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# Non-stationary free-space geodesic dynamics due to the interaction with weak gravitational waves

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## Abstract

In this paper, the author investigates how non-stationary reference frames affect the observed trajectories of test particles, and how these observed effects might allow for the detection of gravitational waves. Most gravitational wave experiments rely on observing geometric oscillations of the spacetime between stationary test masses (with respect to each other). The results of this study indicate that an impinging gravitational wave, even of very weak amplitude, can cause much more noticeable effects between moving reference frames. The concepts herein are loosely analogous to how the properties of a magnetic field can be obtained from observing a charged particle's cyclotron motion through it. To this end, the linearized geodesic differential equations are solved to obtain the coordinates of a particle along a path, and explicit trajectories are calculated for a wide range of gravitational wave parameters. It is found that the angular deviation in trajectory is dependent on the polarization state of the wave and amplitude, leading to the possibly of extracting these parameters from the deviation for real-world experiments. The approach here represents a first step and as such, there are many simplifying assumptions, which will be relaxed slowly over time in future work.

## 1. Introduction

Gravitational waves, first introduced by Einstein [1, 2] and studied by many afterwards [3–10], continue to be an active topic of research in the current literature, including theoretical investigations using both linearized theory and the exact solutions [11, 12], and of course physical detection for experimental validation of these theories [13]. Most of these detectors are based on measuring the oscillating spacetime between two stationary test masses via interferrometric setups. The resulting spacetime oscillations between the masses are very small and as such, require extremely sensitive and expensive hardware, necessitating the cooperation of nations around the globe for successful implementation and measurement. Therefore, alternative strategies of gravitational wave detection which may produce more pronounced effects and less expensive experiments are highly sought after.

This motivation for this paper is to use explicit geodesic trajectories in the presence of a gravitational wave to probe whether or not it is plausible to study how these trajectories may deviate during a wave event. It has been known that gravitational waves can impart momentum on an object under certain conditions [14, 15], and perhaps if observed long enough, slight deviations in how a particle is moving can infer the earlier presence of a gravitational wave. In other words, instead of relying on stationary test masses, perhaps the observed effects from moving test masses will have more noticeable effects, similar[1] to how the properties of a magnetic field can be obtained by allowing a charged particle to move within the field to produce cyclotron motion. Indeed it has been shown that Lorentz type motions can occur to masses in gravitational fields [16]. The issue is that a moving object with respect to an outside observer would quickly move away from this observer before a wave event could arrive. Perhaps however, planetary geodesic motion can alleviate this issue by allowing the observer to remain on Earth and monitor the object in orbit indefinitely. Another idea would be to observe orbits in far away

[1] But still distinctly different.

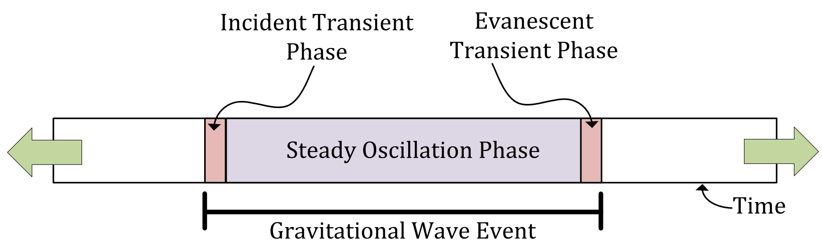


**Figure 1.** Depiction of the time region where the proposed model is valid (shown as PURPLE color). The approach does not take into account transient dynamics from either the incident, or evanescent sides of the wave (shown as RED color).

galaxies to increase the relative velocity between the observer and the particle in motion and enhance the measurement further.

More explicitly, suppose one sends a small cube-satellite into orbit around the Earth. It's trajectory can be determined to a high degree of accuracy using Schwarzchild geodesics, or Newtonian gravity. Now suppose a gravitational wave impinges on the satellite and causes its path to deviate slightly. Since the satellites motion can be monitored indefinitely, over a long period of time, a slow deviation from its predicted orbit can be detected and a gravitational wave can be inferred. If more than one satellite is used, coincident deviation amongst all of them would add confidence to the measurement.

Such an experiment is difficult to solve mathematically in the general case, and this paper represents the first step in probing for plausibility of this idea by setting up a much simpler problem to see if the predicted dynamics occur. Namely, observing particle motion in empty space using linearized gravity. In the most general case, the spacetime geometry is described by the Einstein field equations, which are second order nonlinear differential equations that are very difficult to solve and produce complicated solutions. If however the spacetime curvature can be accurately assumed to be very weak with respect to the flat space background, one can perturbatively expand Einstein's field equations and produce linear wave equations that are much simpler. This becomes particularly helpful when using these simplified metric components to calculate geodesic trajectories.

Linearized theory helps in that the metric components can be represented as simple plane waves, and the metric overall component functions are less complicated. The geodesic equations from the linearized metric are able to be solved, and these solutions show that there is indeed a trajectory deviation from the impinging wave which agrees with previous literature reasonably well [17], and gives plausibility to future study as will be explained in the final section.

Since the solutions here utilize linearized gravity with a monochromatic (single frequency component) assumption, the corresponding physical scenario does not include transient metric dynamics during the initial impingment or the end of the wave event. But rather represents a wave event which extends in time. Comparisons on trajectories can then be made by comparing to completely flat space, as will be shown. Figure 1 shows the locations in time where this model is valid. If one wishes to include transient dynamics, the spectral composition of the wave must be assumed to be comprised of many frequency components that together manifest the transient time domain response on each end of the wave event.

The structure of the paper is as follows; First, the linearized gravity problem is setup, and the corresponding Christoffel symbols and geodesic differential equations are determined. Second the equations are solved, and simplified. Third, sanity checks on the solutions are done by taking the limit as the wave amplitudes and frequency go to zero, which reduces to simple linear motion in flat space. Fourth, explicit trajectories are calculated to show the deviation and the observed oscillatory motion. Finally, conclusions and real-world considerations are presented, as well as a framework for future follow-on studies is given to slowly make the problem more rigorous.

