



# Thermal features of Barrow corrected-entropy black hole formulation

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**Abstract** Through the last years, it was demonstrated that quantum corrections of entropy, represented by logarithmic and power law corrections terms, constituted an association between semi-classical entropic areas and the curvature correction in Einstein–Hilbert’s Lagrangian and vice-versa. Loop quantum gravity approach provided the logarithmic corrections, which arises from quantum and thermal equilibrium fluctuations. On the other hand, Barrow’s entropy was introduced from the fact that the black hole surface can be modified due to quantum gravitational effects. The new exponent  $\Delta$  that appears in Barrow’s entropy is a measure of this perturbation. In this letter we have analyzed the thermodynamical effects of the quantum fluctuations upon the geometry of a Barrow’s black hole. We demonstrated that new formulations of the equipartition law, which corresponds to the horizon energy, can be constructed from both entropic formalisms. Besides, we have calculated the heat capacity for both formulations and we discussed their thermal viability. We have also establish a condition on one of the constant pre-factors of the logarithmic correction.

## 1 Introduction

In current days, we are living under the aftermath of the observations of type Ia supernovae, concerning dark energy (DE), that constructed a Universe with two dark components. The well known dark matter (DM) and DE [1–4]. The first one is a matter without pressure. Its main function is to clarify both the galactic rotation curves and the formation of large-scale framework. About the second one, which is a negative pressure exotic energy, it is utilized to depict the current cosmic-

accelerated expansion. Nevertheless, its origin and nature are not known yet. Even so, there exist several candidates that bring an idea about its behavior and composition [5–8].

On the other hand, the holographic dark energy (HDE) is one relevant and serious DE contestant that were suggested, relied on the well known holographic principle [9, 10]. From the concept of holographic principle, the number of degrees of freedom (DOF) in a bounded system should be finite. It also has an association with the surface of the boundary of the system [11, 12]. When the holographic principle is used in cosmology, we can compute the upper bound of the entropy contained in the universe [13]. Besides, Li [14] questioned if in QFT, the ultraviolet cut-off  $\Lambda$  could be connected to the infrared cut-off  $L$ . The reason would be the fact that the limit set by constructing a black hole (BH), namely, the quantum zero-point energy of a system with size  $L$  can not be bigger than the mass of a BH that has the same size, or,  $L^3 \Lambda^3 \leq (M_p L)^{3/2}$ . We can rewrite this last equation as  $L^3 \rho_\Lambda \leq L M_p^2$ , where  $\rho_\Lambda \sim \Lambda^4$  is the zero-point energy density and the cut-off is  $\Lambda$ . And again, the last equation can be rewritten as  $\rho_\Lambda \leq M_p^2 L^{-2}$  or  $\rho_\Lambda = 3c^2 M_p^2 L^{-2}$ , where  $M_p^2 = (8\pi G)^{-1}$  is the modified Planck mass and  $3c^2$  is constant and it has a convenient function. There is a huge literature concerning the HDE models investigations [15–38].

The Bekenstein–Hawking entropy has an underlying rôle in HDE model, where  $S_{BH} = A/(4G)$  and it is used at the horizon [39]. As a matter of fact,  $A \sim L^2$  is the horizon’s area. Since the HDE model is connected to the area law of entropy, we have that a small adjustment to the area law of entropy will change the HDE model energy density. One correction concerning the area law of entropy is the logarithmic correction [40–42] given by

$$S_{BH} = \frac{A}{4G} + \tilde{\alpha} \ln \frac{A}{4G} + \tilde{\beta}, \quad (1)$$

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where  $\tilde{\alpha}$  and  $\tilde{\beta}$  are dimensionless constant pre-factors and their values are being discussed and not yet confirmed even within loop quantum gravity [43]. Several formulations concerning BH entropy provided the logarithmic correction yielding  $\tilde{\alpha} = -1/2$  or  $-3/2$  as standard values for this coefficient [44, 45]. However, there is no such agreement about regarding the way one might fix the value of the logarithmic pre-factor  $\tilde{\alpha}$ , e.g., with  $\tilde{\beta} = 0$ , because it seems to be a strongly model dependent parameter [46, 47]. The correction terms have an important and underlying function in both the late-time acceleration and early-time inflation of the Universe [48]. It is easy to see that, for  $\tilde{\alpha} = \tilde{\beta} = 0$  we have Bekenstein–Hawking entropy.

