Parallel Session 6

Conformal Field Theory

Organisers

A. Sen *(TIFR)*

L. Alvarez-Gaumé *(CERN)*

Quantum Group Symmetries of Perturbed Conformal Field Theory[1]

André LECLAIR

Newman Lab. of Nuclear Studies,
Cornell University, Ithaca NY 14853, USA.

I exhibit some new quantum symmetries that exist in a wide variety of integrable quantum field theory in $1+1$ dimensions. The conserved charges are generated by non-local currents. The algebras of conserved charges are the q-deformation of Kac-Moody algebras. S-matrices are completely characterized by the symmetry.

Symmetry is an important tool for obtaining non-perturbative information about quantum field theory. In $3+1$ space-time dimensions the possible symmetries of an S-matrix are severely limited by the Coleman-Mandula theorem. In lower dimensions some of the hypotheses of this theorem can be relaxed in a non-trivial way.

In this talk I will describe some recent work done in collaboration with Denis Bernard on quantum symmetries in $1+1$ dimensional integrable quantum field theory [1]. In $1+1$ dimensions, conserved currents can have non-trivial braiding relations. Generally, these braiding properties arise from the non-locality of the currents. This non-locality has the consequence that the action of the conserved charges on multiparticle states is not simply the sum of the action on each particle separately. This is the primary hypothesis of the Coleman-Mandula theorem that is violated for the currents we consider.

I will describe our results in the specific case of the sine-Gordon theory. However, the formulation we developed is applicable to many other theories.

Consider a conformal field theory perturbed by a relevant operator

$$S = S_{\rm CFT} + \frac{\lambda}{2\pi}\int d^2z\ \Phi_{\rm pert.}(z,\overline{z}). \qquad (1)$$

Let $F(z), \overline{F}(\overline{z})$ be chiral primary fields in the conformal field theory, i.e. satisfying $\partial_{\overline{z}}F = \partial_z\overline{F} = 0$. Furthermore, we suppose that these fields are local with respect to the perturbing field $\Phi_{\rm pert.}$. Zamolodchikov has shown that to first order in perturbation theory [2]:

$$\begin{aligned} \partial_{\overline{z}}\ F(z,\overline{z}) &= \lambda \oint_z \frac{dw}{2\pi i}\ \Phi_{\rm pert.}(w,\overline{z})\ F(z) \\ \partial_z\ \overline{F}(z,\overline{z}) &= \lambda \oint_{\overline{z}} \frac{d\overline{w}}{2\pi i}\ \Phi_{\rm pert.}(z,\overline{w})\ \overline{F}(\overline{z}). \end{aligned} \qquad (2)$$

Now consider the sine-Gordon (SG) theory with the Euclidean action

$$S = \frac{1}{4\pi}\int d^2z\ \partial_z\Phi\partial_{\overline{z}}\Phi + \frac{\lambda}{\pi}\int d^2z\ :\cos\left(\widehat{\beta}\Phi\right): \qquad (3)$$

In the massless limit the chiral components of the SG field are

$$\begin{aligned} \phi(x,t) &= \frac{1}{2}\Big(\Phi(x,t) + \int_{-\infty}^{x} dy\ \partial_t\Phi(y,t)\Big) \\ \overline{\phi}(x,t) &= \frac{1}{2}\Big(\Phi(x,t) - \int_{-\infty}^{x} dy\ \partial_t\Phi(y,t)\Big). \end{aligned} \qquad (4)$$

Using the equation of motion (2), we find the following four conserved currents:

$$\partial_{\overline{z}}J_\pm = \partial_z H_\pm \quad ; \quad \partial_z\overline{J}_\pm = \partial_{\overline{z}}\overline{H}_\pm, \quad (5)$$

$$J_\pm(x,t) = \exp\left(\pm\frac{2i}{\widehat{\beta}}\ \phi(x,t)\right) \qquad (6a)$$

$$H_\pm(x,t) = \lambda \exp\left[\pm i\left(\frac{2}{\widehat{\beta}} - \widehat{\beta}\right)\phi(x,t) \mp i\widehat{\beta}\ \overline{\phi}(x,t)\right]$$

$$\overline{J}_\pm(x,t) = \exp\left(\mp\frac{2i}{\widehat{\beta}}\ \overline{\phi}(x,t)\right) \qquad (6b)$$

$$\overline{H}_\pm(x,t) = \lambda \exp\left[\mp i\left(\frac{2}{\widehat{\beta}} - \widehat{\beta}\right)\overline{\phi}(x,t) \pm i\widehat{\beta}\ \phi(x,t)\right].$$

From these currents we define four conserved charges

$$\begin{aligned} Q_\pm &= \frac{1}{2\pi i}\left(\int dz J_\pm + \int d\overline{z} H_\pm\right) \\ \overline{Q}_\pm &= \frac{1}{2\pi i}\left(\int d\overline{z}\overline{J}_\pm + \int dz \overline{H}_\pm\right). \end{aligned} \qquad (7)$$

In addition to the above conserved currents, the SG theory has the conserved topological current

$$\mathcal{J}^\mu(x,t) = \frac{\widehat{\beta}}{2\pi}\ \epsilon^{\mu\nu}\partial_\nu\ \Phi(x,t), \qquad (8)$$

with the topological charge $\mathcal{T} = \int dx \mathcal{J}^0$.

The conserved currents (6) are non-local due to the non-local expressions for the chiral components of the SG field. They satisfy the following braiding relations

$$\begin{aligned} J_\pm(x,t)\ \overline{J}_\mp(y,t) &= q^{-2}\ \overline{J}_\mp(y,t)\ J_\pm(x,t) \\ J_\pm(x,t)\ \overline{J}_\pm(y,t) &= q^{2}\ \overline{J}_\pm(y,t)\ J_\pm(x,t) \end{aligned} \qquad (9)$$

$\forall x, y$, where

$$q = \exp(-2\pi i/\widehat{\beta}^2) = -\exp(-i\pi/\gamma). \qquad (10)$$

Using these braiding relations and the operator product expansion one finds that the charges satisfy the following algebra

$$\begin{aligned} Q_+\ \overline{Q}_+ - q^2\ \overline{Q}_+\ Q_+ &= 0 && (11a) \\ Q_-\ \overline{Q}_- - q^2\ \overline{Q}_-\ Q_- &= 0 && (11b) \\ Q_+\ \overline{Q}_- - q^{-2}\ \overline{Q}_-\ Q_+ &= a\left(1 - q^{2\mathcal{T}}\right) && (11c) \\ Q_-\ \overline{Q}_+ - q^{-2}\ \overline{Q}_+\ Q_- &= a\left(1 - q^{-2\mathcal{T}}\right) && (11d) \\ [\ \mathcal{T}\ ,\ Q_\pm\ ] &= \pm 2\ Q_\pm && (11e) \\ [\ \mathcal{T}\ ,\ \overline{Q}_\pm\ ] &= \pm 2\ \overline{Q}_\pm, && (11f) \end{aligned}$$

where $a \equiv \lambda/2\pi i$.

This algebra is isomorphic to a q-deformation of the infinite dimensional centerless $sl(2)$ Kac-Moody algebra [3][4]. This isomorphism is

$$\begin{aligned} Q_+ &= c\ E_1\ q^{H_1/2} \quad ; \quad Q_- = c\ E_0\ q^{H_0/2} \\ \overline{Q}_- &= c\ F_1\ q^{H_1/2} \quad ; \quad \overline{Q}_+ = c\ F_0\ q^{H_0/2} \\ \mathcal{T} &= H_1 = -H_0 \end{aligned} \qquad (12)$$

where c is a constant ($c^2 = \frac{\lambda}{2\pi i}(q^{-2} - 1)$), and E_i, F_i, H_i are a Chevalley basis for the quantum affine algebra. The last equation above reflects the fact that the center of $\widehat{sl_q(2)}$ is zero. Note that when $q = -i$, ($\widehat{\beta} = 2/\sqrt{3}$), the algebra (11) is a topological extension of the $N = 2$ supersymmetry algebra. This fact was recently used in the study of perturbations of the $N = 2$ superconformal series [1].

One can now use the above non-local currents to obtain interesting non-perturbative information about the theory. The fields that create SG solitons are required to have topological charge ± 1, and to have well-defined transformation properties with respect to the non-local charges. They are given by

$$\begin{aligned} \Psi_\pm(x,t) &= \exp\left(\pm\frac{i}{\widehat{\beta}}\ \phi(x,t)\right) \\ \overline{\Psi}_\pm(x,t) &= \exp\left(\mp\frac{i}{\widehat{\beta}}\ \overline{\phi}(x,t)\right). \end{aligned} \qquad (13)$$

These fields are chosen such that the non-local charges will transform solitons into anti-solitons and vise versa. This fact is ensured by the operator product expansions

$$\begin{aligned} J_+(z)\ \Psi_-(w) &\sim \frac{1}{(z-w)^{2/\widehat{\beta}^2}}\ \Psi_+(w) + \cdots \\ J_-(z)\ \Psi_+(w) &\sim \frac{1}{(z-w)^{2/\widehat{\beta}^2}}\ \Psi_-(w) + \cdots \end{aligned} \qquad (14)$$

The non-local currents and soliton fields are all characterized by non-trivial Lorentz spin:

$$\frac{1}{\gamma} \equiv \text{spin}\,(Q_\pm) = -\text{spin}\,(\overline{Q}_\pm) = \frac{2}{\widehat{\beta}^2} - 1 \quad (15)$$

$$\text{spin}\,(\Psi_\pm) = -\text{spin}\,(\overline{\Psi}_\pm) = \frac{1}{2\widehat{\beta}^2}. \quad (16)$$

Note that the soliton fields only have spin $1/2$ when $\widehat{\beta} = 1$, which is known to correspond to a free theory. Thus one concludes that the interpretation of the SG solitons as the Thirring fermions is not a unique one. This is perfectly consistent with the fact that the Lorentz spin of a field is not an observable property of a particle in $1+1$ dimensions.

In order to obtain non-perturbative information about the S-matrix one must determine how the non-local charges are represented on asymptotic multisoliton states. We find the following representation on one-particle states:

$$\begin{aligned} Q_\pm &= c\, e^{\theta/\gamma}\, E_\pm\, q^{\pm H/2} \\ \overline{Q}_\pm &= c\, e^{-\theta/\gamma}\, E_\pm\, q^{\mp H/2} \\ \mathcal{T} &= H \end{aligned} \quad (17)$$

where c is a constant, $H = \text{diag}(+1, -1)$ and $E_\pm$ are the Pauli spin matrices $\sigma_\pm$.

To find the action on multiparticle states one must take into account the braiding relations between the currents and the soliton fields:

$$\begin{aligned} J_\pm(x,t)\, \overline{\Psi}_\mathcal{T}(y,t) &= q^{\pm\mathcal{T}}\, \overline{\Psi}_\mathcal{T}(y,t)\, J_\pm(x,t) \\ \overline{J}_\pm(x,t)\, \Psi_\mathcal{T}(y,t) &= q^{\mp\mathcal{T}}\, \Psi_\mathcal{T}(y,t)\, \overline{J}_\pm(x,t) \end{aligned} \quad (18)$$

$\forall x, y$. The action on two particle states defines the comultiplication Δ:

$$\begin{aligned} \Delta\,(Q_\pm) &= Q_\pm \otimes 1 + q^{\pm H} \otimes Q_\pm & (19a) \\ \Delta\,(\overline{Q}_\pm) &= \overline{Q}_\pm \otimes 1 + q^{\mp H} \otimes \overline{Q}_\pm & (19b) \\ \Delta\,(H) &= H \otimes 1 + H \otimes 1. & (19c) \end{aligned}$$

The last relation follows from the additivity of the topological charge $\mathcal{T}$.