## 2. Current methods of wave detection, and motivation for proposed approach

Over the decades, there have been a number of techniques developed to detect gravitational waves [2, 18, 19]. The first of which was a resonant mass antenna, often called a Weber bar antenna, after its creator Joseph Weber [20]. The premise of this antenna is to rely on the gravitational wave to excite the natural resonant frequency of a solid, isolated mass. Measurement of this vibrational mode would indicate the presense of a graviational wave. It is generally accepted that these devices are not sensitive enough to measure the very small strains a gravitational wave event would induce in the mass.

The next type of wave detector utilizes the very regular electromagnetic pulses emitted from pulsars. As a wave passes by, it will stretch and compress the spacetime between the event and the Earth, and this will cause small changes in the periodicity of pulsar events. By observing the regular beats of many pulsars over a long period of time, the presence of the wave can be inferred. This method however is limited to very low frequency waves (nanoHertz regime) and therefore have a very long integration time required to obtain results [21].

The best known approach for detection of gravitational waves utilizes a laser interferometer. This method is responsible for the very first ever gravitational wave detection at the Laser Interferometer Gravitational Wave Observatory (LIGO), therefore it has been experimentally proven to be a viable technique and continues to be in use [13].

An issue with interferometers, and gravitational wave detectors in general is that the length oscillation imposed by the wave is extremely small, on the order of $10^{-23}$–$10^{-18}$ meters, thus detection is extremely challenging. Interferometers in particular, utilize the effects of geodesic deviation. A laser is split and each output branch of the laser is sent towards a mirror very far away. The two arms are tuned to produce a known interference at the detector and the small space oscillations from the wave will cause the path lengths of the arms, and therefore the interference to change, which will yield a detection. The mirrors act as test masses that are stationary with respect to one another and are traveling along two different, but nearby geodesic paths. Therefore, one can utilize the theory of geodesic deviation to determine how these two nearby geodesics change with respect to one another.

In general, it can be shown that for an initial separation of $\Delta_0$ between two test masses, the oscillation, or deviation $\Delta$ between the masses from the gravitational wave event is found to be of the general form: [22]

$$\Delta(x^{\mu}) \sim h\Delta_0 e^{ik_{\mu}x^{\mu}} \tag{1}$$

where $h$ is a component of the linearized metric tensor (more details on this in the upcoming sections), the complex exponential term describes the oscillation from the wave, $k_{\mu}$ is the wave four vector, and $x^{\mu}$ are the coordinates. In general, the exact deviation depends on the wave polarization, but these details are not important for this discussion.

The noteworthy observation of the above expression is that, in general, one wishes to maximize $\Delta(x^{\mu})$ so as to make the measurement more experimentally tractable. The experimenter only has control over the distance between the test masses $\Delta_0$ to maximize the measurement, as $h$ describes the (very small) amplitude of the incident wave and is the quantity being measured. Thus, experimental efforts focus on making the arms of the interferometer as long as possible to maximize these effects.

It would be most desirable if there were more the experimenter could do to impose a stronger measurement response from the incident wave. This motivates the discussion of this paper. The proposed approach presented here can be understood by the following analogy.

Suppose one wishes to measure the strength of a magnetic flux density field $B$. The experimenter knows that if you launch charged particles with mass $m$ and charge $Q$, with some velocity component $v_{\perp}$ perpendicular to the field, the charge will undergo cyclotron motion. More specifically, its trajectory will produce curved paths, or spirals. The radius of this spiral path is called the cyclotron radius $r$, and is equal to [23]:

$$r = \frac{mv_{\perp}}{|Q|B} \tag{2}$$

Therefore, if one can measure this radius $r$ by observing the particle trajectory, the only unknown becomes the strength of the field $B$, which can be found. The other parameters in the above equation can be modified to maximize the measurement response depending on the value of $B$. For example, to increase the observed radius, the velocity or mass of the particle can be increased, or its charge can be decreased.

In regard to gravitational waves, perhaps if the experimenter and a test mass are moving with respect to one another, the measurement will include velocity terms as well as distance terms, and there will be more ability to influence the measurement response from the wave event. This in indeed what will be found, as well as amplitude contributions from the wave's frequency. All of these effects together cause the particle trajectory with respect to an outside observer to deviate in path and in an oscillatory manner. The exact deviation and magnitude can be used to determine the wave polarization. As was mentioned previously, in order for the particle to not fly off and away with respect to an observer, one can observe the trajectory over time of a particle in orbit, or perhaps even of orbits very far away in other galaxies, so as to increase the relative velocity and enhance the measurement.

## 3. Linearized gravity, and the lowest order geodesic solutions

In this section, linearized gravity is briefly reviewed to set to stage to solve the resulting equations. These details will become important for later arguments when analyzing explicit particle trajectories.

Linearized gravity assumes the spacetime curvative is small enough in magnitude to allow us to perturbatively expand the metric up to linear order:

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} \tag{3}$$

where $g^{\mu\nu}$ are the components of the metric tensor, $\eta^{\mu\nu}$ are the components of the flat space metric tensor, and $h^{\mu\nu}$ is the metric perturbation which gives the spacetime curvature. Such a linearization is very reasonable as we assume that the incident gravitational waves are from some far away source and are thus, very weak. One can show that when changing coordinates such that $(x')^{\mu} = x^{\mu} + \zeta^{\mu}$, the transformed metric $(h')_{\mu\nu}$ is given by:

$$(h')_{\mu\nu} = h_{\mu\nu} + \partial_{\nu}\zeta_{\mu} + \partial_{\mu}\zeta_{\nu} \tag{4}$$

where all components are functions of the new coordinates $x'$. From here, the *trace-reversed* metric perturbation is defined as $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\alpha}{}_{\alpha}$. This yields the following relationship:

$$(\bar{h}')^{\mu\nu} = \bar{h}^{\mu\nu} + \partial^{\nu}\zeta^{\mu} + \partial^{\mu}\zeta^{\nu} - \eta^{\mu\nu}\partial_{\alpha}\zeta^{\alpha} \tag{5}$$

Taking the partial derivative gives:

$$\partial_{\mu}(\bar{h}')^{\mu\nu} = \partial_{\mu}\bar{h}^{\mu\nu} + \partial_{\mu}\partial_{\nu}\zeta^{\nu} \tag{6}$$