A general sign from almost all formulations of quantum gravity is that the geometry of space-time will be formed by quantum fluctuations near Planck scale. In such scenario, it would not be feasible to investigate the geometric structures below Planck scale [49, 50]. These quantum fluctuations could be the origin of the well known virtual BHs [51]. In [52–54] the authors investigated the connection between gravitational dynamics and thermodynamics in brane world scenarios.

To organize the ideas here, we have organized the issues in the following manner, in Sect. 2 we made a brief review of the Barrow entropy and our previous results. After that, in Sect. 3, we provided the logarithm correction to Barrow formulation and in Sect. 4, we discussed the results and made our final remarks.

## 2 Barrow black hole entropy

Recently, Barrow [55] analyzed the scenario where quantum gravitational effects could cause about some intricate, fractal structure on the BH surface. It changes its actual horizon area, which in turn leads us to a new BH entropy relation, namely,

$$S_B = \left( \frac{A}{A_o} \right)^{1+\frac{\Delta}{2}}, \quad (2)$$

where  $A$  is the usual horizon area and  $A_o$  the Planck area. The quantum gravitational perturbation is represented by the new exponent  $\Delta$ . There are some characteristic values for  $\Delta$ . For example, when  $\Delta = 0$  we have the simplest horizon construction. In this case we obtain the well known Bekenstein–Hawking entropy. On the other hand, when  $\Delta = 1$  we have the so-called maximal deformation. Hence, although  $\Delta$  is not, obviously, a quantum quantity, it is the “quantum” effect present in Barrow’s entropy expression.

Let us state now some physical motivations of Barrow’s formula. Although Barrow BH entropy was constructed as a toy model, there is some theoretical evidences that support

his ideas. In [56], the authors constructed Barrow holographic energy. They used the standard holographic principle at a cosmological structure and Barrow’s entropy, instead of the well known Bekenstein–Hawking one. The authors demonstrated precisely that Barrow holographic dark energy can depict the Universe thermal history, with the sequence of matter and dark energy eras [56]. A dark energy EoS parameter was obtained where the  $\Delta$ -exponent affects this EoS and several dark energy scenarios were obtained as functions of  $\Delta$ -value. In another support of Barrow’s ideas, in [57], the authors analyzed the validity of the generalized second law of thermodynamics using the Barrow entropy. The sum of the entropy inside the apparent horizon plus the entropy of the horizon itself is always a non-decreasing function of time. Hence, also concerning Barrow’s entropy, the generalized second law is confirmed. As another theoretical evidence, in [58], the authors discussed modified cosmological scenarios using Barrow entropy. The Friedmann equations were obtained when  $\Delta = 0$ . The new terms obtained constitute an effective dark energy sector. They lead us to intriguing phenomenological behavior and, for  $\Delta = 0$ , we observe that the  $\Lambda$ CDM concordance model is recovered. Experimentally, in [59] the authors used observational data from Supernovae (SNIa) Pantheon sample, together with direct measurements of the Hubble parameter from the cosmic chronometers sample to obtain constraints concerning the scenario of Barrow holographic dark energy. Barrow entropy is similar to Tsallis entropy (see [60]), although the physical framework is radically different. As a matter of fact, the equipartition theorem is central in Tsallis entropy. We strongly believe that, although being a toy model, all these evidences at least motivates, consistently, that Barrow’s new concept of entropy deserves some profound investigation.

In our last work [61], we have computed the equipartition law compatible with Barrow’s BH entropy and its heat capacity confirming the BH unstableness even in this not fixed  $\Delta$  scenario. In this paper we will analyze the thermodynamical features of the logarithmically corrected entropy and its Barrow’s version. We will calculate the temperature, the heat capacity, its physical implications and a condition on one of the pre-factors,  $\tilde{\alpha}$ , for both scenarios. To accomplish the task, we will use thermodynamic functions, which are normally used in BHs physics, they are entropy and temperature. The thermodynamics of BHs were constructed upon the basis of the ideas of both the entropy and temperature of BH [39, 62–65]. The temperature of a BH horizon is directly proportional to its surface gravity. In Einstein gravity theory, the BH horizon entropy is proportional to its horizon area, i.e., the BH entropy area law. Throughout this letter we will use  $\hbar = c = k_B = 1$ . In the context of the usual BH area entropy law,  $S = A/4G$ . We will also assume that the number  $N$  of DOF of the horizon satisfy the standard equipartition law [66]

$$M = \frac{1}{2}NT, \quad (3)$$

where  $T$  is the temperature and  $M$  is the BH mass and we assume here that there are no interaction among the DOF of a BH.