It is remarkable that the soliton S-matrix is completely determined by its symmetry with respect to the above charges:

$$[\check{S}, \Delta(Q_\pm)] = [\check{S}, \Delta(\overline{Q}_\pm)] = [\check{S}, \Delta(H)] = 0. \quad (20)$$

The minimal solution to the above equations is the known result [5].

To summarize, I have shown that the SG theory can be non-perturbatively defined by using the non-local symmetry. It is interesting that a manifestly local theory such as the SG theory can be characterized by such symmetries. These results generalize to the affine-Toda theories and numerous others.

Acknowledgements

I would like to thank my collaborator D. Bernard on this work, and the organizers of this session for the opportunity to present these results here.

References

[1] D. Bernard and A. LeClair, Phys. Lett. B247 (1990) 309; 'Quantum Group Symmetries and Non-Local Currents in 2D QFT', in preparation.

[2] A. B. Zamolodchikov, Int. Journ. of Mod. Phys. A4 (1989) 4235; in Adv. Studies in Pure Math. vol. 19 (1989) 641.

[3] M. Jimbo, Lett. Math. Phys. 10 (1985) 63; Lett. Math. Phys. 11 (1986) 247; Commun. Math. Phys. 102 (1986) 537.

[4] V. G. Drinfel'd, Sov. Math. Dokl. 32 (1985) 254; Sov. Math. Dokl. 36 (1988) 212.

[5] A. B. Zamolodchikov and Al. B. Zamolodchikov, Annals Phys. 120 (1979) 253.

CONTINUOUS AND DISCRETE SYMMETRY FROM CONFORMAL FIELD THEORY

L. DOLAN

Department of Physics, University of North Carolina
Chapel Hill, North Carolina 27599-3255, USA

ABSTRACT

Locality is used to give an explicit construction of twisted conformal field theory. This fixes both the continuous and discrete symmetries of the theory. In general, all weight one conformal fields close to form an affine Kac-Moody algebra, whose zero modes generate the continuous symmetry group. For the Z_2-twisted bosonic theory associated with the Leech lattice, there is no continuous symmetry and the discrete symmetry is the largest finite simple group, the Monster F_1. For Z_N-twisted fermion conformal fields, the weight one-half and a subset of the weight one fields form a twisted super Kac-Moody algebra, whose semidirect product with the super Virasoro algebra has zero mode commutators equivalent to those of the untwisted super Kac-Moody algebra.

1. Introduction

Both continuous and discrete symmetries of the dynamics of particle interactions are fixed by conformal field theory (CFT), when string theory is used to describe nature. In this talk, the concise framework of consistent twisted conformal field theory is reviewed[1]. The explicit construction of the Z_2-bosonic theory associated with a d-dimensional momentum lattice, provides in the case of the Leech lattice, the natural module[2] of the Monster group[3]. It's triality element is identified, and is seen to be a generic feature of twisted conformal field theory[4]. The construction of the vertex operators, i.e. the conformal fields, for all the states is then extended to Z_2-twisted fermionic conformal field theory. The weight one-half states and a subset of all the weight one states in a Z_N-twisted CFT form a twisted super Kac-Moody algebra, whose semidirect product with the super Virasoro algebra is shown to have zero mode commutators *identical* to those of the untwisted super algebra. This follows from the fact that, in general, in the Z_N-twisted sectors, the intertwining relations of the vertex operators requires a shift in the definition of not only the Virasoro generator (which is familiar from the Ramond, i.e. Z_2-twisted sector), but also the super Virasoro generator and the Kac-Moody generators associated with the Cartan subalgebra[5].

This analysis is useful in studying the detailed properties of the vertex operators in superconformal field theory. In particular, a viable low-energy phenomenology predicted by Type II superstrings in four-dimensions would be extremely economical, and thereby have a good chance to offer a precise connection[6−10] between string theory and the standard model. The spontaneous breakdown of space-time supersymmetry and the associated non-vanishing vacuum expectation value of the dilaton field provides a possible resolution of the presently[11] "missing quark doublet" in Type II. Even without supersymmetry breaking, a more thorough investigation and the explicit construction of the conformal fields in the context of a consistent local theory may well indicate that one should take the string more seriously, i.e. that the (supersymmetric) standard model is the ground state of a conformal field theory. In this case, the discrete symmetries of the CFT will be responsible for the absence of baryon and lepton violating interactions[12], since unlike the conventional standard model, it is known that its supersymmetric versions require additional symmetries to eliminate such interactions.

2. Twisted Bosonic CFT

The twisted conformal field theory $\widetilde{\mathcal{H}}(\Lambda)$ associated with a lattice Λ of dimension d is defined for a Z_2 reflection twist by keeping the $\theta = 1$ subset of the states created by integrally-moded bosonic operators a^j_m, $1 \leq j \leq d$, $m \in \mathbf{Z}$ from momentum states $|\lambda\rangle, \lambda \in \Lambda$. Here $\theta|\lambda\rangle = |-\lambda\rangle$ and $\theta a^j_m \theta^{-1} = -a^j_m$. To this we add in the $\theta = 1$ subspace of the space $\mathcal{H}_T(\Lambda)$ generated from an irreducible representation space $\mathcal{X}_0(\Lambda)$ for the gamma matrix algebra $\{\gamma_\lambda : \lambda \in \Lambda\}$ associated with Λ, by half-integrally moded oscillators c^j_r, $1 \leq j \leq d$, $r \in \mathbf{Z} + \frac{1}{2}$. In this case, the involution θ is defined by $\theta c_r^j \theta^{-1} = -c_r^j$. The oscillators satisfy the commutation relations $[a_m^i, a_n^j] = m\delta_{m,-n}\delta^{ij}$, $[c_r^i, c_s^j] = r\delta_{r,-s}\delta^{ij}$, and $[a_m^i, c_r^i] = 0$. From the locality requirement of Eq.(12), we find that the twisted CFT is bosonic and meromorphic provided that $d = \dim\Lambda$ is a multiple of eight and both $\sqrt{2}\Lambda^*$ and Λ are even, a condition implied by self-duality of the lattice. If Λ is the $d = 24$ Leech lattice, then $\widetilde{\mathcal{H}}(\Lambda)$ is the natural module for the Monster group.

In the twisted CFT, the untwisted and twisted sectors of $\widetilde{\mathcal{H}}(\Lambda)$ are the subspaces $\mathcal{H}^+(\Lambda)$ and $\mathcal{H}_T^+(\Lambda)$ on which $\theta = 1$. If $\psi \in \mathcal{H}^+(\Lambda)$, $\mathcal{V}(\psi, z)$ maps $\mathcal{H}^+(\Lambda) \to \mathcal{H}^+(\Lambda)$ and $\mathcal{H}_T^+(\Lambda) \to \mathcal{H}_T^+(\Lambda)$ whereas $\mathcal{V}(\chi, z)$ maps $\mathcal{H}^+(\Lambda) \to \mathcal{H}_T^+(\Lambda)$ and $\mathcal{H}_T^+(\Lambda) \to \mathcal{H}^+(\Lambda)$ if $\chi \in \mathcal{H}_T^+(\Lambda)$. Thus we can write these vertex operators in matrix form

$$\mathcal{V}(\psi, z) = \begin{pmatrix} V(\psi, z) & 0 \\ 0 & V_T(\psi, z) \end{pmatrix} \qquad (1)$$

$$\mathcal{V}(\chi, z) = \begin{pmatrix} 0 & \overline{W}(\chi, z) \\ W(\chi, z) & 0 \end{pmatrix} \qquad (2)$$

In this notation, the vertex operators of the twisted CFT $\widetilde{\mathcal{H}}(\Lambda)$ are given by, for the untwisted states: $\psi = \left(\prod_{a=1}^M a_{-m_a}^{j_a}\right)|\lambda\rangle$,

$$\begin{aligned} V(\psi, z) &= \sum_{\lambda' \in \Lambda} \langle \lambda' | : e^{F(-z)} : |\psi\rangle \sigma_{\lambda'} \qquad (3) \\ &= : \left(\prod_{a=1}^M \frac{i}{(m_a - 1)!} \frac{d^{m_a} X^{j_a}(z)}{dz^{m_a}} \right) \exp\{i\lambda \cdot X(z)\} \sigma_\lambda : \end{aligned}$$

and

$$\begin{aligned} V_T(\psi, z) &= V_T^0(e^{\Delta(z)}\psi, z) \\ &= \sum_{\lambda' \in \Lambda} \gamma_{\lambda'} \langle \lambda' | : e^{B(-z)} : e^{A(-z)} |\psi\rangle \, , \qquad (4) \end{aligned}$$

where

$$\begin{aligned} V_T^0(\psi, z) &= \sum_{\lambda' \in \Lambda} (4z)^{-\frac{1}{2}\lambda'^2} \gamma_{\lambda'} \langle \lambda' | : e^{B(-z)} : |\psi\rangle \\ &=: \left(\prod_{a=1}^M \frac{i}{(m_a - 1)!} \frac{d^{m_a} R^{j_a}(z)}{dz^{m_a}} \right) \exp\{i\lambda \cdot R(z)\} : \\ &\quad \cdot (4z)^{-\frac{1}{2}\lambda^2} \gamma_\lambda \qquad (5) \end{aligned}$$

and

$$X^j(z) = q^j - ip^j \log z + i \sum_{n \neq 0} \frac{a_n^j}{n} z^{-n} \, , \qquad (6)$$

$$R(z) = i \sum_{r=-\infty}^{\infty} \frac{c_r}{r} z^{-r} \, ; \qquad (7)$$

and for the twisted states: $\chi = \left(\prod_{a=1}^M c_{-m_a}^{j_a}\right) \mathcal{X}_0$, the analogue of the fermion emission operator is

$$W(\chi, z) = e^{zL_{-1}^c} \tilde{W}(\psi, z) \, , \qquad (8)$$

where

$$\tilde{W}(\chi, z) = \sum_{\lambda \in \Lambda} \gamma_\lambda \langle \lambda | : e^{B(z)} : e^{A(z)} |\chi\rangle \, , \qquad (9)$$

and

$$\overline{W}(\chi, z) = z^{-2h_\chi} W(e^{z^* L_1^c} \overline{\chi}, 1/z^*)^\dagger \, . \qquad (10)$$

In the above expressions we define

$$\Delta(z) = \frac{1}{2} \sum_{\substack{m,n \geq 0 \\ (m,n) \neq (0,0)}} \binom{-\frac{1}{2}}{m} \binom{-\frac{1}{2}}{n} \frac{z^{-m-n}}{m+n} a_m \cdot a_n \, . \qquad (11)$$

Expressions for A(z), B(z), and F(z) are also written as bilinears in oscillators and are given in Ref.[1,5]. Note that the special state ψ_L is given in these CFT's by $\frac{1}{2} a_{-1} \cdot a_{-1} |0\rangle$ and that its vertex operator is $L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} : a_m \cdot a_{n-m} :$ from Eq.(3), and $L_n^c = \frac{1}{2} \sum_{m=-\infty}^{\infty} : c_m \cdot c_{n-m} : + \frac{d}{16}$ from Eq.(4). The cocycle operators σ_λ and γ_λ on the untwisted and twisted sectors respectively are defined and discussed comprehensively in Ref. [1]. The locality relation satisfied by the vertex operators is:

$$\mathcal{V}(\psi, z) \mathcal{V}(\phi, \zeta) = \mathcal{V}(\phi, \zeta) \mathcal{V}(\psi, z) \, . \qquad (12)$$

3. Twisted Fermionic CFT

The space of states for the Z_2-twisted fermionic theory, $\widetilde{\mathcal{H}}$, is obtained by starting with the states of the untwisted Neveu-Schwarz theory, $\mathcal{H}$, adding in a twisted Ramond sector, $\mathcal{H}_T$, and keeping only the subspace of each defined by $\theta = 1$, with $\theta^2 = 1$. The states of the untwisted theory are generated by the action of d infinite sets of half-integrally moded oscillators, b_s^j, $1 \leq j \leq d$, on the vacuum state, Ψ_0. The twisted sector is obtained from the action of d infinite sets of integrally moded oscillators, d_n^j, on the twisted ground states which form a $2^{d/2}$ irreducible representation, $\mathcal{X}_0$, of the gamma matrix Clifford algebra, $\{\gamma^j\}$. The involution θ is defined

$\theta b^i_s \theta^{-1} = -b^i_s$, and on the twisted space, $\mathcal{H}_T$, by $\theta|0\rangle^{\pm}_R = \pm|0\rangle_R$, $\theta d^i_n\theta^{-1} = -d^i_n$, where $\mathcal{X}_0 = |0\rangle^+_R + |0\rangle^-_R$ and we are assuming d is a multiple of 8, (which is necessary for the spectrum of L^d_0 to contain half-integral values).