Thus, in order to impose the Lorentz gauge condition $\partial_{\mu}(\bar{h}')^{\mu\nu} = 0$, we must choose coordinate such that they satisfy the following wave equation:

$$\partial_{\mu}\partial_{\nu}\zeta^{\nu} = -\partial_{\mu}\bar{h}^{\mu\nu} \tag{7}$$

Using this gauge condition in the linearized Einstein field equations leads to flat space wave equations describing the metric perturbation with non-zero, wave-like solutions for the 00, 0$i$, and $ij$ components. Imposing an additional contribution $\rho$ to $\zeta$ such that $\partial_{\alpha}\partial^{\alpha}\rho^{\mu} = 0$ leaves the underlying physics unchanged but simplifies the solutions to [16, 22]:

$$\begin{aligned} h^{00} &= 0 \\ h^{0i} &= 0 \\ h^{ij} &= \mathrm{Re}\left(\int A^{ij}(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega\tau)}d^3k\right) \end{aligned} \tag{8}$$

where we have dropped the prime and bar notation to reduce clutter. Here, roman indices represent spatial components, $\omega$ is the angular frequency, $\mathbf{x} = (x^1, x^2, x^3)$, and $A^{ij}(\mathbf{k})$ are the wave magnitudes in the integral expansion which are functions of the wave vector $\mathbf{k}$. For further simplicity, we (i). assume that there is only one gravitational wave incident on the particle and this wave is monochromatic (contains only one frequency), and (ii). rotate the coordinate system such that the incident wave is traveling in the +z direction. Since $|\mathbf{k}|$ is null, it must be the case that:

$$k^{\mu} = (\omega,\ 0,\ 0,\ k^3) = (\omega,\ 0,\ 0,\ \omega) \tag{9}$$

where we have set the speed of light $c$ equal to 1 and will restore it at the end of the calculations. Due to the gauge condition $\partial_{\mu}h^{\mu\nu} = 0$, we have:

$$\mathrm{Re}\,(A_{ij}k^{i}e^{\mathbf{x}\cdot\mathbf{k}}) = 0 \rightarrow -A_{0j}k^0 + A_{3j}k^3 = 0 \rightarrow A_{0j} = A_{3j} = 0 \tag{10}$$

Thus, in the matrix representation, we have:

$$A_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{11}$$

where the two diagonal terms are equal and opposite because the metric perturbation is, by construction, trace-free. The two remaining degrees of freedom shown here are relabeled as $A_{11} = A_{+}$ and $A_{12} = A_{\times}$ to define the wave's polarization state (Plus or Cross polarized).

Since the new coordinate contribution $\zeta$ solves the wave equation, this implies wave-like coordinates are imposed. It turns out, that in our chosen coordinate system, whereby the wave is traveling along the $+z$ direction, the Lorentz gauge $\bar{h}^{\mu\nu}$ is already satisfied, and there is no need to impose these wave-like coordinates. This detail will become important later when analyzing the test mass trajectories, as we can be assured that the wave-like motion is not due to the coordinates imposed.

We wish to determine the dynamics of a particle given an incident gravitational wave. We start with the geodesic equation:

$$\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\alpha\beta}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau} = 0 \tag{12}$$

where $\tau$ is the proper time of the particle which represents an affine parametrization, and $\Gamma^{\mu}{}_{\alpha\beta}$ are the Christoffel symbols. The Christoffel symbols are given by:

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha h_{\beta\nu} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}) \tag{13}$$

Performing the calculation on each component gives:

$$\Gamma^{0}{}_{00} = \frac{1}{2}\eta^{0\nu}(\partial_0 h_{0\nu} + \partial_0 h_{0\nu} - \partial_\nu h_{00}) = 0 \tag{14}$$

$$\Gamma^{0}{}_{i0} = \Gamma^{0}{}_{0i} = \frac{1}{2}\eta^{0\nu}(\partial_i h_{0\nu} + \partial_0 h_{i\nu} - \partial_\nu h_{i0}) = \frac{1}{2}\eta^{0j}\partial_0 h_{ij} = 0 \tag{15}$$

$$\Gamma^{i}{}_{00} = \frac{1}{2}\eta^{i\nu}(\partial_0 h_{0\nu} + \partial_0 h_{0\nu} - \partial_\nu h_{00}) = 0 \tag{16}$$

$$\Gamma^{i}{}_{j0} = \Gamma^{i}{}_{0j} = \frac{1}{2}\eta^{i\nu}(\partial_j h_{0\nu} + \partial_0 h_{j\nu} - \partial_\nu h_{j0}) = \frac{1}{2}\eta^{i\nu}\partial_0 h_{j\nu} = \frac{1}{2}\partial_0 h_{ji} \tag{17}$$

$$\Gamma^{0}{}_{ij} = \frac{1}{2}\eta^{0\nu}(\partial_i h_{j\nu} + \partial_j h_{i\nu} - \partial_\nu h_{ij}) = \frac{1}{2}\partial_0 h_{ij} \tag{18}$$

$$\Gamma^{i}{}_{jk} = \frac{1}{2}\eta^{i\nu}(\partial_j h_{k\nu} + \partial_k h_{j\nu} - \partial_\nu h_{jk}) = \frac{1}{2}(\partial_j h_{ki} + \partial_k h_{ji} - \partial^i h_{jk}) \tag{19}$$