Our main target will be the Schwarzschild BH entropy, which will depict the horizon. Following Barrow deformed entropy, given in Eq. (2) for BHs [67] it is given by

$$S_B = \left(\frac{A}{4G}\right)^{1+\frac{\Delta}{2}}, \quad (4)$$

where  $G$  is the gravitation constant,  $4G$  is the Planck area and  $A$  is the standard horizon area. In BH physics, the area  $A$  of the horizon can be associated with the source mass  $M$  through the relation

$$A = 16\pi G^2 M^2. \quad (5)$$

It will be assumed that the number of DOF,  $N$ , of the horizon obey the standard equipartition theorem in Eq. (3) above [66]. In [61], we substituted the area in Eq. (5) into Eq. (4), we had that

$$S_B = \left(\frac{16\pi G^2 M^2}{4G}\right)^{1+\frac{\Delta}{2}} \Rightarrow S_B = \left(4\pi G M^2\right)^{1+\frac{\Delta}{2}}. \quad (6)$$

The temperature is given by

$$\frac{1}{T} = \frac{\partial S(M)}{\partial M}, \quad (7)$$

where the expressions representing both BH entropy and temperature have a kind of universality since both the horizon area and surface gravity are geometric quantities altogether, determined by the space-time geometry [68].

Now, using Eq. (6)

$$\frac{1}{T} = (\Delta + 2) \left(4\pi G\right)^{1+\frac{\Delta}{2}} M^{1+\Delta}. \quad (8)$$

If the area increases by a scaling  $A \rightarrow \kappa A$ , where  $\kappa \geq 1$ , consequently the BH's Hawking lifetime  $t_{BH}$ , fall as

$$t_{BH} \propto \frac{M^3}{\kappa^2} \propto \frac{1}{T^3 \kappa^2} \quad (9)$$

since we saw in the last equation that  $T \propto M^{-1}$  and  $\kappa$  increases. Together with an increasing temperature, if there is not upper bound on  $\kappa$ , primordial BHs will explode very rapidly [55].

We will use that the number of DOF,  $N$ , in the horizon can be obtained by [69]

$$N = 4S, \quad (10)$$

where  $S$  is the specific entropy that describes the horizon. Hence, using Eqs. (6) and (10) we have that

$$\frac{N}{4} = \left(4\pi G\right)^{1+\frac{\Delta}{2}} M^{2+\Delta}, \quad (11)$$

but, substituting the result in Eq. (8) for the temperature into Eq. (11), we have that

$$M = \frac{1}{2} \left(1 + \frac{\Delta}{2}\right) N T, \quad (12)$$

which corresponds to the horizon energy in Barrow's entropy model. From the last equation we can notice the appearance of an extra term  $\Delta/2$  in the usual equipartition theorem, Eq. (3). When we make  $\Delta = 0$  we recover the usual equipartition law.

At this point, it is important to clarify our proposition here. It is easy to understand that, for each entropy formulation we have two main quantities, namely, the number of DOF and the equipartition law. This last one is the energy of the event horizon. The standard entropy is the Bekenstein–Hawking one. Hence, we know that, for  $N = 4S$  DOF we have that  $M = 1/2 NT$ . So, for other formulations we have to change one of these two quantities. Let us explain better by taking Barrow entropy for example. It can be shown [70] that its number of DOF is  $N = 2(2 + \Delta) S_{\text{Barrow}}$ . However, if we calculate the respective temperature, the final result is that  $M = 1/2 NT$ , independent of  $\Delta$ , and we have the same final result for any DOF expression relative to a certain entropy. But, on the other hand, if we keep the same number of DOF relative to Bekenstein–Hawking entropy, of course we will have a new expression for the equipartition law. And it is exactly what we are doing here, we are following this second path. Namely, we will find a new equipartition law for each entropy formulation. In other words, considering the same number of DOF for Bekenstein–Hawking entropy, we will have different event horizon energies. In one of our previous works [71] we have obtained different horizon energies relative to different entropy formulations.