This theory consists of fermionic and bosonic fields. As in the bosonic case, the conformal field theories discussed here are defined on the complex plane, or rather the Riemann sphere, and are *chiral, i.e.* holomorphic. In this case, the intertwining relation (12) is generally defined by

$$\mathcal{V}(\psi,z)\mathcal{V}(\phi,\zeta) = \epsilon_{\psi\phi}\mathcal{V}(\phi,\zeta)\mathcal{V}(\psi,z) \qquad (13)$$

in the sense of analytic continuation, where $\epsilon_{\psi\phi} = 1$ if either of the states ψ or ϕ are bosons, and $\epsilon_{\psi\phi} = -1$ if both of them are fermions. We will construct (in the F_1- picture) the vertex operators $\mathcal{V}(\psi,z)$, i.e. conformal fields which are in one-to-one correspondence with a basis of states for the theory:

$$\mathcal{V}(\psi,z)|0\rangle = e^{zL_{-1}}\psi. \qquad (14)$$

In the language of superconformal field theory, these vertex operators are the lower components of the superfields. Here $|0\rangle \equiv \Psi_0$ is the vacuum and L_{-1} one of the moments of the special vertex operator $V(\psi_L,z) = \sum_n L_n z^{-n-2}$, which satisfy the Virasoro algebra: $[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{24}m(m^2-1)\delta_{m,-n}$, where m,n run over the integers, $L^\dagger_n = L_{-n}$, and $L_n|0\rangle = 0$ for $n \geq -1$.

The oscillators satisfy the anti-commutation relations $\{b^i_r, b^j_s\} = \delta^{ij}\delta_{r,-s}$, $\{d^i_m, d^j_n\} = \delta^{ij}\delta_{m,-n}$, and $\{b^i_r, d^j_n\} = 0$, where $b^{j\,\dagger}_s = b^j_{-s}$, $b^j_s|0\rangle = 0$, $s > 0$, $d^{j\,\dagger}_n = d^j_{-n}$, $d^j_n|0\rangle_R = 0$, $n > 0$. In these theories, the special state ψ_L is given by $\frac{1}{2}b_{-\frac{3}{2}}\cdot b_{-\frac{1}{2}}|0\rangle$ and that its vertex operator is defined by $L_n = \frac{1}{2}\sum_{s=-\infty}^{\infty}(\frac{1}{2}n - s) : b_s\cdot b_{n-s} :$ in the untwisted sector from Eq. (15), and by $L^d_n = \frac{1}{2}\sum_{m=-\infty}^{\infty}(\frac{1}{2}n - m) : d_m \cdot d_{n-m} : + \frac{d}{16}\delta_{n0}$ in the twisted sector from Eq. (16).

The states in the Neveu-Schwarz sector are given by $\psi = \left(\prod_{a=1}^{M} b^{j_a}_{-s_a}\right)|0\rangle$, where each s_a is a positive half-odd integer, and the product is understood to be written down in a definite order, *e.g.* left to right, in order to avoid a sign ambiguity,, and each oscillator occurs at most once. The vertex operators for these states with $(m_a = s_a - \frac{1}{2})$ are given by

$$\begin{aligned} V(\psi,z) &= :\left(\prod_{a=1}^{M}\frac{1}{m_a!}\frac{d^{m_a}b^{j_a}(z)}{dz^{m_a}}\right): \\ &= \langle 0'| : e^{F(-z)} : |\psi\rangle, \end{aligned} \qquad (15)$$

where we have introduced the Neveu-Schwarz fermion conformal fields $b^j(z) = \sum_{s=-\infty}^{\infty} b^j_s z^{-s-\frac{1}{2}}$ and

$$\begin{aligned} V_T(\psi,z) &= V^0_T(e^{\Delta(z)}\psi, z)) \\ &= \langle 0| : e^{B(-z)} : e^{A(-z)}|\psi\rangle, \end{aligned} \qquad (16)$$

where

$$\begin{aligned} V^0_T(\psi,z) &=: \left(\prod_{a=1}^{M}\frac{1}{(m_a)!}\frac{d^{m_a}d^{j_a}(z)}{dz^{m_a}}\right): \\ &= \langle 0| : e^{B(-z)} : |\psi\rangle, \end{aligned} \qquad (17)$$

with the Ramond fermion fields defined as

$$d^j(z) = \sum_{n=-\infty}^{\infty} d^j_n z^{-n-\frac{1}{2}}. \qquad (18)$$

For the twisted states $\chi = \left(\prod_{a=1}^{M} d^{j_a}_{-m_a}\right)|0\rangle^{\pm}_R$, the fermion emission operator is

$$W(\chi,z) = e^{zL^d_{-1}}\tilde{W}(\psi,z), \qquad (19)$$

where

$$\tilde{W}(\chi,z) = \langle 0| : e^{B(z)} : e^{A(z)}|\chi\rangle. \qquad (20)$$

In the above expressions, define $A(z) = \Delta(-z)$ where

$$\Delta(z) = \frac{1}{4}\sum_{r,s>0}\binom{-\frac{1}{2}}{r-\frac{1}{2}}\binom{-\frac{1}{2}}{s-\frac{1}{2}}\frac{r-s}{r+s}z^{-r-s}b_r\cdot b_s. \qquad (21)$$

Similar expressions for B(z) and F(z) are also written as bilinears in oscillators and are given in Ref.[5]. In general these "lower component" vertex operators are not meromorphic, for *eg.* $V_T(b^j_{-\frac{1}{2}}|0\rangle, z) = d^j(z) = \sum_n d^j_n z^{-n-\frac{1}{2}}$. Therefore although the intertwining relation is satisfied: $V_T(b^j_{-\frac{1}{2}}|0\rangle, z)W(|0\rangle^+_R,\zeta) = W(|0\rangle^+_R,\zeta)V(b^j_{-\frac{1}{2}}|0\rangle, z)$, the operator product expansion $\tilde{W}(|0\rangle^+_R,\zeta)V(b^j_{-\frac{1}{2}}|0\rangle$ is double valued.

4. Twisted Super Kac-Moody Algebra

In fermionic conformal field field theory, the weight one-half and a subset of the weight one fields may form the "lower" and "upper" components of massless superfields, which are the vertex operators for the massless states in the F_1 and F_2-pictures respectively. The moments of these conformal fields generate a super Kac-Moody algebra which forms a semi-direct product with the super Virasoro algebra. In the presence of Z_N-twisted fermionic fields, this set of operators will close to form a twisted super Kac-Moody algebra, whose semi-direct product with the super Virasoro algebra is shown to have zero mode commutators identical to those of the untwisted super algebra.[5]

In this case, the weight one-half fields are given by $h^i(e^{2\pi i}z) = e^{i\pi\omega}h^i(z)$ and $h^\alpha(e^{2\pi i}z) = e^{i\pi\omega}e^{-2\pi i\lambda\cdot\alpha}h^\alpha(z)$, where $\omega = 0,1$ for Neveu-Schwarz and Ramond sectors respectively; i, α label the Cartan subalgebra and the roots, respectively of a dimension d semi-simple Lie algebra g; and the Z_N-twist is labelled by the vector λ such that $-\frac{1}{2} \leq \lambda\cdot\alpha \leq \frac{1}{2}$. In analogy with Eq.(21) we find

$$\Delta(z) = \frac{1}{4}\sum_{\substack{r,s>0\\ i}} \begin{pmatrix} -\frac{1}{2} \\ r-\frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ s-\frac{1}{2} \end{pmatrix} \frac{r-s}{r+s} z^{-r-s} b^i_r b^i_s$$
$$-\frac{1}{2}\sum_{\substack{r,s>0\\ \alpha}} (\lambda\cdot\alpha) \begin{pmatrix} \lambda\cdot\alpha-\frac{3}{2} \\ r-\frac{1}{2} \end{pmatrix} \begin{pmatrix} -\lambda\cdot\alpha-\frac{1}{2} \\ s-\frac{1}{2} \end{pmatrix} \frac{z^{-r-s}}{r+s} b^\alpha_r b^{-\alpha}_s \tag{22}$$

A realization of the twisted super Kac-Moody and super Virasoro algebras is given by the vertex operators $H^i(z)$, $E^\alpha(z)$, $L(z)$, and $G(z)$ as follows. The necessity for the shift in the definition of $H^i(z)$ is seen from its identification as $\mathcal{V}(\frac{1}{2}\sum_\alpha \alpha^i b^\alpha_{-\frac{1}{2}} b^{-\alpha}_{-\frac{1}{2}}|0\rangle, z) \equiv H^i(z)$, and from (22) and (16), where $h(z)$ replaces $d(z)$. Here $N(\alpha,\gamma)$ are the structure constants of g in the Cartan-Weyl basis, and $\sum_\alpha \alpha^i\alpha^j = c_\psi\delta^{ij}$. For $\omega = 0$:

$$H^i(z) = \frac{1}{2}\sum_\alpha \alpha^i h^\alpha(z)h^{-\alpha}(z) - \frac{1}{2z}\sum_\alpha \alpha^i\alpha\cdot\lambda \tag{23}$$

$$E^\alpha(z) = \sum_j \alpha^j h^j(z)h^\alpha(z) + \frac{1}{2}\sum_{\substack{\gamma\\ \alpha-\gamma\, root}} N(-\gamma,\alpha)h^\gamma(z)h^{\alpha-\gamma}(z) \tag{24}$$

$$L(z) = \frac{1}{2}\sum_i \frac{dh^i(z)}{dz}h^i(z) + \frac{1}{2}\sum_\alpha \frac{dh^\alpha(z)}{dz}h^{-\alpha}(z) + \frac{1}{4z^2}\sum_\alpha (\alpha\cdot\lambda)^2 \tag{25}$$

$$G(z) = \frac{1}{2\sqrt{c_\psi/2}}\Big(\sum_{i,\alpha} \alpha^i h^i(z)h^\alpha(z)h^{-\alpha}(z) + \frac{1}{3}\sum_{\substack{\alpha,\gamma\\ \alpha-\gamma\, root}} N(-\gamma,\alpha)h^\gamma(z)h^{\alpha-\gamma}(z)h^{-\alpha}(z) - \frac{1}{z}\sum_{i,\alpha}\alpha^i\alpha\cdot\lambda h^i(z)\Big) \tag{26}$$

5. References

1. L. Dolan, P. Goddard, and P. Montague, Nucl. Phys. **B338**, 529 (1990).
2. I. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster* (Academic Press, New York, 1988).
3. R. Griess, Invent. Math. **69**, 1 (1982).
4. L. Dolan, P. Goddard, and P. Montague, Phys. Lett **B236**, 165 (1990).
5. L. Dolan, to appear in *Strings '90*, Texas A&M March 1990; and a review on twisted conformal field theory in Int.Jour.of Mod. Phys. **A**.
6. R. Bluhm and L. Dolan, Phys. Lett. **B169**, 347 (1986).
7. L. Dolan, in *Proceedings of the XXIII International Conference on High Energy Physics. Loken, S., ed. Singapore: World Scientific, pp. 346-350, 1987.*
8. R. Bluhm, L. Dolan, And P. Goddard, Nucl. Phys. **B289**, 364 (1987).
9. R. Bluhm, L. Dolan, And P. Goddard, Nucl. Phys. **B309**, 330 (1988).
10. For a review and other references see *Current Physics - Sources and Comments, Vol. 4: Superstring Construction*, Schellekens, B., ed. Amsterdam: North-Holland, 1989.
11. L. Dixon, V. Kaplunovsky, and C. Vafa, Nucl. Phys. **B294**, 43 (1987).
12. L. Dolan and S. Lau, in preparation.