Having these on hand, the resulting differential equations from equation (12) can be found:

$$\frac{i}{\omega}\frac{d^2x^0}{d\tau^2} + \frac{1}{2}A_+\left[\left(\frac{dx^1}{d\tau}\right)^2 - \left(\frac{dx^2}{d\tau}\right)^2\right] + A_\times\frac{dx^1}{d\tau}\frac{dx^2}{d\tau} = 0 \tag{20}$$

$$\begin{aligned}\frac{1}{i\omega}\frac{d^2x^1}{d\tau^2} - A_+\left[2\frac{dx^0}{d\tau}\frac{dx^1}{d\tau} + \frac{dx^3}{d\tau}\frac{dx^1}{d\tau} + \frac{1}{2}\left(\frac{dx^2}{d\tau}\right)^2 - \left(\frac{dx^1}{d\tau}\right)^2\right]\\ +A_\times\left[-2\frac{dx^0}{d\tau}\frac{dx^2}{d\tau} + \frac{dx^3}{d\tau}\frac{dx^2}{d\tau} - \frac{3}{2}\frac{dx^1}{d\tau}\frac{dx^2}{d\tau}\right] = 0\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{1}{i\omega}\frac{d^2x^2}{d\tau^2} + A_+\left[\frac{1}{2}\left(\frac{dx^2}{d\tau}\right)^2 - \left(\frac{dx^1}{d\tau}\right)^2 - \frac{dx^3}{d\tau}\frac{dx^2}{d\tau} + 2\frac{dx^0}{d\tau}\frac{dx^2}{d\tau}\right]\\ +A_\times\left[\frac{dx^3}{d\tau}\frac{dx^1}{d\tau} - \frac{3}{2}\frac{dx^1}{d\tau}\frac{dx^2}{d\tau} - 2\frac{dx^0}{d\tau}\frac{dx^1}{d\tau}\right] = 0\end{aligned} \tag{22}$$

$$\frac{1}{i\omega}\frac{d^2x^3}{d\tau^2} + A_+\left[\frac{1}{2}\left(\frac{dx^2}{d\tau}\right)^2 - \left(\frac{dx^1}{d\tau}\right)^2\right] - \frac{3}{2}A_\times\frac{dx^1}{d\tau}\frac{dx^2}{d\tau} = 0 \tag{23}$$

Next we realize that the metric components do not depend on $x^1$ or $x^2$, but only $x^3$ and $\tau$. This allows us to determine two Killing vectors and use symmetry to immediately find two of the components. These Killing vectors are given by:

$${}_1\Omega = \langle 0, 1, 0, 0\rangle \quad {}_2\Omega = \langle 0, 0, 1, 0\rangle \tag{24}$$

where we have labeled each vector with a left-sided abstract index, so-as to allow the same symbol for each Killing vector. We can now use Killing's equation to more easily determine the coordinate solutions. Killing's equation states that if one can find Killing vectors ${}_{1,2}\Omega_\nu$, the quantity ${}_{1,2}\Omega_\nu u^\nu$ is a constant of motion and a conserved quantity. For the problem considered here, we have chosen a set of coordinates which results in the wave propagating in a single direction with respect to the test mass, and therefore the resulting metric is independent of the other two spatial coordinates. In more complicated problems, such as when there is more than one wave propagating towards the mass simultaneously from different directions, implementing Killing's equation will be more difficult, as the metric will in general be a function of all coordinates. For problems involving Schwartzchild spacetimes, there are symmetries along the time and azimuthal directions. The presence of a gravitational wave in these scenarios (which will be explored in future work) may however eliminate these symmetries. At any rate, utilizing Killing's equation here results in:

$$g_{\mu\nu}\ {}_{1,2}\Omega^\mu u^\nu = \chi_{1,2} \tag{25}$$

where $\chi_1$ and $\chi_2$ are conserved quantities in the manifold's tangent space (velocity) related to the momentum. Expanding this out gives:

$$\begin{aligned}g_{1\nu}u^\nu = (1 + A_+e^{i\omega(x^3/c-\tau)})\frac{dx^1}{d\tau} + (1 + A_\times e^{i\omega(x^3/c-\tau)})\frac{dx^2}{d\tau} = \chi_1\\ g_{2\nu}u^\nu = (1 + A_\times e^{i\omega(x^3/c-\tau)})\frac{dx^1}{d\tau} + (1 - A_+e^{i\omega(x^3/c-\tau)})\frac{dx^2}{d\tau} = \chi_2\end{aligned} \tag{26}$$

We note that these equations imply conservation of momentum in the 1 and 2 coordinate directions. If the wave amplitude goes to zero, both equations reduce to the same conservation law. The effect of the wave breaks the symmetry between the two component momentum conservation. Note also that the wave will not impart momentum on the particle if the interaction time is equal to the period of oscillation, as this will cause the exponential terms to be zero. Thus, the ability for the wave to impart momentum is intrinsically linked to this interaction time. This agrees with previous literature on the topic stating that momentum is only transferred if its time average over the interaction is non-zero [15].

Determining the value of the constants of motion $\chi_{1,2}$ is a matter of substituting the initial conditions. For time, this is $\tau = 0$. For the spatial component $x^3$, this is just equal to the initial position $X^3$. However, we must find values for the initial coordinate velocities $\frac{dx^{1,2}(\tau)}{d\tau}$. These represent the observed velocity of the particles from an external book keeper, but with respect to the particle's clock. It is more desirable to relate this to the book keeper's observed velocity with respect to his/her own clock. The oscillating acceleration makes this relationship non-trivial, unlike special relativity. However, it will be shown soon that the time dilation between the particles clock and the book keepers clock is so small that the velocities with respect to either clock can be approximated to be equal. However, the solution given below allows one to implicitly determine the time-dilation, even though there is no trivial Lorentz factor relationship. Thus, we set these initial conditions equal to the following:

$$\frac{dx^1(0)}{d\tau} = v^1, \qquad \frac{dx^2(0)}{d\tau} = v^2 \tag{27}$$

where $v^{1,2}$ is again to be generally understood as the initial velocity seen by the book keeper using the particle's clock, but can be approximated to be with respect to the book keepers clock. This gives for the constants of motion:

$$\chi_1 = (1 + A_+ e^{ik^3X^3})v^1 + (1 + A_\times e^{ik^3X^3})v^2 \tag{28}$$

$$\chi_2 = (1 + A_\times e^{ik^3X^3})v^1 + (1 - A_+ e^{ik^3X^3})v^2 \tag{29}$$

In general, the coefficients $A_\times$ and $A_+$ are complex in nature. Thus, the value of these coefficients can possibly introduce phase terms in the resulting solutions. In other words:

$$\tilde{A}_+ = |A_+|e^{i\phi_+} = A_+ e^{i\phi_+} \tag{30}$$

$$\tilde{A}_\times = |A_\times|e^{i\phi_\times} = A_\times e^{i\phi_\times} \tag{31}$$

where $\phi_{+,\times}$ are the phases associated with the plus and cross polarization components respectively.[2] The system of equations in (26) can be easily solved for the respective derivatives to obtain:

