving the BGR, see, M.L.Ge et al, Inter J.Mod Phys, 4(89)3351, J.Phys.23A 605(90), 23A 2273(90).