507

CALABI–YAU MODULI SPACE, EFFECTIVE LAGRANGIANS AND ALGEBRAIC GEOMETRY

S. FERRARA
CERN, Geneva, Switzerland

ABSTRACT

Recent results on the geometry of Calabi–Yau moduli space and the computation of effective couplings for superstrings compactified on such manifolds are reported.

In this talk I will report on recent progress that has been made in the construction of effective theories for four dimensions, resulting from Calabi–Yau (CY) [1] compactifications of critical superstrings. Although the physically more interesting superstrings are the heterotic ones [2], I will talk about the application of cohomology theory and the methods of algebraic geometry for type II superstrings where the interplay of geometry, topology, and local space–time supersymmetry is deeper.

The massless modes for type II strings on (2,2) vacua [3] are suitably described, in the point-field theory limit, by the Kaluza–Klein compactification of D=10, N=2 supergravity on CY vacua [4–6].

For a CY space with SU(3) holonomy, type II superstrings give N=2 space– time supersymmetry in four dimensions, and therefore the massless multiplets are assigned to N=2 supergravity multiplets, which are of three kinds: the graviton multiplet, vector multiplets, and hypermultiplets [4–6].

There are four bosonic degrees of freedom in each of these multiplets: helicities $\lambda = \pm 2$ (graviton) and $\lambda = \pm 1$ (graviphoton) in the graviton multiplet, two scalars and a vector, and four scalars respectively for each vector multiplet and hypermultiplet.

The number of matter (massless) vector multiplets is $h_{(1,1)}$ in type IIA superstrings and $h_{(2,1)}$ in type IIB superstrings, whilst the number of hypermultiplets is $h_{(2,1)}$ is type IIA and $h_{(1,1)}$ in type IIB [1] superstrings.

From results of matter couplings in N=2 supergravity, it follows that vector multiplets contain scalars that are coordinates of a special Kähler manifold ([4] to [8]), whilst hypermultiplets contain scalars that are coordinates of a quaternionic manifold with a special value of the curvature [9].

The four degrees of freedom of each matter multiplets come, at the string level, from the N–S sector (two) and from the R–R sector (two).

The massless scalars coming from the N–S sector correspond to the moduli fields and are common to both type II and heterotic strings. This is why the target-space σ-model for the moduli is constrained to obey the special geometry of N=2 supergravity. Moreover, since the quaternionic geometry of hypermultiplets in type IIA (type IIB) is related to the special geometry of vector multiplets in the chirality-reversed theory (A $\rightarrow$ B), it follows that the quaternionic σ-model is also restricted [5].

The class of quaternionic σ-models compatible with CY compactifications has been called dual quaternionic manifolds, and it was argued that their metric can be explicitly constricted by using three-dimensional duality for N=2 vector multiplets dimensionally reduced [5, 10] from D=4 to D=3.

We may also wonder how general these results are, since we are dealing with compactification at the level of a point-field theory rather than at the string level.

[1]Note that $h_{(1,1)}$ and $h_{(2,1)}$ are the two independent Hodge numbers of a CY threefold.

Since field theory compactification is a reliable approximation only if the size of the 'internal space' is much larger than the string size, we may expect that a field theory discussion only gives reliable formulae for large values of the moduli parameters for the Kähler class deformations since these parameters are related to the size deformations of the internal manifold.

At the string level, these parameters [the (1,1) moduli] (divided by the string size) appear as (the inverse of) coupling constants of the world-sheet σ-model on the CY manifold, and this microscopic interpretation of the moduli parameters allows us to make some general statements [4, 11]. The point-field theory limit is reliable only for a weakly coupled σ-model, and therefore for large values of the (1,1) moduli parameters.

What about a strongly coupled σ-model? The (1,1) moduli parameters form supermultiplets with the internal components of the antisymmetric tensor $B_{\mu\nu}$, with which there is an associated Peccei–Quinn symmetry that is due to the σ-model interpretation of the B-couplings as a topological term.

This symmetry is exact at each finite order of σ-model perturbation theory and is broken only by world-sheet non-perturbative (instantons) phenomena, which are related to non-trivial maps of the world-sheet on the (target-space) CY manifold [12].

Recent investigations of the (2,2) orbifolds [13, 14] and mirror CY manifolds [15–17] have led to the conclusion that these non-perturbative effects are controlled by the quantum duality symmetries of superstrings, which are a generalization of the $R \rightarrow \alpha'/R$ symmetry of toroidal compactifications [18].

What about the other types of moduli that exist in CY manifolds, namely the deformations of the complex structure?

Owing to the product structure of the moduli space for (1,1) and (2,1) moduli—which is also a consequence of their multiplet assignment in type II superstrings—the (2,1) moduli and their σ-model interactions do not depend on the σ-model coupling constants and are expected to be correctly computed for the large size of the manifold [11]. This means that string tree-level results for (2,1) moduli can be exactly computed in the point-field theory limit. We may also say that for (2,1) moduli the classical and quantum moduli spaces coincide, whilst they usually differ for (1,1) moduli.

These results also imply that the field theoretic Yukawa couplings in heterotic strings are exact (up to string-loop non-perturbative phenomena) for $\overline{27}$ families, whilst they get non-perturbative world-sheet corrections for 27 families.

This relation is due to the fact that Yukawa couplings $C_{ijk}(\phi)$ for 27 (or $\overline{27}$) families are geometrical quantities of the moduli space, in the sense that the Kähler curvature of the special geometry of the two (Hodge) Kähler spaces for the (1,1) and (2,1) moduli satisfy the identity [7, 8, 19]

$$R_{i\bar{j}l\bar{k}} = G_{i\bar{j}}G_{l\bar{k}} + G_{i\bar{k}}G_{l\bar{j}} - \mathrm{e}^{2K} C_{ilp} G^{-1p\bar{p}}\, \overline{C}_{\bar{j}\bar{k}\bar{p}} \tag{1}$$

where C_{ilp} is a (completely symmetric tensor) holomorphic section of the line-bundle associated with the U(1) Kähler connection Q in such a way that $\mathrm{d}Q$ is the Kähler form J:

$$Q = i(K_{,i}\, \mathrm{d}x^i - K_{,\bar{i}}\, \mathrm{d}\bar{x}^i), \tag{2}$$

$$G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K, \quad J = \mathrm{d}Q = iG_{i\bar{j}} \mathrm{d}x^i \wedge \mathrm{d}\bar{x}^{\bar{j}}. \tag{3}$$

The relation between (2,2) conformal field theories and CY compactifications gives a precise meaning to the concept of 'quantum' CY space.

In conformal field-theory language, the (1,1) and (2,1) moduli are associated with (2,2) chiral–chiral and chiral–antichiral primary fields [20], so that the two moduli differ only by the sign of the right U(1) charge of the N=2 superconformal algebra. The recent discovery of 'isomorphic' superconformal field theories [15], in which the role of (1,1) and (2,1) moduli is reversed, has led to the concept of a mirror manifold of a given CY manifold, in which the even and odd harmonic-form cohomology classes are interchanged. Note that this is possible only because $h_{(0,0)} = h_{(3,3)} = h_{(3,0)} = h_{(0,3)} = 1$ and $h_{(1,1)} = h_{(2,2)}$.

As a result of the previous observations, the existence of mirror manifolds implies that exact results for the (1,1) moduli of a given CY threefold can be obtained from results for the (2,1) moduli of its mirror image. Therefore quantum stringy effects and quantum duality symmetries are related, through the mirror construction, to classical properties of the manifolds of complex structure deformations, i.e. the moduli space for the (2,1) moduli [16].

Let us apply the deformation theory of the complex structure to the construction of effective Lagrangians of a given CY manifold and its mirror image.

In type II theories, the (2,1) moduli correspond to vector multiplets in type IIB theory and to hypermultiplets in type IIA.

These results, for the mirror manifolds, will also hold provided A–B are also interchanged.

Let us call ψ^α the moduli coordinate for complex structure deformations. The special geometry, derived from N=2 supergravity, implies that there must exist [8] $h+1$ holomorphic sections $L^I(\psi)$ $(I = 0, ..., h)$ of the line-bundle associated with the Hodge–Kähler space [20], whose curvature satisfies Eq. (1).

These holomorphic sections are related to the 'periods' of a (3,0)-form $\Omega(\psi)$, which varies holomorphically with the moduli [6, 21].

Let us call α_A, β^B $(A, B = 0, ..., h_{2,1})$ a (moduli-independent) cohomology basis that is dual to the homology cycles $\mathcal{A}^A, \mathcal{B}_A$:

$$\begin{aligned} \textstyle\int_{\mathcal{A}^B} \alpha_A &= \textstyle\int_{C_3} \alpha_A \wedge \beta^B = -\int_{\mathcal{B}_A} \beta^B = \\ & -\textstyle\int_{C_3} \beta^B \wedge \alpha_A = \delta_A^B. \end{aligned} \tag{4}$$

Then

$$L^A(\psi) = \textstyle\int_{\mathcal{A}^A} \Omega(\psi) = \int_{C_3} \Omega(\psi) \wedge \beta^A, \tag{5}$$

$$F_A(\psi) = i\textstyle\int_{\mathcal{B}_A} \Omega(\psi) = i\int_{C_3} \Omega(\psi) \wedge \alpha_A, \tag{6}$$

from which follows that

$$\Omega(\psi) = L^A(\psi)\alpha_A + iF_A(\psi)\beta^A. \tag{7}$$

Moreover, from the condition

$$\textstyle\int_{C_3} \Omega(\psi) \wedge \frac{\partial \Omega}{\partial \psi^\alpha} = 0 \tag{8}$$

it follows that

$$F_A = \frac{\partial F}{\partial L^A} \qquad \text{with} \qquad F(\lambda L^A) = \lambda^2 F(L^A). \tag{9}$$

Since Ω is a holomorphic section of a line-bundle, i.e. it is defined up to a holomorphic rescaling,

$$\Omega(\psi) \to \Omega(\psi)\ e^{f(\psi)}, \qquad \text{under} \qquad L^A \to L^A e^f. \tag{10}$$

A special gauge, which defines a standard supergravity basis, is a choice of coordinates for which [8]

$$\frac{\partial X^A}{\partial \psi^\alpha} = \delta_\alpha^A, \qquad \text{with} \qquad X^A = L^A / L^0. \tag{11}$$

The Kähler potential for the (3,1) moduli space is the log of the scalar product of the three-form Ω:

$$K(X, \overline{X}) = -\log i \textstyle\int \Omega \wedge \overline{\Omega}, \tag{12}$$

$$\begin{aligned} i\textstyle\int \Omega \wedge \overline{\Omega} &= L^A \overline{F}_A + \overline{L}^A F_A = 4\overline{L}NL \\ &= 2(f + \bar{f}) - (f_A - \bar{f}_A)(X^A - \overline{X}^A), \\ & \quad f(X) = L^{0^{-2}} F(L), \end{aligned} \tag{13}$$

$$N_{AB} = \frac{1}{4}\left(\frac{\partial^2 F}{\partial L^A \partial L^B} + \frac{\partial^2 \overline{F}}{\partial \overline{L}^A \partial \overline{L}^B}\right).$$

For the (2,1) forms we have

$$\Phi_I = \Omega_I - K_I \Omega, \quad \Omega_I = \frac{\partial \Omega}{\partial L^J}, \tag{14}$$

Where [16]

$$K_I = \frac{(\Omega_I, \overline{\Omega})}{(\Omega, \overline{\Omega})} = \frac{(N\overline{L})_I}{\overline{L}NL}, \quad L^I \Phi_I = 0.$$

It then follows that

$$\begin{aligned} G_{I\bar{J}} &= \frac{\partial}{\partial L^I} \frac{\partial}{\partial \overline{L}^J} K \\ &= \frac{1}{\overline{L}NL}\left[N_{IJ} - \frac{(N\overline{L})_I (NL)_J}{\overline{L}NL}\right] \\ &= \frac{1}{(\Omega, \overline{\Omega})} \textstyle\int \varphi_I \wedge \bar{\varphi}_J = -\partial_I \partial_{\bar{J}} \log \overline{L}NL. \end{aligned} \tag{15}$$

An important point coming from Eqs. (4) to (7) is that the cohomology basis is fixed up to a (rigid) symplectic rotation Sp $(b_3; Z)$.