The heat capacity is given by

$$C = - \frac{[S'_{BH}(M)]^2}{S''_{BH}(M)}, \quad (13)$$

where the prime means a derivative relative to  $M$  and a negative heat capacity indicates that the temperature of the system, the horizon temperature, increases as it evaporates and the energy decreases, due to the Hawking radiation, which

means a thermodynamical unstableness. In fact, it is expected in BH scenarios since BHs are unstable thermodynamically. So, following [61], substituting Barrow's entropy in Eq. (6) into Eq. (13) we have that

$$C_B = - \frac{\left[ (4\pi G)^{1+\frac{\Delta}{2}} (2 + \Delta) \right]}{1 + \Delta} M^{2+\Delta}, \quad (14)$$

which means that, for stability, we must have that

$$\frac{2 + \Delta}{1 + \Delta} < 0 \implies -2 < \Delta < -1 \implies C_B > 0. \quad (15)$$

Hence, for thermodynamical coherence with the equipartition law, we have that  $\Delta$  has to be in the interval  $-2 < \Delta < -1$ . But the interval for  $\Delta$  is  $0 \leq \Delta \leq 1$  [55, 61]. Therefore, Barrow's BH is unstable, as it is expected.

For  $\Delta = 0$  in Eq. (14), for a smooth spacetime structure, we have that

$$C_B = -8\pi GM^2, \quad (16)$$

which reproduces the usual value of the heat capacity of a BH and it means, as well known, that a BH is thermally unstable. The negative heat capacity in this scenario means that a slight drop in BH's temperature will cause an extra drop as the energy keeps being absorbed.

For  $\Delta = 1$  in Eq. (14) we have that

$$C_B = -12(\pi G)^{3/2} M^3. \quad (17)$$

which corresponds to the heat capacity of a maximal deformation of spacetime, i.e., for the most intricate.

As we said before,  $\Delta$  represents the quantum fluctuations and the fractal feature of spacetime, which are motivated by LQG [55]. At the same time, we know that the introduction of quantum effects, motivated by LQG caused by thermal equilibrium and quantum fluctuations, conduct us to the curvature correction in Einstein's action known as the logarithmic entropy-correction described above in Eq. (1). Hence, we see that it is completely adequate to carry out the logarithmic correction concerning Barrow's entropy since both are connected by LQG fractal and quantum features. From now on, we will demonstrate this last assertion and its consequences.

### 3 Barrow logarithm corrected-entropy

Now, our main target here is to consider the logarithmic correction, from Eq. (1) and using Eq. (5) we can write that

$$S = 4\pi GM^2 + \tilde{\alpha} \ln(4\pi GM^2) + \tilde{\beta}. \quad (18)$$

and substituting this equation into Eq. (7) we can calculate the temperature, which is

$$T = \frac{M}{2} \left( \frac{1}{\tilde{\alpha} + 4\pi GM^2} \right), \quad (19)$$

and the number of DOF is

$$N = 4S = 16\pi GM^2 + 4\tilde{\alpha} \ln(4\pi GM^2) + 4\tilde{\beta}. \quad (20)$$

It is very important to stress that, as we said before,  $\tilde{\alpha}$  and  $\tilde{\beta}$  can have negative values, but we are considering that these negativeness does not affect the positiveness of  $S$ ,  $T$  and  $N$ .

From Eq. (19) we can write that

$$M = (8\pi GM^2 + 2\tilde{\alpha})T \implies 4\pi GM^2 = \frac{M}{2T} - \tilde{\alpha}, \quad (21)$$

and from Eq. (20) we have

$$\begin{aligned} \frac{N}{4} &= 4\pi GM^2 + \tilde{\alpha} \ln(4\pi GM^2) + \tilde{\beta} \\ &\cong 4\pi GM^2 + \tilde{\alpha} - \frac{\tilde{\alpha}}{4\pi GM^2} + \tilde{\beta}, \end{aligned} \quad (22)$$

where the second and higher orders of  $1/(4\pi GM^2)$  of the logarithm expansion were neglected since the most popular values of the pre-factors are small values around unity [43]. Besides, since the equipartition law is an average value of energy, we can also use an approximate value for the energy of the logarithmically corrected-entropy.

Substituting Eq. (21) into Eq. (22) we can write that

$$\left( \frac{M}{2T} \right)^2 - \left( \tilde{\alpha} - \frac{N}{4} - \tilde{\beta} \right) \left( \frac{M}{2T} \right) + \tilde{\alpha} \left( \frac{N}{4} - \tilde{\beta} - 1 \right) \cong 0, \quad (23)$$

which means that

$$M \cong \left[ \frac{N}{4} + \tilde{\alpha} - \tilde{\beta} \pm \sqrt{\left[ \frac{N}{4} - (\tilde{\alpha} + \tilde{\beta}) \right]^2 + 4\tilde{\alpha}} \right] T. \quad (24)$$