$$\frac{dx^1(\tau)}{d\tau} = \frac{e^{i\omega(x^3/c-\tau)}(\tilde{A}_+\chi_1 + \tilde{A}_\times\chi_2) - \chi_1 + \chi_2}{(\tilde{A}_+^2 + \tilde{A}_\times^2)e^{2i\omega(x^3/c-\tau)} + 2\tilde{A}_\times e^{i\omega(x^3/c-\tau)}} \tag{32}$$

$$\frac{dx^2(\tau)}{d\tau} = \frac{e^{i\omega(x^3/c-\tau)}(\tilde{A}_\times\chi_1 - \tilde{A}_+\chi_2) + \chi_1 - \chi_2}{(\tilde{A}_+^2 + \tilde{A}_\times^2)e^{2i\omega(x^3/c-\tau)} + 2\tilde{A}_\times e^{i\omega(x^3/c-\tau)}} \tag{33}$$

In their current form, these cannot be integrated directly since we do not know the functional form of the $x^3$ coordinate, which depends on the proper time. However, gravitational waves in this context (far field and weak in magnitude) predominantly cause spacetime oscillations in the transverse directions of propagation, and not along the propagation. Therefore, it can be reasonably assumed that the $x^3$ coordinate approximately follows simple linear motion given by $x^3(\tau) = v^3\tau + X^3$, where $v^3$ is the velocity in this direction. This gives:

$$\frac{dx^1(\tau)}{d\tau} = \frac{e^{-i(\omega-k^3v^3)\tau}e^{ik^3X^3}(\tilde{A}_+\chi_1 + \tilde{A}_\times\chi_2) - \chi_1 + \chi_2}{(\tilde{A}_+^2 + \tilde{A}_\times^2)e^{-2i(\omega-k^3v^3)\tau}e^{2ik^3X^3} + 2\tilde{A}_\times e^{-i(\omega-k^3v^3)\tau}e^{ik^3X^3}} \tag{34}$$

$$\frac{dx^2(\tau)}{d\tau} = \frac{e^{-i(\omega-k^3v^3)\tau}e^{ik^3X^3}(\tilde{A}_\times\chi_1 - \tilde{A}_+\chi_2) + \chi_1 - \chi_2}{(\tilde{A}_+^2 + \tilde{A}_\times^2)e^{-2i(\omega-k^3v^3)\tau}e^{2ik^3X^3} + 2\tilde{A}_\times e^{-i(\omega-k^3v^3)\tau}e^{ik^3X^3}} \tag{35}$$

One can see that the act of moving in the direction of the wave propagation will induce a shift in observed angular frequency, namely $\omega_{obs} = \omega - k^3v^3$. With this assumption on the $x^3$ coordinate, direct integration can now be performed, starting with the $x^1$ coordinate:

[2] One should replace $A_+$ and $A_\times$ with these complex relationships in the above equations for $\chi_1$ and $\chi_2$ and all others.

$$x^1(\tau) = X^1 + \mathrm{Re}\Bigg\{\frac{1}{4\tilde{A}_\times^2(k^3v^3-\omega)}[(\tilde{A}_+ + \tilde{A}_\times)(\tilde{A}_+(\chi_1-\chi_2) + \tilde{A}_\times(\chi_1+\chi_2))$$
$$\times\,(-\tau(\omega - k^3v^3) + i\ln(2\tilde{A}_\times + (\tilde{A}_+^2 + \tilde{A}_\times^2)e^{ik^3X^3}e^{-i\tau(\omega-k^3v^3)}))$$
$$-\,i(\tilde{A}_+ + \tilde{A}_\times)(\tilde{A}_+(\chi_1-\chi_2) + \tilde{A}_\times(\chi_1+\chi_2))\ln(2\tilde{A}_\times + (\tilde{A}_+^2 + \tilde{A}_\times^2)e^{ik^3X^3})$$
$$-\,2i\tilde{A}_\times(\chi_1-\chi_2)e^{-ik^3X^3}e^{i\tau(\omega-k^3v^3)} + 2i\tilde{A}_\times(\chi_1-\chi_2)e^{-ik^3X^3}]\Bigg\} \quad (36)$$

where $X^1$ is the initial position of the particle in motion in the $x^1(\tau)$ direction. Performing similar steps for the $x^2(\tau)$ coordinate gives:

$$x^2(\tau) = X^2 + \mathrm{Re}\Bigg\{\frac{1}{4\tilde{A}_\times^2(k^3v^3-\omega)}[(\tilde{A}_\times - \tilde{A}_+)(\tilde{A}_+(\chi_1-\chi_2) + \tilde{A}_\times(\chi_1+\chi_2))$$
$$\times\,(-\tau(\omega - k^3v^3) + i\ln(2\tilde{A}_\times + (\tilde{A}_+^2 + \tilde{A}_\times^2)e^{ik^3X^3}e^{-i\tau(\omega-k^3v^3)}))$$
$$+\,i(\tilde{A}_+ - \tilde{A}_\times)(\tilde{A}_+(\chi_1-\chi_2) + \tilde{A}_\times(\chi_1+\chi_2))\ln(2\tilde{A}_\times + (\tilde{A}_+^2 + \tilde{A}_\times^2)e^{ik^3X^3})$$
$$+\,2i\tilde{A}_\times(\chi_1-\chi_2)e^{-ik^3X^3}e^{i\tau(\omega-k^3v^3)} - 2i\tilde{A}_\times(\chi_1-\chi_2)e^{-ik^3X^3}]\Bigg\} \quad (37)$$

where $X^2$ is the initial position of the particle in motion in the $x^2(\tau)$ direction. In the above solutions for $x^{1,2}(\tau)$, although it is not explicitly clear, the solutions can be interpreted to be of the following form:

$$x^{1,2}(\tau) = \text{Constant}+\text{Oscillatory Term} + \text{Linear Term} \quad (38)$$

Where the linear term describes the straight line trajectory of the particle. In both cases, this linear term receives contributions from the natural log function. This can be roughly understood by imagining that the natural logarithm removes the complex exponential inside of it, leaving its linear argument. Of course, the nature of the logarithm argument makes this concept more complicated, as it may also produce oscillatory terms, but never-the-less, contributions to the linear term emerge from this sort of behavior.