This is the discrete version, since α, β are elements of integral cohomology, of the possible duality transformations of vector multiplets in

N=2 supergravity [5]. More generally, one can say that symplectic transformations of the form

$$\delta L = (B - iD\mathcal{F})L, \qquad (\mathcal{F} = F_{IJ}), \qquad (16)$$

are the coordinate transformations that preserve the special gauge. Note that the cohomology basis transforms as [2]

$$\delta\alpha = -B^T\alpha - C\beta\ , \quad \delta\beta = D\alpha + B\beta, \qquad (17)$$

where B is a $(h_{21}+1)\times(h_{21}+1)$ real matrix and C,D are $(h_{21}+1)(h_{21}+1)$ symmetric real matrices. Equation (17) follows from the fact that $(-iF_I, L^I)$ must transform as in Eq. (16) to make Ω invariant, which implies Eq. (17) by virtue of the fact that $L^I F_{IJ} = F_J$. It is important to give the transformations of the relevant quantities under symplectic changes of the cohomology basis [17, 23]:

$$\begin{aligned} \delta\Phi &= -(B^T - i\mathcal{F}D)\Phi, \quad \delta\Omega = 0, \\ \delta\mathcal{F} &= -iC - B^T\mathcal{F} - \mathcal{F}B + i\mathcal{F}D\mathcal{F}, \qquad (18) \\ \delta K &= 0. \end{aligned}$$

Another relevant quantity for the vector couplings and for the quaternionic manifold is the matrix

$$\mathcal{N}_{IJ} = \frac{1}{4}\overline{\mathrm{F}}_{IJ} - \frac{(NL)_I(NL)_J}{LNL}, \qquad (19)$$

which transforms as

$$\delta\mathcal{N} = i\frac{C}{4} - B^T\mathcal{N} - \mathcal{N}(B - 2iD\mathcal{N}). \qquad (20)$$

The N=2 vector multiplet and hypermultiplet couplings for (2,1) moduli arise from a self-dual five-form in the type IIB theory and from a real three-form in the type IIA theory [23].

The contribution of the massless modes to these two forms on $M_4 \times C_3$ is given by the real part of the following complex forms [17, 22]:

$$\mathcal{F}^- = iF^{-I}(b_I{}^J\Phi_J + c_I\overline{\Omega}), \qquad (21)$$

where F^{-I} is an anti-self-dual two-form in M_4, the field strength of the massless vector fields' superpartners of the (2,1) moduli and the graviphoton, coming from the R–R sector of the type IIB theory; and

$$A^- = \psi_I(d^{IJ}\Phi_J + a^I\overline{\Omega}), \qquad (22)$$

[2]Although we use 'small symplectic transformations' which are not appropriate for integral cohomology, the results we get for the relevant quantities are unaffected by this limitation.

where ψ_I are complex scalar fields (zero-forms) in M_4, and are related to the massless modes coming from the R–R sector of the type IIA theory.

The real self-dual five-form and real three-forms are simply given by Re $\mathcal{F}^-$ and Re A^-.

The coefficients b, c, d, and a are functions of the (2,1) moduli of complex structure deformations and are fixed by the invariance of $\mathcal{F}^-$ and A^- under symplectic changes of the three-form cohomology basis [Eq. (17)]. Their expression [17, 23] is

$$\begin{aligned} b_I{}^J &= \delta_I{}^J - \frac{(N\overline{L})_I\overline{L}^J}{(\overline{L}N\overline{L})}, \quad c_I = \frac{(N\overline{L})_I}{\overline{L}N\overline{L}}, \\ d^{IJ} &= (N^{-1})^{IJ}, \quad a^I = \frac{-L^I}{\overline{L}NL}. \qquad (23) \end{aligned}$$

Equations (21) and (22) are invariant under symplectic changes provided the R–R fields transform as follows:

$$\begin{aligned} \delta F^- &= F^-(B^T - 4i\overline{\mathcal{N}}D), \\ \delta\psi &= -\psi(B + 4iD\mathcal{N}), \qquad (24) \end{aligned}$$

where $\mathcal{N}$ is the matrix given by Eq. (19).

The five-form Re $\mathcal{F}^-$ is conserved in M^4:

$$\partial_\mu \ \mathrm{Re}\ \mathcal{F}^-_{\mu\nu} = 0. \qquad (25)$$

Its periods, along the homology cycles, give the Bianchi identities and the equations of motions of N=2 vector fields coupled to the (2,1) moduli scalars, as is obvious from the identity:

$$\mathcal{F}^- = iF^{-I}\alpha_I - 4i\overline{\mathcal{N}}_{IJ}F^{-J}\beta^I. \qquad (26)$$

For the type IIA theory, by using Eq. (22) and integrating over the CY internal space, it can be shown that we get the same non-linear couplings as are obtained from three-dimensional duality [10], thus giving an explicit construction of the dual quaternionic manifolds [5] in superstring theory.

REFERENCES

1. E. Calabi, *in* Algebraic geometry and topology (Princeton Univ. Press, Princeton, NJ, 1957), p. 78.
   S.T. Yau, Proc. Nat. Acad. Sci. **74** (1977) 1798.
2. P. Candelas, G. Horowitz, A. Strominger and E. Witten, Nucl. Phys. **B258** (1985) 46.
3. W. Boucher, D. Friedan and A. Kent, Phys. Lett. **172B** (1986) 316.
   A. Sen, Nucl. Phys. **B278** (1986) 289 and **B284** (1987) 423.
   L. Dixon, D. Friedan, E. Martinec and S.H. Shenker, Nucl. Phys. **B282** (1987) 13.
   T. Banks and L. Dixon, Nucl. Phys. **B307** (1988) 93.
   D. Gepner, Nucl. Phys. **B296** (1988) 757 and Phys. Lett. **199B** (1987) 380.
   W. Lerche, D. Lüst and A.N. Schellekens, Nucl. Phys. **B287** (1987) 477.
4. N. Seiberg, Nucl. Phys. **B303** (1988) 206.
5. S. Cecotti, S. Ferrara and L. Girardello, Int. J. Mod. Phys. **4** (1989) 2475 and Phys. Lett. **B213** (1988) 443.
6. S. Ferrara and A. Strominger, *in* Strings 89, eds. Arnowitt, Bryan, Duff, Nanopoulos and Pope (World Scientific, Singapore, 1989), p. 245.
7. E. Cremmer, C. Kounnas, A. van Proeyen, J.P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, Nucl. Phys. **B250** (1985) 385.
8. A. Strominger, Commun. Math. Phys. **133** (1990) 163.
   L. Castellani, R. D'Auria and S. Ferrara, Phys. Lett. **B241** (1990) 57, and Classical and Quantum Gravity **7** (1990) 1767.
9. J. Bagger and E. Witten, Nucl. Phys. **B222** (1983) 1.
10. S. Ferrara and S. Sabharwal, Nucl. Phys. **B332** (1990) 317.
11. M. Dine, P. Huet and N. Seiberg, Nucl. Phys. **B322** (1989) 301.
12. M. Dine, N. Seiberg, X.G. Wen and E. Witten, Nucl. Phys. **B278** (1986) 769 and **B289** (1987) 319.
    M. Dine and N. Seiberg, Nucl. Phys. **B301** (1988) 357.
13. E.J. Chan, J. Mas, J. Lauer and H.P. Nilles, Nucl. Phys. **B233** (1989) 141; and Munich preprint MPI–PAE/Pth 59/89 (1989).
14. S. Ferrara, D. Lüst and S. Theisen, Phys. Lett. **B233** (1989) 147 and **B242** (1990) 39.
15. B.R. Green and R. Plessler, Harvard Univ. preprint HUTP–89/B241 (1989) and HUTP–89/AO 43 (1989).
    P. Aspinswall, C. Lütken and G. Ross, Univ. Oxford preprint OUTP–90–05P (1990).
    P. Aspinswall and C. Lütken, Univ. Oxford preprint OUTP–90–12P (1990).
    M. Cvetič, J. Molera and B. Ovrut, Pennsylvania State Univ. preprint UPR–0426T (1990).
16. P. Candelas and X. de la Ossa, Univ. Texas (Austin) preprint UTTG–07–90 (1990).
    P. Candelas, X. de la Ossa, P. Green and L. Parkes, Univ. Texas (Austin) preprint UTTG–25–90 (1990).
17. S. Ferrara, M. Bodner and A.C. Cadavid, Phys. Lett. **B247** (1990) 25.
18. V.P. Nair, A. Shapere, A. Strominger and F. Wilczek, Nucl. Phys. **B287** (1987) 402.
    A. Shapere and F. Wilczek, Nucl. Phys. **B320** (1989) 669.
    A. Giveon, E. Rabinovici and G. Veneziano, Nucl. Phys. **B322** (1989) 167.
    S. Ferrara, D. Lüst, P. Shapere and S. Theisen, Phys. Lett. **B225** (1989) 363.
    A. Giveon and M. Porrati, Phys. Lett. **B246** (1990) 54.
19. L. Dixon, V.S. Kaplunovsky and J. Louis, Nucl. Phys. **B239** (1990) 27.
20. E. Witten and J. Bagger, Phys. Lett. **B115** (1982) 202.

21. P. Candelas, P. Green and T. Hübsch, *in* Strings 88, eds. Gates, Preitschof and Seigel (World Scientific, Singapore, 1989); and Univ. Texas (Austin) preprint UTTG-10-89 (1989).

22. M. Bodner, A.C. Cadavid and S. Ferrara, preprint UCLA/90/TEP/58, CERN–TH.5887/90 (1990).

23. M.B. Green and J.H. Schwarz, Phys. Lett. **B122** (1983) 143.
J.H. Schwarz, Nucl. Phys. **B226** (1983) 269.
J.H. Schwarz and P. West, Phys. Lett. **B126** (1983) 301.
P. Howe and P. West, Nucl. Phys. **B238** (1984) 181.