Hence, we have two options, but we know that, for  $\tilde{\alpha} = \tilde{\beta} = 0$ , we must have  $M = 1/2NT$ . Substituting these zero values for  $\tilde{\alpha}$  and  $\tilde{\beta}$  in both roots we have that the correct root is

$$M \cong \left[ \frac{N}{4} + \tilde{\alpha} - \tilde{\beta} + \sqrt{\left[ \frac{N}{4} - (\tilde{\alpha} + \tilde{\beta}) \right]^2 + 4\tilde{\alpha}} \right] T, \quad (25)$$

which corresponds to the horizon energy for a logarithmically corrected-entropy model. The other root, the negative one, results  $M = 0$ .

For the heat capacity in Eq. (13), using Eq. (18)

$$C = -2 \frac{(4\pi GM^2 + \tilde{\alpha})^2}{4\pi GM^2 - \tilde{\alpha}} , \quad (26)$$

which means that  $\tilde{\alpha} < 4\pi GM^2$  reflects the thermodynamical unstableness of a BH and established a condition on  $\tilde{\alpha}$ .

Let us construct a corrected-entropy version of Barrow's BH formulation, hence,

$$S = \left(\frac{A}{4G}\right)^{1+\frac{\Delta}{2}} + \tilde{\alpha}_1 \ln\left(\frac{A}{4G}\right) + \tilde{\beta} , \quad (27)$$

where  $\tilde{\alpha}_1 = \tilde{\alpha}\left(1 + \frac{\Delta}{2}\right)$ . Using Eq. (5), we have that

$$S = \left(4\pi GM^2\right)^{1+\frac{\Delta}{2}} + \tilde{\alpha}_1 \ln\left(4\pi GM^2\right) + \tilde{\beta} . \quad (28)$$

The temperature is given by

$$\begin{aligned} T &= \frac{M}{2} \left( \frac{1}{\tilde{\alpha}_1 + 2^{2+\Delta}(\pi G)^{1+\frac{\Delta}{2}} M^{2+\Delta}} \right) \\ \implies M &= 2 \left( \tilde{\alpha}_1 + 2^{2+\Delta}(\pi G)^{1+\frac{\Delta}{2}} M^{2+\Delta} \right) T , \end{aligned} \quad (29)$$

which means that we can write conveniently that

$$\frac{M}{2T} - \tilde{\alpha}_1 = 2^{2+\Delta}(\pi G)^{1+\frac{\Delta}{2}} M^{2+\Delta} . \quad (30)$$

The number of DOF is

$$N = 4 \left[ \left(4\pi GM^2\right)^{1+\frac{\Delta}{2}} + \tilde{\alpha} \ln\left(4\pi GM^2\right)^{1+\frac{\Delta}{2}} + \tilde{\beta} \right] , \quad (31)$$

where, using the same logarithm expansion and neglecting the  $1/M^2$  higher order terms, using Eq. (30) and after some algebra we have that

$$\begin{aligned} \left(\frac{M}{2T}\right)^2 - \Lambda_1 \frac{M}{2T} + \Lambda_2 &\cong 0 \\ \implies M &\cong \left[ \Lambda_1 \pm \sqrt{\Lambda_1 - 4\Lambda_2} \right] T , \end{aligned} \quad (32)$$

where

$$\Lambda_1 = \frac{N}{4} + \tilde{\alpha}(1 + \Delta) - \tilde{\beta} , \quad (33)$$

and

$$\Lambda_2 = \tilde{\alpha}_1 \left(1 + \frac{N}{4} - \tilde{\alpha} - \tilde{\beta}\right) - \tilde{\alpha} , \quad (34)$$

where Eq. (32) represents the horizon approximate energy for Barrow's logarithmically corrected-entropy model. For  $\tilde{\alpha} = \tilde{\beta} = 0$  we have  $\Lambda_1 = \frac{N}{4}$  and  $\Lambda_2 = 0$ . Considering the positive root of Eq. (32) we have  $M \cong \frac{1}{2}NT$ , and the negative one,  $M = 0$ . It is important to remember that  $\tilde{\alpha}_1 = \tilde{\alpha}\left(1 + \frac{\Delta}{2}\right)$ , so for  $\tilde{\alpha} = 0$  we have  $\Lambda_2 = 0$  independent of the value of Barrow's exponent  $\Delta$ .