Since we have already determined the approximate solution to $x^3$, the last step is to obtain the temporal coordinate solution $x^0(\tau)$. To do so, we use the results already obtained in equation (20) integrate twice, imposing $c$ for the temporal velocity initial condition, and $X^0$ for the initial time. Doing these calculations yield:

$$x^0(\tau) = \mathrm{Re}\Bigg\{\kappa_1\tau + \kappa_2 + \frac{1}{16\tilde{A}_\times^3(\omega-k^3v^3)^2}\Bigg[e^{-2ik^3X^3}\Bigg(\frac{1}{2}e^{2ik^3X^3}($$
$$+\,2\omega(\tilde{A}_+^2 + \tilde{A}_\times^2)(\tilde{A}_+(\chi_1-\chi_2) + \tilde{A}_\times(\chi_1+\chi_2))^2\Bigg(i\mathrm{Li}_2\Bigg(-\frac{2\tilde{A}_\times e^{i(\tau\omega - k^3(\tau v^3+X^3))}}{\tilde{A}_+^2+\tilde{A}_\times^2}\Bigg)$$
$$+\,i\ln(e^{-ik^3X^3}((\tilde{A}_+^2+\tilde{A}_\times^2)e^{ik^3(\tau v^3+X^3)} + 2\tilde{A}_\times e^{i\tau\omega}))$$
$$+\,\tau\Bigg(\omega + (k^3v^3-\omega)\Bigg(\ln\Bigg(1 + \frac{2\tilde{A}_\times e^{i(\tau\omega-k^3(\tau v^3+X^3))}}{\tilde{A}_+^2+\tilde{A}_\times^2}\Bigg)$$
$$-\,\ln(\tilde{A}_+^2 + \tilde{A}_\times(\tilde{A}_\times + 2e^{i(\tau\omega-k^3(\tau v^3+X^3))}))\Bigg)\Bigg)\Bigg)\Bigg) + i\tilde{A}_\times^2\omega(\chi_1-\chi_2)^2e^{2i\tau(\omega-k^3v^3)})\Bigg]\Bigg\} \quad (39)$$

where

$$\kappa_1 = c + \frac{1}{16c\tilde{A}_\times(\omega-k^3v^3)}\Bigg[i\omega\Bigg(\frac{i(\tilde{A}_+^2+\tilde{A}_\times^2)^2e^{ik^3X^3}(\tilde{A}_+(\chi_1-\chi_2)+\tilde{A}_\times(\chi_1+\chi_2))^2}{\tilde{A}_\times^2(2\tilde{A}_\times + (\tilde{A}_+^2+\tilde{A}_\times^2)e^{ik^3X^3})}$$
$$+\,\frac{i(\tilde{A}_+^2+\tilde{A}_\times^2)(\tilde{A}_+(\chi_1-\chi_2)+\tilde{A}_\times(\chi_1+\chi_2))^2\ln(\tilde{A}_+^2+\tilde{A}_\times(\tilde{A}_\times+2e^{-ik^3X^3}))}{\tilde{A}_\times^2}$$
$$-\,2i(\chi_1-\chi_2)^2e^{-2ik^3X^3})\Bigg] \quad (40)$$

$$\kappa_2 = X^0 - \frac{1}{16c\tilde{A}_\times^3(\omega-k^3v^3)^2}[e^{-2ik^3X^3}(\omega(\tilde{A}_+^2+\tilde{A}_\times^2)e^{2ik^3X^3}(\tilde{A}_+(\chi_1-\chi_2)+\tilde{A}_\times(\chi_1+\chi_2))^2$$
$$\times\Bigg(i\mathrm{Li}_2\Bigg(-\frac{2\tilde{A}_\times e^{-ik^3X^3}}{\tilde{A}_+^2+\tilde{A}_\times^2}\Bigg) + i\ln(e^{-ik^3X^3}(2\tilde{A}_\times + (\tilde{A}_+^2+\tilde{A}_\times^2)e^{ik^3X^3}))\Bigg) + i\tilde{A}_\times^2\omega(\chi_1-\chi_2)^2\Bigg)\Bigg] \quad (41)$$

## 4. Limiting cases of very weak and no gravitational wave

A basic check of the above solutions is to determine if the geodesic trajectories reduce to basic linear motion in the limit where there is no gravitational wave. Starting with the $x^1(\tau)$ coordinate, one can suppose that in the case of no gravitational wave, both polarization components can be set equal to each other, namely $\tilde{A}_\times = \tilde{A}_+ = A$, and then one can take the limit as $A \to 0$. This gives:

$$\begin{aligned}\bar{x}^1(\tau) &= \lim_{A\to 0}(x^1(\tau)|_{\tilde{A}_\times=\tilde{A}_+=A})\\ &= \tau(v^1+v^2) + X^1 + \frac{v^2\sin(\tau(\omega - k^3v^3))}{k^3v^3-\omega}\end{aligned} \tag{42}$$

This expression by itself is remarkable because it implies that in the limit of extremely weak gravitational waves, where the amplitude is so small to be considered zero, its oscillation will still cause noticeable trajectory deviations and oscillations to an external observer observing a particle in motion. We know from section 2 that due to our original problem setup, it was unnecessary to impose wavy coordinates to enforce the Lorentz condition, therefore these wave effects are not due to the coordinates. We denote this limiting equation with an over-bar as shown above. To obtain simple linear motion, we simply need to now take the limit as the angular frequency $\omega$ goes to zero, namely:

$$\lim_{\omega\to 0}(\bar{x}^1(\tau)) = X^1 + v^1\tau \tag{43}$$

This is simple linear motion. If the particle approaches the speed of light in the $x^3$ direction ($v^3 \to c$) for $\bar{x}^1$, one also reduces to simple linear motion. This is because the particle will now be co-moving with the gravitational wave, and the spacetime oscillations can no longer take place. If the particle approaches the speed of light in the other direction ($v^3 \to -c$), the frequency of oscillations will double, namely:

$$\lim_{v^3\to -c}(\bar{x}^1(\tau)) = \tau(v^1+v^2) + X^1 + \frac{v^2\sin(2\omega\tau)}{2\omega} \tag{44}$$

Similar steps can be taken with the $x^2(\tau)$ coordinate which yields:

$$\bar{x}^2(\tau) = X^2 + v^2\mathrm{sinc}\left(\left(1-\frac{v^3}{c}\right)\omega\tau\right)\tau \tag{45}$$

where the result has been manipulated to show explicitly that the observed $v^2$ velocity is modulated by the wave. This sort of behavior also occurs in the $x^1$ coordinate above. Taking $\omega \to 0$ gives:

$$\lim_{\omega\to 0}(\bar{x}^2(\tau)) = X^2 + v^2\tau \tag{46}$$

By contrast,the $x^0$ coordinate does not produce any interesting behavior as only $A \to 0$, it simply reduces to linear motion given by $\bar{x}^0 = c\tau + X^0$. There is no need to take the limit as $\omega \to 0$.