ALL ORDER RESULTS IN STRING THEORY

ENRIQUE ALVAREZ

Theory Division, CERN, 1211 Geneva 23, Switzerland

ABSTRACT

A review is given of some results which can be proved to be valid to all orders in string theory. They include R-duality and β-duality, for toroidal compactifications and thermal strings, respectively, and the critical behaviour, that is, the Hagedorn temperature. Some remarks on non-perturbative effects are also included.

1. R-Duality and β-duality

A most remarkable property of strings is the so-called duality transformation which, in its simplest form, relates physical quantities computed for a toroidal compactification at some radius R, with the same quantities computed at another radius, $\alpha\prime/R$. A related (although slightly more complicated due to the different GSO projections one has to perform at finite temperature) symmetry exists between the free energy computed at a temperature β, and the same quantity computed at a temperature π^2/β.

This symmetry stems from the exchange of winding and momentum modes, and seems thus a very "stringy" property. Its physical meaning could be related with the "generalized string uncertainty principle", posited in [1]; in addition, it could be that this symmetry is spontaneously broken due to gaugino condensation, which is of the utmost importance from the phenomenological point of view (cf. [2] for a discussion of this possibility).

In [3] we succeeded in packing together all soliton contributions, (both for the free energy and for the simplest d-dimensional flat toroidal compactification) without background fields at genus g, in a theta function of order $2g$. By exploiting the well-known properties of the thetas, we were able to show that duality is an exact property of string perturbation theory; to be explicit, if we define the formal series:

$$F(\kappa, R) \equiv \sum_{g=0}^{\infty} \kappa^{2g-2} F_g(R) \quad (1)$$

$$F(\kappa, \beta) \equiv \sum_{g=0}^{\infty} \kappa^{2g-2} F_g(\beta) \quad (2)$$

then the duality transformations are:

$$F(\kappa, R) = F(\kappa^*, R^*) \quad (3)$$

$$\kappa^* \equiv \kappa \alpha\prime^{d/2}/R^d \quad (4)$$

$$R^* \equiv \alpha\prime/R \quad (5)$$

and

$$F(\kappa, \beta) = \pi^2/\beta^2 F(\kappa^*, \beta^*) \quad (6)$$

$$\kappa^* \equiv \kappa\pi/\beta \quad (7)$$

$$\beta^* \equiv \pi^2/\beta \quad (8)$$

2. Critical Behaviour

The density of states of any string theory grows exponentially with the energy; this fact alone implies that strings cannot be at equilibrium at temperatures greater than a critical one, called Hagedorn temperature. When interactions are included, however, most of the excited states are unstable, and the physical question becomes a quantitative one, as to whether they live enough as to persist between two successive interactions. To be precise, a state with width $\Gamma(m)$ has a probability $\exp -\beta\Gamma(m)$ of survive during one mean free time (of order β). This means that if we define

$$\tau \equiv \lim_{m\to\infty} m/\Gamma(m) \tag{9}$$

then, when $\tau = 0$, the interactions render the states so unstable that no critical temperature exists. On the other hand, if τ diverges, we expect that the critical temperature stays unchanged. The marginal case corresponds to $\tau = O(1)$; we expect then numerical modifications to the values of the critical temperature. Remarkably, the widths for closed string states have been estimated in [4] ,with the (numerical) result that $\tau = \infty$. This is in perfect concordance with our own results, to be found in the references [5] , which prove for the bosonic string, and strongly suggest for the heterotic string, that the critical temperature remains the same to all orders in string perturbation theory.

3. Non-perturbative results

We know that the predictions of string perturbation theory cannot be trusted in general, because the perturbative series is divergent and non even Borel summable [6]. Unfortunately, we do not even know whether there is a region (like the asymptotically free regime in QCD) in which these predictions give a good indication of the ingoing physics.

Once we have identified a symmetry of string perturbation theory, it is very important to check whether non-perturbative contributions are likely to break it or not.This research was undertaken for duality in [7], and further pursued for toy models in [8], but the results are inconclusive for the time being. We have been able to show, in particular, that it is possible to define the Ising model in a random lattice in such a way as to preserve Kramers-Warnier duality. This property is of course non-universal (it is already so in the simpler case of standard, non-random lattices), which means that we have to " fine tune " the potential to preserve duality. These results cannot be, unfortunately carried over the corresponding problem in string theory, although work on this is in progress.

Gross and Klebanov [7] claim that the discrete definitions which seem most natural not only break duality (through a Kosterlitz-Thouless phase transition), but also fail to give a correct description of some (apparently) well-established results of string perturbation theory. It is possible in most cases, however, to perform "ad hoc" modifications of the discrete action so as to preserve the perturbative symmetries.

More work is needed, however, before the physical meaning of these non perturbative effects can be unravelled.

ACKNOWLEDGEMENTS

I am grateful to my collaborators, J.L.F. Barbón, T. Ortín and M.A.R. Osario for the good times we had working together, and to L. Alvarez Gaumé, M. Bowick, L. Ibáñez and G. Veneziano for many useful discussions.

REFERENCES

[1] G. Veneziano, CERN preprint CERN TH-5581 (1989);
David J. Gross, Princeton preprint PUPT-1108 (1988).

[2] A. Font, L.E. Ibanez, C. Lüst and F. Quevedo, CERN preprint, CERN-TH-5726/90 (1990).

[3] E. Alvarez and M.A.R. Osorio, *Phys. Rev.* **D40** (1989) 1150.

[4] R.B. Wilkinson, N. Turok and D. Mitchell, Imperial College preprint: TP/88-89-23 (year??).

[5] E. Alvarez, *Phys. Lett.* **B210** (1988) 73.

[6] D.J.Gross and V. Periwal, *Phys. Rev. Lett.* **60** (1988) 2105.

[7] D. Gross and I. Klebanov, Princeton preprint PUPT-1172 (1990);
Z. Yang, *Phys. Lett.* **B243** (1990) 365.

[8] E. Alvarez and J.L.F. Barb ón, work in progress.

VORTICES, MONOPOLES AND LIOUVILLE THEORY

Burt A. OVRUT and Steven THOMAS*

Department of Physics, University of Pennsylvania,
Philadelphia, Pennsylvania 19104-6396, USA

ABSTRACT

We consider applications of recent results describing Kosterlitz-Thouless type phase transitions of vortices and monopoles defined on a two dimensional space of spherical topology. It is argued that bosonic Liouville theory, and its $N = 1$ supersymmetric generalization, are in the strong coupling phase unless $D < 1$. For the $N = 2$ theory, however, it is unlikely that such a restriction is necessary; the theory being in the weak phase for any value of D.

Recently there has been much investigation into the phase structure of Liouville theory and its supersymmetric generalizations.[1,2] In this talk we shall examine the possible role that worldsheet vortices and monopoles play in explaining certain aspects of this phase structure. In reference 3, the theory of vortices and monopoles on a 2-sphere and the corresponding Kosterlitz-Thouless[4] (K-T) phase transition was presented. The critical temperatures at which vortex-antivortex (or monopole – antimonopole) pairs dissociate was shown to equal that of a planar system. This was in spite of the fact that a number of new features, resulting from the presence of a non-flat metric and compact topology, had to be taken into account. In this talk we shall use these results in an attempt to understand the interesting phase structure of bosonic Liouville theory[1,2] and its $N = 1$ and $N = 2$ supersymmetric extensions.

We begin by recalling some relevant facts about the theory of vortices and monopoles moving on a 2-sphere of constant curvature,[3] which we shall need later. Let X be a scalar field on S^2 that is defined modulo $2\pi n$ (that is, X is an angular field) whose action is

$$S[X] = \frac{1}{2}\int d^2x\sqrt{g}g^{\alpha\beta}\partial_\alpha X\partial_\beta X \tag{1}$$

with $g_{\alpha\beta}$ the constant curvature metric on S^2. It is convenient to stereographically project S^2 onto the complex plane in which case

$$ds^2 = \left(1 + \frac{|z|^2}{4r^2}\right)^{-2} dzd\bar{z} \tag{2}$$

where $z, \bar{z}$ are complex coordinates and r is the radius of the sphere. The angular function given by

$$X = q\left\{\mathcal{I}m\ln(z - z_1) - \mathcal{I}m\ln(z - z_2)\right\} \quad , q \in \mathcal{Z} \tag{3}$$

represents the simplest vortex distribution on S^2, where $\mathcal{Z}$ is the set of integers. This is because, unlike the Euclidean plane where one may have a single vortex centered at z_1 described by $q\mathcal{I}m\ln(z - z_1)$, on S^2 there must also be a point z_2 at which an antivortex is present. This can be thought of as vortex charge conservation on S^2. As well as the vortex-antivortex configuration of eq. (3), it is also possible to consider monopole-antimonopole configurations

$$X' = q'\left[\ln\left\{\frac{|z - z_1|}{(1 + \frac{|z_1|^2}{4r^2})^{1/2}}\right\} - \ln\left\{\frac{|z - z_2|}{(1 + \frac{|z_2|^2}{4r^2})^{1/2}}\right\}\right] \tag{4}$$

* Work supported in part by the D.O.E. Contract No. DOE-AC02-76-ERO-3071 and NATO Grant Number 860684.

Again (4) is the simplest monopole distribution one can allow on S^2 for similar reasons to those given above. Unlike the vortex charge q in eq. (3), which must be quantized due to the periodic nature of X, q' in (4) is not, a priori, quantized.

If we think of configurations like those in eqs. (3) and (4) as being due to thermal excitations in the system described by (1), then the grand canonical partition function is found to be

$$Z = \sum_{M,N=0}^{\infty} \frac{(\tilde{z})^N(\tilde{z}')^M}{N!M!} \sum_{q_i,q_i'} \Pi_{i=1}^{N} \int \frac{d^2z_i\sqrt{|g(z_i)|}}{4\pi\epsilon^2r^2} \times \int \frac{d^2z_j'\sqrt{|g(z_j')|}}{4\pi\epsilon^2r^2} e^{-\beta S(z_1,\ldots z_N;z_1',\ldots z_M')} \quad (5)$$

where $|q_i| = q$, $\sum_{i=1}^{N} q_i = 0$ for any N, $|q_i'| = q'$, $\sum_{j=1}^{M} q_j' = 0$ for any M, and

$$S(z_1,\ldots z_N;\ z_1',\ldots z_M') = -\pi\left[\sum_{i,j=1}^{N} q_iq_j \ln\left\{\frac{|z_i-z_j|}{2r(1+\frac{|z_i|^2}{4r^2})^{1/2}(1+\frac{|z_j|^2}{4r^2})^{1/2}}\right\} + \sum_{i,j=1}^{M} q_i'q_j' \ln\left\{\frac{|z_i'-z_j'|}{2r(1+\frac{|z_i'|^2}{4r^2})^{1/2}(1+\frac{|z_j'|^2}{4r^2})^{1/2}}\right\} + 2i\sum_{i=1}^{N}\sum_{j=1}^{M} q_iq_j' \mathcal{I}m \ln(z_i - z_j')\right] \quad (6)$$

In equation (5), $\tilde{z}$ and $\tilde{z}'$ are the fugacities for vortex and monopole pair creation respectively, $\beta = \frac{1}{kT}$, and ϵ is an infinitesimal angular cutoff around each vortex. Single-valuedness of $e^{\beta S(z_1\ldots z_M')}$, as one rotates a monopole of point z_j' through an angle 2π about a vortex at z_i, requires the quantization condition

$$2\pi\beta q_j' = e_j \ , \ \ e_j \in \mathcal{Z} \quad (7)$$

It can be shown[3] that the partition function (5) can be expressed as a functional integral over an effective theory

$$Z = \int [d\tilde{X}] e^{-\beta S_{eff}(\tilde{X})} \ , \ \ \tilde{X} = \beta^{1/2}X \quad (8)$$

where

$$\beta S_{eff}(\tilde{X}) = \int d^2z \Big\{ 2\partial_z\tilde{X}\partial_{\bar{z}}\tilde{X} + \frac{1}{4\pi r^2}\Big[\tilde{z}\epsilon^{\alpha/2-2}(2\sqrt{|g(z)|})^{1-\alpha/4} : \cos 2\pi\beta^{1/2}q(\tilde{X}(z)+\tilde{X}(\bar{z})) : + \tilde{z}'\epsilon^{\alpha'/2-2}(2\sqrt{|g(z)|})^{1-\alpha'/4} : \cos\frac{e}{\beta^{1/2}}(\tilde{X}(z)-\tilde{X}(\bar{z})) : \Big]\Big\} \quad (9)$$

and

$$\alpha = 2\pi\beta q^2 \ \text{ and } \ \alpha' \equiv \frac{e^2}{2\pi\beta} \ . \quad (10)$$

Although we do not explicitly indicate it, we shall assume that path integrals, such as the one given in equation (8), are normalized by dividing by the partition function corresponding to no vortices or monopoles. The relative signs within the argument of each cosine term follows from the fact that the first such term in (9) arises from summing over vortex-antivortex pairs and so must be periodic in $X = X(z) + X(\bar{z})$. The second term arises from summation over monopole-antimonopole pairs which can occur even if X is not a periodic variable, hence the relative $-$ sign. We note that even though this second term is periodic in $X(z) - X(\bar{z})$ this does not imply periodicity in X.