Computing the heat capacity for the entropy in Eq. (27), its value is,

$$C = \frac{\left[ 2\tilde{\alpha}_1 + (2 + \Delta)(4\pi G)^{1+\frac{\Delta}{2}} M^{2+\Delta} \right]^2}{2\tilde{\alpha}_1 - (2 + \Delta)(1 + \Delta)(4\pi G)^{1+\frac{\Delta}{2}} M^{2+\Delta}} , \quad (35)$$

where the numerator is all positive, so the negative sign, or not, has to come from the denominator. Hence, for a positive heat capacity we must have

$$\tilde{\alpha} > (1 + \Delta)(4\pi G)^{1+\frac{\Delta}{2}} M^{2+\Delta} , \quad (36)$$

since  $\tilde{\alpha}$  is close to unity, this condition can not be obeyed and we have an unstableness scenario which is expected. However, it is very well known that Schwarzschild black holes have negative heat capacity, i.e., they are unstable. Consequently, we have in fact an unstableness condition on  $\tilde{\alpha}$ , which is

$$\tilde{\alpha} < (1 + \Delta)(4\pi G)^{1+\frac{\Delta}{2}} M^{2+\Delta} . \quad (37)$$

Let us analyze some special cases of these conditions.

For  $\Delta = 0$ , the Bekenstein–Hawking entropy case, the heat capacity in Eq. (36) is given by

$$C = 2 \frac{\left[ \tilde{\alpha} + 4\pi GM^2 \right]^2}{\tilde{\alpha} - 4\pi GM^2} , \quad (38)$$

which confirms the condition from Eq. (26) for unstableness.

For  $\Delta = 1$ , fractal case scenario,

$$C = 3 \frac{\left[ \tilde{\alpha} + (4\pi G)^{3/2} M^3 \right]^2}{\tilde{\alpha} - (4\pi G)^{3/2} M^3} , \quad (39)$$

and the fractality condition changes the unstableness condition on  $\tilde{\alpha}$ , which is now,  $\tilde{\alpha} < (4\pi G)^{3/2} M^3$ .

We can see clearly that considering the standard Barrow's entropy expression, when the fractal effect factor  $\Delta$  is zeroed, i.e., when it is withdrawn from the expression in Eq. (12), we obtain the standard equipartition theorem, which is a classical expression. However, from Eqs. (32) to (34), it is also clear that we do not need to touch the fractal factor in the corrected equipartition law to obtain the standard classical one. Hence,

if  $\Delta$  represents also the quantum fluctuations, the classical or semi-classical approximation procedure here does not mean to zero directly the “quantum” object, as usual. Notice that we are not saying that  $\Delta$  is a quantum object, which is not, of course. But it represents the effect of quantum fluctuations.

On the other hand, since this “quantum” object is directly connected to  $\tilde{\alpha}$ , via the value of  $\tilde{\alpha}_1$  and its effects on Eq. (32), we can say that in the case of logarithmically corrected-Barrow’s entropy, the  $\tilde{\alpha}$  can be associated, in some still unknown way, with the fractality of spacetime. We believe that this is a new physical interpretation for the  $\tilde{\alpha}$ -parameter. Or  $\tilde{\alpha}$  can also be responsible for the quantum to classical approximation, which is also a new interpretation for  $\tilde{\alpha}$ .

#### 4 Conclusions and final remarks

To conclude, we can mention that Barrow’s entropy originates from the fact that the BH surface can be perturbed by the so-called quantum gravitational effects. In other words, we could expect that quantum fluctuations of space-time can cause a modification of the topology of spacetime at the Planck scale. The result would be a foam-like framework called the spacetime foam. Hence, considering Barrow’s formulation, we can measure its deviation from the Bekenstein–Hawking entropy through a new exponent  $\Delta$ , where  $\Delta = 0$  means Bekenstein–Hawking entropy, and  $\Delta = 1$  means the most intricate case.

In this work we have investigated the logarithmic correction of both Bekenstein–Hawking and Barrow’s entropies. We calculated the expression of the equipartition law, which corresponds to the horizon energy in both entropic models. After that, to observe the application of the thermodynamical coherence of the models, we have calculated the heat capacity of both systems, which must be a positive quantity. Concerning both formulations, we have analyzed the positivity condition based upon the value of the  $\tilde{\alpha}$ -parameter. Since the interval known for the validity of  $\Delta$  is positive, the results obtained showed that both BH’s models are unstable, as it is expected. We obtained conditions on  $\tilde{\alpha}$  for some different scenarios, confirming that the value of the pre-factor is dependent of the model.

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