## 5. Analysis of the dynamics

In this section the trajectories of an observed particle are studied for various parametrizations of the given independent variables. Figure 2 shows the two transverse coordinates and the time coordinate of the moving mass as seen by a stationary observer for one set of parametric inputs, along with a comparison of simple linear motion and of the limiting equations given by equations (42) and (45) for the $x^1$ and $x^2$ coordinates respectively. The figure also shows the two trajectories in the three spatial dimensions. As can be seen, in comparison to linear motion, the $x^1$ and $x^2$ coordinates show a significant deviation in their trajectory. In contrast, the $x^0$ coordinate shows negligible difference compared to linear motion. The stepped motion of the $x^0$ coordinate in comparison to linear motion is likely a mathematical artifact do to (i). the range limitations from the arg(...) function phase ambiguities, and (ii). limitations in numerical precision.

Although all of the coordinate solutions are functions of the angular frequency of the wave $\omega$ and the initial positions and velocities, changing the value of these variables does not change the overall behavior of the plots in interesting ways other than introducing more or less cycles over time, or by changing the magnitude of these cycles. It is more interesting and useful to study how the solutions change as one changes the incident wave amplitudes and phases for the different polarization modes. The results of these types of sweeps are shown in figure 3. In these sweeps, the angular deviation was calculated by simply assuming that the space is approximately flat, and was therefore done by simply creating two vectors that follow the trajectory of the oscillating and simple linear motion path in the three spatial coordinates, and then using the flat space inner product to find the angle between the vectors. Since the local curvature is so small due to the amplitude of the

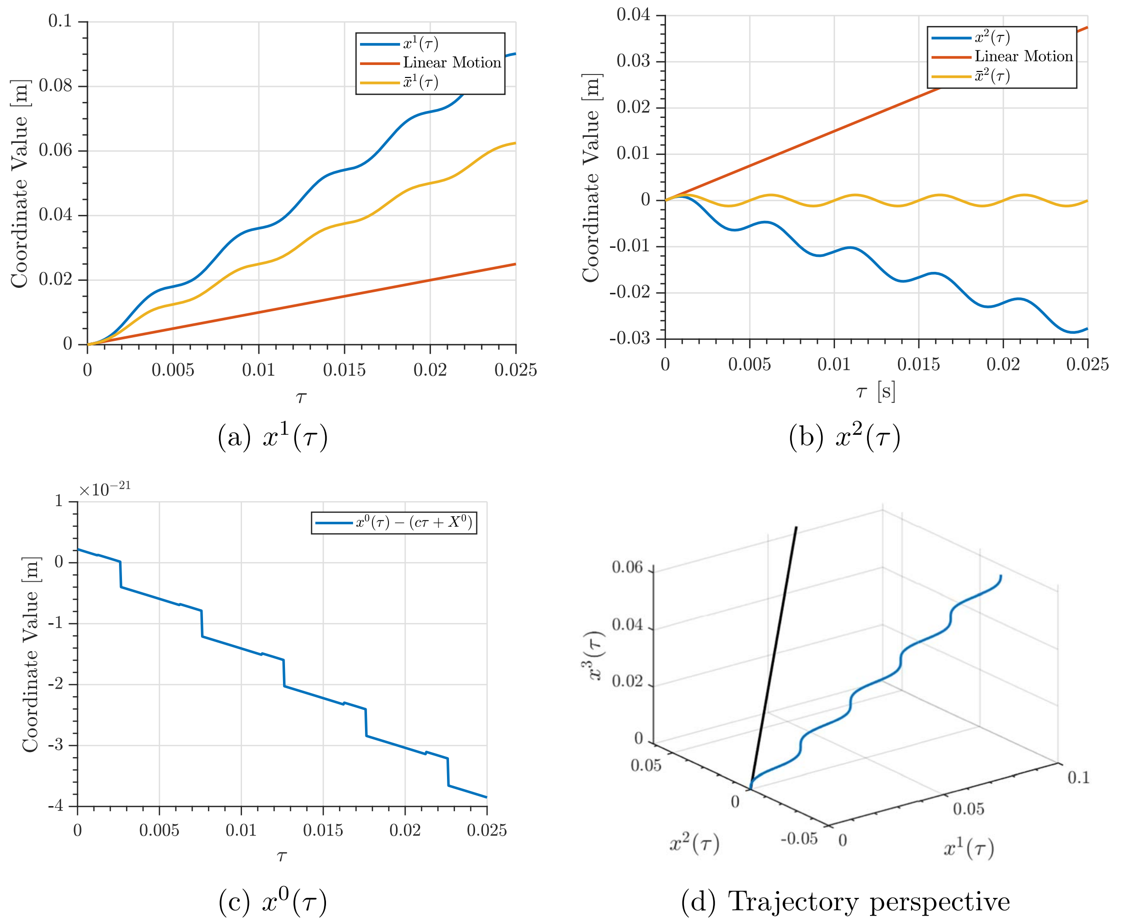


**Figure 2.** Example of the the trajectory deviations observed for one particular parametrization of variables. (a and b) Comparisons are between basic linear motion and the full solutions $x^{\mu}$, as well as the limiting expressions when the wave amplitude goes to zero $\bar{x}^{\mu}$; (c) The $x^0$ component changes very little, so this plot shows the difference between the full solution and basic linear motion. The steps are likely artifacts due to i) the limited range of the arg(...) function and ii) numerical limitations in precision; (d) Comparison of the perturbed trajectory (blue) to linear motion (black) in 3D space. Parameters in this example are: $|\tilde{A}_+| = 0.4 \times 10^{-21}$, $|\tilde{A}_\times| = 0.15 \times 10^{-21}$, $\phi_\times = \pi/2$, $\phi_+ = \pi/4$, $v^1 = 1$ m s$^{-1}$, $v^2 = 1.5$ m s$^{-1}$, $v^3 = 2.5$ m s$^{-1}$, $f = 200$ Hz, $X^{1,2,3} = 0$.