The effect that summation of vortices and monopoles has on the stability of the vacuum of (1) can be summarized by the conformal dimensions (Δ_q, Δ_q) and (Δ_e, Δ_e) of the two kinds of cosine operators in (9) with respect to the free stress tensors $T(z)$ and $\bar{T}(\bar{z})$. One finds

$$\Delta_q = \frac{\alpha}{4} \qquad \Delta_e = \frac{\alpha'}{4} \quad (11)$$

The K-T phase transition induced by either vortices or monopoles (or both) occurs when $\Delta_q = 1$ or $\Delta_e = 1$ [3]. When either $\Delta < 1 (> 1)$ the corresponding operator deformations are relevant (irrelevant) respectively. For relevant deformations, the system is dissociated into free charges while, conversely, irrelevant operators imply a neutral dipole gas. In the former case, fugacities are driven to large values by renormalization group transformations and the vacuum of (1) is destabilized (strong coupling). In the latter they are driven to zero (weak coupling). Before moving on to discuss Liouville theory we comment that the partition function (8) is invariant under vortex-monopole duality[5] defined by

$$\pi\beta \leftrightarrow \frac{1}{4\pi\beta} \ , \ q_i \leftrightarrow e_i \tag{12}$$

Now we wish to see if the ideas presented above have any bearing on the structure of Liouville theory. We shall concern ourselves with the theory defined on a world sheet of spherical topology (tree level) and with the cosmological constant set to zero. These are just the conditions considered in ref. 1, in which the Liouville action (in the conformal gauge) is written as

$$S_L = \left(\frac{25-D}{96\pi}\right) \int d^2x\sqrt{g}\left\{g^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma + \sigma R\right\} \tag{13}$$

where $g_{\alpha\beta}$ is the (fixed) world sheet metric and R the scalar curvature. The normalization of the action (13) ensures that the contribution of σ to the background field Weyl anomaly cancels that from the matter and ghost sectors[1]. Note that the kinetic term for σ is only positive definite for $D < 25$. In the region $D > 25$, which is what we shall now discuss, one way to maintain boundedness of the action is to perform the analytic continuation $\sigma \to i\sigma$. Such a continuation has recently been discussed by Das *et al* [6] where they try to interpret σ as a time coordinate. In this context, the above analytic continuation might be thought of as a Wick rotation to Euclidean time. The action for σ then becomes

$$S_L = \left(\frac{D-25}{96\pi}\right) \int d^2x\sqrt{g}\left\{g^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma - i\sigma R\right\} \tag{14}$$

The continuation described above has produced a very interesting result, namely that e^{-S_L} is periodic for $\sigma \to \sigma \bmod 2\pi n\left(\frac{12}{D-25}\right)$. Here we used the fact that $\frac{1}{4\pi}\int d^2x\sqrt{g}R = 2$, the Euler number of the sphere. Hence, for $D > 25$, σ acts like a compactified coordinate. It is convenient to rescale σ so that it becomes an angular field defined modulo $2\pi n$, like the field X described earlier. Therefore, define

$$\tilde{\sigma} = \left(\frac{D-25}{12}\right)\sigma \tag{15}$$

The action for $\tilde{\sigma}$ is then

$$S_L(\tilde{\sigma}) = \frac{3}{2\pi(D-25)} \times \int d^2x\sqrt{g}\left\{g^{\alpha\beta}\partial_\alpha\tilde{\sigma}\partial_\beta\tilde{\sigma} - \frac{i}{8\pi}\tilde{\sigma}R\right\} \tag{16}$$

Since $\tilde{\sigma}$ is now an angular field we can consider the contributions of both monopole and vortex configurations, equations (3) and (4) respectively, to the partition function of $\tilde{\sigma}$. Their action may be determined by expanding (16) about $\tilde{\sigma}_{cl} + (\tilde{\sigma}_v + \tilde{\sigma}_q)$ where $\tilde{\sigma}_{cl}$ are solutions to the $\tilde{\sigma}$ equations of motion derived from (14). The field $\tilde{\sigma}_q$ represents quantum (or 'spin wave') fluctuations and $\tilde{\sigma}_v$ represents monopole and vortex configurations. The combination $\tilde{\sigma}_v + \tilde{\sigma}_q$ occurs only quadratically in (16) since $\tilde{\sigma}_{cl}$ minimizes $S_L(\tilde{\sigma})$. Moreover, the spin wave excitation decouples from $\tilde{\sigma}_v$ for the same reason that they do in planar systems[3], since the quadratic part of (16) is classically Weyl invariant and has the same form on the sphere or Euclidean plane. (The decoupling is basically due to the fact that $\tilde{\sigma}_q$ is in a topologically trivial sector compared to $\tilde{\sigma}_v$). Therefore the action of $\tilde{\sigma}_v$ is determined to be

$$S_L(\tilde{\sigma}_v) = \frac{3}{2\pi(D-25)} \int d^2x\sqrt{g}g^{\alpha\beta}\partial_\alpha\tilde{\sigma}_v\partial_\beta\tilde{\sigma}_v \tag{17}$$

Comparing this action to that of $\beta S(X)$ given earlier (the Liouville partition function is defined by the functional integral of $e^{-S_L(\tilde{\sigma})}$) we may define an inverse 'temperature' for this system, given by

$$\beta_L = \frac{3}{\pi(D-25)} \ , \ D > 25 \tag{18}$$

Contributions from vortices and monopoles to the Liouville partition function may be summed up as described earlier. The condition that either vortices or monopoles induce a K-T type phase transition are found to be

$$\frac{\alpha_L}{4} = \frac{\pi}{2}\beta_L q^2 = 1 \ , \ q \in \mathcal{Z} \tag{19}$$

$$\frac{\alpha_L'}{4} = \frac{e^2}{8\pi\beta_L} = 1 \ , \ e \in \mathcal{Z} \tag{20}$$

for vortices and monopoles respectively, where in (19) and (20) we have also used the quantization condition (7), with β replaced by β_L. For example, taking $q = \pm 1$, (19) is solved by $D = \frac{53}{2}$ while taking $e = \pm 1$, (20) is solved with $D = 49$. Given that $\frac{\alpha_L}{4}$ and $\frac{\alpha_L'}{4}$ are the dimensions of the vortex and monopole operators, it is clear that there is no region for $D > 25$ where both these operators are irrelevant so that the system appears strongly coupled. This picture is not altered by considering higher values of e and q.

What can we conclude from these results about the region $D < 25$? The only consistent way to make the action (16) valid for this region seems to be again analytic continuation $\tilde{\sigma} \to i\tilde{\sigma}$. This renders $\tilde{\sigma}$ to be a strictly single valued field, so in particular no vortex configuration in $\tilde{\sigma}$ are allowed, only those of monopoles. Even in the absence of vortices, we will assume that the fundamental charge q' on monopoles for $D < 25$ still obeys the quantization condition (7) with β replaced by $\beta_L = 3/\pi(D-25)$ (the sign change in β is irrelevant in (7) since both e and q' can be positive or negative). The condition that monopoles induce a K-T phase transition for $D < 25$ is then given by

$$\frac{3}{2(25-D)} \ (q')^2 = \ 1 \ , \ D < 25 \tag{21}$$

or, using the quantization condition (7),

$$\frac{1}{8}\frac{(25-D)}{3}e^2 = 1 \ , \ D < 25 \ . \tag{22}$$

Taking $e^2 = 1$, we find that $D = 1$ solves (22). Moreover, unlike the situation for $D > 25$, for $D < 1$ there is a stable region where monopoles are bound into dipoles. For $D > 1$ they dissociate and there is instability. Remarkably this stability restriction, $D < 1$, coincides with that found by other methods in the quantization of Liouville theory[1] or of 2−d gravity in the light cone gauge.[2]

At this point it is worth mentioning that other authors have considered divergent field configurations (or 'spikes') in Liouville theory[7,8] and have argued that they are suppressed when $D < 1$. From our point of view these 'spikes' are monopole configurations in the variable $\tilde{\sigma}$. In reference 8 the spikes are in σ with the action as given in equation (13), and it had to be argued that the charge 2 spikes (which diverge as $D \to 1$) were somehow more important than others. In fact, this situation is far from satisfactory because it was not shown that such configurations were topologically stable. Our monopoles have the advantage of being stable (due to charge quantization) and of producing a phase transition at $D = 1$ for the lowest value of $|e|$, namely 1. Johnston in reference 9 has also shown that these spike configurations persist at the 1-loop level in the bosonic and $N = 1$ super Liouville theories.

We end this talk by presenting the results of extending the previous ideas to $N = 1$ and $N = 2$ super Liouville theory[1]. The technical details may be found elsewhere.[9] It turns out that summing over vortices and monopoles produces effective Lagrangians that are of the $N = 1$ and $N = 2$ supersymmetric Sine-Gordon type. For the $N = 1$ case, the super K-T phase transition occurs when

$$\frac{1}{8}\left(\frac{9-D}{2}\right)e^2 = \frac{1}{2} \ , \ e \in \mathcal{Z} \tag{23}$$

For $|e| = 1$ this is again satisfied for $D = 1$. If $D < 1$ we have weak coupling while for $D > 1$ the theory is in a strong coupling phase.

In the $N = 2$ case we find qualitatively different results. Now, $D = 1$ is the equivalent point to $D = 25$ in the bosonic Liouville model. On either side of $D = 1$ we find the theory stable to vortices and/or monopoles. That is, the theory is in a weak coupling regime for any value of D. Again, the stability requirements that $D < 1$ in the $N = 1$ theory and no apparent restriction on D in $N = 2$ theory coincides with restrictions on D obtained by quantization of super Liouville theory[1] or 2−d supergravity in the light-cone gauge.[2]

References

1. J. Distler and H. Kawai, Nucl. Phys. **B321** (1989) 509; J. Distler, Z. Hlousek and H. Kawai, Int. Jour. Mod. Phys. **A5** (1990) 391.
2. A.M. Polyakov, Mod. Phys. Lett. **A2** (1987) 893; V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, Mod. Phys. Lett. **A3** (1988) 819; M.T. Grisaru and R.M. Xu, Phys. Lett. **205B** (1988) 486.
3. B.A. Ovrut and S. Thomas, University of Pennsylvania Preprints UPR-0418T, UPR-0419T.
4. J.M. Kosterlitz and D.J. Thouless, J. Phys. **C6** (1973) 1181; V.L. Berenzinski, Sov. Phys. JETP **32** (1970) 493, **34** (1971) 610.
5. J. Cardy, Nucl. Phys. **B205** (1982) 17; J. Cardy and E. Rabinovici, Nucl. Phys. **B205** (1982) 1.
6. S.R. Das, S. Naik and S.R. Wadia, Mod. Phys. Lett. **A4** (1989) 1043.
7. M. Cates, Europhys. Lett. **8** (1988) 719.
8. D. Johnston, Phys. Rev. **D40** (1989) 3401; Phys. Lett. **B241** (1990) 41.
9. B.A. Ovrut and S. Thomas preprint in preperation.