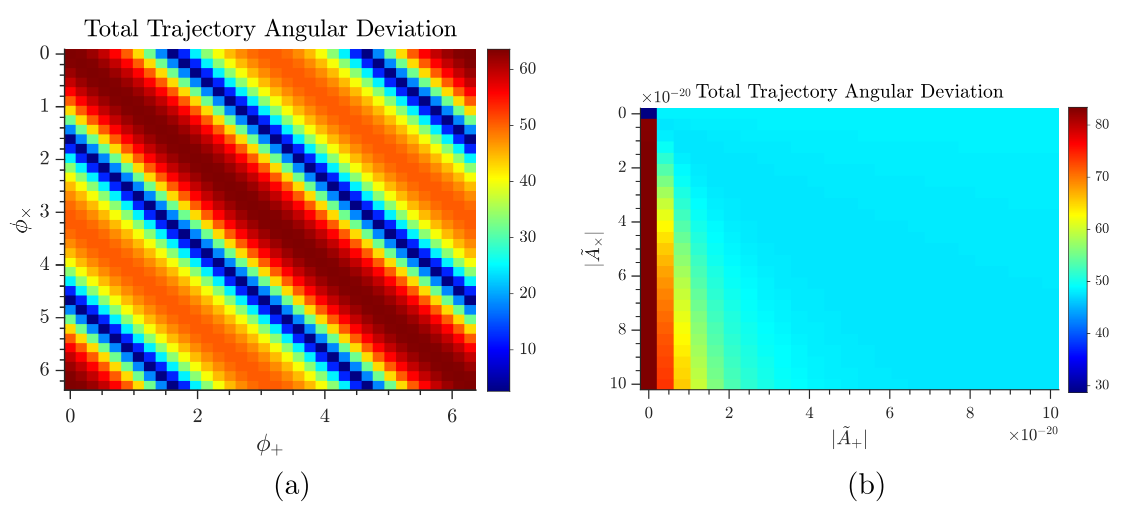


**Figure 3.** Trajectory angular deviation (in units of degrees) as a function of (a) wave phases for each polarization basis, and (b) wave amplitudes for each polarization basis. The parameters for both (a) and (b) were the same as described in figure 2 except for the sweeping variables.

gravitational wave, this methodology should lead to results with little error. Such a method would not be appropriate for highly curved manifolds.

As can be seen in figures 2 and 3, the impinging wave will impart a trajectory deviation on the particle, and depending on the polarization, amplitude, and phase properties of the gravitational wave, this will impart variations on these trajectories. There will however exist ambiguities in the mapping between these properties and the observed deviation. These ambiguities are seen explicitly in figure 3(a) where there are multiple points with the same color. This issue can be alleviated by having multiple test masses of differing velocities, and observing trajectory deviations of all of them at once.

## 6. Conclusions, real world considerations, and future work

In this paper, the geodesic equation was solved using the metric solution for monochromatic gravitational waves. It was found that the impinging gravitational wave causes spacetime oscillations in the particles motion as seen by an external observer. The oscillations are in the directions transverse to the direction of wave propagation, as is normally found for stationary particles in the conventional analysis of analyzing geodesic deviations or line element intervals. However, the particles motion also causes deviations in the temporal direction, but they are very small in magnitude. Most importantly, it was found that the particles motion with respect to an external observer causes the gravitational wave to impart a change in its trajectory, which might be able to be utilized to infer the presence of a gravitational wave.

As mentioned earlier, this work represents the first step in determining if it is possible to observe these deviations over time to infer the presence of a gravitational wave. There are a few options in how to proceed next. One could perform an identical analysis but instead use a Schwartzchild background metric to describe an orbit rather than motion in free-space. Another option is to do this analysis using *pp*-wave solutions, as this would represent the exact solution. Although the expected gravitational wave amplitudes justify the use of linearized theory, the perturbative expansion of the non-linear equations do not guarantee a completely accurate representation of the resulting particle dynamics.

The results herein present a number of questions regarding real-world implications. Most importantly, they suggest that (i) gravitational waves with very small amplitudes can lead to significant trajectory deviations on moving particles with respect to an external observer, and (ii) even if the wave amplitude is infinitely small, an effect is still observed (although much smaller). How can this be true if this has never been observed?

There can be a number of reasons why this effect has yet to be observed; the first of which would be that most background gravitational waves are low frequency in nature (as low as nanohertz). Such a low frequency, from the equations shown here, would require very long interaction times, with no other external influences to have a noticeable effect. Second, if the objects in question are bathed in a constant impingement of other low (or high) frequency gravitational waves from all directions, the net effect would tend towards zero. Very small particles however may not be so unaffected, as the constant perturbation of waves from all directions may cause its space time motion to be affected in non-trivial ways, since these perturbations and the imparted wave momentum may be large with respect to its size.[3] Finally, the equations here represented a very special, and unrealistic case for the sake of simplicity and first steps. Namely, a monochromatic wave. A real-world gravitational wave would have many frequency components (although would certainly have dominant frequencies). It is unclear how these other wave modes would change the behavior described in this paper. One could suppose that for a strong gravitational wave event, there would be enough of an amplitude difference between the event and the background noise to cause a noticeable trajectory deviation effect if observed under the right circumstances.

At any rate, such an effect as described herein, would be difficult to detect because the particle must already be in motion before and during the wave event, and therefore would propagate away from the observer before a wave could arrive. This is why observing orbiting bodies about earth or in far away galaxies may be plausible for detection, but will require more complicated calculations. All of these considerations will be explored for future work.

## Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

[3] This will be investigated in future work.

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