FEYNMAN PATH INTEGRAL FOR THE SUM OF KAC-MOODY CHARACTERS

BELAL E. BAAQUIE
Physics Department, National University of Singapore
Kent Ridge, Singapore 0511

ABSTRACT

A functional differential realization of the Kac-Moody (KM) algebra yeild a Feynman path integral for the sum over the integrable irreps of the global (and local) characters. A semi-classical expansion yields the Weyl-Kac demoninator as the leading term.

1. INTRODUCTION

Let G be a compact Lie group. The KM-algebra is given by

$$[K_\alpha(\sigma), K_\beta(\sigma')] = ic_{\alpha\beta\gamma}K_\gamma(\sigma)\delta(\sigma-\sigma') + i\frac{k}{2\pi}\delta_{\alpha\beta}\delta'(\sigma-\sigma) \quad (1)$$

where $\sigma \in S^1$.

Let $\theta_\alpha : S^1 \to G$ be the coordinates of the KM-group manifold. Then [1]

$$K_\alpha(\sigma) = f_{\alpha\beta}(\sigma)\frac{\delta}{i\delta\theta_\beta(\sigma)} + \frac{k}{2\pi}F_{\alpha\beta}(\sigma)\theta'_\beta(\sigma) \quad (2)$$

: Exact realization

where

$$f^{-1}_{\alpha\beta} = e_{\alpha\beta} = (ad\theta)^{-1}(1 - e^{ad\theta}) \quad (3)$$

and from cohomology [2]

$$F = (ad\theta)^{-1}(f - 1) \quad (4)$$

The point-split Virasoro generator is given by

$$L_0 = \frac{1}{2k+c_A}\int d\sigma(K_\sigma K_{\sigma+\varepsilon} + h.c.) + \frac{c}{2\varepsilon^2} - \frac{c}{24} \quad (5)$$

where

$c = 2k\dim G/(2k + c_A)$

Let

$T_\alpha = \int d\sigma K_\alpha(\sigma)$: generators of G

Then $[T_\alpha, L_0] = 0$ and (T_I, L_0), $I \in$ center of G is the CSA of the KM-algebra.

Let $\lambda_\alpha(\sigma): S^1 \to G$; an element of the KM-group, dense in the neigborhood of the identity, is given by

$$e^{i\lambda K} \equiv \exp\{i\int d\sigma\ \lambda_\alpha(\sigma)K_\alpha(\sigma)\} \quad (6)$$

The *global* KM-character for an irrep space H_p is ($Re_\tau > 0$)

$$ch[\lambda,\tau,p,k] = tr_{Hp}(e^{i\lambda K}\ e^{-\tau L_0}) \quad (7)$$

The *local* KM-characters is defined for constant maps $\lambda_\alpha(\sigma) = \lambda_\alpha$: coords of G, and

$$ch(\lambda,\tau,p,k) = tr_{Hp}(e^{i\lambda_\alpha T_\alpha}\ e^{-\tau L_0}) \quad (8)$$

To obtain a path integral for ch, we need a coordinate representation of a complete basis of Hp; this seems to be intractable. Consider instead the Hilbert space

$H = \bigoplus_{p:p\psi\leq k} H_p$: irreps of KM-algebra

Then we have the completeness equation [3]

$$1_H = \int D\theta\ |\theta\rangle\langle\theta| \quad (9)$$

where the coordinate eignestate is

$|\theta\rangle = \prod_{\alpha\sigma}|\theta_\alpha(\sigma)\rangle$

Define

$$Z = tr_H(e^{i\lambda K}\ e^{-\tau L_0}) \quad (10)$$

$$= \sum_{p:p\psi\leq k} ch[\lambda,p,k,\tau] \quad (11)$$

2. PATH INTEGRAL FOR GLOBAL CHARACTERS

From completeness equation (9)

$$Z = \int D\theta\ \langle\theta|e^{-\tau L_0}\ e^{i\lambda K}|\theta\rangle \quad (12)$$

Note

$$e^{i\lambda K}|\theta\rangle = e^{i\omega_1[\lambda,\theta]}|\theta.\lambda\rangle \quad (13)$$

where $\theta.\lambda$ is pointwise multiplication in G and [2]

$$\omega_1[\lambda,\theta] = \frac{k}{2\pi}\int_0^\tau dt d\sigma\ \lambda_\alpha(\sigma)F_{\alpha\beta}(t\lambda.\theta)(t\lambda.\theta)'_\beta$$

: 1 co-chain (14)

Hence

$$Z = \int D\theta\ e^{i\omega_1[\lambda,\theta]}\ \langle\theta|e^{-\tau L_0}|\theta.\lambda\rangle \quad (15)$$

Since L_0 is bounded from below

$$\langle\theta|e^{-\tau L_0}|\theta.\lambda\rangle = e^{\frac{c\tau}{24} - \frac{c\tau}{2\varepsilon^2}}\int D\theta e^S \quad (16)$$

with $\theta(0,\sigma) = \theta.\lambda$, $\theta(\tau,\sigma) = \theta$ and S, defined on $(0,\tau) \times S^1$, given by

$$S = -\frac{2k+c_A}{8\pi}\int dtd\sigma \dot{\theta}_\alpha e_{\alpha\beta} e_{\gamma\beta}\dot{\theta}_\gamma + \frac{ik}{2\pi}\int dtd\sigma\ \dot{\theta}_\alpha e_{\alpha\beta} F_{\beta\gamma}\theta'_\gamma \quad (17)$$

Note that S is non-polynomial and that
1) There is no kinetic term for
2) Coupling in σ-direction is topological
3) The last term is point-split
4) S can be considered defined on Riemann surface

3. GLOBAL U(1) KM-CHARACTERS

$c = 1$, $c_A = 0$

$$\omega_1[\lambda,\theta] = \frac{k}{2\pi}\int d\sigma\ \lambda(\sigma)\theta'(\sigma) \quad (18)$$

and

$$S = -\frac{k}{4\pi}\int dtd\sigma(\dot{\theta}_\sigma^2 - i\dot{\theta}_\sigma\theta'_{\sigma+\varepsilon}) \quad (19)$$

Since U(1) KM has no null vectors

$$Z = \sum_{p=-\infty}^{+\infty} ch[\lambda,p,k,\tau] \quad (20)$$

$$\text{Let } \lambda(\sigma) = \lambda_0 + \sum_{n=0} e^{in\sigma}\lambda_n \quad (21)$$

The path integral yields [4]

$$ch[\lambda,p,k,\tau] = \frac{1}{\eta(\tau)} e^{ip\lambda_0} e^{-\frac{\tau p^2}{2}} \exp\{-\frac{k}{2}\sum_{n=1}^{\infty} a_n(\tau)|\lambda_n|^2\} \quad (22)$$

where

$$a_n(\tau) = n\sinh(n\tau)/\{\cosh(n\tau) - 1\} \quad (23)$$

It is an intractable problem to obtain this result solely from the KM-algebra. For $\lambda = \lambda_0$, we recover the local U(1) result.

4. PATH INTEGRAL FOR LOCAL KM-CHARACTERS

Since $[T_a,L_0] = 0$, we can fold-in the $\exp(i\lambda_\alpha T_\alpha)$ term into the time-slicing to obtain a simpler form for Z_{Local}, namely

$$Z_{Local} = e^{\frac{c\tau}{24}-\frac{c\tau}{2\varepsilon^2}}\int D\theta e^{S_L}$$

where S_L is defined on a torus and

$$S_L = -\frac{2k+c_A}{8\pi}\int dtd\sigma(\theta_\alpha e_{\alpha\beta} - \frac{1}{\tau}\lambda_\beta)^2 + \frac{ik}{2\pi}\int dtd\sigma\ \dot{\theta}_\alpha F_{\alpha\beta}\theta'_\beta \quad (24)$$

On exacty performing the path integration, we should obtain an independent derivation of the Weyl-Kac formula given by

$$ch[\lambda,p,k,\tau] = \frac{1}{\Pi}\sum_\ell N(\ell,k,p) e^{\frac{\tau c_2(\ell)}{2k+c_A}}\chi_\ell(\lambda) \quad (25)$$

where

$$\Pi = \prod_{n=1}^{\infty}\det(1 - e^{-n\tau}ad\lambda) \quad (26)$$

Note that Π^{-1} is the generating function for the number of states for a given level of the Verma module.
Recall that Weyl character $\chi_\ell(\lambda)$ has the denominator $\prod_{\alpha>0}\sin(\lambda\alpha/2)$ where α are the positive roots of G.

The Weyl-Kac denominator D is given by

$$D = \prod_{\alpha>0}\sin(\lambda\alpha/2)\prod_{n=1}^{\infty}\det(1 - e^{-n\tau}Ad\lambda) \quad (27)$$

Hence, the semi-classical limit $(k \to \infty)$ for Z_{Local} will yeild D^{-1} as the k-independent term. We obtain, for $k \to \infty$

$$S = -\frac{k}{2\tau}\lambda^2 - \frac{k}{4\pi}\int dtd\sigma(\theta_\alpha^2 - \frac{1}{\tau}\dot{\theta}_\alpha c_{\alpha i\beta}\theta_i\lambda_\beta + i\dot{\theta}_a\theta'_a) + O(\theta^3) \quad (28)$$

In the Weyl-Cartan basis

$$S = -\frac{k}{2\tau}\lambda^2 - \frac{k}{4\pi}\int\{\sum_I \theta_I^2 + \sum_{\alpha>0}\dot{\theta}_{-\alpha}\dot{\theta}_\alpha + \frac{i}{\tau}\sum_{\alpha>0}\lambda\alpha\dot{\theta}_{-\alpha}\theta_\alpha + i\sum_I\dot{\theta}_I\theta'_I + i\sum_{\alpha>0}\dot{\theta}_{-\alpha}\theta'_\alpha\} \quad (29)$$

Path integration yields as expected
5

$$Z_{Local} \approx \frac{e^{-k\lambda^2/2\tau}}{D} \quad (30)$$

The singularities of the path integration are exactly cancelled by the prefactor $e^{-\frac{c\tau}{2\varepsilon^2}}$. The null vectors are subtracted out, as an expansion in 1/k, by the non-linear terms in the action.

5. CONCLUSIONS

1. Can one project out a particular character ch_p from Z? This is possible for U(1).
2. Can one constract modular invariant theories from the global characters?
3. To give an exact derivation, using the semi-classical expansion, of the Weyl-Kac formula from the path integral.

REFERENCES

1. B.E. Baaguie, *Phys.Lett.* B177 (1986) 310; *Nucl.Phys.* B295 (1988) 188.
2. B.E. Baaquie, NUS Report No NUSHEP89/06.
3. D. Gepner and E. Witten, *Nucl.Phys.* B278 (1986) 493.
4. B.E. Baaquie, NUS Report No NUSHEP90/02.
5. B.E. Baaquie, NUS Report No NUSHEP90/01.